Problem Set 4 Checkpoint Solutions

Checkpoint Problem: Paradoxical Sets (25 Points if Submitted)

i. Prove that if A and B are sets where $A \subseteq B$, then $|A| \le |B|$. Although this probably makes intuitive sense, to formally prove this result, find an injection $f : A \to B$ and prove that your function is injective.

Proof: Let A and B be sets where $A \subseteq B$. We will prove $|A| \le |B|$ by exhibiting an injective function $f : A \to B$. Because $A \subseteq B$, any $a \in A$ also satisfies $a \in B$. Therefore, the function f defined as $f(a) = a$ is a legal function from A to B, since for any $a \in A$ we have that $f(a) = a \in B$.

To see that f is injective, consider any $a_1, a_2 \in A$ where $f(a_1) = f(a_2)$. We will prove $a_1 = a_2$. Since $f(a_1) = f(a_2)$ and $f(a) = a$, we get $a_1 = a_2$, as required. Thus f is injective, so $|A| \le |B|$. ■

Why we asked this question: In one sense, this result is obvious: if A is a subset of B, then *of course* B has to be at least as large as A. However, using our formal definition of set cardinality, in order to prove this it's necessary to find an injection $f : A \to B$.

We asked this question because we wanted you to explicitly come up with an injective function from A to B rather than to just describe why the result should be true in natural language. Coming up with the injection is tricky – we know pretty much *nothing* about A and B, with the only fact we can assume being that $A \subseteq B$. However, this constraint actually helps us. Since the only thing we know about A and B is that every $a \in A$ satisfies $a \in B$, it's natural to map any element to itself. We wanted you to have to specify this by writing something to the effect of $f(a) = a$ so that we could see that you were comfortable defining your own functions.

A very common mistake on this problem was to try to number the elements of A as $a_1, a_2, \ldots, a_{|A|}$ and the elements of B as $b_1, b_2, \ldots, b_{|B|}$, then to define $f(a_k) = b_k$. This approach will work as long as A and B are finite sets (or countably infinite sets, sets with cardinality $|\mathbb{N}|$), but it doesn't work if $|A|$ or $|B|$ are larger than $|\mathbb{N}|$. The reason for this is that if $|A|$ is larger than $|\mathbb{N}|$, then it's impossible to number the elements in A using the natural numbers simply because there aren't enough natural numbers to do this. The function $f(a) = a$, on the other hand, doesn't have this problem because there's no dependence on the number of elements in A or B.

ii. Using your result from (i), prove that if $\mathscr{U}$ exists at all, then $|\wp(\mathscr{U})| \le |\mathscr{U}|$.

Proof: We will prove $\wp(\mathscr{U}) \subseteq \mathscr{U}$, from which the result follows. Since $\mathscr{U}$ is the universal set which contains all objects, we see that any $x \in \wp(\mathscr{U})$ also satisfies $x \in \mathscr{U}$. Thus $\wp(\mathscr{U}) \subseteq \mathscr{U}$, and by our result from (i) we see that $|\wp(\mathscr{U})| \le |\mathscr{U}|$. ■

Why we asked this question: We asked this question to make sure that you're still comfortable distinguishing between the element-of relation $\in$ and the subset-of relation $\subseteq$.

Below is an *incorrect* proof of this result:

> *Proof:* Since $\mathscr{U}$ is the universal set which contains all objects, we know $\wp(\mathscr{U}) \in \mathscr{U}$. Therefore, $|\wp(\mathscr{U})| \le |\mathscr{U}|$, as required. ■

This proof hinges on the following claim:

$$\text{If } A \in B, \text{ then } |A| \le |B|$$

Here, A is $\wp(\mathscr{U})$ and B is $\mathscr{U}$. Unfortunately, this claim is false. Consider the set $\mathbb{N}$, which is infinite, and the set $\{\mathbb{N}\}$, which is finite. It's true that $\mathbb{N} \in \{\mathbb{N}\}$, but it's not the case that $|\mathbb{N}| \le |\{\mathbb{N}\}|$.

To correctly prove this result, it's necessary to show that $\wp(\mathscr{U}) \subseteq \mathscr{U}$ by showing that every element of the set $\wp(\mathscr{U})$ is also an element of the set $\mathscr{U}$, as we've done in our above proof.

iii. Using your result from (ii) and Cantor's Theorem, prove that $\mathscr{U}$ cannot exist.

Proof: By contradiction; suppose $\mathscr{U}$ exists. By our result from (ii), we know $|\wp(\mathscr{U})| \le |\mathscr{U}|$. However, by Cantor's Theorem, we know that $|\mathscr{U}| < |\wp(\mathscr{U})|$, contradicting our earlier statement. We have reached a contradiction, so our assumption must have been wrong. Thus $\mathscr{U}$ does not exist. ■

Why we asked this question: This proof is alarmingly short. We've primarily used Cantor's Theorem to show that there are unsolvable problems, but here we're using it to show a much deeper result: there are some collections of objects that can't be grouped together into sets! We hoped that by writing this proof, you would get an appreciation for some of the mysteries of set theory.