Confidence Intervals and Hypothesis Testing
Pen and Paper Questions

ANSWERS

(1)

In order to define the confidence interval, we need to find the values at the lower and upper boundaries of a range, symmetrical with respect to the measured mean, in the *sampling distribution*. The range must include a percentage of values such that the percentage is equal to the confidence level, which in our case is 95%. The upper and lower boundaries are, hence, values on the x-axis of the distribution that 'fence out' the remaining 5% of possible measured means (2.5% on each tail of the distribution).

We can use the standard normal (z) distribution table as the sample size is greater than 30. Also, as we have a one-tailed table, we need to look up the p-value corresponding to one tail, which is 0.025.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which $P(Z > z) = p$, where the distribution of Z is $N(0, 1)$.

p	z	p	z	p	z	p	z	p	z
.50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
.45	0.126	.14	1.080	.024	1.977	.009	2.366	$.0^{3}3$	3.432
.40	0.253	.13	1.126	.023	1.995	.008	2.409	$.0^{3}2$	3.540
.35	0.385	.12	1.175	.022	2.014	.007	2.457	$.0^{3}1$	3.719
.30	0.524	.11	1.227	.021	2.034	.006	2.512	$.0^{4}5$	3.891

The z-value (an x-axis value in the standardised normal distribution) that is lower than 2.5% of all possible values is 1.96 (see the picture). As the standardised normal distribution is symmetrical around the mean of 0, the value that is higher than 2.5% of all possible values is -1.96.

The other calculation that we must do is to find the *standard error*, which is the standard deviation of the sampling distribution. It is equal to $\frac{s}{\sqrt{n}}$, where s is the standard deviation.

Finally, we can say, with a confidence of 95%, that the mean amount of shampoo in the bottles is in the range:

$$200ml \pm \frac{2ml}{10} \times 1.96 \qquad \text{or} \qquad 200ml \pm 0.39ml$$

(2)

In this hypothesis test the null hypothesis is that the mean value of the amount of shampoo in the bottles is 200ml:

$$H_0 : \mu = 200ml$$

$$H_a : \mu \neq 200ml$$

H_a is the alternative hypothesis, that the mean is different from 200ml.

For the hypothesis test we need to identify the lower and upper boundaries of the *sampling distribution* range symmetrical around the measured mean that 'fences out' the percentage of values equal to the significance level, which in our case is 1%. Similarly to the look-up in the previous problem, for the upper boundary we need to find the z-value that corresponds to a p-value of 0.005 (0.5%), since the lower boundary will 'fence out' the other 0.5% of the 1% in the negative (left) end tail of the distribution.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which $P(Z > z) = p$, where the distribution of Z is $N(0, 1)$.

p	z	p	z	p	z	p	z	p	z
.50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
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.30	0.524	.11	1.227	.021	2.034	.006	2.512	$.0^{4}5$	3.891
.25	0.674	.10	1.282	.020	2.054	.005	2.576	$.0^{4}1$	4.265
.24	0.706	.09	1.341	.019	2.075	.004	2.652	$.0^{5}5$	4.417
.23	0.739	.08	1.405	.018	2.097	.003	2.748	$.0^{5}1$	4.753
.22	0.772	.07	1.476	.017	2.120	.002	2.878	$.0^{6}5$	4.892
.21	0.806	.06	1.555	.016	2.144	.001	3.090	$.0^{6}1$	5.199
.20	0.842	.050	1.645	.015	2.170	$.0^{3}9$	3.121	$.0^{7}5$	5.327
.19	0.878	.045	1.695	.014	2.197	$.0^{3}8$	3.156	$.0^{7}1$	5.612
.18	0.915	.040	1.751	.013	2.226	$.0^{3}7$	3.195	$.0^{8}5$	5.731
.17	0.954	.035	1.812	.012	2.257	$.0^{3}6$	3.239	$.0^{8}1$	5.998
.16	0.994	.030	1.881	.011	2.290	$.0^{3}5$	3.291	$.0^{9}5$	6.109

The z-value we are looking for is 2.576. This is the upper boundary of the 'acceptable' range.

Now we calculate the z-value corresponding to the measured mean, in order to see where it falls with respect to the acceptable range. This we call the test statistic and calculate as:

$$T = \frac{\bar{x} - \mu_0}{\frac{S}{\sqrt{n}}}$$

where $\bar{x}$ is the mean calculated from the sample (197ml), μ_0 is the hypothesized population mean (200ml) and the fraction acting as nominator is the *standard error* i.e. the standard deviation of the sampling distribution.

We calculate the statistic:

$$T = \frac{197ml - 200ml}{\frac{2ml}{\sqrt{100}}} = 15$$

Note that the units (ml) have been cancelled out and that we have a dimensionless number, i.e. a z-value, which has a standardised normal distribution.

The last thing we need to do is to see whether our calculated test statistic falls within the 'acceptable' range:

$-2.576 \leq T \leq 2.576$ **?**

$-2.576 \leq 15 \leq 2.576$ **FALSE!**

It does not fall in the range, so we *reject* the null hypothesis. This means that there is statistical evidence that the mean volume of shampoo poured into the bottles is different from 200ml.

(3)

As the sample is small and we do not know the standard deviation, we will be using the t-distribution and the sample standard deviation, S. Calculate the mean.

$$\bar{x} = \frac{19 + 20 + 24}{3} = 21$$

Calculate the sample standard deviation.

$$S = \sqrt{\frac{\sum_i^n (x_i - \bar{x})^2}{n-1}} \sqrt{\frac{(19-21)^2 + (20-21)^2 + (24-21)^2)}{3-1}} = 2.65$$

Calculate the standard error.

$$SE = \frac{S}{\sqrt{n}} = 1.53$$

Calculate df.

$$df = n - 1 = 3 - 1 = 2$$

Look up the two-tailed t-table for $df = 2$ and p-value $1 - 0.975 = 0.025$.

The value should be **6.2**.

De-normalise the half-interval.

$$HI = 1.53 \times 6.2 = 9.47$$

State the confidence interval.

21 ± 9.47, with a confidence of 97.5%

(4)

As the sample is small and we do not know the standard deviation, we will be using the t-distribution and the sample standard deviation, S.

Hypotehsis statement:

$H_0 : \mu \geq 22$

$H_a : \mu < 22$

Calculate the mean.

$$\bar{x} = \frac{19 + 20 + 21 + 21 + 24}{5} = 21$$

Calculate the sample standard deviation.

$$S = \sqrt{\frac{\sum_i^n (x_i - \bar{x})^2}{n-1}}$$

$$S = \sqrt{\frac{(19-21)^2 + (20-21)^2 + (21-21)^2 + (21-21)^2 + (24-21)^2)}{5-1}} = 1.87$$

Calculate the standard error: $SE = \frac{S}{\sqrt{n}} = \frac{1.87}{\sqrt{5}} = 0.84$

Calculate the test statistic: $t = \frac{\bar{x} - \mu_0}{SE} = \frac{21-22}{0.84} = -1.19$

Calculate degrees of freedom: $df = n - 1 = 5 - 1 = 4$

The required test is one-tailed but the table is two-tailed. For the required one-tailed significance level of 0.025 we need to look up the **critical value for p=0.05 in the two-tailed table**.

The critical value is **-2.78** (we use a negative value because we are interested in the lower tail).

Test the statistic against the critical value: as -1.19 is not less than -2.78, at the 0.025 significance level we fail to reject the null hypothesis that the mean age of the concert goers is 22 or above.