ICON4Py and IBM documentation

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Abstract

Work in progress documentation of the immersed boundary method implementation in ICON.

1 Equations

1.1 Normal wind

Advection

wind and kinetic energy

$$ {v_t}^{\mathrm{n}}_{\mathrm{e\,k}} = \sum_{e2c2e} \overset{\mathrm{rbf}}{\chi} {v_n}^{\mathrm{n}}_{\mathrm{e\,k}} \tag{1} $$

$$ {K_h}^{\mathrm{n}}_{\mathrm{e\,k}} = \frac{1}{2}\left({{v_n}^{\mathrm{n}}_{\mathrm{e\,k}}}^{2} + {{v_t}^{\mathrm{n}}_{\mathrm{e\,k}}}^{2}\right) \tag{2} $$

$$ {K_h}^{\mathrm{n}}_{\mathrm{c\,k}} = \sum_{c2e} \overset{\mathrm{hor}}{\chi} {K_h}^{\mathrm{n}}_{\mathrm{e\,k}} \tag{3} $$

$$ {v_n}^{\mathrm{n}}_{\mathrm{e\,k-1/2}} = \overset{\mathrm{lev}}{\chi} {v_n}^{\mathrm{n}}_{\mathrm{e\,k}} + (1-\overset{\mathrm{lev}}{\chi}) {v_n}^{\mathrm{n}}_{\mathrm{e\,k-1}} \tag{4} $$

$$ \tag{5} $$

vorticity

$$ \zeta^{\mathrm{n}}_{\mathrm{v\,k}} = \sum_{v2e} \overset{\mathrm{rot}}{\chi} {v_n}^{\mathrm{n}}_{\mathrm{e\,k}} \tag{6} $$

contravariant-corrected vertical wind

$$ {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{e\,k}} = {v_n}^{\mathrm{n}}_{\mathrm{e\,k}} \frac{\partial z}{\partial n} + {v_t}^{\mathrm{n}}_{\mathrm{e\,k}} \frac{\partial z}{\partial t} \tag{7} $$

$$ {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k}} = \sum_{c2e} \overset{\mathrm{hor}}{\chi} {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{e\,k}} \tag{8} $$

$$ {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k-1/2}} = \overset{\mathrm{lev}}{\chi} {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k}} + (1-\overset{\mathrm{lev}}{\chi}) {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k-1}} \tag{9} $$

$$ (w^{\mathrm{n}}_{\mathrm{c\,k-1/2}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k-1/2}}) = \begin{cases} w^{\mathrm{n}}_{\mathrm{c\,k-1/2}}, & \mathrm{k} \in [0, \mathtt{flat\_lev}+1) \\ w^{\mathrm{n}}_{\mathrm{c\,k-1/2}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k-1/2}}, & \mathrm{k} \in [\mathtt{flat\_lev}+1, \mathtt{num\_lev}) \\ 0, & \mathrm{k} = \mathtt{num\_lev} \end{cases} \tag{10} $$

$$ (w^{\mathrm{n}}_{\mathrm{c\,k}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k}}) = \frac{1}{2}\left[(w^{\mathrm{n}}_{\mathrm{c\,k-1/2}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k-1/2}}) + (w^{\mathrm{n}}_{\mathrm{c\,k+1/2}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k+1/2}})\right] \tag{11} $$

$$ \tag{12} $$

sum all contributions

$$ \begin{aligned} \mathrm{adv}(v_n)^{\mathrm{n}}_{\mathrm{e\,k}} &= \frac{\partial K_h}{\partial n} + v_t\,(\zeta + f) + \frac{\partial v_n}{\partial z}(w - w_{\mathrm{cc}}) \\ &= \Delta_{e2c} \overset{\mathrm{grad}}{\chi} {K_h}^{\mathrm{n}}_{\mathrm{c\,k}} + {K_h}^{\mathrm{n}}_{\mathrm{e\,k}} \Delta_{e2c} \overset{\mathrm{grad}}{\chi} \\ &+ {v_t}^{\mathrm{n}}_{\mathrm{e\,k}} (f_{\mathrm{e}} + 1/2 \sum_{e2v} \zeta^{\mathrm{n}}_{\mathrm{v\,k}}) \\ &+ \frac{{v_n}^{\mathrm{n}}_{\mathrm{e\,k-1/2}} - {v_n}^{\mathrm{n}}_{\mathrm{e\,k+1/2}}}{\Delta z_{\mathrm{k}}} \sum_{e2c} \overset{\mathrm{hor}}{\chi} (w^{\mathrm{n}}_{\mathrm{c\,k}} - {w_{\mathrm{cc}}}^{\mathrm{n}}_{\mathrm{c\,k}}) \end{aligned} \tag{13} $$

Exner

$$ {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}} = (1+\gamma){\pi'}^{\mathrm{n}}_{\mathrm{c\,k}} - \gamma {\pi'}^{\mathrm{n-1}}_{\mathrm{c\,k}} \tag{14} $$

$$ {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k-1/2}} = \overset{\mathrm{lev}}{\chi} {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}} + (1-\overset{\mathrm{lev}}{\chi}) {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k-1}}, \quad \mathrm{k} \in [\max(1, \mathtt{flat\_lev}), \mathtt{num\_lev}) \tag{15} $$

$$ {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,num\_lev-1/2}} = \sum_{\mathrm{k}=\mathtt{num\_lev}-1}^{\mathtt{num\_lev}-3} \overset{\mathrm{lev}}{\chi}_{\mathrm{k}} {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}} \tag{16} $$

horizontal gradient (at constant height) of π' on flat and non-flat levels

$$ \left.\frac{\partial \pi'}{\partial n}\right.^{\tilde{\mathrm{n}}}_{\mathrm{e\,k}} = \overset{\mathrm{grad}}{\chi} \Delta_{e2c} {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}}, \quad \mathrm{k} \in [0, \mathtt{flat\_lev}) \tag{17} $$

$$ \begin{aligned} \left.\frac{\partial \pi'}{\partial n}\right.^{\tilde{\mathrm{n}}}_{\mathrm{e\,k}} &= \left.\frac{\partial \pi'}{\partial n}\right|_s - \left.\frac{\partial h}{\partial n}\right|_s \frac{\partial \pi'}{\partial z}, \quad \mathrm{k} \in [\mathtt{flat\_lev}, \mathtt{flat\_gradp\_lev}] \\ &= \frac{1}{d_{12}} \Delta_{e2c} {\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}} - \frac{\partial h}{\partial n} \sum_{e2c} \overset{\mathrm{hor}}{\chi} \left.\frac{\partial \pi'}{\partial z}\right.^{\tilde{\mathrm{n}}}_{\mathrm{c\,k}} \end{aligned} \tag{18} $$

$$ \begin{aligned} \left.\frac{\partial \pi'}{\partial n}\right.^{\tilde{\mathrm{n}}}_{\mathrm{e\,k}} &= \frac{1}{d_{12}} ({\pi'}^{*}_{\mathrm{c}_1} - {\pi'}^{*}_{\mathrm{c}_0}), \quad \mathrm{k} \in [\mathtt{flat\_gradp\_lev}+1, \mathtt{num\_lev}) \\ &= \frac{1}{d_{12}} \Delta_{e2c} \left[{\pi'}^{\tilde{\mathrm{n}}}_{\mathrm{c\,k^*}} + (h^* - h_k) \left(\left.\frac{\partial \pi'}{\partial z}\right.^{\tilde{\mathrm{n}}}_{\mathrm{c\,k^*}} + (h^* - h_k) \frac{1}{2} \left.\frac{\partial^2 \pi'}{\partial z^2}\right.^{\tilde{\mathrm{n}}}_{\mathrm{c\,k^*}} \right) \right] \end{aligned} \tag{19} $$

Hydrostatic correction term to $\frac{\partial \pi'}{\partial n}$.

$$\theta_{v_{\mathrm{c}_i\,\mathrm{k}}} = \theta_{v_{\mathrm{c}_i\,\mathrm{k}^*}} + (h^* - h_k)\frac{\theta_{v_{\mathrm{c}_i\,\mathrm{k}^*-1/2}} - \theta_{v_{\mathrm{c}_i\,\mathrm{k}^*+1/2}}}{\Delta z_{\mathrm{k}^*}} \tag{20}$$

$$\frac{g}{c_{pd}{\theta_v}^2}\frac{\partial \theta_v}{\partial n}_{\mathrm{e}} = \frac{g}{c_{pd}}\frac{1}{d_{12}}4\frac{\theta_{v_{\mathrm{c}_1\,\mathrm{k}}} - \theta_{v_{\mathrm{c}_0\,\mathrm{k}}}}{(\theta_{v_{\mathrm{c}_1\,\mathrm{k}}} + \theta_{v_{\mathrm{c}_0\,\mathrm{k}}})^2} \tag{21}$$

$$\frac{\partial \pi'^{\tilde{\mathrm{n}}}}{\partial n}_{\mathrm{e\,k}} = \frac{\partial \pi'^{\tilde{\mathrm{n}}}}{\partial n}_{\mathrm{e\,k}} + \frac{g}{c_{pd}{\theta_v}^2}\frac{\partial \theta_v}{\partial n}_{\mathrm{e}}(h_k - h_{k^*}), \quad \mathrm{e} \in \overset{\partial\pi/\partial n}{\mathrm{IDX}} \tag{22}$$

Miura scheme
${\theta_v}^{\mathrm{n}}_{\mathrm{e\,k}}$ is computed with the Miura scheme §1.6

Final update

$$v_n{}^{\mathrm{n}+1^*}_{\mathrm{e\,k}} = v_n{}^{\mathrm{n}}_{\mathrm{e\,k}} - \Delta t\left(\mathrm{adv}(v_n)^{\mathrm{n}}_{\mathrm{e\,k}} + c_{pd}{\theta_v}^{\mathrm{n}}_{\mathrm{e\,k}}\frac{\partial \pi'^{\tilde{\mathrm{n}}}}{\partial n}_{\mathrm{e\,k}}\right) \tag{23}$$

1.2 Vertical wind

Exner

$$\left.\frac{\partial \pi'^{\tilde{n}}}{\partial z}\right|_{\mathrm{c\,k}} \approx \frac{\pi'^{\tilde{n}}_{\mathrm{c\,k-1/2}} - \pi'^{\tilde{n}}_{\mathrm{c\,k+1/2}}}{\Delta z_{\mathrm{k}}} \tag{24}$$

1.3 Exner

Coefficients

$$\alpha_{k??} = \eta\rho^{n}_{c\,k-1/2}\theta'^{\,n}_{v\,c\,k-1/2} \tag{25}$$

$$\beta_{k??} = \Delta t\frac{R_d}{c_{vd}}\frac{\Gamma^{n}_{k}}{\Delta z} = \Delta t\frac{R_d}{c_{vd}}\frac{\pi'^{\,n}_{c\,k}}{\rho^{n}_{c\,k}\theta'^{\,n}_{v\,c\,k}}\frac{1}{\Delta z} \tag{26}$$

Average normal wind
"needed to obtain a nearly 2nd order accurate divergence (Zangl 2015)"

$$\texttt{z\_vn\_avg}_{ek} = \sum_{e2c2eo}\overset{\text{hor}}{\chi} v_{n}{}^{n+1*}_{e\,k} \tag{27}$$

Miura scheme
$\rho^{n}_{e\,k}$ is computed with the Miura scheme §1.6

Explicit term

$$\texttt{z\_theta\_v\_fl\_e}_{ek} = \texttt{mass\_fl\_e}_{ek}\texttt{z\_theta\_v\_e}_{ek} = \rho^{n}_{e\,k}\texttt{z\_vn\_avg}_{ek}/\Delta z\texttt{z\_theta\_v\_e}_{ek} \tag{28}$$

$$\texttt{z\_flxdiv\_theta}_{ck} = \sum_{c2e}\overset{\text{div}}{\chi}\texttt{z\_theta\_v\_fl\_e}_{ek} \tag{29}$$

$$\begin{aligned} Z^{\pi\,\text{expl}} = \pi'^{\,n}_{c\,k} - \beta_{k??}\Big(&\texttt{z\_flxdiv\_theta}_{ck} + \theta'^{\,n}_{v\,c\,k-1/2}\texttt{z\_contr\_w\_fl\_l}_{k-1/2} \\ &-\theta'^{\,n}_{v\,c\,k+1/2}\texttt{z\_contr\_w\_fl\_l}_{k+1/2}\Big) + \Delta t\texttt{ddt\_exner\_phy} \end{aligned} \tag{30}$$

Final update

$$\pi'^{\,n+1^*}_{c\,k} = Z^{\pi\,\text{expl}} + \pi_{0\,c\,k} - \beta\left(\alpha_{k-1/2}w^{n+1^*}_{c\,k-1/2} - \alpha_{k+1/2}w^{n+1^*}_{c\,k+1/2}\right) \tag{31}$$

1.4 Density

Average normal wind
"needed to obtain a nearly 2nd order accurate divergence (Zangl 2015)"

$$\texttt{z\_vn\_avg}_{\mathrm{e}\mathrm{k}} = \sum_{e2c2eo} \overset{\mathrm{hor}}{\chi} v_{n}{}^{\mathrm{n+1*}}_{\mathrm{e\ k}} \tag{32}$$

Miura scheme
$\rho^{\mathrm{n}}_{\mathrm{e\ k}}$ is computed with the Miura scheme §1.6

Explicit term

$$\texttt{mass\_fl\_e}_{\mathrm{e}\mathrm{k}} = \rho^{\mathrm{n}}_{\mathrm{e\ k}} \texttt{z\_vn\_avg}_{\mathrm{e}\mathrm{k}} / \Delta z \tag{33}$$

$$\texttt{z\_flxdiv\_mass}_{\mathrm{c}\mathrm{k}} = \sum_{c2e} \overset{\mathrm{div}}{\chi} \texttt{mass\_fl\_e}_{\mathrm{e}\mathrm{k}} \tag{34}$$

$$Z^{\rho\ \mathrm{expl}} = \rho^{\mathrm{n}}_{\mathrm{c\ k}} - \frac{\Delta t}{\Delta z} \left(\texttt{z\_flxdiv\_mass}_{\mathrm{c}\mathrm{k}} + \texttt{z\_contr\_w\_fl\_l}_{\mathrm{c}\mathrm{k}-1/2} - \texttt{z\_contr\_w\_fl\_l}_{\mathrm{c}\mathrm{k}+1/2} \right) \tag{35}$$

Final update

$$\rho^{\mathrm{n+1*}}_{\mathrm{c\ k}} = Z^{\rho\ \mathrm{expl}} - \eta \frac{\Delta t}{\Delta z} \left(\rho^{\mathrm{n}}_{\mathrm{c\ k-1/2}} w^{\mathrm{n+1*}}_{\mathrm{c\ k-1/2}} - \rho^{\mathrm{n}}_{\mathrm{c\ k+1/2}} w^{\mathrm{n+1*}}_{\mathrm{c\ k+1/2}} \right) \tag{36}$$

1.5 Virtual potential temperature

$$\theta'^{\,\mathrm{n+1}^*}_{v\,\mathrm{c}\,\mathrm{k}} = \frac{\rho^{\mathrm{n}}_{\mathrm{c}\,\mathrm{k}}\theta'^{\,\mathrm{n}}_{v\,\mathrm{c}\,\mathrm{k}}}{\rho^{\mathrm{n+1}^*}_{\mathrm{c}\,\mathrm{k}}}\left(\left(\frac{\pi'^{\,\mathrm{n+1}^*}_{\mathrm{c}\,\mathrm{k}}}{\pi'^{\,\mathrm{n}}_{\mathrm{c}\,\mathrm{k}}}-1\right)\frac{c_{vd}}{R_d}+1\right) \tag{37}$$

1.6 Miura scheme

No artificial diffusion term is needed for the scalar quantities because the advection of ρ and θ_v is discretized with an upwind-biased second-order accurate scheme following [Miura, 2007].

$$\rho^{\mathrm{n}}_{\mathrm{e\,k}} = \rho_{\mathrm{e\,k}} + \rho'^{\mathrm{n}}_{\mathrm{c0/1\,k}} + \Delta^{\hat{x}}_{\mathrm{c-btrj}} \frac{\partial \rho'^{\mathrm{n}}}{\partial x}_{\mathrm{c\,k}} + \Delta^{\hat{y}}_{\mathrm{c-btrj}} \frac{\partial \rho'^{\mathrm{n}}}{\partial y}_{\mathrm{c\,k}} \tag{38}$$

$$\theta_{v\,\mathrm{e\,k}}^{\mathrm{n}} = \theta_{v0\,\mathrm{e\,k}} + \theta'^{\mathrm{n}}_{v\,\mathrm{c0/1\,k}} + \Delta^{\hat{x}}_{\mathrm{c-btrj}} \frac{\partial \theta'^{\mathrm{n}}_v}{\partial x}_{\mathrm{c\,k}} + \Delta^{\hat{y}}_{\mathrm{c-btrj}} \frac{\partial \theta'^{\mathrm{n}}_v}{\partial y}_{\mathrm{c\,k}} \tag{39}$$

$$\frac{\partial \rho'^{\mathrm{n}}}{\partial x}_{\mathrm{c\,k}} = \sum\nolimits_{c2e2co} \texttt{geofac\_grg\_x}\ \rho'^{\mathrm{n}}_{\mathrm{c\,k}} \tag{40}$$

$$\frac{\partial \rho'^{\mathrm{n}}}{\partial y}_{\mathrm{c\,k}} = \sum\nolimits_{c2e2co} \texttt{geofac\_grg\_y}\ \rho'^{\mathrm{n}}_{\mathrm{c\,k}} \tag{41}$$

$$\frac{\partial \theta'^{\mathrm{n}}_v}{\partial x}_{\mathrm{c\,k}} = \sum\nolimits_{c2e2co} \texttt{geofac\_grg\_x}\ \theta'^{\mathrm{n}}_{v\,\mathrm{c\,k}} \tag{42}$$

$$\frac{\partial \theta'^{\mathrm{n}}_v}{\partial y}_{\mathrm{c\,k}} = \sum\nolimits_{c2e2co} \texttt{geofac\_grg\_y}\ \theta'^{\mathrm{n}}_{v\,\mathrm{c\,k}} \tag{43}$$

The immersed boundary method (IBM) handles these operations by zeroing the gradient computation in cells adjacent to masked cells—those indicated with a dotted pattern in figure 1.

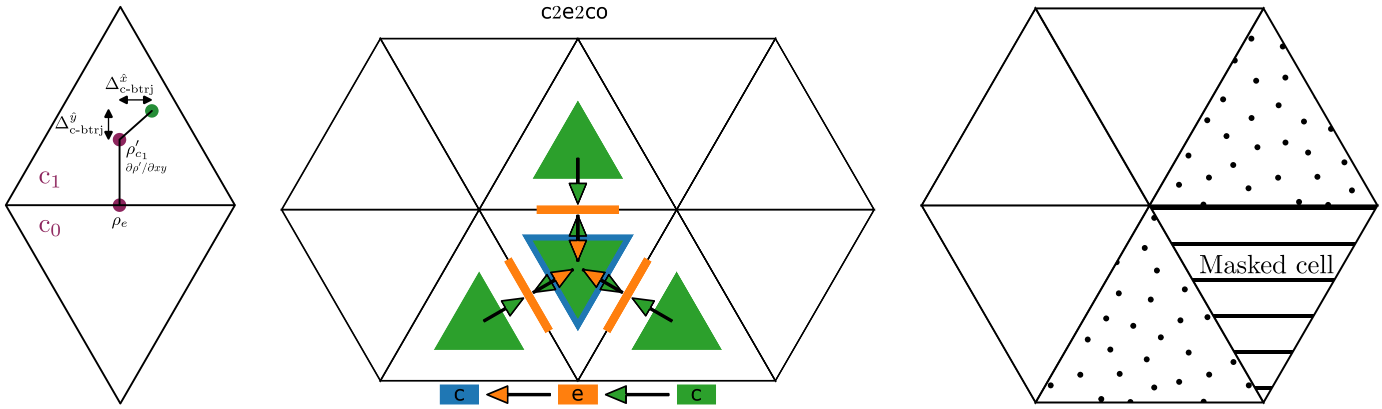

Figure 1: Miura scheme definition and offset provider used for the horizontal derivative computations

1.7 Diffusion

e2v interpolation

$$u_{\mathrm{v}} = \sum_{v2e} \overset{\mathrm{rbf}}{\chi}_1 v_{n_{\mathrm{e}}} \tag{44}$$

$$v_{\mathrm{v}} = \sum_{v2e} \overset{\mathrm{rbf}}{\chi}_2 v_{n_{\mathrm{e}}} \tag{45}$$

Smagorinsky coefficient

$$\begin{aligned} K_{h\ \mathrm{e\,k}}^{\mathrm{smag}^{\mathrm{n}}} &= \min\left(\texttt{smag\_limit}, \max\left(0, \texttt{diff\_multfac\_smag}\sqrt{*} - \texttt{smag\_offset}\right)\right) \\ * &= \left(\frac{-\left(u_{\mathrm{v2}}\hat{n_x}_{\mathrm{v2}} + v_{\mathrm{v2}}\hat{n_y}_{\mathrm{v2}}\right) + \left(u_{\mathrm{v3}}\hat{n_x}_{\mathrm{v3}} + v_{\mathrm{v3}}\hat{n_y}_{\mathrm{v3}}\right)}{\ell_{\mathrm{vv}}} - \frac{\left(-\left(u_{\mathrm{v0}}\hat{t_x}_{\mathrm{v0}} + v_{\mathrm{v0}}\hat{t_y}_{\mathrm{v0}}\right) + \left(u_{\mathrm{v1}}\hat{t_x}_{\mathrm{v1}} + v_{\mathrm{v1}}\hat{t_y}_{\mathrm{v1}}\right)\right)\hat{t}_{\mathrm{e}}}{\ell_{\mathrm{e}}}\right)^2 \\ &+ \left(\frac{\left(-\left(u_{\mathrm{v0}}\hat{n_x}_{\mathrm{v0}} + v_{\mathrm{v0}}\hat{n_y}_{\mathrm{v0}}\right) + \left(u_{\mathrm{v1}}\hat{n_x}_{\mathrm{v1}} + v_{\mathrm{v1}}\hat{n_y}_{\mathrm{v1}}\right)\right)\hat{t}_{\mathrm{e}}}{\ell_{\mathrm{e}}} + \frac{-\left(u_{\mathrm{v2}}\hat{t_x}_{\mathrm{v2}} + v_{\mathrm{v2}}\hat{t_y}_{\mathrm{v2}}\right) + \left(u_{\mathrm{v3}}\hat{t_x}_{\mathrm{v3}} + v_{\mathrm{v3}}\hat{t_y}_{\mathrm{v3}}\right)}{\ell_{\mathrm{vv}}}\right)^2 \end{aligned} \tag{46}$$

Nabla 2

$$\nabla^2(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}} = 4\left[\frac{u_{\mathrm{v0}}\hat{n_x}_{\mathrm{v0}} + v_{\mathrm{v0}}\hat{n_y}_{\mathrm{v0}} + u_{\mathrm{v1}}\hat{n_x}_{\mathrm{v1}} + v_{\mathrm{v1}}\hat{n_y}_{\mathrm{v1}} - 2v_{n_{\mathrm{e}}}}{\ell_{\mathrm{e}}^2} + \frac{u_{\mathrm{v2}}\hat{n_x}_{\mathrm{v2}} + v_{\mathrm{v2}}\hat{n_y}_{\mathrm{v2}} + u_{\mathrm{v3}}\hat{n_x}_{\mathrm{v3}} + v_{\mathrm{v3}}\hat{n_y}_{\mathrm{v3}} - 2v_{n_{\mathrm{e}}}}{\ell_{\mathrm{vv}}^2}\right] \tag{47}$$

$$\nabla^2(w)_{\mathrm{c\,k-1/2}} = \sum_{c2e2co} \overset{\mathrm{hor}}{\chi} w_{\mathrm{c\,k-1/2}} \tag{48}$$

$$\nabla^2(\theta_v'{}^{\mathrm{n}})_{\mathrm{c\,k}} = \sum_{c2e}\left(\overset{\mathrm{hor}}{\chi} K_{h\ \mathrm{e\,k}}^{\mathrm{smag}} \overset{\mathrm{grad}}{\chi} \Delta_{e2c} \theta'_{v\,\mathrm{c\,k}}{}^{\mathrm{n}}\right) \tag{49}$$

e2v interpolation

$$\nabla^2(v_n)_{\mathrm{v}^x} = \sum_{v2e} \overset{\mathrm{rbf}}{\chi}_1 \nabla^2(v_n)_{\mathrm{e}} \tag{50}$$

$$\nabla^2(v_n)_{\mathrm{v}^y} = \sum_{v2e} \overset{\mathrm{rbf}}{\chi}_2 \nabla^2(v_n)_{\mathrm{e}} \tag{51}$$

Nabla 4

$$\texttt{nabv\_tang} = \nabla^2(v_n)_{\mathrm{v}^x{}_0}\hat{n_x}_{\mathrm{v0}} + \nabla^2(v_n)_{\mathrm{v}^y{}_0}\hat{n_y}_{\mathrm{v0}} + \nabla^2(v_n)_{\mathrm{v}^x{}_1}\hat{n_x}_{\mathrm{v1}} + \nabla^2(v_n)_{\mathrm{v}^y{}_1}\hat{n_y}_{\mathrm{v1}} \tag{52}$$

$$\texttt{nabv\_norm} = \nabla^2(v_n)_{\mathrm{v}^x{}_2}\hat{n_x}_{\mathrm{v2}} + \nabla^2(v_n)_{\mathrm{v}^y{}_2}\hat{n_y}_{\mathrm{v2}} + \nabla^2(v_n)_{\mathrm{v}^x{}_3}\hat{n_x}_{\mathrm{v3}} + \nabla^2(v_n)_{\mathrm{v}^y{}_3}\hat{n_y}_{\mathrm{v3}} \tag{53}$$

$$\nabla^4(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}} = 4\left[\frac{\texttt{nabv\_norm} - 2\nabla^2(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}}}{\ell_{\mathrm{vv}}^2} + \frac{\texttt{nabv\_tang} - 2\nabla^2(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}}}{\ell_{\mathrm{e}}^2}\right] \tag{54}$$

Final updates

$$v_{n_{\mathrm{e\,k}}}^{d} = v_{n_{\mathrm{e\,k}}}^{\mathrm{n}} + a_e\left(K_{h\ \mathrm{e\,k}}^{\mathrm{smag}}\nabla^2(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}} - \texttt{diff\_multfac\_vn}a_e\nabla^4(v_n{}^{\mathrm{n}})_{\mathrm{e\,k}}\right) \tag{55}$$

$$w^{d}_{\mathrm{c\,k-1/2}} = w^{\mathrm{n}}_{\mathrm{c\,k-1/2}} - \texttt{diff\_multfac\_w}a_c^2 \sum_{c2e2co} \overset{\mathrm{hor}}{\chi} \nabla^2(w)_{\mathrm{c\,k-1/2}} \tag{56}$$

$$\theta_{v_{\mathrm{c\,k}}}^{d} = \theta_{v_{\mathrm{c\,k}}}^{\mathrm{n}} + a_c\nabla^2(\theta_v'{}^{\mathrm{n}})_{\mathrm{c\,k}} \tag{57}$$

$$\pi^{d}_{\mathrm{c\,k}} = \pi^{\mathrm{n}}_{\mathrm{c\,k}}\left(1 + \frac{R_d}{c_{vd}}\left(\frac{\theta_{v_{\mathrm{c\,k}}}^{d}}{\theta_{v_{\mathrm{c\,k}}}^{\mathrm{n}}} - 1\right)\right) \tag{58}$$

1.8 Don't remember what these are for

$$\frac{1}{2} \left. \frac{\partial^2 \pi'^{\tilde{\mathrm{n}}}}{\partial z^2} \right|_{\mathrm{c\,k}} = -\frac{1}{2} \left((\theta'^{\mathrm{n}}_{v\,\mathrm{c\,k}-1/2} - \theta'^{\mathrm{n}}_{v\,\mathrm{c\,k}+1/2}) \frac{1}{\Delta z \theta_{v0}} \left. \frac{\mathrm{d}\pi_0}{\mathrm{d}z} \right|_{\mathrm{c\,k}} + \theta'^{\mathrm{n}}_{v\,\mathrm{c\,k}} \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{1}{\theta_{v0}} \frac{\mathrm{d}\pi_0}{\mathrm{d}z} \right)_{\mathrm{c\,k}} \right), \quad \mathrm{k} \in [\texttt{flat\_gradp\_lev}, \texttt{num\_lev}) \tag{59}$$

$$\frac{\mathrm{d}\pi_0}{\mathrm{d}z} = -\frac{g}{c_{pd}\theta_{v0}} \tag{60}$$

References

[Miura, 2007] Miura, H. (2007). An Upwind-Biased Conservative Advection Scheme for Spherical Hexagonal–Pentagonal Grids. *Monthly Weather Review*, 135(12):4038–4044.