Statistical Inference - Confidence Intervals and Hypothesis Tests

ANSWERS

Exercise 1 - Confidence interval

In order to define the confidence interval, we need to find the values at the lower and upper boundaries of a range, symmetrical with respect to the measured mean, in the *sampling distribution*. The range must include a percentage of values such that the percentage is equal to the confidence level, which in our case is 95%. The upper and lower boundaries are, hence, values on the x-axis of the distribution that 'fence out' the remaining 5% of possible measured means (2.5% on each tail of the distribution). The boundary values can be read out of the percentage point table for a suitable distribution. In this case, the sample is large ($n \geq 30$), which means that we can use the standard normal distribution (the z-distribution). This table is given in the appendix in the lab sheet. If the sample were small ($n < 30$), we would use a t-distribution table for degrees of freedom $df = n - 1$.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which $P(Z > z) = p$, where the distribution of Z is $N(0, 1)$.

p	z	p	z	p	z	p	z	p	z
.50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
.45	0.126	.14	1.080	.024	1.977	.009	2.366	$.0^{3}3$	3.432
.40	0.253	.13	1.126	.023	1.995	.008	2.409	$.0^{3}2$	3.540
.35	0.385	.12	1.175	.022	2.014	.007	2.457	$.0^{3}1$	3.719
.30	0.524	.11	1.227	.021	2.034	.006	2.512	$.0^{4}5$	3.891

The z-value (an x-axis value in the standardised normal distribution) that is lower than 2.5% of all possible values is 1.96 (see the picture). As the standardised normal distribution is symmetrical around the mean of 0, the value that is higher than 2.5% of all possible values is -1.96.

The other calculation that we must do is to find the *standard error*, which is the standard deviation of the sampling distribution. It is equal to $\frac{s}{\sqrt{n}}$, where s is the standard deviation.

Finally, we can say, with a confidence of 95%, that the mean amount of shampoo in the bottles is in the range:

$$200ml \pm \frac{2ml}{10} \times 1.96$$

or

$$200ml \pm 0.39ml$$

Exercise 2 - Hypothesis test

In this hypothesis test the null hypothesis is that the mean value of the amount of shampoo in the bottles is 200ml:

$$H_0 : \mu = 200ml$$

$$H_a : \mu \neq 200ml$$

H_a is the alternative hypothesis, that the mean is different from 200ml.

For the hypothesis test we need to identify the lower and upper boundaries of the *sampling distribution* range symmetrical around the measured mean that 'fences out' the percentage of values equal to the significance level, which in our case is 1%. Similarly to the look-up in the previous problem, for the upper boundary we need to find the z-value that corresponds to a p-value of 0.005 (0.5%), since the lower boundary will 'fence out' the other 0.5% of the 1% in the negative (left) end tail of the distribution. If we were dealing with a small sample ($n < 30$), we would use the t-distribution table instead of a z-distribution table.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which $P(Z > z) = p$, where the distribution of Z is $N(0, 1)$.

p	z	p	z	p	z	p	z	p	z
.50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
.45	0.126	.14	1.080	.024	1.977	.009	2.366	$.0^{3}3$	3.432
.40	0.253	.13	1.126	.023	1.995	.008	2.409	$.0^{3}2$	3.540
.35	0.385	.12	1.175	.022	2.014	.007	2.457	$.0^{3}1$	3.719
.30	0.524	.11	1.227	.021	2.034	.006	2.512	$.0^{4}5$	3.891
.25	0.674	.10	1.282	.020	2.054	.005	2.576	$.0^{4}1$	4.265
.24	0.706	.09	1.341	.019	2.075	.004	2.652	$.0^{5}5$	4.417
.23	0.739	.08	1.405	.018	2.097	.003	2.748	$.0^{5}1$	4.753
.22	0.772	.07	1.476	.017	2.120	.002	2.878	$.0^{6}5$	4.892
.21	0.806	.06	1.555	.016	2.144	.001	3.090	$.0^{6}1$	5.199
.20	0.842	.050	1.645	.015	2.170	$.0^{3}9$	3.121	$.0^{7}5$	5.327
.19	0.878	.045	1.695	.014	2.197	$.0^{3}8$	3.156	$.0^{7}1$	5.612
.18	0.915	.040	1.751	.013	2.226	$.0^{3}7$	3.195	$.0^{8}5$	5.731
.17	0.954	.035	1.812	.012	2.257	$.0^{3}6$	3.239	$.0^{8}1$	5.998
.16	0.994	.030	1.881	.011	2.290	$.0^{3}5$	3.291	$.0^{9}5$	6.109

The z-value we are looking for is 2.576. This is the upper boundary of the 'acceptable' range.

Now we calculate the z-value corresponding to the measured mean, in order to see where it falls with respect to the acceptable range. This we call the test statistic and calculate as:

$$T = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

where $\bar{x}$ is the mean calculated from the sample (197ml), μ_0 is the hypothesized population mean (200ml) and the fraction acting as nominator is the *standard error* i.e. the standard deviation of the sampling distribution.

We calculate the statistic:

$$T = \frac{197ml - 200ml}{\frac{2ml}{\sqrt{100}}} = -15$$

Note that the units (ml) have been cancelled out and that we have a dimensionless number, i.e. a z-value, which has a standardised normal distribution.

The last thing we need to do is to see whether our calculated test statistic falls within the 'acceptable' range:

$-2.576 \leq T \leq 2.576$ **?**

$-2.576 \leq -15 \leq 2.576$ **FALSE!**

It does not fall in the range, so we *reject* the null hypothesis.