Indices

1 Laws of Indices

In this session we will look at indices, or powers. Both names may be used.

Indices are basically a shorthand way of writing multiplications of the same number
Consider the following
What if we are asked to evaluate "$3z^2$ multiplied by$(2z)^2$?"
We can of course do this by writing everything out explicitly and seeing what the result is, however this can be quite time consuming.
To speed things up we have a set of rules for index notation which allow us to answers these questions in a much quicker manner, which we will now introduce.

1.1 The First Law

Suppose we have x^2 and we want to multiply it by x^4. Writing this out explicitly we have

$$x^2 \times x^4 = x \times x \times x \times x \times x \times x$$

which is clearly equal to x^6.
This suggests our first rule which states that:

If we have $x^a \times x^b$ we add the indices to get:

$$x^a \times x^b = x^{a+b}$$

Examples Evaluate

$$2^2 \cdot 2^3 \qquad a^7 \cdot a^{35} \qquad b^2 \cdot b^3 \cdot b^4$$

1.2 The Second Law

Suppose we have x^6 and we want to divide it x^2. Writing this out explicitly we have

$$x^6 \div x^2 = \frac{x \cdot x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$$

If we divide out the common factors of x i.e.

$$x^6 \div x^2 = \left(\frac{x \cdot x}{x \cdot x}\right) \cdot x \cdot x \cdot x \cdot x = x \cdot x \cdot x \cdot x = x^4$$

This suggests our second rule which states that:

If we have $x^a \div x^b$ we subtract the indices to get:

$$x^a \div x^b = x^{a-b}$$

Examples
Evaluate

$$2^7 \div 2^3 \qquad a^{19} \div a^7$$

1.3 The Third Law

Suppose we have x^2 and we want to raise it to the power of 3 . This means:

$$(x^2)^3 = x^2 \cdot x^2 \cdot x^2$$

Using the first rule we simply add the indices together giving:

$$(x^2)^3 = x^6$$

Now notice that $2 \cdot 3 = 6$. This suggests the third rule:

$$(x^a)^b = x^{ab}$$

Examples
Evaluate

$$(2^2)^3 \qquad (a^4)^7$$

1.4 The Fourth Law

Consider x^4 divided by itself, anything divided by itself must be 1, i.e.

$$x^4 \div x^4 = 1$$

However if we apply the second rule to this expression we end up with:

$$x^4 \div x^4 = x^{4-4} = x^0$$

We appear to have obtained a different answer. The only logical conclusion to this is that

$$x^0 = 1$$

This means that any number raised to the power of 0 is 1, i.e.

$$3^0 = 1 \qquad (2000)^0 = 1 \qquad (-4)^0 = 1.$$

1.5 The Fifth Law

Let us return to the division powers of x again. This time lets consider

$$x^2 \div x^6 = \frac{x \;\cdot x}{x \cdot x \cdot x \cdot x \cdot x \cdot x}$$

Once more factorising out whats in common on the top and the bottom we have

$$x^2 \div x^6 = \frac{x \cdot x}{x \cdot x} \cdot \frac{1}{x \cdot x \cdot x \cdot x} = \frac{1}{x^4}$$

However, by using our second rule we also have

$$x^2 \div x^6 = x^{2-6} = x^{-4}$$

Therefore, the only logical conclusion is that

$$x^{-4} = \frac{1}{x^4}$$

In general we have that

$$x^{-a} = \frac{1}{x^a}$$

Examples
Re-write the following as proper fractions

$$2^{-2} \qquad 3^{-3}$$

Consider the fraction $\frac{1}{x^{-2}}$, using the above rule this means

$$\frac{1}{\frac{1}{x^2}} = 1 \cdot \frac{x^2}{1} = x^2$$

In general we have

$$\frac{1}{x^{-a}} = x^a$$

1.6 The Sixth Law

The sixth and final rule has to do with fractional powers, consider the equation

$$3^2 = 9$$

Let's raise both sides of this equation to the power of 1/2, we then have that

$$(3^2)^{1/2} = 9^{1/2}$$

Applying the third rule to the left hand side gives us

$$3 = 9^{1/2}$$

We know however that 3 is the square root of 9, therefore we have the result

$$x^{1/2} = \sqrt{x}$$

Similarly

$$x^{1/3} = \sqrt[3]{x}$$

and in general

$$x^{1/a} = \sqrt[a]{x}$$

This is the sixth rule of indices

Examples
Evaluate

$$a) 36^{1/2} \qquad b) 27^{1/3} \qquad c) 16^{1/4}$$

1.7 Summary of Laws of Indices

1st Rule: $x^a \cdot x^b = x^{a+b}$

2nd Rule: $x^a \div x^b = x^{a-b}$

3rd Rule: $(x^a)^b = x^{ab}$

4th Rule: $x^0 = 1$

5th Rule: $x^{-a} = \frac{1}{x^a}$

6th Rule: $x^{1/a} = \sqrt[a]{x}$.

1.8 Final point

What if we are asked to evaluate an expression like $16^{3/4}$?

Well using the 3rd rule above we can write this as

$$16^{3/4} = (16^{1/4})^3 = 2^3 = 8$$

Note we could also have answered this question in the following manner

$$16^{3/4} = (16^3)^{1/4} = 4096^{1/4} = 8$$

both methods are equally valid, however as is clear from this example the first method is usually the easiest, as it involves recognising powers of smaller numbers (i.e. it is easier to recognize that the fourth root of 16 is 2 rather than spotting that $8^4 = 4096$)

In general we have that

$$x^{a/b} = (x^a)^{1/b} = \sqrt[b]{x^a}$$

or

$$x^{a/b} = (x^{1/b})^a = (\sqrt[b]{x})^a$$

Examples
Evaluate

$$a) 9^{5/2} \qquad b) 125^{2/3} \qquad c) 8^{-2/3}$$

Exercises
Write the following in a single index form

$$a) 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \qquad b) 6^7 \cdot 6^9 \qquad c) \frac{x^7}{x^{19}} \qquad d) \sqrt[3]{x^5}$$

Simplify each of the following expressions

$$a) 3x^2 \cdot 2x^4 \qquad b) 12x^8 \cdot \frac{1}{3x^2} \qquad c) \sqrt{a} \cdot a^3 \qquad d) \frac{\sqrt[3]{x^2}}{\sqrt[4]{x}}$$

$$e) \left(\frac{x^2y}{z^3}\right)^{\frac{1}{3}} \times \left(\frac{zy}{x^2}\right)^{\frac{3}{2}} \qquad f) \sqrt[3]{a^2b}(a^{\frac{1}{2}}c)^{\frac{-2}{3}}\frac{1}{\sqrt{cb^3}}$$