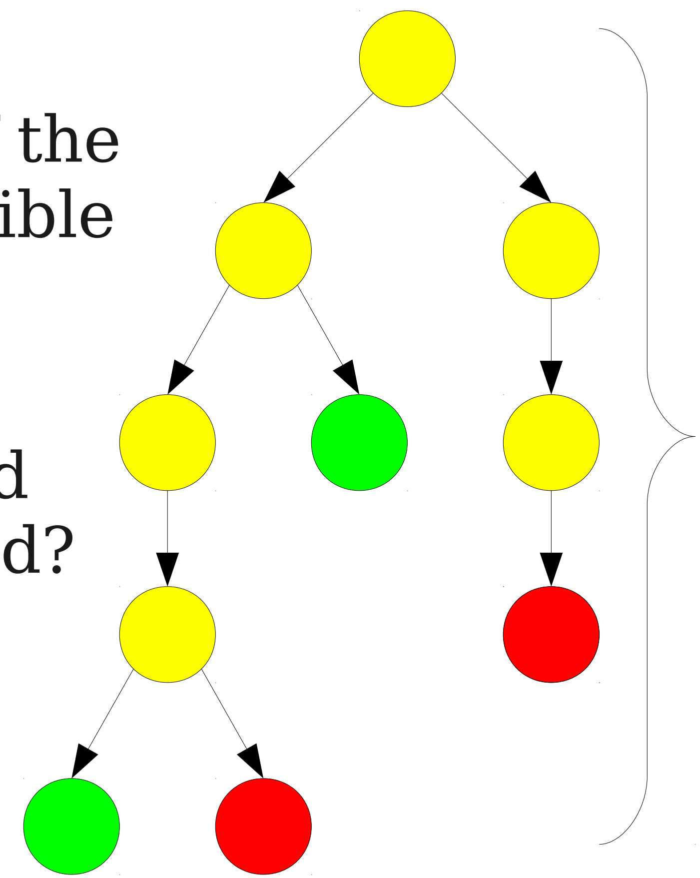
NP-Completeness

Recap from Last Time

Analyzing NTMs

- When discussing deterministic TMs, the notion of time complexity is (reasonably) straightforward.
- **Recall:** One way of thinking about nondeterminism is as a tree.
- The time complexity is the height of the tree (the length of the **longest** possible choice we could make).
- Intuition: If you ran all possible branches in parallel, how long would it take before all branches completed?



The Complexity Class **NP**

- The complexity class **NP** (**nondeterministic polynomial time**) contains all problems that can be solved in polynomial time by an NTM.
- Formally:

NP = { *L* | There is a nondeterministic TM that decides *L* in polynomial time. }

Another View of **NP**

- ***Theorem:*** $L \in$ **NP** iff there is a *deterministic* TM V with the following properties:
  - $w \in L$ iff there is some $c \in \Sigma^*$ such that V accepts $\langle w, c \rangle$.
  - V runs in time polynomial in $|w|$.
- Some terminology:
  - A TM V with the above property is called a **polynomial-time verifier for *L***.
  - The string c is called a **certificate** for w.
  - You can think of V as checking the certificate that proves $w \in L$.

NP and Reductions

Polynomial-Time Reductions

- Suppose that we know that $B \in \mathbf{NP}$.
- Suppose that $A \leq_{\mathrm{P}} B$ and that the reduction f can be computed in time $\mathrm{O}(n^k)$.

A

Solvable?

B

Solvable by NTM in $\mathrm{O}(n^r)$

Polynomial-Time Reductions

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Input size: $\boldsymbol{n}$

A

Solvable?

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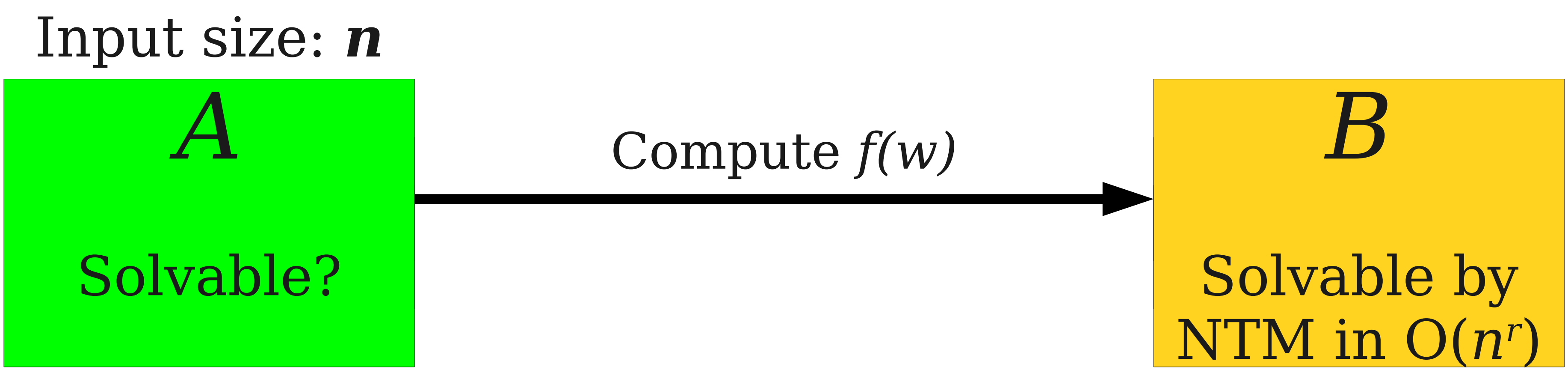

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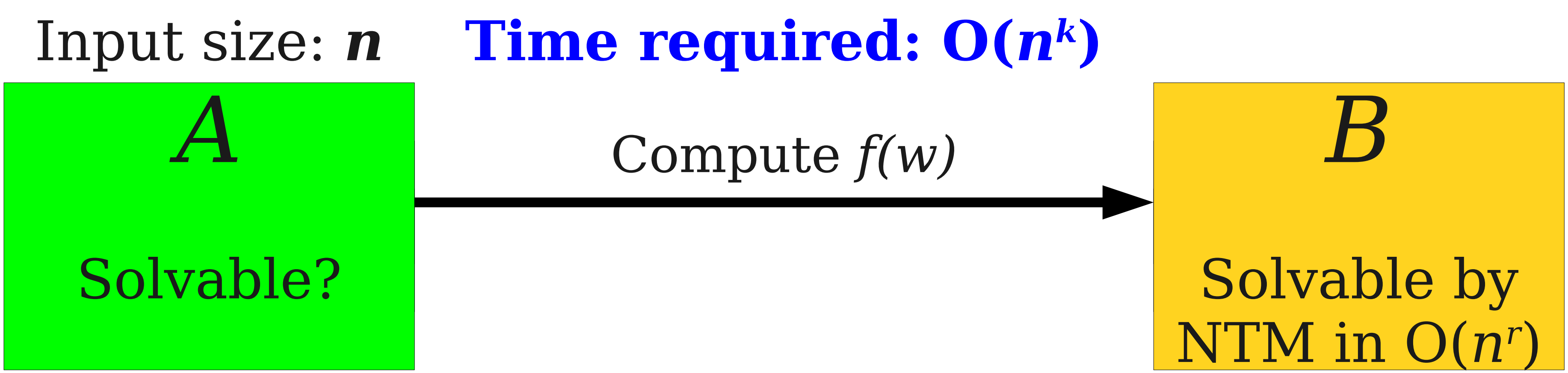

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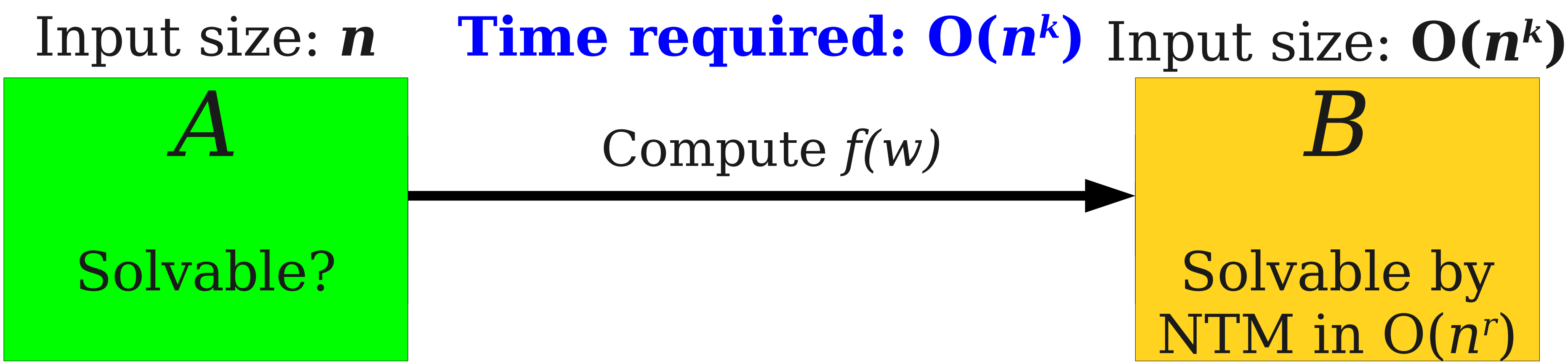

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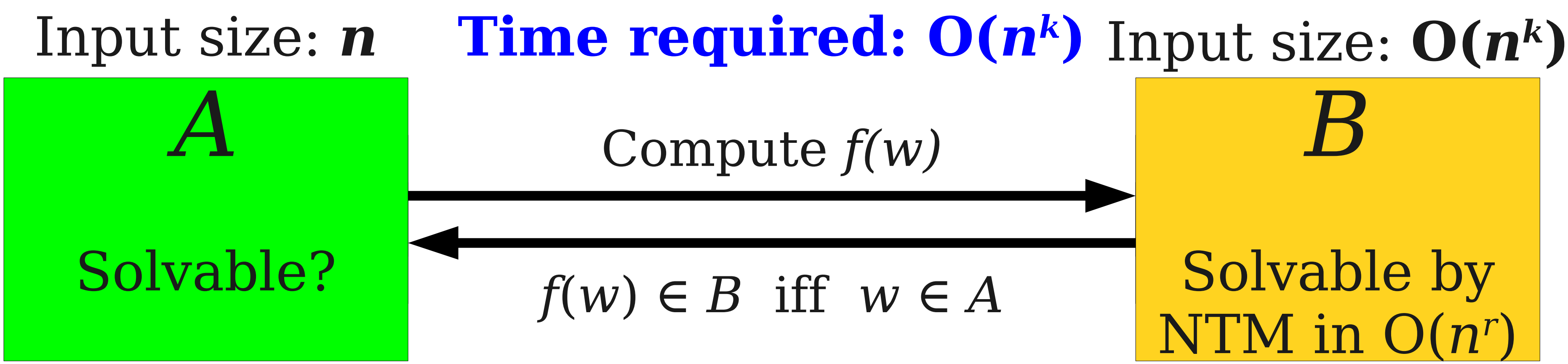

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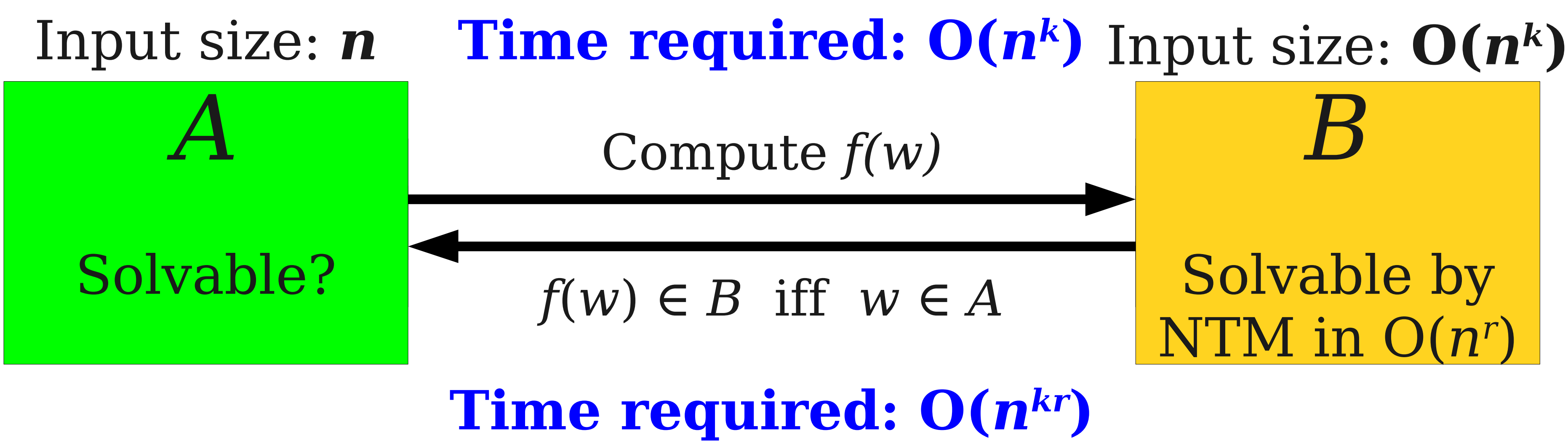

Polynomial-Time Reductions

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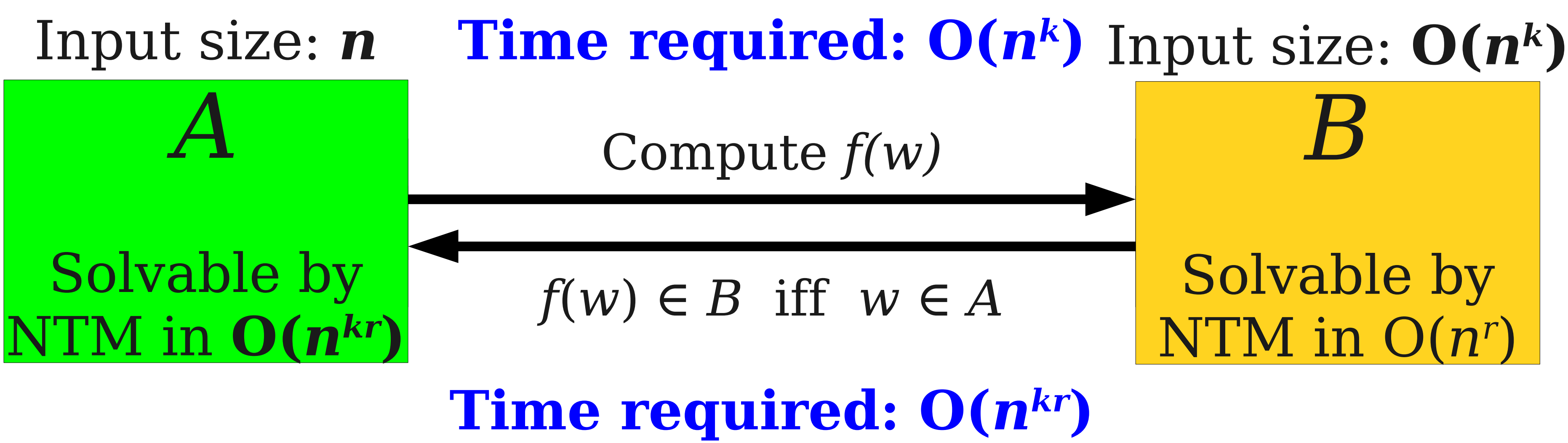
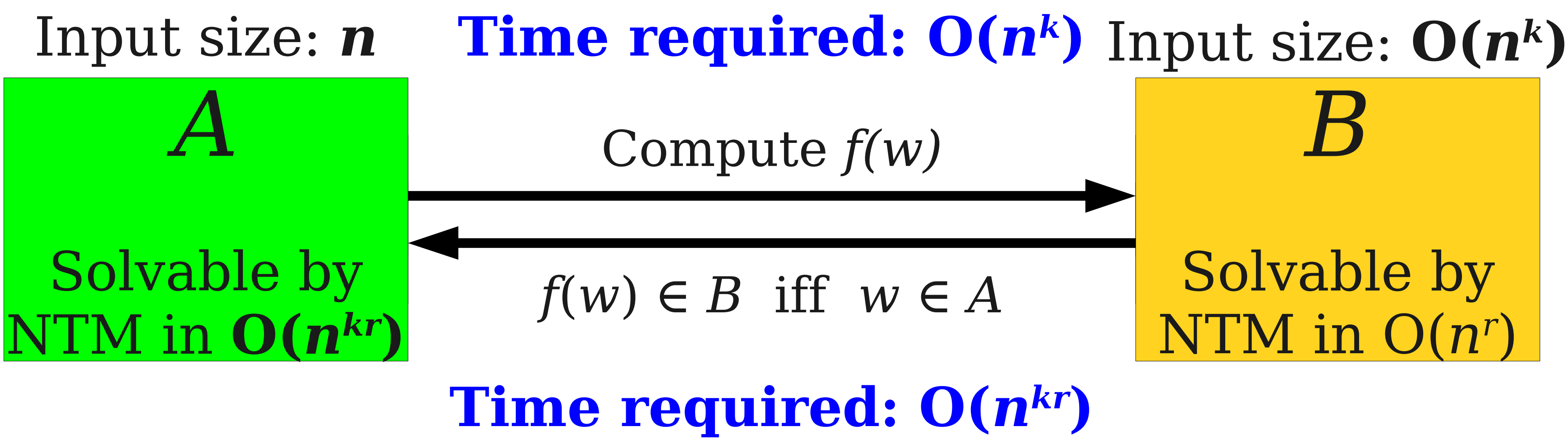


Polynomial-Time Reductions

- Suppose that we know that $B \in \mathbf{NP}$.
- Suppose that $A \leq_P B$ and that the reduction f can be computed in time $O(n^k)$.
- Then $A \in \mathbf{NP}$ as well.



A Sample Reduction

$$U = \{1, 2, 3, 4, 5, 6\}$$

$$S = \left\{ \begin{matrix} \{1, 2, 5\}, \{2, 5\}, \{1, 3, 6\}, \\ \{2, 3, 4\}, \{4\}, \{1, 5, 6\} \end{matrix} \right\}$$

Let U be a set of elements (the **universe**) and $S \subseteq \wp(U)$. An **exact covering** of U is a collection of sets $I \subseteq S$ such that every element of U belongs to exactly one set in I.

$$U = \{1, 2, 3, 4, 5, 6\}$$

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Let U be a set of elements (the **universe**) and $S \subseteq \wp(U)$. An **exact covering** of U is a collection of sets $I \subseteq S$ such that every element of U belongs to exactly one set in I.

Exact Covering

- Given a universe U and a set $S \subseteq \wp(U)$, the exact covering problem is

Does S contain an exact covering of U?

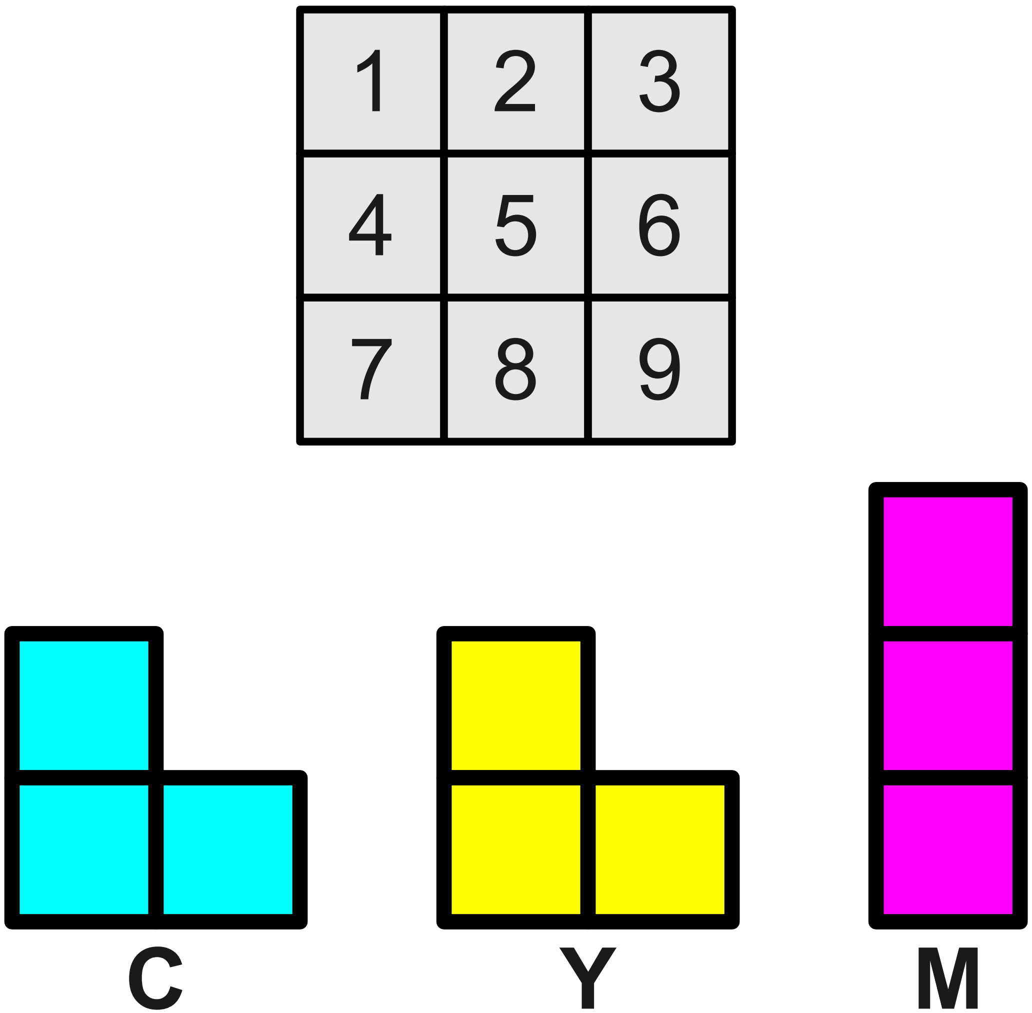
- As a formal language:

$$\textbf{EXACT-COVER} = \{ \langle U, S \rangle \mid S \subseteq \wp(U) \text{ and } S \text{ contains an exact covering of } U \}$$

EXACT-COVER $\in$ **NP**

- Here is a polynomial-time verifier for *EXACT-COVER*:
- V = "On input $\langle U, S, I\rangle$, where U, S, and I are sets:
  - Verify that every set in S is a subset of U.
  - Verify that every set in I is an element of S.
  - Verify that every element of U belongs to an element of I.
  - Verify that every element of U belongs to at most one element of I."

Applications of Exact Covering



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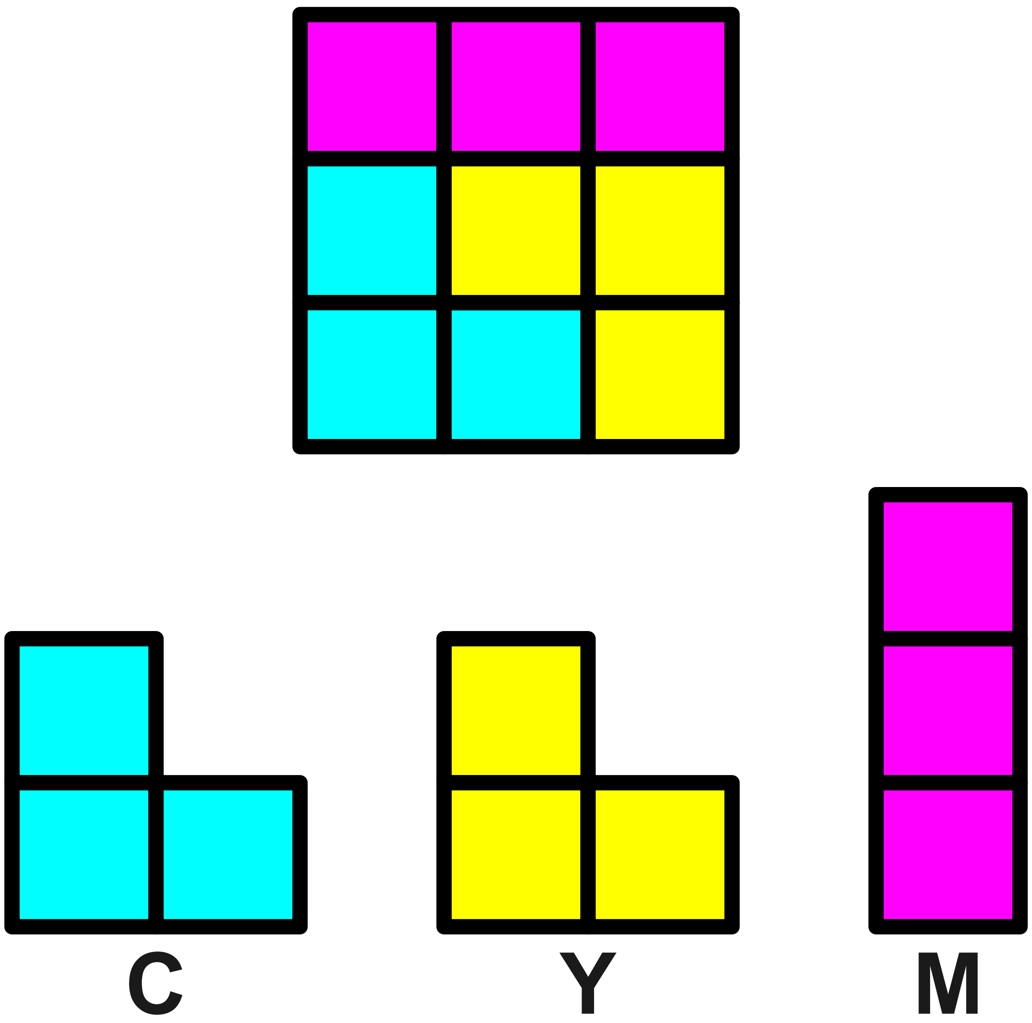

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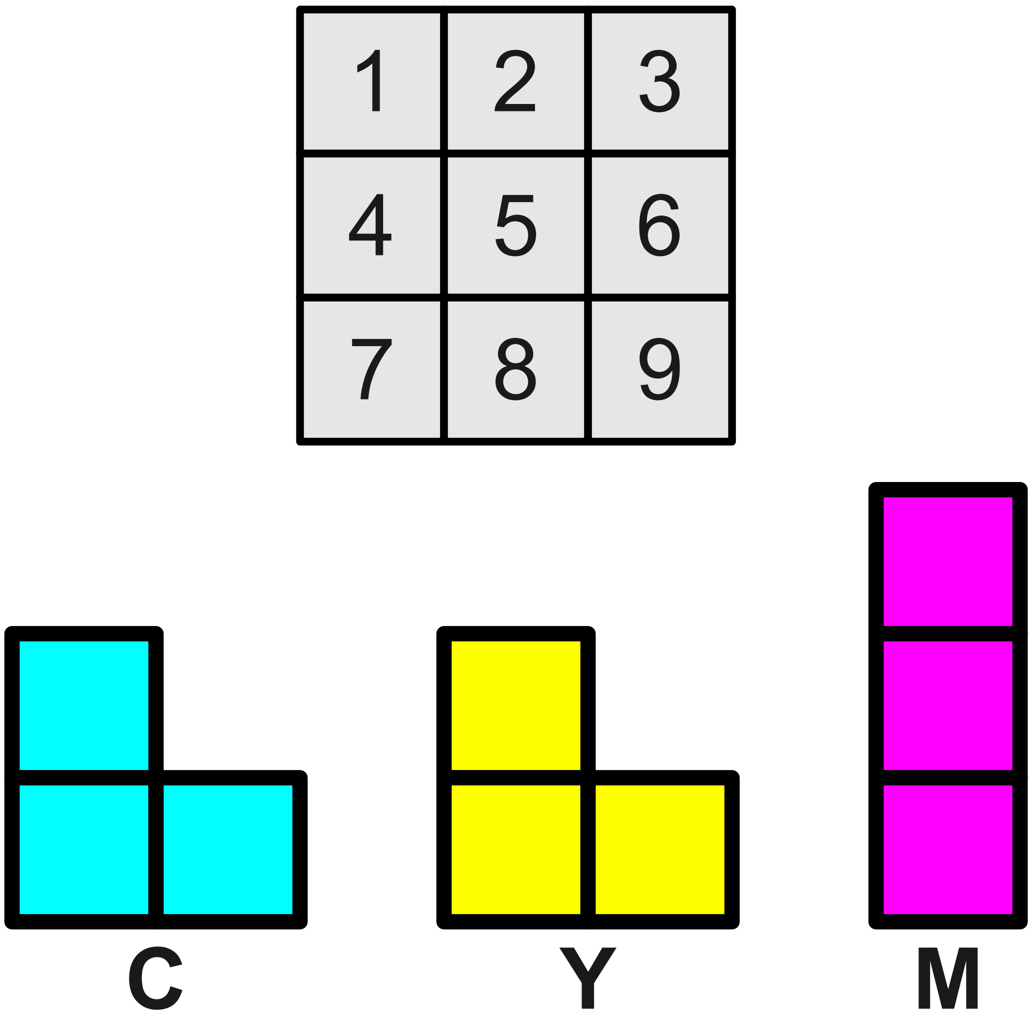

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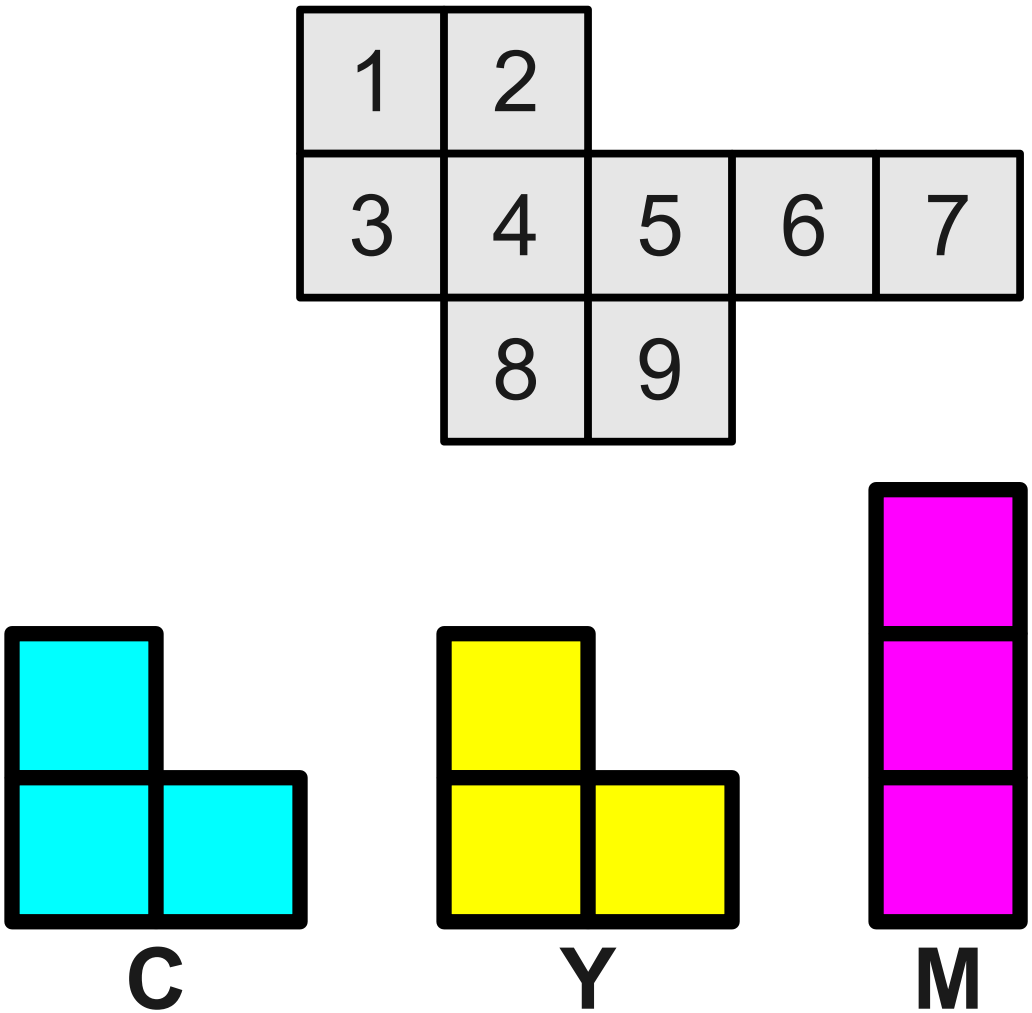

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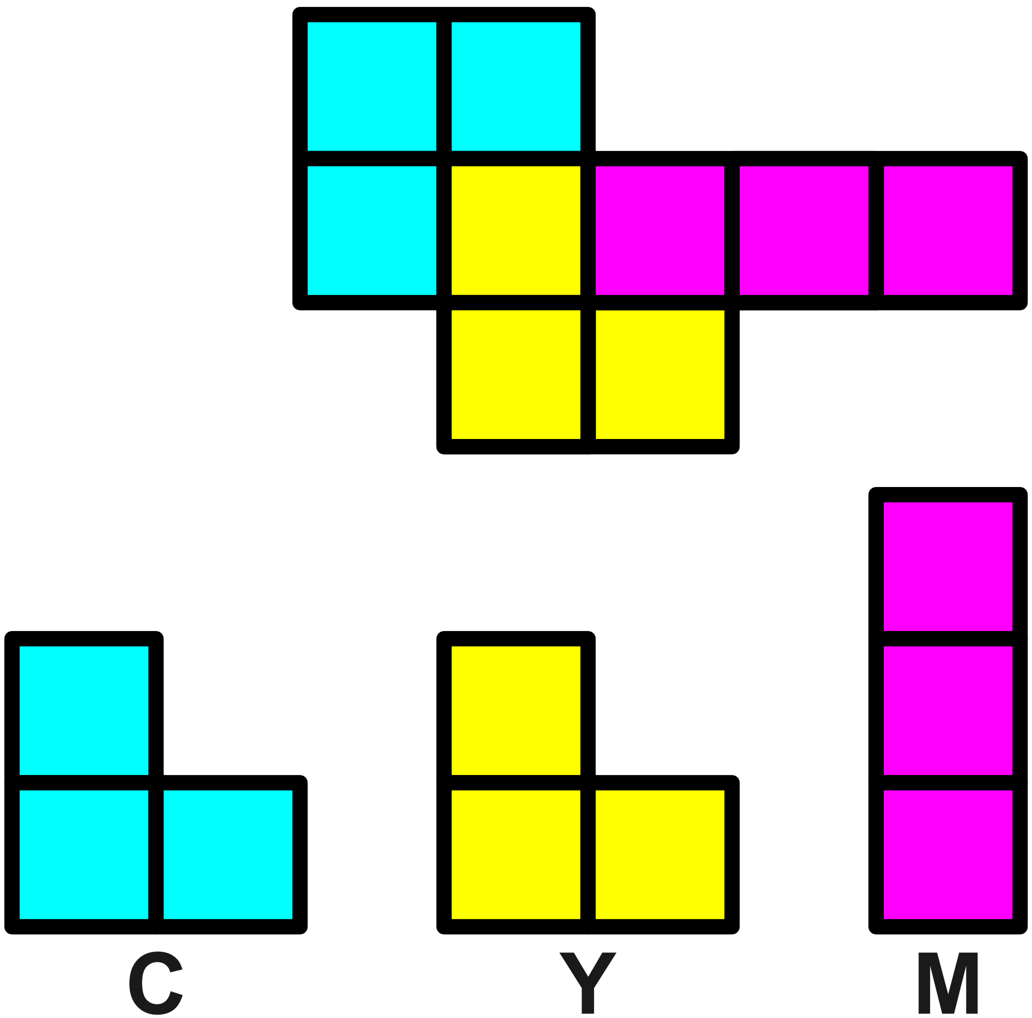

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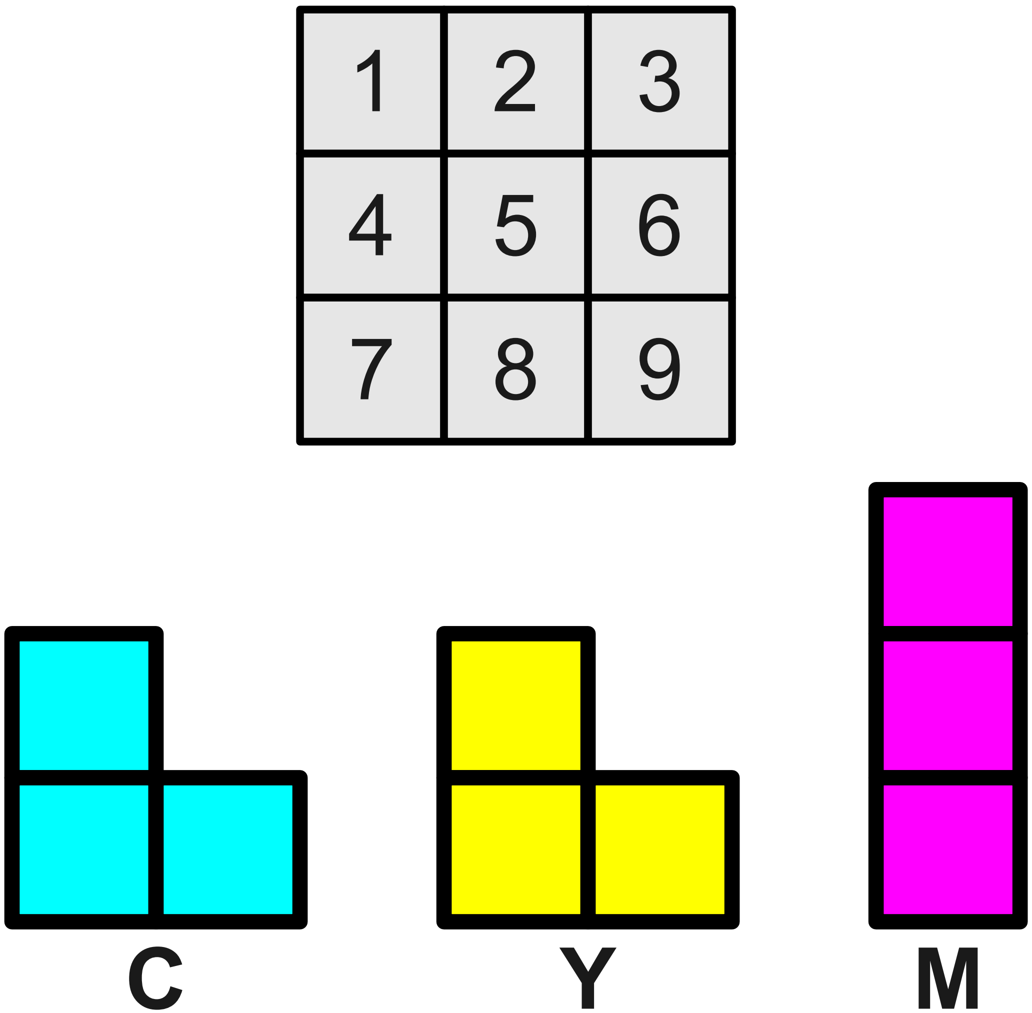

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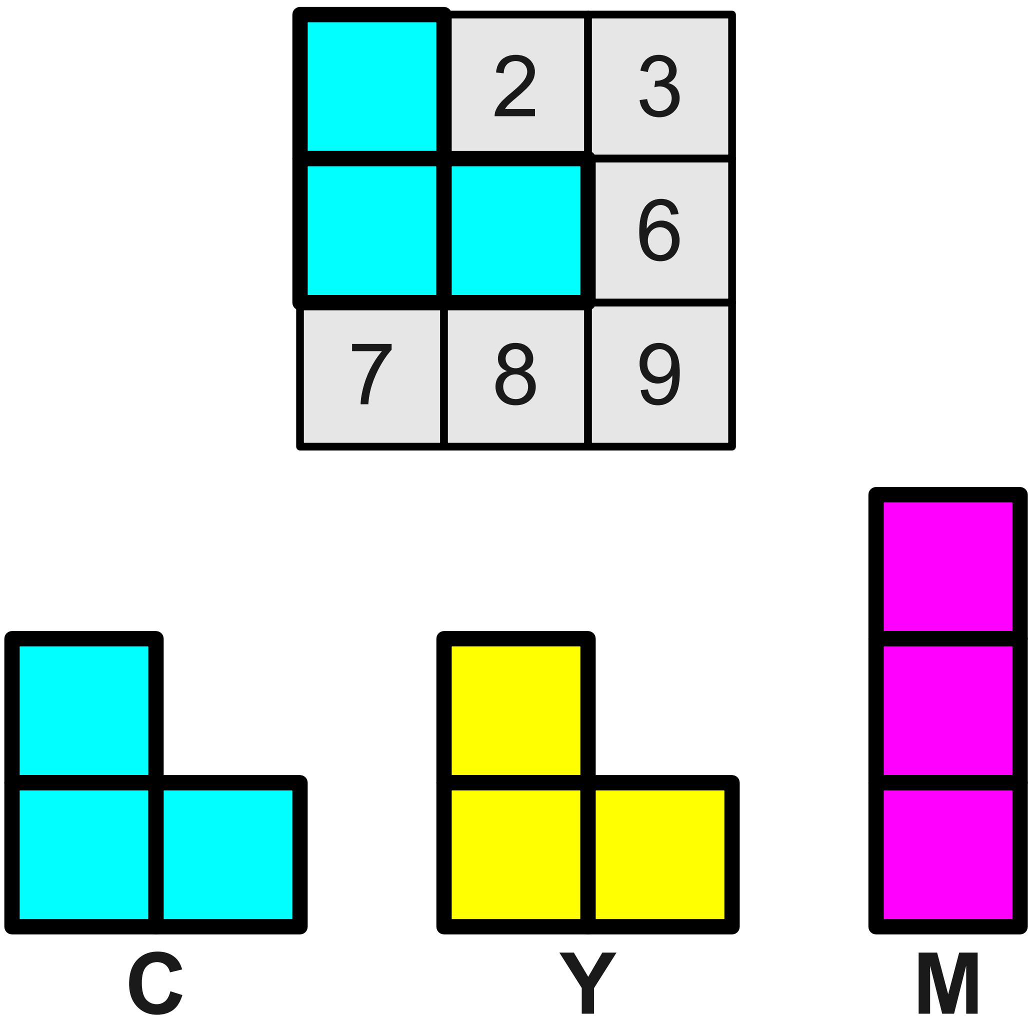

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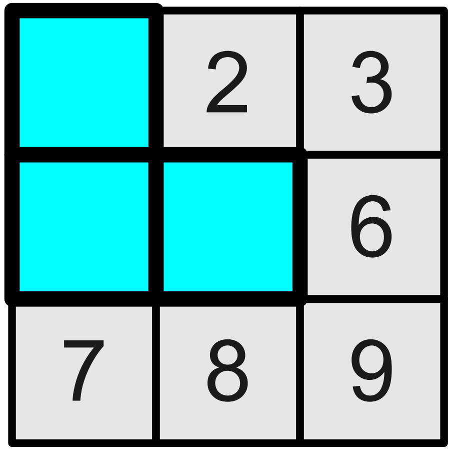
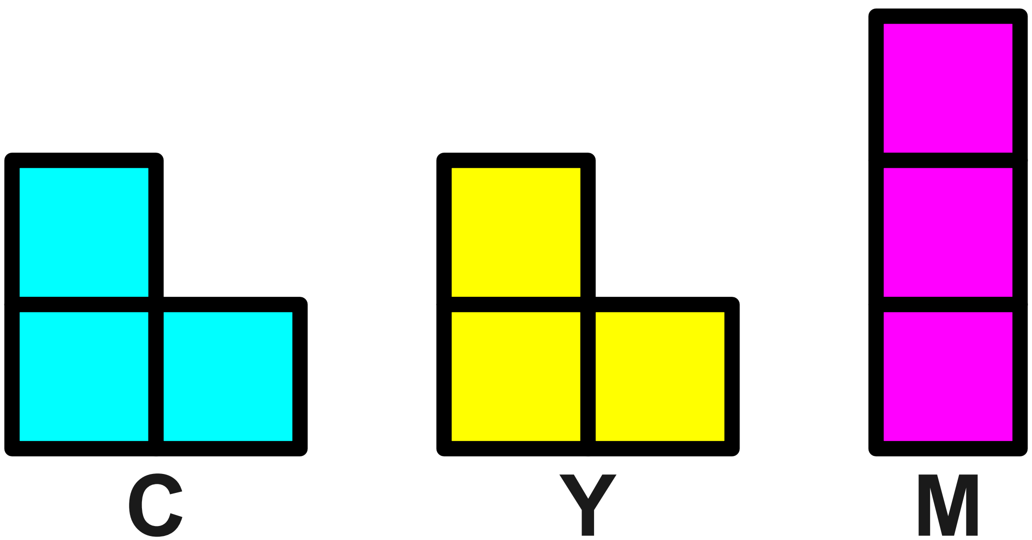


{ C, 1, 4, 5 }



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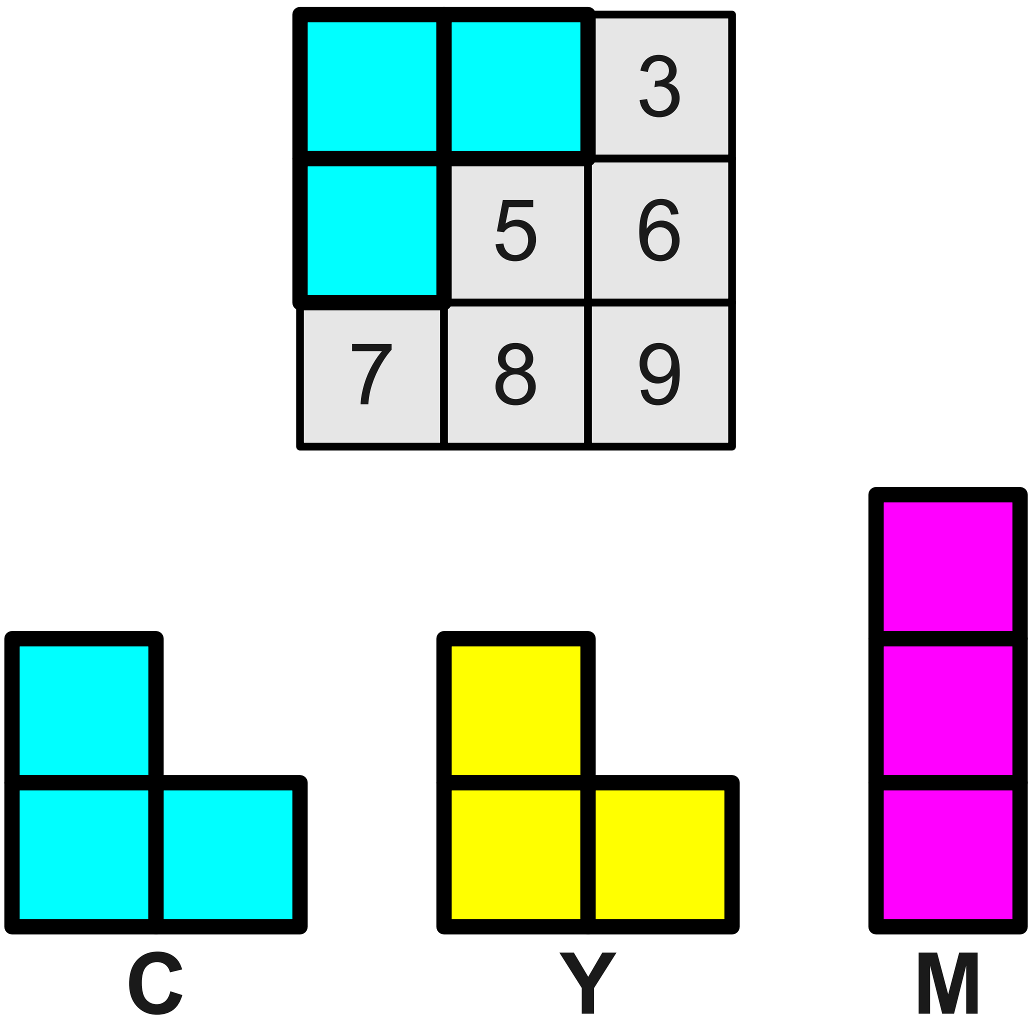

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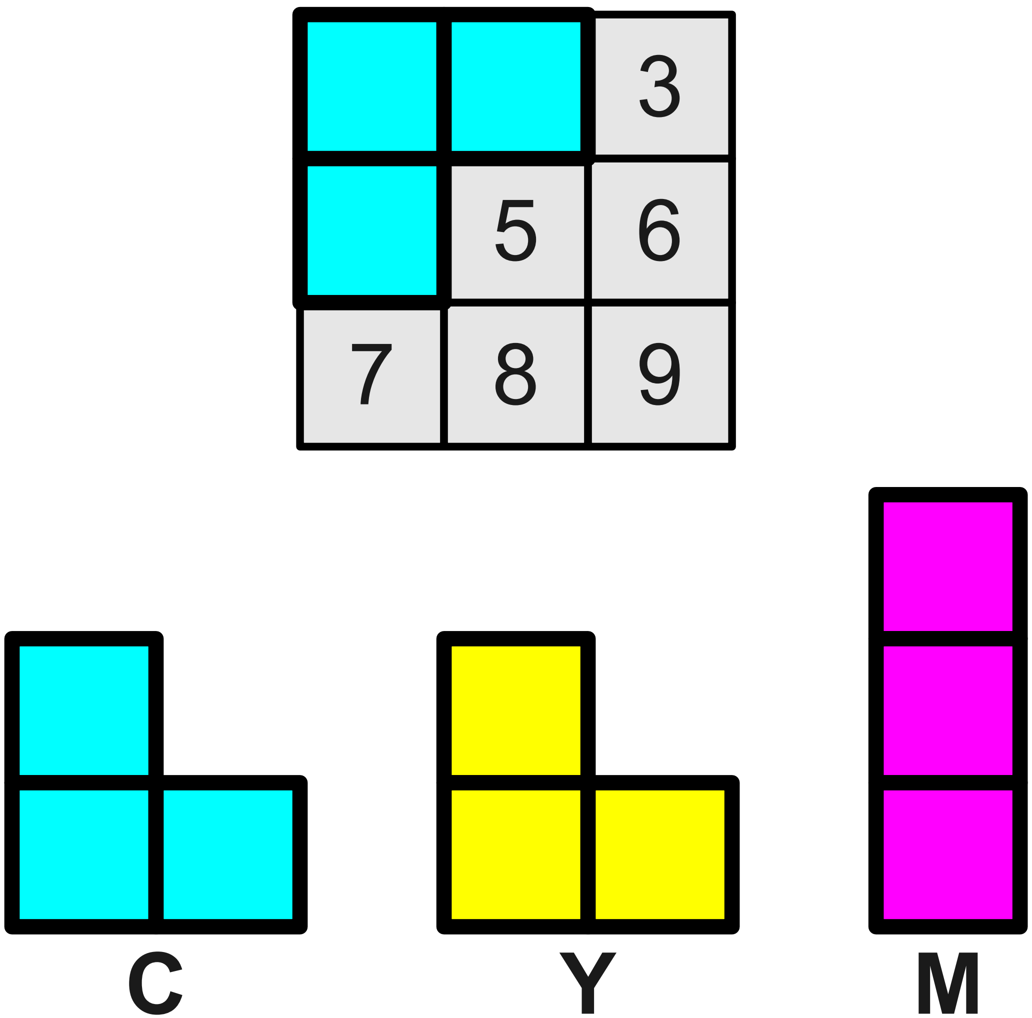

{ C, 1, 4, 5 }
{ C, 1, 2, 4 }

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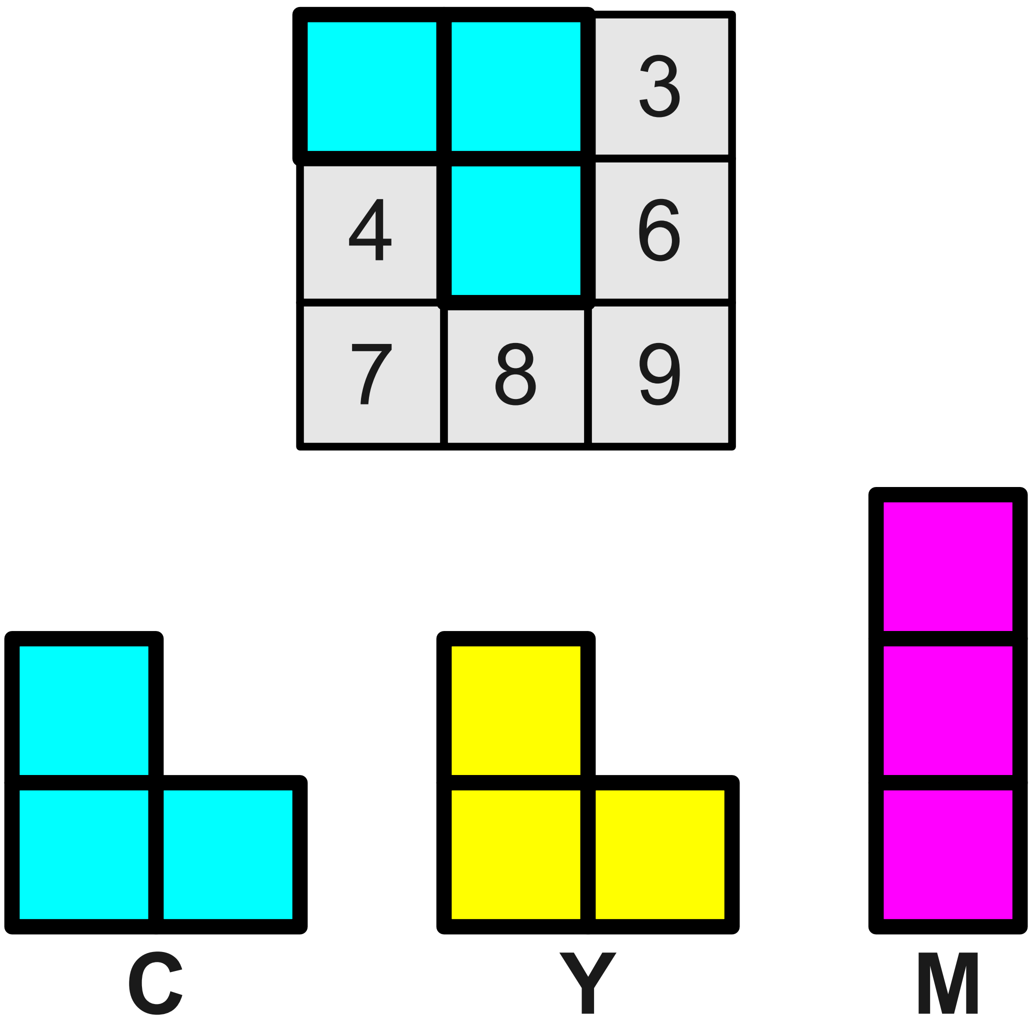

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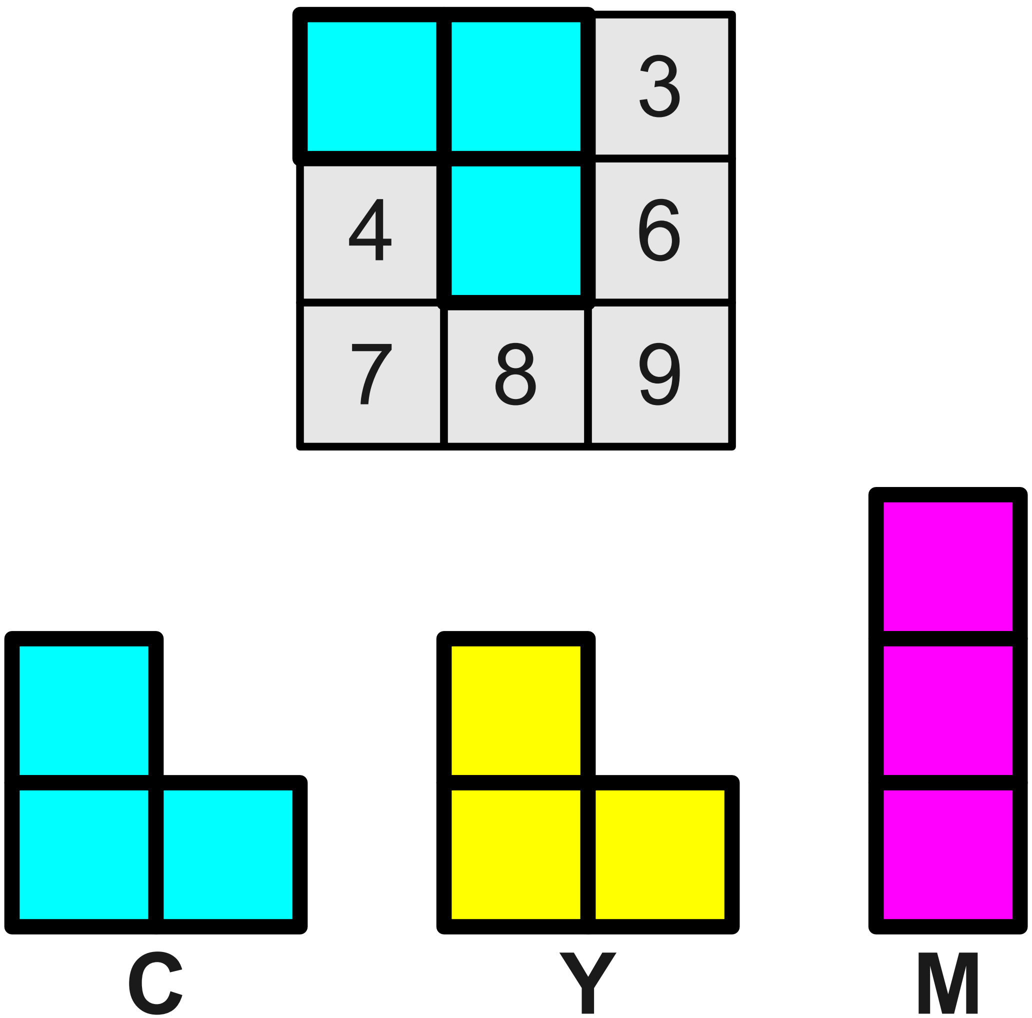

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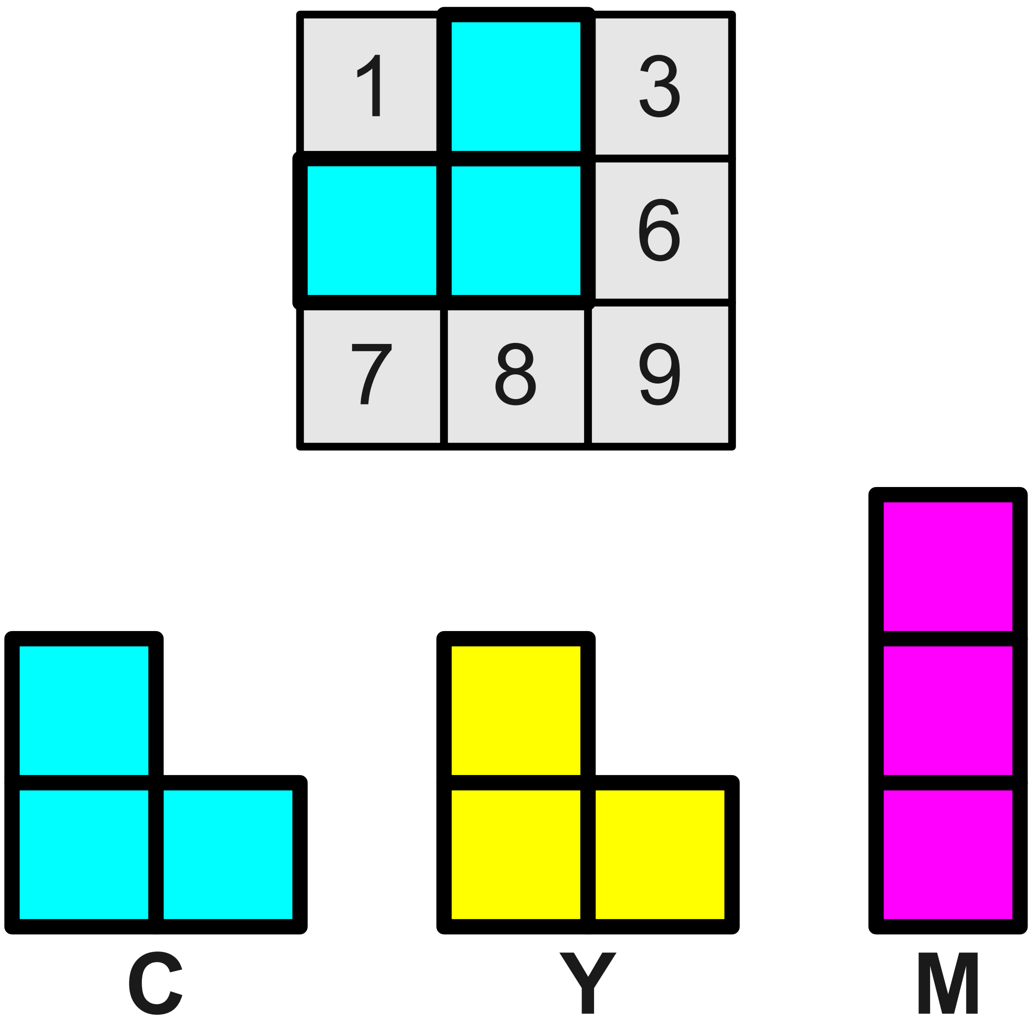

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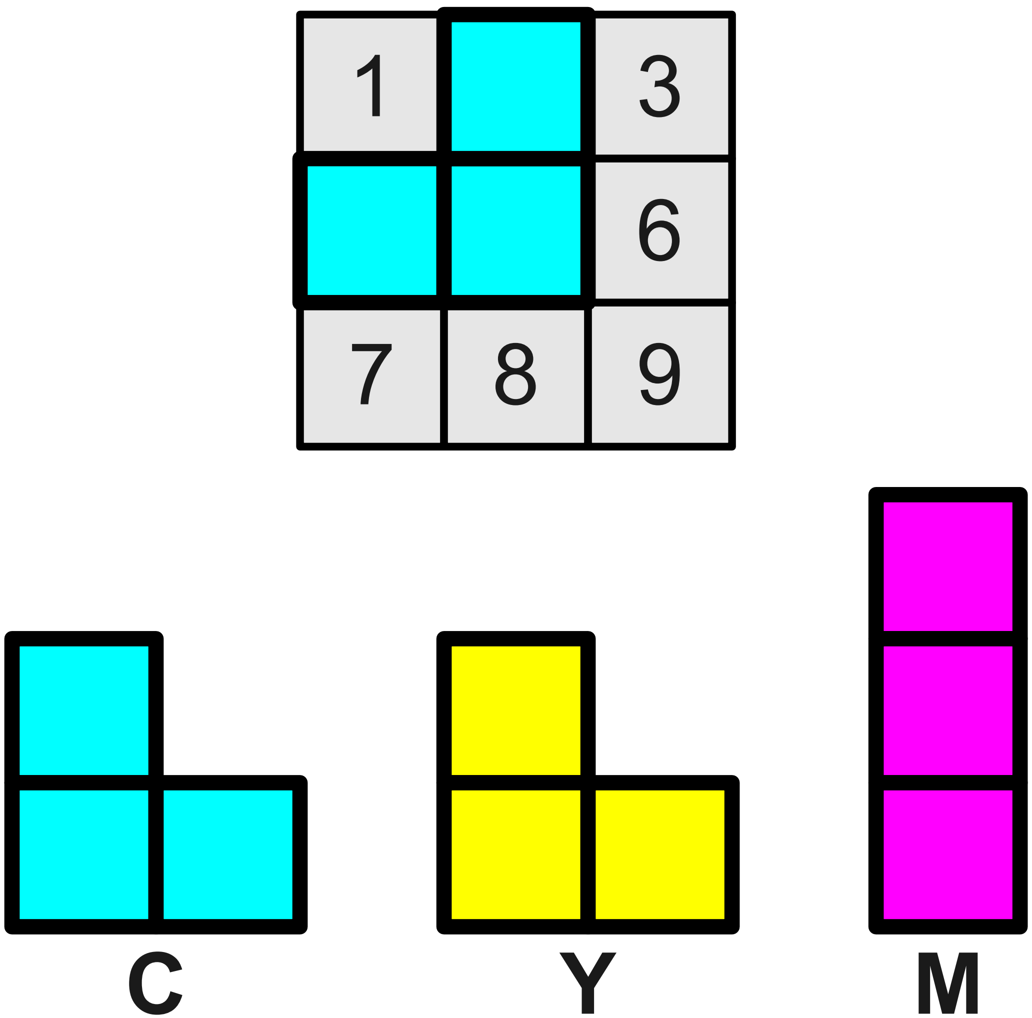

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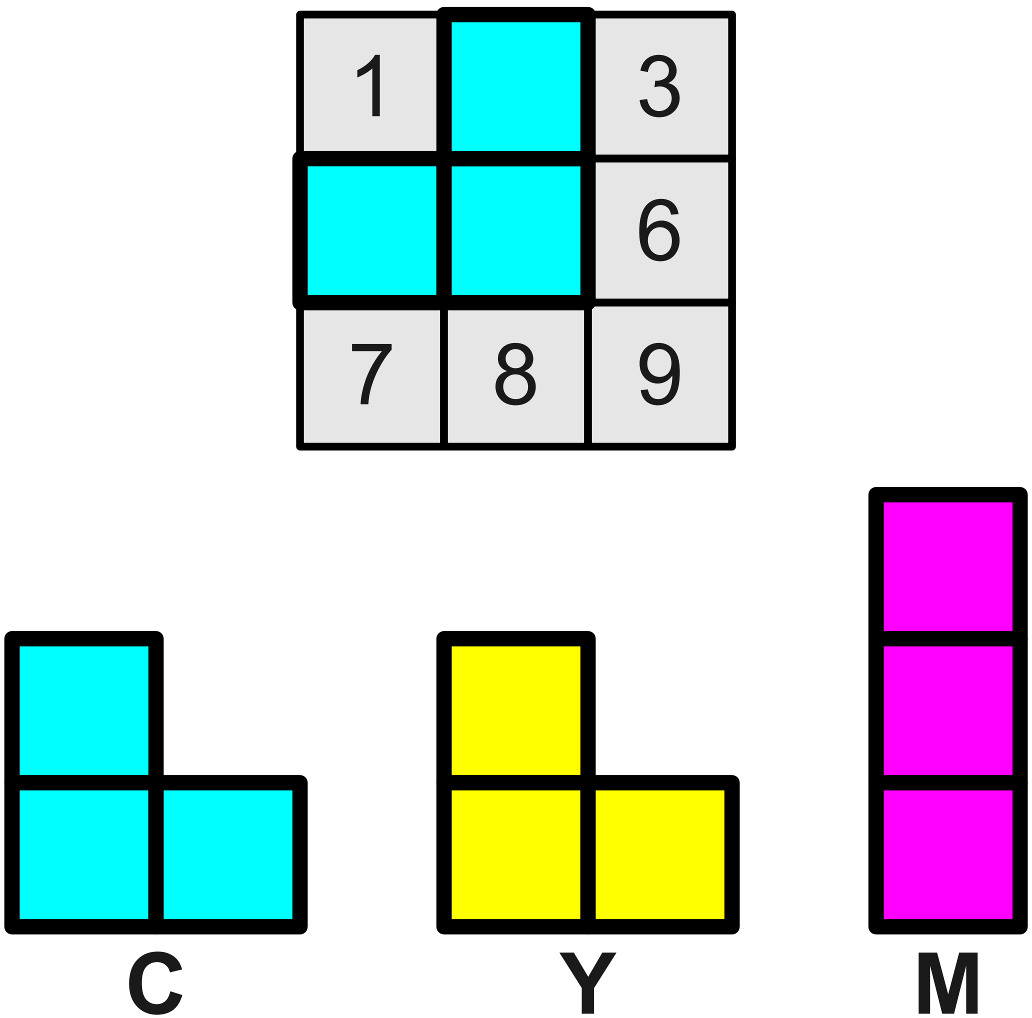

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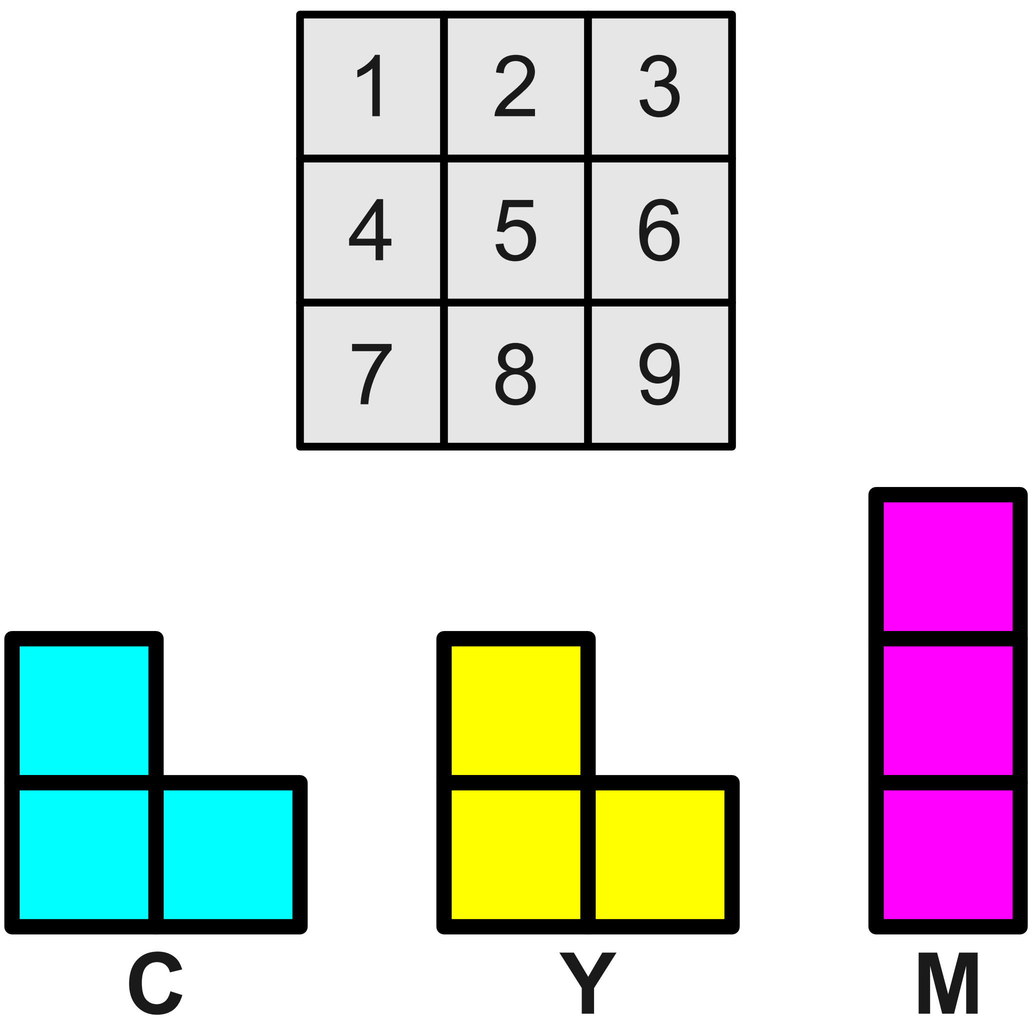

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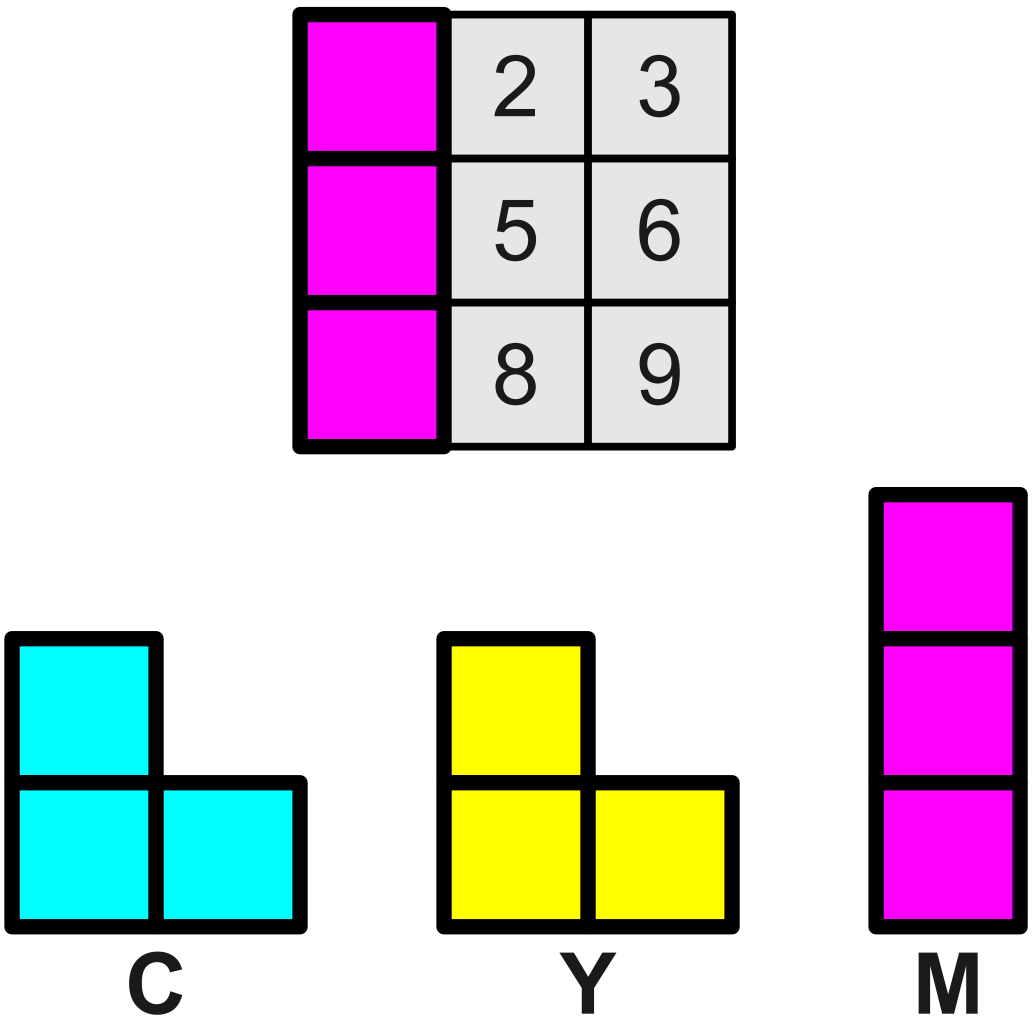

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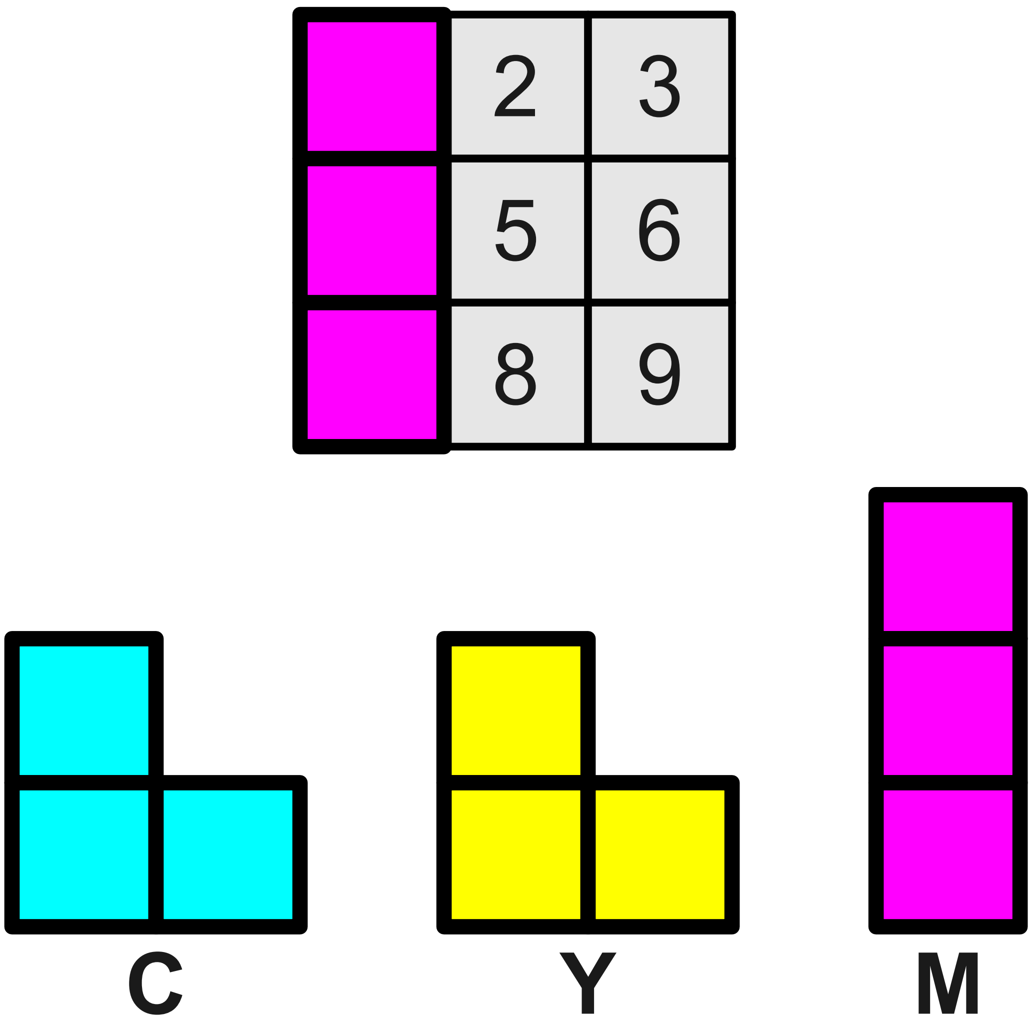
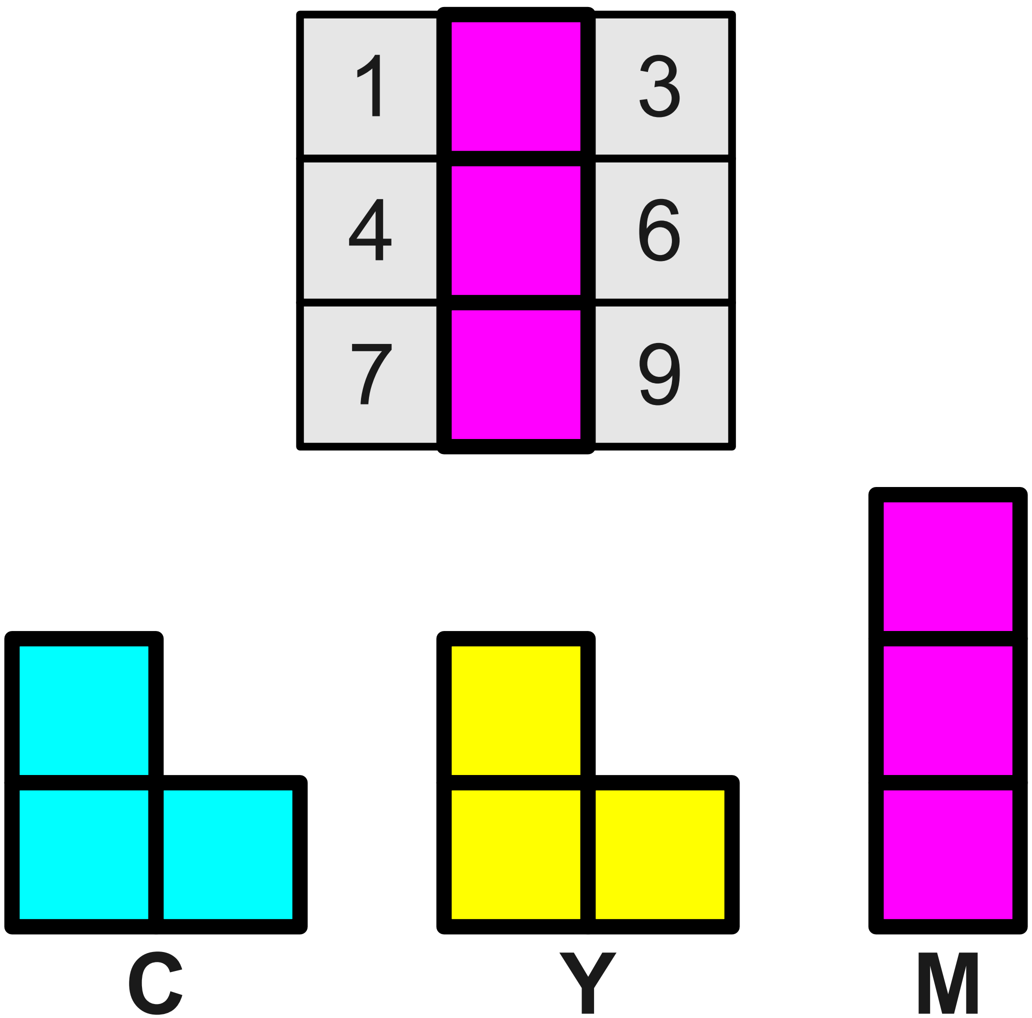


{ C, 1, 4, 5 }
{ C, 1, 2, 4 }
{ C, 1, 2, 5 }
{ C, 2, 4, 5 }

...

{ M, 1, 4, 7 }

Applications of Exact Covering



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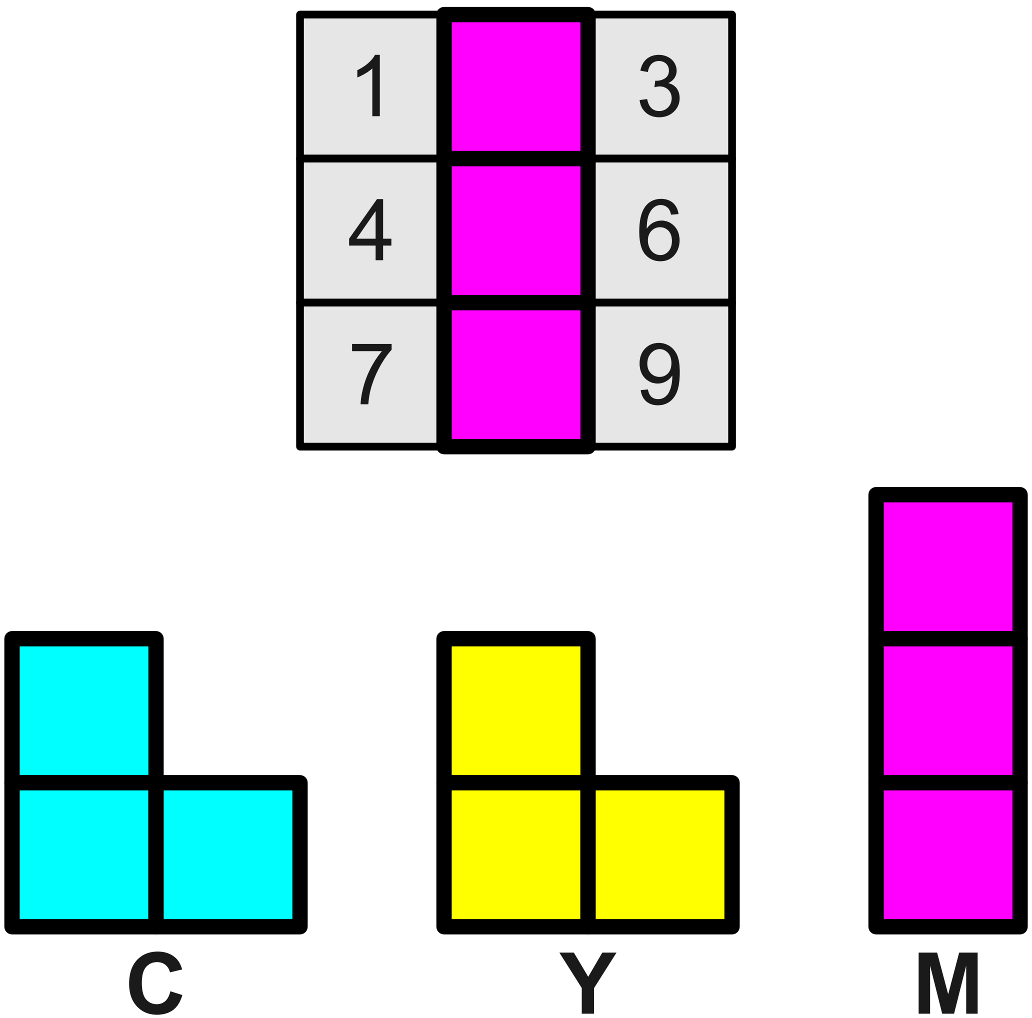

{ C, 1, 4, 5 }
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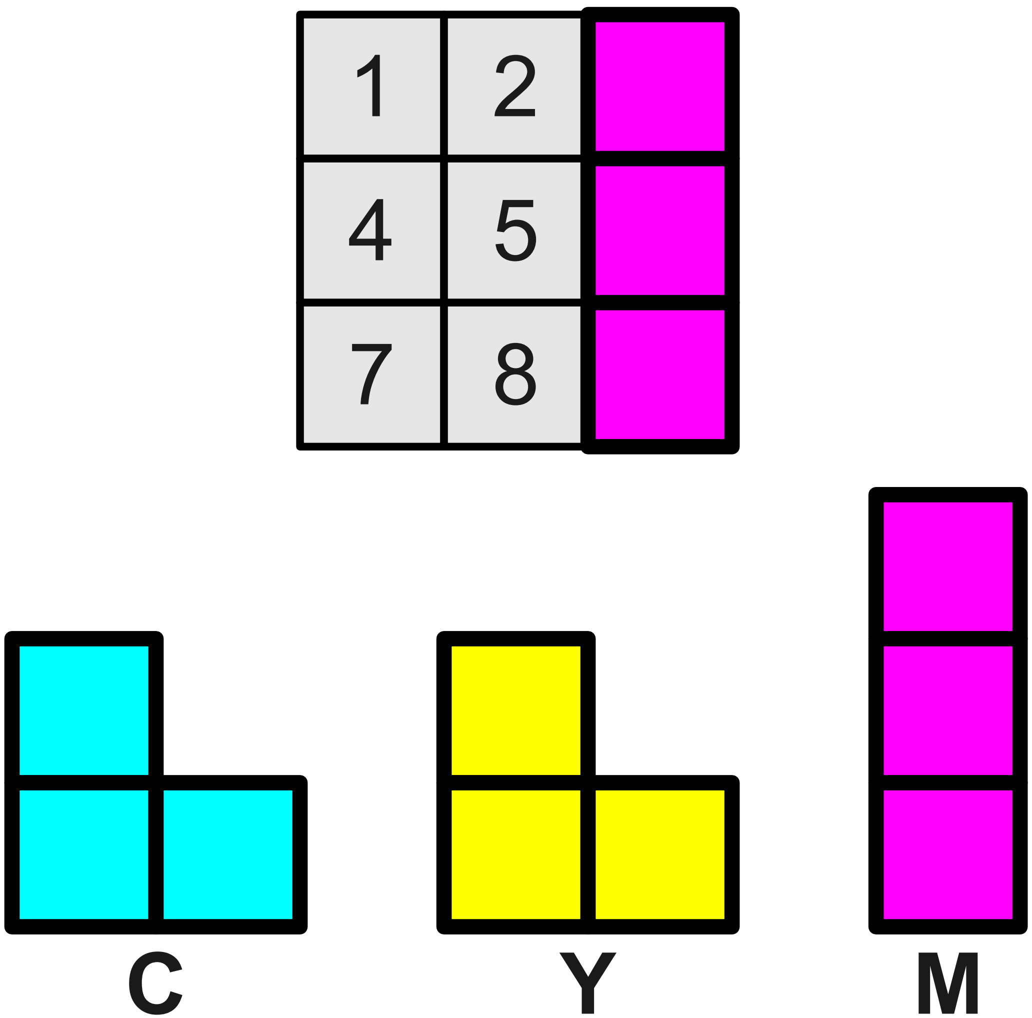
{ M, 2, 5, 8 }

Applications of Exact Covering



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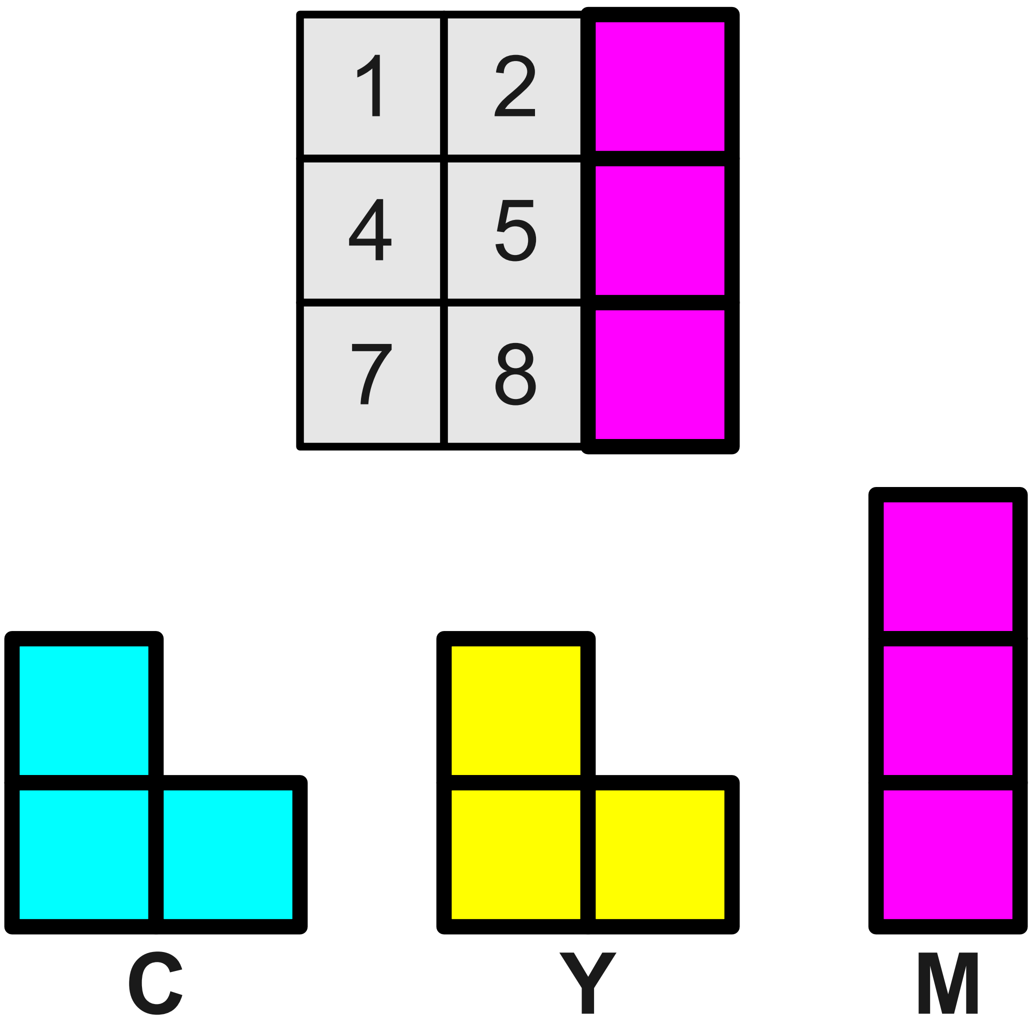

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{ M, 2, 5, 8 }
{ M, 3, 6, 9 }

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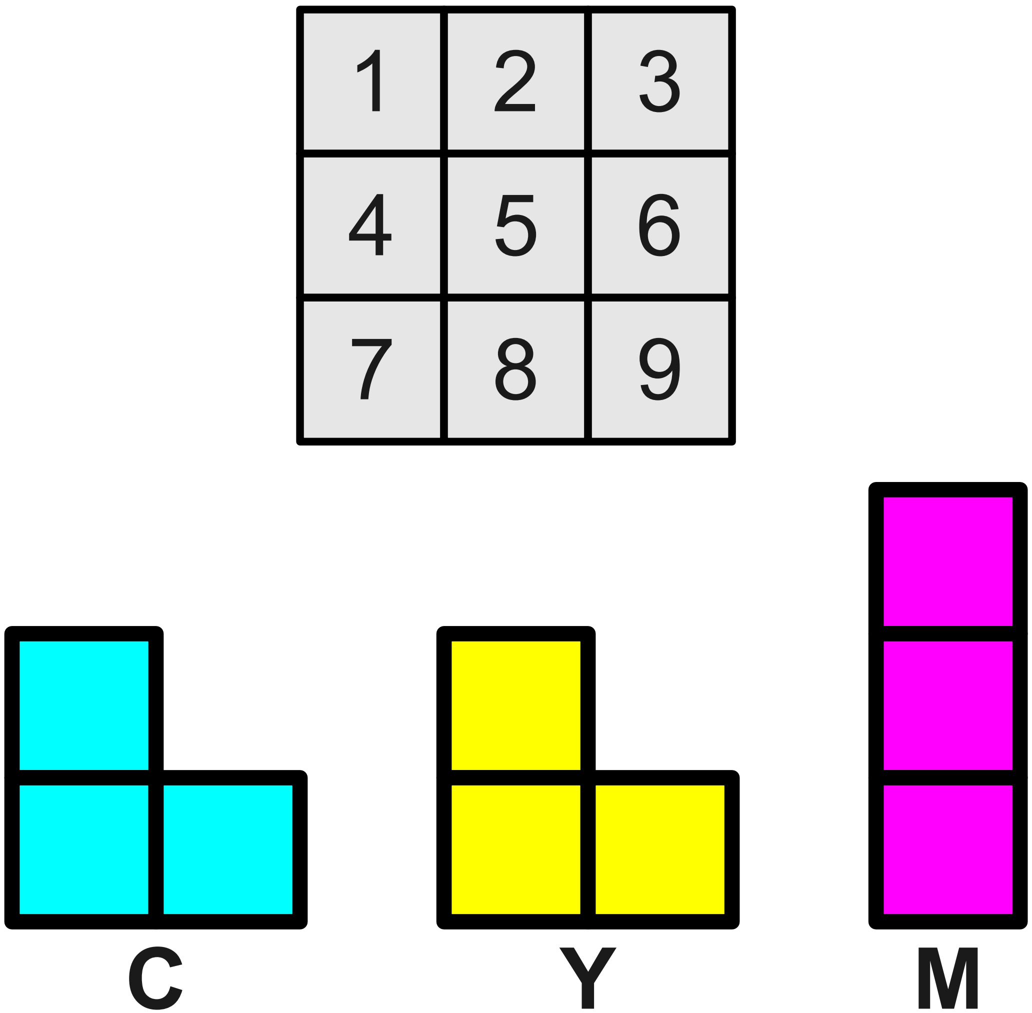

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...

{ M, 1, 4, 7 }
{ M, 2, 5, 8 }
{ M, 3, 6, 9 }

...

Trust me, these reductions matter.

We'll see why in a few minutes.

The

Most Important Question

in

Theoretical Computer Science

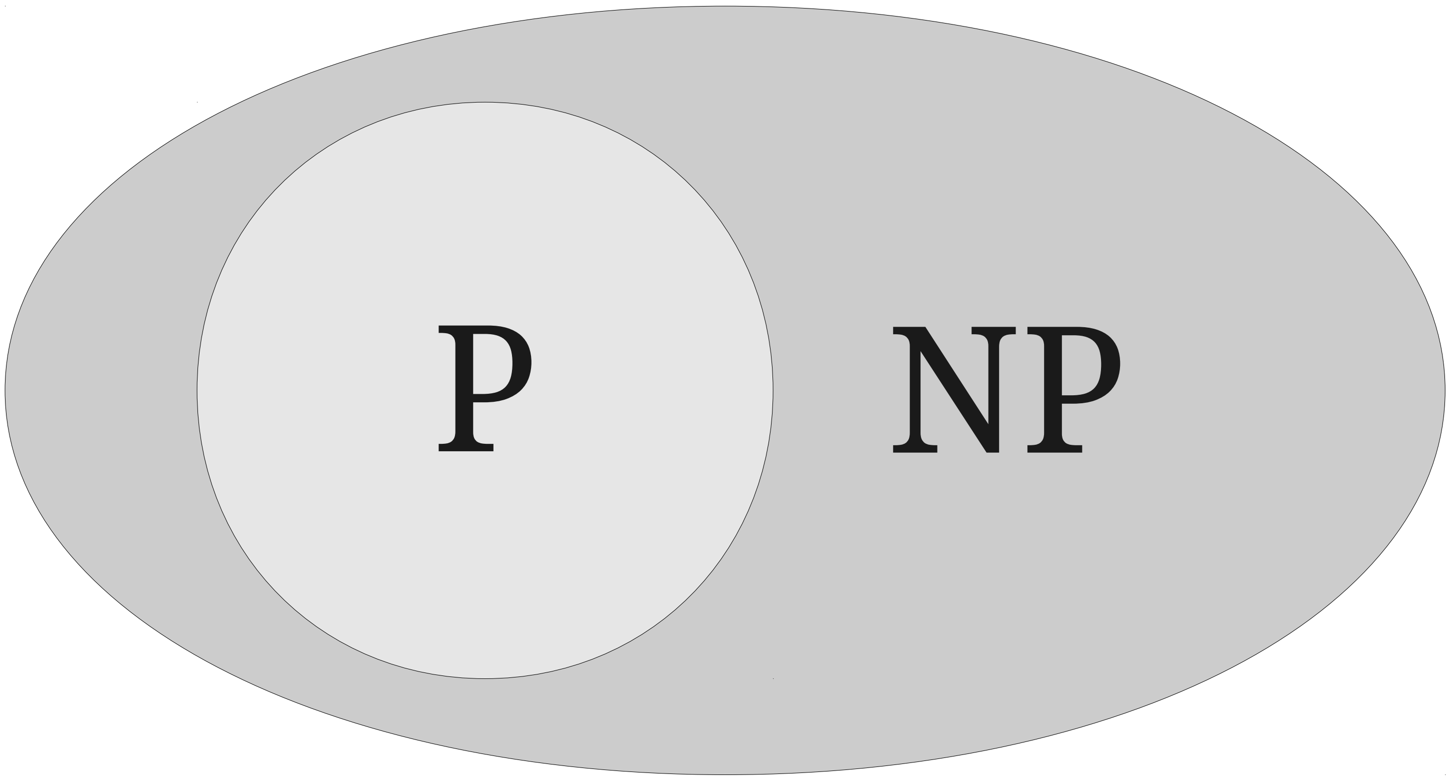
What is the connection between **P** and **NP**?

$$\mathbf{P} = \{\ L \mid \text{There is a polynomial-time decider for } L\ \}$$

$$\mathbf{NP} = \{\ L \mid \text{There is a nondeterministic polynomial-time decider for } L\ \}$$

$$\mathbf{P} \subseteq \mathbf{NP}$$

Which Picture is Correct?



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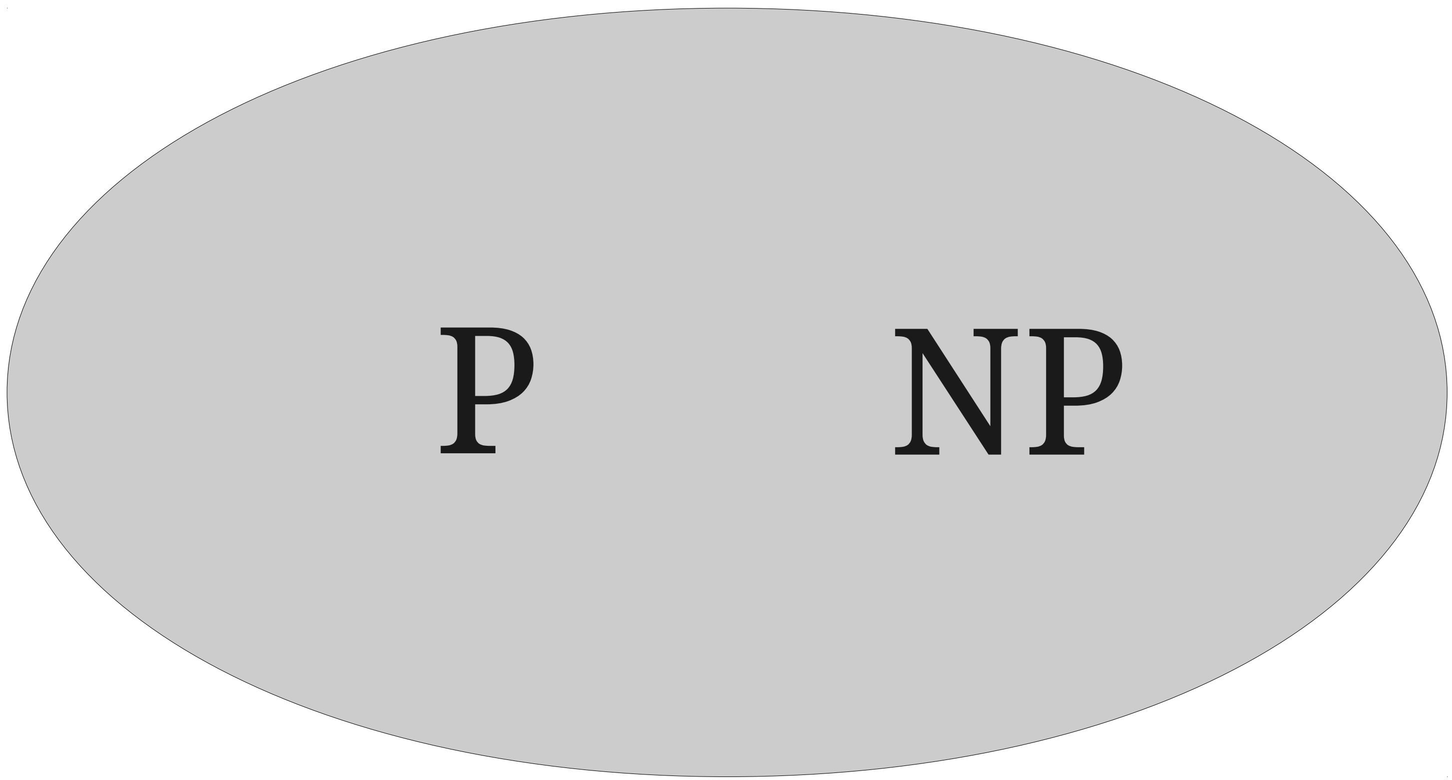

Does **P** = **NP**?

$\mathbf{P} \stackrel{?}{=} \mathbf{NP}$

- The $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ question is the most important question in theoretical computer science.
- With the verifier definition of **NP**, one way of phrasing this question is

If a solution to a problem can be **verified** efficiently, can that problem be **solved** efficiently?

- An answer either way will give fundamental insights into the nature of computation.

Why This Matters

- The following problems are known to be efficiently verifiable, but have no known efficient solutions:
  - Determining whether an electrical grid can be built to link up some number of houses for some price (Steiner tree problem).
  - Determining whether a simple DNA strand exists that multiple gene sequences could be a part of (shortest common supersequence).
  - Determining the best way to assign hardware resources in a compiler (optimal register allocation).
  - Determining the best way to distribute tasks to multiple workers to minimize completion time (job scheduling).
  - **And many more**.
- If **P = NP**, **all** of these problems have efficient solutions.
- If **P ≠ NP**, **none** of these problems have efficient solutions.

Why This Matters

- If **P = NP**:
  - A huge number of seemingly difficult problems could be solved efficiently.
  - Our capacity to solve many problems will scale well with the size of the problems we want to solve.
- If **P ≠ NP**:
  - Enormous computational power would be required to solve many seemingly easy tasks.
  - Our capacity to solve problems will fail to keep up with our curiosity.

What We Know

- Resolving $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$ has proven ***extremely difficult.***
- In the past 35 years:
  - Not a single correct proof either way has been found.
  - Many types of proofs have been shown to be insufficiently powerful to determine whether $\mathbf{P} = \mathbf{NP}$.
  - A majority of computer scientists believe $\mathbf{P} \neq \mathbf{NP}$, but this isn't a large majority.
- Interesting read: Interviews with leading thinkers about $\mathbf{P} \stackrel{?}{=} \mathbf{NP}$:
  - http://web.ing.puc.cl/~jabaier/iic2212/poll-1.pdf

The Million-Dollar Question

The Clay Mathematics Institute has offered a **$1,000,000 prize** to anyone who proves or disproves **P = NP**.

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Time-Out For Announcements

Please evaluate this course in Axess.

Your feedback really does make a difference.

Final Exam Logistics

- Final exam is this upcoming Monday, December 9th from 12:15PM – 3:15PM.
- Room information TBA; we're still finalizing everything.
- Exam is cumulative, but focuses primarily on material from DFAs onward.
  - Take a look a the practice exams for a sense of what the coverage will be like.

Practice Finals

- We have three practice exams available right now:
  - An **extra credit** practice exam worth +5 EC points.
  - Two actual final exams from previous quarters, which are good for studying but not worth any extra credit.
- Solutions to the two additional practice finals will be released Wednesday.
- **Please take the additional final exams under realistic conditions** so that you can get a sense of where you stand. Most of the problems are “nondeterministically trivial.”

A Note on Honesty and Integrity

Review Sessions

- We will be holding at least one final exam review session later this week.
- We will announce date and time information once it's finalized.
- Feel free to show up with any questions you'd like answered!

Casual CS Dinner

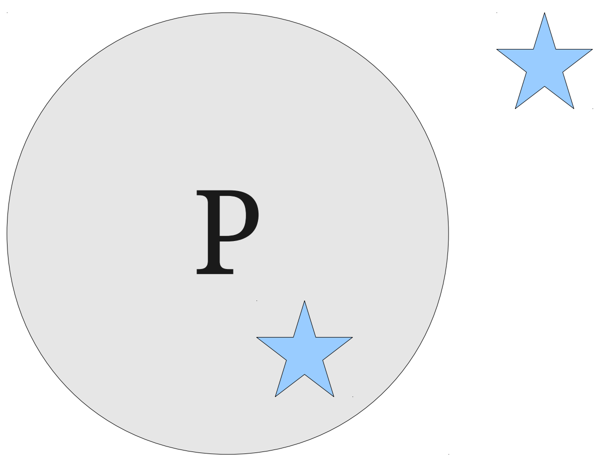
- The second biquarterly Casual CS Dinner for Women in CS is **tonight** at 6PM on the fifth floor of Gates.
- Everyone is welcome!
- RSVP appreciated; check the email sent to the CS103 list.

Back to CS103!

NP-Completeness

Polynomial-Time Reductions

- If $L_1 \leq_P L_2$ and $L_2 \in \mathbf{P}$, then $L_1 \in \mathbf{P}$.



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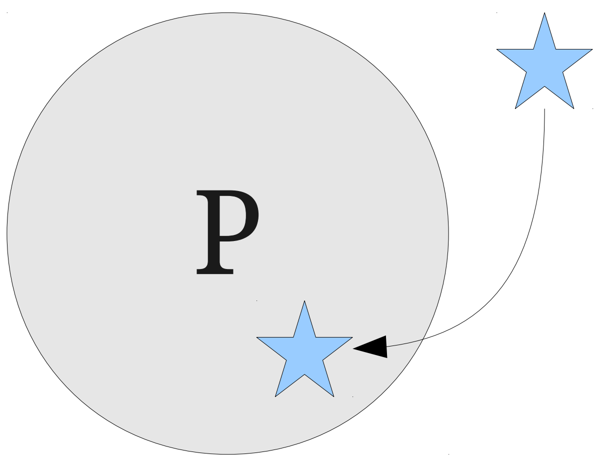

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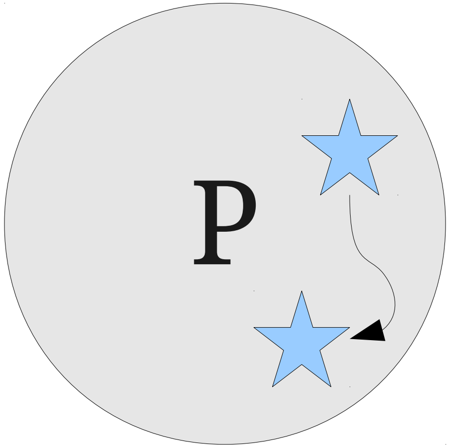
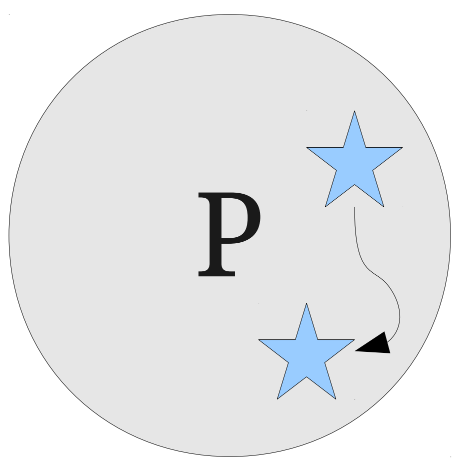


Polynomial-Time Reductions

- If $L_1 \leq_{\mathrm{P}} L_2$ and $L_2 \in \mathbf{P}$, then $L_1 \in \mathbf{P}$.
- If $L_1 \leq_{\mathrm{P}} L_2$ and $L_2 \in \mathbf{NP}$, then $L_1 \in \mathbf{NP}$.



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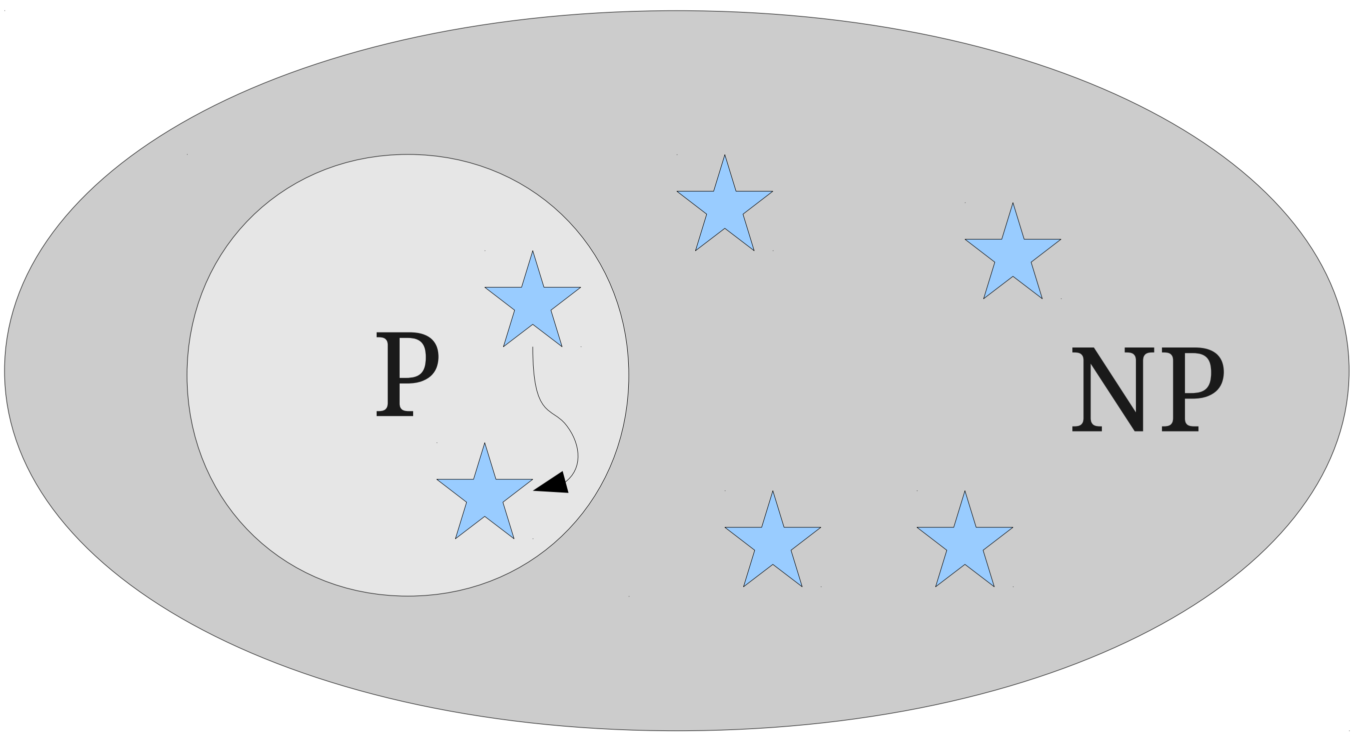

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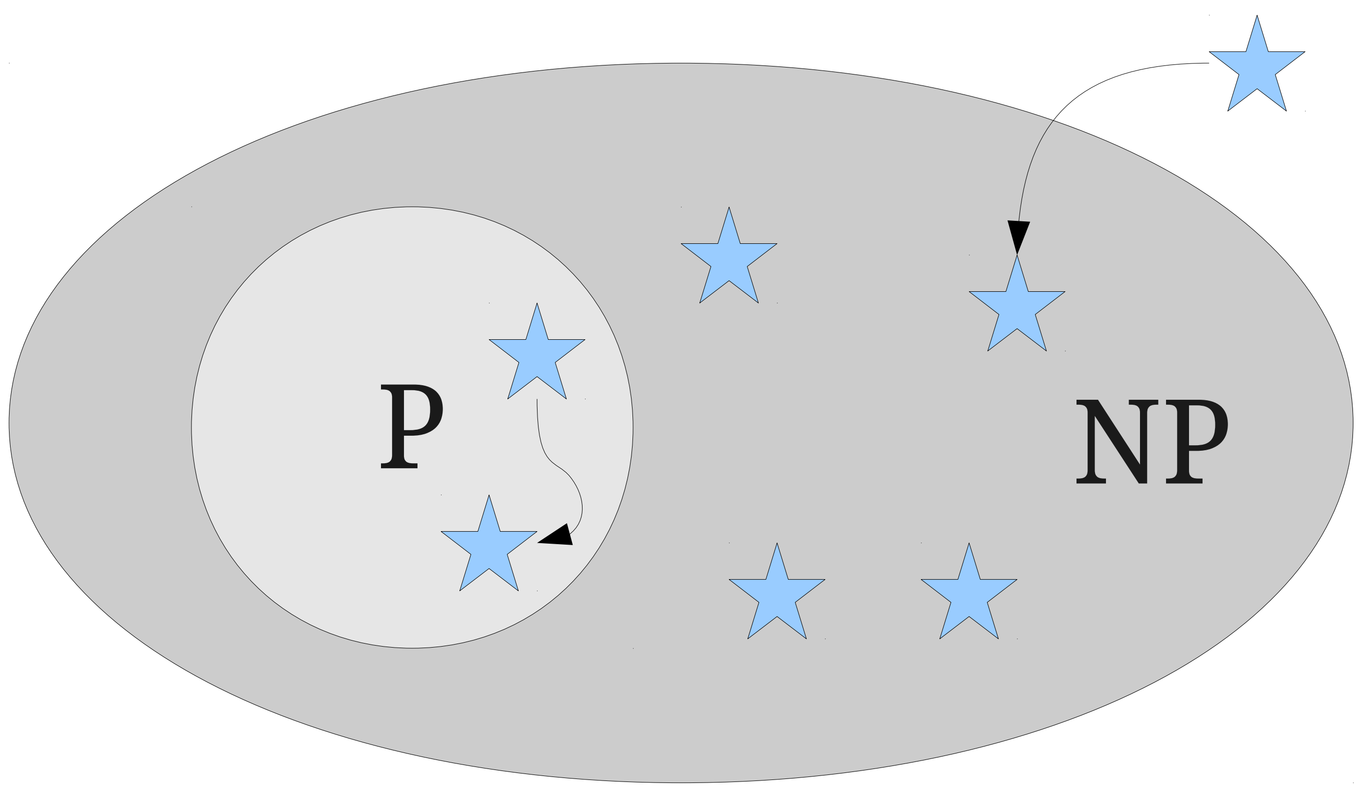

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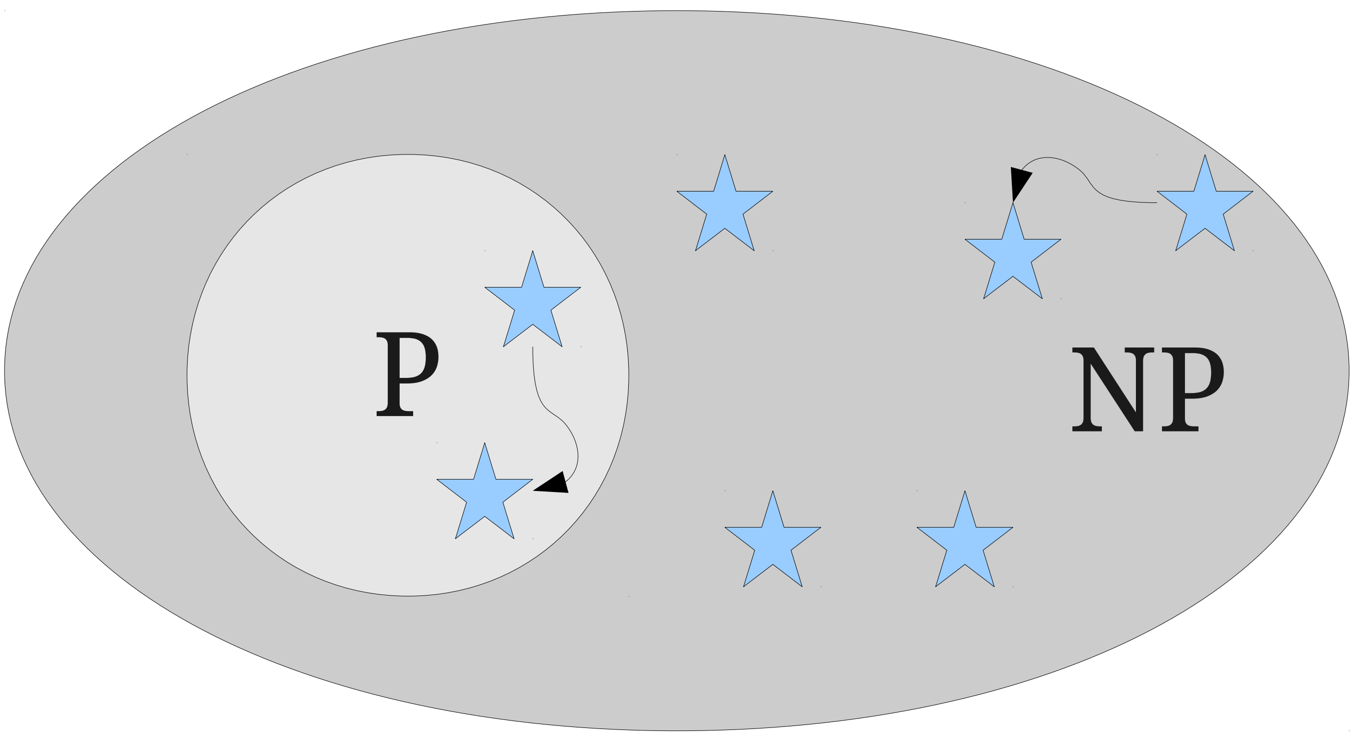

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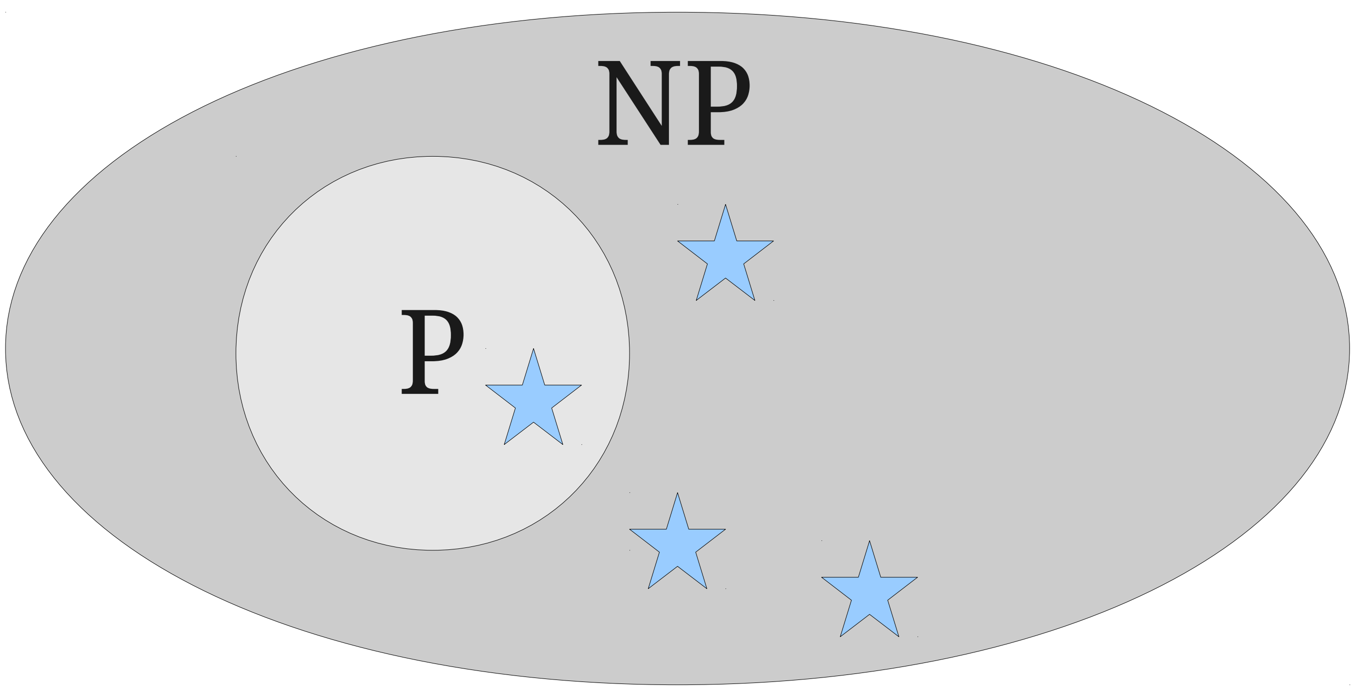

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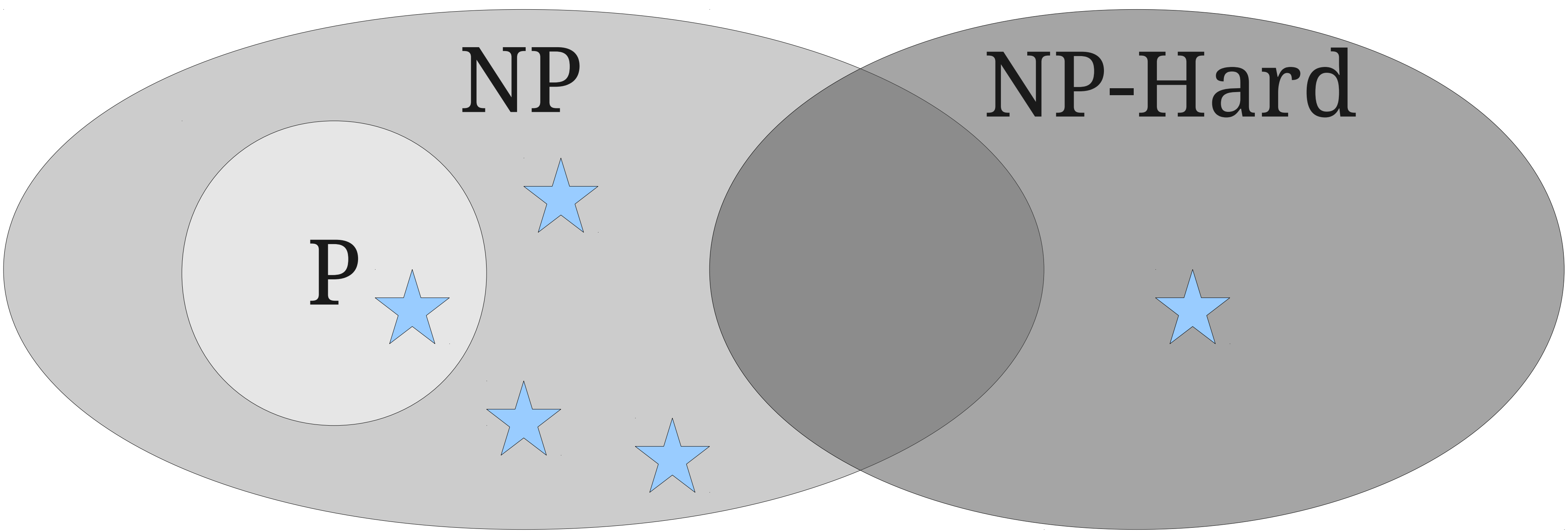

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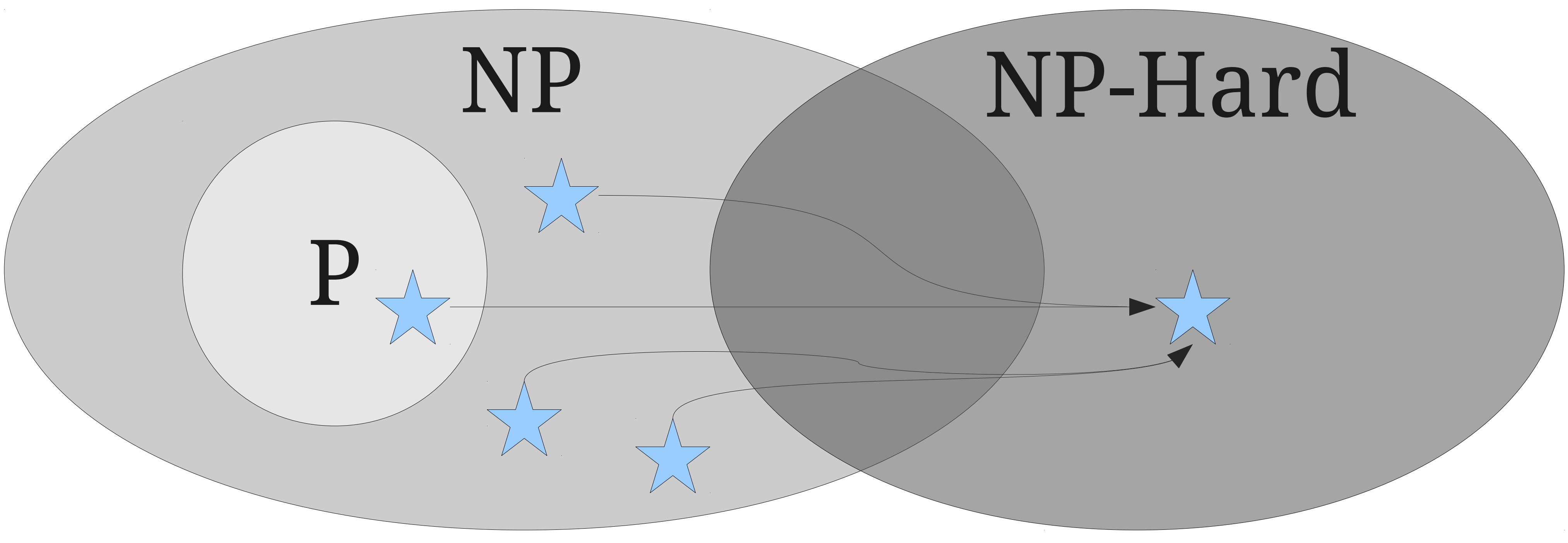

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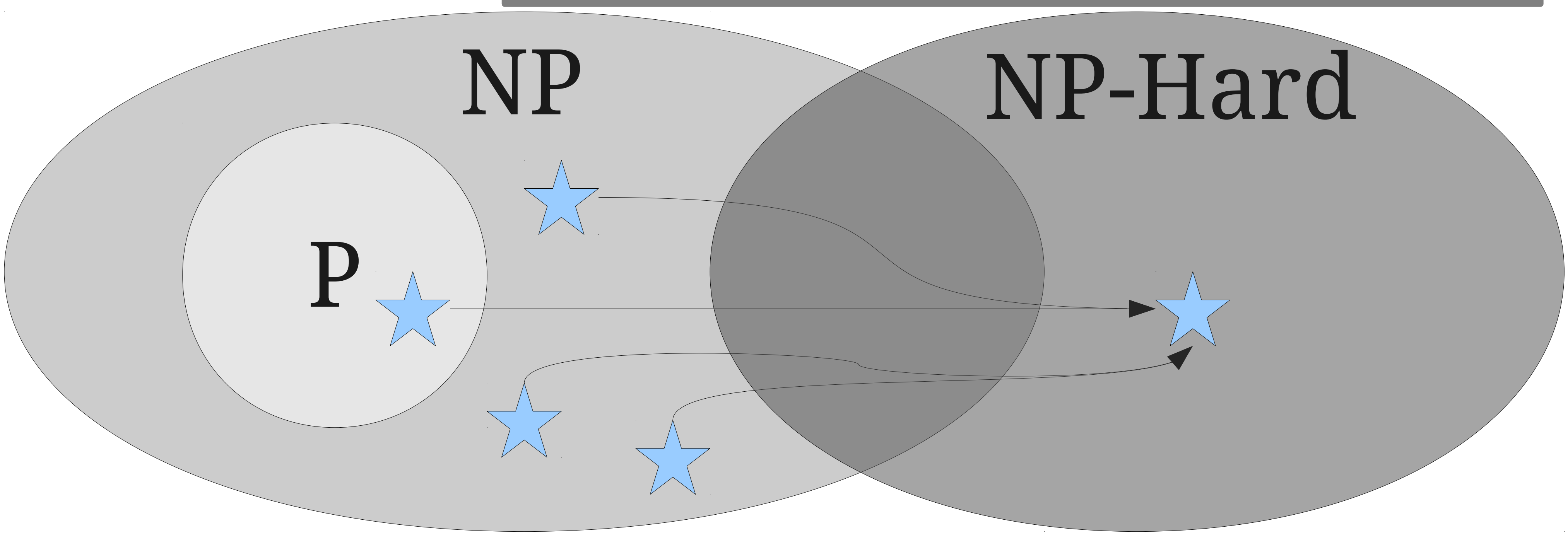
Intuitively: L has to be at least as hard as every problem in **NP**, since an algorithm for L can be used to decide all problems in **NP**.



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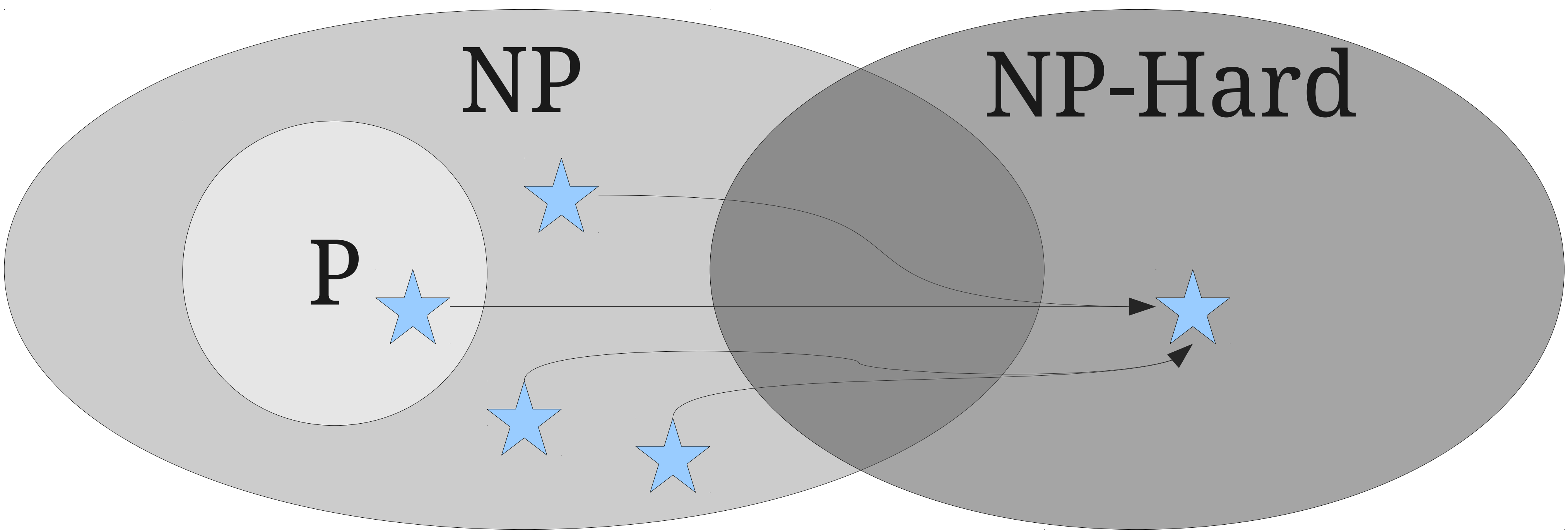

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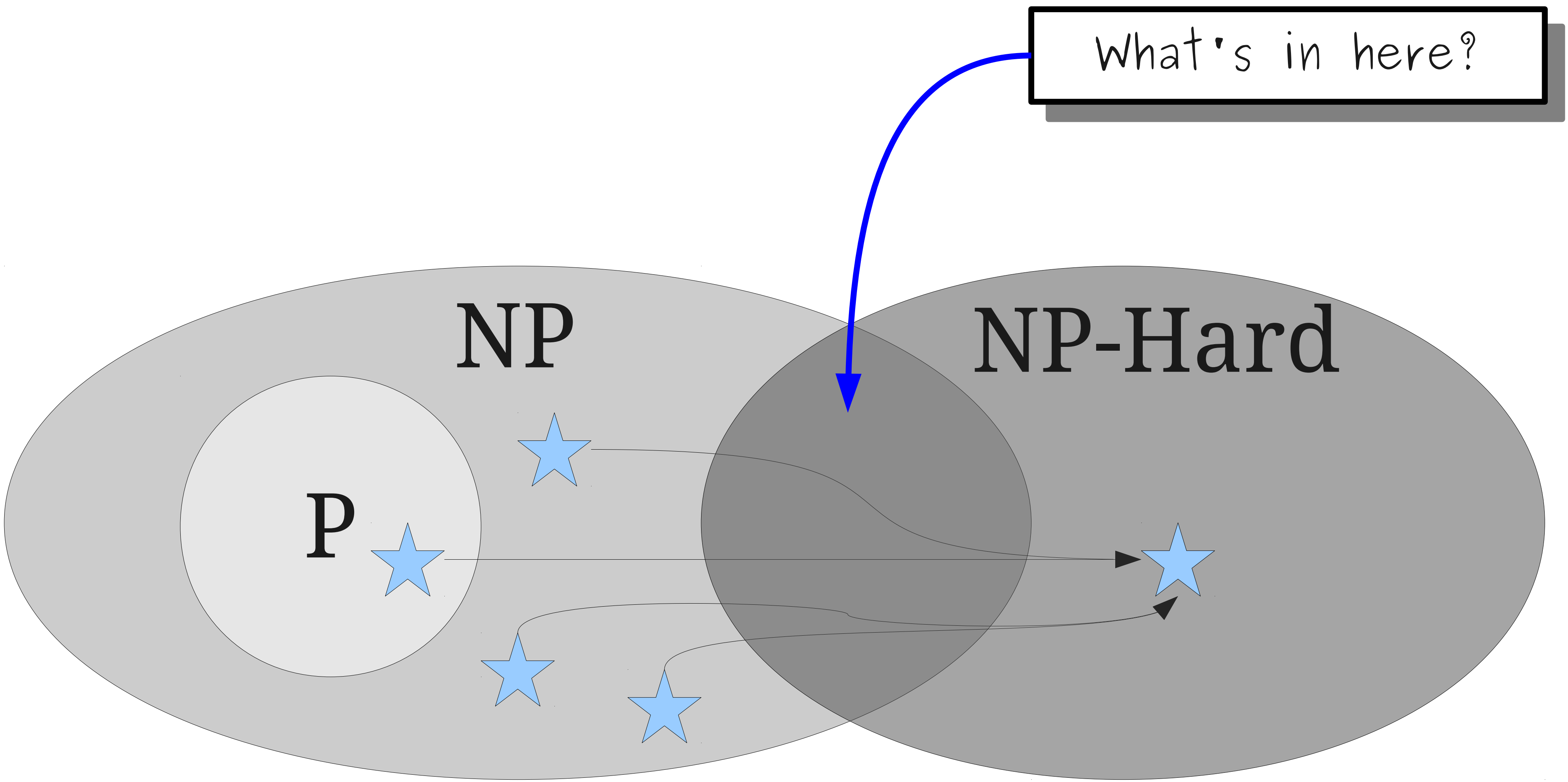

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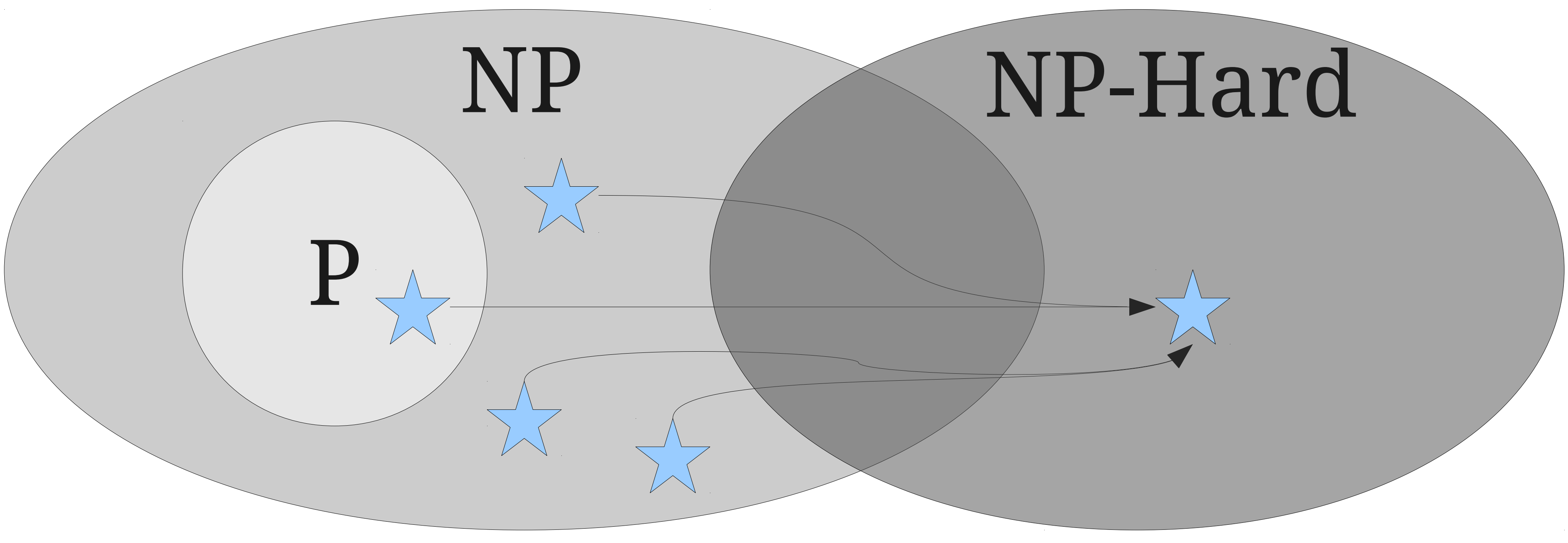
- A language L is called **NP-hard** iff for *every* $L' \in$ **NP**, we have $L' \leq_P L$.
- A language in L is called **NP-complete** iff L is **NP**-hard and $L \in$ **NP**.
- The class **NPC** is the set of **NP**-complete problems.



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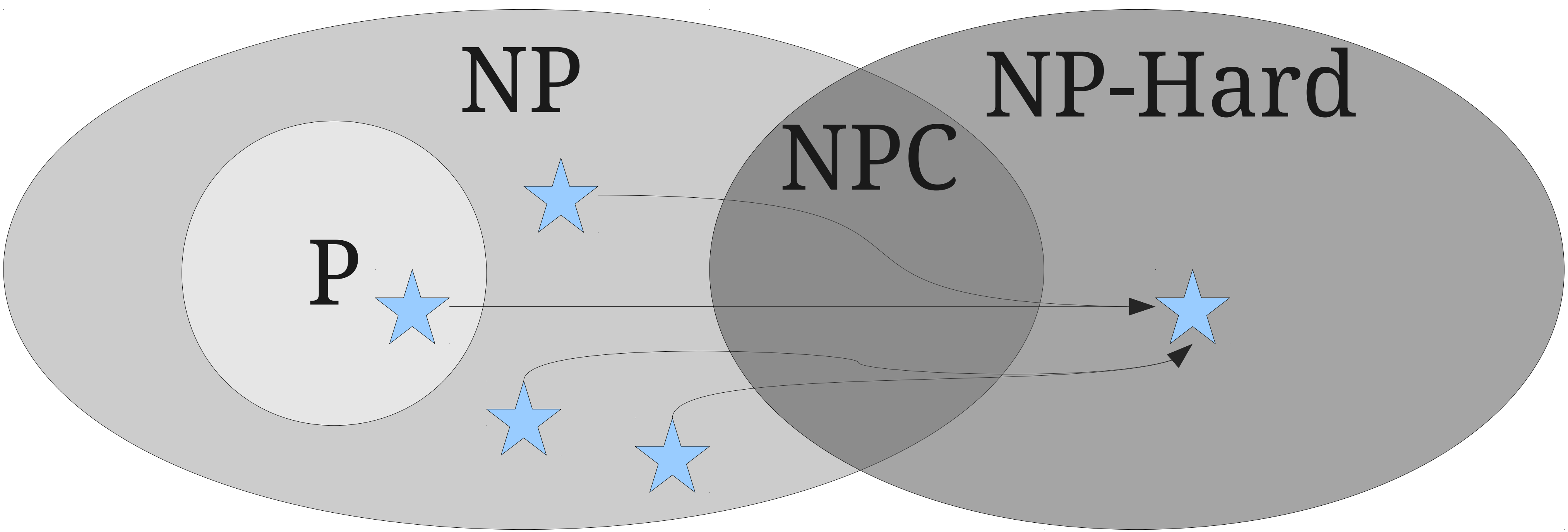

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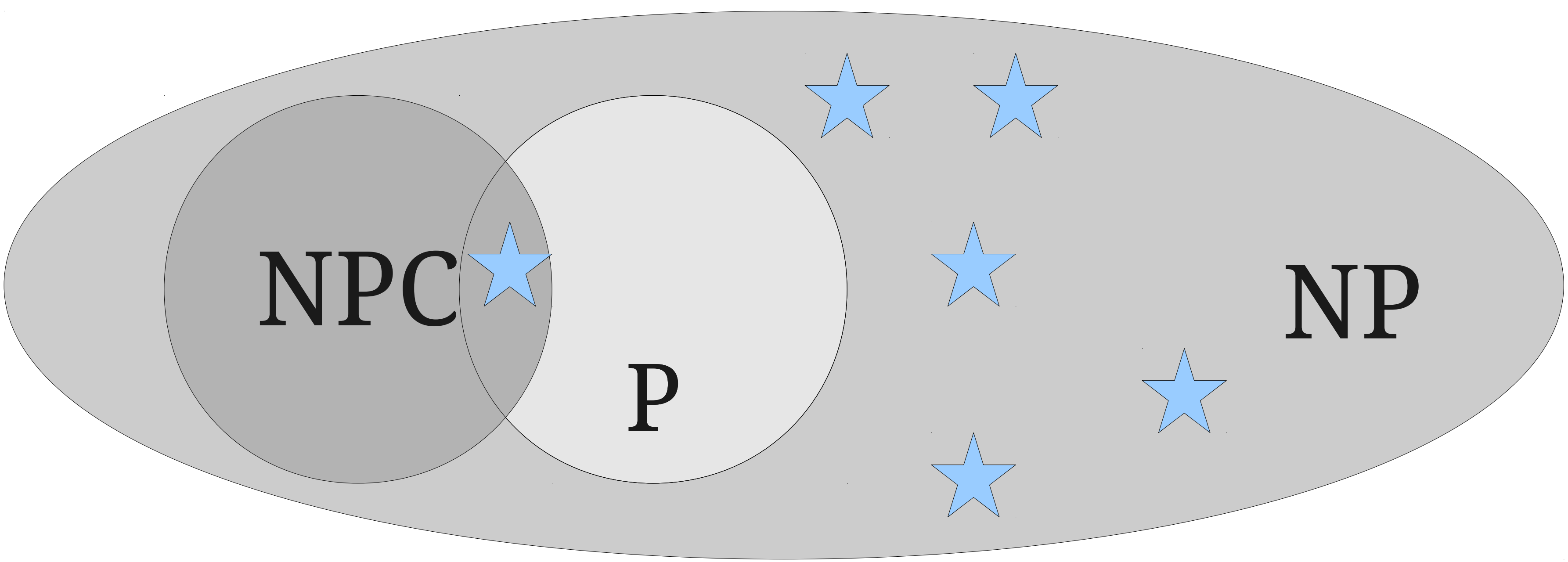

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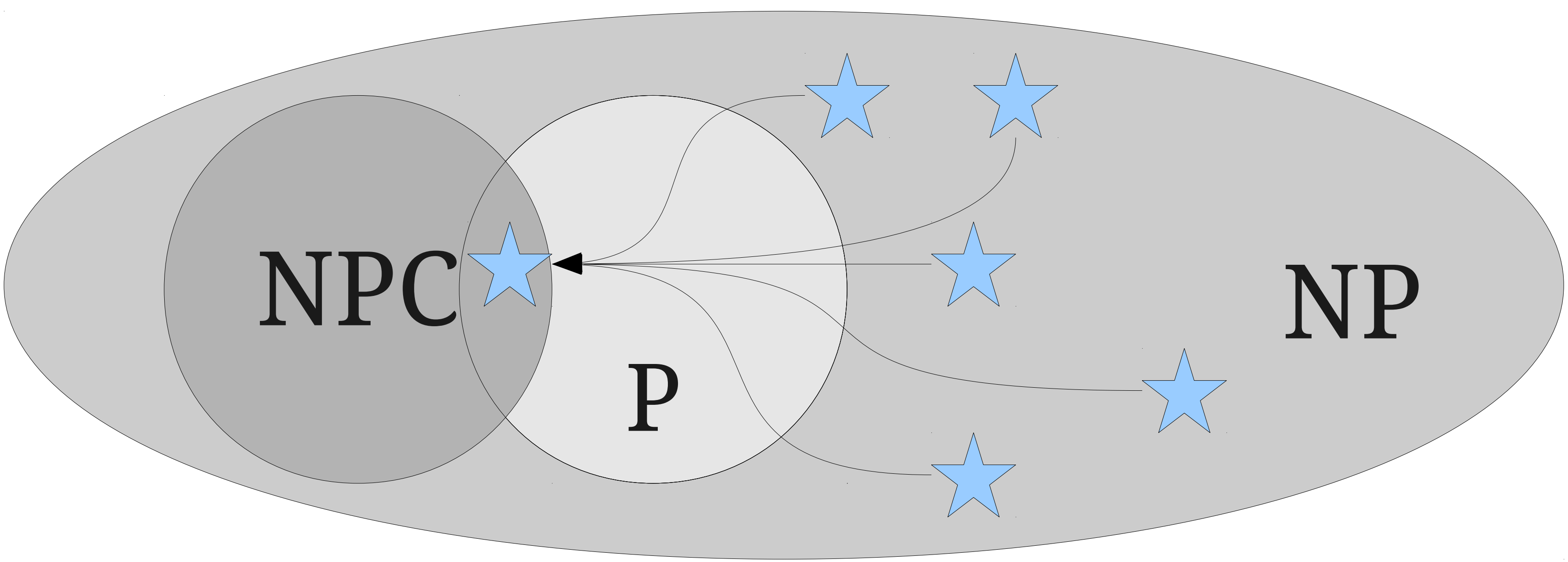

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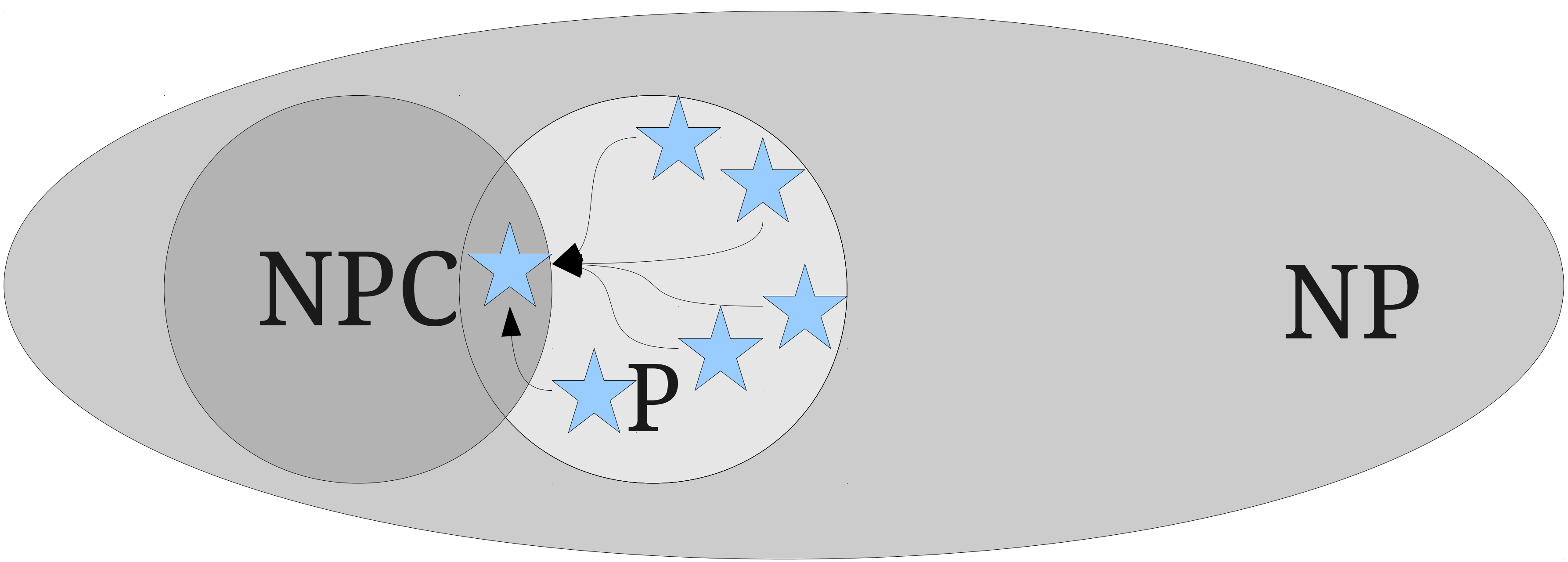

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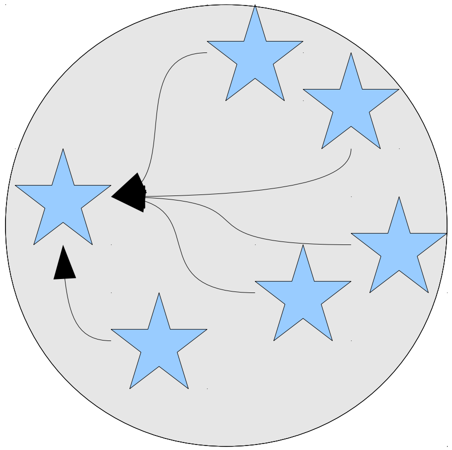
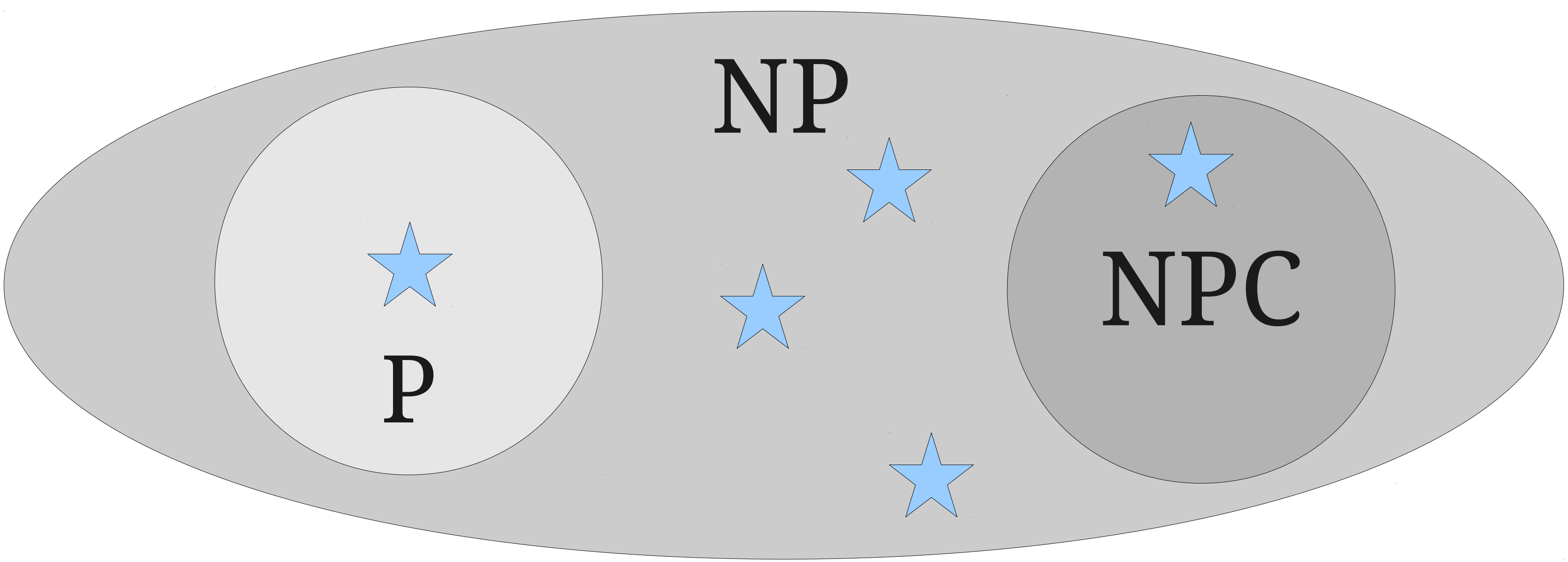


P = NP

The Tantalizing Truth

Theorem: If *any* **NP**-complete language is not in **P**, then **P ≠ NP**.

Proof: If $L \in$ **NPC**, then $L \in$ **NP**. Thus if $L \notin$ **P**, then $L \in$ **NP** – **P**.
This means that **NP** – **P** ≠ Ø, so **P** ≠ **NP**. ■



A Feel for **NP**-Completeness

- If a problem is **NP**-complete, then under the assumption that **P ≠ NP**, there cannot be an efficient algorithm for it.
- In a sense, **NP**-complete problems are the hardest problems in **NP**.
- All known **NP**-complete problems are enormously hard to solve:
  - All known algorithms for **NP**-complete problems run in worst-case exponential time.
  - Most algorithms for **NP**-complete problems are infeasible for reasonably-sized inputs.

How do we even know NP-complete problems exist in the first place?

Satisfiability

- A propositional logic formula φ is called **satisfiable** if there is some assignment to its variables that makes it evaluate to true.
  - $p \land q$ is satisfiable.
  - $p \land \neg p$ is unsatisfiable.
  - $p \to (q \land \neg q)$ is satisfiable.
- An assignment of true and false to the variables of φ that makes it evaluate to true is called a **satisfying assignment**.

SAT

- The **boolean satisfiability problem** (**SAT**) is the following:

Given a propositional logic formula φ, is φ satisfiable?

- Formally:

$$\textbf{SAT} = \{ \langle \varphi \rangle \mid \varphi \textbf{ is a satisfiable PL formula} \}$$

Theorem (Cook-Levin): SAT is **NP**-complete.

A Simpler **NP**-Complete Problem

Literals and Clauses

- A **literal** in propositional logic is a variable or its negation:
    - x
    - $\neg y$
    - But not $x \wedge y$.
- A **clause** is a many-way OR (*disjunction*) of literals.
    - $\neg x \vee y \vee \neg z$
    - x
    - But not $x \vee \neg(y \vee z)$

Conjunctive Normal Form

- A propositional logic formula φ is in **conjunctive normal form** (**CNF**) if it is the many-way AND (*conjunction*) of clauses.
  - $(x \vee y \vee z) \wedge (\neg x \vee \neg y) \wedge (x \vee y \vee z \vee \neg w)$
  - $x \vee z$
  - But not $(x \vee (y \wedge z)) \vee (x \vee y)$
- Only legal operators are ¬, ∨, ∧.
- No nesting allowed.

The Structure of CNF

$$(x \lor y \lor \neg z) \land (\neg x \lor \neg y \lor z) \land (\neg x \lor y \lor \neg z)$$

Each clause must have
at least one
true literal in it.

The Structure of CNF

$$(x \lor y \lor \neg z) \land (\neg x \lor \neg y \lor z) \land (\neg x \lor y \lor \neg z)$$

We should pick at least one true literal from each clause

The Structure of CNF

$$(x \lor y \lor \neg z) \land (\neg x \lor \neg y \lor z) \land (\neg x \lor y \lor \neg z)$$

… subject to the constraint that we never choose a literal and its negation

3-CNF

- A propositional formula is in **3-CNF** if
  - It is in CNF, and
  - Every clause has *exactly* three literals.
- For example:
  - $(x \lor y \lor z) \land (\neg x \lor \neg y \lor z)$
  - $(x \lor x \lor x) \land (y \lor \neg y \lor \neg x) \land (x \lor y \lor \neg y)$
  - But not $(x \lor y \lor z \lor w) \land (x \lor y)$
- The language **3SAT** is defined as follows:

3SAT = { ⟨φ⟩ | φ is a satisfiable 3-CNF formula }

Theorem: 3SAT is **NP**-Complete

Using the Cook-Levin Theorem

- When discussing decidability, we used the fact that $A_{TM} \notin \mathbf{R}$ as a starting point for finding other undecidable languages.
  - **Idea:** Reduce A_{TM} to some other language.
- When discussing **NP**-completeness, we will use the fact that 3SAT ∈ **NPC** as a starting point for finding other **NPC** languages.
  - **Idea**: Reduce 3SAT to some other language.

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Theorem: Let L_1 and L_2 be languages. If $L_1 \leq_P L_2$ and L_1 is **NP**-hard, then L_2 is **NP**-hard.

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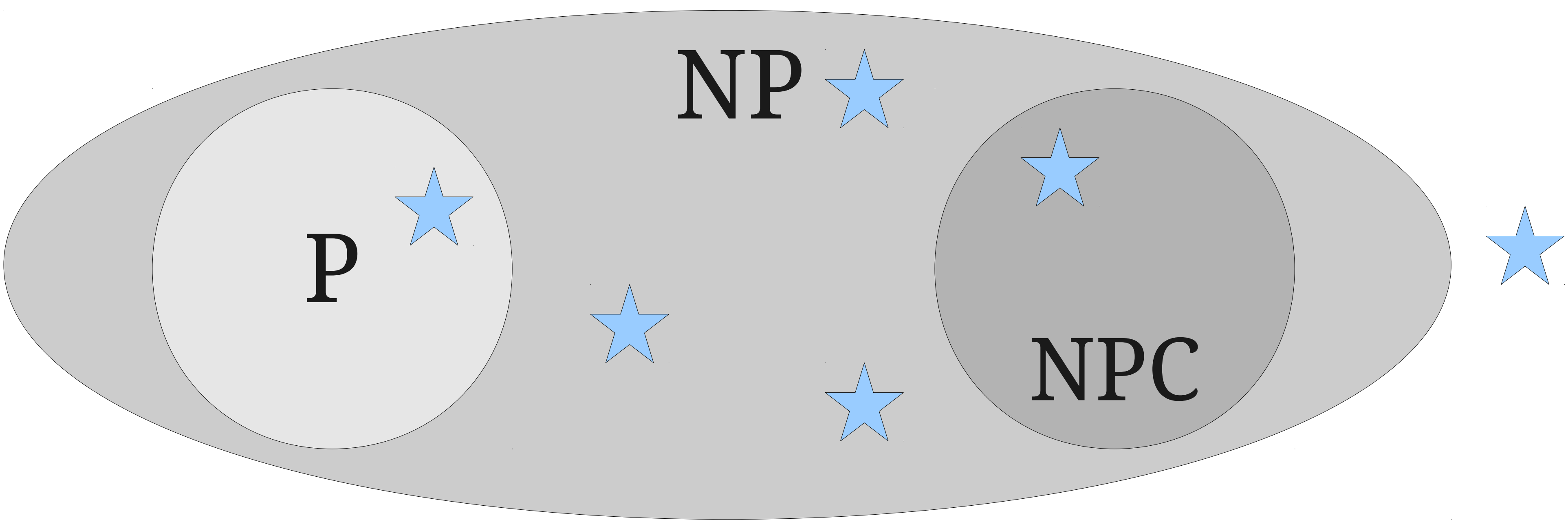

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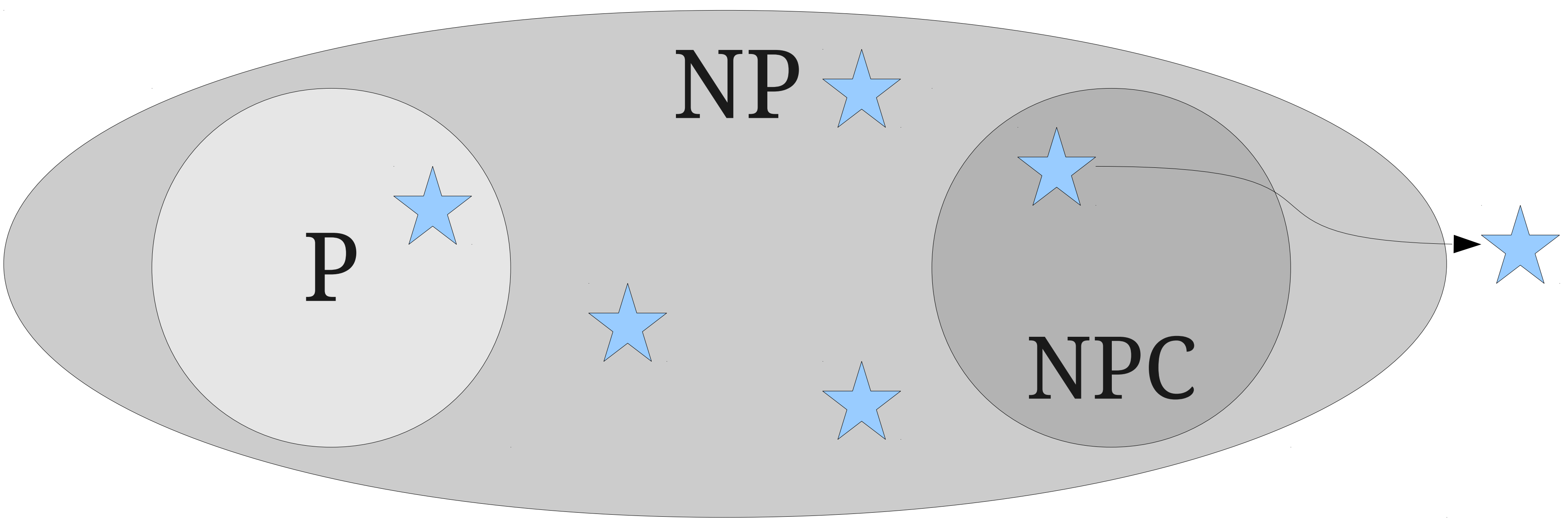

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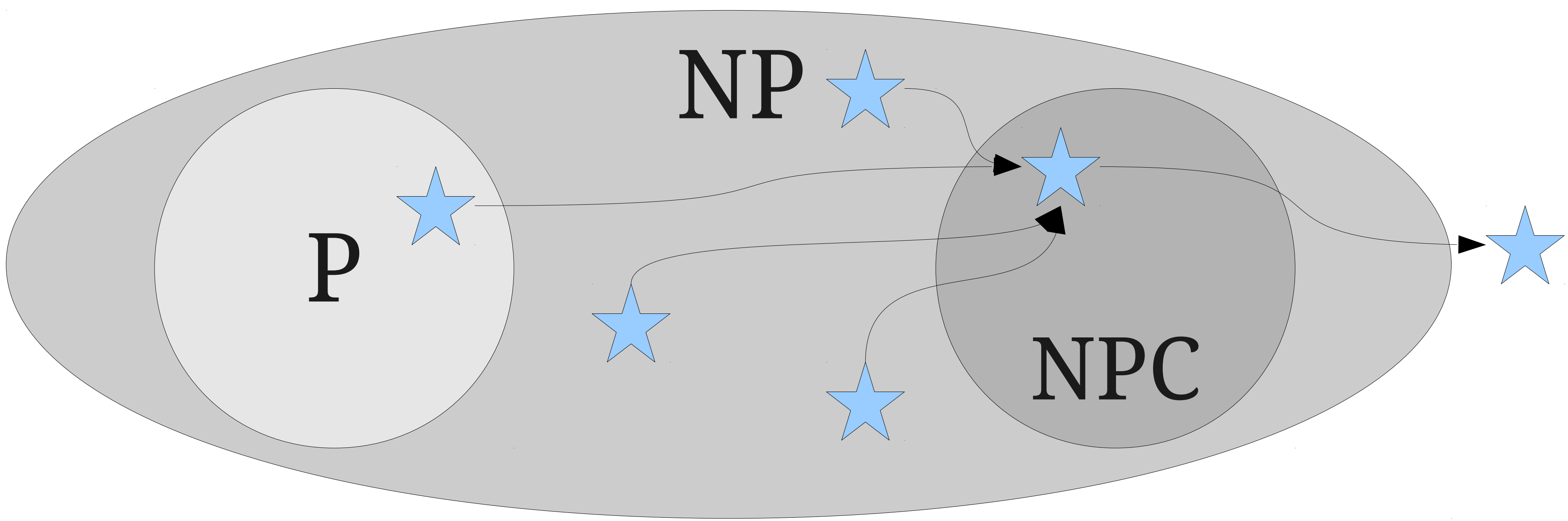

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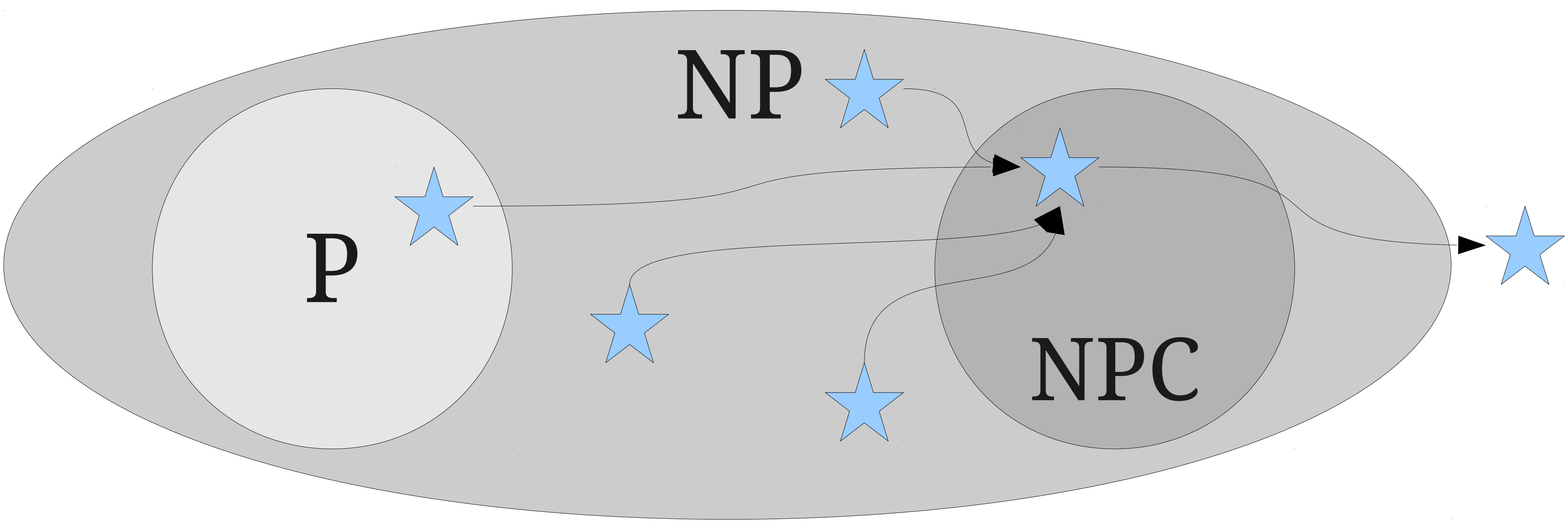

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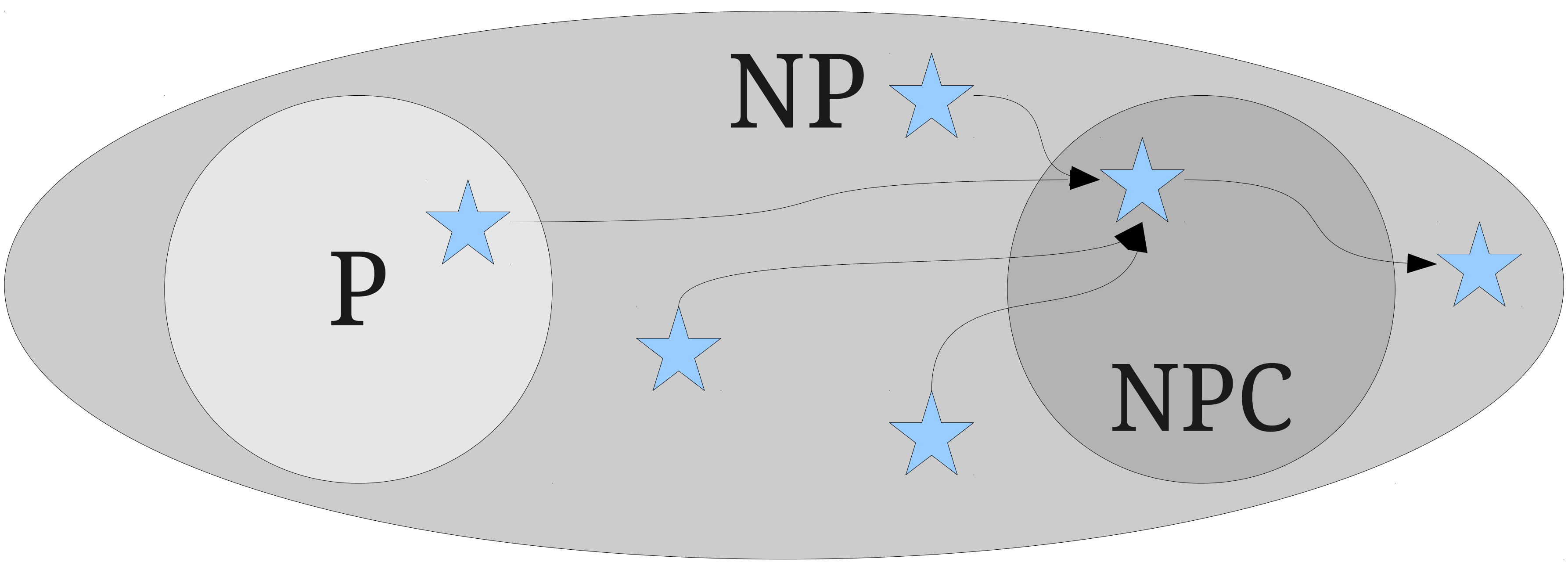

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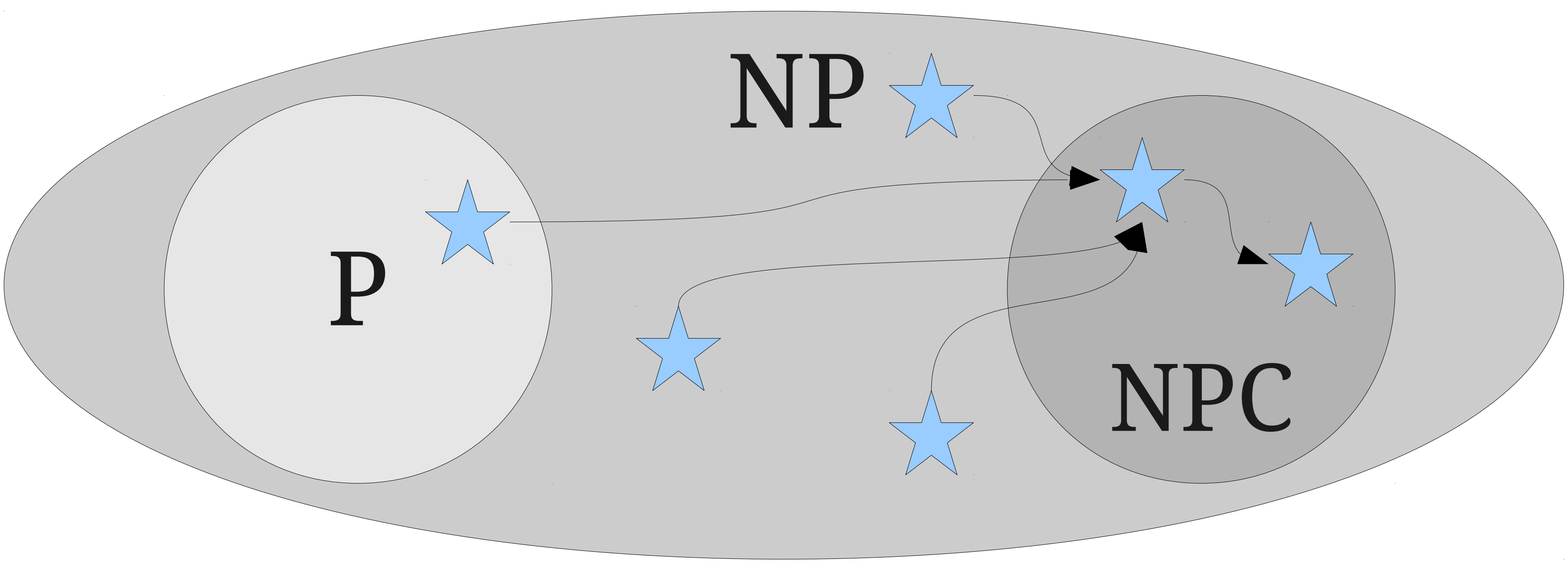

Next Time

- **More NP-Complete Problems**
  - Independent Sets
  - Graph Coloring
- **Applied Complexity Theory (ITA)**
  - Why does all of this matter?