

# Pairings for Beginners Notes

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## Elliptic curve group law explicit derivations

### 0.1 General case where $P \neq Q$

Consider an elliptic curve  $E$  defined over a field  $K$ . Let  $P = (x_1, y_1)$ ,  $Q = (x_2, y_2)$  be two points on  $E/K$  where  $P \neq Q$ . Let  $y = \lambda x + v$  be the tangent line to  $E$  where

$$\lambda = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad v = y_2 - \lambda x_2$$

Let  $R = (x_3, y_3) = P + Q$ . We know that  $P, Q, R$  define points on  $E$ , so by definition  $y_i^3 = x_i^3 + ax_i + b$  where  $i \in \{1, 2, 3\}$ , and by substituting  $\lambda x + v$  into the curve equation, we have

$$\begin{aligned} (\lambda x + v)^2 &= x^3 + ax + b \\ 0 &= (x^3 + ax + b) - (\lambda x + v)^2 \\ &= (x - x_1)(x - x_2)(x - x_3) \end{aligned}$$

The last polynomial comes from the fact that we know that there are 3 points such that the curve equation is satisfied, so the resulting polynomial will have 3 roots equivalent to  $x_1, x_2, x_3$ . Expanding  $(x - x_1)(x - x_2)(x - x_3)$  yields

$$x^3 - (x_1 + x_2 + x_3)x^2 - (x_1 + x_2 + x_3)x + x(x_1x_2 + x_1x_3 + x_2x_3)$$

and expanding  $(x^3 + ax + b) - (\lambda x + v)^2$  yields

$$x^3 - \lambda^2 x^2 + (a - 2\lambda v)x + b - v^2$$

We see clearly that the coefficient of  $x^2$  determines  $x_3$  via  $\lambda^2 = (x_1 + x_2 + x_3)$ , so

$$x_3 = \lambda^2 - x_1 - x_2$$

From here, we can find  $y_3$  via substituting  $x_3$  into the tangent line equation  $y = \lambda x + v$  above and negating the result due to chord-and-tangent, which yields

$$y_3 = -(\lambda x + v)$$

Thus, we have in the general case when  $P \neq Q$  that

$$\begin{aligned} x_3 &= \lambda^2 - x_1 - x_2 \\ y_3 &= -(\lambda x + v) \end{aligned} \tag{1}$$

## 0.2 General case where $P = Q$

When  $P = Q$ , then the tangent line becomes the derivative of the curve equation. We calculate the derivative via implicit differentiation as follows:

$$\frac{dy}{dx}(y^2) = \frac{dy}{dx}(x^3 + ax + b)$$

$$2y \frac{dy}{dx} = 3x^2 + a$$

$$\frac{dy}{dx} = \frac{3x^2 + a}{2y}$$

Substitute  $x_1$  into  $\lambda = dy/dx$  and we obtain

$$\lambda = \frac{3x_1^2 + a}{2y_1}$$

Again, we refer to  $y = \lambda x + v$  as defined in the previous subsection and substitute this into the curve equation to obtain

$$\begin{aligned} y^2 &= x^3 + ax + b \\ (\lambda x + v)^2 &= x^3 + ax + b \\ 0 &= (x^3 + ax + b) - (\lambda x + v)^2 \\ &= (x - x_1)^2(x - x_3) \end{aligned}$$

The derivation is similar to the general case, but since  $P = Q$ , we have  $x_1 = x_2$  and  $y_1 = y_2$ , so we obtain the cubic polynomial in the last step of the derivation with a repeated root at  $x = x_1$ . Expanding this polynomial yields

$$x^3 - (2x_1 - x_3)x^2 + x_1^2x + 2x_1x_3x - x_1^2x_3$$

and expanding  $(x^3 + ax + b) - (\lambda x + v)^2$  yields

$$x^3 - \lambda^2 x^2 + (a - 2\lambda v)x + b - v^2.$$

Again, we see that the value of  $x_3$  is determined by the coefficient of  $x^2$ . We have that

$$\begin{aligned} \lambda^2 &= 2x_1 - x_3 \\ \implies x_3 &= \lambda^2 - 2x_1 \end{aligned}$$

To find  $y_3$ , we just substitute  $x_3$  into the tangent line  $\lambda x + v$  and reflect it about the  $x$ -axis via the chord-and-tangent rule to get  $y = -(\lambda x_3 + v)$ . Thus, we have in the general case that

$$\begin{aligned} x_3 &= \lambda^2 - 2x_1 \\ y_3 &= -(\lambda x_3 + v) \end{aligned} \tag{2}$$

### 0.3 Special cases

There are a few special cases to consider for point addition. With  $O$  being the point at infinity, we have  $P + O = O + P = P$ . If  $x_1 = x_2$  and  $y_2 = -y_1$ , then  $P = -Q$ , so  $P + Q = P - P = O$ . If  $y_2 = y_1 \neq 0$ , then  $P = Q$  and we can refer to the point doubling formulas in the previous sections. As Costello notes, these cases tend not to arise in optimized cryptographic protocols since the parameters can be chosen in a manner that avoids them.

## Group law optimizations via projective coordinates

In the previous section, we gave explicit derivations for the group laws in affine space, but in practice the group laws would be computed in projective space instead due to the fact that this avoids field inversions, which are computationally expensive. Recall that we can map all points from affine to projective space via  $(x, y) \mapsto (\lambda x, \lambda y, \lambda)$  and map back to affine space via  $(X, Y, Z) \mapsto (X/Z, Y/Z)$ . When  $P \neq Q$ , we can make the substitution  $X_i/Z_i, Y_i/Z_i$  for  $i \in \{1, 2, 3\}$  into the affine group law equations for  $P + Q$  to obtain

$$\frac{X_3}{Z_3} = \left( \frac{Y_2}{Z_2} - \frac{Y_1}{Z_1} \right)^2 - \frac{X_1}{Z_1} - \frac{X_2}{Z_2} \quad \frac{Y_3}{Z_3} = \left( \frac{Y_2}{Z_2} - \frac{Y_1}{Z_1} \right)^2 \left( \frac{X_1}{Z_1} - \frac{X_3}{Z_3} \right) - \frac{Y_1}{Z_1}$$

Expanding by a common denominator  $Z_3$  such that the denominators of both expressions divide  $Z_3$  yields

$$\begin{aligned} X_3 &= (X_1 Z_2 - X_2 Z_1)(Z_1 Z_2 (Y_1 Z_2 - Y_2 Z_1)^2 - (X_1 Z_2 - X_2 Z_1)^2 (X_1 Z_2 + X_2 Z_1)) \\ Y_3 &= Z_1 Z_2 (X_2 Y_1 - X_1 Y_2)(X_1 Z_2 - X_2 Z_1)^2 - \\ &\quad (Y_1 Z_2 - Y_2 Z_1)((Y_1 Z_2 - Y_2 Z_1)^2 Z_1 Z_2 - (X_1 Z_2 + X_2 Z_1)(X_1 Z_2 - X_2 Z_1)^2) \\ Z_3 &= Z_1 Z_2 (X_1 Z_2 - X_2 Z_1)^3 \end{aligned} \tag{3}$$

When  $P = Q$ , we repeat a similar process but via substituting  $X_i/Z_i, Y_i/Z_i$  into the derivative of the curve equation as follows:

$$\frac{X_3}{Z_3} = \left( \frac{3X_1^2 + aZ_1^2}{2Y_1 Z_1} \right)^2 - \frac{2X_1}{Z_1} \quad \frac{Y_3}{Z_3} = \left( \frac{3X_1^2 + aZ_1^2}{2Y_1 Z_1} \right) \left( \frac{X_1}{Z_1} - \frac{X_3}{Z_3} \right) - \frac{Y_1}{Z_1}$$

It's clear that we only need to find a value of  $Z_3$  such that the denominator of the right hand side of  $Y_3/Z_3$  divides  $Z_3$ . Thus, we set  $Z_3 = 8Y_1^3 Z_1^3$ , which yields

$$\begin{aligned} X_3 &= 2a^2 Y_1 Z_1^5 + 12a X_1^2 Y_1 Z_1^3 + 18 X_1^4 Y_1 Z_1 - 16 X_1 Y_1^3 Z_1^2 \\ Y_3 &= -a^3 Z_1^6 - 9a^2 X_1^2 Z_1^4 - 27a X_1^4 Z_1^2 + 12a X_1 Y_1^2 Z_1^3 - 27 X_1^6 + 36 X_1^3 Y_1^2 Z_1 - 8 Y_1^4 Z_1^2 \\ Z_3 &= 8 Y_1^3 Z_1^3 \end{aligned} \tag{4}$$

**Note:** The next few chapters are less computationally heavy and more theoretical, so I switched to bullets to better organize information and key points.

## Torsion, point counting, endomorphisms

We outline some noteworthy points in this section of Pairings for Beginners:

### 0.4 Structure of $E(\mathbb{F}_q)$

- $E(\mathbb{F}_q)$  is the group of  $K$ -rational points on  $E$ . The  $r$ -torsion of  $E(\mathbb{F}_q)$  is the set of points  $Q \subseteq E(\mathbb{F}_q)$  such that for all points  $P \in Q$ ,  $[r]P = O$ . By Lagrange's theorem, to get a point of order  $r|105$ , we can multiply each point by  $\#E(\mathbb{F}_q)/r$ , called the cofactor of  $r$ .
- It turns out that  $E(\mathbb{F}_q)$  is isomorphic to a cyclic group or the direct product of two cyclic groups  $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$  with  $n_1 | n_2$ . We prefer  $E(\mathbb{F}_q)$  to be as cyclic as possible and  $\#E(\mathbb{F}_q)$  to be as close to prime as possible since the runtime of discrete logarithm algorithms is bottlenecked by the largest prime factor of  $\#E(\mathbb{F}_q)$ .
- By the Hasse bound, we have that  $-2\sqrt{q} \leq \#E(\mathbb{F}_q) - (q+1) \leq 2\sqrt{q}$ ; no proof is given, but the intuition for why  $\#E(\mathbb{F}_q) \approx q+1$  comes from the fact that  $x^3 + ax + b$  is a quadratic residue modulo  $q$  for around half the values of  $x$  less than  $q$ , which yields 2 points  $(x, \pm\sqrt{x^3 + ax + b})$  unless  $x^3 + ax + b = 0$ , which yields 1 point.

### 0.5 The Frobenius endomorphism

- The Frobenius endomorphism is the mapping

$$\pi : E \rightarrow E, \quad (x, y) \mapsto (x^q, y^q)$$

When  $E$  is defined over  $\mathbb{F}_q$  where  $q = p^n$  for some positive integer  $n$ , the Frobenius endomorphism fixes all elements of  $E(\mathbb{F}_q)$ . In fact,  $E$  is defined over  $\mathbb{F}_q$  if and only if it is invariant under  $\pi$ . To show this, realize that by Lagrange's theorem, all elements of  $\mathbb{F}_q$  have order dividing  $q-1$ , so for any  $x \in \mathbb{F}_q$ ,  $x^q = x^{q-1} \cdot x = 1 \cdot x = x$ . We can also take the  $k$ -th iteration of the Frobenius endomorphism, which is defined as

$$\pi^k : E \rightarrow E, \quad (x, y) \mapsto (x^{q^k}, y^{q^k})$$

When  $E$  is defined over  $\mathbb{F}_{q^k}$ ,  $\pi^k$  fixes exactly those elements of  $E(\mathbb{F}_{q^k})$ . We can see this by extending our argument above to  $\mathbb{F}_{q^k}$ , which we claim is an algebraic extension of  $\mathbb{F}_q$ . In particular, every element of  $\mathbb{F}_{q^k}$  is a root of the polynomial  $x^{q^k} - x$ , and this is again easy to see by Lagrange's theorem; the multiplicative order of any  $x \in \mathbb{F}_{q^k}$  must divide  $q^k - 1$ , so  $x^{q^k-1} = 1$ , and  $x(x^{q^k-1}) = x^{q^k} = x$ . Finally, when  $E$  is defined over the algebraic closure of  $\mathbb{F}_q$  denoted as  $\bar{\mathbb{F}}_q$ , the values of  $(x, y) \in E(\bar{\mathbb{F}}_q)$  come from  $\bar{\mathbb{F}}_q$ , so  $\pi^k$  acts nontrivially only on  $(x, y) \in E(\bar{\mathbb{F}}_q) \setminus E(\mathbb{F}_{q^k})$ .

- In general,  $\#E(\mathbb{F}_q) = \# \ker([1] - \pi) = \deg([1] - \pi)$  where  $[n]$  denotes the multiplication-by- $n$  map on  $E$ .

## Divisors

The following notations are used extensively in the next few sections of Pairings for Beginners, so we define them here.

- A divisor  $D$  on an elliptic curve  $E$  is a multi-set of points denoted as

$$D = \sum_{P \in E(\mathbb{F}_q)} n_P(P)$$

where  $n_P$  is the multiplicity of a point  $P$  in  $D$ . Note that divisor sums are denoted by  $(\cdot)$  and point addition / multiplication by  $[\cdot]$ .

- The set of all divisors  $\text{Div}_{\bar{\mathbb{F}}_q}(E)$  forms a group with addition defined naturally and the identity element being the zero divisor with all  $n_P = 0$ .
- The degree of a divisor  $D$  is defined as

$$\text{Deg}(D) = \sum_{P \in E(\bar{\mathbb{F}}_q)} n_P$$

and the support of  $D$  is the set

$$\text{supp}(D) = \{P \in E(\bar{\mathbb{F}}_q) \mid n_P \neq 0\}$$

As an example, let  $D_1 = 2(P) - 3(Q)$ ,  $D_2 = 3(Q) + (R) - S$  where  $P, Q, R, S \in E(\bar{\mathbb{F}}_q)$ . Then  $\text{Deg}(D_1) = 2 - 3 = -1$ , and  $\text{Deg}(D_2) = 3 + 1 - 1 = 3$ , and more generally  $\text{Deg}(D_1 + D_2) = \text{Deg}(D_1) + \text{Deg}(D_2)$  for any divisors  $D_1, D_2$ . Furthermore,  $\text{supp}(D_1) = \{P, Q\}$ ,  $\text{supp}(D_2) = \{Q, R, S\}$ , and  $\text{supp}(D_1 + D_2) = \{P, R, S\}$ .

- Associating a divisor with a function  $f$  on  $E$  is a convenient way to count intersection points of  $f$  (and their multiplicities) with  $E$ . The divisor of a function is denoted as

$$(f) = \sum_{P \in E(\mathbb{F}_q)} \text{ord}_P(f)(P)$$

where  $\text{ord}_P(f)$  is the multiplicity of  $f$  at  $P$ , which is positive if  $f$  has a zero at  $P$  and negative if  $f$  has a pole at  $P$ . The algebra between function divisors is a consequence of the algebra between functions. In particular,  $(fg) = (f) + (g)$  and  $(f/g) = (f) - (g)$ .

### 0.6 The divisor class group