Agent based Optimally Weighted Kalman Consensus Filter over a Lossy Network

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***Abstract*—We consider a distributed agent based state estimation problem where each agent has a local underdetermined observation space and is interested in a subset of states. At a particular time instant, each agent predicts its state and makes intermediate correction based on its local measurements. Information about the corrected state elements are then exchanged among neighboring (i.e., agents who share at least one state element) agents. Based on the final processing of the exchanged information, an agent based optimally weighted Kalman consensus Filter is formed in the light of well-established theory of distributed Kalman filtering. The optimal weighting of consensus is derived through Lyapunov function based stability analysis of corresponding estimation error. The effect of communication is also investigated by introducing random failures in the communication link among neighboring agents. The corresponding bounds on the degree of consensus and inter-agent link failure rate are also derived for stable implementation of the agent based dynamic estimation. The proposed filter is applied to a custom built 2-agent system to conform the desired optimal limit for the degree of consensus based information exchange both under perfect and lossy communication network.**

***Index Terms*—Kalman consensus filter, binary projection matrix, random link failure, Lyapunov stability.**

I. Introduction

Distributed Kalman filtering can not only reduce the communication bottleneck but also the computational burden of a centralized Kalman filter [1]. In a distributed implementation, the objective of each sensor or agent is to have updated status of the overall system through local prediction and necessary correction based on the type of information (measurement and/or predicted state values) exchanged among its neighbors. Among numerous variations of this fundamental concept, Kalman consensus filter (KCF) [2], [3] and diffusion Kalman filter (DKF) [4] are worth mentioning. In order to assess the importance of reliable neighborhood communication, these filters are also investigated by incorporating random link failure in consensus approach [5], [6] and having noisy communication link for stationary diffusive estimation of a single parameter [7].

Dynamic state estimation in a large scale cyber-physical system (CPS) (e.g., Smart Grid) presents some unique challenges namely, (1) dimension of the state vector is quite large, (2) sensors are spatially distributed over a large area. Consequently, sensors are required to be fully connected in order to regularly store and update the global state vector and estimation error covariance matrix. Additionally, the number of communication links, computational memory and hardware requirements increase with the increase in active sensor nodes. Thus, it may be impractical to track the high dimensional state vector in its entirety at each communicable sensor [8]. This constraint can be overcome by shrinking the original observation space of each sensor solely based on the characteristics of state dynamics [9], [10], [11], [12]. On the contrary, in a practical CPS, the observation space of a sensor may be coupled to limited set of specific state elements. Some state elements may even be coupled to two or more sensors' observation space. Under these circumstances, each sensor may be relieved to track only the pertinent state elements. In this way, the sensors are acting as agents and the underlying system can be referred to as a multi-agent system. A typical example is the radial network of power system being estimated using agent based integration of consensus and innovation [13]. The underlying concept is put to the agent based formulation of KCF and DKF (referred to AKCF and ADKF, respectively) in a recent work [14]. In this scenario, the active sensors (referred to as agents) collect measurements and participate in neighborhood-predicted-state-information-exchange unless the communication link is broken. It is mentioned that, for the custom multi-agent systems considered therein, AKCF performs better than ADKF and is more robust to lossy communication network. However, no strategy of choosing consensus weights is reported. Also, the filter stability is not investigated under lossy communication network. In this paper, we propose a strategy of choosing optimal consensus weights in AKCF formulation. A collective measure of agent-wise estimation error dynamics is formed. Through Lyapunov stability analysis of the error dynamics, the expression of optimal consensus weights and the upper bound on the degree of consensus are derived. Additionally, the bound on communication failure rate is derived beyond which the filter becomes unstable.

A. Contributions

Our contributions to Agent based Kalman consensus filter are summarized below,

- Perform Lyapunov stability analysis from the AKCF estimation error dynamics.
- Find the expression of optimal consensus weight in the form of $\epsilon\mathbf{A}$, where ϵ is a positive scalar value and $\mathbf{A}$ is

system specific and timely updated matrix.
- Find an upper bound of ϵ that ensures filter stability.
- Incorporate random link failure among neighboring agents and find the theoretical bound on link failure rate till which the estimation error won't diverge.

The rest of the paper is organized as follows: Section II describes the system model and the AKCF algorithm. The main results are represented in Section III in terms of a lemma and a theorem along with the proofs. Section IV discusses about the lossy communication network and presents a corollary of the previous theorem. A case study is reported in Section V to conform with the theoretical bounds. Concluding remarks and direction to future works are given in Section VI.

II. System Model

We consider a system whose dynamics can be modeled in discrete time as 1^{st} order Gauss-Markov Process, i.e.,

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{w}_{t-1}; t = 0, 1, 2, ... \tag{1}$$

where, the overall system state is represented by the n-dimensional state vector $\mathbf{x}_t$ at time instant t. The initial values of the state vector elements at $t = -1$ follow Gaussian distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$. Unlike [9]- [12], the state transition matrix $\mathbf{F} \in \mathbb{R}^{n\times n}$ is a general square matrix with eigenvalues lying within a unit circle. The process noise $\mathbf{w}_t \sim \mathcal{N}(\mathbf{0}, \mathbf{Q})$.The underlying physical system is observed by N agents. The linear observation model for the k^{th} agent is,

$$\mathbf{y}_{t,k} = \mathbf{H}_k\mathbf{x}_{t,k} + \mathbf{v}_{t,k}; k = 1, 2, ..., N \tag{2}$$

where, the n_k-dimensional vector $\mathbf{x}_{t,k}$ is Agent k's local state vector $-$ a subset of $\mathbf{x}_t$. The observation matrix $\mathbf{H}_k \in \mathbb{R}^{m_k\times n_k}$ is full row rank (i.e.,$m_k \leq n_k$). The measurement noise $\mathbf{v}_{t,k} \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k)$ and also independent of process noise. Unlike conventional distributed Kalman filtering, a particular agent k attempts to estimate only the local state vector $\mathbf{x}_{t,k}$ instead of $\mathbf{x}_t$. To incorporate this scenario in the local system dynamics, we introduce a binary projection matrix $\mathbf{T}_k$ such that,

$$\mathbf{x}_{t,k} = \mathbf{T}_k\mathbf{x}_t; k = 1, 2, ..., N \tag{3}$$

The projection matrix $\mathbf{T}_k$ is an $n_k\times n$ matrix and rank deficient ($n_k < n$). Therefore, exact recovery of the global set of states $\mathbf{x}_t$ from individual agent's local state vector $\mathbf{x}_{t,k}$ is not possible and also not required in current scenario. Nevertheless, the projection matrix also follows Lemma 1.

Lemma 1. *For the binary projection matrix $\mathbf{T}_k$ of dimension $n_k \times n$ ($n_k < n$), $\mathbf{T}_k\mathbf{T}_k^\top = \mathbf{I}_{n_k}$. Here, $\mathbf{I}_{n_k}$ is an identity matrix of dimension n_k.*

Proof: An alternate representation of the projection matrix is, $\mathbf{T}_k = \begin{bmatrix}\mathbf{I}_{n_k} & \mathbf{O}\end{bmatrix}\mathcal{P}$. Here, $\mathbf{O}$ is $n_k \times (n - n_k)$ matrix of zeros and $\mathcal{P}$ is an $n \times n$ permutation matrix. Using the properties of permutation matrices [15],

$$\begin{aligned}\mathbf{T}_k\mathbf{T}_k^\top &= \begin{bmatrix}\mathbf{I}_{n_k} & \mathbf{O}\end{bmatrix}\mathcal{P}\mathcal{P}^\top\begin{bmatrix}\mathbf{I}_{n_k}^\top \\ \mathbf{O}^\top\end{bmatrix} \\ &= \begin{bmatrix}\mathbf{I}_{n_k} & \mathbf{O}\end{bmatrix}\mathbf{I}_n\begin{bmatrix}\mathbf{I}_{n_k}^\top \\ \mathbf{O}^\top\end{bmatrix} = \mathbf{I}_{n_k}\end{aligned}$$

■

Using the projection matrix $\mathbf{T}_k$, the system dynamics of k^{th} agent can be mapped from equation (1),

$$\mathbf{x}_{t,k} = \mathbf{T}_k\mathbf{F}\mathbf{x}_{t-1} + \mathbf{w}_{t-1,k} \tag{4}$$

where, $\mathbf{w}_{t-1,k} = \mathbf{T}_k\mathbf{w}_{t-1}$. Therefore, $\mathbf{w}_{t-1,k} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k)$ with $\mathbf{Q}_k = \mathbf{T}_k\mathbf{Q}\mathbf{T}_k^\top$. It should be noted that, the local system dynamics for k^{th} agent should reflect the corresponding observation model of equation (2). As a consequence, the desired 1^{st} order Gauss-Markov process of k^{th} agent can be expressed as,

$$\mathbf{x}_{t,k} = \mathbf{F}_k\mathbf{x}_{t-1,k} + \mathbf{w}_{t-1,k} \tag{5}$$

The above model conforms with equation (4) if and only if $\mathbf{F}_k\mathbf{T}_k = \mathbf{T}_k\mathbf{F}$. Or, using Lemma 1, $\mathbf{F}_k = \mathbf{T}_k\mathbf{F}\mathbf{T}_k^\top$. It is important to note that, this irreversible mapping is intended only to design the agent based Kalman filter (Algorithm 1), and subsequent derivation of theoretical bounds for filter stability. Later in section IV, a case study is carried out to verify these bounds, where equations (1) (global system dynamics) and (3) (local state mapping) serve as true state vector for individual agents.

The static set of *physical* neighbors for the k^{th} agent is defined based on the overlap/sharing of state elements. Mathematically,

$$\mathcal{S}_k = \{i : \mathbf{P}_{i,k}\mathbf{S}_i\mathbf{x}_{t,i} \text{ projects onto } \mathbf{L}_{i,k}\mathbf{S}_k\mathbf{x}_{t,k}, \forall t\}.$$

Here, $\mathbf{P}_{i,k}$ and $\mathbf{L}_{i,k}$ are $n_k^s \times n_i^s$ and $n_k^s \times n_k^s$ binary projection matrices, respectively. n_k^s is the number of state elements shared between agent k and its neighbors. Similar definition applies for n_i^s. When agent i sends state information to agent k, the projection matrix $\mathbf{P}_{i,k}$ is multiplied to the sender agent's vector of shared states whereas $\mathbf{L}_{i,k}$ is used with receiver agent's shared state vector. By default, $\mathbf{P}_{k,k} = \mathbf{L}_{k,k} = \mathbf{I}_{n_k^s}$. The other binary matrices $(\mathbf{S}_i, \mathbf{S}_k)$ are required to extract the shared state elements for the respective agents. In addition to that, a reordering binary matrix $\mathbf{O}_k^s$ is used to assign the shared state elements to their original position in the k^{th} agent's state vector with the entries of unshared elements kept zero. Similar operation of extracting and reordering can be defined for completely unshared state elements for any agent. Fig. 1 illustrates these operations for shared and unshared state elements of agent k. The decomposition into shared and unshared portion and recombining them should constitute the original state vector for the k^{th} agent. Thus, mathematically $\mathbf{O}_k^s\mathbf{S}_k + \mathbf{O}_k^u\mathbf{U}_k = \mathbf{I}_{n_k}$, where $\mathbf{I}_{n_k}$ denotes an identity matrix of dimension n_k.

The agent based dynamic state estimation procedure is developed with minimum mean square error (MSE) as the metric of interest. The state vector estimated by the k^{th} agent at discrete time instant i is defined as,

$$\hat{\mathbf{x}}_{i,k|j} = \mathbb{E}\left[\mathbf{x}_{i,k}|\mathbf{y}_{0,k}, \mathbf{y}_{1,k}, ..., \mathbf{y}_{j,k}\right] \tag{6}$$

The corresponding error covariance matrix is,

$$\mathbf{M}_{i,k|j} = \mathbb{E}\left[(\mathbf{x}_{i,k} - \hat{\mathbf{x}}_{i,k|j})(\mathbf{x}_{i,k} - \hat{\mathbf{x}}_{i,k|j})^\top\right] \tag{7}$$

Using these definitions, the agent based Kalman consensus filtering is described in Algorithm 1. At the initialization step, $\boldsymbol{\mu}_k = \mathbf{T}_k\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_k = \mathbf{T}_k\boldsymbol{\Sigma}\mathbf{T}_k^\top$. Steps 2 to 6 are performed according to traditional Kalman filtering. The standard form calculation of Kalman gain (step 4) is feasible since each agent k has to bear $\mathcal{O}(n_k^3)$ complexity as compared to $\mathcal{O}(n^3)$ in conventional distributed Kalman filtering. At the 7^{th} step of estimation, correction is made to individual agents' shared state elements by using $\hat{\mathbf{b}}_{t,k}$ and exchanged information from the neighbors. Finally, the estimates of shared and unshared state elements are combined in a single vector for each agent. It should be noted that, unlike conventional algorithms of distributed Kalman consensus filter [2], [3], the agents are not required to aggregate information about their neighbors' measurements or individual error covariance matrices.

Algorithm 1 AKCF

1: $\hat{\mathbf{x}}_{-1,k|-1} = \boldsymbol{\mu}_k, \mathbf{M}_{-1,k|-1} = \boldsymbol{\Sigma}_k$ ▷ Initialization
2: $\hat{\mathbf{x}}_{t,k|t-1} = \mathbf{F}_k\hat{\mathbf{x}}_{t-1,k|t-1}$ ▷ Predict State
3: $\mathbf{M}_{t,k|t-1} = \mathbf{F}_k\mathbf{M}_{t-1,k|t-1}\mathbf{F}_k^\top + \mathbf{Q}_k$ ▷ Update Error Covariance
4: $\mathbf{K}_{t,k} = \mathbf{M}_{t,k|t-1}\mathbf{H}_k^\top\left(\mathbf{H}_k\mathbf{M}_{t,k|t-1}\mathbf{H}_k^\top + \mathbf{R}_k\right)^{-1}$ ▷ Kalman Gain
5: $\mathbf{M}_{t,k|t} = \left(\mathbf{I}_{n_k} - \mathbf{K}_{t,k}\mathbf{H}_k\right)\mathbf{M}_{t,k|t-1}$ ▷ Correct Error Covariance
6: $\hat{\mathbf{b}}_{t,k} = \hat{\mathbf{x}}_{t,k|t-1} + \mathbf{K}_{t,k}\left(\mathbf{y}_{t,k} - \mathbf{H}_k\hat{\mathbf{x}}_{t,k|t-1}\right)$ ▷ Intermediate Correction
7: $\hat{\mathbf{x}}^s_{t,k|t} = \mathbf{S}_k\hat{\mathbf{b}}_{t,k}$ $+ \mathbf{W}^s_{t,k}\sum_{i\in\mathcal{S}_k}\left(\mathbf{P}_{i,k}\mathbf{S}_i\hat{\mathbf{x}}_{t,i|t-1} - \mathbf{L}_{i,k}\mathbf{S}_k\hat{\mathbf{x}}_{t,k|t-1}\right)$ ▷ Shared Element Correction through Inter-Agent Information Exchange
8: $\hat{\mathbf{x}}_{t,k|t} = \mathbf{O}^s_k\hat{\mathbf{x}}^s_{t,k|t} + \mathbf{O}^u_k\mathbf{U}_k\hat{\mathbf{b}}_{t,k}$ ▷ Combining the Shared and Unshared Parts

III. Main Results

We present the primary outcome of this research starting with an important lemma, which will be used in upcoming theorems and subsequent sections.

Lemma 2. *Consider two symmetric positive definite matrices* $\mathbf{G}$ *and* $\mathbf{L}$. *Let* $\epsilon \in \mathbb{R}$. *Then the positive definiteness of the matrix quantity* $\left(\mathbf{G} - \epsilon^2\mathbf{L}\right)$ *is guaranteed, when* ϵ *follows the illustration in Fig. 2.*

Proof: Let, $\mathbf{B} = \mathbf{G} - \epsilon^2\mathbf{L}$. Therefore, $\mathbf{B}$ will be positive definite if and only if for all non zero vectors $\mathbf{u}$, the quadratic form, $\mathbf{u}^\top\mathbf{B}\mathbf{u} > 0$. The primary decomposition of this form reveals the required inequality, $\mathbf{u}^\top(\epsilon^2\mathbf{L})\mathbf{u} < \mathbf{u}^\top\mathbf{G}\mathbf{u}$. Because of the symmetric matrices, the quadratic values $\mathbf{u}^\top(\cdot)\mathbf{u}$ lie on

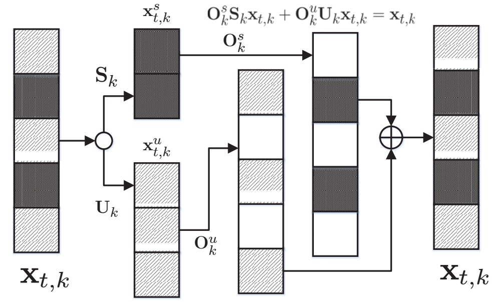

Fig. 1. Extracting and Reordering of Shared (darker shade) and Unshared (lighter shade) State Elements.

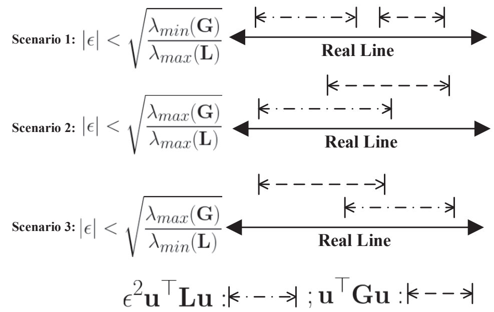

Fig. 2. Three Scenarios for Lemma 2

the real line whose range is defined according to Rayleigh-Ritz inequality. Thus, for the two sides of the above equation we have the following range of real quadratic values, $\lambda_{min}(\mathbf{G})\mathbf{u}^\top\mathbf{u} \leq \mathbf{u}^\top\mathbf{G}\mathbf{u} \leq \lambda_{max}(\mathbf{G})\mathbf{u}^\top\mathbf{u}$, and $\epsilon^2\lambda_{min}(\mathbf{L})\mathbf{u}^\top\mathbf{u} \leq \mathbf{u}^\top(\epsilon^2\mathbf{L})\mathbf{u} \leq \epsilon^2\lambda_{max}(\mathbf{L})\mathbf{u}^\top\mathbf{u}$. The relative positioning of the two ranges of quadratic values over the real line results in three possible scenarios, which satisfy the condition of positive definiteness. These are illustrated in Fig.2. Therefore, positive definiteness will be ensured, when in scenario 1, $\epsilon^2\lambda_{max}(\mathbf{L})\mathbf{u}^\top\mathbf{u} < \lambda_{min}(\mathbf{G})\mathbf{u}^\top\mathbf{u}$, scenario 2, $\epsilon^2\lambda_{max}(\mathbf{L})\mathbf{u}^\top\mathbf{u} < \lambda_{max}(\mathbf{G})\mathbf{u}^\top\mathbf{u}$ and in scenario 3, $\epsilon^2\lambda_{min}(\mathbf{L})\mathbf{u}^\top\mathbf{u} < \lambda_{max}(\mathbf{G})\mathbf{u}^\top\mathbf{u}$. Rearranging the inequalities and taking square roots on both sides complete the proof. ■

In addition to this, we define the following matrices according to AKCF steps in Algorithm 1,
$\mathcal{C}_{t,k} = \left(\mathbf{I}_{n_k} - \mathbf{K}_{t,k}\mathbf{H}_k\right)\mathbf{F}_k$; $\mathcal{D}_{t,k} = \mathcal{C}^{-1}_{t,k}\mathbf{M}_{t,k|t}\left(\mathcal{C}^{-1}_{t,k}\right)^\top$; $\mathbf{G}_{t,k} = \mathbf{M}^{-1}_{t-1,k|t-1} - \mathcal{D}^{-1}_{t,k}$;
and,

$$[\mathbb{A}^F]_{(k,l)^{th}block} = \begin{cases} -\mathbf{O}^s_k\mathbf{P}_{l,k}\mathbf{S}_l\mathbf{F}_l; & l \in \mathcal{S}_k \\ \sum_{i\in\mathcal{S}_k}\mathbf{O}^s_k\mathbf{L}_{i,k}\mathbf{S}_k\mathbf{F}_k; & l = k \\ \mathbf{0}_{n_k\times n_l}; & otherwise. \end{cases} \tag{8}$$

The following theorem gives an optimal expression of the consensus weight $\mathbf{W}^s_{t,k}$ in association with a tuning parameter ϵ in order to set the degree of participation in consensus.

Theorem 1. *The error dynamics of the Kalman consensus filter for multi-agent system described in Algorithm 1 is globally asymptotically stable if,* $\mathbf{W}^s_{t,k} =$

$\epsilon\left(\mathbf{O}_k^s\right)^{\dagger}\mathbf{M}_{t,k|t-1}\left(\mathbf{F}_k^{-1}\right)^{\top}\mathbf{O}_k^s;\forall k,\forall t$*; and the sufficient conditions on the level of consensus to guarantee stability are,*

$$\text{Scenario 1:}\quad |\epsilon|<\sqrt{\min_k\lambda_{min}(\mathbf{G}_{t,k})/\lambda_{max}\left(\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\right)}$$
$$\text{Scenario 2:}\quad |\epsilon|<\sqrt{\max_k\lambda_{max}(\mathbf{G}_{t,k})/\lambda_{max}\left(\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\right)}$$
$$\text{Scenario 3:}\quad |\epsilon|<\sqrt{\max_k\lambda_{max}(\mathbf{G}_{t,k})/\lambda_{min}\left(\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\right)}$$

Proof: To investigate the error dynamics, an estimation error vector associated with each agent is defined in two steps. At first, error in estimating the shared states and those of unshared ones are defined. For shared states, the error is, $\boldsymbol{\eta}_{t,k}^s=\hat{\mathbf{x}}_{t,k|t}^s-\mathbf{S}_k\mathbf{x}_{t,k}$. And for the unshared elements, $\boldsymbol{\eta}_{t,k}^u=\mathbf{U}_k\left(\hat{\mathbf{b}}_{t,k}-\mathbf{x}_{t,k}\right)$. Here, $\mathbf{x}_{t,k}$ is the true state vector of the k^{th} agent at discrete time instant t. At the second step, the estimation error of k^{th} agent is defined as, $\boldsymbol{\eta}_{t,k}=\mathbf{O}_k^s\boldsymbol{\eta}_{t,k}^s+\mathbf{O}_k^u\boldsymbol{\eta}_{t,k}^u$. Therefore, the error dynamics for the k^{th} agent can be derived from equations (2), (5), steps 2,6,7 and 8 of Algorithm 1,

$$\begin{aligned}\boldsymbol{\eta}_{t,k}&=\mathcal{C}_{t,k}\boldsymbol{\eta}_{t-1,k}+\mathbf{O}_k^s\mathbf{W}_{t,k}^s\mathbf{z}_{t-1,k}\\&+\mathbf{K}_{t,k}\mathbf{v}_{t,k}-\left(\mathbf{I}_{n_k}-\mathbf{K}_{t,k}\mathbf{H}_k\right)\mathbf{w}_{t-1,k}\\&-\mathbf{O}_k^s\mathbf{W}_{t,k}^s\sum_{i\in\mathcal{S}_k}\left(\mathbf{P}_{i,k}\mathbf{S}_i\mathbf{w}_{t-1,i}-\mathbf{L}_{i,k}\mathbf{S}_k\mathbf{w}_{t-1,k}\right)\end{aligned}\tag{9}$$

where, $\mathbf{z}_{t-1,k}=\sum_{i\in\mathcal{S}_k}\left(\mathbf{P}_{i,k}\mathbf{S}_i\mathbf{F}_i\boldsymbol{\eta}_{t-1,i}-\mathbf{L}_{i,k}\mathbf{S}_k\mathbf{F}_k\boldsymbol{\eta}_{t-1,k}\right)$ and $\mathbf{P}_{i,k}\mathbf{S}_i\mathbf{x}_{t,i}=\mathbf{L}_{i,k}\mathbf{S}_k\mathbf{x}_{t,k},\forall i\in\mathcal{S}_k,\forall k,\forall t$. From equation (9) it can be observed that, the measurement and process noise terms (i.e.,$\mathbf{v}$ and $\mathbf{w}$) are random inputs characterized by zero mean Gaussian probability distribution. Therefore, for k^{th} agent, the stability of equation (9) is governed by the following homogeneous equation [16],

$$\boldsymbol{\eta}_{t,k}=\mathcal{C}_{t,k}\boldsymbol{\eta}_{t-1,k}+\mathbf{O}_k^s\mathbf{W}_{t,k}^s\mathbf{z}_{t-1,k}\tag{10}$$

The optimum choice for $\mathbf{W}_{t,k}^s$ is derived using the Lyapunov stability criteria [3], [16]. The corresponding Lyapunov function for the k^{th} agent, $V_k(t)=\boldsymbol{\eta}_{t,k}^{\top}\mathbf{M}_{t,k|t}^{-1}\boldsymbol{\eta}_{t,k}$. Hence, the change in $V_k(t)$ can be written as,

$$\begin{aligned}\delta V_k&=V_k(t)-V_k(t-1)\\&=-\boldsymbol{\eta}_{t-1,k}^{\top}\mathbf{G}_{t,k}\boldsymbol{\eta}_{t-1,k}\\&+2\boldsymbol{\eta}_{t-1,k}^{\top}\left(\mathcal{C}_{t,k}^{\top}\mathbf{M}_{t,k|t}^{-1}\mathbf{O}_k^s\mathbf{W}_{t,k}^s\right)\mathbf{z}_{t-1,k}\\&+\mathbf{z}_{t-1,k}^{\top}\left(\mathbf{O}_k^s\mathbf{W}_{t,k}^s\right)^{\top}\mathbf{M}_{t,k|t}^{-1}\mathbf{O}_k^s\mathbf{W}_{t,k}^s\mathbf{z}_{t-1,k}\end{aligned}\tag{11}$$

From the second term of the above equation, $\mathbf{W}_{t,k}^s=\epsilon\left(\mathbf{O}_k^s\right)^{\dagger}\mathbf{M}_{t,k|t}\left(\mathcal{C}_{t,k}^{-1}\right)^{\top}\mathbf{O}_k^s;\epsilon\in\mathbb{R}$. Using step 5 of Algorithm 1, $\mathbf{W}_{t,k}^s=\epsilon\left(\mathbf{O}_k^s\right)^{\dagger}\mathbf{M}_{t,k|t-1}\left(\mathbf{F}_k^{-1}\right)^{\top}\mathbf{O}_k^s$. Consequently, the third term reduces to $\epsilon^2\left(\mathbf{O}_k^s\mathbf{z}_{t-1,k}\right)^{\top}\mathcal{D}_{t,k}\left(\mathbf{O}_k^s\mathbf{z}_{t-1,k}\right)$. We get the change in Lyapunov function for the multi-agent system by accumulating δV_k for all agents, $\delta V=\sum_{k=1}^N\delta V_k$ [3]. Therefore,

$$\begin{aligned}\delta V&=-\sum_{k=1}^N\boldsymbol{\eta}_{t-1,k}^{\top}\mathbf{G}_{t,k}\boldsymbol{\eta}_{t-1,k}+2\epsilon\sum_{k=1}^N\boldsymbol{\eta}_{t-1,k}^{\top}\left(\mathbf{O}_k^s\mathbf{z}_{t-1,k}\right)\\&+\epsilon^2\sum_{k=1}^N\left(\mathbf{O}_k^s\mathbf{z}_{t-1,k}\right)^{\top}\mathcal{D}_{t,k}\left(\mathbf{O}_k^s\mathbf{z}_{t-1,k}\right)\\&=-\boldsymbol{\eta}_{t-1}^{\top}\mathbb{G}_t\boldsymbol{\eta}_{t-1}-2\epsilon\boldsymbol{\eta}_{t-1}^{\top}\mathbb{A}^F\boldsymbol{\eta}_{t-1}\\&+\epsilon^2\boldsymbol{\eta}_{t-1}^{\top}\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\boldsymbol{\eta}_{t-1}\end{aligned}\tag{12}$$

where, $\boldsymbol{\eta}_t=\left[\boldsymbol{\eta}_{t,1}^{\top}\cdots\boldsymbol{\eta}_{t,N}^{\top}\right]^{\top}$, $\mathbb{D}_t=Blockdiag\left[\mathcal{D}_{t,1}\cdots\mathcal{D}_{t,N}\right]$, $\mathbb{G}_t=Blockdiag\left[\mathbf{G}_{t,1}\cdots\mathbf{G}_{t,N}\right]$ and $\mathbb{A}^F$ is defined in equation (8). From the properties of block-diagonal matrices, $\lambda_{min}(\mathbb{G}_t)=\min_k\lambda_{min}(\mathbf{G}_{t,k})$. In addition to that, the term $\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F$ exhibits a weighted symmetry and thus have real and positive eigenvalues. Hence, the error dynamics of the multi-agent system will be globally asymptotically stable if and only if $\delta V<0$ i.e., the matrix $\left(\mathbb{G}_t-\epsilon^2\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\right)$ is positive definite. Using Lemma 2, this condition gives the bounds on $|\epsilon|$ in three scenarios. ■

Theorem 1 is system specific as can be seen from the three possible upper bounds in the degree of consensus participation.

IV. Effect of Lossy Communication Network

In the proposed method of agent based filtering, it is evident from Algorithm 1 that only the information about relevant state elements are being exchanged among neighbors at the final correction steps. Therefore, the inter-agent two way information exchange plays an important role in agent based Kalman filtering and can be hampered if the underlying communication link fails. In smart grid perspective, this is crucial, since, an authentic, reliable and timely sharing of state variable information can ensure stable operation of the grid with the desired level of satisfaction [17].

These circumstances can be simulated by introducing random link failures (RLF). Mathematically, effect of RLF can be analyzed by inserting Bernoulli random variables $\zeta_{i,k}(t)$ in step 7 of Algorithm 1. These random variables have the following probability mass functions,

$$\zeta_{i,k}(t)=\begin{cases}0 & \text{with Prob.}\quad\rho_{i,k}\\1 & \text{with Prob.}\quad 1-\rho_{i,k}\end{cases}\quad;\forall i\in\mathcal{S}_k,\forall k.\tag{13}$$

Here, $\rho_{i,k}$ represents the probability of failure to send information from agent i to a neighbor agent k. From the properties of Bernoulli random variables, $\mathbb{E}[\zeta_{i,k}(t)]=1-\rho_{i,k}$. Consequently, the right hand side of step 7 in Algorithm 1 becomes,

$$\mathbf{S}_k\hat{\mathbf{b}}_{t,k}+\mathbf{W}_{t,k}^s\sum_{i\in\mathcal{S}_k}\zeta_{i,k}(t)\left(\mathbf{P}_{i,k}\mathbf{S}_i\hat{\mathbf{x}}_{t,i|t-1}-\mathbf{L}_{i,k}\mathbf{S}_k\hat{\mathbf{x}}_{t,k|t-1}\right)\tag{14}$$

Thus, it is evident that only the consensus part of equation (14) is affected by communication network. Additionally, the introduction of Bernoulli random variable $\zeta_{i,k}(t)$ randomizes equation (8) in the following way,

$$[\mathbb{A}_t^F]_{(k,l)^{th}block} = \begin{cases} -\zeta_{l,k}(t)\mathbf{O}_k^s\mathbf{P}_{l,k}\mathbf{S}_l\mathbf{F}_l; & l \in \mathcal{S}_k \\ \sum_{i\in\mathcal{S}_k}\zeta_{i,k}(t)\mathbf{O}_k^s\mathbf{L}_{i,k}\mathbf{S}_k\mathbf{F}_k; & l = k \\ \mathbf{0}_{n_k\times n_l}; & otherwise. \end{cases} \tag{15}$$

The following assumptions are made in analyzing the lossy network effect over the agent based estimation problem:

Assumption 1 : The random events $\zeta_{i,k}(t)$ and $\zeta_{k,i}(t)$ are independent of each other $\forall i \in \mathcal{S}_k; \forall k$.

Assumption 2 : The random events $\zeta_{i,k}(t)$ and $\zeta_{j,l}(t)$ are independent of each other $\forall i \neq j; \forall k \neq l$.

Now, we represent a corollary of Theorem 1 for the optimal choice of $\mathbf{W}_{t,k}^s$ under lossy communication network.

Corollary 1. *If the probability of failure for any neighboring agent pair (i,k) in an N-agent system follows assumptions 1,2 and equals to ρ, the stochastic error dynamics of the associated Kalman Consensus filter is globally asymptotically stable if, $\mathbf{W}_{t,k}^s = \epsilon\left(\mathbf{O}_k^s\right)^\dagger \mathbf{M}_{t,k|t-1}\left(\mathbf{F}_k^{-1}\right)^\top \mathbf{O}_k^s; \forall k, \forall t$; and the sufficient conditions on the level of consensus to guarantee stability are,*

Scenario 1: $\quad |\epsilon| < \sqrt{\frac{\min_k \lambda_{min}(\mathbf{G}_{t,k})}{\mathbb{E}\left[\lambda_{max}\left(\mathbb{A}_t^{F\top}\mathbb{D}_t\mathbb{A}_t^F\right)\right]}}$

Scenario 2: $\quad |\epsilon| < \sqrt{\frac{\max_k \lambda_{max}(\mathbf{G}_{t,k})}{\mathbb{E}\left[\lambda_{max}\left(\mathbb{A}_t^{F\top}\mathbb{D}_t\mathbb{A}_t^F\right)\right]}}$

Scenario 3: $\quad |\epsilon|(1-\rho) < \sqrt{\frac{\max_k \lambda_{max}(\mathbf{G}_{t,k})}{\lambda_{min}\left(\mathbb{A}^{F\top}\mathbb{D}_t\mathbb{A}^F\right)}}$

Proof: From equations (2), (5) and (14) and steps 2,6 of Algorithm 1, we express the homogeneous equation governing the error dynamics of the k^{th} agent,

$$\boldsymbol{\eta}_{t,k} = \mathcal{C}_{t,k}\boldsymbol{\eta}_{t-1,k} + \mathbf{O}_k^s\mathbf{W}_{t,k}^s\underline{\mathbf{z}}_{t-1,k} \tag{16}$$

where, $\boldsymbol{\eta}_{t,k}$ is defined as in the proof of Theorem 1 and $\underline{\mathbf{z}}_{t-1,k} = \sum_{i\in\mathcal{S}_k}\zeta_{i,k}(t)\left(\mathbf{P}_{i,k}\mathbf{S}_i\mathbf{F}_i\boldsymbol{\eta}_{t-1,i} - \mathbf{L}_{i,k}\mathbf{S}_k\mathbf{F}_k\boldsymbol{\eta}_{t-1,k}\right)$. The stochastic nature introduced by RLF is reflected through an *underline* embedded below the $\mathbf{z}$ vector. Furthermore, if every agent pair (i,k) within the neighborhood fails to communicate with the same probability ρ, then $\rho_{i,k} = \rho, \forall i \in \mathcal{S}_k; \forall k$. Hence, from equation (15), $\mathbb{E}\left(\mathbb{A}_t^F\right) = (1-\rho)\mathbb{A}^F$. The corresponding change in k^{th} agent's stochastic Lyapunov function is given by,

$$\begin{aligned}\delta\underline{V}_k &= \underline{V}_k(t) - \underline{V}_k(t-1) \\ &= -\boldsymbol{\eta}_{t-1,k}^\top\mathbf{G}_{t,k}\boldsymbol{\eta}_{t-1,k} \\ &\quad + 2\boldsymbol{\eta}_{t-1,k}^\top\left(\mathcal{C}_{t,k}^\top\mathbf{M}_{t,k|t}^{-1}\mathbf{O}_k^s\mathbf{W}_{t,k}^s\right)\underline{\mathbf{z}}_{t-1,k} \\ &\quad + \underline{\mathbf{z}}_{t-1,k}^\top\left(\mathbf{O}_k^s\mathbf{W}_{t,k}^s\right)^\top\mathbf{M}_{t,k|t}^{-1}\mathbf{O}_k^s\mathbf{W}_{t,k}^s\underline{\mathbf{z}}_{t-1,k}\end{aligned} \tag{17}$$

The above equation have similar form as in equation (11). Hence, $\mathbf{W}_{t,k}^s$ is the same as in Theorem 1. Consequently, the third term reduces to $\epsilon^2\left(\mathbf{O}_k^s\underline{\mathbf{z}}_{t-1,k}\right)^\top\mathcal{D}_{t,k}\left(\mathbf{O}_k^s\underline{\mathbf{z}}_{t-1,k}\right)$. It is observed that the Lyapunov function becomes stochastic because of $\underline{\mathbf{z}}_{t-1,k}$. The collected stochastic Lyapunov dynamics for the multi-agent system, $\delta\underline{V} = \sum_{k=1}^N \delta\underline{V}_k$, or, equivalently,

$$\begin{aligned}\delta\underline{V} &= -\sum_{k=1}^N \boldsymbol{\eta}_{t-1,k}^\top\mathbf{G}_{t,k}\boldsymbol{\eta}_{t-1,k} + 2\epsilon\sum_{k=1}^N \boldsymbol{\eta}_{t-1,k}^\top\mathbf{O}_k^s\underline{\mathbf{z}}_{t-1,k} \\ &\quad + \epsilon^2\sum_{k=1}^N\left(\mathbf{O}_k^s\underline{\mathbf{z}}_{t-1,k}\right)^\top\mathcal{D}_{t,k}\mathbf{O}_k^s\underline{\mathbf{z}}_{t-1,k} \\ &= -\boldsymbol{\eta}_{t-1}^\top\mathbb{G}_t\boldsymbol{\eta}_{t-1} - 2\epsilon\boldsymbol{\eta}_{t-1}^\top\mathbb{A}_t^F\boldsymbol{\eta}_{t-1} \\ &\quad + \epsilon^2\boldsymbol{\eta}_{t-1}^\top\underline{\mathbb{L}}_t\boldsymbol{\eta}_{t-1}\end{aligned} \tag{18}$$

where, $\underline{\mathbb{L}}_t = \mathbb{A}_t^{F\top}\mathbb{D}_t\mathbb{A}_t^F$ and $\mathbb{G}_t$, $\mathbb{D}_t$ are the block-diagonal matrices as defined in previous theorem. Thus, the stochastic error dynamics of the multi-agent system will be globally asymptotically stable if and only if $\delta\underline{V} < 0$, i,e., the random matrix $\left(\mathbb{G}_t - \epsilon^2\underline{\mathbb{L}}_t\right)$ is positive definite. Our goal is to express the sufficient condition of positive definiteness in terms of eigenvalue ratios according to Lemma 2. Now, the random nature of $\underline{\mathbb{L}}_t$ would result in random ratios of eigenvalues. Therefore, we are going to use the average eigenvalue ratios. Since, the matrices $\mathbb{G}_t$, $\underline{\mathbb{L}}_t$ are real, symmetric, positive definite and independent of each other,

$$\mathbb{E}\left[\lambda(\mathbb{G}_t)/\lambda(\underline{\mathbb{L}}_t)\right] = \lambda(\mathbb{G}_t)\mathbb{E}\left[1/\lambda(\underline{\mathbb{L}}_t)\right] > \lambda(\mathbb{G}_t)/\mathbb{E}[\lambda(\underline{\mathbb{L}})]$$

The above inequality is obtained from the properties of arithmetic and harmonic means of positive datasets. Consequently, the more strict bound is, $\epsilon^2 < \lambda(\mathbb{G}_t)/\mathbb{E}[\lambda(\underline{\mathbb{L}})]$. Now, from Jensen's inequality, $\lambda_{max}\left(\mathbb{E}[\underline{\mathbb{L}}_t]\right) \leq \mathbb{E}\left[\lambda_{max}(\underline{\mathbb{L}}_t)\right]$ and $\lambda_{min}\left(\mathbb{E}[\underline{\mathbb{L}}_t]\right) \geq \mathbb{E}\left[\lambda_{min}(\underline{\mathbb{L}}_t)\right]$. Using these properties and assumptions 1 and 2, we obtain sufficient bounds on ϵ^2 under the three scenarios.

Taking square roots on both sides of these strict inequalities complete the proof. ■

Intuitively, more consideration is expected in agent-wise consensus as the inter-agent communication link continues to become more and more prone to failure. This can be observed from Corollary 1 through the proportional relationship between the degree of participation in consensus "ϵ" and the rate of link failure "ρ".

V. Case Study

The optimal AKCF is applied to a custom 2-agent system "SYS" as illustrative example. The set of state elements for the 1^{st} agent is $\{a,b\}$ and that for the 2^{nd} agent $\{b,c\}$. It is clear that the state element b is shared between the two agents and rest of the state elements are strictly local to the respective agents. The system parameters described in Section II and corresponding extraction, reordering and projection matrices are given below.

- Global state transition matrix: $\mathbf{F} =$

$$\begin{bmatrix} 0.95 & 0 & 0 \\ 1 & 0.9 & 0 \\ 1 & 1 & 0.8 \end{bmatrix}$$

- Process noise: $\mathbf{Q} = diag(1.8, 0.9, 0.5)$

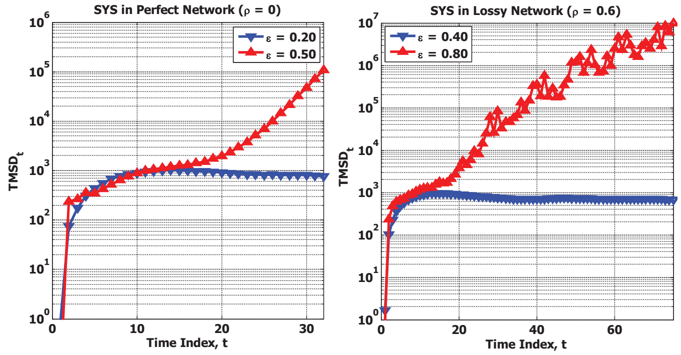

Fig. 3. AKCF Stability for SYS in Perfect and Lossy Network.

- State initialization:
  $\boldsymbol{\mu} = \begin{bmatrix}10 & 5 & 8\end{bmatrix}^\top$; $\boldsymbol{\Sigma} = diag(0.8, 0.2, 0.5)$
- Agent 1:
  $\mathbf{H}_1 = \begin{bmatrix}2 & 0\end{bmatrix}$; $\mathbf{R}_1 = 0.0648$; $\mathbf{S}_1 = \begin{bmatrix}0 & 1\end{bmatrix}$; $\mathbf{O}_1^s = \mathbf{S}_1^\top$; $\mathbf{U}_1 = \begin{bmatrix}1 & 0\end{bmatrix}$; $\mathbf{O}_1^u = \mathbf{U}_1^\top$; $\mathbf{P}_{2,1} = 1$; $\mathbf{L}_{2,1} = 1$;
  $\mathbf{T}_1 = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}$
- Agent 2: $\mathbf{H}_2 = \begin{bmatrix}3 & 0\end{bmatrix}$; $\mathbf{R}_2 = 0.05$; $\mathbf{S}_2 = \mathbf{U}_1$; $\mathbf{O}_2^s = \mathbf{O}_1^u$; $\mathbf{U}_2 = \mathbf{S}_1$; $\mathbf{O}_2^u = \mathbf{O}_1^s$; $\mathbf{P}_{1,2} = 1$; $\mathbf{L}_{1,2} = 1$;
  $\mathbf{T}_2 = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

The parameters are so chosen that SYS follows Scenario 2 of Lemma 2. Table I lists the corresponding steady-state upper bounds of $|\epsilon|$ under perfect and lossy communication network following Theorem 1 and Corollary 1, respectively.

TABLE I
STEADY STATE BOUNDS FOR SYS

	ρ	$\vert\epsilon\vert$ **Upper bound**
Perfect Network	0	0.3849
Lossy Network	0.2	0.4103
	0.4	0.4410
	0.6	0.4791
	0.8	0.5279

The stability performance of the proposed optimal AKCF is investigated by running 1000 independent Monte Carlo trials over SYS at each time step. These trials simulate the total mean squared deviation, (TMSD), which has the following definition,

$$\mathbf{TMSD}_t = \sum_{k=1}^{2} \mathbb{E}\left[\boldsymbol{\eta}_{t,k}^\top \boldsymbol{\eta}_{t,k}\right] \tag{19}$$

where, the estimation error, $\boldsymbol{\eta}_{t,k} = \hat{\mathbf{x}}_{t,k|t} - \mathbf{T}_k \mathbf{x}_t$, the true states $\mathbf{x}_t$ being obtained according to equation (1). TMSD is used as an indication of AKCF stability under different degree of consensus and communication reliability. The filter is stable if TMSD values converge over time. In Fig. 3 this metric shows the stability performance under perfect and lossy network. Consistent with Theorem 1, it is observed that the filter becomes unstable when $\epsilon > 0.3849$ under perfect communication. Additionally, as defined in Corollary 1, stability is preserved under lossy network as long as the the degree of participation in consensus does not exceed 0.4791 given the rate of link failure is 6 out of 10.

VI. Conclusions and Future Work

An agent based optimally weighted Kalman consensus filter is proposed. Through Lyapunov stability analysis, the optimal degree of participation in consensus is derived under both perfect and lossy communication network. Future work will investigate the impact on stability when agents are given the option to either aggregate their measurements and/or exchange information.

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