Statistical Inference - Confidence Intervals and Hypothesis Tests

ANSWERS

Exercise 1 - Confidence interval

In order to define the confidence interval, we need to find the values at the lower and upper boundaries of a range, symmetrical with respect to the measured mean, in the sampling distribution. The range must include a percentage of values such that the percentage is equal to the confidence level, which in our case is 95%. The upper and lower boundaries are, hence, values on the x-axis of the distribution that 'fence out' the remaining 5% of possible measured means (2.5%) on each tail of the distribution. The boundary values can be read out of the percentage point table for a suitable distribution. In this case, the sample is large $(n \ge 30)$, which means that we can use the standard normal distribution (the z-distribution). This table is given in the appendix in the lab sheet. If the sample were small (n < 30), we would use a t-distribution table for degrees of freedom df = n - 1.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which P(Z > z) = p, where the distribution of Z is N(0, 1).

p	Z	p	Z	p	Z	p	Z	p	Z
.50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
.45	0.126	.14	1.080	.024	1.977	.009	2.366	$.0^{3}$ 3	3.432
.40	0.253	.13	1.126	.023	1.995	.008	2.409	$.0^{3}2$	3.540
.35	0.385	.12	1.175	.022	2.014	.007	2.457	$0^{3}1$	3.719
.30	0.524	.11	1.227	.021	2.034	.006	2.512	0^45	3.891

The z-value (an x-axis value in the standardised normal distribution) that is lower than 2.5% of all possible values is 1.96 (see the picture). As the standardised normal distribution is symmetrical around the mean of 0, the value that is higher than 2.5% of all possible values is -1.96.

The other calculation that we must do is to find the *standard error*, which is the standard deviation of the sampling distribution. It is equal to $\frac{s}{\sqrt{n}}$, where s is the standard deviation.

Finally, we can say, with a confidence of 95%, that the mean amount of shampoo in the bottles is in the range:

$$200ml \pm \frac{2ml}{10} \times 1.96$$

or

 $200ml \pm 0.39ml$

Exercise 2 - Hypothesis test

In this hypothesis test the null hypothesis is that the mean value of the amount of shampoo in the bottles is 200ml:

 $H_0: \mu = 200ml$

 $H_a: \mu \neq 200ml$

 H_a is the alternative hypothesis, that the mean is different from 200ml.

For the hypothesis test we need to identify the lower and upper boundaries of the *sampling distribution* range symmetrical around the measured mean that 'fences out' the percentage of values equal to the significance level, which in our case is 1%. Similarly to the look-up in the previous problem, for the upper boundary we need to find the z-value that corresponds to a p-value of 0.005~(0.5%), since the lower boundary will 'fence out' the other 0.5% of the 1% in the negative (left) end tail of the distribution. If we were dealing with a small sample (n < 30), we would use the t-distribution table instead of a z-distribution table.

Upper-tail percentage points of the standard normal distribution

The table gives the values of z for which P(Z > z) = p, where the distribution of Z is N(0, 1).

p	Z	p	Z	p	Z	p	Z	p	Z
50	0.000	.15	1.036	.025	1.960	.010	2.326	$.0^{3}4$	3.353
45	0.126	.14	1.080	.024	1.977	.009	2.366	$.0^{3}3$	3.432
40	0.253	.13	1.126	.023	1.995	.008	2.409	$.0^{3}2$	3.540
35	0.385	.12	1.175	.022	2.014	.007	2.457	$.0^{3}1$	3.719
30	0.524	.11	1.227	.021	2.034	.006	2.512	$.0^{4}5$	3.891
25	0.674	10	1 202	020	2.054	005	2 576	041	4.265
25	0.6/4	.10	1.282	.020	2.054				
24	0.706	.09	1.341	.019	2.075	.004	2.652		4.417
23	0.739	.08	1.405	.018	2.097	.003	2.748	$.0^{5}1$	4.753
22	0.772	.07	1.476	.017	2.120	.002	2.878	$.0^{6}5$	4.892
21	0.806	.06	1.555	.016	2.144	.001	3.090	$.0^{6}1$	5.199
				William Bridge					
20	0.842	.050	1.645	.015	2.170	0^{39}	3.121	$.0^{7}5 $	5.327
19	0.878	.045	1.695	.014	2.197	$.0^{3}8$	3.156	$.0^{7}1$	5.612
18	0.915	.040	1.751	.013	2.226	$.0^{3}7$	3.195	0.085	5.731
17	0.954	.035	1.812	.012	2.257	$.0^{3}6$	3.239	0.081	5.998
16	0.994	.030	1.881	.011	2.290	$.0^{3}5$	3.291	.095	6.109
	50 45 40 35 330 25 224 223 222 19 18 17	50 0.000 45 0.126 40 0.253 35 0.385 30 0.524 25 0.674 24 0.706 23 0.739 22 0.772 21 0.806 20 0.842 19 0.878 18 0.915 17 0.954	50 0.000 .15 45 0.126 .14 40 0.253 .13 35 0.385 .12 30 0.524 .11 25 0.674 .10 24 0.706 .09 23 0.739 .08 22 0.772 .07 21 0.806 .06 20 0.842 .050 19 0.878 .045 18 0.915 .040 17 0.954 .035	50 0.000 .15 1.036 45 0.126 .14 1.080 40 0.253 .13 1.126 35 0.385 .12 1.175 30 0.524 .11 1.227 25 0.674 .10 1.282 24 0.706 .09 1.341 23 0.739 .08 1.405 22 0.772 .07 1.476 21 0.806 .06 1.555 20 0.842 .050 1.645 19 0.878 .045 1.695 18 0.915 .040 1.751 17 0.954 .035 1.812	50 0.000 .15 1.036 .025 45 0.126 .14 1.080 .024 40 0.253 .13 1.126 .023 35 0.385 .12 1.175 .022 30 0.524 .11 1.227 .021 25 0.674 .10 1.282 .020 24 0.706 .09 1.341 .019 23 0.739 .08 1.405 .018 22 0.772 .07 1.476 .017 21 0.806 .06 1.555 .016 20 0.842 .050 1.645 .015 19 0.878 .045 1.695 .014 18 0.915 .040 1.751 .013 17 0.954 .035 1.812 .012	50 0.000 .15 1.036 .025 1.960 45 0.126 .14 1.080 .024 1.977 40 0.253 .13 1.126 .023 1.995 35 0.385 .12 1.175 .022 2.014 30 0.524 .11 1.227 .021 2.034 25 0.674 .10 1.282 .020 2.054 24 0.706 .09 1.341 .019 2.075 23 0.739 .08 1.405 .018 2.097 22 0.772 .07 1.476 .017 2.120 21 0.806 .06 1.555 .016 2.144 20 0.842 .050 1.645 .015 2.170 19 0.878 .045 1.695 .014 2.197 18 0.915 .040 1.751 .013 2.226 17 0.954 .035	50 0.000 .15 1.036 .025 1.960 .010 45 0.126 .14 1.080 .024 1.977 .009 40 0.253 .13 1.126 .023 1.995 .008 35 0.385 .12 1.175 .022 2.014 .007 30 0.524 .11 1.227 .021 2.034 .006 25 0.674 .10 1.282 .020 2.054 .006 24 0.706 .09 1.341 .019 2.075 .004 23 0.739 .08 1.405 .018 2.097 .003 22 0.772 .07 1.476 .017 2.120 .002 21 0.806 .06 1.555 .016 2.144 .001 20 0.842 .050 1.645 .015 2.170 .0³8 18 0.915 .040 1.751 .013 2.226 <th>50 0.000 .15 1.036 .025 1.960 .010 2.326 45 0.126 .14 1.080 .024 1.977 .009 2.366 40 0.253 .13 1.126 .023 1.995 .008 2.409 35 0.385 .12 1.175 .022 2.014 .007 2.457 30 0.524 .11 1.227 .021 2.034 .006 2.512 25 0.674 .10 1.282 .020 2.054 .006 2.576 24 0.706 .09 1.341 .019 2.075 .004 2.652 23 0.739 .08 1.405 .018 2.097 .003 2.748 22 0.772 .07 1.476 .017 2.120 .002 2.878 21 0.806 .06 1.555 .016 2.144 .001 3.090 20 0.842 .045 <t< th=""><th>50 0.000 .15 1.036 .025 1.960 .010 2.326 .0³4 45 0.126 .14 1.080 .024 1.977 .009 2.366 .0³3 40 0.253 .13 1.126 .023 1.995 .008 2.409 .0³2 35 0.385 .12 1.175 .022 2.014 .007 2.457 .0³1 30 0.524 .11 1.227 .021 2.034 .006 2.512 .0⁴5 25 0.674 .10 1.282 .020 2.054 .005 2.576 .0⁴5 24 0.706 .09 1.341 .019 2.075 .004 2.652 .0⁵5 23 0.739 .08 1.405 .018 2.097 .003 2.748 .0⁵1 22 0.772 .07 1.476 .017 2.120 .002 2.878 .0⁶5 21 0.806 .06 <</th></t<></th>	50 0.000 .15 1.036 .025 1.960 .010 2.326 45 0.126 .14 1.080 .024 1.977 .009 2.366 40 0.253 .13 1.126 .023 1.995 .008 2.409 35 0.385 .12 1.175 .022 2.014 .007 2.457 30 0.524 .11 1.227 .021 2.034 .006 2.512 25 0.674 .10 1.282 .020 2.054 .006 2.576 24 0.706 .09 1.341 .019 2.075 .004 2.652 23 0.739 .08 1.405 .018 2.097 .003 2.748 22 0.772 .07 1.476 .017 2.120 .002 2.878 21 0.806 .06 1.555 .016 2.144 .001 3.090 20 0.842 .045 <t< th=""><th>50 0.000 .15 1.036 .025 1.960 .010 2.326 .0³4 45 0.126 .14 1.080 .024 1.977 .009 2.366 .0³3 40 0.253 .13 1.126 .023 1.995 .008 2.409 .0³2 35 0.385 .12 1.175 .022 2.014 .007 2.457 .0³1 30 0.524 .11 1.227 .021 2.034 .006 2.512 .0⁴5 25 0.674 .10 1.282 .020 2.054 .005 2.576 .0⁴5 24 0.706 .09 1.341 .019 2.075 .004 2.652 .0⁵5 23 0.739 .08 1.405 .018 2.097 .003 2.748 .0⁵1 22 0.772 .07 1.476 .017 2.120 .002 2.878 .0⁶5 21 0.806 .06 <</th></t<>	50 0.000 .15 1.036 .025 1.960 .010 2.326 .0³4 45 0.126 .14 1.080 .024 1.977 .009 2.366 .0³3 40 0.253 .13 1.126 .023 1.995 .008 2.409 .0³2 35 0.385 .12 1.175 .022 2.014 .007 2.457 .0³1 30 0.524 .11 1.227 .021 2.034 .006 2.512 .0⁴5 25 0.674 .10 1.282 .020 2.054 .005 2.576 .0⁴5 24 0.706 .09 1.341 .019 2.075 .004 2.652 .0⁵5 23 0.739 .08 1.405 .018 2.097 .003 2.748 .0⁵1 22 0.772 .07 1.476 .017 2.120 .002 2.878 .0⁶5 21 0.806 .06 <

The z-value we are looking for is 2.576. This is the upper boundary of the 'acceptable' range.

Now we calculate the z-value corresponding to the measured mean, in order to see where it falls with respect to the acceptable range. This we call the test statistic and calculate as:

$$T = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}}$$

where \bar{x} is the mean calculated from the sample (197ml), μ_0 is the hypothesized population mean (200ml) and the fraction acting as nominator is the *standard error* i.e. the standard deviation of the sampling distribution.

We calculate the statistic:

$$T = \frac{197ml - 200ml}{\frac{2ml}{\sqrt{100}}} = -15$$

Note that the units (ml) have been cancelled out and that we have a dimensionless number, i.e. a z-value, which has a standardised normal distribution.

The last thing we need to do is to see whether our calculated test statistic falls within the 'acceptable' range:

$$-2.576 \le T \le 2.576$$
 ?

$$-2.576 \le -15 \le 2.576$$
 FALSE!

It does not fall in the range, so we reject the null hypothesis.