# Discussion Solutions 3

## Problem One: Graphs

Let $G = (V, E)$ be an undirected graph with no self-loops. Prove that if the degree of every node in $G$ is $|V| / 2$ or greater, then $G$ is connected.

*Proof:* Consider any $u, v \in V$. We will prove $u \leftrightarrow v$.

> If the edge $\{u, v\} \in E$, then we are done because there is a direct edge from $u$ to $v$. Otherwise, $\{u, v\} \notin E$. Since there are no self-loops in $G$ and $\{u, v\} \notin E$, there are a total of $|V| - 2$ nodes in $G$ that can have direct edges to either $u$ or $v$. Since the degrees of $u$ and $v$ are at least $|V| / 2$, there are a total of $|V|$ edges leaving $u$ and $v$. By the pigeonhole principle, since there are $|V|$ edges leaving $u$ and $v$ and only $|V| - 2$ possible endpoints for those edges, there must be some node $z$ such that $\{u, z\} \in E$ and $\{v, z\} \in E$. Therefore, there is a path $u, z, v$ from $u$ to $v$, so $u \leftrightarrow v$, as required. ■

**Why we asked this question:** This problem combines two concepts we've discussed in lecture: graph connectivity and the pigeonhole principle. To use the pigeonhole principle, though, we need to first rule out nodes $u$ and $v$ as buckets, which we do by using that there are no self-loops and special-casing the scenario where there's a direct edge between $u$ and $v$.

## Problem Two: The Pigeonhole Principle

Prove that if you pick any five points in the unit square, there must be some pair of points chosen whose distance from one another is at most $\sqrt{2}/2$ .

*Proof:* Subdivide the unit square into four squares of side length ½. Distribute the five points into the unit square. By the pigeonhole principle, there must be at least two points in the same square. The maximum distance these two points can be from one another is $\sqrt{2}/2$ , which occurs if the points are in opposite corners of the square; any other positions place the points closer together. ■

**Why we asked this question:** As with all pigeonhole proofs, this proof is all about finding the right buckets and the right objects. Once you see that you're supposed to subdivide the square into four smaller squares, the rest of the math clicks into place.

## Problem Three: Binary Relations

Consider the following relation $R$ defined over $\mathbb{N}^2$:

$$(a, b)R(c, d) \quad \text{iff} \quad a + d = b + c$$

i. Prove that $R$ is an equivalence relation.

*Proof:* We will prove that $R$ is reflexive, symmetric, and transitive.

To see that $R$ is reflexive, consider any $(a, b) \in \mathbb{N}^2$. We will prove $(a, b)R(a, b)$, meaning that $a + b = b + a$. This holds because addition is commutative.

To see that $R$ is symmetric, consider any $(a, b), (c, d) \in \mathbb{N}^2$ where $(a, b)R(c, d)$. We will prove that $(c, d)R(a, b)$. Since $(a, b)R(c, d)$, we know $a + d = b + c$, so, $c + b = d + a$. Therefore, $(c, d)R(a, b)$, as required.

To see that $R$ is transitive, consider any $(a, b), (c, d), (e, f) \in \mathbb{N}^2$ such that $(a, b)R(c, d)$ and $(c, d)R(e, f)$. We will prove that $(a, b)R(e, f)$. Since $(a, b)R(c, d)$, we see $a + d = b + c$. Rearranging, we get $a - b = c - d$. Since $(c, d)R(e, f)$, we see $c + f = d + e$. Rearranging this yields $c - d = e - f$. Therefore, $a - b = e - f$, so $a + d = b + e$. Thus $(a, b)R(e, f)$, as required.

Since $R$ is reflexive, symmetric, and transitive, $R$ is an equivalence relation. ■

ii. What are the equivalence classes of $R$?

For any pair $(a, b) \in \mathbb{N}^2$, the equivalence class $[(a, b)]_R$ is the set

$$[(a, b)]_R = \{ (c, d) \in \mathbb{N}^2 \mid (a, b)R(c, d) \}$$

Equivalently:

$$[(a, b)]_R = \{ (c, d) \in \mathbb{N}^2 \mid a + d = b + c \}$$

Equivalently:

$$[(a, b)]_R = \{ (c, d) \in \mathbb{N}^2 \mid a - b = c - d \}$$

In other words, the equivalence class of $(a, b)$ is the set of all pairs of natural numbers where the difference between the first and second elements is equal to the difference between $a$ and $b$. Therefore, the equivalence classes of $R$ are sets of pairs of natural numbers where the difference between the first and second elements are the same.

**Why we asked this question:** The equivalence relation $R$ you saw in this problem is used as a building block for defining the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$. In axiomatic set theory, the integers are *defined* to be the equivalence classes of $\mathbb{N}$ under the relation $R$.