

ICON4Py and IBM documentation

Jacopo Canton

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Abstract

Work in progress documentation of the immersed boundary method implementation in ICON.

1 Equations

1.1 Normal wind

Advection

wind and kinetic energy

$$v_{t_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} = \sum_{e2c2e}^{\text{rbf}} \chi v_{n_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \quad (1)$$

$$K_{h_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} = \frac{1}{2} (v_{n_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}^2} + v_{t_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}^2}) \quad (2)$$

$$K_{h_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}} = \sum_{c2e}^{\text{hor}} \chi K_{h_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \quad (3)$$

$$v_{n_{\mathbf{e}\mathbf{k}-1/2}}^{\mathbf{n}} = \chi v_{n_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} + (1 - \chi) v_{n_{\mathbf{e}\mathbf{k}-1}}^{\mathbf{n}} \quad (4)$$

$$(5)$$

vorticity

$$\zeta_{\mathbf{v}\mathbf{k}}^{\mathbf{n}} = \sum_{v2e}^{\text{rot}} \chi v_{n_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \quad (6)$$

contravariant-corrected vertical wind

$$w_{cc_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} = v_{n_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \frac{\partial z}{\partial n} + v_{t_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \frac{\partial z}{\partial t} \quad (7)$$

$$w_{cc_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}} = \sum_{c2e}^{\text{hor}} \chi w_{cc_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \quad (8)$$

$$w_{cc_{\mathbf{c}\mathbf{k}-1/2}}^{\mathbf{n}} = \chi w_{cc_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}} + (1 - \chi) w_{cc_{\mathbf{c}\mathbf{k}-1}}^{\mathbf{n}} \quad (9)$$

$$(w_{\mathbf{c}\mathbf{k}-1/2}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}-1/2}}^{\mathbf{n}}) = \begin{cases} w_{\mathbf{c}\mathbf{k}-1/2}^{\mathbf{n}}, & \mathbf{k} \in [0, \text{flat_lev} + 1) \\ w_{\mathbf{c}\mathbf{k}-1/2}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}-1/2}}^{\mathbf{n}}, & \mathbf{k} \in [\text{flat_lev} + 1, \text{num_lev}) \\ 0, & \mathbf{k} = \text{num_lev} \end{cases} \quad (10)$$

$$(w_{\mathbf{c}\mathbf{k}}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}}) = \frac{1}{2} [(w_{\mathbf{c}\mathbf{k}-1/2}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}-1/2}}^{\mathbf{n}}) + (w_{\mathbf{c}\mathbf{k}+1/2}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}+1/2}}^{\mathbf{n}})] \quad (11)$$

$$(12)$$

sum all contributions

$$\begin{aligned} \text{adv}(v_n)_{\mathbf{e}\mathbf{k}}^{\mathbf{n}} &= \frac{\partial K_h}{\partial n} + v_t (\zeta + f) + \frac{\partial v_n}{\partial z} (w - w_{cc}) \\ &= \Delta_{e2c}^{\text{grad}} \chi K_{h_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}} + K_{h_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} \Delta_{e2c}^{\text{grad}} \chi \\ &\quad + v_{t_{\mathbf{e}\mathbf{k}}}^{\mathbf{n}} (f_{\mathbf{e}} + 1/2 \sum_{e2v} \zeta_{\mathbf{v}\mathbf{k}}^{\mathbf{n}}) \\ &\quad + \frac{v_{n_{\mathbf{e}\mathbf{k}-1/2}}^{\mathbf{n}} - v_{n_{\mathbf{e}\mathbf{k}+1/2}}^{\mathbf{n}}}{\Delta z_{\mathbf{k}}} \sum_{e2c}^{\text{hor}} \chi (w_{\mathbf{c}\mathbf{k}}^{\mathbf{n}} - w_{cc_{\mathbf{c}\mathbf{k}}}^{\mathbf{n}}) \end{aligned} \quad (13)$$

Exner

$$\pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}} = (1 + \gamma) \pi'_{\mathbf{c}\mathbf{k}}^{\mathbf{n}} - \gamma \pi'_{\mathbf{c}\mathbf{k}}^{\mathbf{n}-1} \quad (14)$$

$$\pi'_{\mathbf{c}\mathbf{k}-1/2}^{\bar{\mathbf{n}}} = \chi \pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}} + (1 - \chi) \pi'_{\mathbf{c}\mathbf{k}-1}^{\bar{\mathbf{n}}}, \quad \mathbf{k} \in [\max(1, \text{flat_lev}), \text{num_lev}) \quad (15)$$

$$\pi'_{\mathbf{c}\text{num_lev}-1/2}^{\bar{\mathbf{n}}} = \sum_{\mathbf{k}=\text{num_lev}-1}^{\text{num_lev}-3} \chi_{\mathbf{k}}^{\text{lev}} \pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}} \quad (16)$$

horizontal gradient (at constant height) of π' on flat and non-flat levels

$$\frac{\partial \pi'_{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e}\mathbf{k}}} = \chi^{\text{grad}} \Delta_{e2c} \pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}}, \quad \mathbf{k} \in [0, \text{flat_lev}) \quad (17)$$

$$\begin{aligned} \frac{\partial \pi'_{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e}\mathbf{k}}} &= \left. \frac{\partial \pi'}{\partial n} \right|_s - \left. \frac{\partial h}{\partial n} \right|_s \frac{\partial \pi'}{\partial z}, \quad \mathbf{k} \in [\text{flat_lev}, \text{flat_gradp_lev}] \\ &= \frac{1}{d_{12}} \Delta_{e2c} \pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}} - \frac{\partial h}{\partial n} \sum_{e2c}^{\text{hor}} \chi \frac{\partial \pi'_{\bar{\mathbf{n}}}}{\partial z_{\mathbf{c}\mathbf{k}}} \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial \pi'_{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e}\mathbf{k}}} &= \frac{1}{d_{12}} (\pi'_{\mathbf{c}\mathbf{k}}^* - \pi'_{\mathbf{c}\mathbf{k}}^0), \quad \mathbf{k} \in [\text{flat_gradp_lev} + 1, \text{num_lev}) \\ &= \frac{1}{d_{12}} \Delta_{e2c} \left[\pi'_{\mathbf{c}\mathbf{k}}^{\bar{\mathbf{n}}} + (h^* - h_{\mathbf{k}}) \left(\frac{\partial \pi'_{\bar{\mathbf{n}}}}{\partial z_{\mathbf{c}\mathbf{k}}^*} + (h^* - h_{\mathbf{k}}) \frac{1}{2} \frac{\partial^2 \pi'_{\bar{\mathbf{n}}}}{\partial z_{\mathbf{c}\mathbf{k}}^{*2}} \right) \right] \end{aligned} \quad (19)$$

Hydrostatic correction term to $\frac{\partial \pi'}{\partial n}$.

$$\theta_{v_{\mathbf{c}_i \mathbf{k}}} = \theta_{v_{\mathbf{c}_i \mathbf{k}^*}} + (h^* - h_k) \frac{\theta_{v_{\mathbf{c}_i \mathbf{k}^* - 1/2}} - \theta_{v_{\mathbf{c}_i \mathbf{k}^* + 1/2}}}{\Delta z_{\mathbf{k}^*}} \quad (20)$$

$$\frac{g}{c_{pd} \theta_v^2} \frac{\partial \theta_v}{\partial n_{\mathbf{e}}} = \frac{g}{c_{pd}} \frac{1}{d_{12}} 4 \frac{\theta_{v_{\mathbf{c}_1 \mathbf{k}}} - \theta_{v_{\mathbf{c}_0 \mathbf{k}}}}{(\theta_{v_{\mathbf{c}_1 \mathbf{k}}} + \theta_{v_{\mathbf{c}_0 \mathbf{k}}})^2} \quad (21)$$

$$\frac{\partial \pi'^{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e} \mathbf{k}}} = \frac{\partial \pi'^{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e} \mathbf{k}}} + \frac{g}{c_{pd} \theta_v^2} \frac{\partial \theta_v}{\partial n_{\mathbf{e}}} (h_k - h_{k^*}), \quad \mathbf{e} \in \text{IDX} \quad (22)$$

Miura scheme

$\theta_{v_{\mathbf{e} \mathbf{k}}}^{\mathbf{n}}$ is computed with the Miura scheme §1.6

Final update

$$v_{n_{\mathbf{e} \mathbf{k}}}^{\mathbf{n}+1*} = v_{n_{\mathbf{e} \mathbf{k}}}^{\mathbf{n}} - \Delta t \left(\text{adv}(v_n)_{\mathbf{e} \mathbf{k}}^{\mathbf{n}} + c_{pd} \theta_{v_{\mathbf{e} \mathbf{k}}}^{\mathbf{n}} \frac{\partial \pi'^{\bar{\mathbf{n}}}}{\partial n_{\mathbf{e} \mathbf{k}}} \right) \quad (23)$$

1.2 Vertical wind

Exner

$$\frac{\partial \pi'^{\bar{n}}}{\partial z_{\textcolor{brown}{c}\textcolor{brown}{k}}} \approx \frac{\pi'^{\bar{n}}_{\textcolor{brown}{c}\textcolor{brown}{k}-1/2} - \pi'^{\bar{n}}_{\textcolor{brown}{c}\textcolor{brown}{k}+1/2}}{\Delta z_{\textcolor{brown}{k}}}\tag{24}$$

1.3 Exner

Coefficients

$$\alpha_{k??} = \eta \rho_{\mathbf{c}_{k-1/2}}^{\mathbf{n}} \theta'_{v_{\mathbf{c}_{k-1/2}}}^{\mathbf{n}} \quad (25)$$

$$\beta_{k??} = \Delta t \frac{R_d}{c_{vd}} \frac{\Gamma_{\mathbf{k}}^{\mathbf{n}}}{\Delta z} = \Delta t \frac{R_d}{c_{vd}} \frac{\pi_{\mathbf{c}_{\mathbf{k}}}^{\mathbf{n}}}{\rho_{\mathbf{c}_{\mathbf{k}}}^{\mathbf{n}} \theta'_{v_{\mathbf{c}_{\mathbf{k}}}}^{\mathbf{n}}} \frac{1}{\Delta z} \quad (26)$$

Average normal wind

“needed to obtain a nearly 2nd order accurate divergence (Zangl 2015)”

$$\mathbf{z_vn_avg}_{\mathbf{ek}} = \sum_{e2c2eo}^{\text{hor}} \chi v_{n_{\mathbf{e}_{\mathbf{k}}}}^{n+1*} \quad (27)$$

Miura scheme

$\rho_{\mathbf{e}_{\mathbf{k}}}^{\mathbf{n}}$ is computed with the Miura scheme §1.6

Explicit term

$$\mathbf{z_theta_v_fle}_{\mathbf{ek}} = \text{mass_fle}_{\mathbf{ek}} \mathbf{z_theta_v_e}_{\mathbf{ek}} = \rho_{\mathbf{e}_{\mathbf{k}}}^{\mathbf{n}} \mathbf{z_vn_avg}_{\mathbf{ek}} / \Delta z \mathbf{z_theta_v_e}_{\mathbf{ek}} \quad (28)$$

$$\mathbf{z_flxdiv_theta}_{\mathbf{ck}} = \sum_{c2e}^{\text{div}} \chi \mathbf{z_theta_v_fle}_{\mathbf{ek}} \quad (29)$$

$$\begin{aligned} Z^{\pi \text{ expl}} = \pi_{\mathbf{c}_{\mathbf{k}}}^{\mathbf{n}} - \beta_{k??} \Big(\mathbf{z_flxdiv_theta}_{\mathbf{ck}} + \theta'_{v_{\mathbf{c}_{k-1/2}}}^{\mathbf{n}} \mathbf{z_contr_w_fl_l}_{\mathbf{k}-1/2} \\ - \theta'_{v_{\mathbf{c}_{k+1/2}}}^{\mathbf{n}} \mathbf{z_contr_w_fl_l}_{\mathbf{k}+1/2} \Big) + \Delta t \text{ddt_exner_phy} \end{aligned} \quad (30)$$

Final update

$$\pi_{\mathbf{c}_{\mathbf{k}}}^{n+1*} = Z^{\pi \text{ expl}} + \pi_{0_{\mathbf{c}_{\mathbf{k}}}} - \beta \left(\alpha_{\mathbf{k}-1/2} w_{\mathbf{c}_{k-1/2}}^{n+1*} - \alpha_{\mathbf{k}+1/2} w_{\mathbf{c}_{k+1/2}}^{n+1*} \right) \quad (31)$$

1.4 Density

Average normal wind

“needed to obtain a nearly 2nd order accurate divergence (Zangl 2015)”

$$\mathbf{z_vn_avg}_{\mathbf{ek}} = \sum_{\mathbf{e2c2eo}}^{\text{hor}} \chi v_{n_{\mathbf{ek}}}^{n+1*} \quad (32)$$

Miura scheme

$\rho_{\mathbf{ek}}^n$ is computed with the Miura scheme §1.6

Explicit term

$$\mathbf{mass_fl_e}_{\mathbf{ek}} = \rho_{\mathbf{ek}}^n \mathbf{z_vn_avg}_{\mathbf{ek}} / \Delta z \quad (33)$$

$$\mathbf{z_flxdiv_mass}_{\mathbf{ck}} = \sum_{\mathbf{c2e}}^{\text{div}} \chi \mathbf{mass_fl_e}_{\mathbf{ek}} \quad (34)$$

$$Z^{\rho \text{ expl}} = \rho_{\mathbf{ek}}^n - \frac{\Delta t}{\Delta z} (\mathbf{z_flxdiv_mass}_{\mathbf{ck}} + \mathbf{z_contr_w_fl_l}_{\mathbf{ck}-1/2} - \mathbf{z_contr_w_fl_l}_{\mathbf{ck}+1/2}) \quad (35)$$

Final update

$$\rho_{\mathbf{ek}}^{n+1*} = Z^{\rho \text{ expl}} - \eta \frac{\Delta t}{\Delta z} (\rho_{\mathbf{ek}-1/2}^n w_{\mathbf{ek}-1/2}^{n+1*} - \rho_{\mathbf{ek}+1/2}^n w_{\mathbf{ek}+1/2}^{n+1*}) \quad (36)$$

1.5 Virtual potential temperature

$$\theta'_{v_{\text{c k}}^{\text{n}+1*}} = \frac{\rho_{\text{c k}}^{\text{n}} \theta'_{v_{\text{c k}}^{\text{n}}}}{\rho_{\text{c k}}^{\text{n}+1*}} \left(\left(\frac{\pi'_{\text{c k}}^{\text{n}+1*}}{\pi'^{\text{n}}_{\text{c k}}} - 1 \right) \frac{c_{vd}}{R_d} + 1 \right) \quad (37)$$

1.6 Miura scheme

$$\rho_{\mathbf{e}\mathbf{k}}^{\mathbf{n}} = \rho_{\mathbf{e}\mathbf{k}} + \rho'_{\mathbf{c}0/1\mathbf{k}}^{\mathbf{n}} + \Delta_{\mathbf{c}-\text{btrj}}^{\hat{x}} \frac{\partial \rho'_{\mathbf{n}}}{\partial x_{\mathbf{c}\mathbf{k}}} + \Delta_{\mathbf{c}-\text{btrj}}^{\hat{y}} \frac{\partial \rho'_{\mathbf{n}}}{\partial y_{\mathbf{c}\mathbf{k}}} \quad (38)$$

$$\theta_{v\mathbf{e}\mathbf{k}}^{\mathbf{n}} = \theta_{v0\mathbf{e}\mathbf{k}} + \theta'_{v\mathbf{c}0/1\mathbf{k}}^{\mathbf{n}} + \Delta_{\mathbf{c}-\text{btrj}}^{\hat{x}} \frac{\partial \theta'_{\mathbf{n}}}{\partial x_{\mathbf{c}\mathbf{k}}} + \Delta_{\mathbf{c}-\text{btrj}}^{\hat{y}} \frac{\partial \theta'_{\mathbf{n}}}{\partial y_{\mathbf{c}\mathbf{k}}} \quad (39)$$

$$\frac{\partial \rho'_{\mathbf{n}}}{\partial x_{\mathbf{c}\mathbf{k}}} = \sum_{c2e2co} \text{geofac_grg_x} \rho'_{\mathbf{c}\mathbf{k}}^{\mathbf{n}} \quad (40)$$

$$\frac{\partial \rho'_{\mathbf{n}}}{\partial y_{\mathbf{c}\mathbf{k}}} = \sum_{c2e2co} \text{geofac_grg_y} \rho'_{\mathbf{c}\mathbf{k}}^{\mathbf{n}} \quad (41)$$

$$\frac{\partial \theta'_{\mathbf{n}}}{\partial x_{\mathbf{c}\mathbf{k}}} = \sum_{c2e2co} \text{geofac_grg_x} \theta'_{v\mathbf{c}\mathbf{k}}^{\mathbf{n}} \quad (42)$$

$$\frac{\partial \theta'_{\mathbf{n}}}{\partial y_{\mathbf{c}\mathbf{k}}} = \sum_{c2e2co} \text{geofac_grg_y} \theta'_{v\mathbf{c}\mathbf{k}}^{\mathbf{n}} \quad (43)$$

1.7 Diffusion

e2v interpolation

$$u_v = \sum_{v2e}^{\text{rbf}} \chi_1 v_{n_e} \quad (44)$$

$$v_v = \sum_{v2e}^{\text{rbf}} \chi_2 v_{n_e} \quad (45)$$

Smagorinsky coefficient

$$\begin{aligned} K_h^{\text{smag}^n}_{e\mathbf{k}} &= \min(\text{smag_limit}, \max(0, \text{diff_multfac_smag}\sqrt{*} - \text{smag_offset})) \\ * &= \left(\frac{-(u_{v2} \hat{n}_{xv2} + v_{v2} \hat{n}_{yv2}) + (u_{v3} \hat{n}_{xv3} + v_{v3} \hat{n}_{yv3})}{\ell_{vv}} - \frac{-(u_{v0} \hat{t}_{xv0} + v_{v0} \hat{t}_{yv0}) + (u_{v1} \hat{t}_{xv1} + v_{v1} \hat{t}_{yv1})}{\ell_e} \hat{t}_e \right)^2 \\ &+ \left(\frac{-(u_{v0} \hat{n}_{xv0} + v_{v0} \hat{n}_{yv0}) + (u_{v1} \hat{n}_{xv1} + v_{v1} \hat{n}_{yv1})}{\ell_e} \hat{t}_e + \frac{-(u_{v2} \hat{t}_{xv2} + v_{v2} \hat{t}_{yv2}) + (u_{v3} \hat{t}_{xv3} + v_{v3} \hat{t}_{yv3})}{\ell_{vv}} \right)^2 \end{aligned} \quad (46)$$

Nabla 2

$$\nabla^2(v_n^{\mathbf{n}})_{e\mathbf{k}} = 4 \left[\frac{u_{v0} \hat{n}_{xv0} + v_{v0} \hat{n}_{yv0} + u_{v1} \hat{n}_{xv1} + v_{v1} \hat{n}_{yv1} - 2v_{n_e}}{\ell_e^2} + \frac{u_{v2} \hat{n}_{xv2} + v_{v2} \hat{n}_{yv2} + u_{v3} \hat{n}_{xv3} + v_{v3} \hat{n}_{yv3} - 2v_{n_e}}{\ell_{vv}^2} \right] \quad (47)$$

$$\nabla^2(w)_{c\mathbf{k}-1/2} = \sum_{c2e2co}^{\text{hor}} \chi w_{c\mathbf{k}-1/2} \quad (48)$$

$$\nabla^2(\theta'_v)^{\mathbf{n}}_{e\mathbf{k}} = \sum_{c2e} \left(\chi K_h^{\text{smag}^{\text{hor}}}_{e\mathbf{k}} \chi^{\text{grad}} \Delta_{e2c} \theta'_{v\mathbf{k}}^{\mathbf{n}} \right) \quad (49)$$

e2v interpolation

$$\nabla^2(v_n)_{v\mathbf{x}} = \sum_{v2e}^{\text{rbf}} \chi_1 \nabla^2(v_n)_e \quad (50)$$

$$\nabla^2(v_n)_{v\mathbf{y}} = \sum_{v2e}^{\text{rbf}} \chi_2 \nabla^2(v_n)_e \quad (51)$$

Nabla 4

$$\text{nabv_tang} = \nabla^2(v_n)_{v\mathbf{x}0} \hat{n}_{xv0} + \nabla^2(v_n)_{v\mathbf{y}0} \hat{n}_{yv0} + \nabla^2(v_n)_{v\mathbf{x}1} \hat{n}_{xv1} + \nabla^2(v_n)_{v\mathbf{y}1} \hat{n}_{yv1} \quad (52)$$

$$\text{nabv_norm} = \nabla^2(v_n)_{v\mathbf{x}2} \hat{n}_{xv2} + \nabla^2(v_n)_{v\mathbf{y}2} \hat{n}_{yv2} + \nabla^2(v_n)_{v\mathbf{x}3} \hat{n}_{xv3} + \nabla^2(v_n)_{v\mathbf{y}3} \hat{n}_{yv3} \quad (53)$$

$$\nabla^4(v_n^{\mathbf{n}})_{e\mathbf{k}} = 4 \left[\frac{\text{nabv_norm} - 2\nabla^2(v_n^{\mathbf{n}})_{e\mathbf{k}}}{\ell_{vv}^2} + \frac{\text{nabv_tang} - 2\nabla^2(v_n^{\mathbf{n}})_{e\mathbf{k}}}{\ell_e^2} \right] \quad (54)$$

Final updates

$$v_{n_{e\mathbf{k}}}^d = v_{n_{e\mathbf{k}}}^{\mathbf{n}} + a_e (K_h^{\text{smag}}_{e\mathbf{k}} \nabla^2(v_n^{\mathbf{n}})_{e\mathbf{k}} - \text{diff_multfac_vna}_e \nabla^4(v_n^{\mathbf{n}})_{e\mathbf{k}}) \quad (55)$$

$$w_{c\mathbf{k}-1/2}^d = w_{c\mathbf{k}-1/2}^{\mathbf{n}} - \text{diff_multfac_wa}_c^2 \sum_{c2e2co}^{\text{hor}} \chi \nabla^2(w)_{c\mathbf{k}-1/2} \quad (56)$$

$$\theta_{v_{c\mathbf{k}}}^d = \theta_{v_{c\mathbf{k}}}^{\mathbf{n}} + a_c \nabla^2(\theta'_v)^{\mathbf{n}}_{c\mathbf{k}} \quad (57)$$

$$\pi_{c\mathbf{k}}^d = \pi_{c\mathbf{k}}^{\mathbf{n}} \left(1 + \frac{R_d}{c_{vd}} \left(\frac{\theta_{v_{c\mathbf{k}}}^d}{\theta_{v_{c\mathbf{k}}}^{\mathbf{n}}} - 1 \right) \right) \quad (58)$$

1.8 Don't remember what these are for

$$\frac{1}{2} \frac{\partial^2 \pi'^{\bar{n}}}{\partial z^2}_{\mathbf{c}, \mathbf{k}} = -\frac{1}{2} \left((\theta'_{v_{\mathbf{c}, \mathbf{k}-1/2}}^{\mathbf{n}} - \theta'_{v_{\mathbf{c}, \mathbf{k}+1/2}}^{\mathbf{n}}) \frac{1}{\Delta z \theta_{v0}} \frac{d\pi_0}{dz}_{\mathbf{c}, \mathbf{k}} + \theta'_{v_{\mathbf{c}, \mathbf{k}}}^{\mathbf{n}} \frac{d}{dz} \left(\frac{1}{\theta_{v0}} \frac{d\pi_0}{dz}_{\mathbf{c}, \mathbf{k}} \right) \right), \quad \mathbf{k} \in [\text{flat_gradp_lev}, \text{num_lev}] \quad (59)$$

$$\frac{d\pi_0}{dz} = -\frac{g}{c_{pd} \theta_{v0}} \quad (60)$$