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Kalman Filtering with Partial Observation Losses

Xiangheng Liu and Andrea Goldsmith
Department of Electrical Engineering, Stanford University, Stanford, CA 94305-9515, USA.
Email: {liuxh,andrea}@wsl.stanford.edu

***Abstract*— We study the Kalman filtering problem when part or all of the observation measurements are lost in a random fashion. We formulate the Kalman filtering problem with partial observation losses and derive the Kalman filter updates with partial observation measurements. We show that with these partial measurements the Kalman filter and its error covariance matrix iteration become stochastic, since they now depend on the random packet arrivals of the sensor measurements, which can be lost or delayed when transmitted over a communication network. The communication network needs to provide a sufficient throughput for each of the sensor measurements in order to guarantee the stability of the Kalman filter, where the throughput captures the rate of the sensor measurements correctly received. We investigate the statistical convergence properties of the error covariance matrix iteration as a function of the throughput of the sensor measurements. A throughput region that guarantees the convergence of the error covariance matrix is found by solving a feasibility problem of a linear matrix inequality. We also find an unstable throughput region such that the state estimation error of the Kalman filter is unbounded. The expected error covariance matrix is bounded both from above and from below. The results are illustrated with some simple numerical examples.**

I. Introduction

The Kalman filter [1] [2] and its variations are widely used in many different areas. We consider the Kalman filtering problem in a distributed control system setting where different components of the control system communicate over a wireless network. As shown in Figure 1, sensors, plants/actuators and controllers are located at different physical locations and thus require a communication network to exchange critical information for system control. In LQG control, the optimal controller consists of a Kalman filter and a state feedback controller. The Kalman filter uses the sensor measurements to compute the minimum mean square error estimate of the control system state and this state estimate is then used to compute the control command.

$u=-Kx_t$

Wireless networks are playing increasingly important roles in distributed control applications. Wireless technology allows fully mobile operation, fast deployment and flexible installation. However, wireless is a difficult channel due to the limited spectrum, time varying channel gains and interference. Packet losses are inevitable because of collisions and transmission errors. The Kalman filter is a well studied component in control theory with no information loss. In a distributed control system, the Kalman filter collects sensor measurements from different sensors

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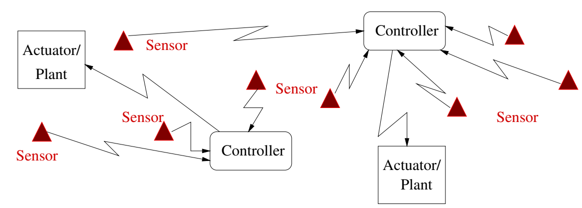

Fig. 1. Distributed Control Systems

and each sensor encodes its own data into an individual packet. The Kalman filter must receive multiple packets to get all observation measurements. However, one or more of the packets can be lost during a given sample period. We are interested in how the Kalman filter updates the state estimate with partial observation losses and how the Kalman filter performs with these random observation losses.

The Kalman filtering problem in the presence of packet losses has recently been studied in [3] [4] [5]. In [3], the statistical convergence properties of the Kalman filter are evaluated assuming the observation measurements are received in full or lost completely. To satisfy this assumption, all the sensor measurements need to be encoded in a single packet and the sensors should be colocated. A critical arrival rate of the observation is shown to exist and both an upper and a lower bound are computed. In [4], [5], the Kalman filter does not update if there is a packet loss. Thus, the sample period becomes random. The convergence properties are discussed under the random sampling, but the results are restricted to scalar systems. In this paper, we generalize [3] by allowing partial packet losses in the observation. Much of this research is along the same line as [3], but allowing partial observation losses introduces new dimensions to the problem. As in [3], the error covariance matrix iteration and the Kalman filter updates are stochastic and depend on the the random arrivals of the sensor measurements.

II. Background

Consider a discrete time linear time-invariant system defined by the system equations:

$$\mathbf{x}_{t+1} = A\mathbf{x}_t + \mathbf{w}_t,$$
$$\mathbf{y}_t = C\mathbf{x}_t + \mathbf{v}_t, \qquad (1)$$

where $\mathbf{x}_t$ is the system state, $\mathbf{y}_t$ is the measurement output, $\mathbf{w}_t$ is the system disturbance, and $\mathbf{v}_t$ is the measurement noise. The subscript t is the time index. Note that all boldface variables in this paper are vectors. Both $\mathbf{w}_t$ and $\mathbf{v}_t$ are assumed to be Gaussian random vectors and their covariance matrices are $Q \geq 0$ and $R > 0$, respectively.

Classical Kalman filtering theory assumes periodic measurement updates, i.e., no packet losses. The estimation error covariance matrices iterate according to the Algebraic Riccati Equation (ARE):

$$P_{t+1} = AP_tA' + Q - AP_tC'(CP_tC' + R)^{-1}CP_tA',$$

where P_t is the minimum mean square error covariance matrix of estimating $\mathbf{x}_t$ based on $\mathbf{y}_0, \cdots, \mathbf{y}_{t-1}$. When (A, Q) is controllable and (A, C) is observable, the ARE converges to a unique positive semidefinite matrix independent of the initial conditions.

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Sinopoli et. al. [3] extend the classical Kalman filtering problem to account for possible observation losses. The observation is either received in full or lost completely. When all the sensor measurements are encoded together and sent over the network in a single packet, the Kalman filter either receives the complete observation if the packet is correctly received or none of the observation if the packet is lost or substantially delayed. The packet delay and the probability of packet loss depend on network conditions such as the channel gain and the network traffic. In control applications, the sensor measurements are delay sensitive and old measurements are often discarded when new measurements are available. Thus, the packet loss is often defined to be delay dependent, that is, a packet is declared lost if it has not been received correctly after a certain time period. In this paper, a packet loss is declared when a packet is not received after one sample period. The random variable γ_t indicates whether the observation at time t is correctly received by the end of the t^{th} sample period. It is assumed that γ_t is i.i.d. Bernouli with $Pr(\gamma_t = 1) = \lambda$. This corresponds to the Binary Symmetric Channel (BSC) with the error probability $1 - \lambda$. For a fixed sampling rate and a given packet size, the throughput of the communication link (from the sensors to the Kalman filter) is the product of the sampling rate, the packet size and the probability λ that a packet is received correctly. For a discrete-time control system, the sampling rate is implicit. We sometimes refer to λ as the link throughput since it is a scaled version of the throughput for a fixed sampling rate and a given packet size. The authors in [3] show that the error covariance matrix iterates according to the following stochastic equation:

$$P_{t+1} = AP_tA' + Q - \gamma_t AP_tC'(CP_tC' + R)^{-1}CP_tA'.$$

Note this is a stochastic iteration due to the random observation losses while in the classical Kalman filter the iteration is deterministic. The convergence of the iteration thus depends on the sample path of $\{\gamma_t\}$. The authors show the existence of a critical value of λ_c such that $\mathrm{E}[P_t]$ is bounded if $\lambda > \lambda_c$ and $\mathrm{E}[P_t]$ goes to infinity as $t \to \infty$ if $\lambda < \lambda_c$. λ_c may not always be found explicitly but both an upper and lower bound can be computed. The steady state estimation error covariance matrix $\lim_{t\to\infty} \mathrm{E}[P_t]$ is bounded both from above and from below for sufficiently high observation arrival rate λ.

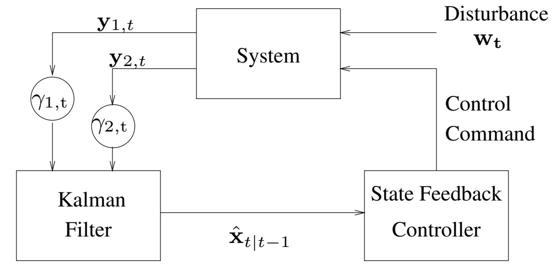

Fig. 2. System Block Diagram

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Our paper extends the results in [3] to allow partial observation losses. We assume $\mathbf{y}_t$ has at least two elements and we partition $\mathbf{y}_t$ into multiple parts where each part can be lost independently. This corresponds to γ_t coming from multiple sensor data packets with some of the packets lost or delayed. For the sake of simplicity, we assume that the observation process $\mathbf{y}_t$ is divided into two parts $\mathbf{y}_{1,t}$ and $\mathbf{y}_{2,t}$, which are sent over two different channels using a shared wireless network. At each step, the Kalman filter may receive $\mathbf{y}_{1,t}$ or $\mathbf{y}_{2,t}$ alone, both, or neither. In the general case, the observation processes can be sent in more than two packets and all the results in this paper extend to this general case.

III. Problem Formulation

We consider a general MIMO discrete time linear system and partition the observation vector $\mathbf{y}_t$ into two parts $[\mathbf{y}_{1,t}; \mathbf{y}_{2,t}]$. The system has the following dynamics:

$$\begin{aligned} \mathbf{x}_{t+1} &= A\mathbf{x}_t + \mathbf{w}_t, \\ \begin{bmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \end{bmatrix} &= \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \mathbf{x}_t + \begin{bmatrix} \mathbf{v}_{1,t} \\ \mathbf{v}_{2,t} \end{bmatrix}, \end{aligned} \tag{2}$$

where $\mathbf{x}_t \in \mathcal{R}^n, \mathbf{y}_{1,t}, \mathbf{v}_{1,t} \in \mathcal{R}^{m_1}$, and $\mathbf{y}_{2,t}, \mathbf{v}_{2,t} \in \mathcal{R}^{m_2}$. The system matrices are of the appropriate dimensions. The covariance matrices of $\mathbf{v}_{1,t}$ and $\mathbf{v}_{2,t}$ are R_{11} and R_{22} respectively. Comparing with the system dynamics in Eqn. (1), we have $\mathbf{y}_t = \begin{bmatrix} \mathbf{y}_{1,t} \\ \mathbf{y}_{2,t} \end{bmatrix}$, $C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and $R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$. We assume the system (A, C) is observable thus the Kalman filter converges without sensor measurement losses.

Our scenario is shown in Figure 2. The measurement outputs $\mathbf{y}_{1,t}$, $\mathbf{y}_{2,t}$ are encoded separately and sent over different wireless channels every time step. We use $\gamma_{i,t}$ to indicate whether $\mathbf{y}_{i,t}$ is received correctly in time step t. We assume $\gamma_{1,t}$ and $\gamma_{2,t}$ are i.i.d. Bernoulli random variables with $\Pr(\gamma_{1,t} = 1) = \lambda_1$ and $\Pr(\gamma_{2,t} = 1) = \lambda_2$. Note that λ_1 and λ_2 represent the percentage of the sensor measurement packets that are correctly received. Also note that λ_i, $i = 1, 2$, is proportional to the link throughput from sensor i to the Kalman filter. We refer to the pair (λ_1, λ_2) as the network throughput, which depends on the channel gains, the network traffic and the network resource allocation (such as power, time slots, etc.).

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We also assume that $\gamma_{1,t}$ and $\gamma_{2,t'}$ are independent for every t and t'. Thus, $\mathbf{y}_{1,t}$ and $\mathbf{y}_{2,t}$ can be independently lost or received. When an observation is lost, it is equivalent to receiving the measurement with an infinite noise variance. The measurement noise $\mathbf{v}_{i,t}$ is assumed to have the following conditional probability density distribution:

$$p(\mathbf{v}_{i,t}|\gamma_{i,t}) \sim \begin{cases} \mathcal{N}(0, R_{ii}) & \text{if } \gamma_{i,t} = 1, \\ \mathcal{N}(0, \sigma_i^2 I) & \text{if } \gamma_{i,t} = 0, \end{cases} \tag{3}$$

where we take $\sigma_i^2 \to \infty$ when the observation $\mathbf{y}_{i,t}$ is lost.

Let $\gamma_t = [\gamma_{1,t}; \gamma_{2,t}]$, $\gamma_0^t = \{\gamma_0, \cdots, \gamma_t\}$, and $\mathbf{y}_0^t = \{\mathbf{y}_0, \cdots, \mathbf{y}_t\}$. We define

$$\begin{aligned} \hat{\mathbf{x}}_{t|t} &\equiv \mathrm{E}[\mathbf{x}_t|\mathbf{y}_0^t, \gamma_0^t], \\ P_{t|t} &\equiv \mathrm{E}[(\mathbf{x}_t - \hat{\mathbf{x}}_{t|t})(\mathbf{x}_t - \hat{\mathbf{x}}_{t|t})'|\mathbf{y}_0^t, \gamma_0^t], \\ \hat{\mathbf{x}}_{t+1|t} &\equiv \mathrm{E}[\mathbf{x}_{t+1}|\mathbf{y}_0^t, \gamma_0^t], \\ P_{t+1|t} &\equiv \mathrm{E}[(\mathbf{x}_{t+1} - \hat{\mathbf{x}}_{t+1|t})(\mathbf{x}_{t+1} - \hat{\mathbf{x}}_{t+1|t})'|\mathbf{y}_0^t, \gamma_0^t]. \end{aligned}$$

The time update of the Kalman filter is independent of the observation process and thus stays the same as in the classical Kalman filter:

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t} &= A\hat{\mathbf{x}}_{t|t}, \\ P_{t+1|t} &= AP_{t|t}A' + Q. \end{aligned} \tag{4}$$

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But the measurement update is now stochastic since the received measurements now depend on the random variables $\gamma_{1,t}$ and $\gamma_{2,t}$.

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1 When $\gamma_{1,t} = 1, \gamma_{2,t} = 1$, the complete observation measurements are received. Thus, the measurement update is the same as the classical Kalman filter:

$$\begin{aligned} &\hat{\mathbf{x}}_{t+1|t+1} = \hat{\mathbf{x}}_{t+1|t} \\ &\quad + P_{t+1|t}C'[CP_{t+1|t}C' + R]^{-1}(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}), \\ &P_{t+1|t+1} = P_{t+1|t} \\ &\quad - P_{t+1|t}C'[CP_{t+1|t}C' + R]^{-1}CP_{t+1|t}. \end{aligned} \tag{5}$$

2 When $\gamma_{1,t} = 0, \gamma_{2,t} = 0$, the optimal measurement update is to run one step open loop. This also corresponds to the case of no observation in [3].

$$\begin{aligned} \hat{\mathbf{x}}_{t+1|t+1} &= \hat{\mathbf{x}}_{t+1|t}, \\ P_{t+1|t+1} &= P_{t+1|t}. \end{aligned} \tag{6}$$

3 When $\gamma_{1,t} = 1, \gamma_{2,t} = 0$, only $\mathbf{y}_{1,t}$ is received by the Kalman filter. The corresponding measurement noise covariance matrix is now

$$\tilde{R} = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & \sigma_2^2 I \end{bmatrix} = R + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2^2 I - R_{22} \end{bmatrix}. \tag{7}$$

With the observation $\mathbf{y}_{1,t}$ only, the Kalman filter updates assuming the noise covariance is $\tilde{R}$:

$$\begin{aligned} &\hat{\mathbf{x}}_{t+1|t+1} = \hat{\mathbf{x}}_{t+1|t} \\ &\quad + P_{t+1|t}C'[CP_{t+1|t}C' + \tilde{R}]^{-1}(\mathbf{y}_{t+1} - C\hat{\mathbf{x}}_{t+1|t}) \\ &P_{t+1|t+1} = P_{t+1|t} \\ &\quad - P_{t+1|t}C'[CP_{t+1|t}C' + \tilde{R}]^{-1}CP_{t+1|t} \end{aligned} \tag{8}$$

Note that

$$\begin{aligned} &C'[CXC' + \tilde{R}]^{-1}C \\ &= C'\left(CXC' + R + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2^2 I - R_{22} \end{bmatrix}\right)^{-1} C \\ &\overset{(a)}{=} C'\left(CXC' + R + \begin{bmatrix} 0 & 0 \\ 0 & \sigma_2^2 I \end{bmatrix}\right)^{-1} C \\ &\overset{(b)}{=} C'\begin{bmatrix} \mathcal{M}_{11} - \mathcal{M}_{12}\mathcal{M}_{22}^{-1}\mathcal{M}_{21} & 0 \\ 0 & 0 \end{bmatrix} C \\ &\overset{(c)}{=} C'\begin{bmatrix} (C_1XC_1' + R_{11})^{-1} & 0 \\ 0 & 0 \end{bmatrix} C \\ &\overset{(d)}{=} C_1'[C_1XC_1' + R_{11}]^{-1}C_1, \end{aligned}$$

where $[CXC' + R]^{-1} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}$, and (a) follows since $\sigma_2^2 \to \infty$, (b) is derived by using the low rank adjustment of the matrix inversion formula [6] and taking $\sigma_2 \to \infty$, (c) is due to the alternative formula of the inverse of a partitioned matrix [6], and (d) is derived by simple multiplication of the partitioned matrices.

Therefore, the measurement update is

$$\begin{aligned} &\hat{\mathbf{x}}_{t+1|t+1} = \hat{\mathbf{x}}_{t+1|t} + P_{t+1|t}C_1' \\ &\quad [C_1P_{t+1|t}C_1' + R_{11}]^{-1}(\mathbf{y}_{1,t+1} - C_1\hat{\mathbf{x}}_{t+1|t}), \\ &P_{t+1|t+1} = P_{t+1|t} \\ &\quad - P_{t+1|t}C_1'[C_1P_{t+1|t}C_1' + R_{11}]^{-1}C_1P_{t+1|t}, \end{aligned} \tag{9}$$

when $\gamma_{1,t} = 1$ and $\gamma_{2,t} = 0$. Note this is equivalent to the classical Kalman filter measurement update if $\mathbf{y}_1$ were the only observation.

Similarly, 4 when $\gamma_{1,t} = 0, \gamma_{2,t} = 1$, the Kalman filter updates as if $\mathbf{y}_2$ were the only observation:

$$\begin{aligned} &\hat{\mathbf{x}}_{t+1|t+1} = \hat{\mathbf{x}}_{t+1|t} + P_{t+1|t}C_2' \\ &\quad [C_2P_{t+1|t}C_2' + R_{22}]^{-1}(\mathbf{y}_{2,t+1} - C_2\hat{\mathbf{x}}_{t+1|t}), \\ &P_{t+1|t+1} = P_{t+1|t} \\ &\quad - P_{t+1|t}C_2'[C_2P_{t+1|t}C_2' + R_{22}]^{-1}C_2P_{t+1|t}. \end{aligned} \tag{10}$$

Let $P_t = P_{t|t-1}$. Combining (4) (5) (6) (9) (10), we get:

$$\begin{aligned} P_{t+1} &= AP_tA' + Q \\ &\quad - \gamma_{1,t}\gamma_{2,t}AP_tC'(CP_tC' + R)^{-1}CP_tA' \\ &\quad - \gamma_{1,t}(1 - \gamma_{2,t})AP_tC_1'(C_1P_tC_1' + R_{11})^{-1}C_1P_tA' \\ &\quad - (1 - \gamma_{1,t})\gamma_{2,t}AP_tC_2'(C_2P_tC_2' + R_{22})^{-1}C_2P_tA'. \end{aligned} \tag{11}$$

The Kalman filter updates become stochastic when part or all of the observation measurements are randomly lost. Due to the stochastic nature, we no longer have a unique deterministic error covariance matrix in the steady state. In the next section, we will focus on the statistical properties of the Kalman filter in the presence of partial observation losses. Our goal is to determine the sufficient information rate to achieve bounded error in the state estimation.

IV. Statistical Properties of the Error Covariance Matrix Iteration

Equation (11) shows that P_{t+1} is a function of random variables $\gamma_{1,t}$ and $\gamma_{2,t}$ given P_t. Thus the sequence $\{P_t\}_{t=0}^{\infty}$

is a random process for a given initial condition P_0. We focus on the statistical properties of the error covariance matrix iteration in Equation (11). Note that P_t is bounded with probability 1 if and only if $\mathrm{E}[P_t]$ is bounded. Therefore, we are interested in the expected error covariance matrix $\mathrm{E}[P_{t+1}] = \mathrm{E}[\mathrm{E}[P_{t+1}|P_t]]$, where the expectation $\mathrm{E}[P_{t+1}|P_t]$ is taken of both $\gamma_{1,t}$ and $\gamma_{2,t}$. Let us define

$$\begin{aligned} g_{\lambda_1\lambda_2}(X) &= AXA' + Q \\ &\quad -\lambda_1\lambda_2 AXC'(CXC'+R)^{-1}CXA' \\ &\quad -\lambda_1(1-\lambda_2)AXC_1'(C_1XC_1'+R_{11})^{-1}C_1XA' \\ &\quad -(1-\lambda_1)\lambda_2 AXC_2'(C_2XC_2'+R_{22})^{-1}C_2XA', \end{aligned} \quad (12)$$

then

$$\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t), \quad (13)$$

and

$$\mathrm{E}[P_{t+1}] = \mathrm{E}[g_{\lambda_1\lambda_2}(P_t)]. \quad (14)$$

In this section, we will first present some useful properties of the function $g_{\lambda_1\lambda_2}(X)$, where $X \geq 0$. All the proofs can be found in [7]. These properties (concavity and mononicity in particular) of the function $g_{\lambda_1\lambda_2}(X)$ allow us to find a lower and an upper bound for the steady state error covariance matrix $\lim_{t\to\infty}\mathrm{E}[P_t]$ that is independent of the initial condition P_0.

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Lemma 1: $g_{\lambda_1\lambda_2}(X)$ is a concave function in X for $X \geq 0$. Thus $\mathrm{E}[g_{\lambda_1\lambda_2}(X)] \leq g_{\lambda_1\lambda_2}(\mathrm{E}[X])$.

Remark: The concavity of the function $g_{\lambda_1\lambda_2}(X)$ is a fundamental property. The inequality in the lemma is derived by directly applying Jensen's inequality. It gives an upper bound of $\mathrm{E}[P_{t+1}]$ as a function of $\mathrm{E}[P_t]$.

Lemma 2: $g_{\lambda_1\lambda_2}(X)$ is a nondecreasing function in X. That is, if $0 \leq X \leq Y$, then $g_{\lambda_1\lambda_2}(X) \leq g_{\lambda_1\lambda_2}(Y)$.

Remark: Let $X_{t+1} = g_{\lambda_1\lambda_2}(X_t)$, $Y_{t+1} = g_{\lambda_1\lambda_2}(Y_t)$ with initial conditions $X_0 \geq 0$ and $Y_0 \geq 0$ respectively. If $X_0 \leq Y_0$, then we have $X_t \leq Y_t$ for every t since $g_{\lambda_1\lambda_2}(X)$ is a nondecreasing function in X. This monotonicity also plays an important role in proving the convergence of the iteration $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$.

Lemma 3: If $0 \leq \lambda_1 \leq 1$ is fixed and $0 \leq \lambda_2^{(1)} \leq \lambda_2^{(2)} \leq 1$, then $g_{\lambda_1\lambda_2^{(1)}}(X) \geq g_{\lambda_1\lambda_2^{(2)}}(X)$. Similarly, for fixed $0 \leq \lambda_2 \leq 1$ and $0 \leq \lambda_1^{(1)} \leq \lambda_1^{(2)} \leq 1$, $g_{\lambda_1^{(1)}\lambda_2}(X) \geq g_{\lambda_1^{(2)}\lambda_2}(X)$.

Remark: This lemma has an intuitive explanation: information never hurts. When the arrival rate of the observation increases, whether it is from $\mathbf{y}_1$ or $\mathbf{y}_2$, the state estimation can only get better. That is, the error covariance becomes smaller as the information rate increases. This property is also useful for determining the instability region of the iteration $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t)$. If the iteration $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1^*\lambda_2^*}(P_t)$ is known to diverge with the arrival rate pair $(\lambda_1^*, \lambda_2^*)$, then the iteration is unstable for any rate pair (λ_1, λ_2) in the rectangular region $0 \leq \lambda_1 \leq \lambda_1^*$ and $0 \leq \lambda_2 \leq \lambda_2^*$.

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Lemma 4: For any $X \geq 0, R > 0$,

$$g_{\lambda_1\lambda_2}(X) \geq (1-\lambda_1)(1-\lambda_2)AXA' + Q. \quad (15)$$

Remark: This lemma provides a universal lower bound of the function $g_{\lambda_1\lambda_2}(X)$. Let $X_{t+1} = g_{\lambda_1\lambda_2}(X_t)$, and $\hat{X}_{t+1} = (1-\lambda_1)(1-\lambda_2)A\hat{X}_tA' + Q$. If both iterations have the same initial condition, then at each time step t, we have $X_t \geq \hat{X}_t$. Therefore, if $\hat{X}_{t+1} = (1-\lambda_1)(1-\lambda_2)A\hat{X}_tA' + Q$ diverges, $X_{t+1} = g_{\lambda_1\lambda_2}(X_t)$ also diverges. This allows us to find a lower bound on the rate pair (λ_1, λ_2) region that leads to unbounded error covariance.

Lemma 5: If $X \geq 0$ is a random variable, then

$$\begin{aligned} (1-\lambda_1)(1-\lambda_2)A\mathrm{E}[X]A' + Q &\leq \mathrm{E}[g_{\lambda_1\lambda_2}(X)] \\ &\leq g_{\lambda_1\lambda_2}(\mathrm{E}[X]). \end{aligned}$$

Remark: This lemma combines Lemma 1 and Lemma 4 and gives both an upper and a lower bound on $\mathrm{E}[g_{\lambda_1\lambda_2}(X)]$. Since $\mathrm{E}[P_{t+1}] = \mathrm{E}[g_{\lambda_1\lambda_2}(P_t)]$, we can now bound $\mathrm{E}[P_{t+1}]$ with functions of $\mathrm{E}[P_t]$.

We now need to define two auxiliary functions that are closely related to $g_{\lambda_1\lambda_2}(X)$. First, we define

$$\begin{aligned} \phi(K, K_1, K_2, X) &= (1-\lambda_1)(1-\lambda_2)(AXA'+Q) \\ &\quad +\lambda_1\lambda_2(FXF'+V) + \lambda_1(1-\lambda_2)(F_1XF_1'+V_1) \\ &\quad +\lambda_2(1-\lambda_1)(F_2XF_2'+V_2), \end{aligned} \quad (16)$$

where $F = A + KC$, $F_1 = A + K_1C_1$, $F_2 = A + K_2C_2$, $V = Q + KRK'$, $V_1 = Q + K_1R_{11}K_1'$, $V_2 = Q + K_2R_{22}K_2'$ and $X \geq 0$. For $X \geq 0$, let $K_x = -AXC'(CXC'+R)^{-1}$, $K_{x1} = -AXC_1'(C_1XC_1'+R_{11})^{-1}$, and $K_{x2} = -AXC_2'(C_2XC_2'+R_{22})^{-1}$. We show that

$$g_{\lambda_1\lambda_2}(X) = \phi(K_x, K_{x1}, K_{x2}, X) \quad (17)$$

by noticing that $(A + K_xC)XC' + K_xR = 0$, $(A + K_{x1}C_1)XC_1' + K_{x1}R_{11} = 0$ and $(A + K_{x2}C_2)XC_2' + K_{x2}R_{22} = 0$. By differentiating the quadratic form $FXF' + V = (A+KC)X(A+KC)' + KRK' + Q$, we know that K_x minimizes $FXF' + V$. Similarly, K_{x1} minimizes $F_1XF_1' + V_1$ and K_{x2} minimizes $F_2XF_2' + V_2$. Therefore,

$$\phi(K_x, K_{x1}, K_{x2}, X) = \min_{K,K_1,K_2} \phi(K, K_1, K_2, X). \quad (18)$$

This implies $g_{\lambda_1\lambda_2}(X) = \min_{K,K_1,K_2}\phi(K, K_1, K_2, X)$. Also, any function $\phi(K, K_1, K_2, X)$ is an upper bound of $g_{\lambda_1\lambda_2}(X)$, that is $g_{\lambda_1\lambda_2}(X) \leq \phi(K, K_1, K_2, X), \forall K, K_1, K_2$.

The second auxiliary function is the linear part of $\phi(K, K_1, K_2, X)$. Let $\mathcal{L}(X) = (1-\lambda_1)(1-\lambda_2)AXA' + \lambda_1\lambda_2FXF' + \lambda_1(1-\lambda_2)F_1XF_1' + \lambda_2(1-\lambda_1)F_2XF_2'$. Clearly, $\mathcal{L}(X)$ is a linear function of X and $\mathcal{L}(X) \geq 0$. The function $\phi(K, K_1, K_2, X)$ is an affine function of X and $\phi(K, K_1, K_2, X) = \mathcal{L}(X) + \bar{V}$, where $\bar{V} = (1-\lambda_1)(1-\lambda_2)Q + \lambda_1\lambda_2V + \lambda_1(1-\lambda_2)V_1 + \lambda_2(1-\lambda_1)V_2$. Note $Q \geq 0$ and $V, V_1, V_2 \geq 0$. Thus $\bar{V} \geq 0$. As we will see in the next section, the properties of these two functions lead to the properties of the iteration $X_{t+1} = g_{\lambda_1\lambda_2}(X_t)$.

Lemma 6: Suppose $\exists\, \bar{Y}$ such that $\bar{Y} > \mathcal{L}(\bar{Y})$, then (a) $\forall W \geq 0$, $\lim_{k\to\infty}\mathcal{L}^k(W) = 0$. (b) Let $V \geq 0$ and consider

$$Y_{k+1} = \mathcal{L}(Y_k) + V$$

with initial condition Y_0, then $\{Y_k\}$ is bounded.

Remark: This lemma gives a condition that the iteration of the linear function $\mathcal{L}(Y)$ converges to 0, which in turn leads to the boundedness of the affine iteration $Y_{k+1} = \mathcal{L}(Y_k) + V$ for arbitrary initial condition Y_0.

Lemma 7: Suppose $\exists \bar{K}, \bar{K}_1, \bar{K}_2$, and $\bar{P} > 0$ such that $\bar{P} > \phi(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{P})$, then $\forall \bar{P}_0, \bar{P}_t = g^t_{\lambda_1\lambda_2}(\bar{P}_0)$ is bounded. That is, $\exists M_{\bar{P}_0}$ dependent on $\bar{P}_0$ such that $\bar{P}_t \leq M_{\bar{P}_0}, \forall t$.

Remark: This lemma is a direct result of Lemma 7. It gives us a condition such that the iteration $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$ can be bounded for arbitrary initial condition $\bar{P}_0$.

Lemma 8: If there exists an $\bar{X}$ such that $\bar{X} \geq g_{\lambda_1\lambda_2}(\bar{X})$, then $\bar{X} > 0$.

Remark: This lemma gives us a connection between the condition $\bar{X} \geq g_{\lambda_1\lambda_2}(\bar{X})$ and the hypothesis in Lemma 7. With Eqn. (17), we know that when $\bar{X} \geq g_{\lambda_1\lambda_2}(\bar{X})$, $\exists \bar{K}, \bar{K}_1, \bar{K}_2$ such that $\bar{X} > \phi(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{X})$. This lemma gives the other condition $\bar{X} > 0$. Thus, we have a condition in terms of the function $g_{\lambda_1\lambda_2}(X)$ to assure the boundedness of the error covariance matrix.

V. Convergence Properties

In this section, we first prove that, under certain conditions, the iteration $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$ converges to a unique positive semidefinite matrix. We then show the existence of a sharp transition curve in the rectangular region $0 \leq \lambda_1 \leq 1$ and $0 \leq \lambda_2 \leq 1$ such that the convergence of the iteration $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(\bar{P}_t)$ changes. In the general case, we can compute both an upper bound and a lower bound for this transition curve. If the rate pair (λ_1, λ_2) falls above the curve given by the upper bound, the iteration is stable and the error covariance matrix is always bounded from above. If the rate pair (λ_1, λ_2) falls below the curve given by the lower bound, the iteration goes unstable and the error covariance matrix goes to infinity eventually. As we shall see in our numerical examples, these two bounds meet for certain special cases. When the rate pair (λ_1, λ_2) guarantees its convergence, we can bound the steady state expected error covariance $\lim_{t\to\infty} \mathrm{E}[P_t]$ from both above and below.

The first theorem proves the convergence of $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$ under a given condition. It also shows the uniqueness of the solution when it does converge.

Theorem 1: Suppose $\exists$ matrices $\bar{K}, \bar{K}_1, \bar{K}_2$ and $\bar{P} > 0$ such that $\bar{P} > \phi(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{P})$, then
(a) $\forall P_0 \geq 0$, the iteration $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$ converges and

$$\lim_{t\to\infty} \bar{P}_t = \lim_{t\to\infty} g^t_{\lambda_1,\lambda_2}(\bar{P}_0) = \bar{P} \tag{19}$$

independent of initial condition $\bar{P}_0$.
(b) $\bar{P}$ is the unique positive semidefinite solution of $\bar{P}_{t+1} = g_{\lambda_1\lambda_2}(\bar{P}_t)$.

The next theorem shows the existence of a sharp transition curve for the stability of the iteration $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t)$.

Theorem 2: Assume (A, Q) is controllable, (A, C) is observable. Fix $0 \leq \lambda_1 \leq 1$. Then if $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t)$ is unstable for $\lambda_2 = 0$ while stable for $\lambda_2 = 1$, then $\exists\ \lambda_{2c}$ with $0 \leq \lambda_{2c} \leq 1$ such that

$$\lim_{t\to\infty} \mathrm{E}[P_t] = +\infty \quad \text{for} \quad 0 \leq \lambda_2 \leq \lambda_{2c},$$

and there exists a positive semidefinite matrix $M_{P_0} > 0$ as a function of the initial condition $P_0 \geq 0$ such that

$$\mathrm{E}[P_t] \leq M_{P_0}\ \forall t \quad \text{for} \quad \lambda_{2c} < \lambda_2 \leq 1.$$

If $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t)$ is unstable for the given λ_1 when $\lambda_2 = 1$, then $\lambda_{2c} = 1$. If $\mathrm{E}[P_{t+1}|P_t] = g_{\lambda_1\lambda_2}(P_t)$ is stable for the given λ_1 when $\lambda_2 = 0$, then $\lambda_{2c} = 0$. We get the same transition curve if we fix λ_2 and vary λ_1.

In the general case, we cannot compute the transition curve explicitly, but we have both an upper and a lower bound. Under special conditions, these two bounds meet and we have an explicit transition curve.

Theorem 3: For a given λ_1, we have an upper bound and a lower bound for λ_{2c}, i.e., $\underline{\lambda}_{2c} \leq \lambda_{2c} \leq \overline{\lambda}_{2c}$, and

$$\begin{aligned}\underline{\lambda}_{2c} &= \arg\inf_{\lambda_2}[\exists \hat{S} > 0 | \hat{S} = (1-\lambda_1)(1-\lambda_2)A\hat{S}A' + Q] \\ &= \max[1 - \tfrac{1}{\alpha^2(1-\lambda_1)}, 0],\end{aligned}$$

where $\alpha = \max_i |\sigma_i|$ and σ_i is the i^{th} eigenvalue of A;

$$\begin{aligned}\overline{\lambda}_{2c} &= \arg\inf_{\lambda_2}[\exists \hat{X} | \hat{X} > g_{\lambda_1\lambda_2}(\hat{X})] \\ &= \arg\inf_{\lambda_2}[\exists \hat{K}, \hat{K}_1, \hat{K}_2, \hat{X} > 0 | \hat{X} > \phi(\hat{K}, \hat{K}_1, \hat{K}_2, \hat{X})].\end{aligned}$$

Again, the result holds similarly if λ_2 is fixed.

The next theorem gives an upper and lower bound of the steady state expected covariance matrix of the Kalman filter with partial observation losses.

Theorem 4: Assume (A, Q) is controllable, (A, C) is observable and A is unstable. For a fixed λ_1, if $\lambda_2 > \overline{\lambda}_{2c}$, then we can find positive semidefinite matrices $\bar{S} \geq 0$ and $\bar{V} \geq 0$ such that

$$0 \leq \bar{S} \leq \lim_{t\to\infty} \mathrm{E}[P_t] \leq \bar{V}, \quad \forall \mathrm{E}[P_0] \geq 0,$$

where $\bar{S} = (1-\lambda_1)(1-\lambda_2)A\bar{S}A' + Q$ and $\bar{V} = g_{\lambda_1\lambda_2}(\bar{V})$.

The lower bound $\bar{S}$ can be computed by standard Lyapunov equation solvers, such as DLYAP in MATLAB. The upper bound $\bar{V}$ can be computed via iterating $V_{t+1} = g_{\lambda_1\lambda_2}(V_t)$ from any initial condition V_0 or by solving a simple semidefinite program (SDP), which will be stated in Theorem 6. The following theorem lets us find $\overline{\lambda}_{2c}$ by solving an Linear Matrix Inequality (LMI).

Theorem 5: If (A, Q) is controllable, then the following three statements are equivalent:
(a) $\exists \bar{X}$ such that $\bar{X} > g_{\lambda_1\lambda_2}(\bar{X})$.
(b) $\exists \bar{K}, \bar{K}_1, \bar{K}_2, \bar{X} > 0$ such that $\bar{X} > \phi(\bar{K}, \bar{K}_1, \bar{K}_2, \bar{X})$.
(c) $\exists \bar{Z}, \bar{Z}_1, \bar{Z}_2$ and $0 < \bar{Y} \leq I$ such that

$$\Psi(Y, Z, Z_1, Z_2) > 0$$

where $\Psi(Y, Z, Z_1, Z_2)$ is defined in Equation (20).

$$\Psi(Y,Z,Z_1,Z_2) = \begin{bmatrix} Y & \sqrt{\lambda_1\lambda_2}(YA+ZC) & \sqrt{\lambda_2(1-\lambda_1)}(YA+Z_2C_2) & \sqrt{\lambda_1(1-\lambda_2)}(YA+Z_1C_1) & \sqrt{(1-\lambda_1)(1-\lambda_2)}YA \\ \sqrt{\lambda_1\lambda_2}(A'Y+C'Z') & Y & 0 & 0 & 0 \\ \sqrt{\lambda_2(1-\lambda_1)}(A'Y+C_2'Z_2') & 0 & Y & 0 & 0 \\ \sqrt{\lambda_1(1-\lambda_2)}(A'Y+C_1'Z_1') & 0 & 0 & Y & 0 \\ \sqrt{(1-\lambda_1)(1-\lambda_2)}A'Y & 0 & 0 & 0 & Y \end{bmatrix}. \quad (20)$$

$$\Gamma(V) = \begin{bmatrix} AVA'+Q-V & \sqrt{\lambda_1\lambda_2}AVC' & \sqrt{\lambda_1(1-\lambda_2)}AVC_1' & \sqrt{\lambda_2(1-\lambda_1)}AVC_2' \\ \sqrt{\lambda_1\lambda_2}CVA' & CVC'+R & 0 & 0 \\ \sqrt{\lambda_1(1-\lambda_2)}C_1VA' & 0 & C_1VC_1'+R_{11} & 0 \\ \sqrt{\lambda_2(1-\lambda_1)}C_2VA' & 0 & 0 & C_2VC_2'+R_{22} \end{bmatrix}. \quad (21)$$

Therefore, if λ_1 is fixed, we can use bisection for the variable λ_2 to find $\bar{\lambda}_{2c}$ by solving a series of LMI feasibility problems. The same curve will be found if we fix λ_2 and solve a series of LMI feasibility problems to find $\bar{\lambda}_{1c}$. The next theorem shows that the upper bound of $\lim_{t\to\infty} \mathrm{E}[P_t]$ can be computed by solving a semidefinite program (SDP).

Theorem 6: If the pair (λ_1, λ_2) satisfies $\Psi(Y,Z,Z_1,Z_2) > 0$, then the matrix $\bar{V}$ such that $\bar{V} = g_{\lambda_1\lambda_2}(\bar{V})$ is given by:

(a) $\lim_{t\to\infty} V_t = \bar{V}$ where $V_0 \geq 0, V_{t+1} = g_{\lambda_1\lambda_2}(V_t)$, or equivalently,

(b) $\begin{cases} \arg\max_V & Trace(V) \\ \text{subject to} & \Gamma(V) \geq 0, \end{cases}$

where $\Gamma(V)$ is defined in Equation (21).

VI. Numerical Example

In this section, we illustrate the results from the previous section with some simple examples. We also use our last numerical example to show that network resources should be allocated with respective to control performance. We first show a couple of special cases in which the sharp transition curve separating the stable and unstable throughput regions can be explicitly found. That is, for a fixed λ_1, the lower bound $\underline{\lambda}_{2c}$ and the upper bound $\bar{\lambda}_{2c}$ coincide. Therefore, $\lambda_{2c} = \underline{\lambda}_{2c} = \bar{\lambda}_{2c}$.

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A. Coinciding the upper and lower bounds

Case 1: When C_1 and C_2 are both square and invertible, the lower bound $\underline{\lambda}_{2c}$ and the upper bound $\bar{\lambda}_{2c}$ are equal for every λ_1. This is because we can choose $K_1 = -AC_1^{-1}, K_2 = -AC_2^{-1}$ and $K = \frac{1}{2}[K_1 \;\; K_2]$ so that $F_1 = A + K_1C_1 = 0, F_2 = A + K_2C_2 = 0$ and $F = A + KC = A + K[C_1; C_2] = 0$. Thus, the LMI in Theorem 5 is equivalent to $X - (1-\lambda_1)(1-\lambda_2)AXA' \geq 0$. Such an $X \geq 0$ exists if and only if $\sqrt{(1-\lambda_1)(1-\lambda_2)}A$ is stable. Therefore, the upper bound and lower bound coincide and $\lambda_{2c} = \max[0, 1 - \frac{1}{\alpha^2(1-\lambda_1)}]$, where $\alpha = \max_i |\sigma_i|$ and σ_i is the i^{th} eigenvalue of matrix A. For example, consider the system $A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $C_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $C = [C_1; C_2]$, $Q = 20\mathrm{I}_2$, and $R = 2.5\mathrm{I}_4$, where I_k is the k by k identity matrix. We plot the upper bound and the lower bound of $\lim_{t\to\infty} \mathrm{E}[P_t]$ in Figure 3. Although not obvious from the figure, there is a

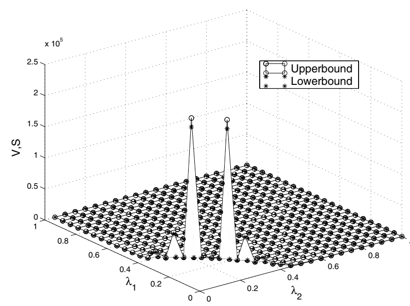

Fig. 3. Upper bound and Lower bound of $\lim_{t\to\infty} \mathrm{E}[P_t]$.

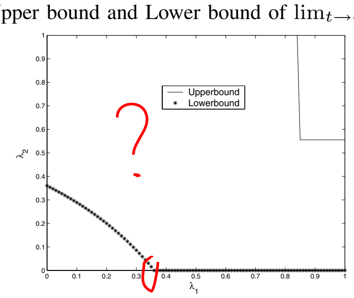

Fig. 4. Upper and Lower bounds of the Stable Throughput Region

gap between the upper and the lower bound and the gap depends on the pair (λ_1, λ_2). Both the upper bound and the lower bound go to infinity as they approach the curve $\lambda_{2c} = \max[0, 1 - \frac{1}{\alpha^2(1-\lambda_1)}]$ for a given λ_1. The unstable rate region is the white area in the (λ_1, λ_2) plane where both the upper bound and the lower bound are infinite.

Case 2: When the system matrix A has a single unstable mode, the lower bound $\underline{\lambda}_{2c}$ and the upper bound $\bar{\lambda}_{2c}$ are equal for every λ_1. This is because we can apply the Kalman decomposition and we can choose K, K_1, K_2 to bring the unstable part of F, F_1, and F_2 to 0. An example system is: $A = \begin{bmatrix} 1.25 & 0 \\ 1 & .9 \end{bmatrix}$, $C_1 = [1 \;\; 0]$, $C_2 = [0 \;\; 1]$, $C = [C_1; C_2]$, $Q = 20\mathrm{I}_2$, and $R = 2.5\mathrm{I}_2$. The plot of the upper and lower bound as a function of the rate pair is very similar to Figure 3 and will not be shown here. The transition curve is known since the lower and the upper bound meet.

B. When the two bounds do not meet

In general, the upper bound and the lower bound are not equal. As a simple example, we consider a second order diagonal system with identity observation matrix C. Thus we can separate it into two scalar systems. Let

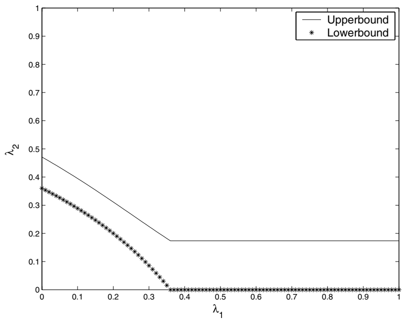

Fig. 5. Upper and Lower bounds of the Stable Throughput Region

$A = \begin{bmatrix} 2.5 & 0 \\ 0 & 1.5 \end{bmatrix}$ and $C = \mathrm{I}_2$. Then $C_1 = [1\ 0]$ and $C_2 = [0\ 1]$. We assume the noise covariance matrix $Q = 20\mathrm{I}_2$ and $R = 2.5\mathrm{I}_2$. The stable and unstable rate regions determined by the upper and lower bounds are shown in Figure 4. As we can see, the two bounds are quite far from each other. The rate pair that falls above the upper bound is stable and the rate pair that falls below the lower bound is unstable. However, there is a large region whose stability cannot be determined with our bounds. Since the two states in this system are completely independent and each of the observations directly measure the states, we can separate the system into two independent scalar systems whose transition rate can be explicitly determined: we need $\lambda_2 > 1 - 1/1.5^2 = 5/9 \approx 0.56$ to guarantee a bounded error in estimating state 2, and $\lambda_1 > 1 - 1/2.5^2 = 0.84$ to guarantee stability in estimating state 1. Therefore, we can find the stable rate region of the system: $\lambda_2 > 0.56$, and $\lambda_1 > 0.84$, which is the same region defined by the upper bound. Thus in this case, the upper bound is tight while the lower bound is loose. This scenario can be generalized to include the block diagonal systems when the observation matrices C_1 and C_2 have completely non-overlapping observable state space.

In the most general case, we cannot determine where the transition curve is. Such a curve exists and it lies somewhere between the lower bound and the upper bound. We now consider a simple system whose stable rate region cannot be found explicitly. Specifically we consider $A = \begin{bmatrix} 1.25 & 0 \\ 1 & 1.1 \end{bmatrix}$, $C_1 = [1\ 0]$, $C_2 = [1\ 1]$, $C = [C_1; C_2]$, $Q = 20\mathrm{I}_2$ and $R = 2.5\mathrm{I}_2$. The stable and unstable throughput regions defined by the upper and lower bounds are plotted in Figure 5. Since (A, C_2) is observable, as long as λ_2 is sufficiently high, the Kalman filter can be stable for any arrival rate λ_1.

C. Network Resource Allocation

For a network with limited resources, such as power and bandwidth, there is always a question of optimized resource allocation. When each user has equal impact on the system, we need to allocate fairly among different users. However, in a networked control system, the goal of the wireless network is to support the control system so that the control system performance can be optimized. Different sensor measurements impact the system performance differently. Consider the previous example and assume we have a limited data rate and the total throughput from both sensors needs to satisfy $\lambda_1 + \lambda_2 = 0.5$. If we allocate all the rate to observation 1, then the system is unstable since (A, C_1) is not observable. On the other hand, if we allocate all the resources to observation 2, the error covariance is bounded at $\lambda_2 = 0.5$. If we have an equal allocation among the two: $\lambda_1 = \lambda_2 = 0.25$, the stability of the Kalman filter cannot be decided. Therefore, it is important to take the control performance as the wireless network design objective in these distributed control systems. Since we only have lower and upper bounds of the error covariance matrix, we cannot determine the optimal allocation but we can avoid allocations that make the system unstable.

VII. Conclusions

We formulate the Kalman filtering problem with random partial observation losses. The error covariance matrix iteration is stochastic and the statistical convergence properties of the iteration are investigated. We find both an upper and lower bound of the steady state expected error covariance. The existence of a sharp transition curve in the rate region for the stability/instability transition is proved and we use the bounds of the error covariance matrix to compute an upper and lower bound of the stability rate region. Numerical results show that the lower and upper bound can meet in certain cases. The results in this paper can also be generalized to cases where the observation data is contained in more than two packets. In the generalization, the Kalman updates upon one or more packet losses can be computed in the same manner since we can always divide the observation into two parts: the received and the lost. The statistical properties of the error covariance matrix iteration stay the same with appropriate extensions and so are the convergence properties. In our future work, we would like to address the Kalman filtering problem with bursty packet losses, which are more realistic in wireless networks.

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