Finite Automata

Part Three

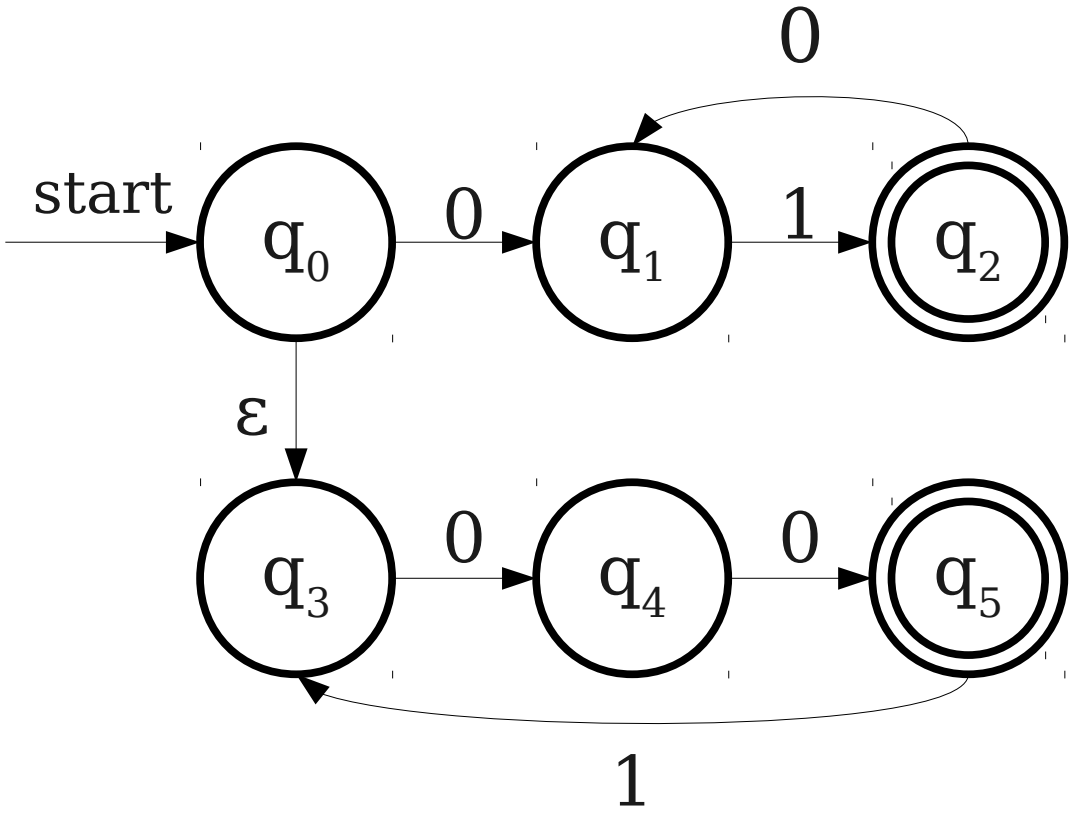
Problem Set Four is due in the box up front.

NFAs

- An **NFA** is a
  - **N**ondeterministic
  - **F**inite
  - **A**utomaton
- Can have zero or more transitions defined for each state/symbol pair.
- An NFA N accepts a string w iff there is some possible series of transitions N can follow that ends in an accepting state.

ε-Transitions

- NFAs have a special type of transition called the **ε-transition**.
- An NFA may follow any number of ε-transitions at any time without consuming any input.



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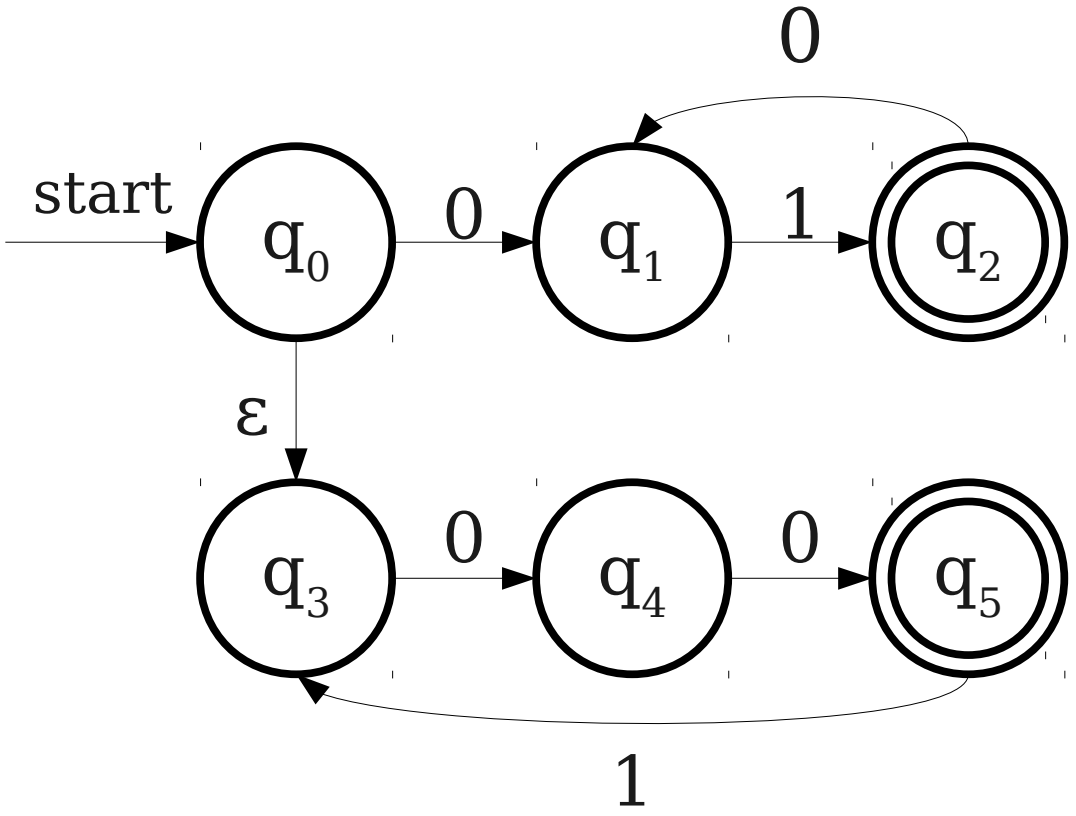

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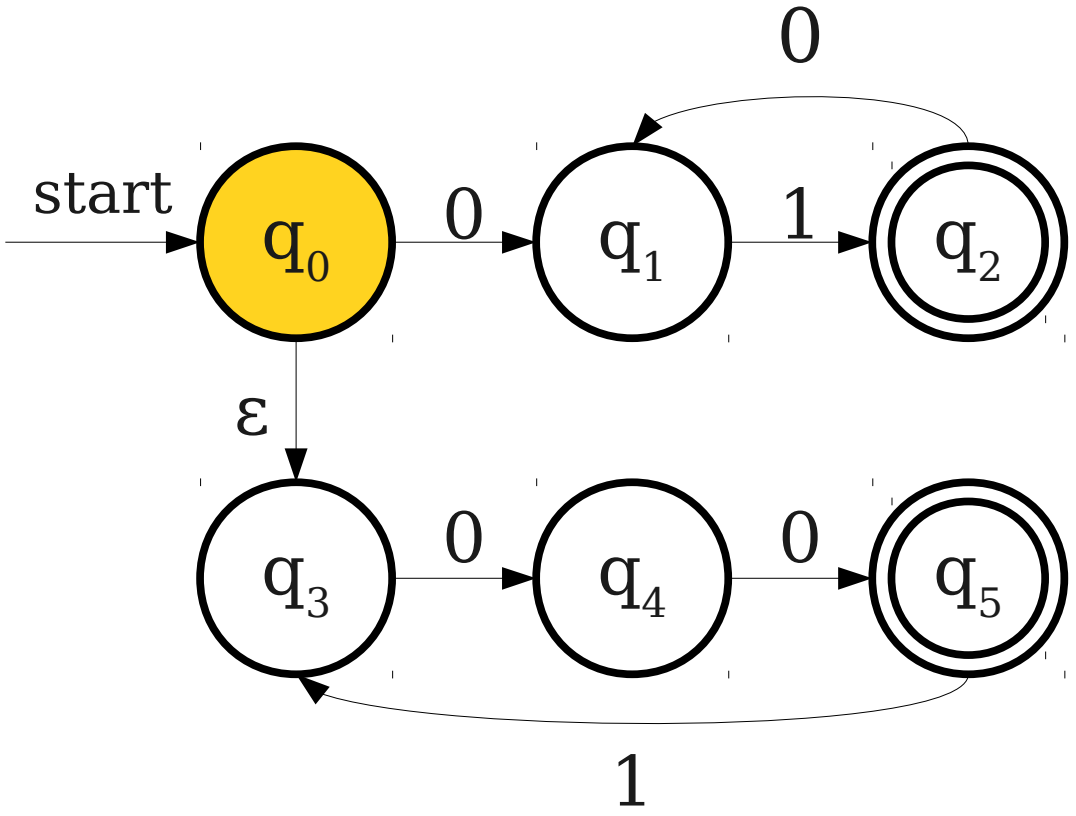

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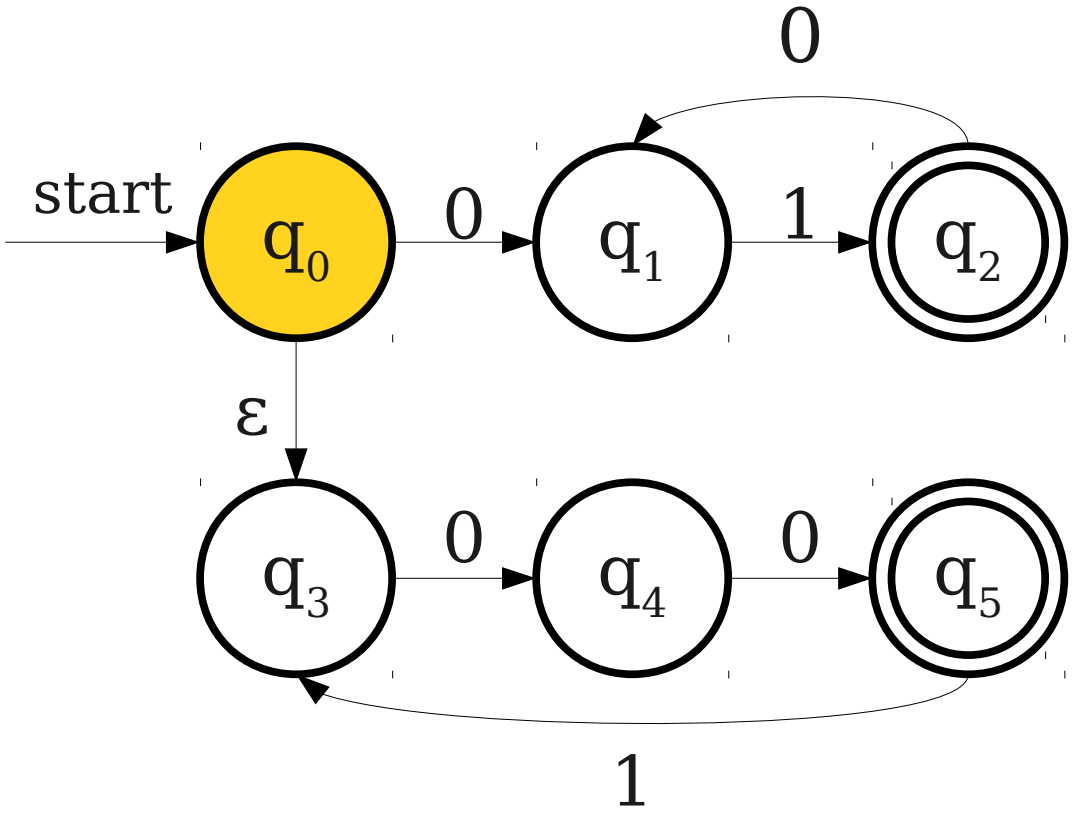

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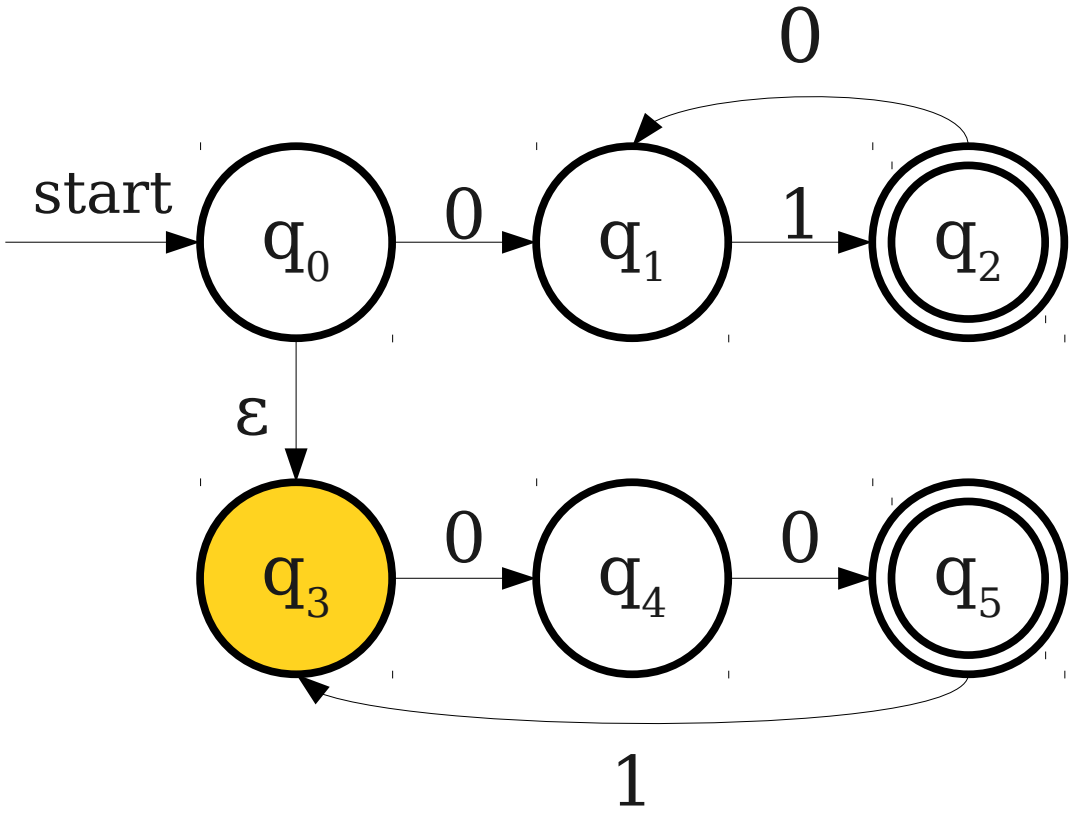

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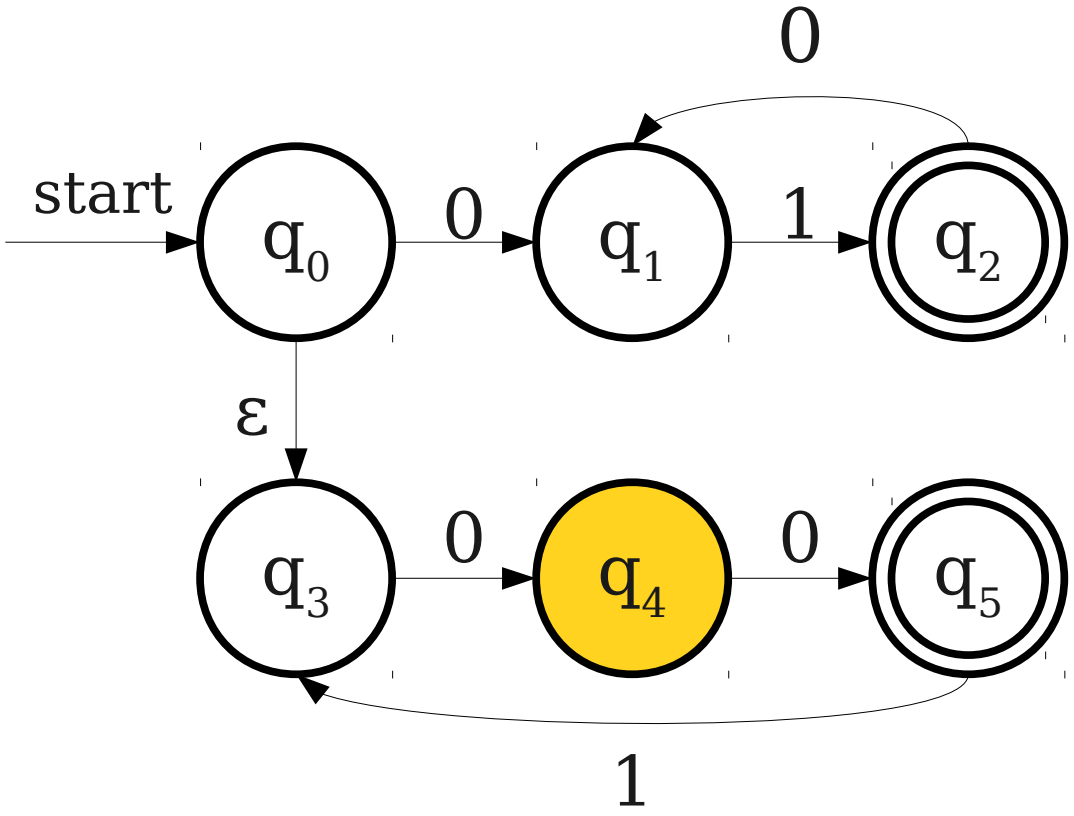

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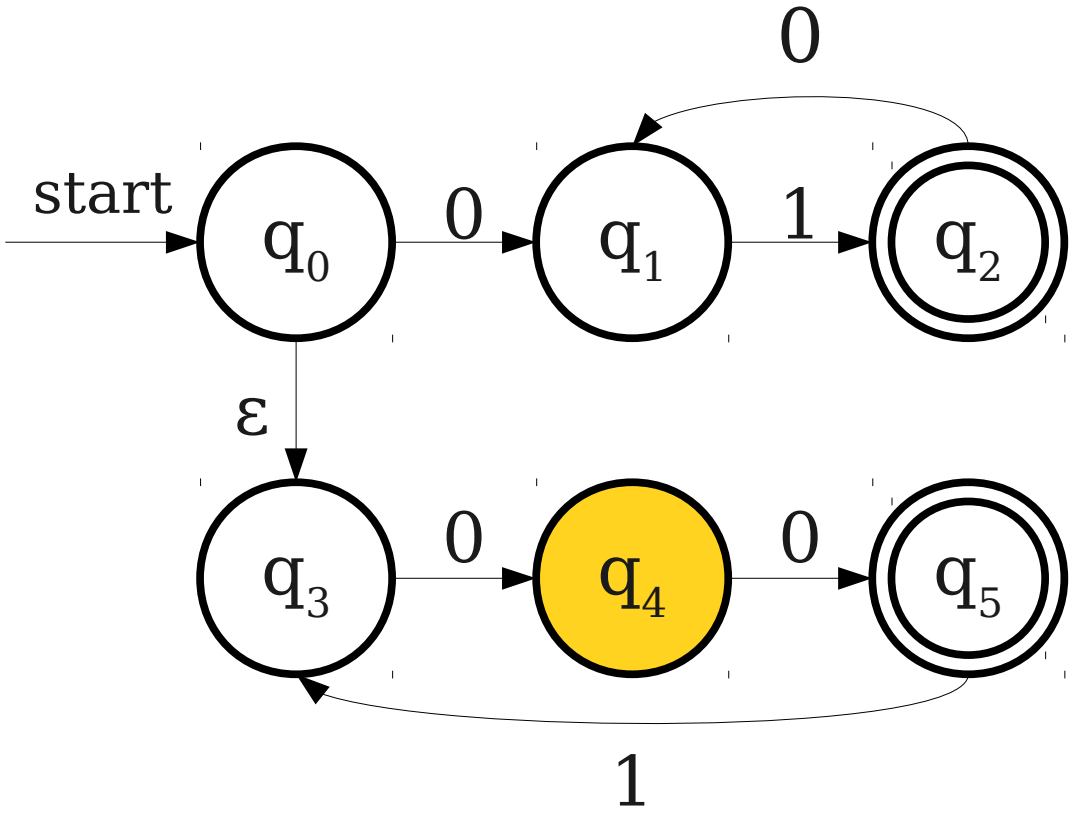

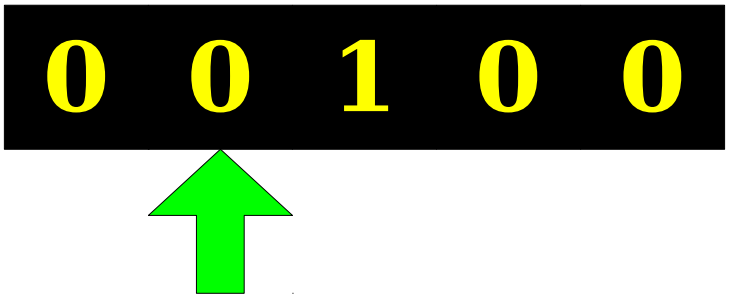

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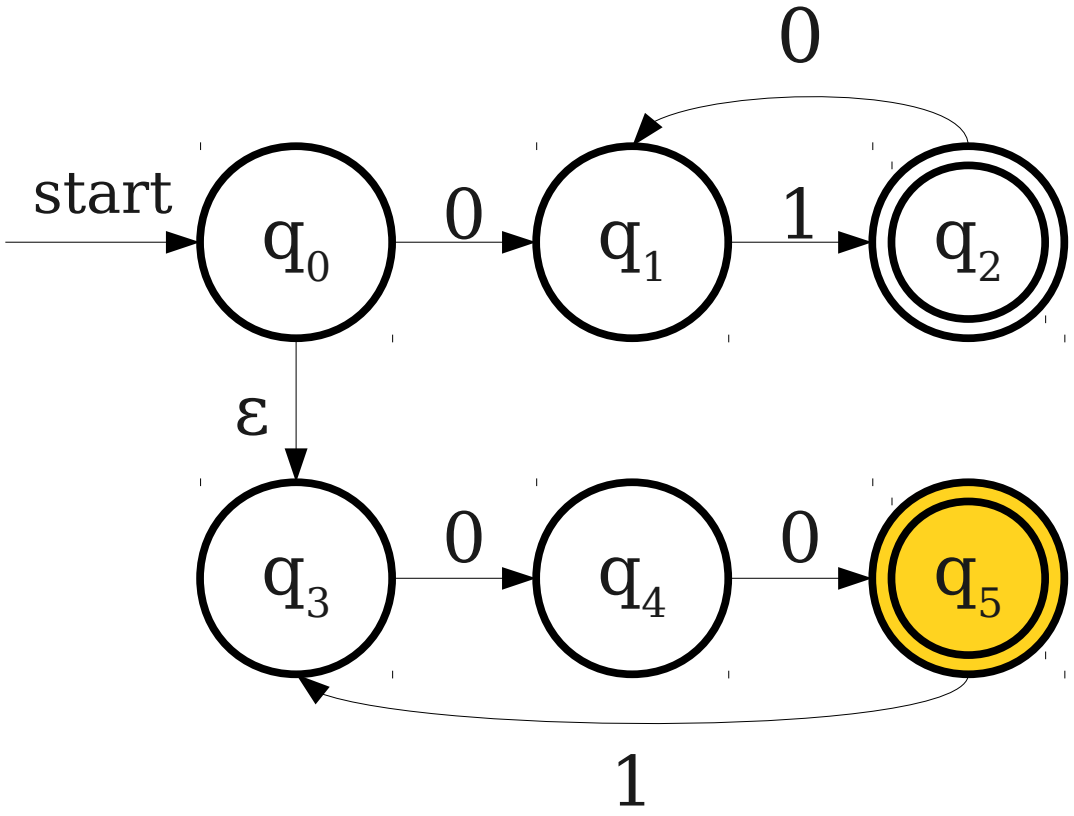

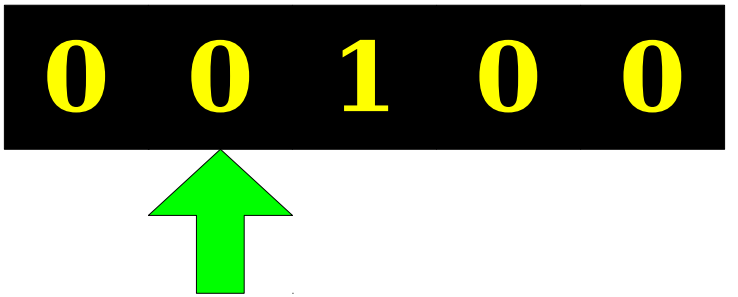

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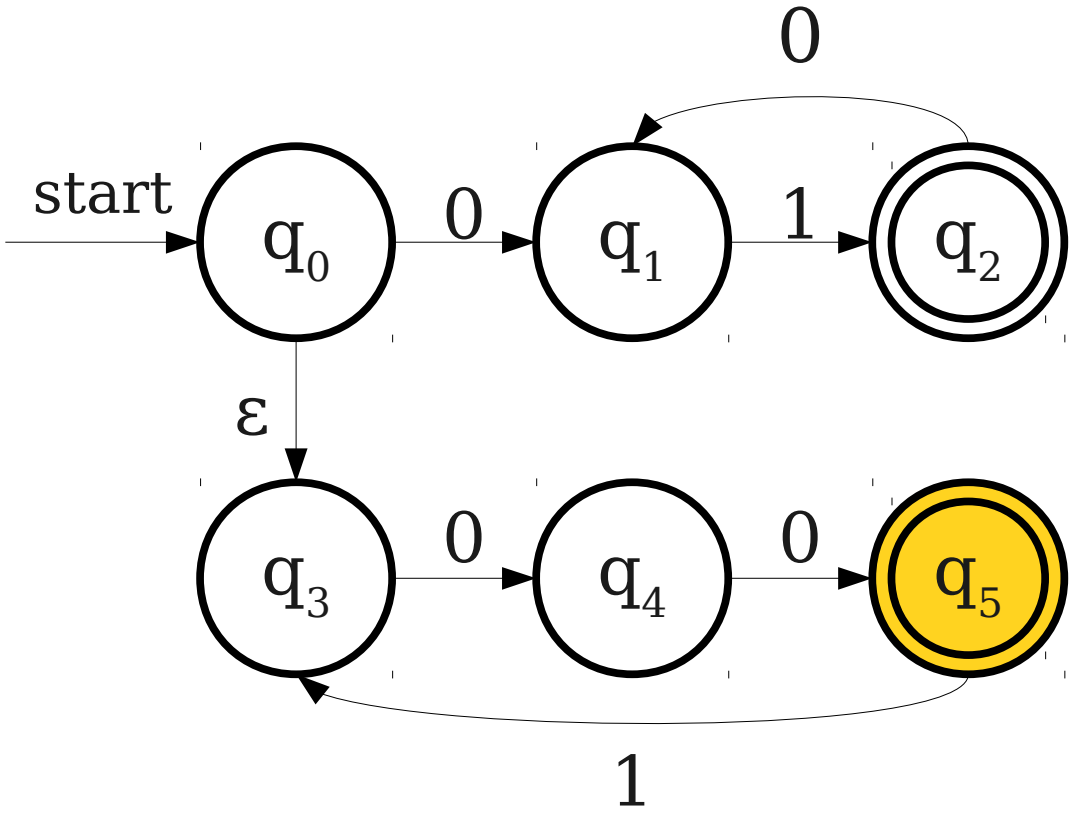

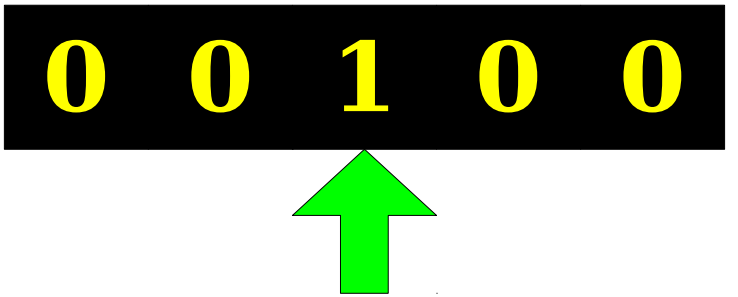

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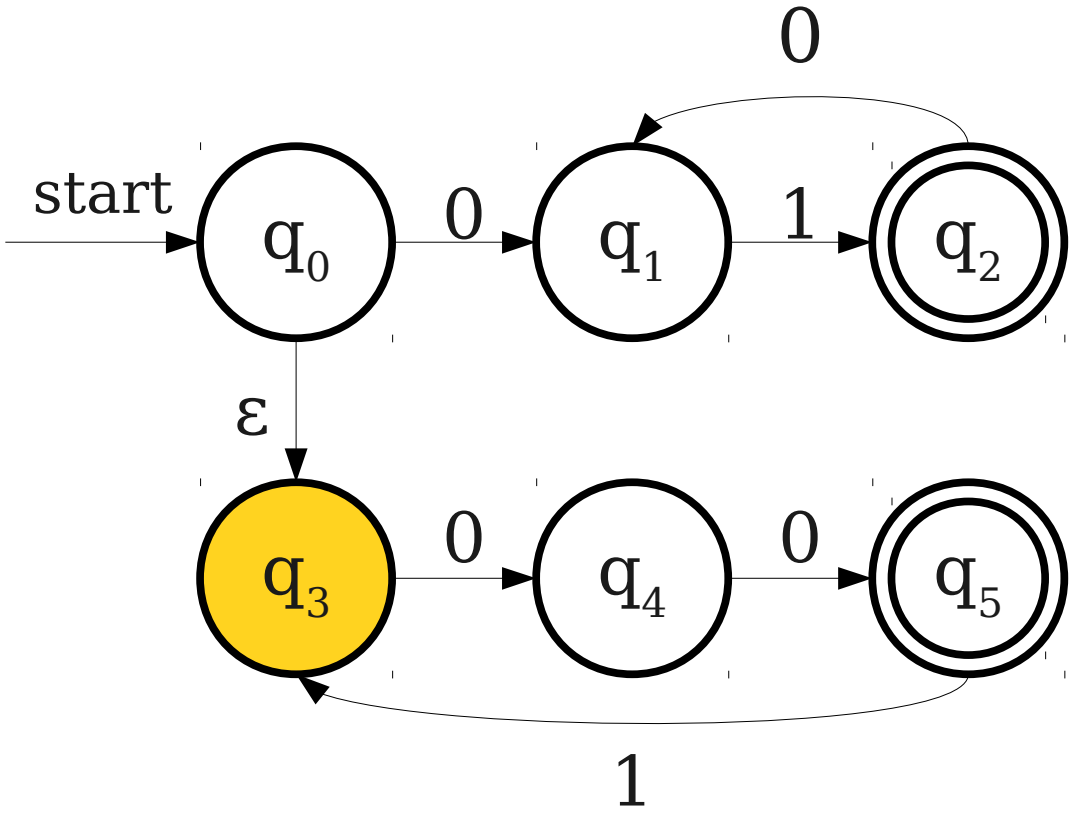

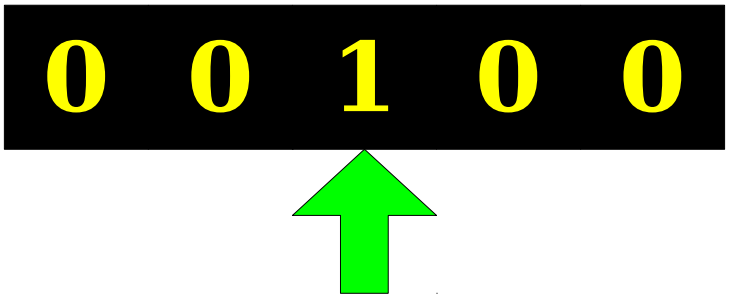

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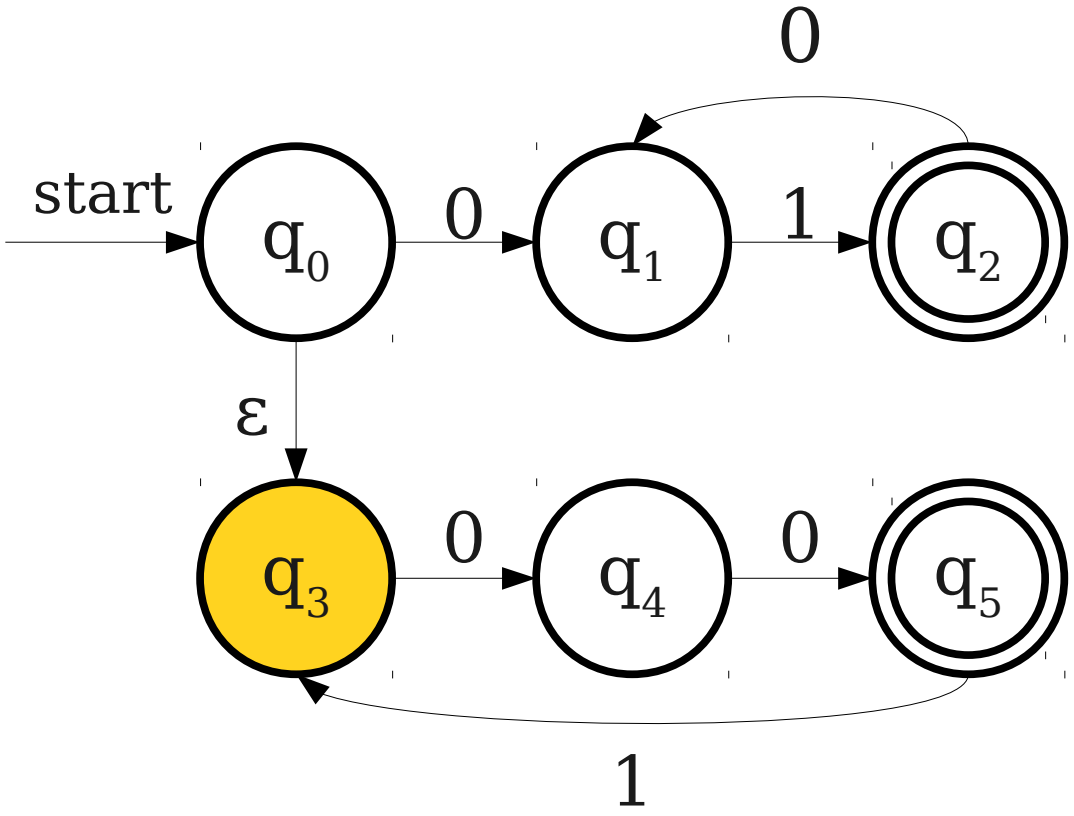

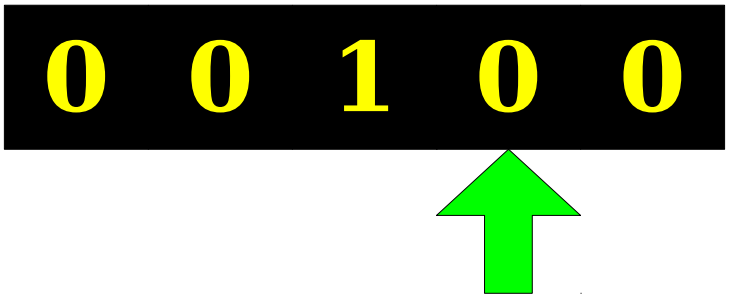

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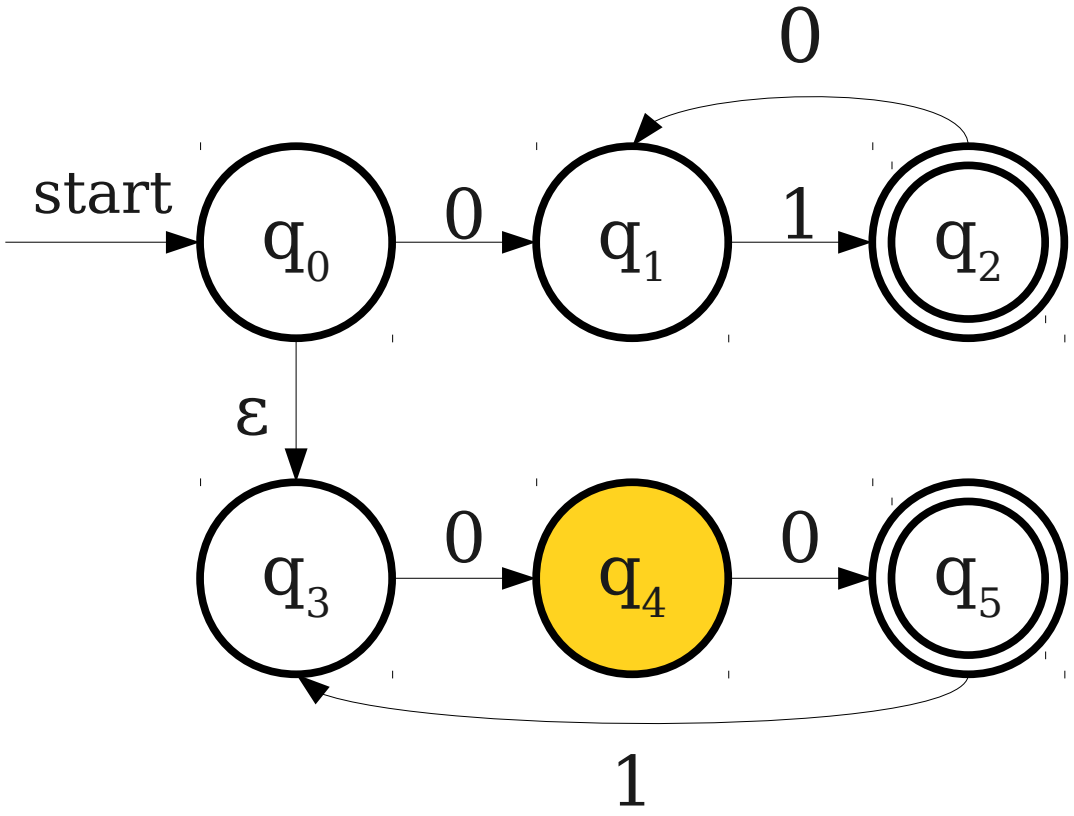

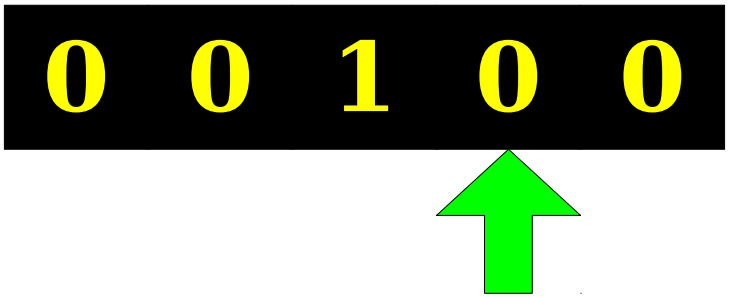

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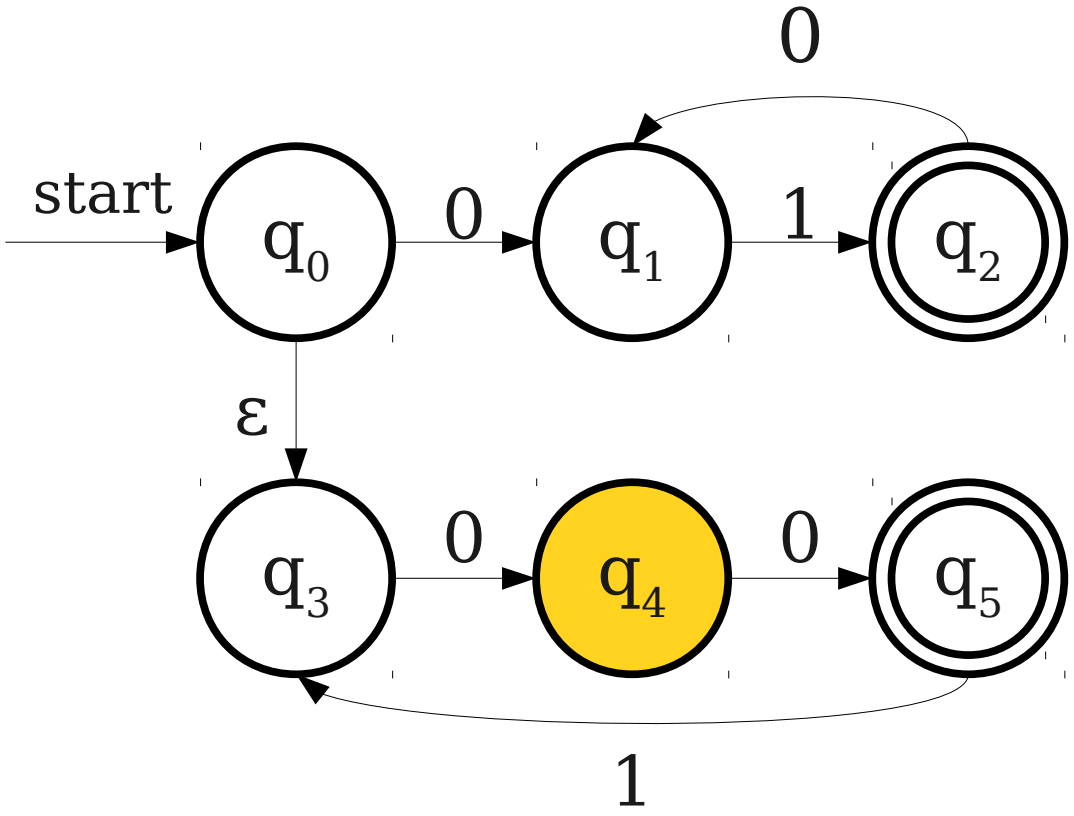

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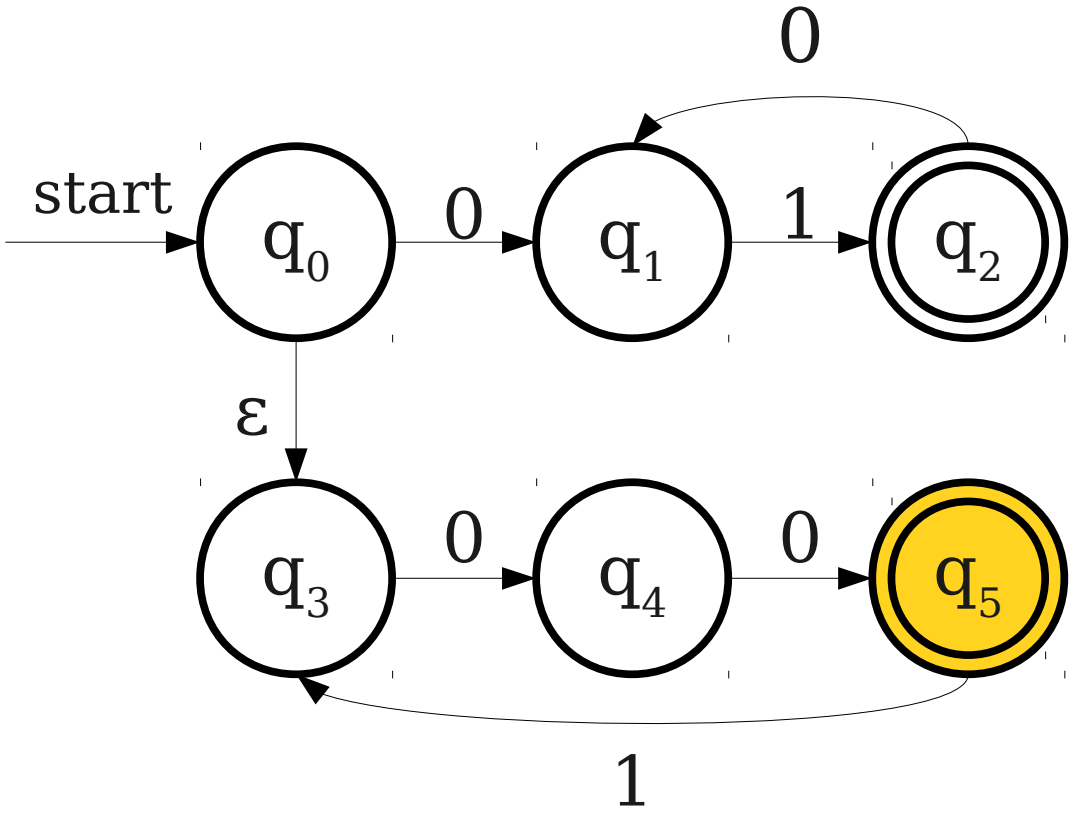

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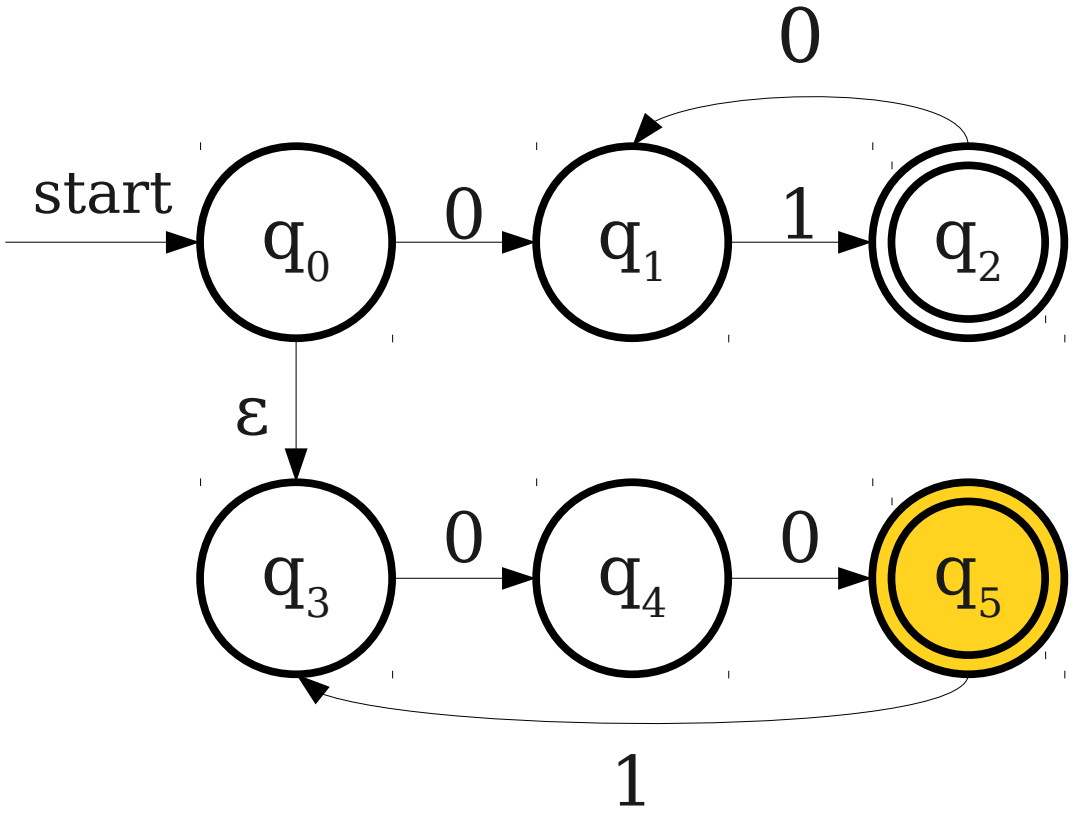

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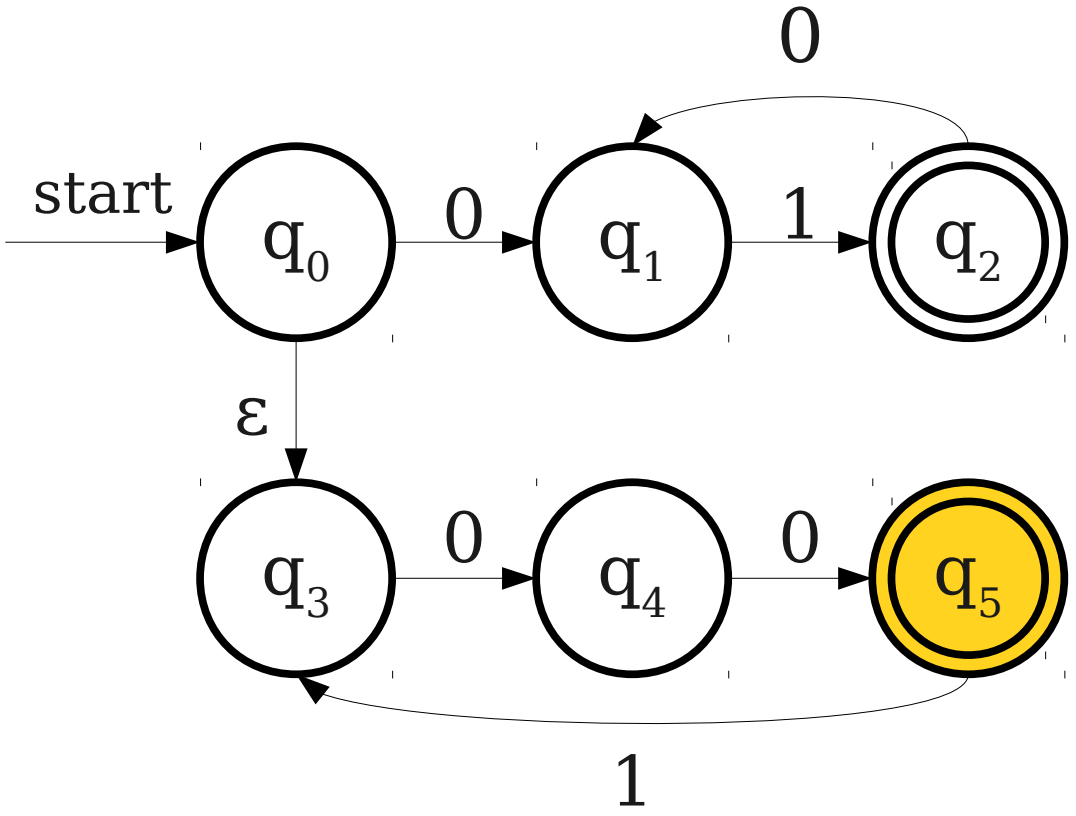


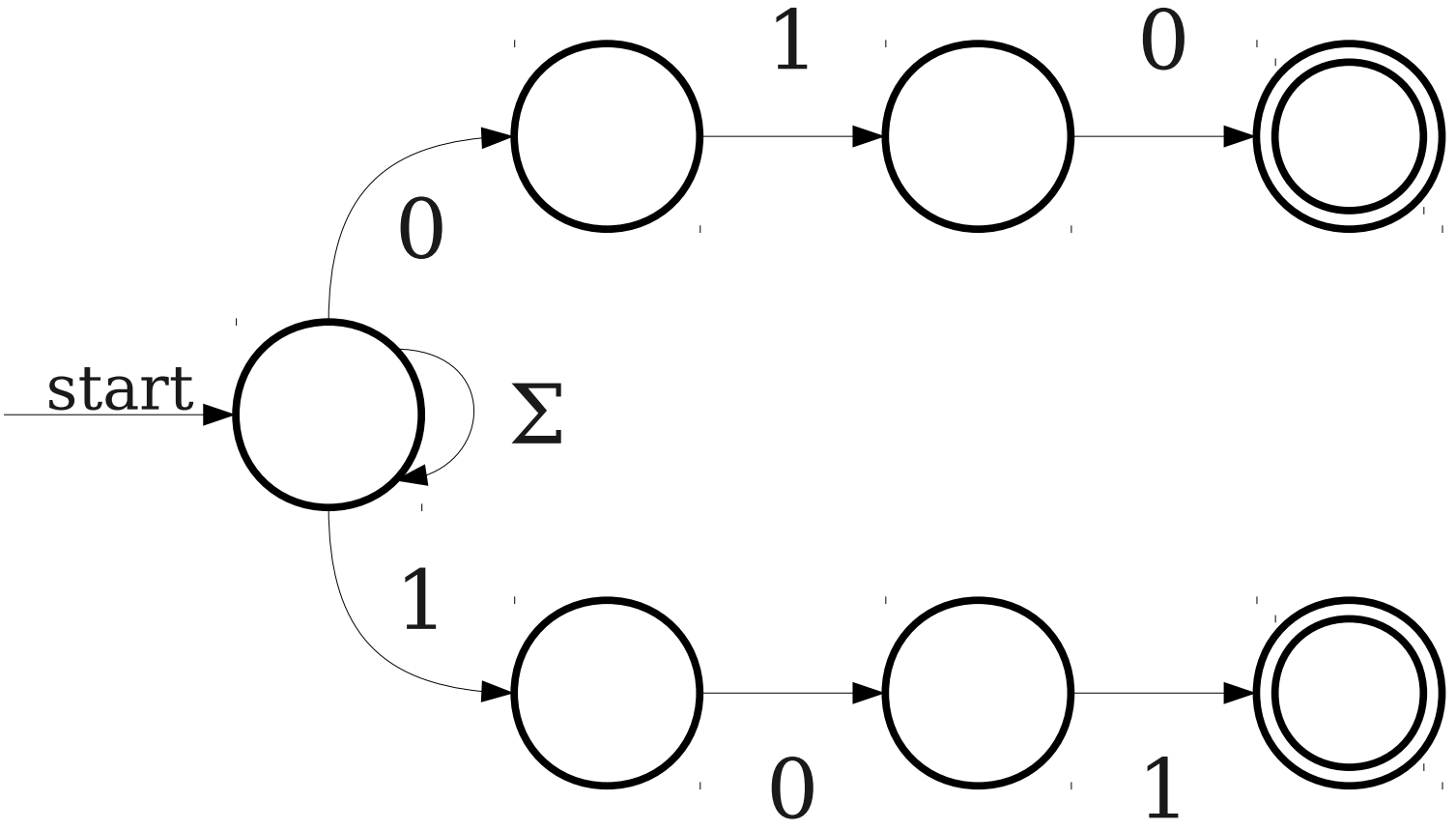
0 0 1 0 0

Designing NFAs

- When designing NFAs, *embrace the nondeterminism!*
- Good model: **Guess-and-check**:
  - Have the machine *nondeterministically guess* what the right choice is.
  - Have the machine *deterministically check* that the choice was correct.

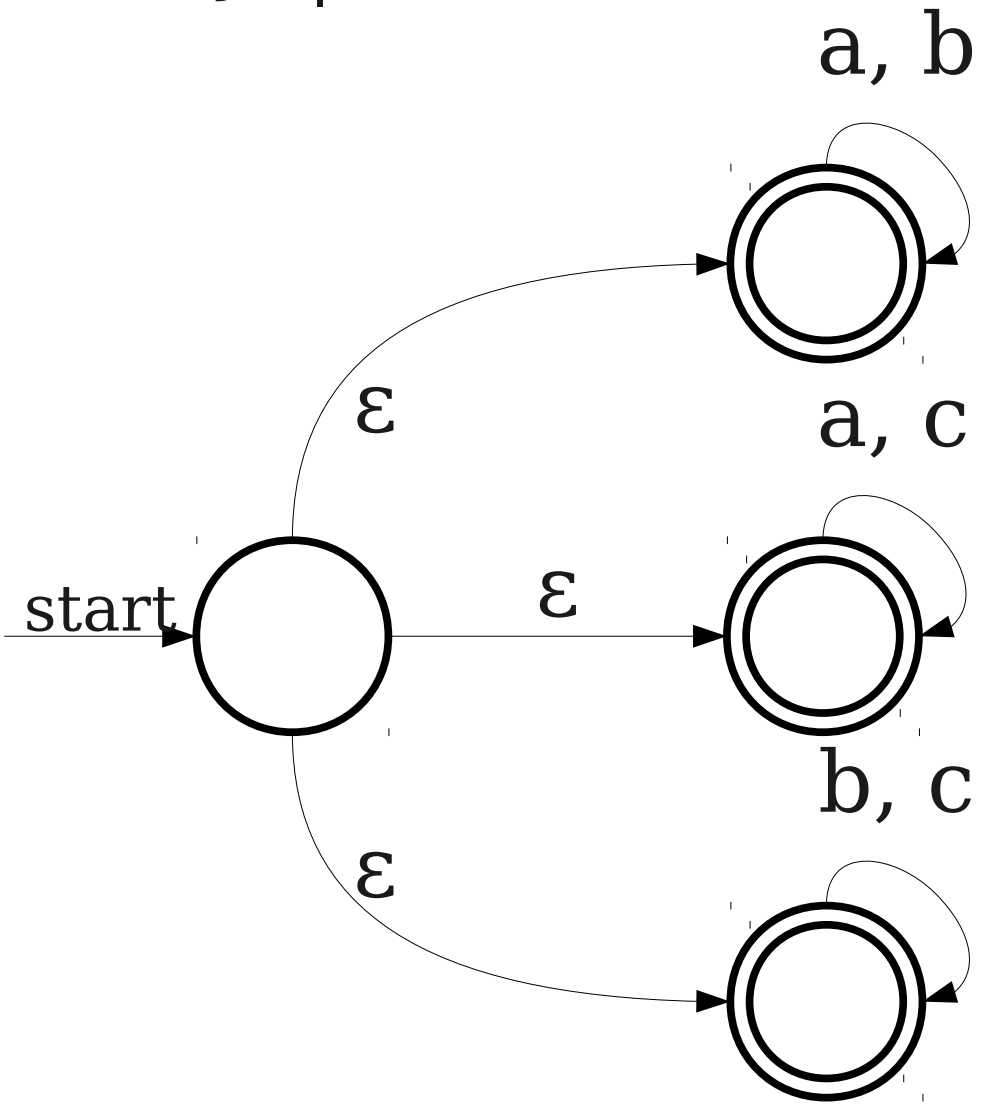
Guess-and-Check

$L = \{ w \in \{\texttt{0}, \texttt{1}\}^* \mid w \text{ ends in } \texttt{010} \text{ or } \texttt{101} \}$



Guess-and-Check

$L = \{ w \in \{\texttt{a}, \texttt{b}, \texttt{c}\}^* \mid$ at least one of **a**, **b**, or **c** is not in $w \}$



New Stuff!

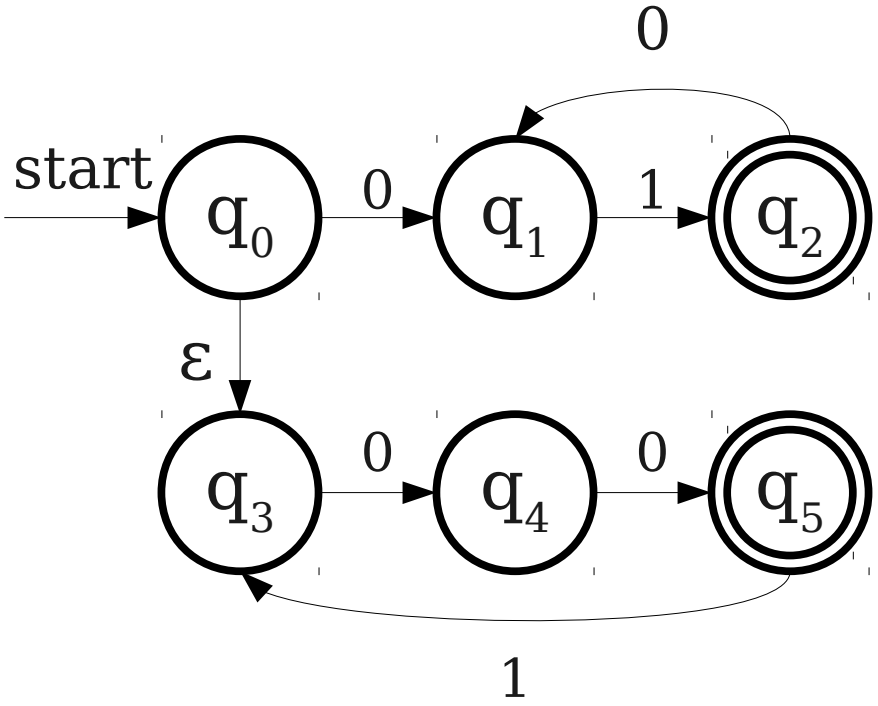
NFAs and DFAs

- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Just use the same set of transitions as before.
- **Question**: Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is **yes!**

Finite Automata

- NFAs and DFAs are *finite* automata; there can only be finitely many states in an NFA or DFA.
- An NFA can be in any combination of its states, but there are only finitely many possible combations.
- **Idea:** Build a DFA where each state of the DFA corresponds to a *set* of states in the NFA.

Simulating an NFA with a DFA



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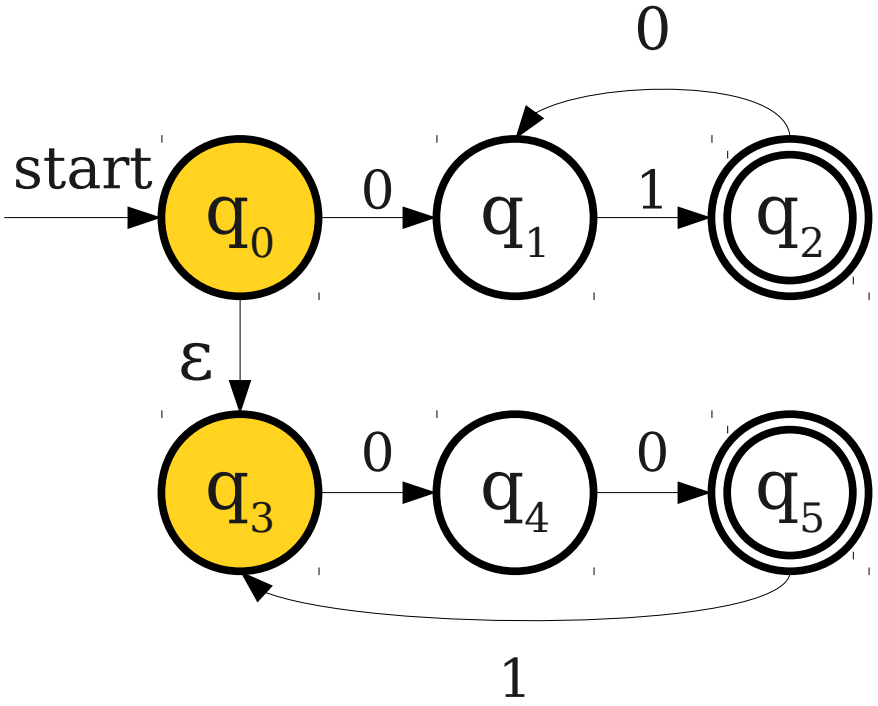

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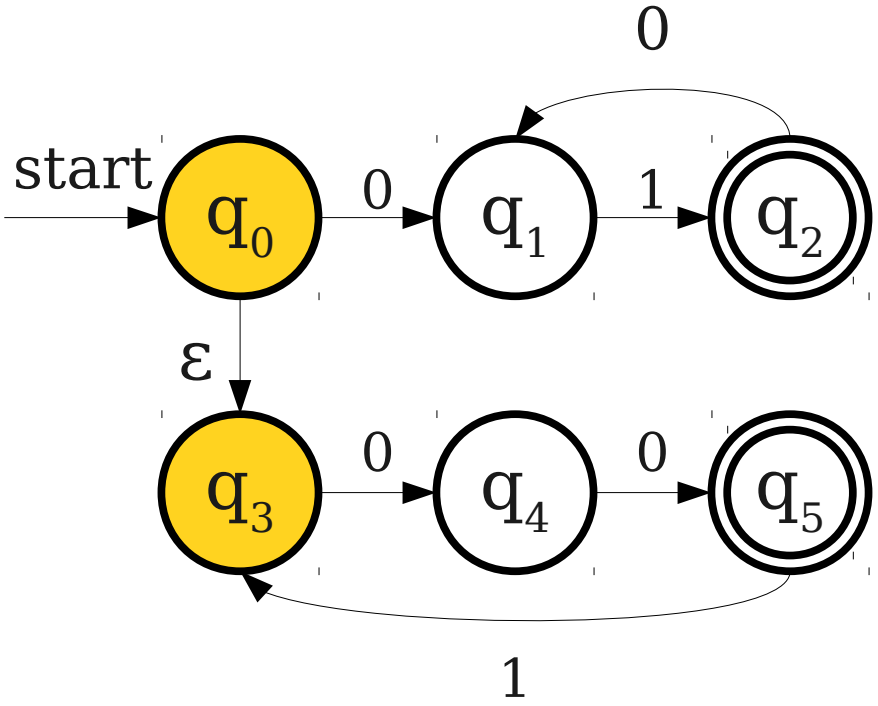

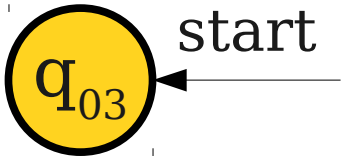

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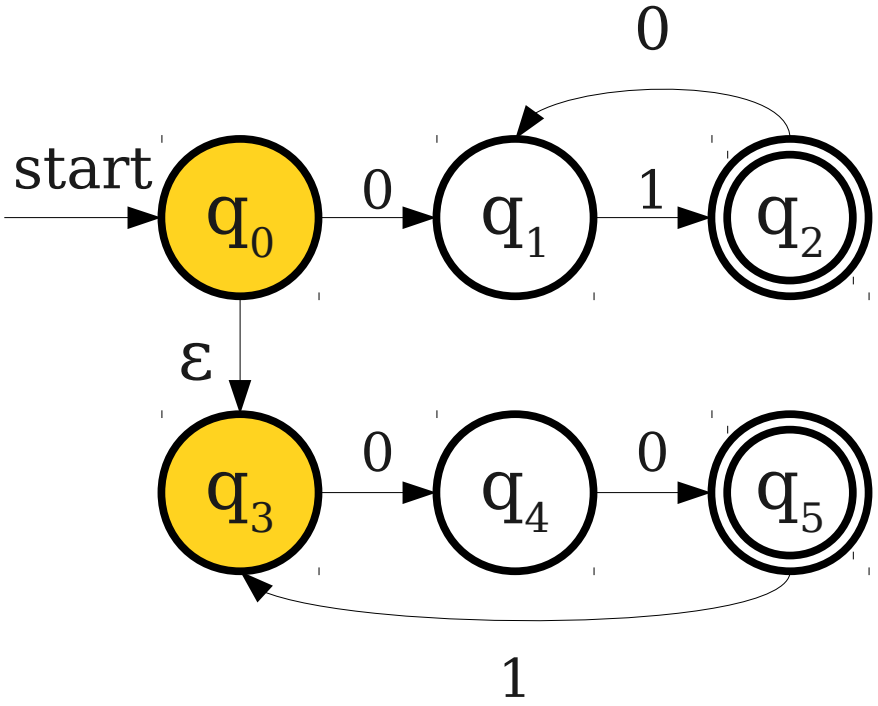

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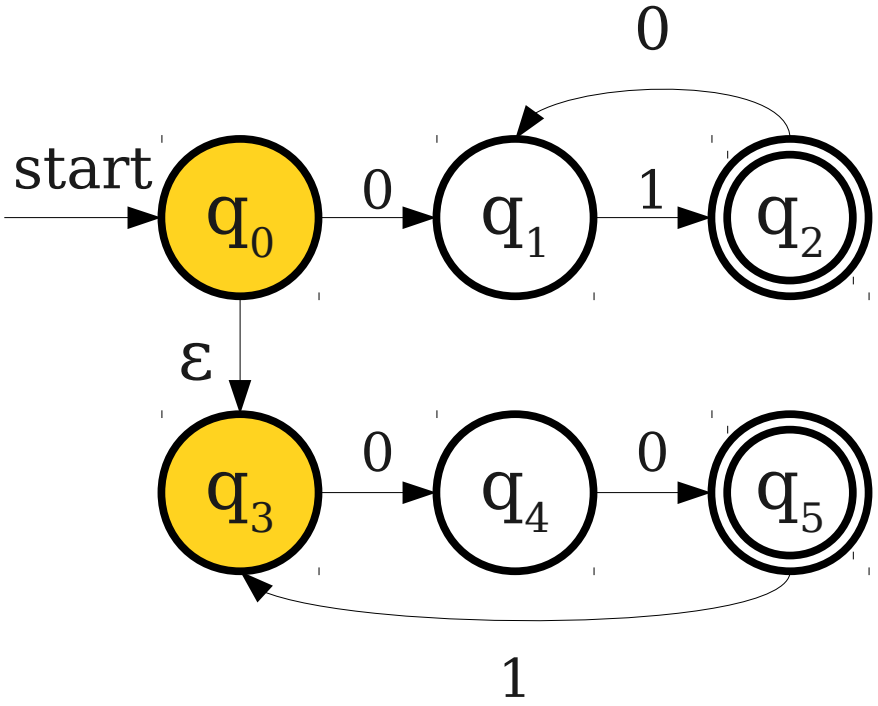

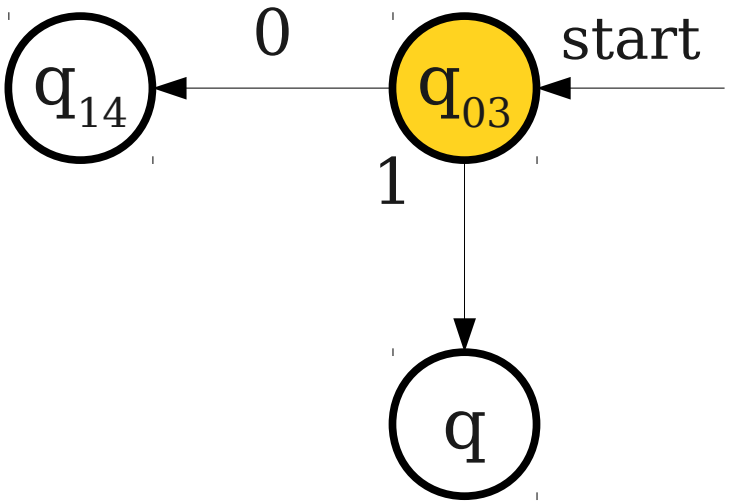

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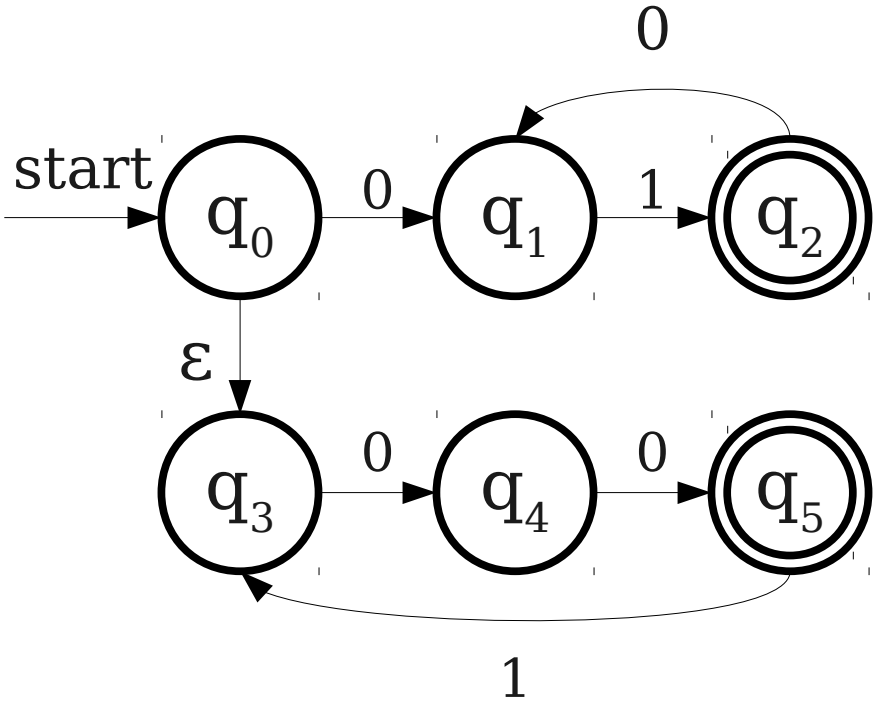

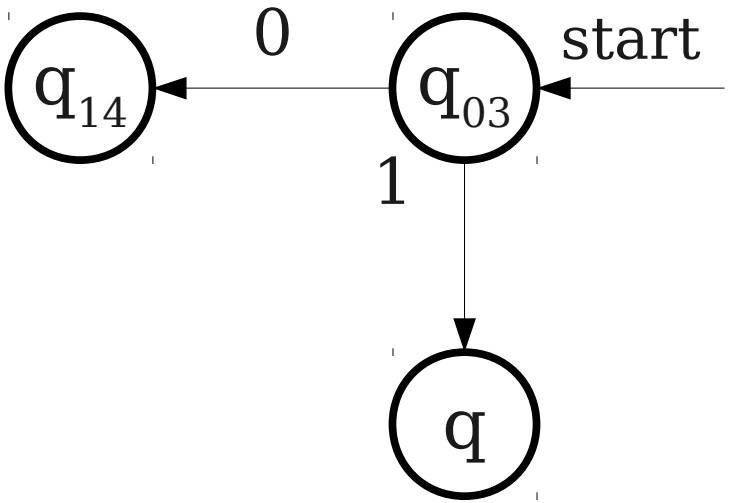

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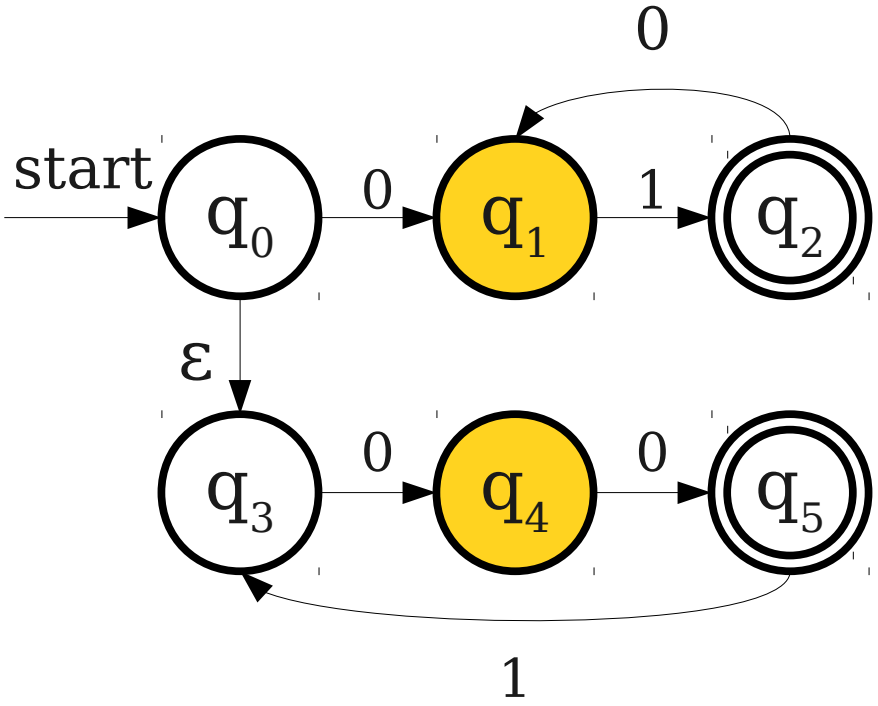

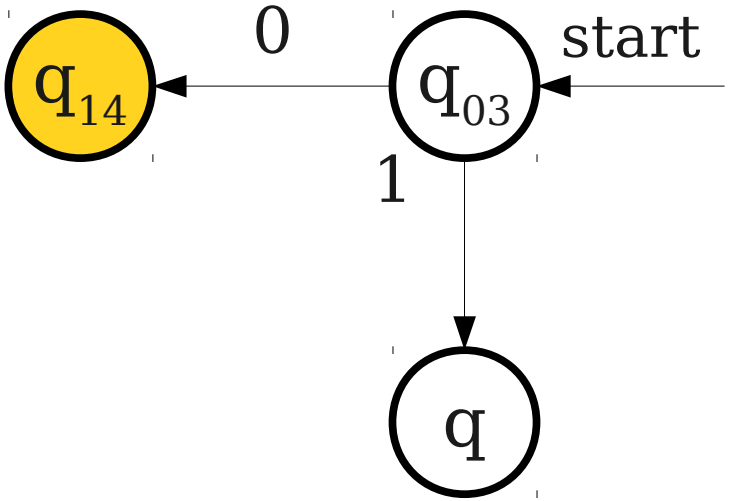

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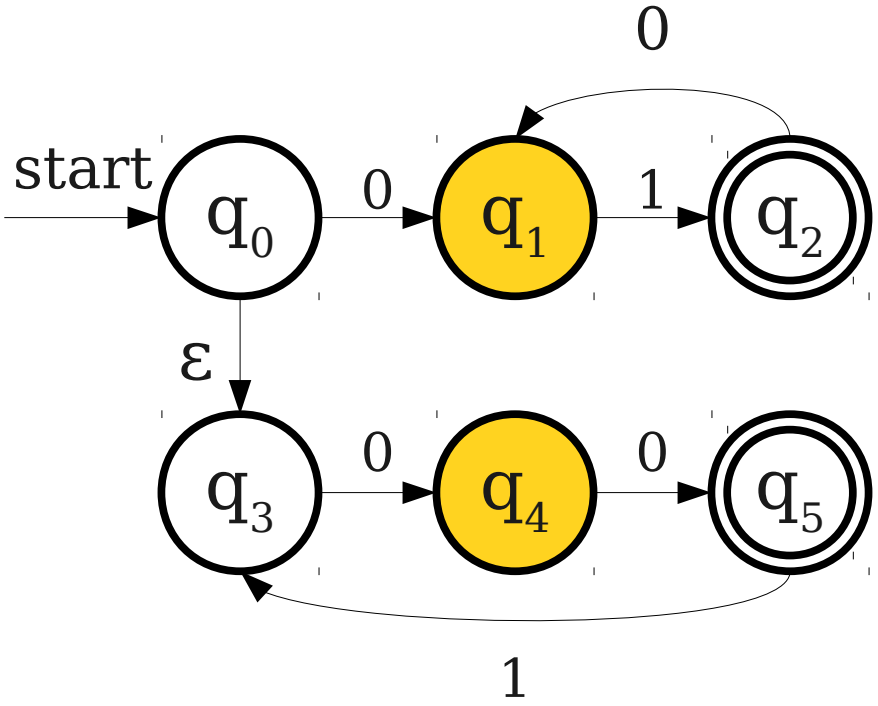

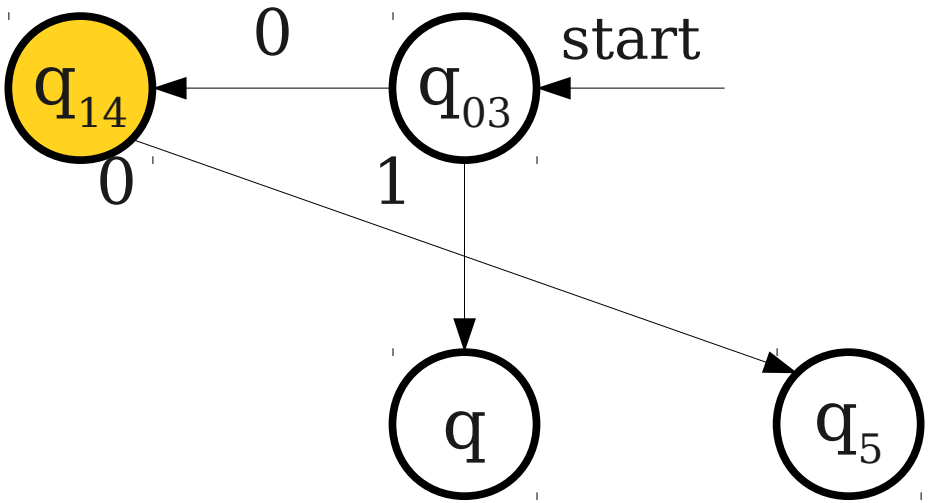

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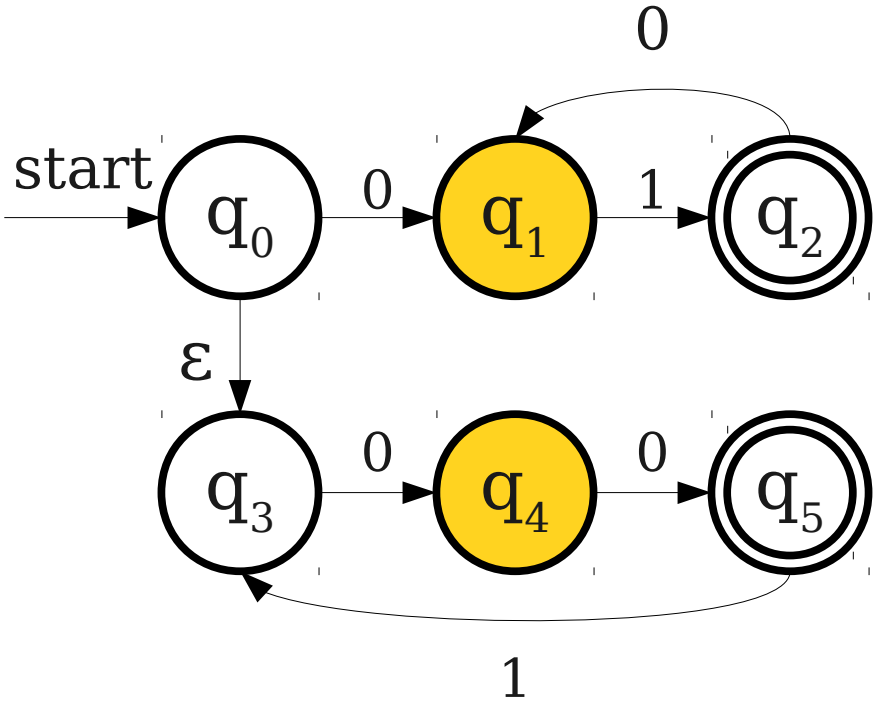

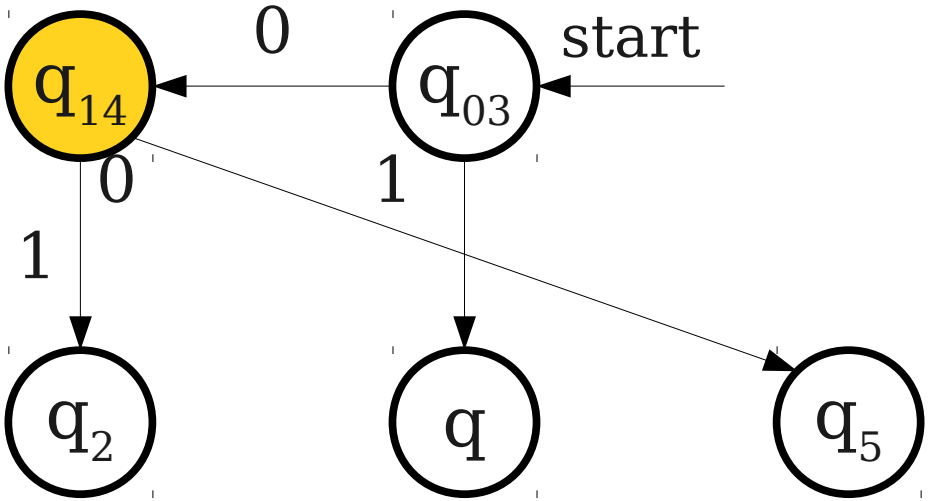

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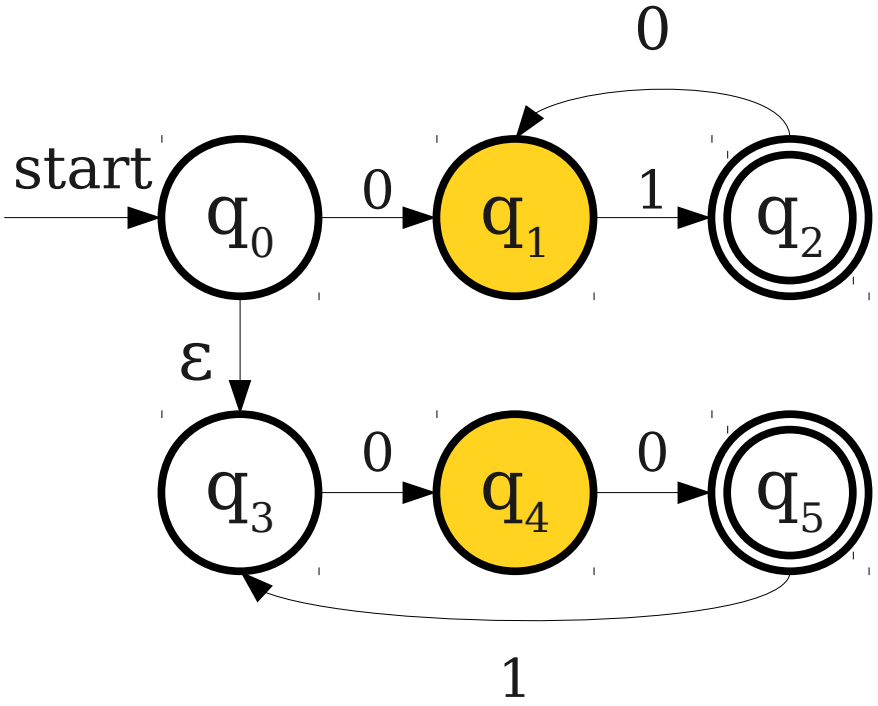

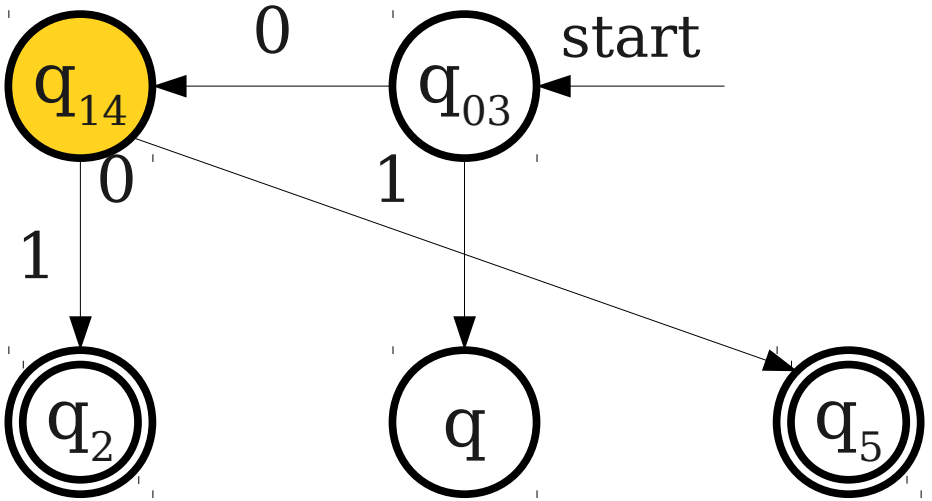

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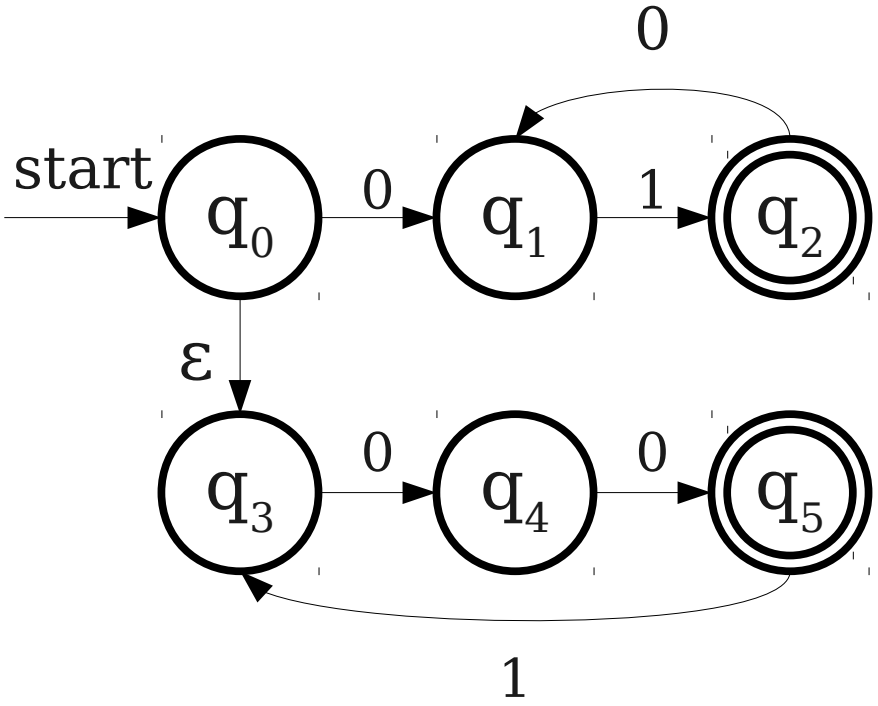

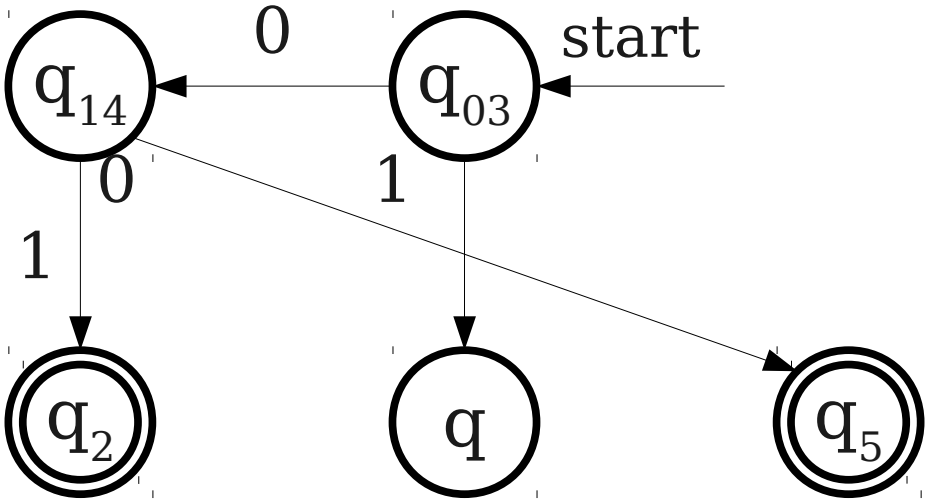

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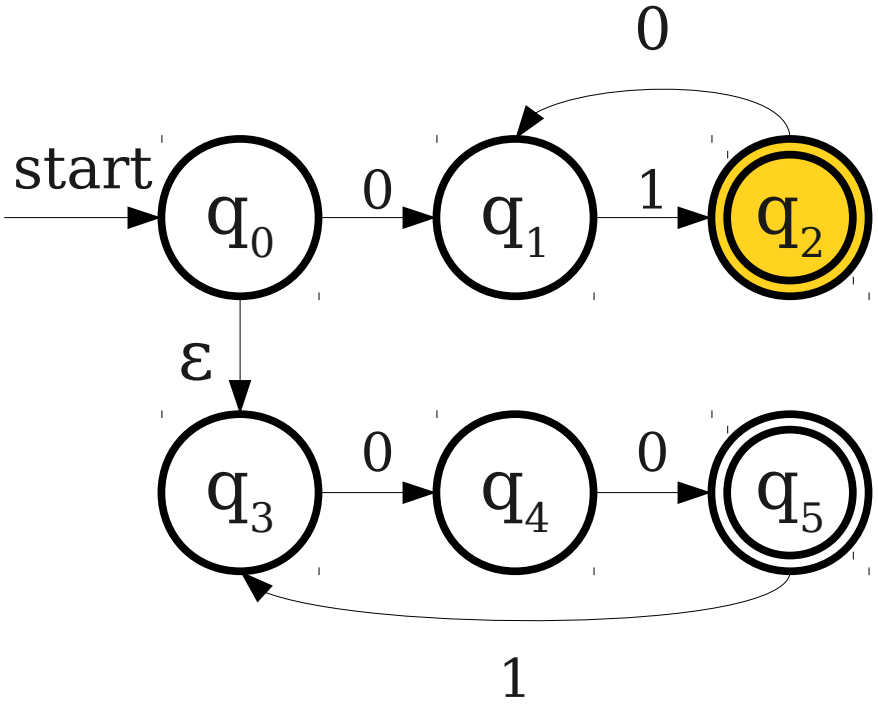

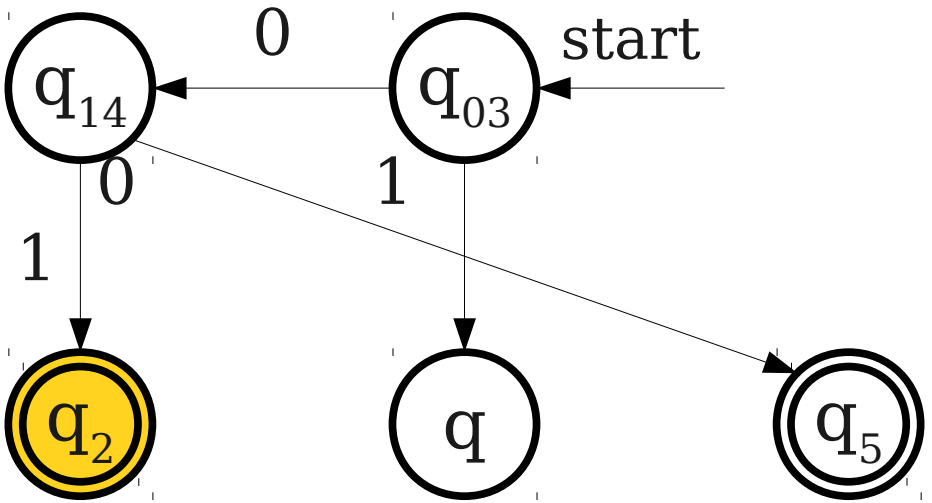

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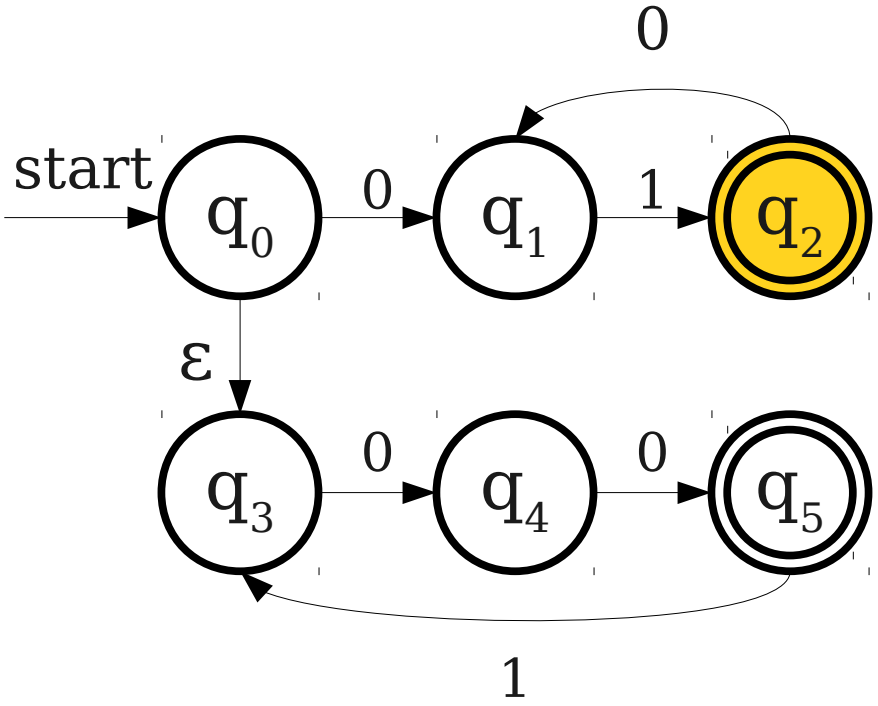

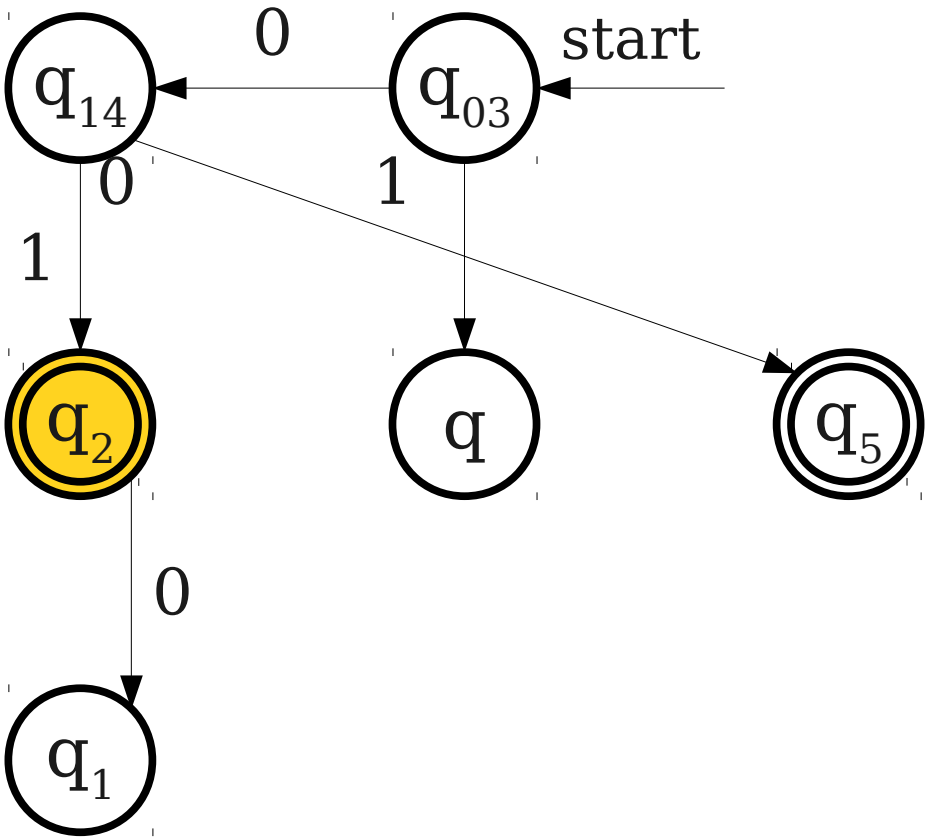

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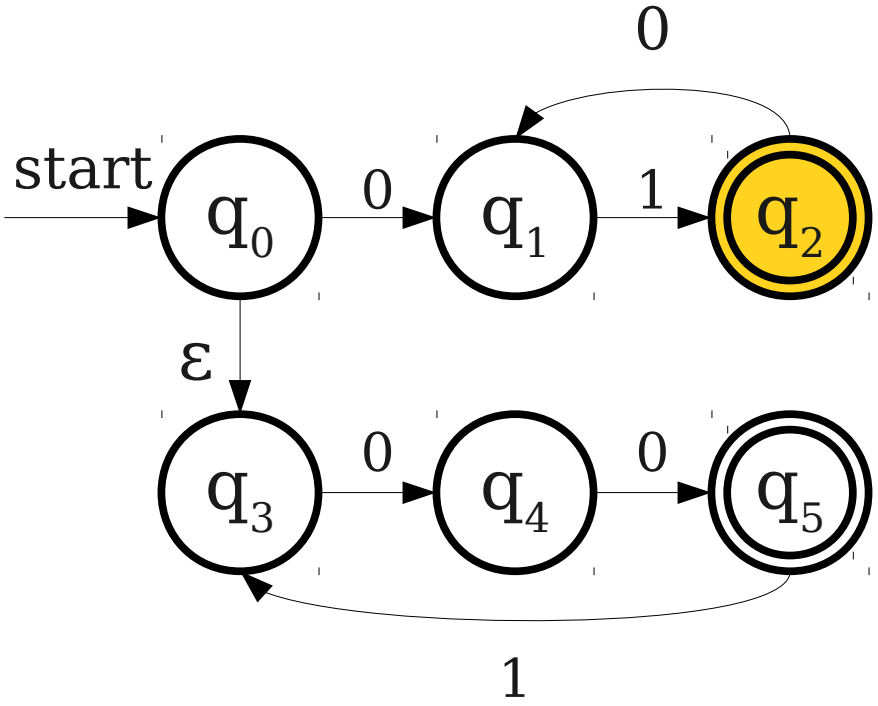

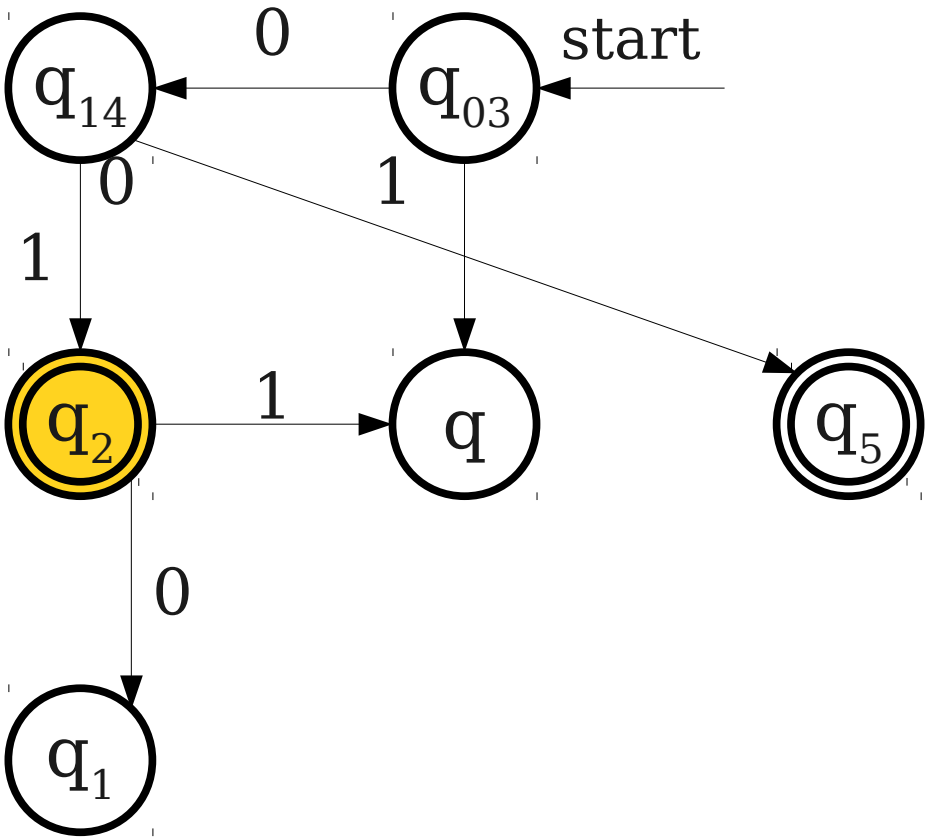

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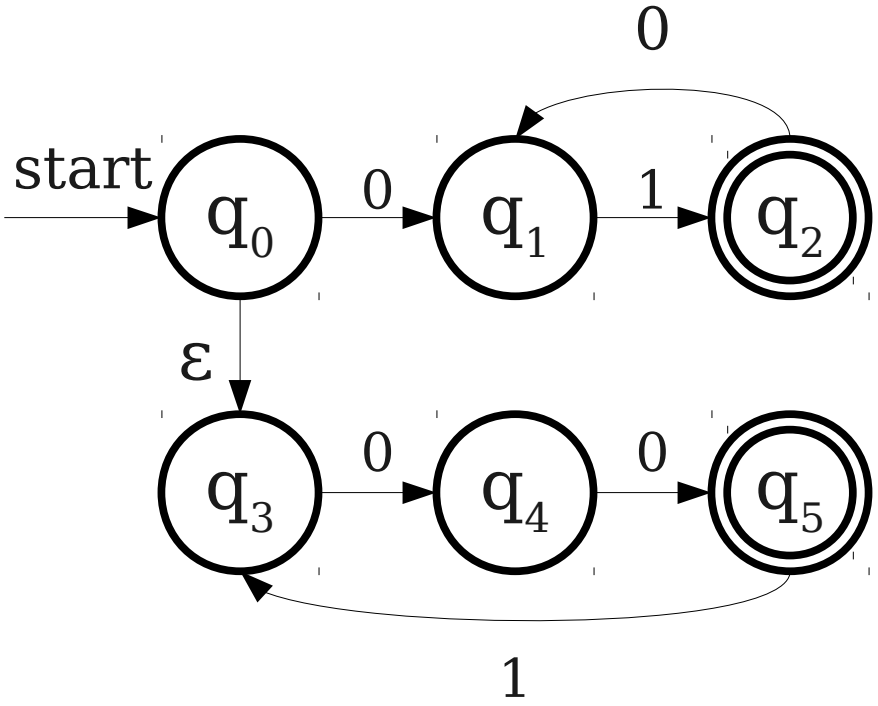

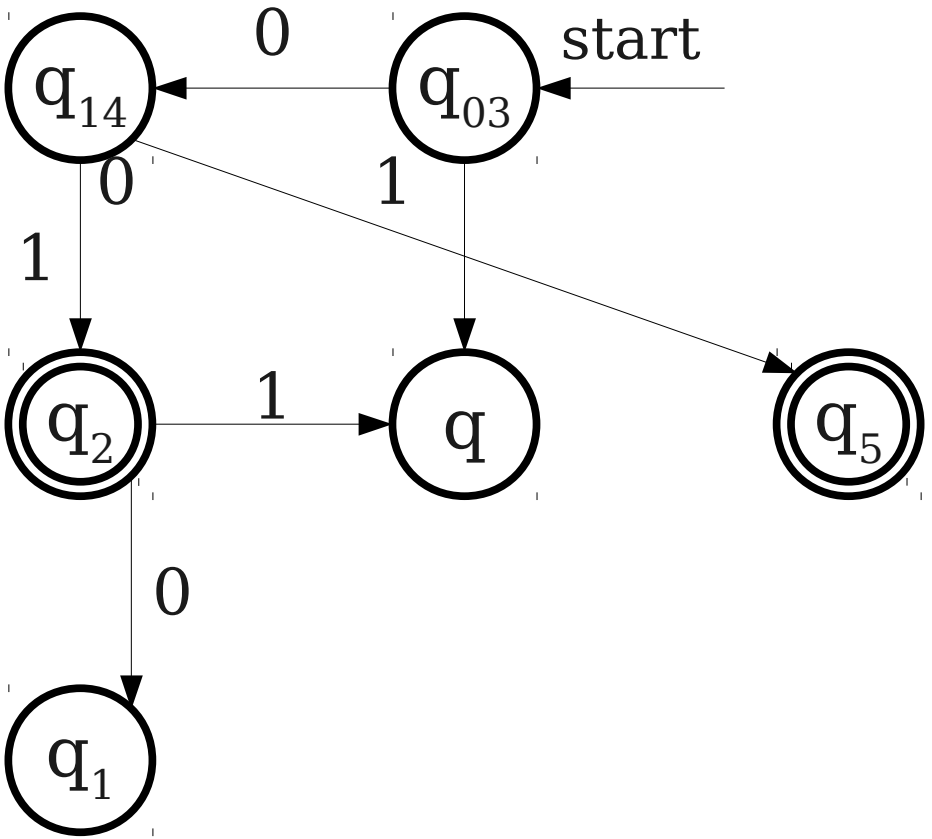

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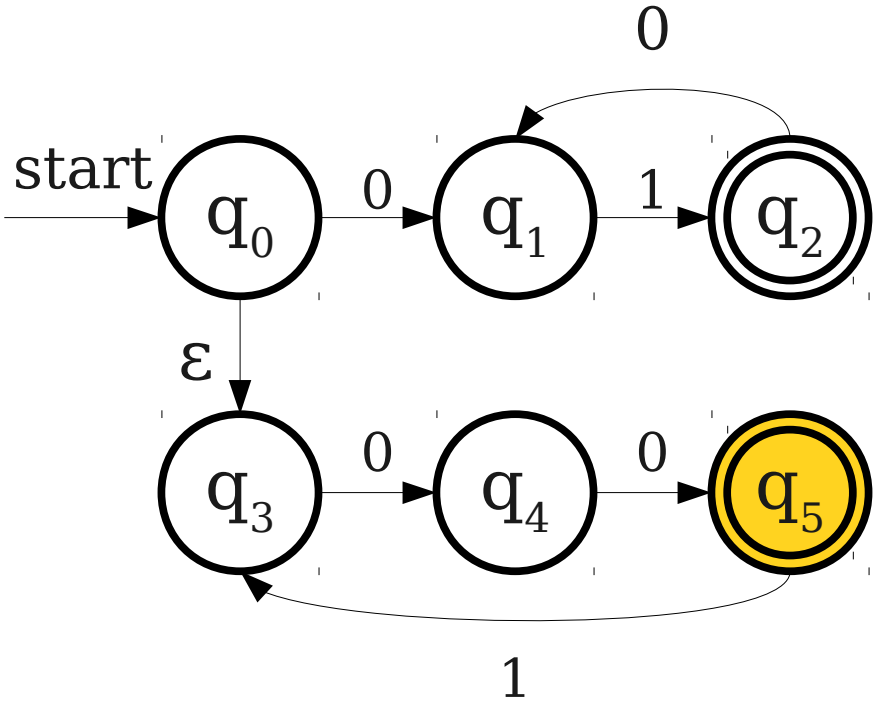

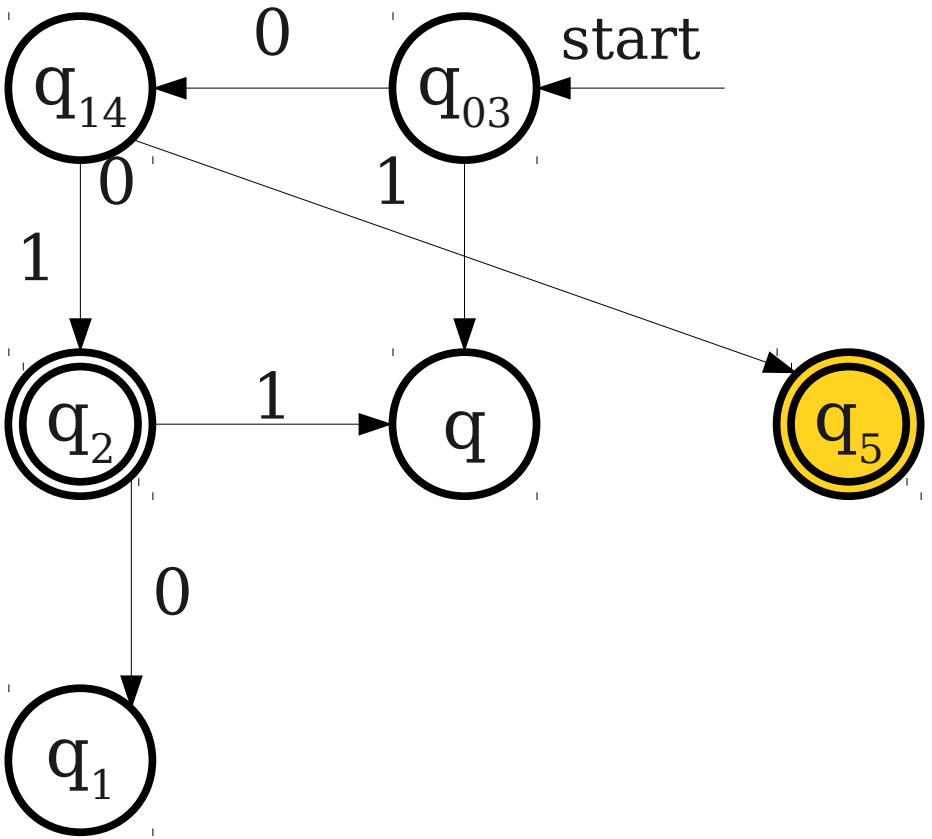

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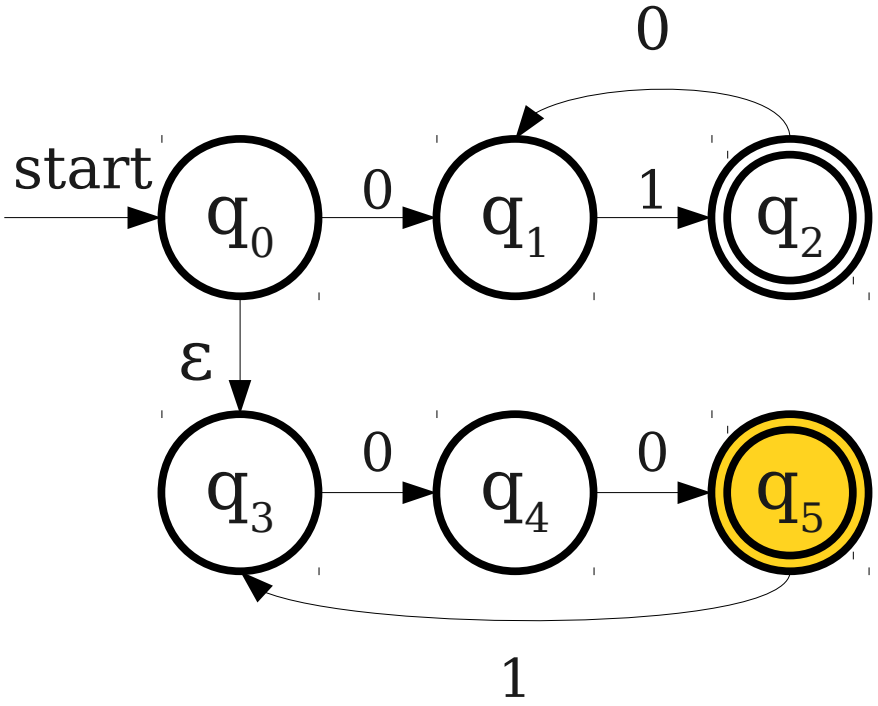

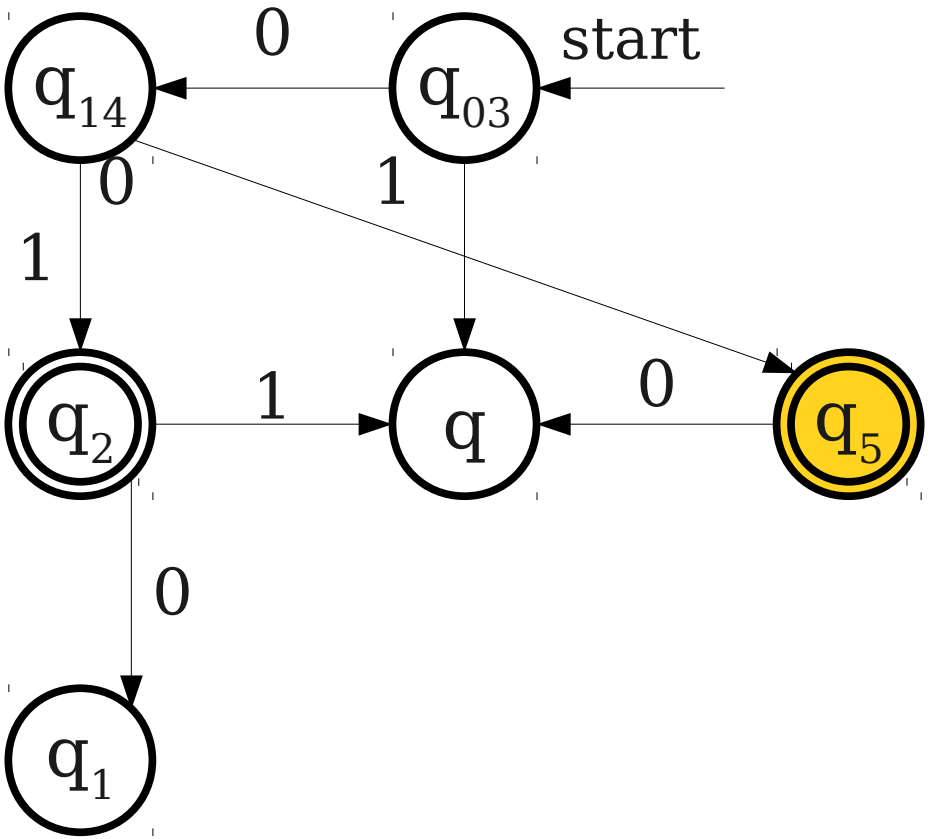

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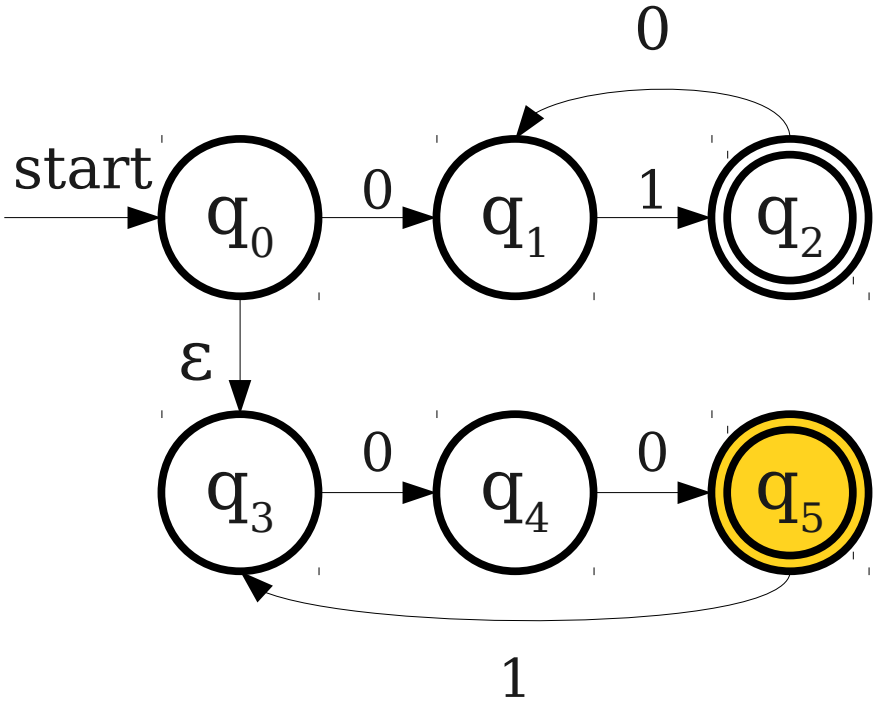

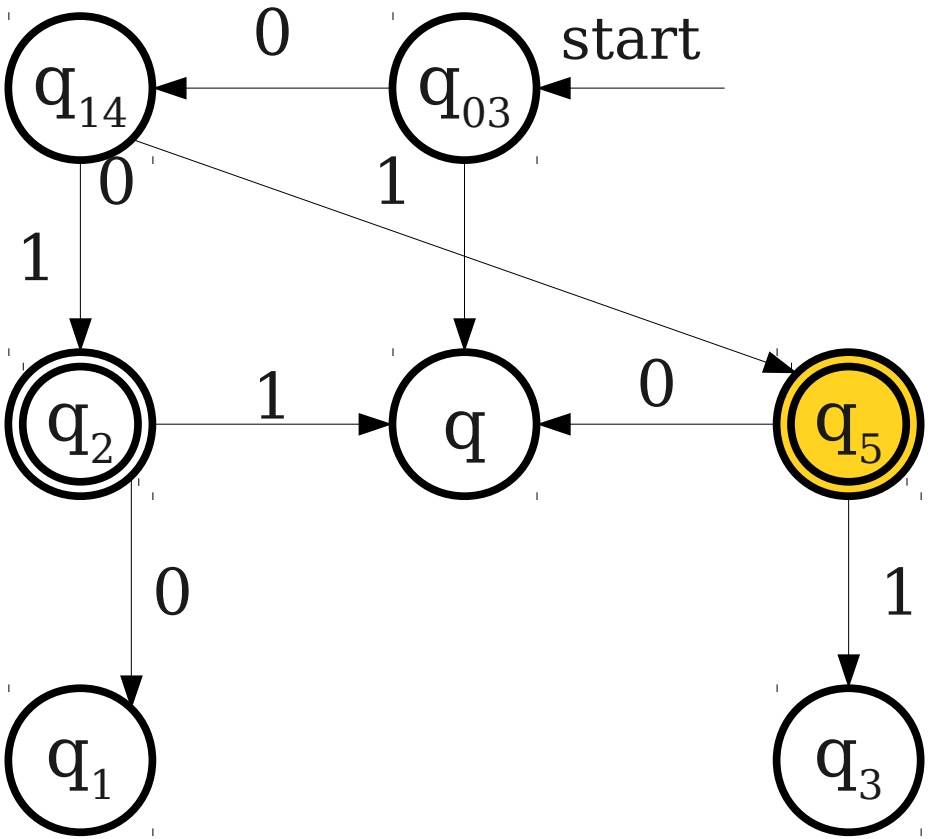

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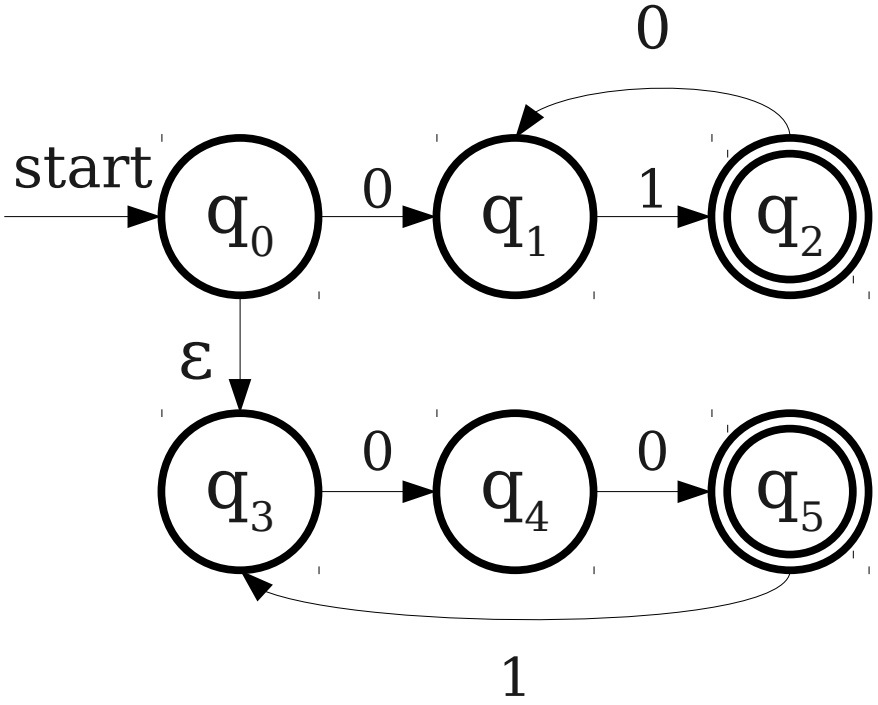

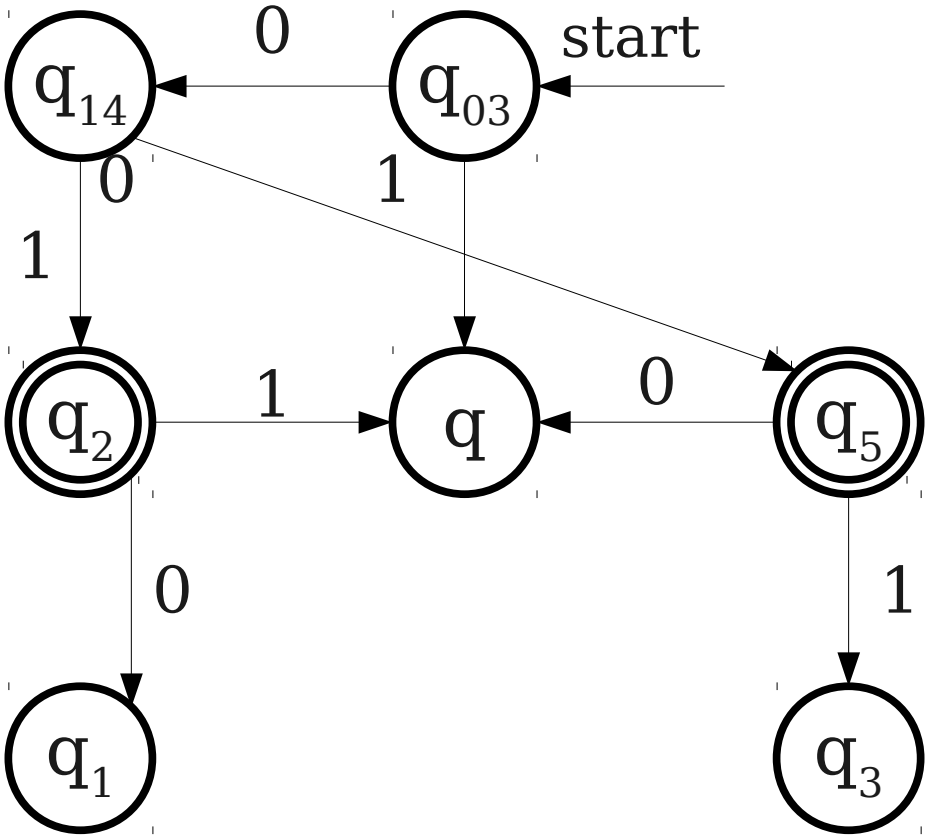

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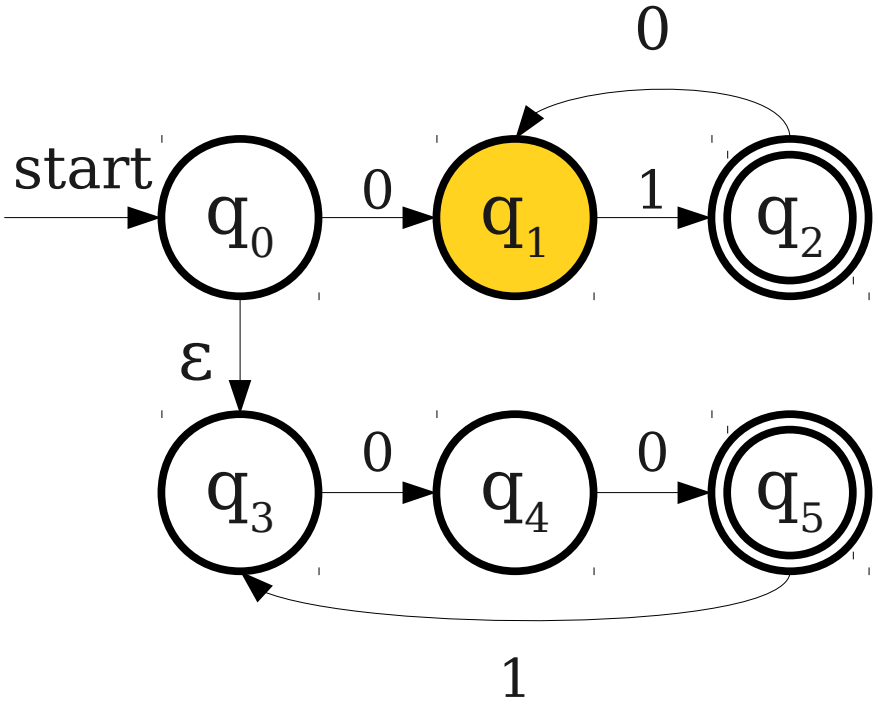

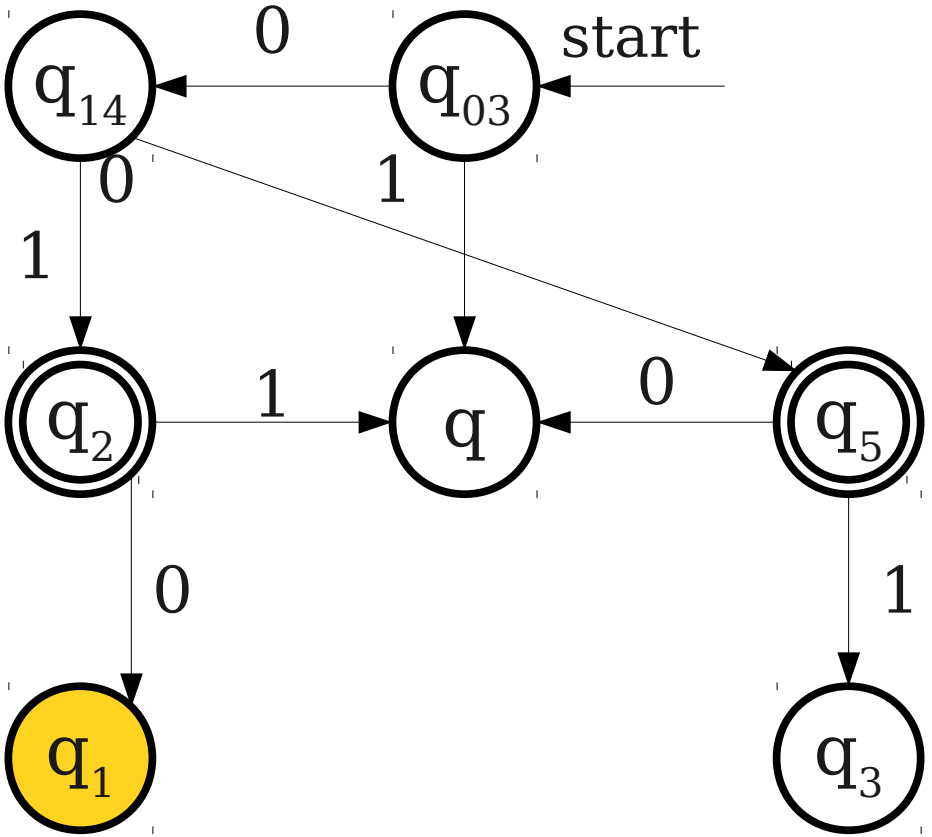

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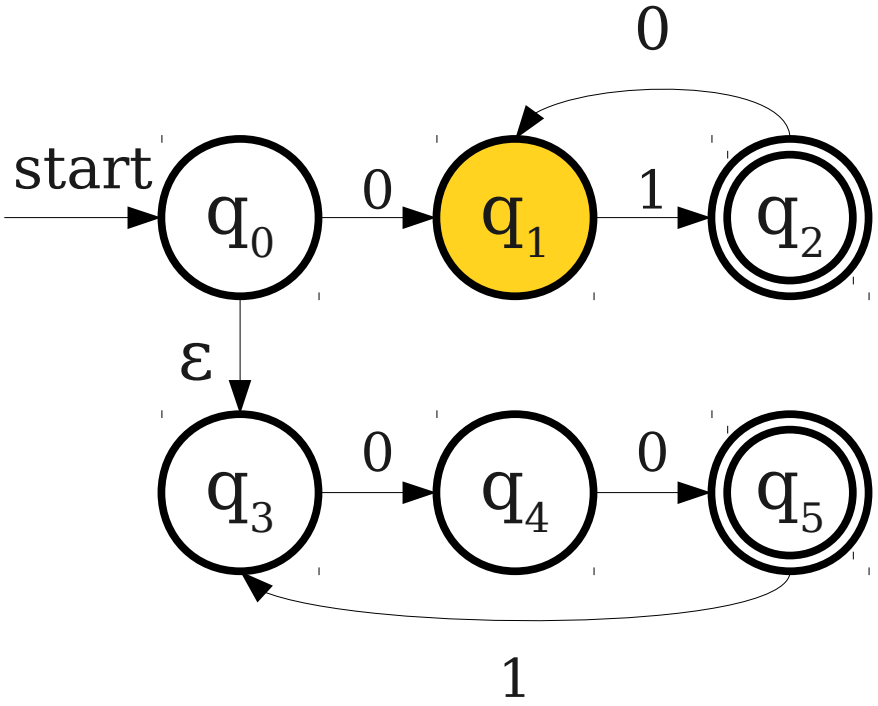

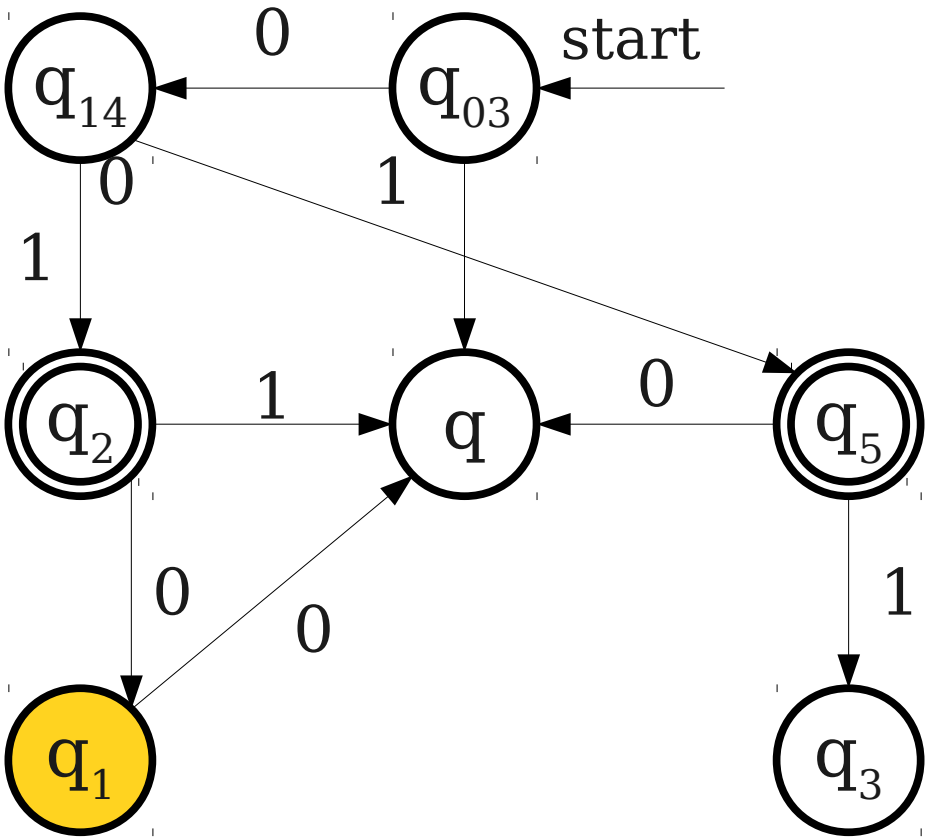

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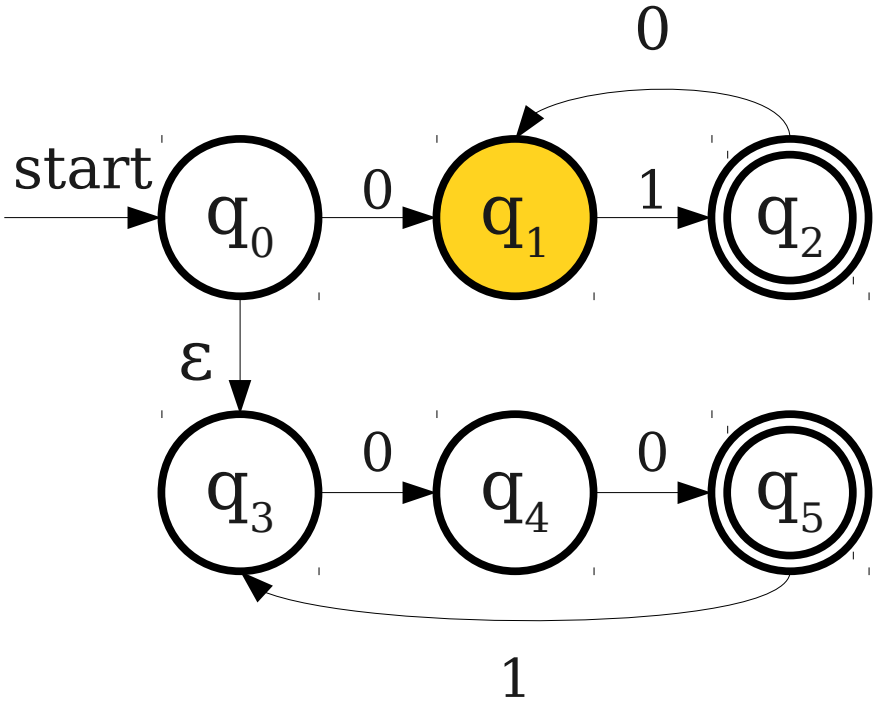

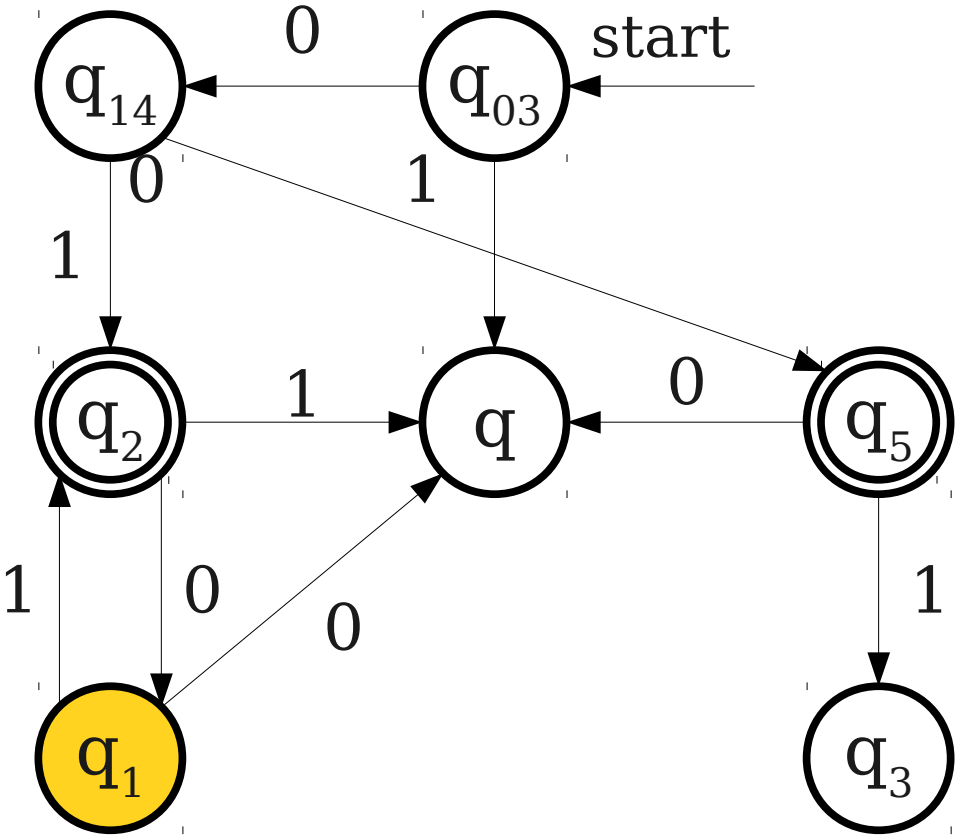

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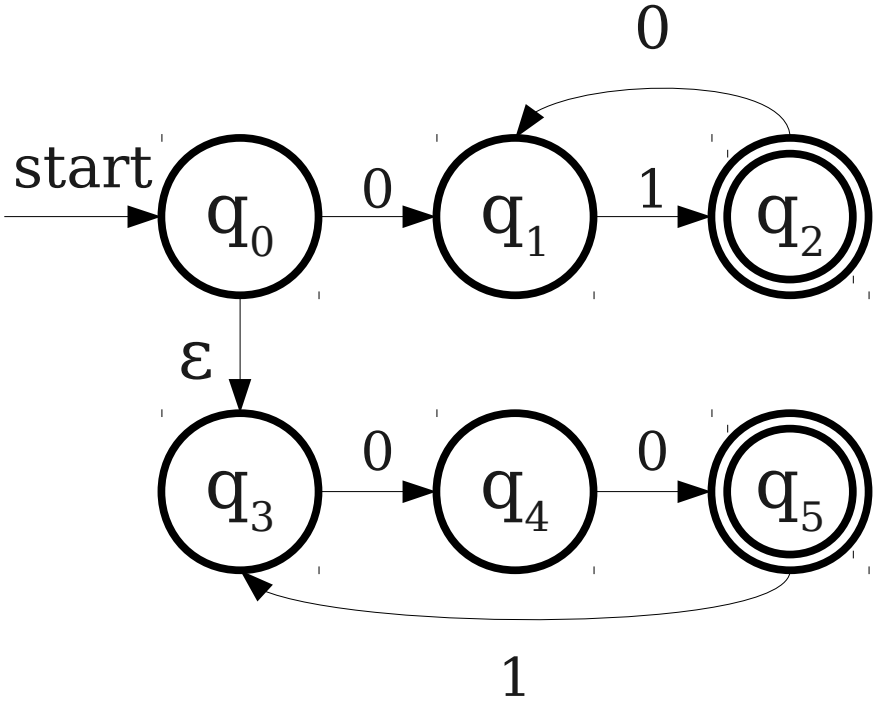

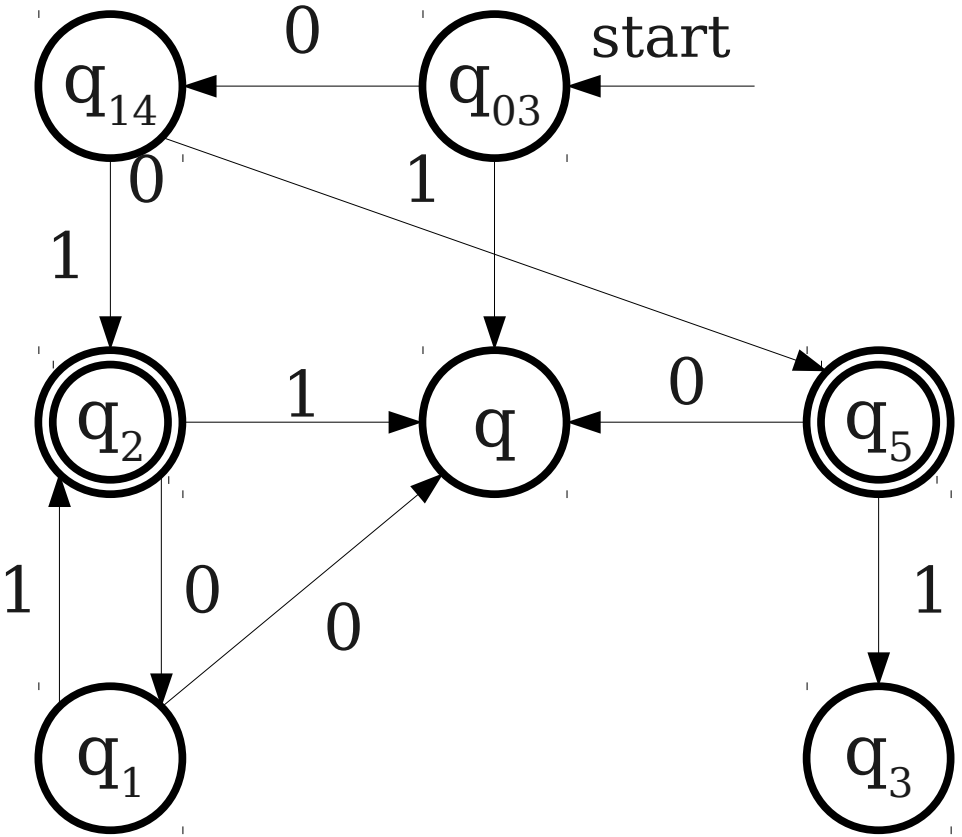

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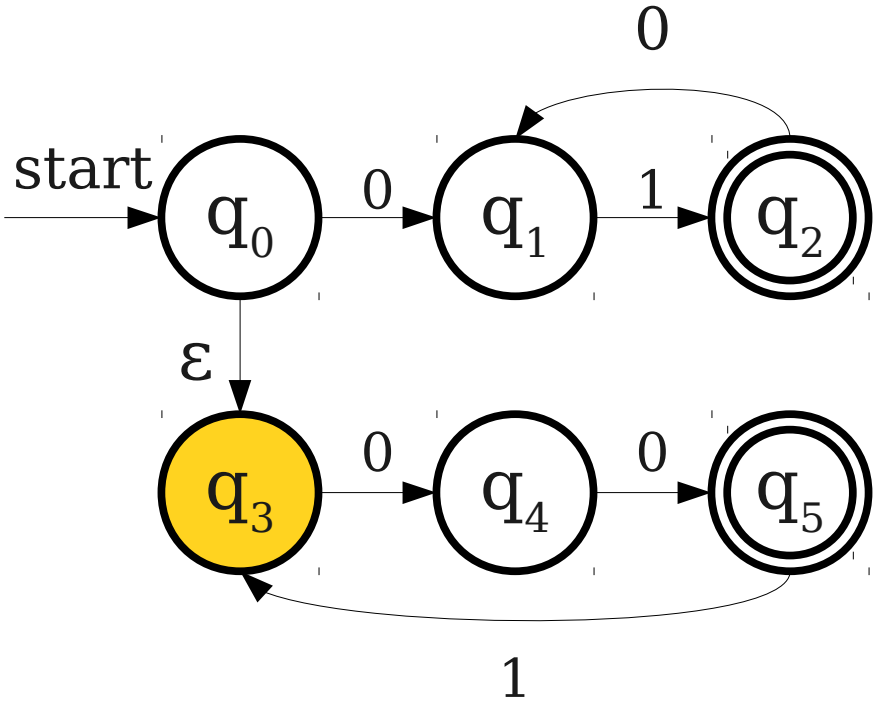

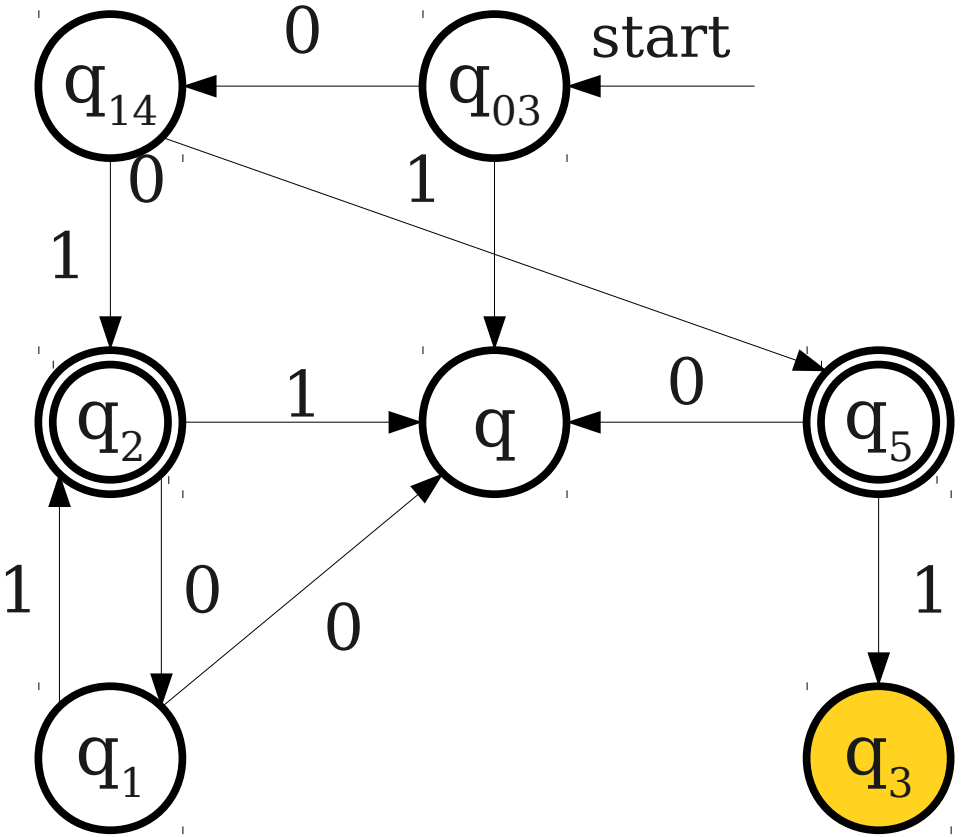

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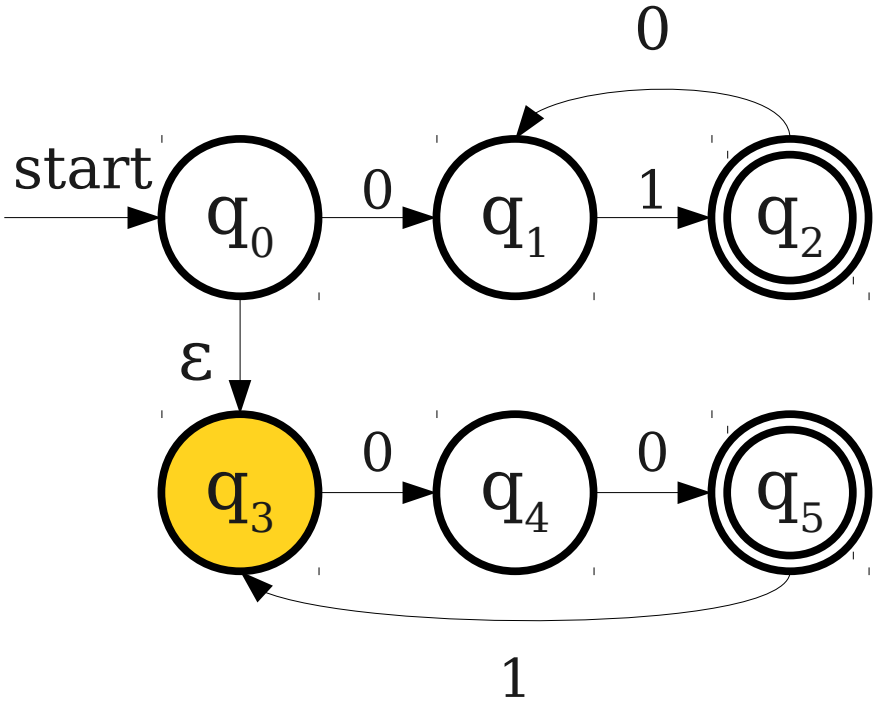

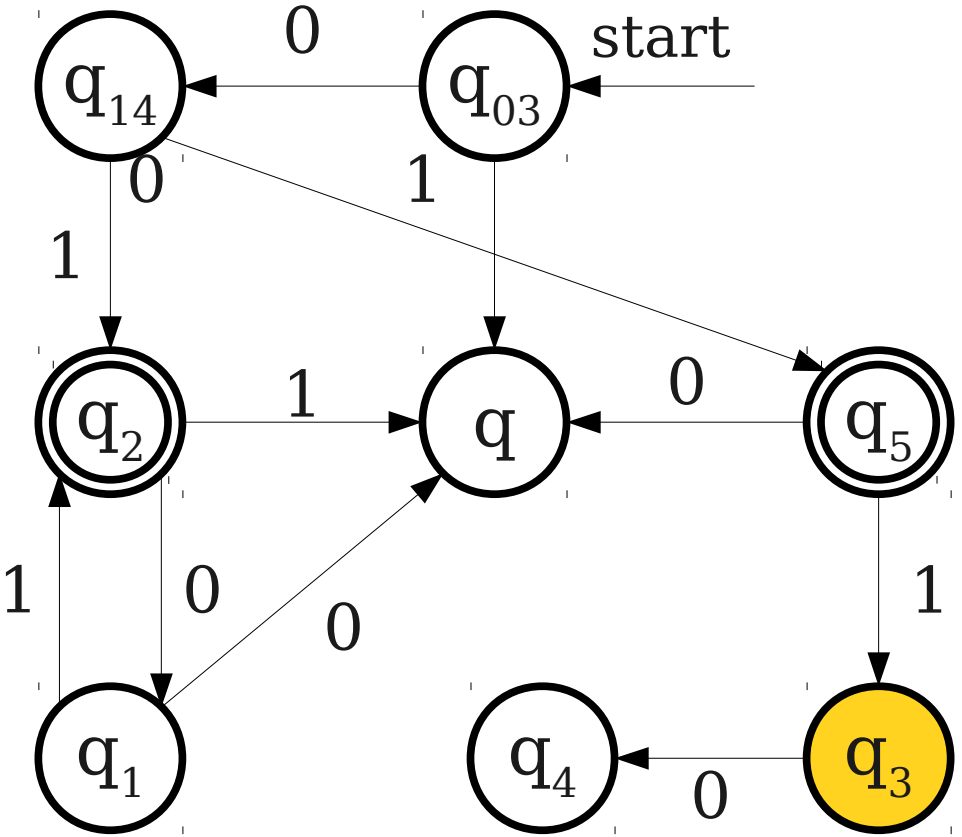

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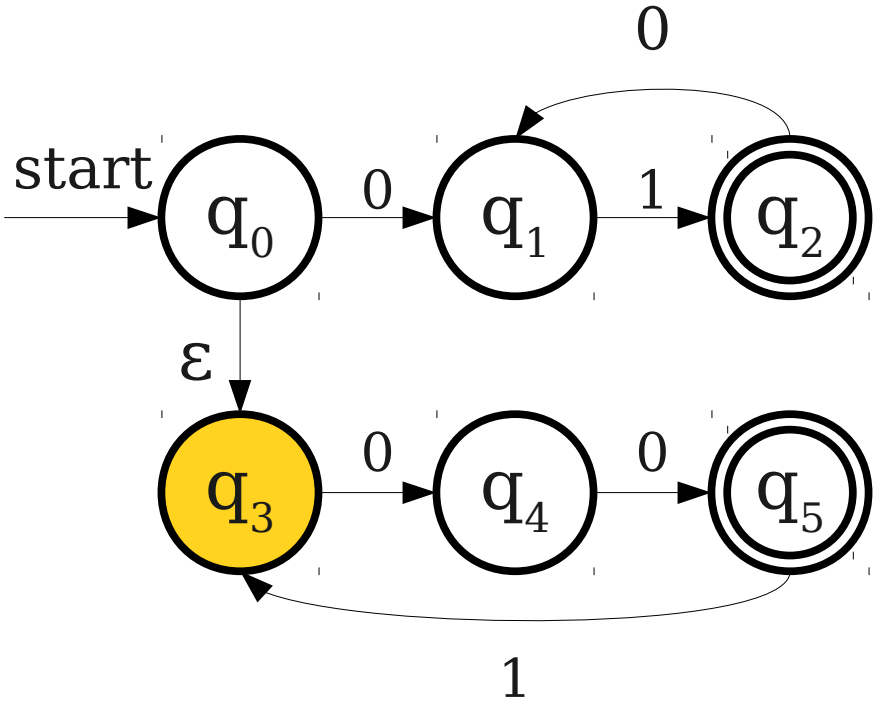

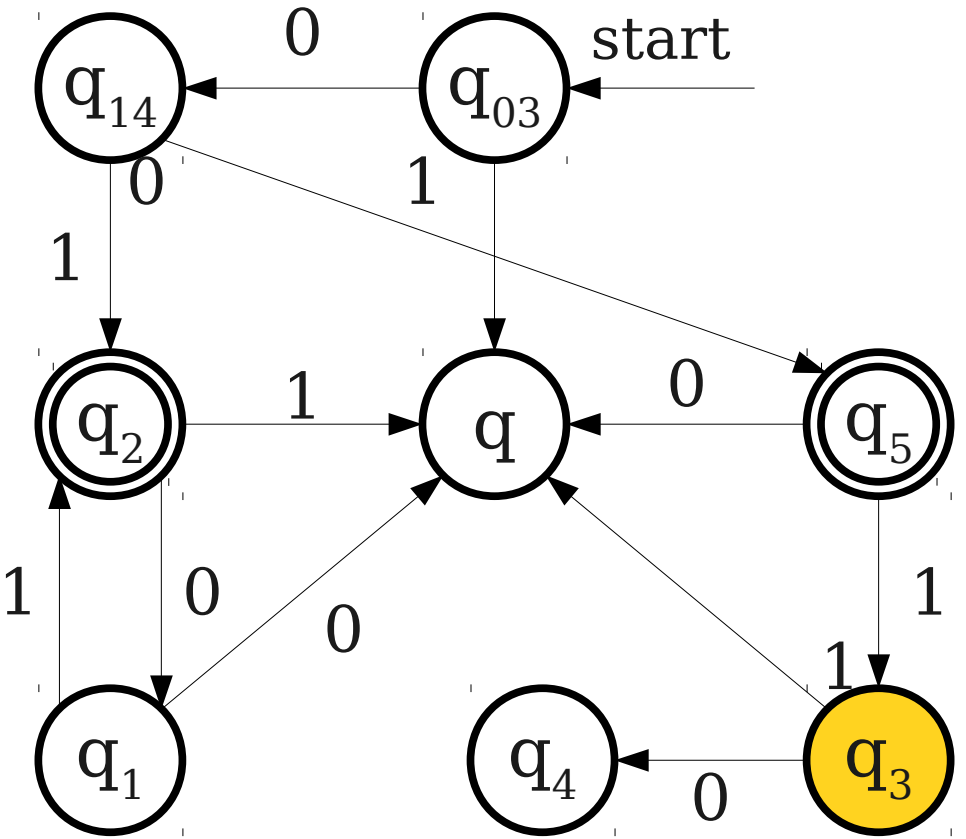

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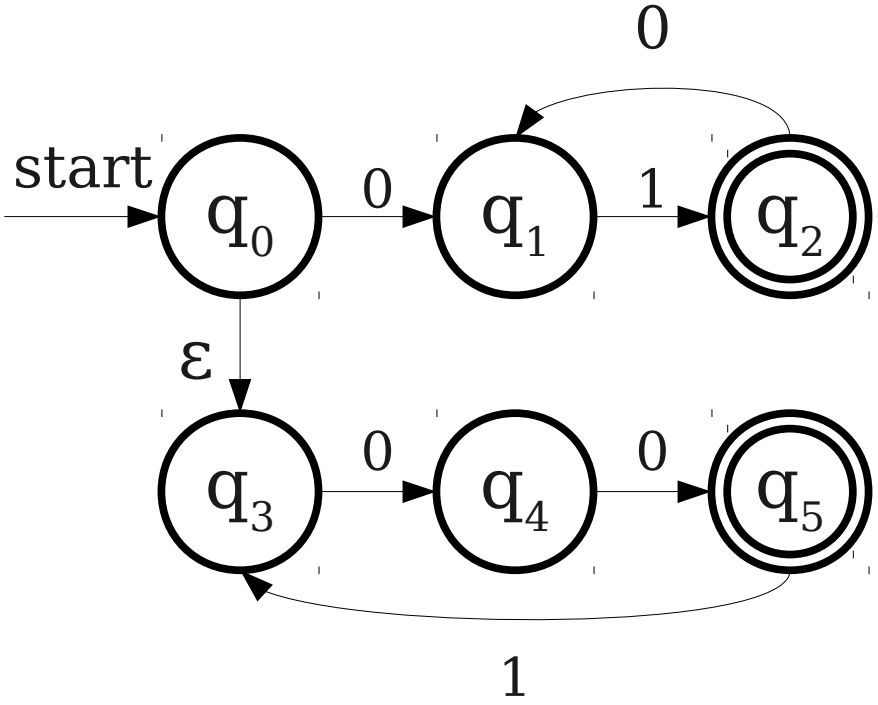

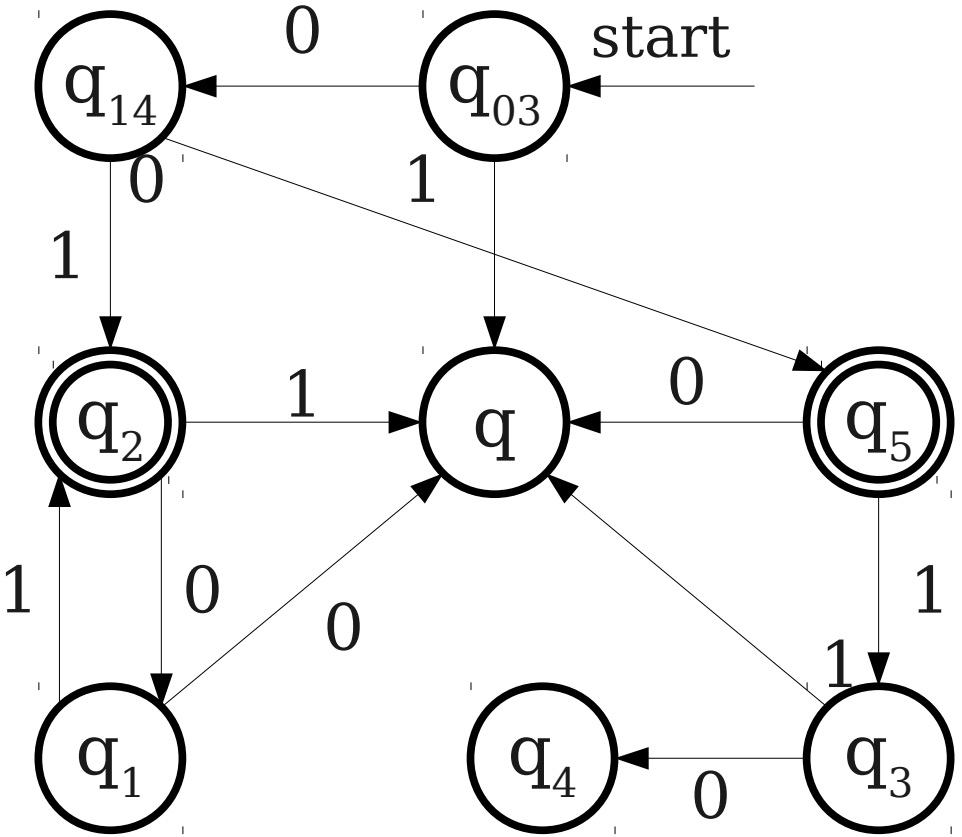

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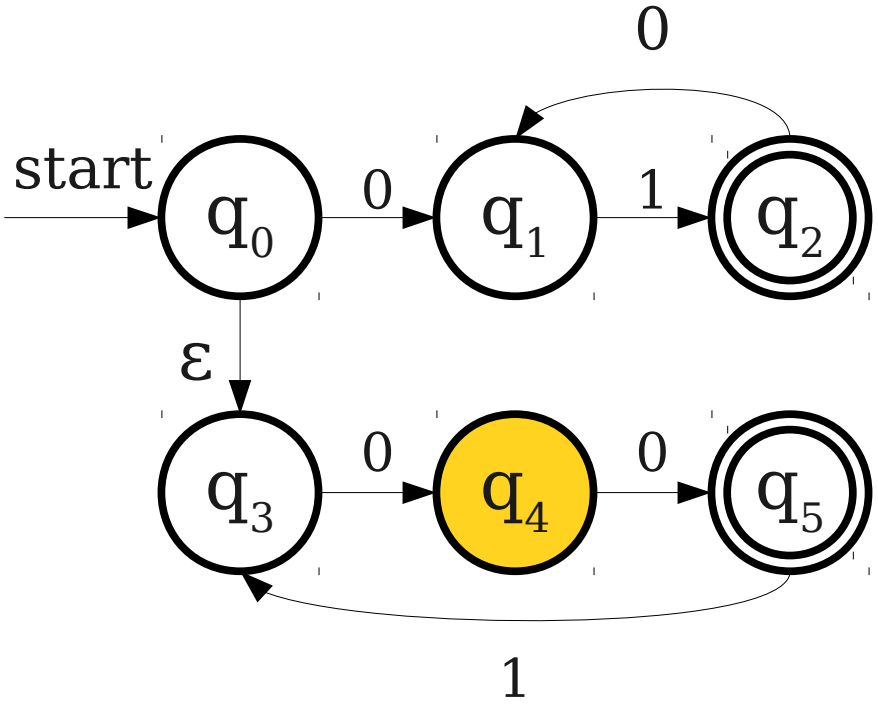

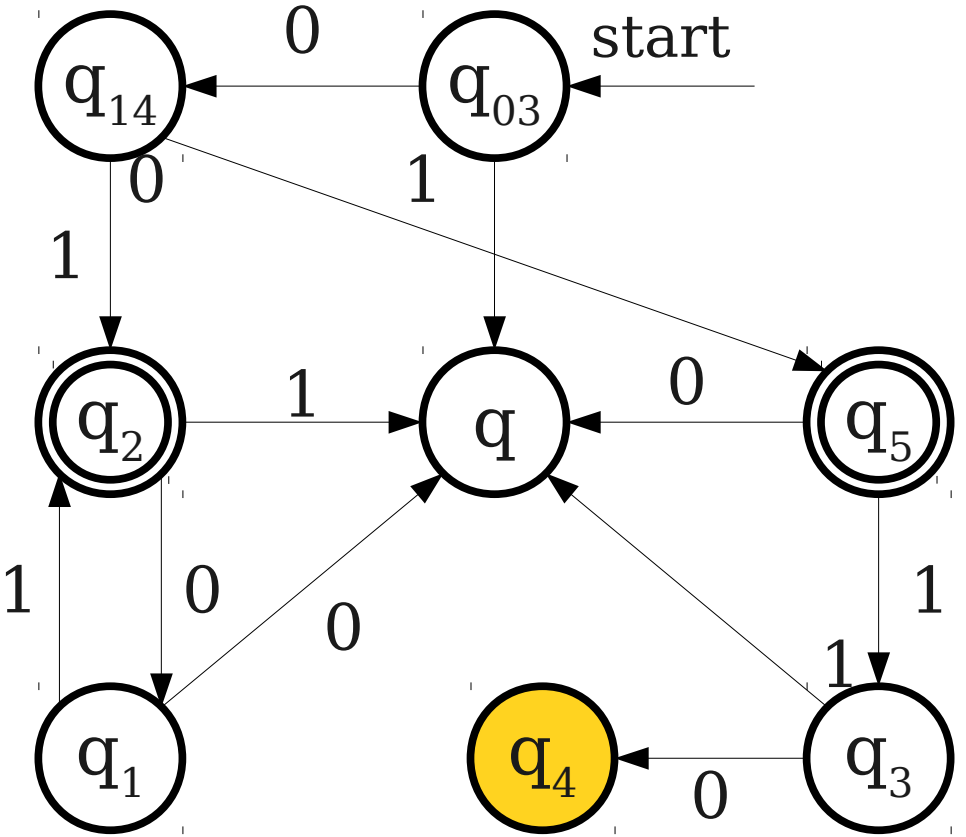

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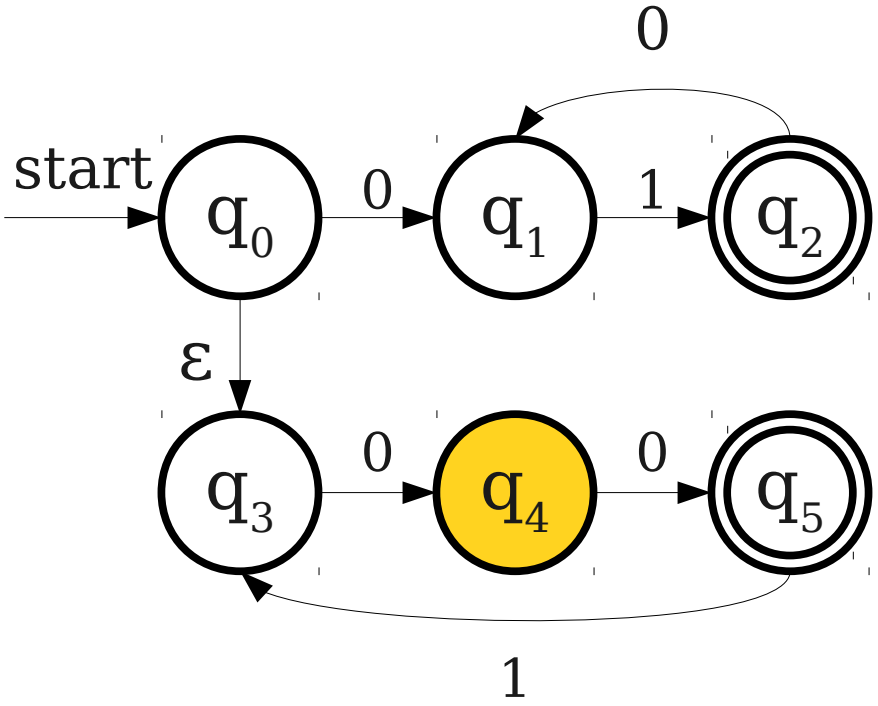

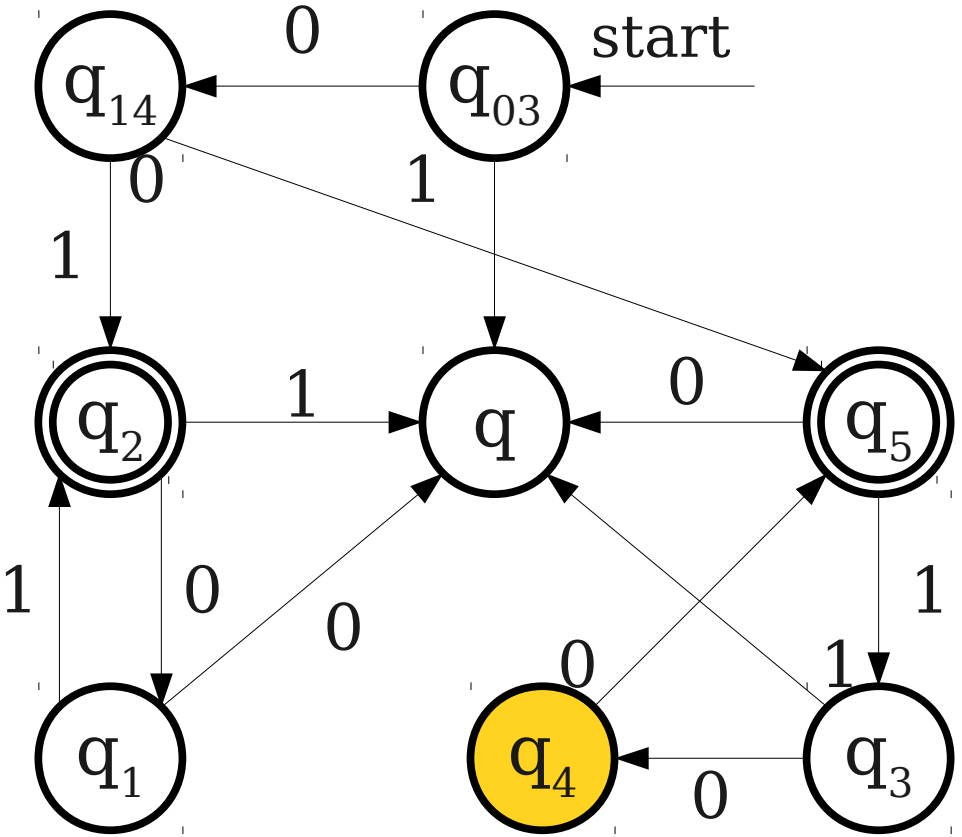

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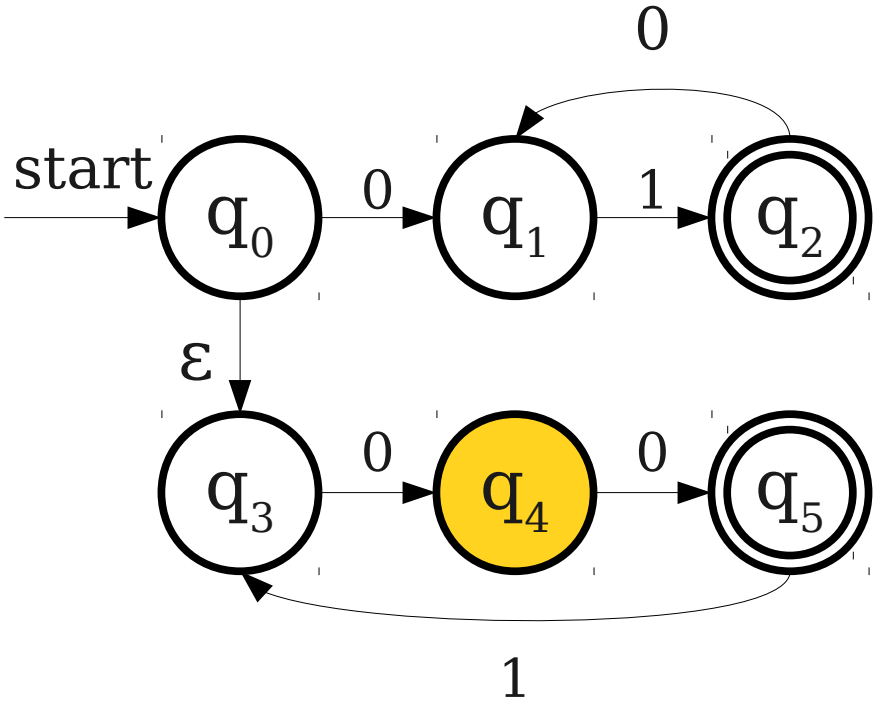

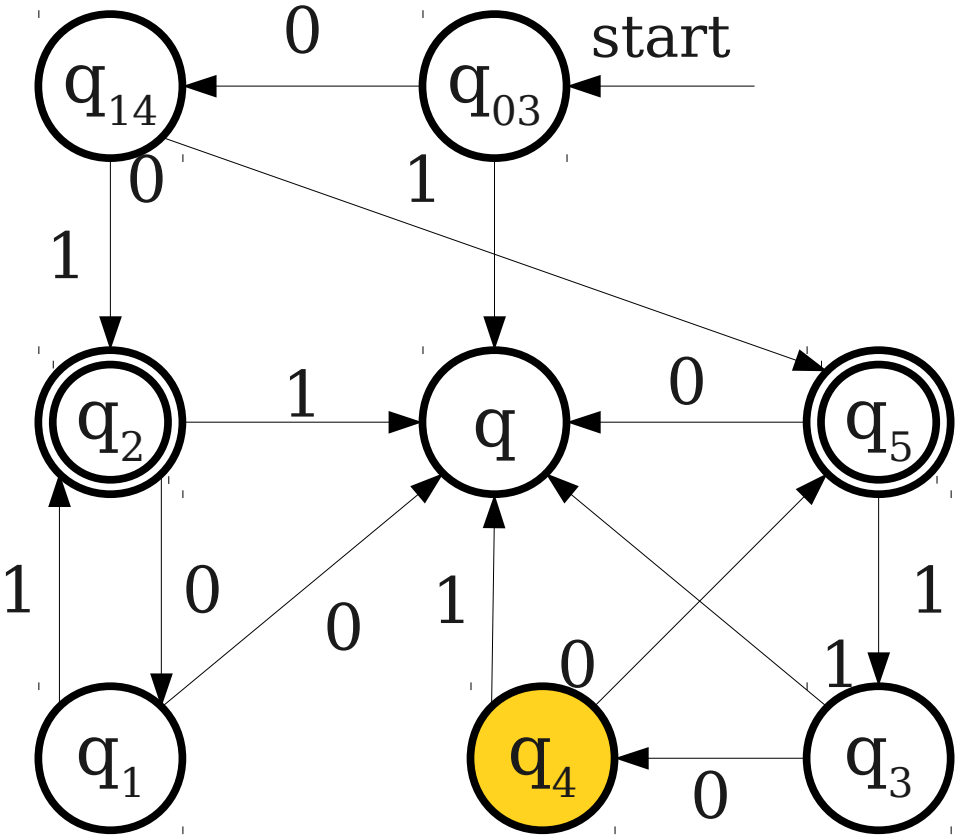

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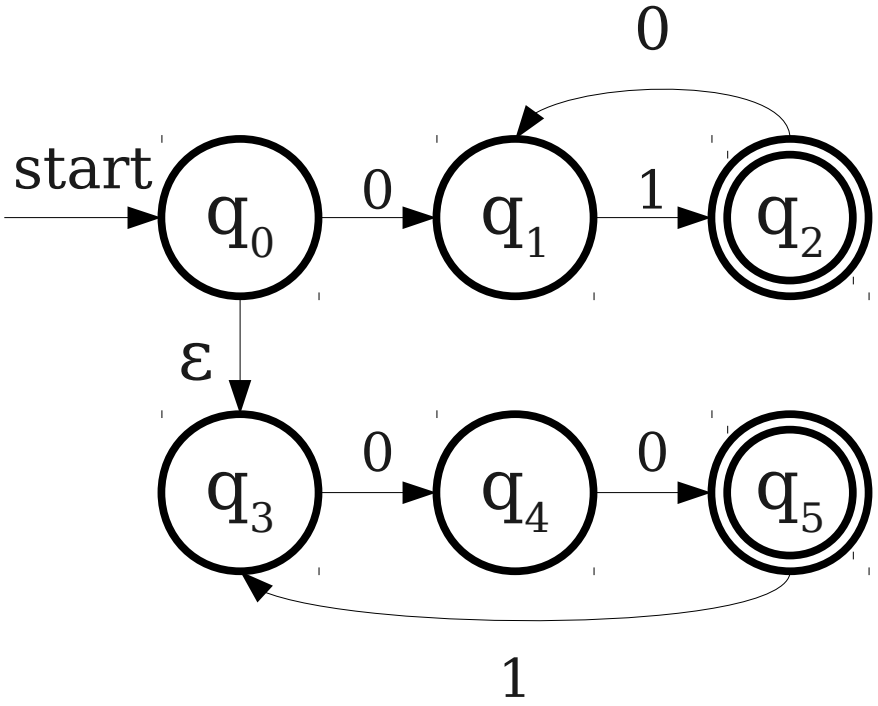

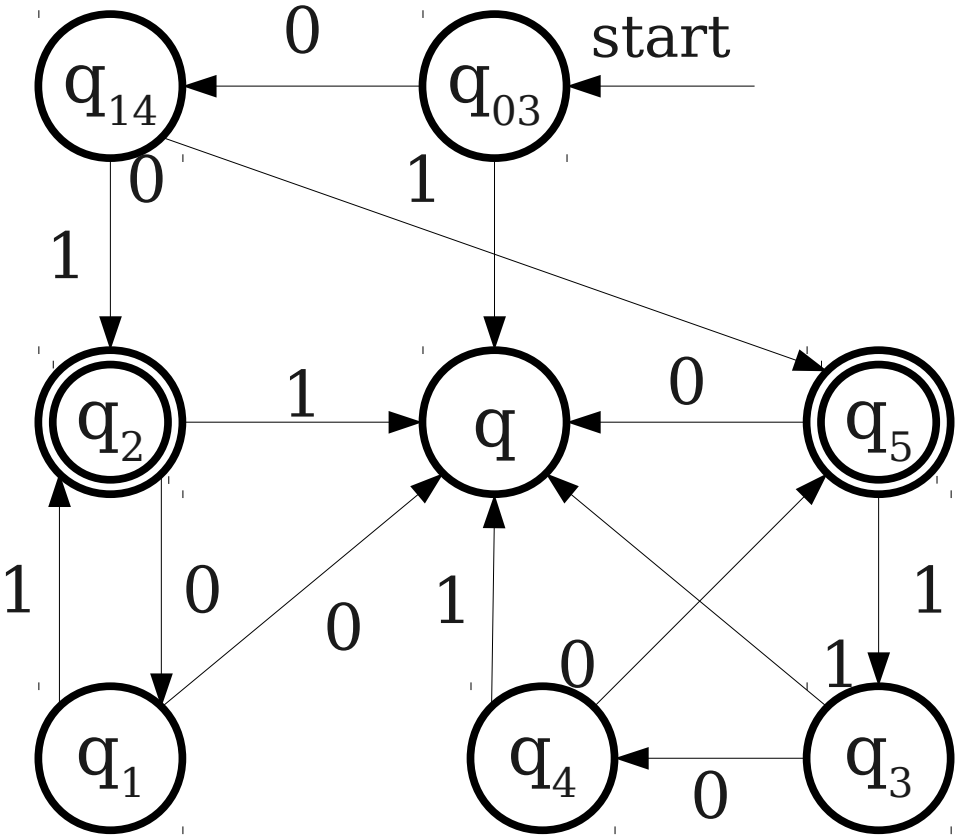

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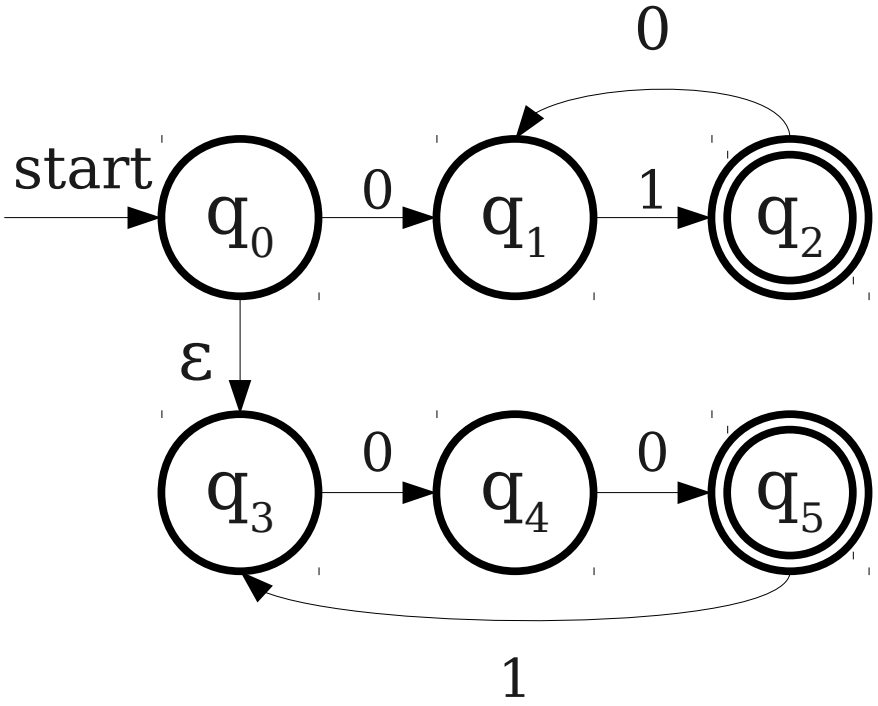

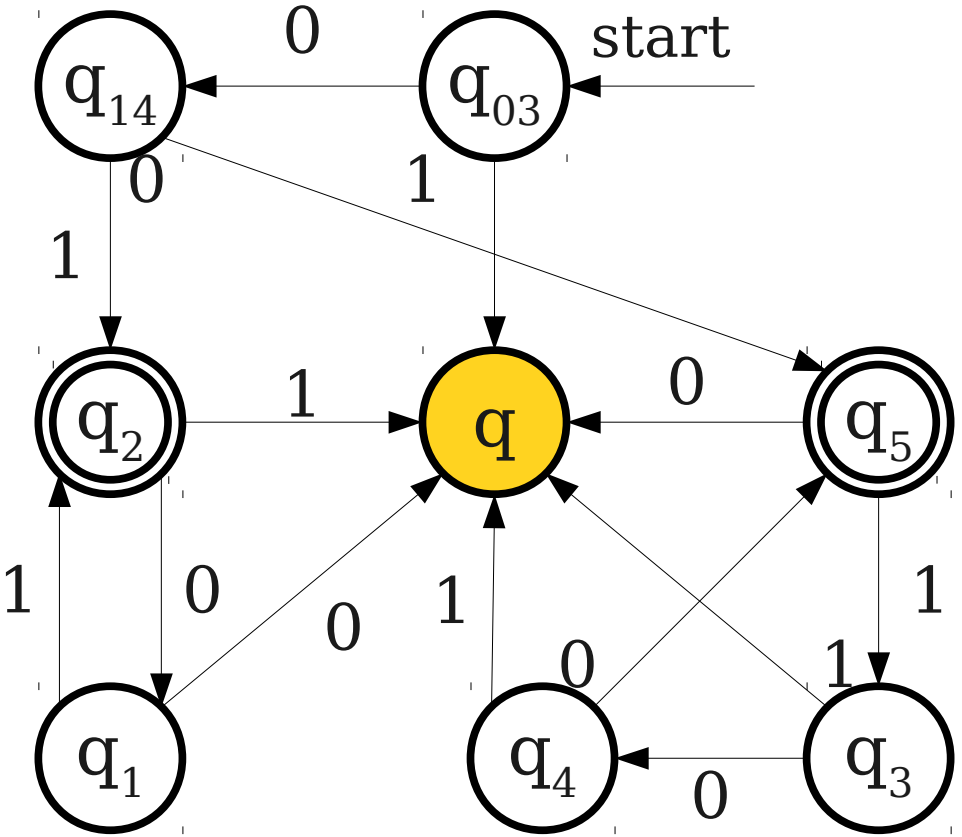

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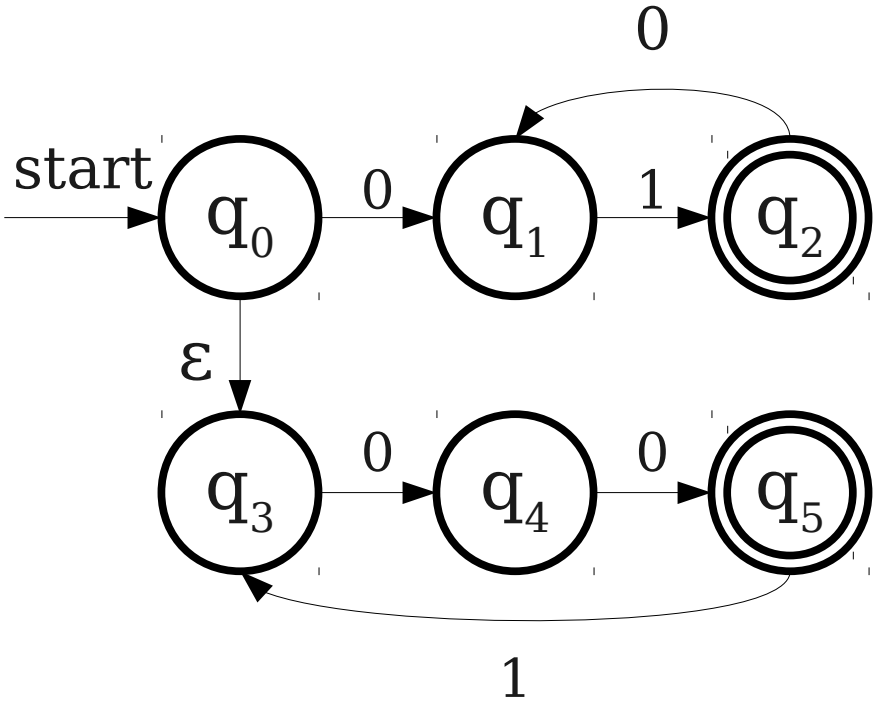

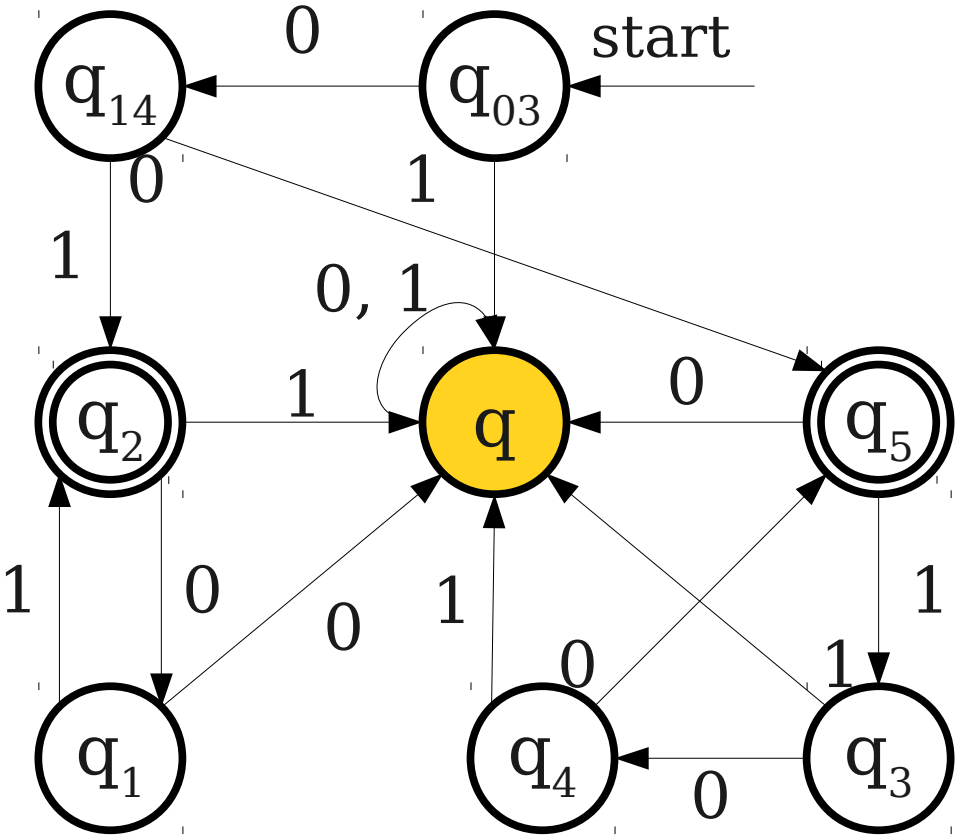

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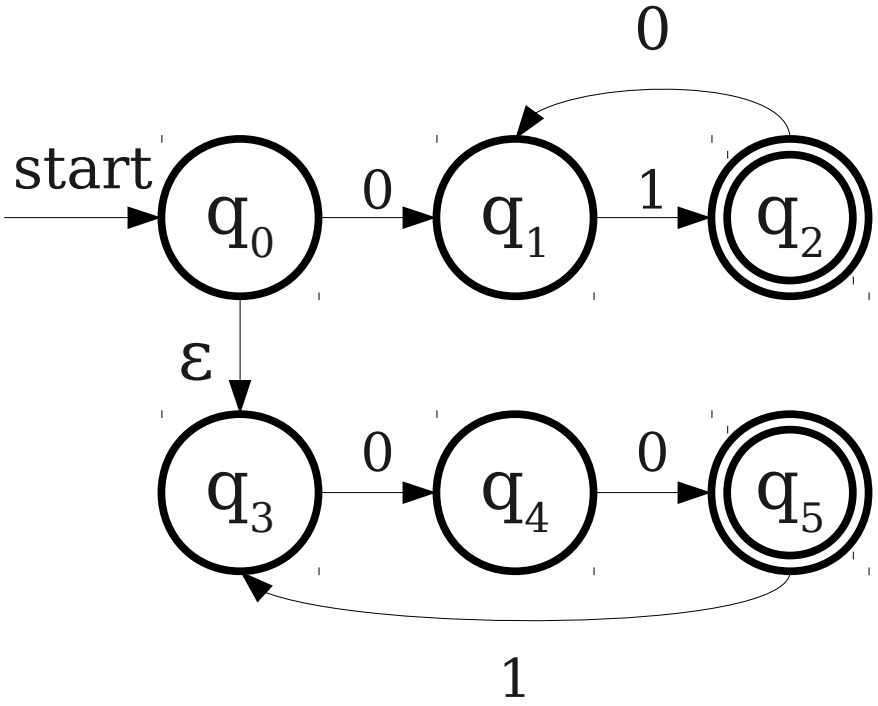

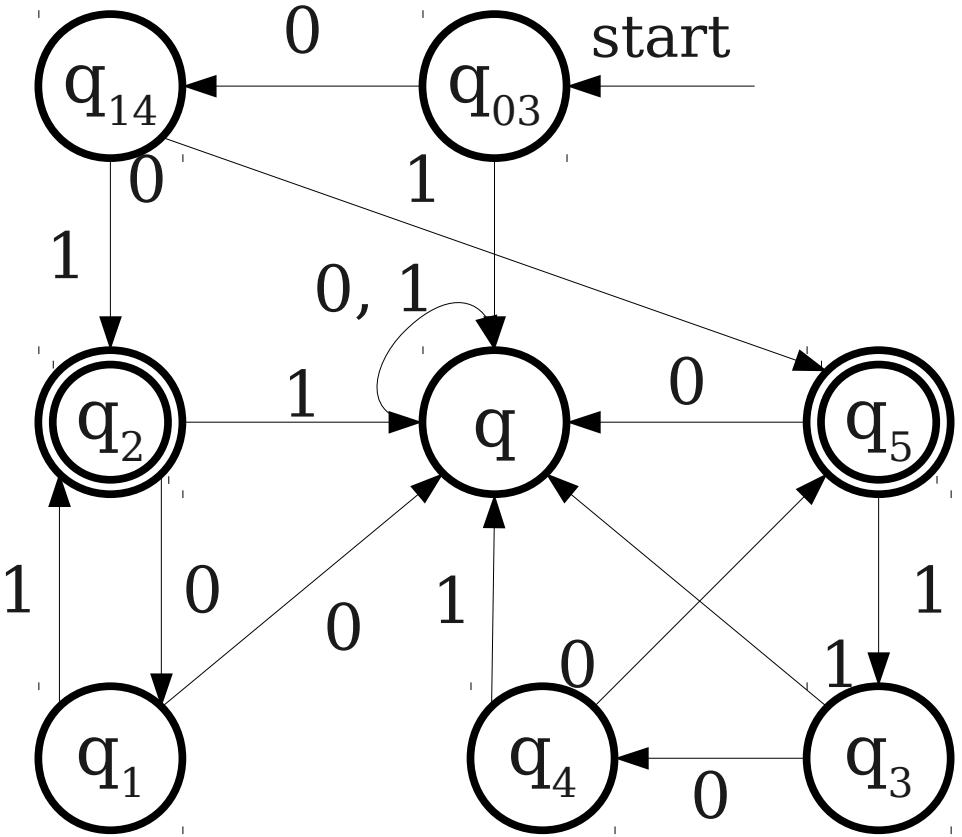

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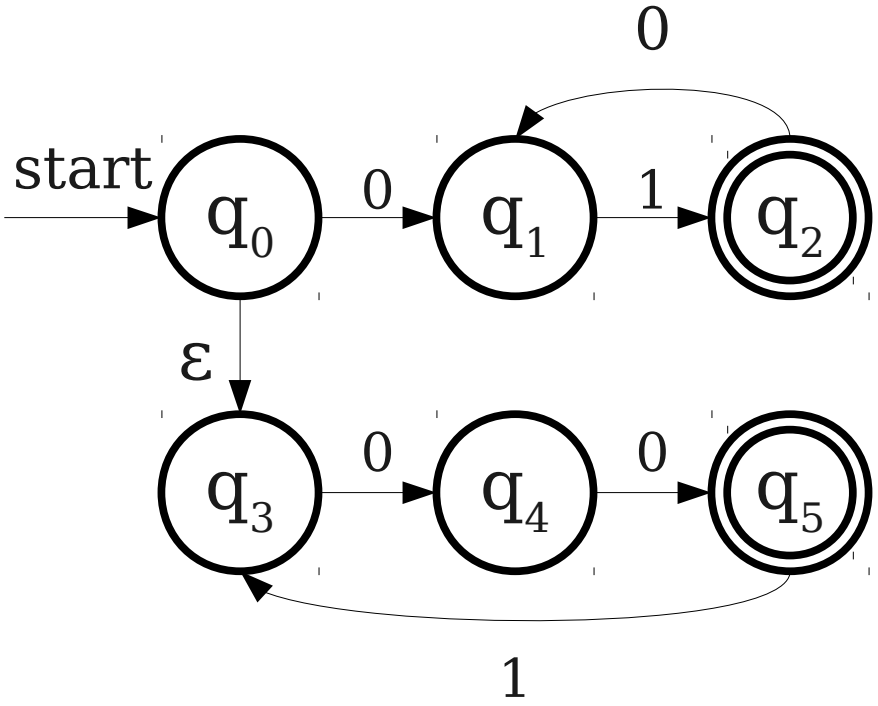

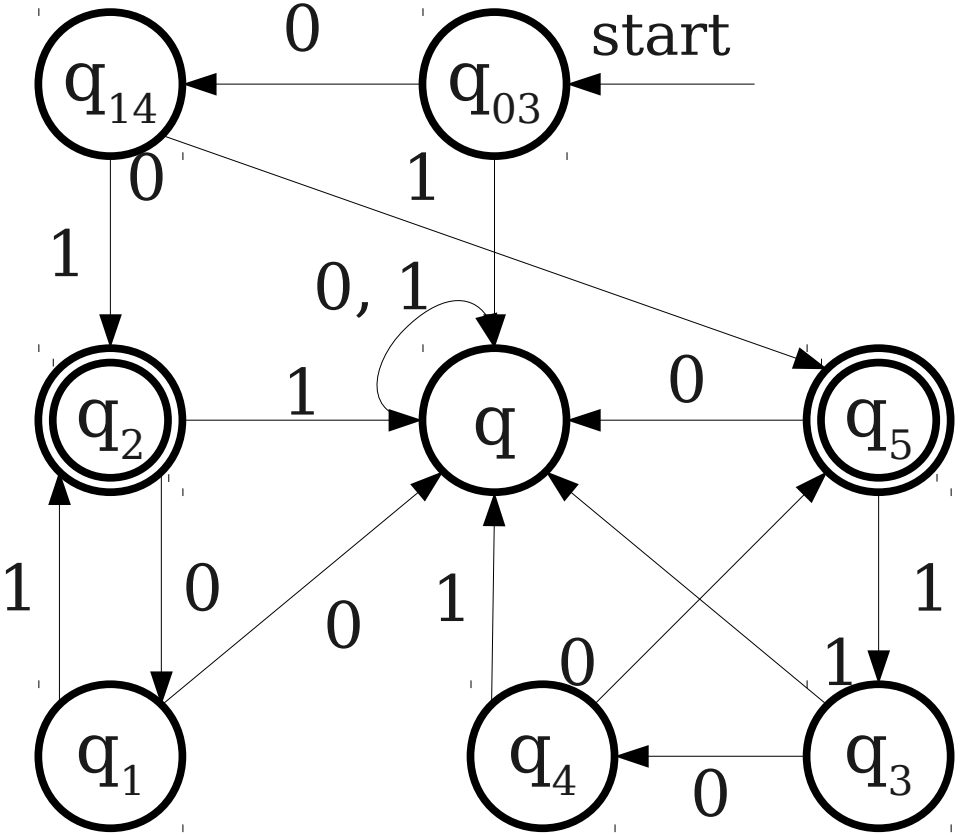

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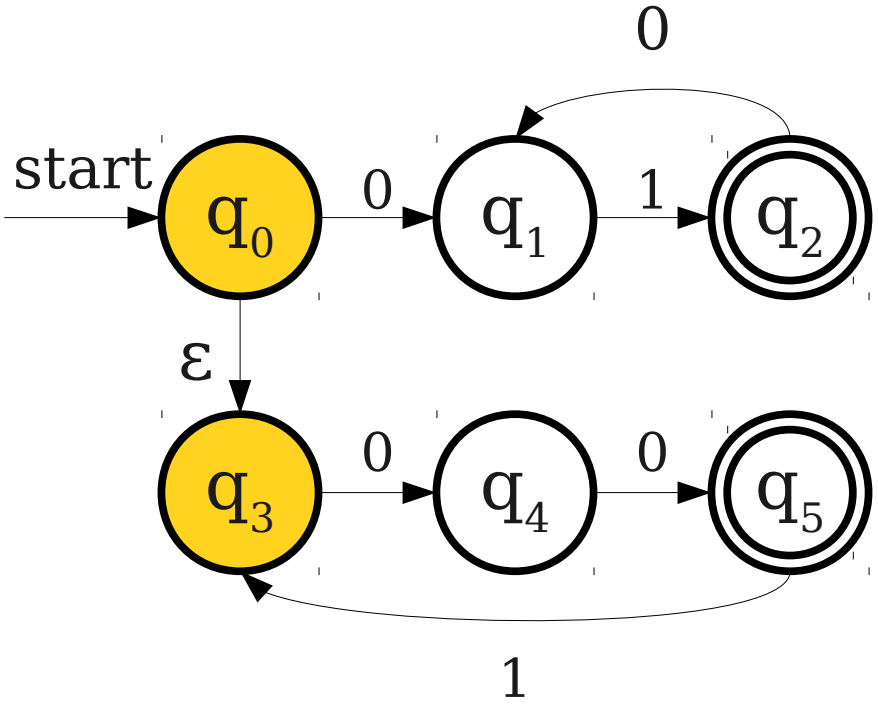

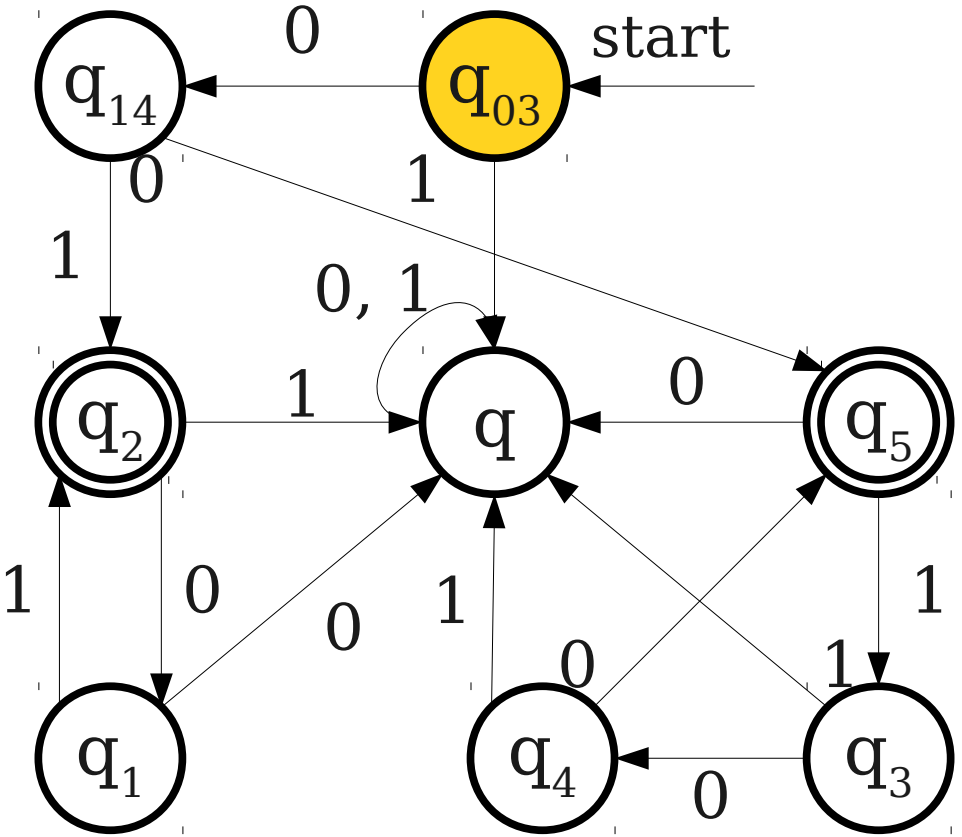

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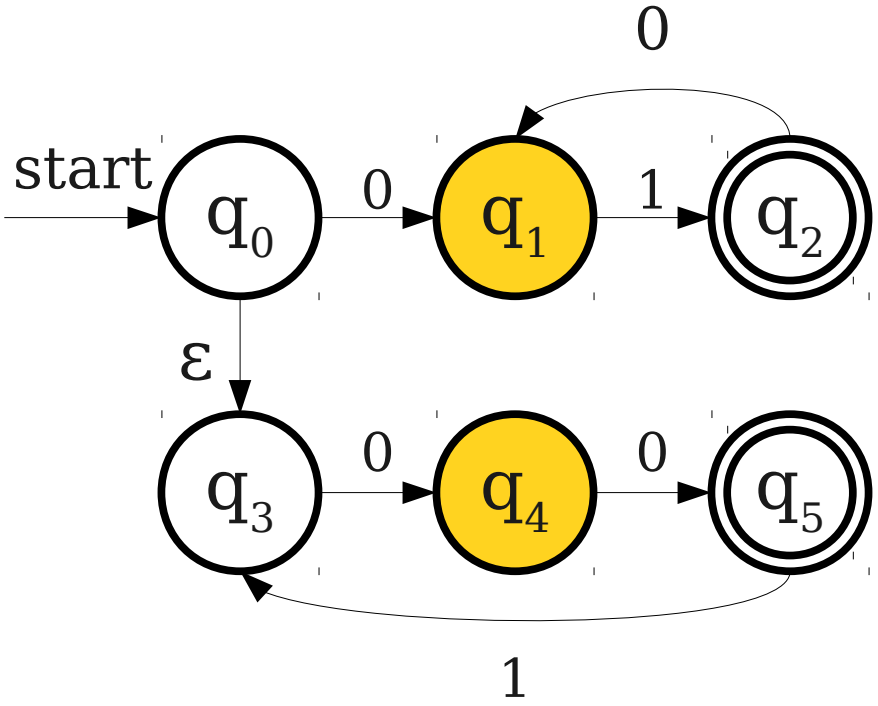

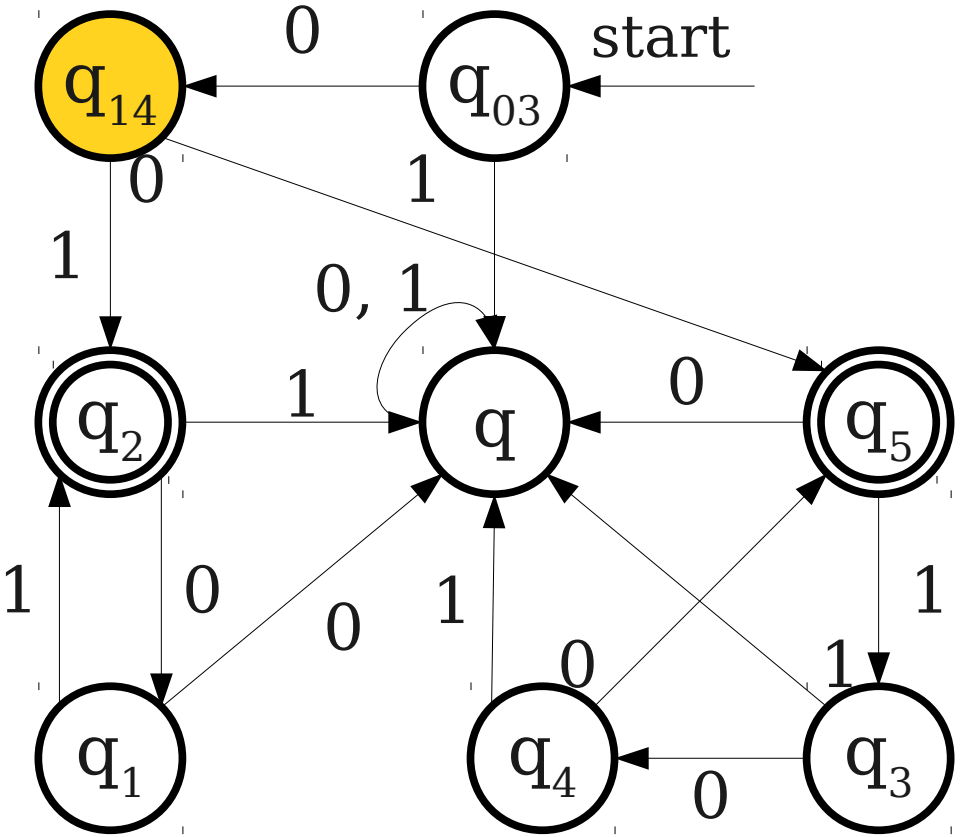

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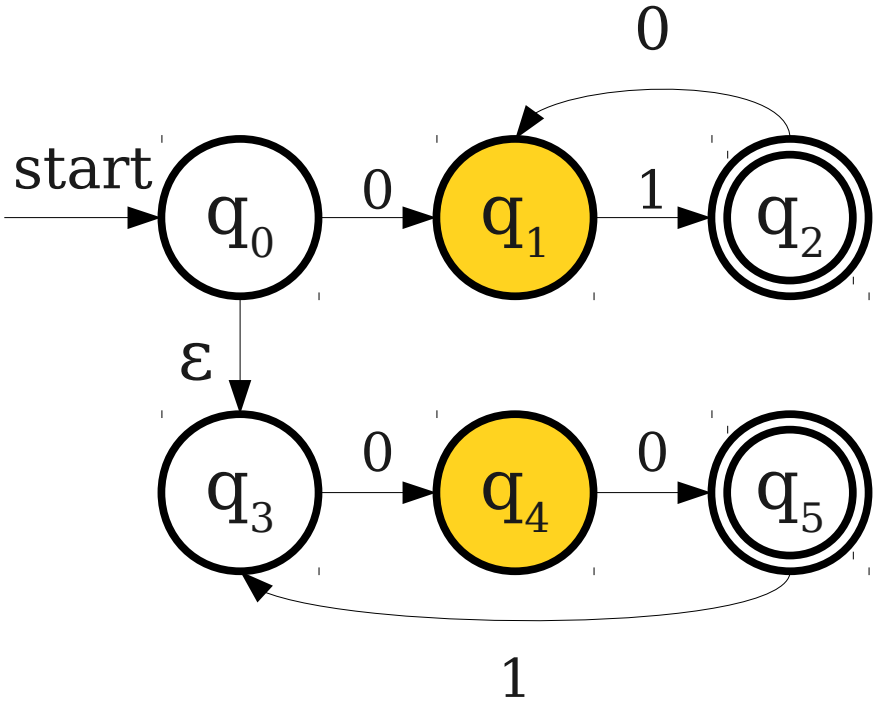

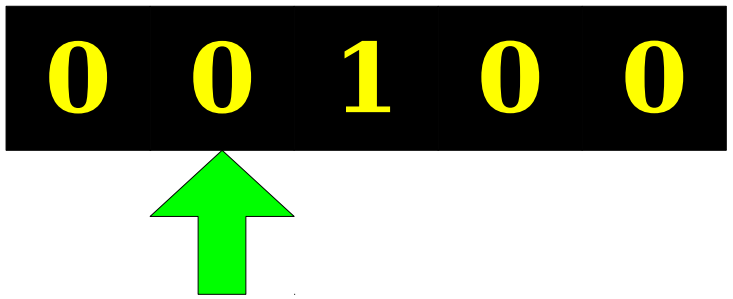

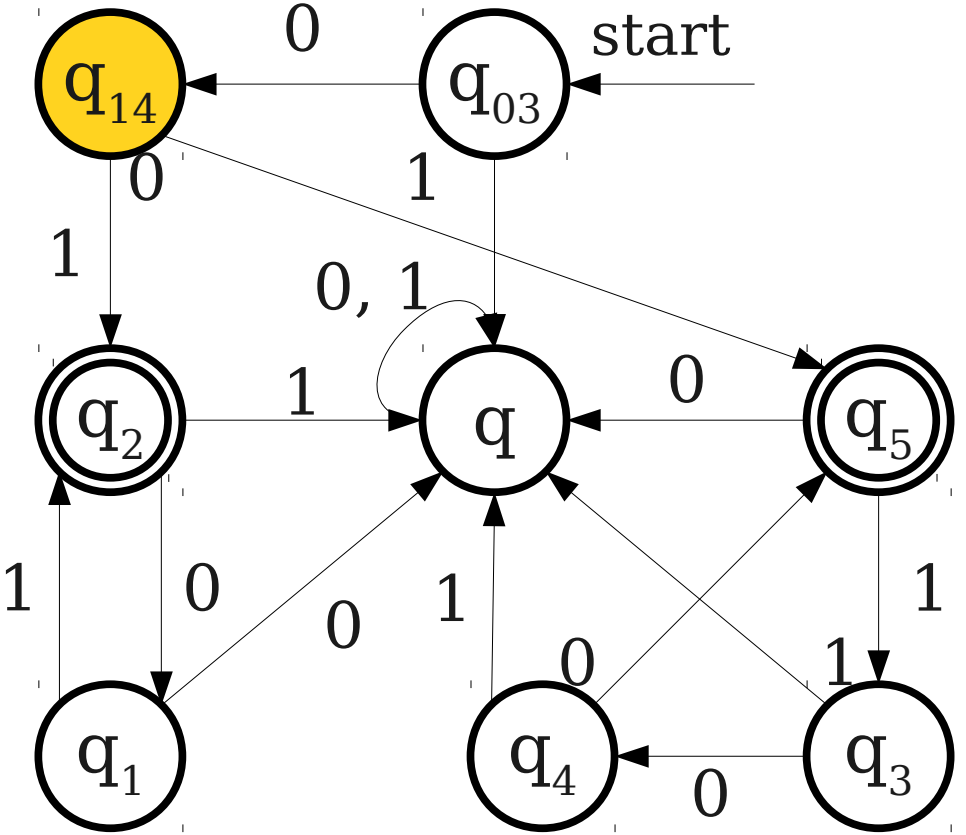

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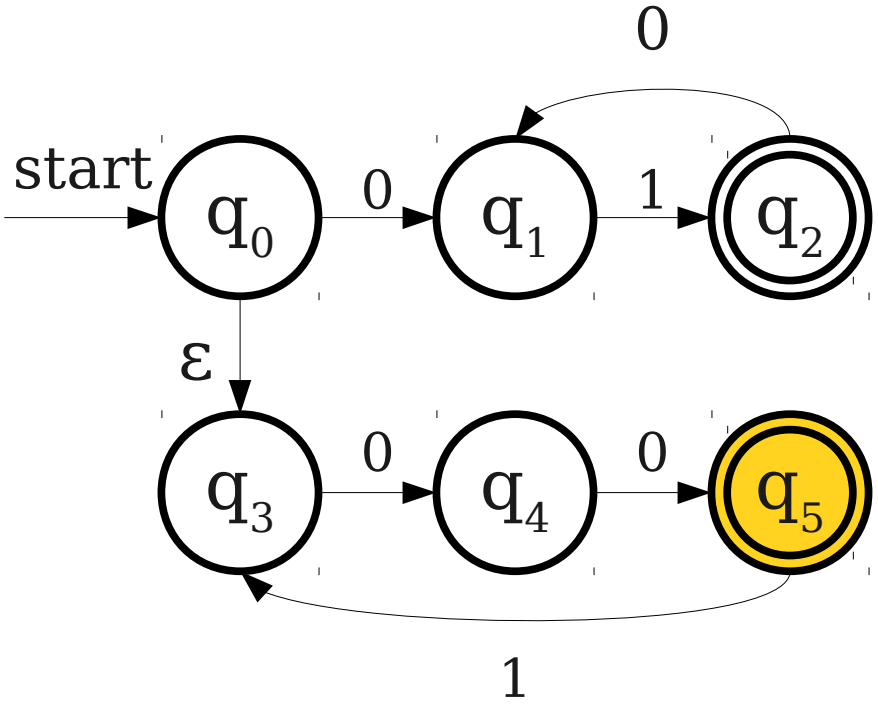

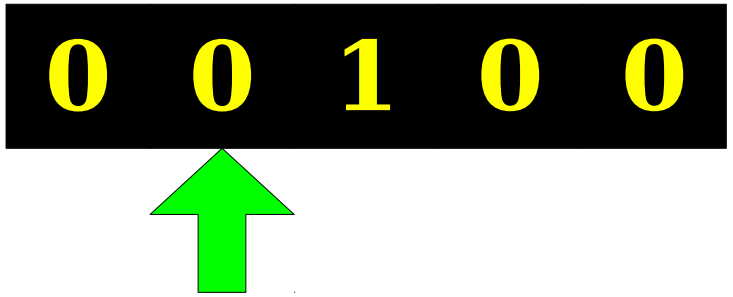

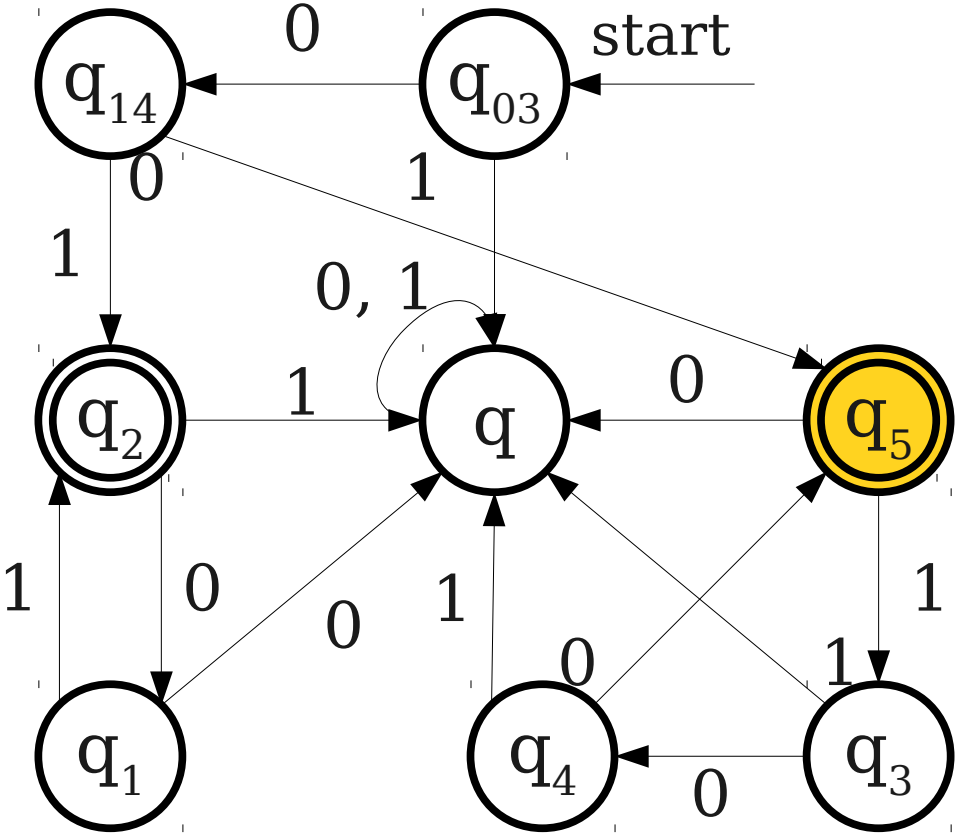

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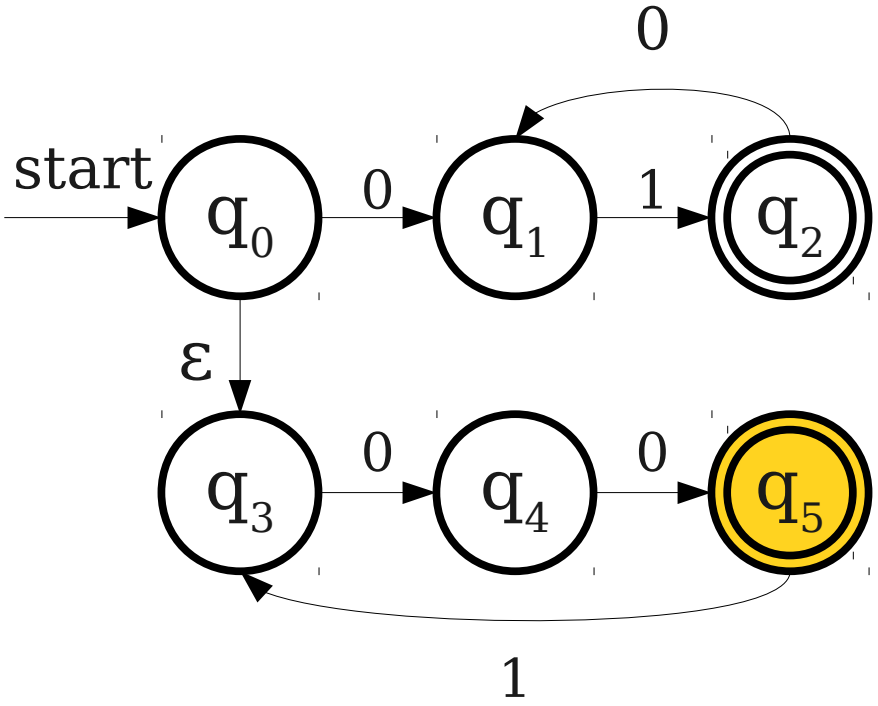

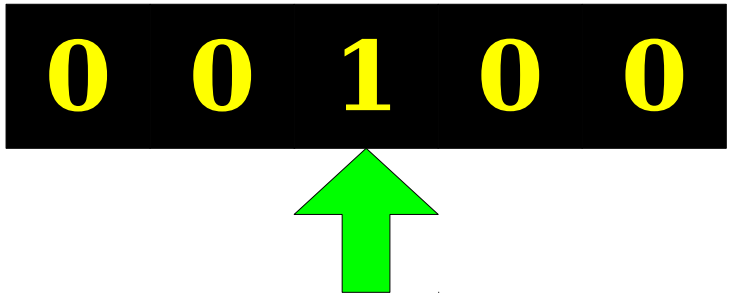

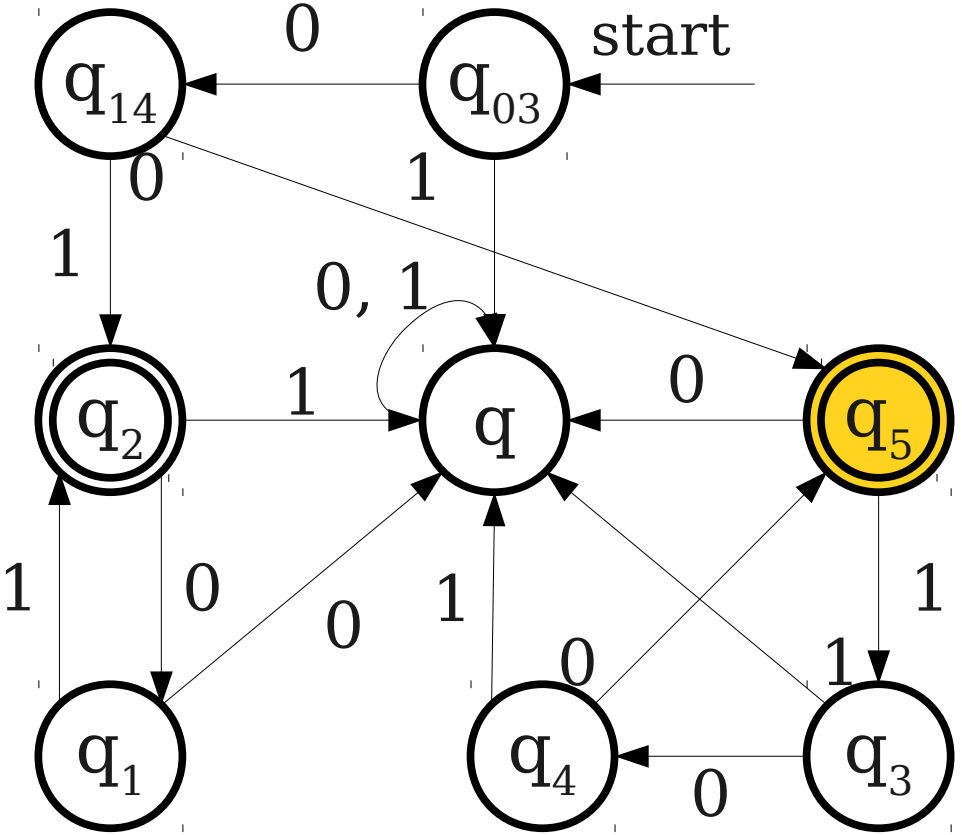

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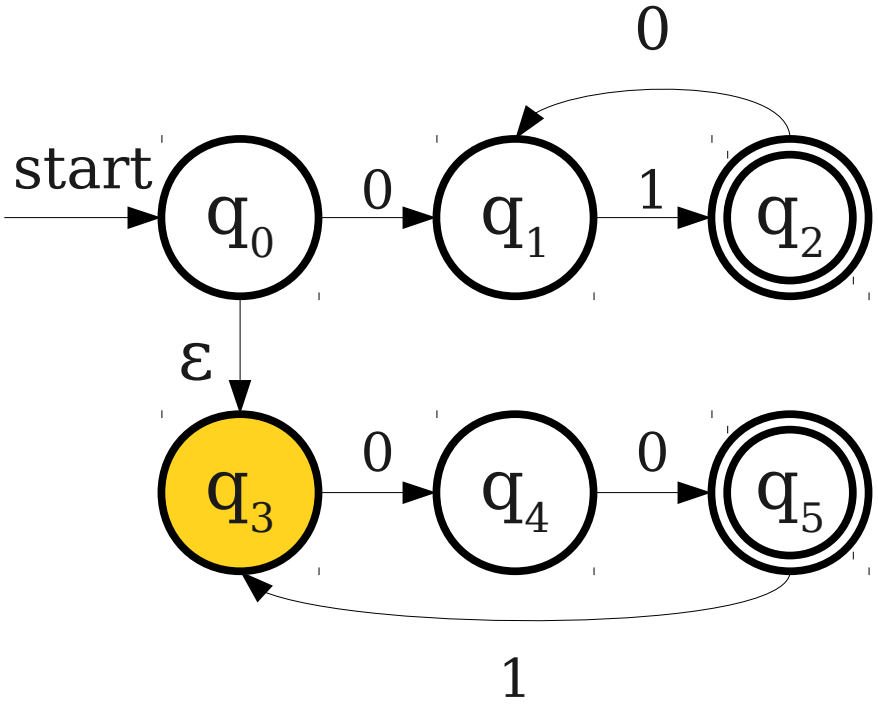

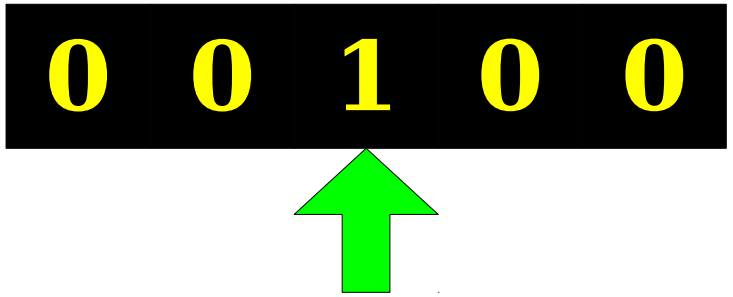

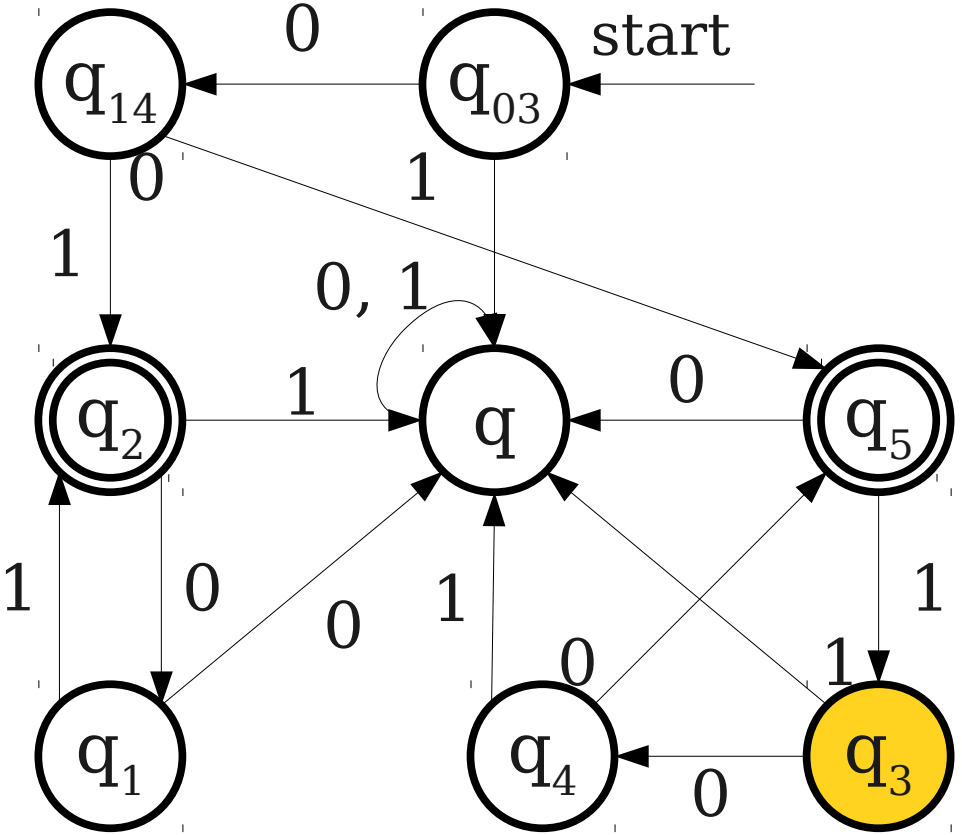

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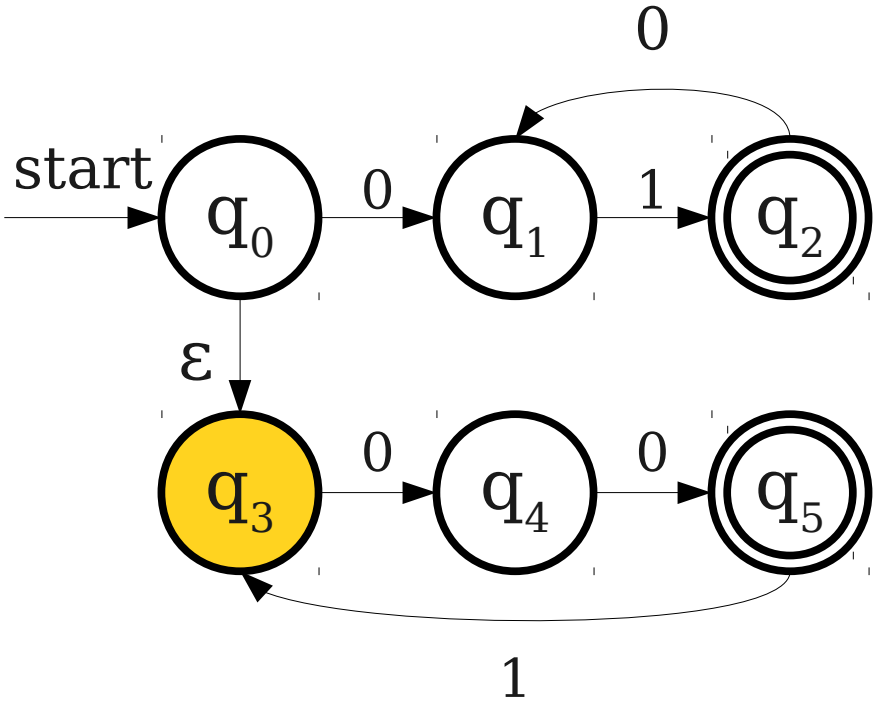

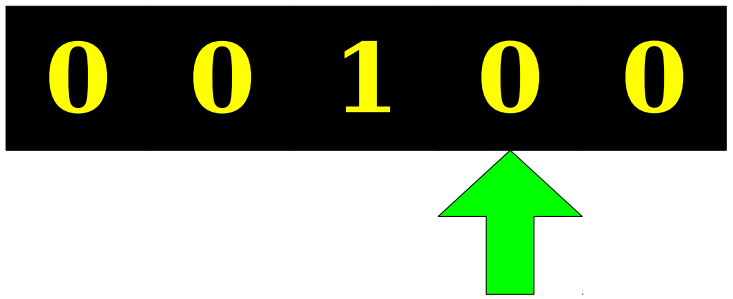

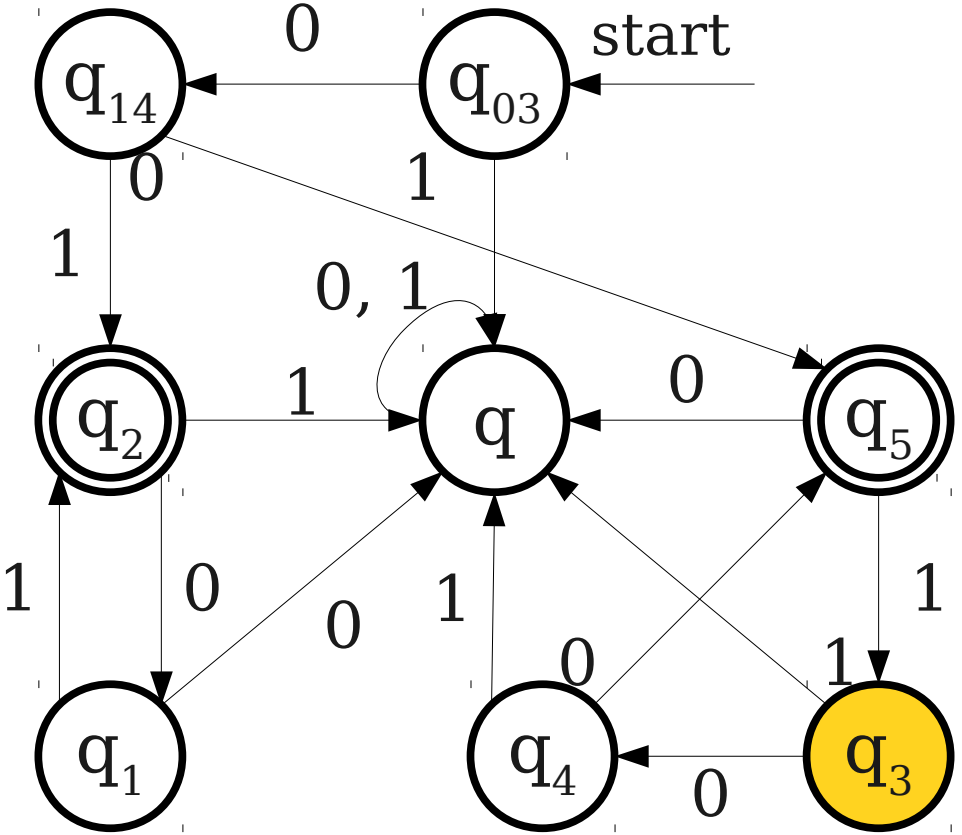

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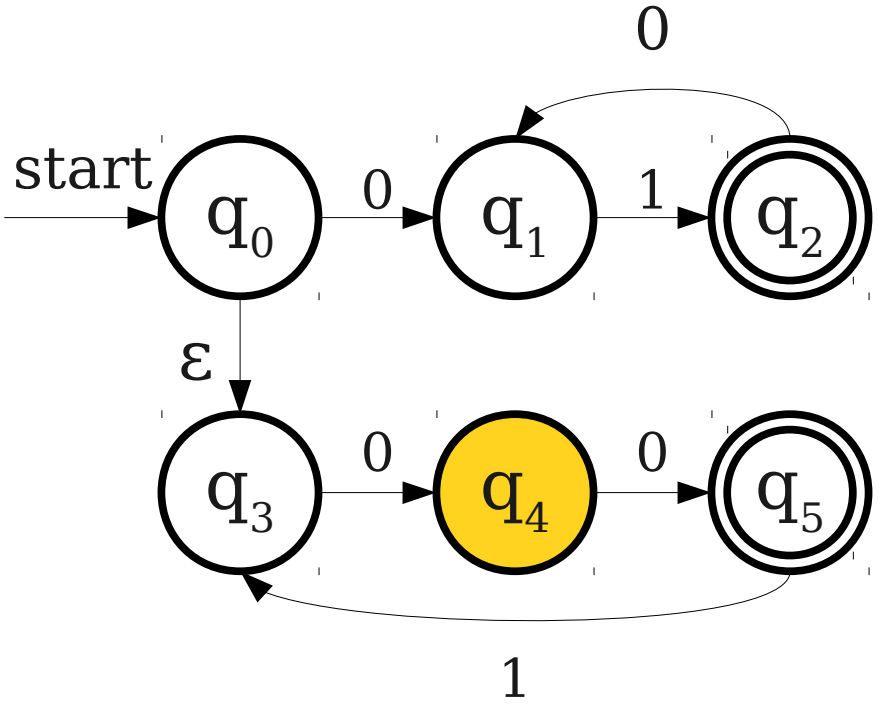

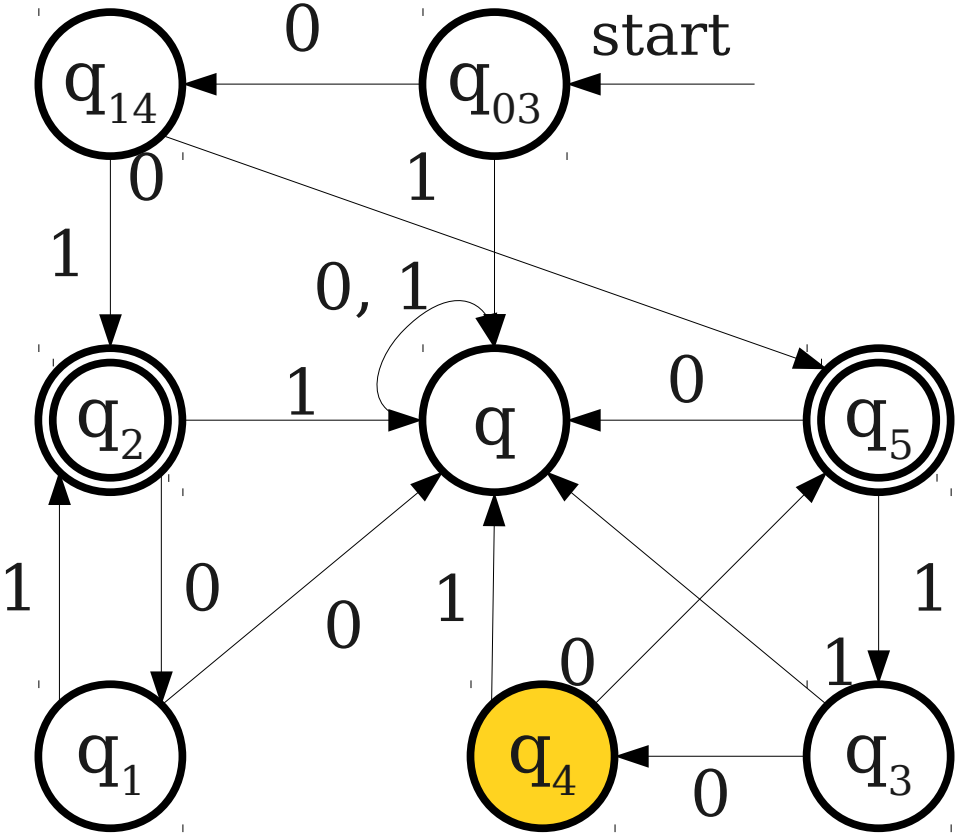

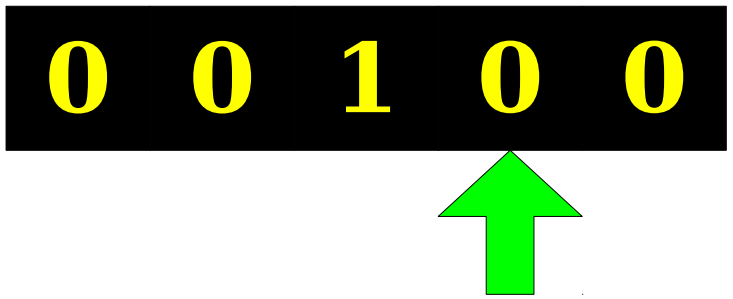

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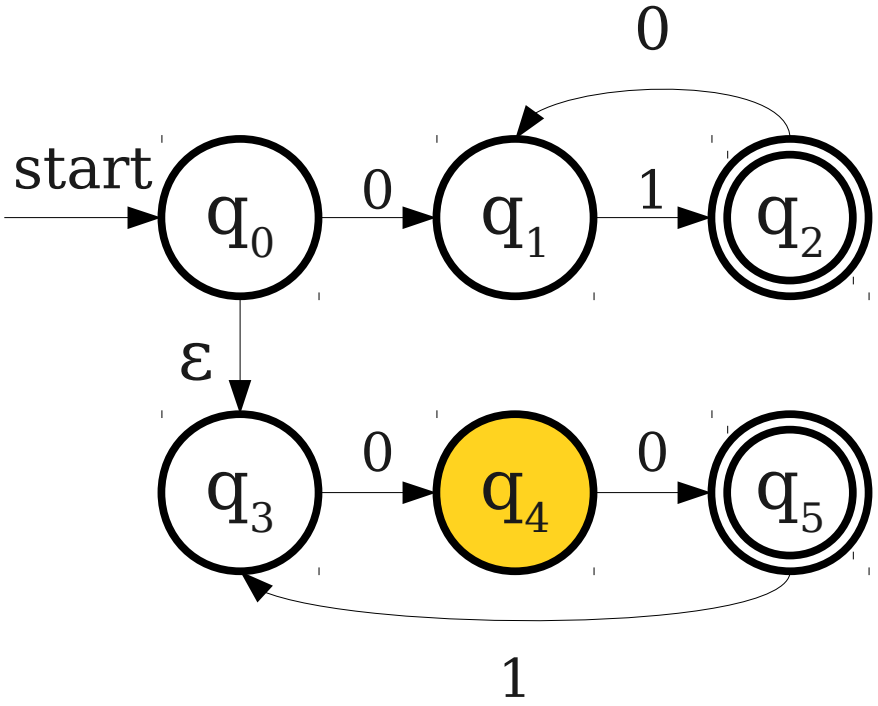

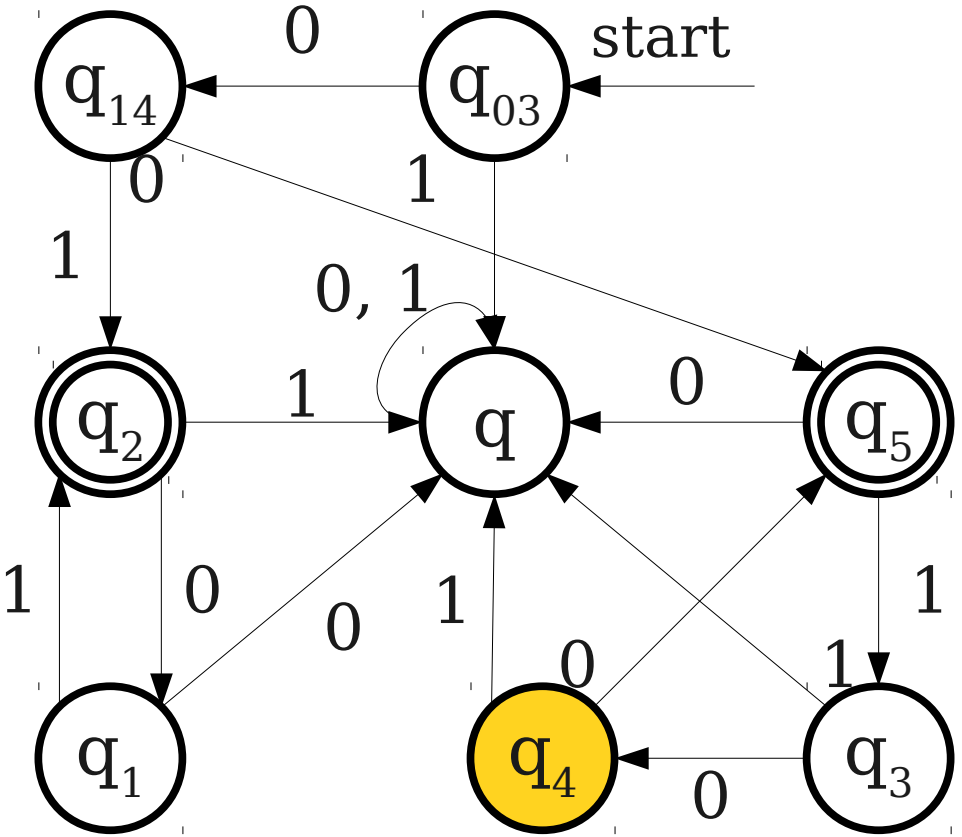

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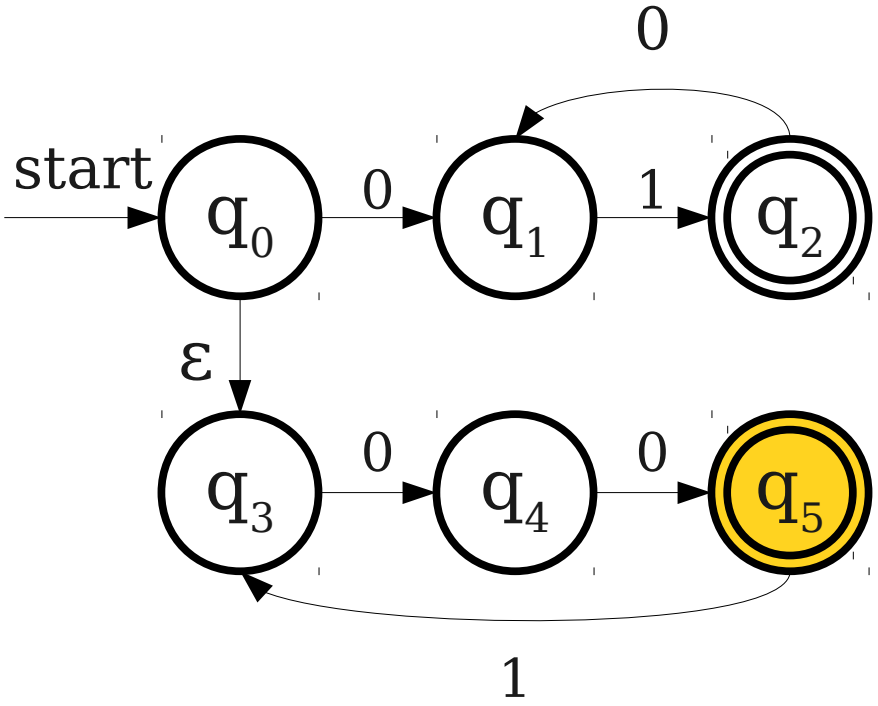

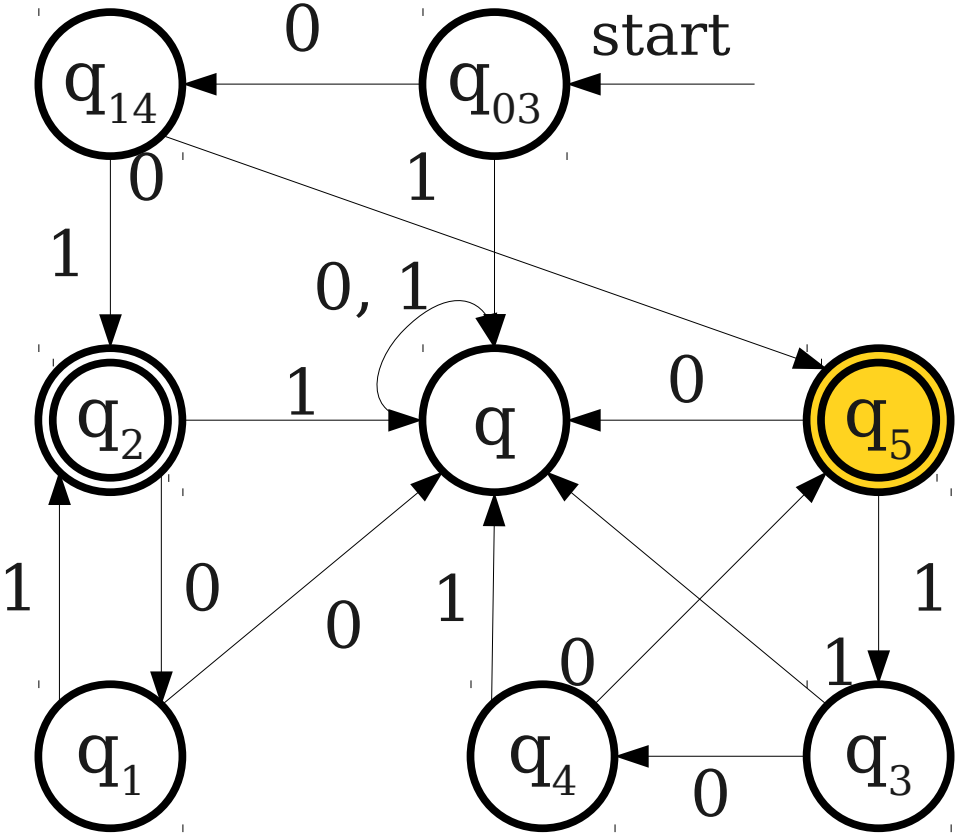

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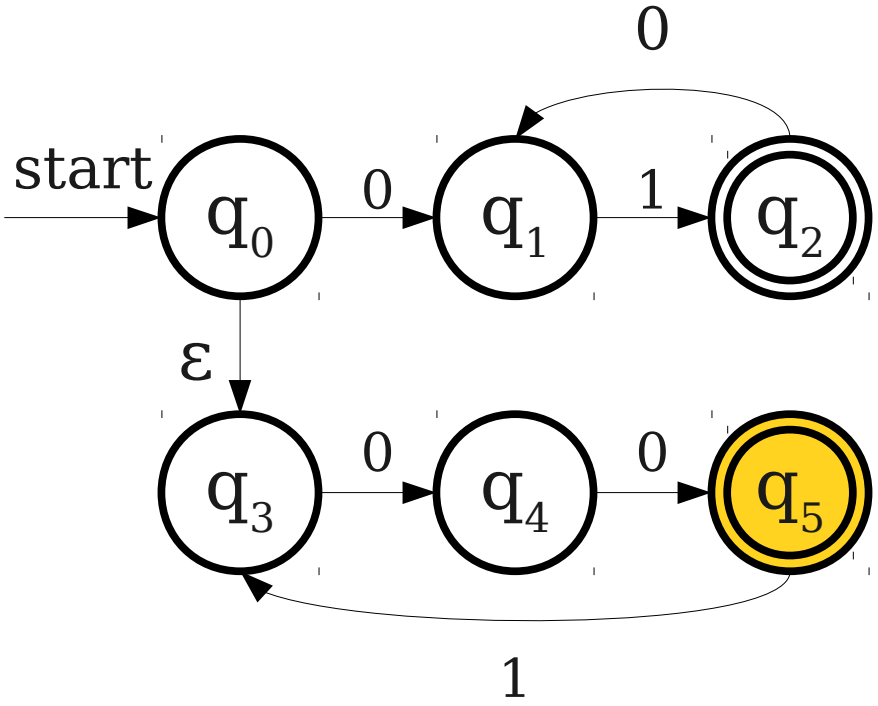

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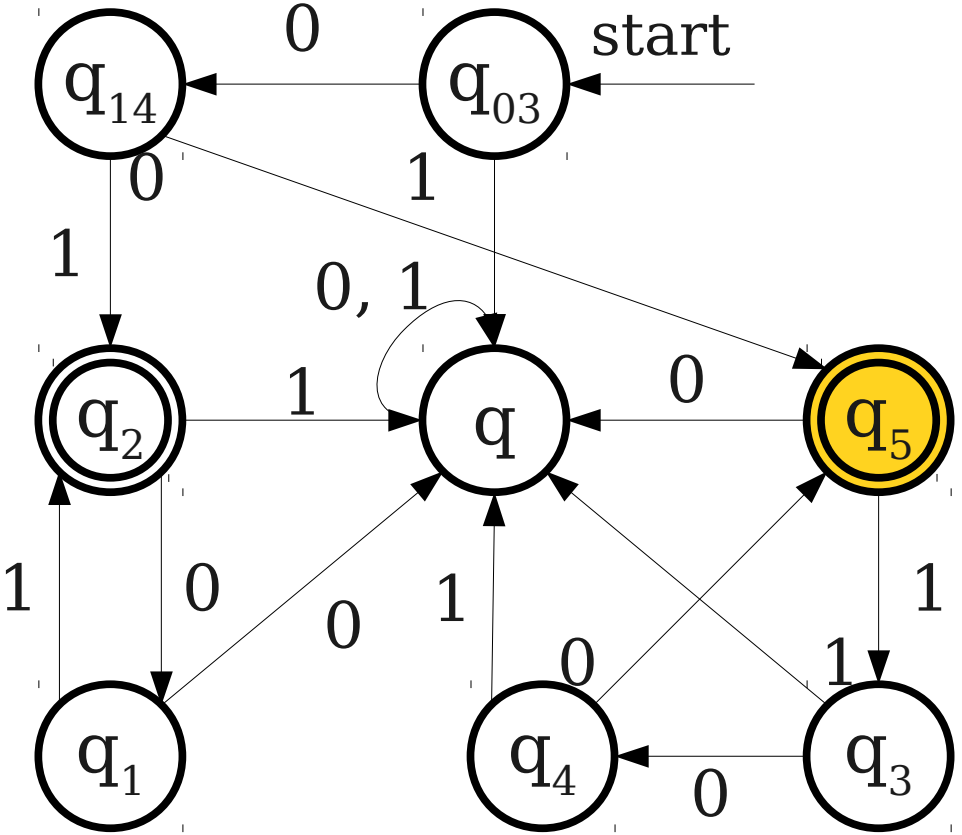

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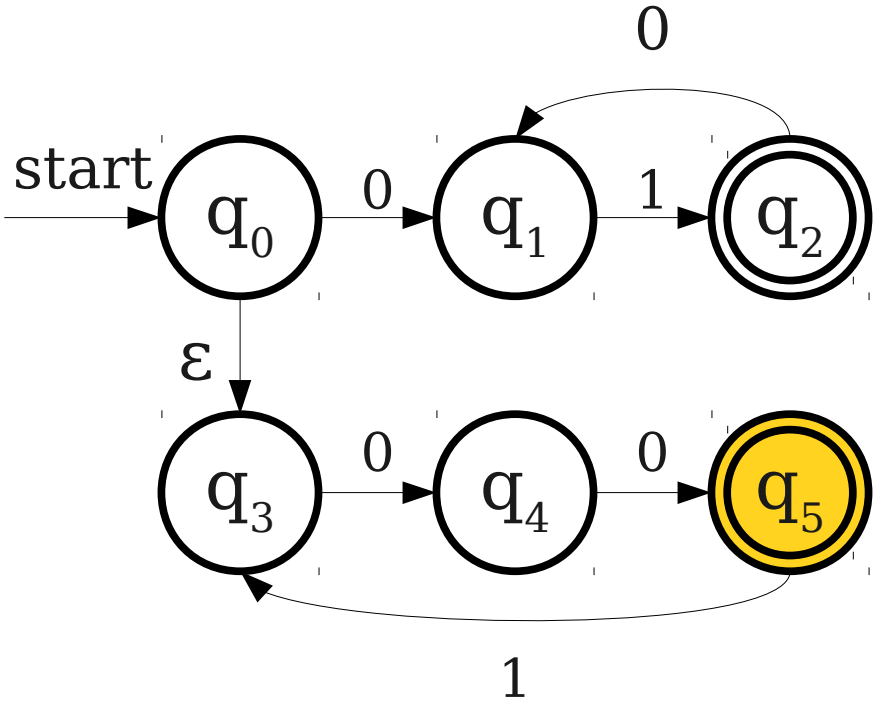

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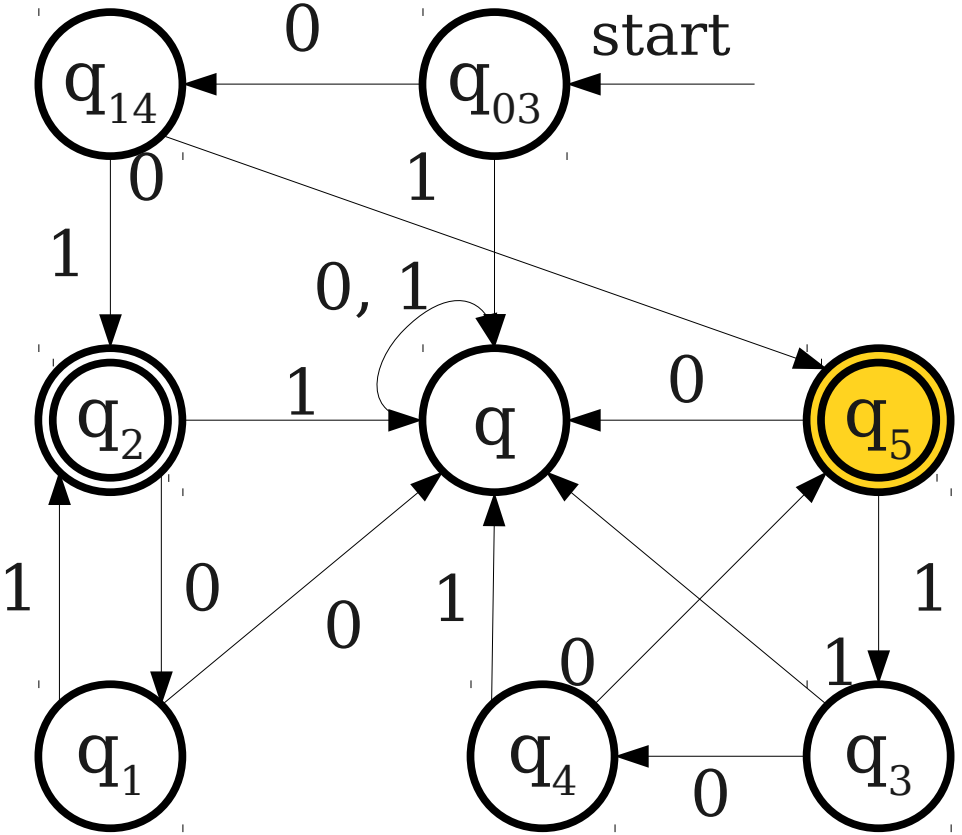

The Subset Construction

- This construction for transforming an NFA into a DFA is called the **subset construction** (or sometimes the **powerset construction**).
  - States of the new DFA correspond to *sets of states* of the NFA.
  - The initial state is the start state, plus all states reachable from the start state via ε-transitions.
  - Transition on state S on character `a` is found by following all possible transitions on `a` for each state in S, then taking the set of states reachable from there by ε-transitions.
  - Accepting states are any set of states where *some* state in the set is an accepting state.
- **Read Sipser for a formal account.**

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- Fact: $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- Interesting challenge: Find a language for which this worst-case behavior occurs (there are infinitely many of them!)

A language L is called a **regular language** iff there exists a DFA D such that $\mathscr{L}(D) = L$.

An Important Result

Theorem: A language L is regular iff there is some NFA N such that $\mathscr{L}(N) = L$.

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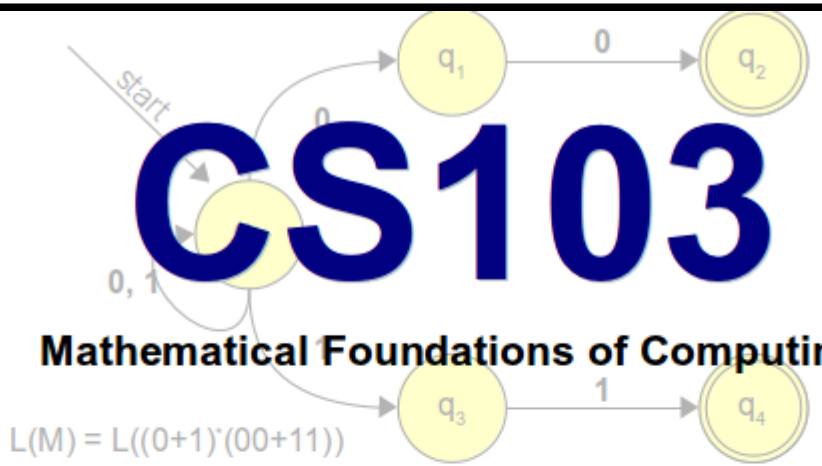
Announcements!



CS103

Mathematical Foundations of Computing

cements

et Five Out
13

Set Five goes out today. This problem set
finite automata, regular languages, and their
s, and will be your first foray into
bility theory. We hope that you enjoy it!

it easier to design and test the automata and

Handouts

00: Course Information
01: Syllabus
02: Problem Set Policies
03: Honor Code
04: Set Theory Definitions
07: Guide to Proofs
14: Practice Midterm 1
14S: Practice Midterm 1 Solns
15: Practice Midterm 2

Resources

Course Reader
Lecture Videos
Theorem and Definition Reference
Office Hours Schedule
Grades
DFA/NFA Designer
Regex Designer

Problem Set Five

- Problem Set Five released, due on **Monday, November 4**.
  - Note the due date is *Monday* rather than *Friday*.
  - Late periods now carry over to *Wednesday* rather than *Monday*.
  - No checkpoint problem.
- Explore finite automata, regular languages, and their properties!

Midterm Logistics

- Midterm is next Tuesday, October 29 from 7PM – 10PM.
  - Covers material up through and including DFAs.
  - Review handout on exam policies and procedures for open-note and limited-computer policies.
- **Alternate exams:** Contact Keith ASAP if you haven't heard back about alternate exams.
- **Review session:** 2:15PM – 4:15PM on Saturday in room **370-370**.
- Have questions for the review session: ask them on Google Moderator!

Your Questions

“I am having trouble being confident in my first order logic translations. Are there ways to self check the translation?

Also, is it possible to release some more English-to-first-order-logic translation problems as practice?”

“Diagonalization is a really cool and powerful proof technique, but are there other ways to show that that infinite sets have different cardinalities? What happens if the problem does not easily lend itself to diagonal arguments?”

Back to Automata...

Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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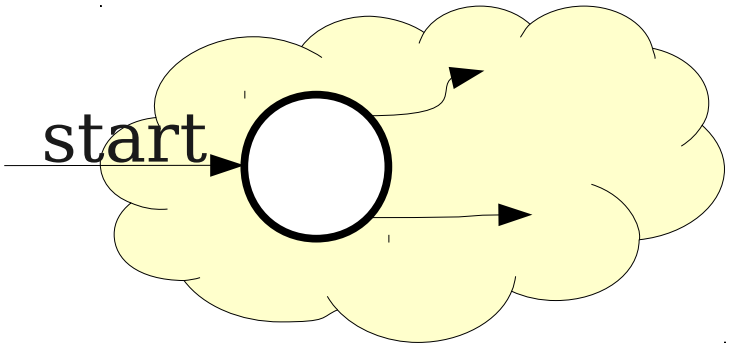
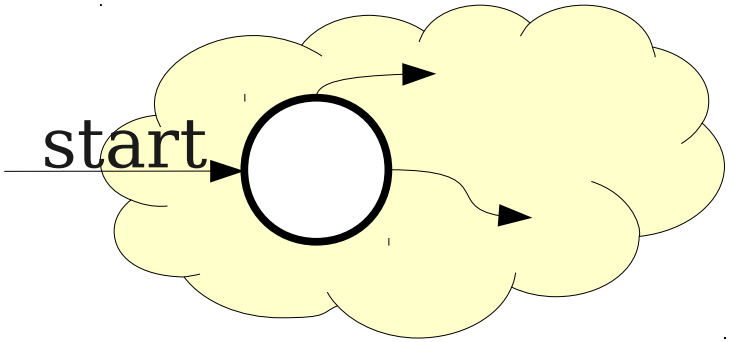


Machine for L_1



Machine for L_2

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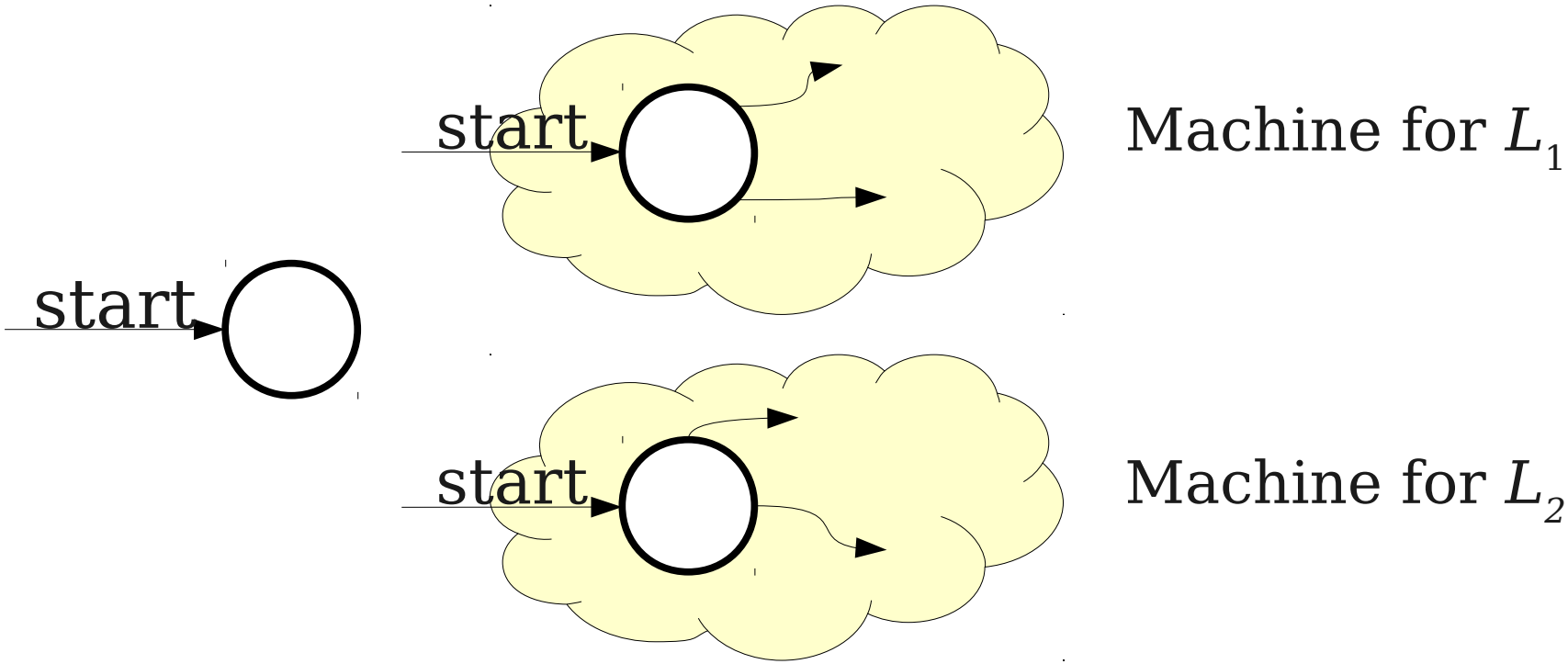

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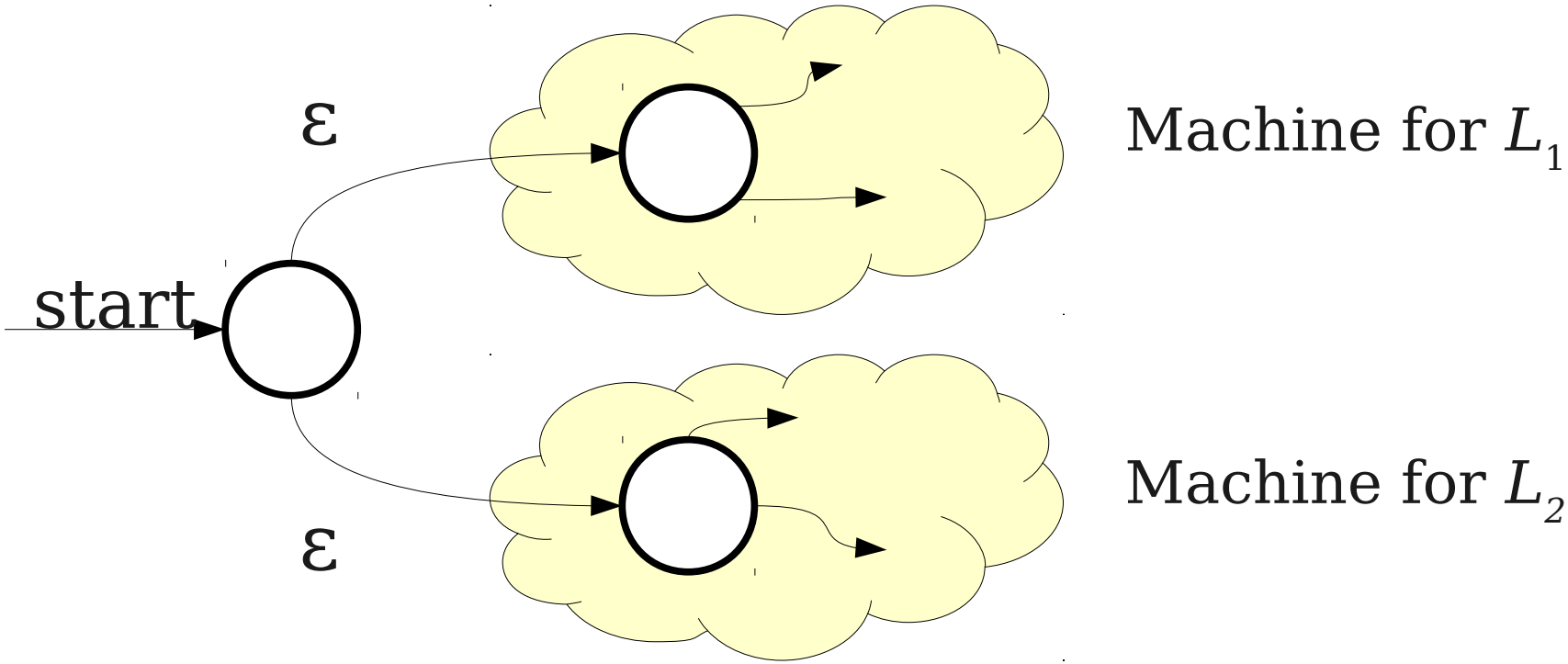

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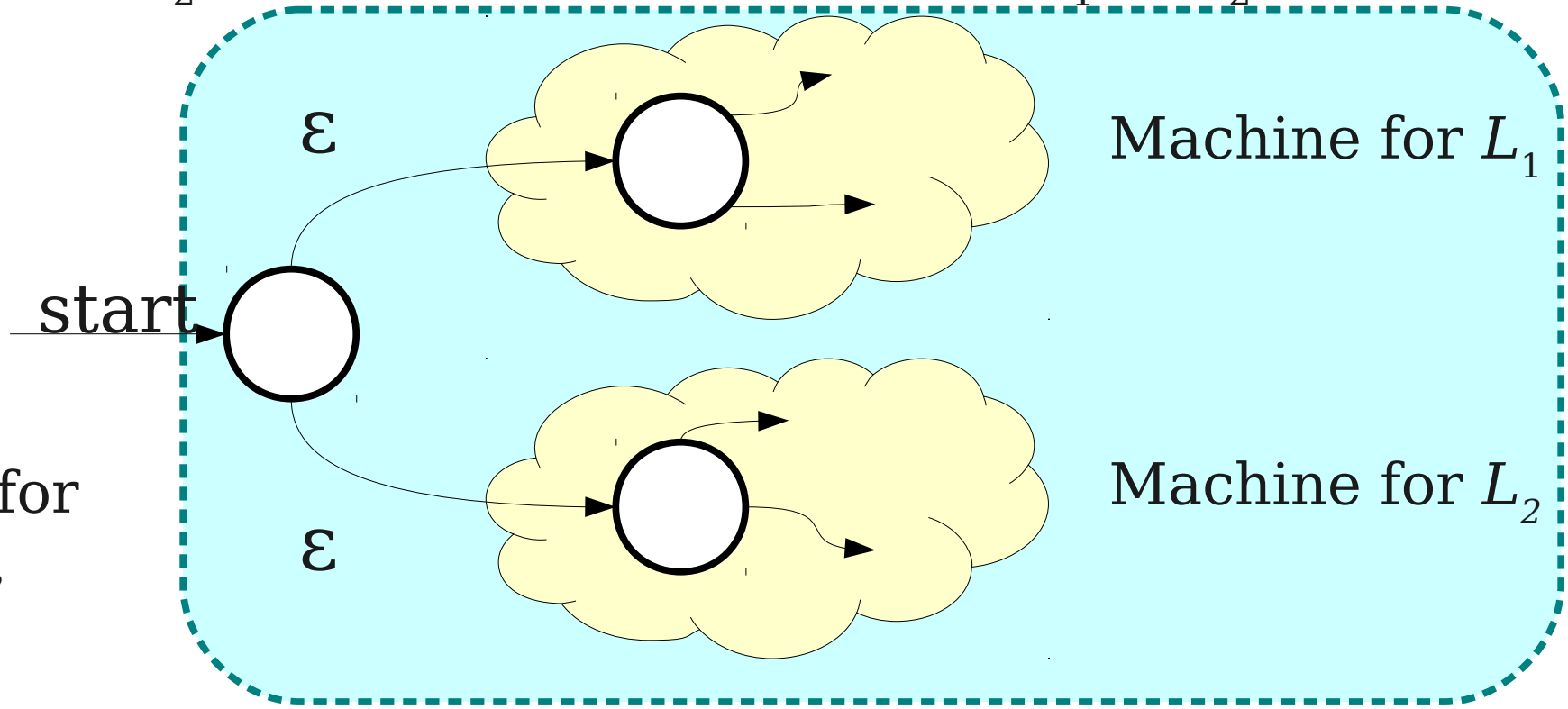

Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

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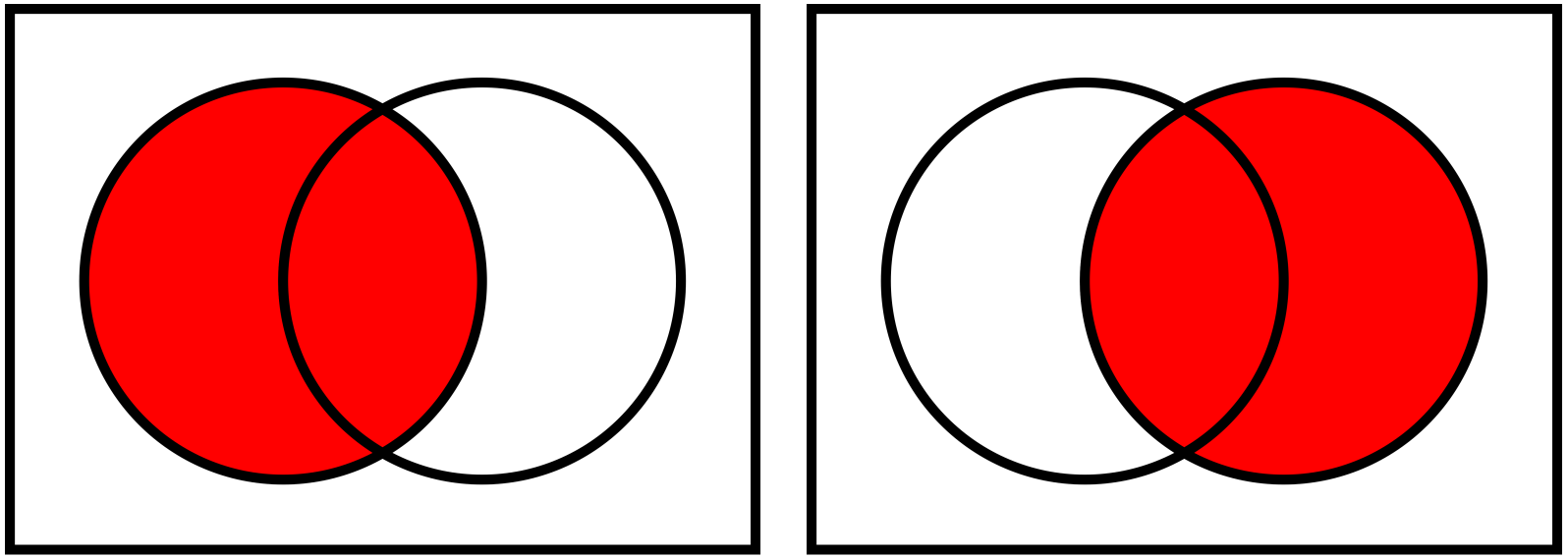

L_1 L_2

The Intersection of Two Languages

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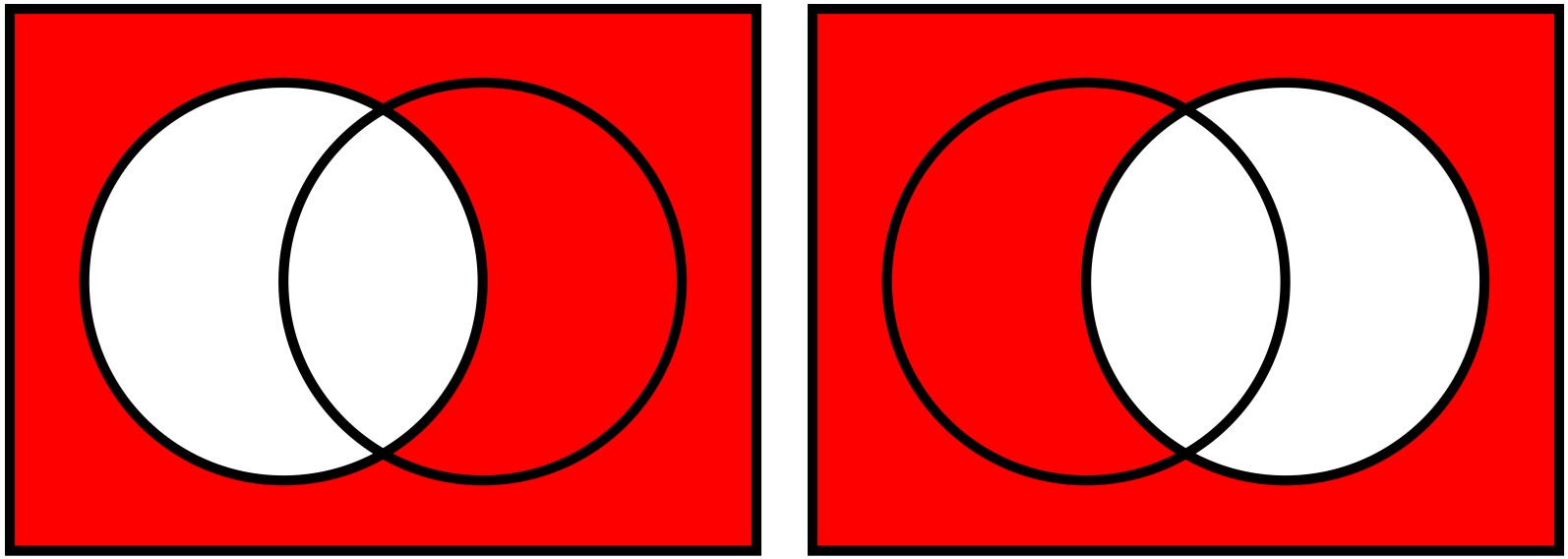

$\overline{L}_1$ $\overline{L}_2$

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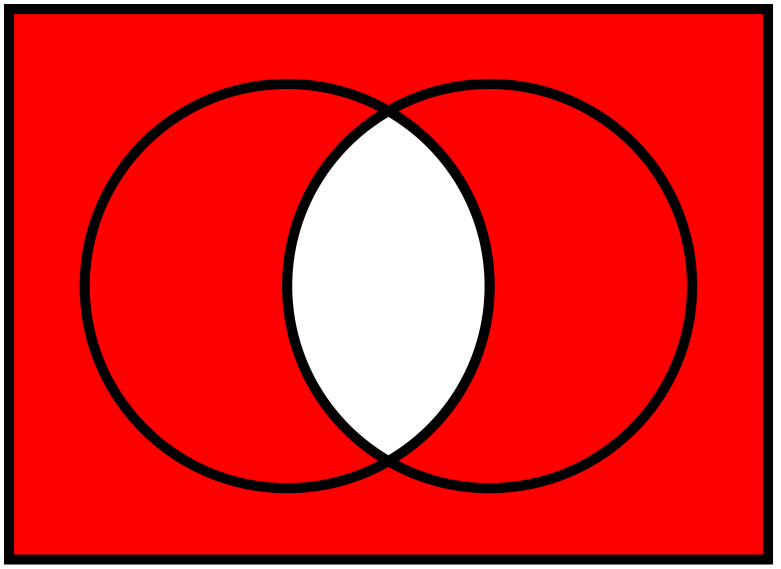

$$\overline{L}_1 \cup \overline{L}_2$$

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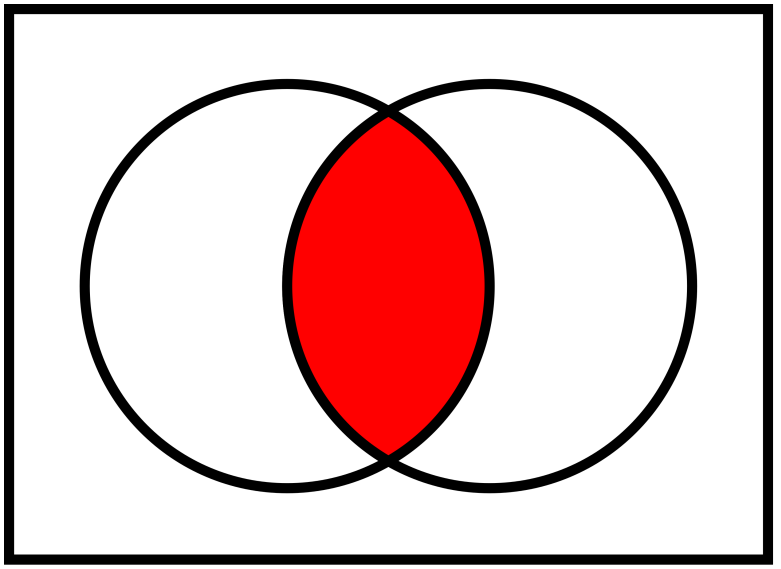

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

- The **concatenation** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \wedge x \in L_2 \}$$

- The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.
- Conceptually similar to the Cartesian product of two sets, only with strings.

Concatenation Example

- Let Σ = { **a**, **b**, …, **z**, **A**, **B**, …, **Z** } and consider these languages over Σ:
  - ***Noun*** = { **Puppy**, **Rainbow**, **Whale**, … }
  - ***Verb*** = { **Hugs**, **Juggles**, **Loves**, … }
  - ***The*** = { **The** }
- The language ***TheNounVerbTheNoun*** is

  { **ThePuppyHugsTheWhale**,
  TheWhaleLovesTheRainbow,
  TheRainbowJugglesTheRainbow, … }

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?

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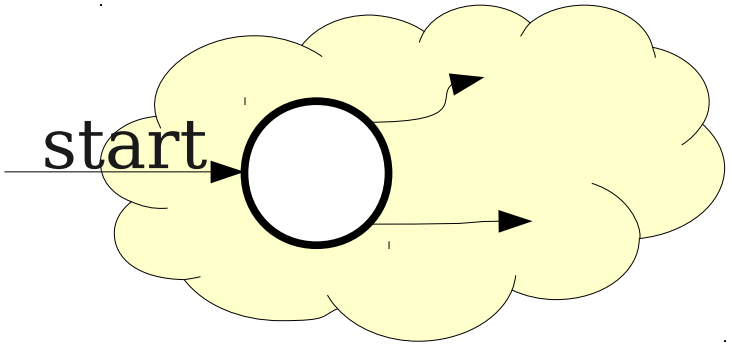
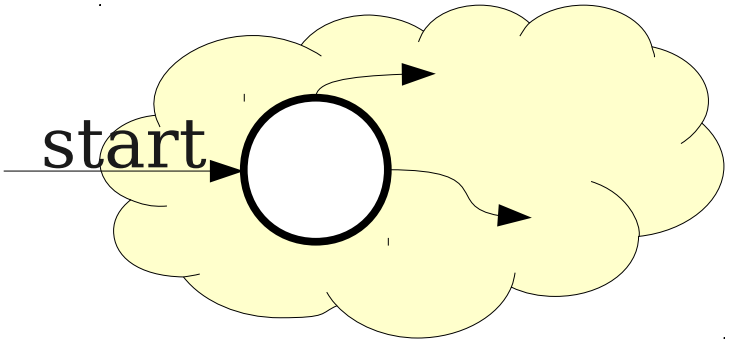


Machine for L_1



Machine for L_2

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
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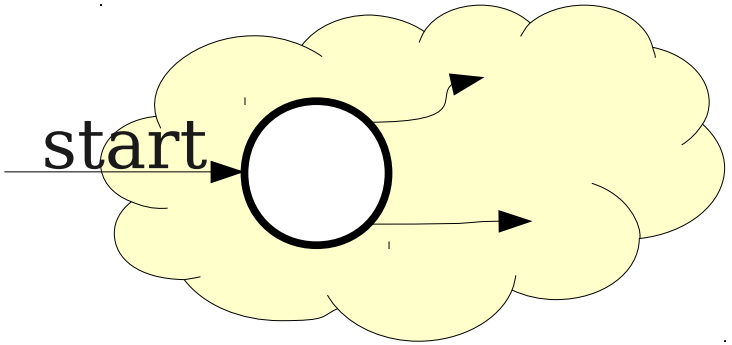
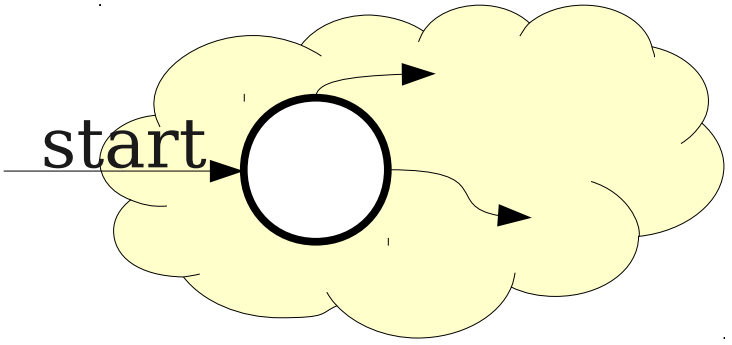


Machine for L_1



Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
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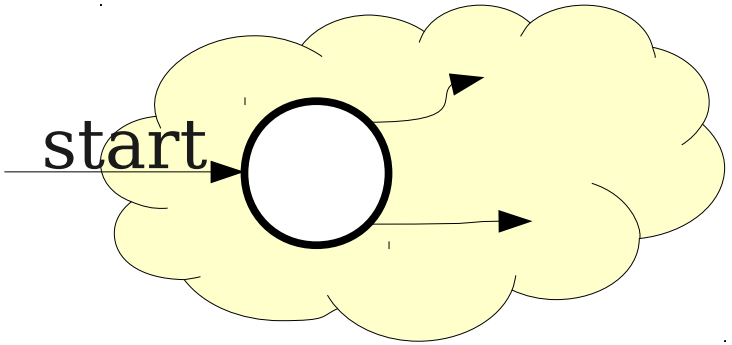

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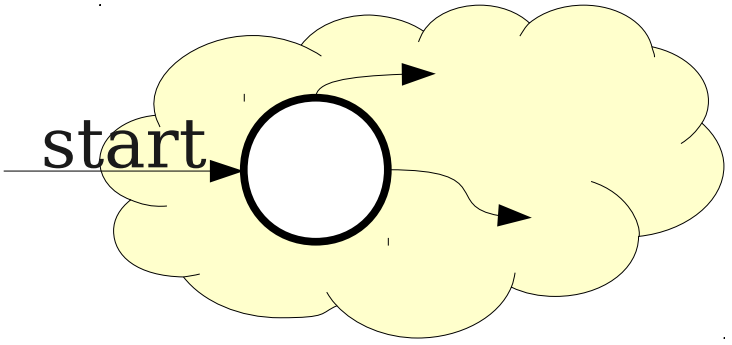

Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

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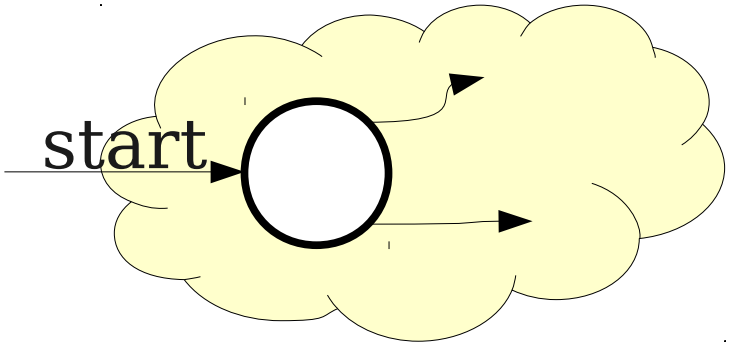
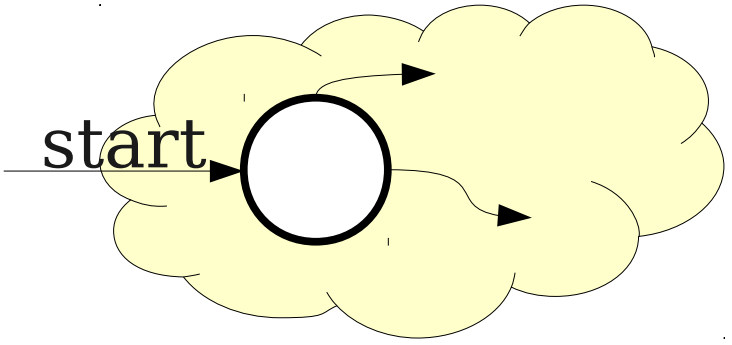


Machine for L_1



Machine for L_2

b	o	o	k

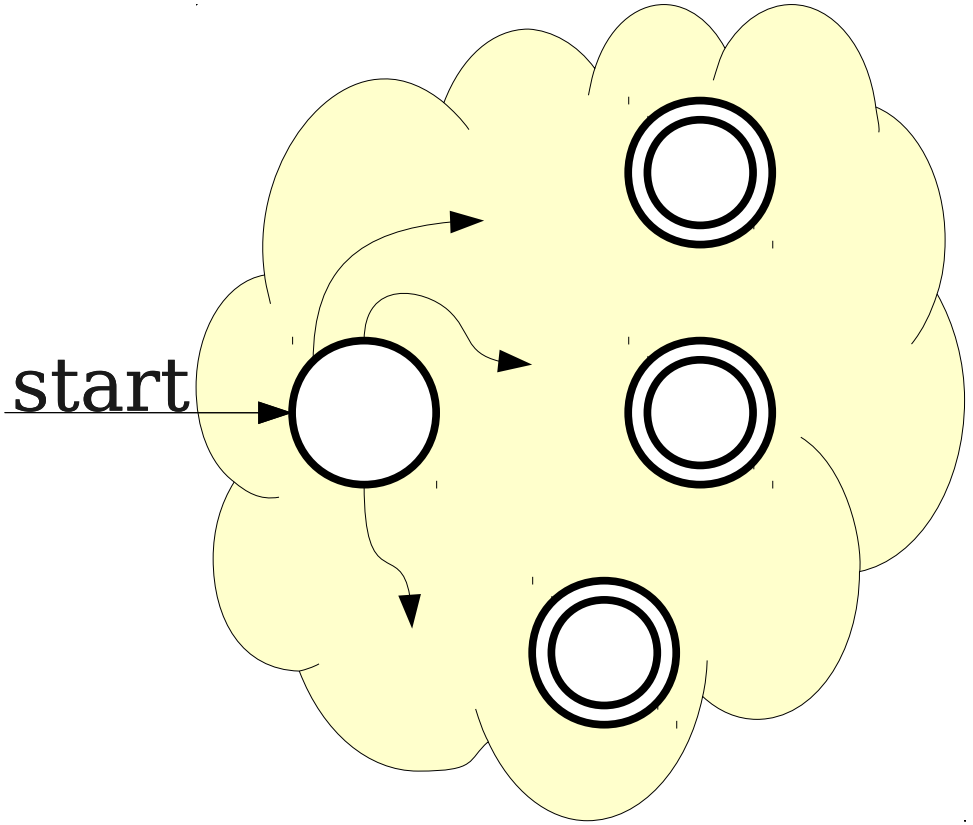
k	e	e	p	e	r

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- **Idea**: Run the automaton for L_1 on w, and whenever L_1 reaches an accepting state, optionally hand the rest off w to L_2.
  - If L_2 accepts the remainder, then L_1 accepted the first part and the string is in L_1L_2.
  - If L_2 rejects the remainder, then the split was incorrect.

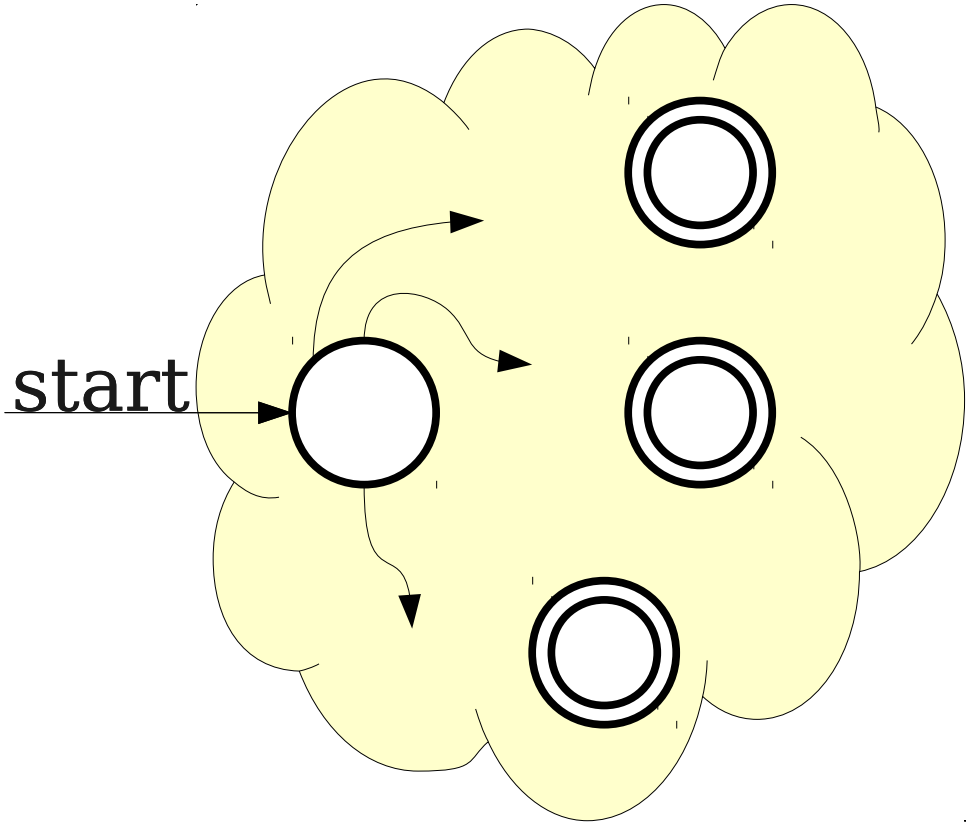
Concatenating Regular Languages

Concatenating Regular Languages

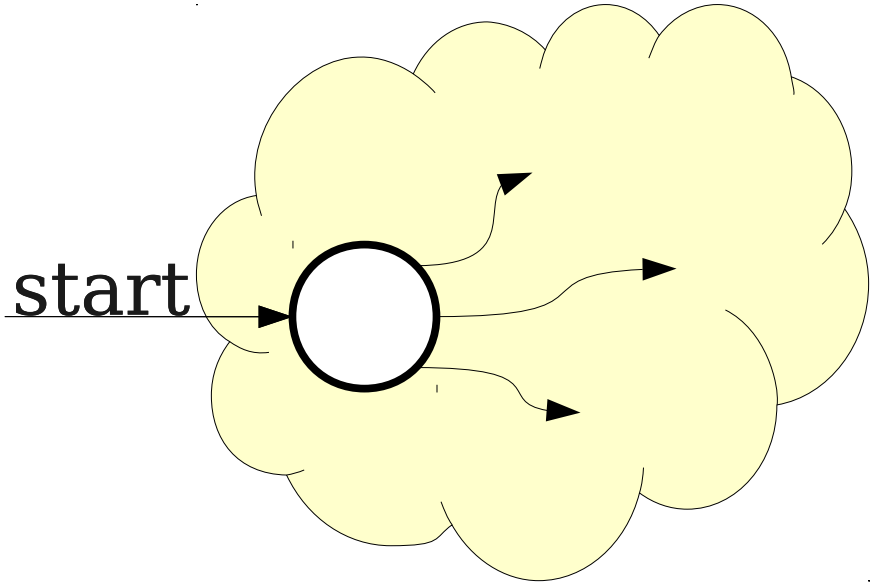


Machine for L_1

Concatenating Regular Languages

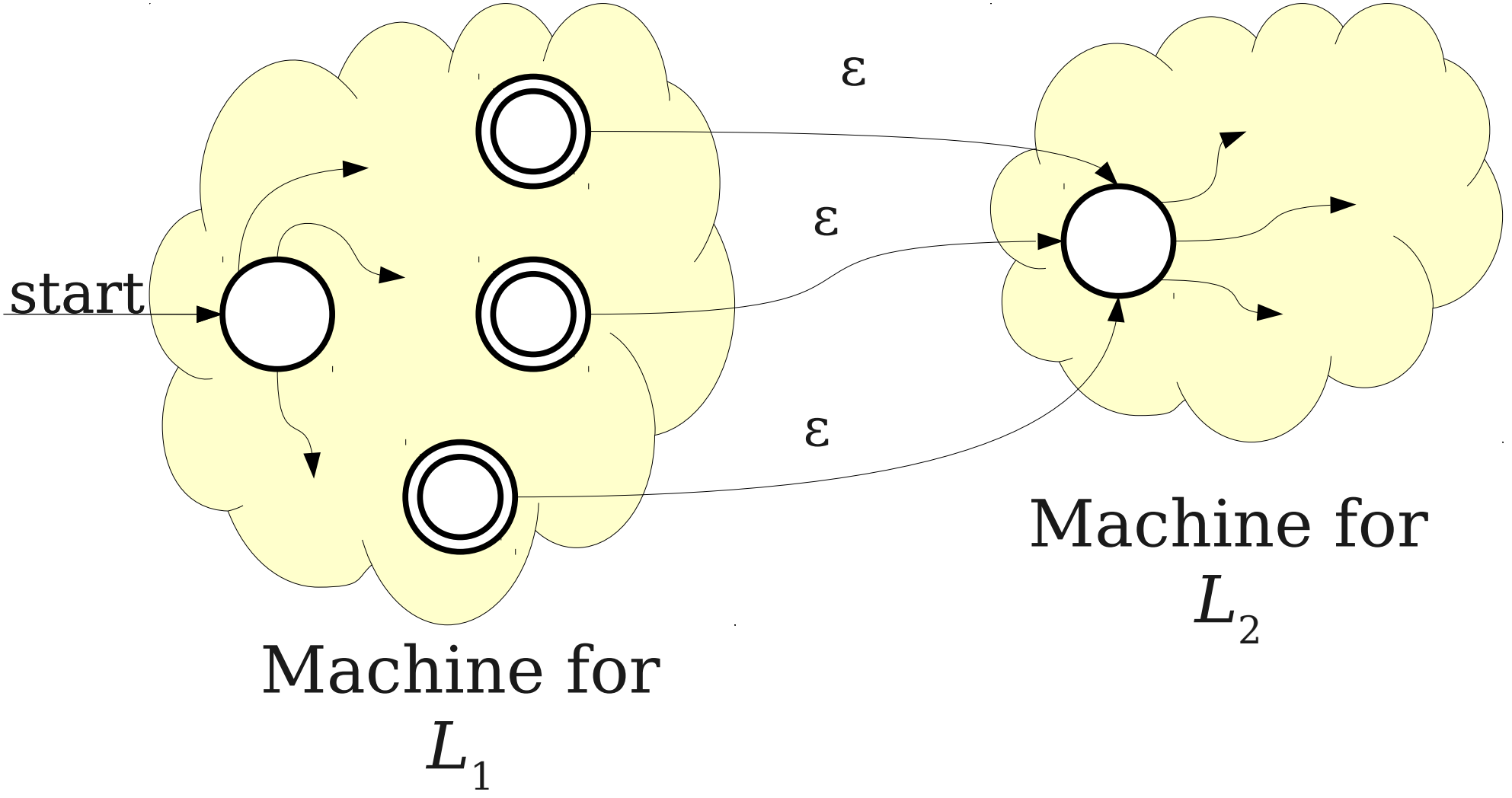


Machine for L_1



Machine for L_2

Concatenating Regular Languages



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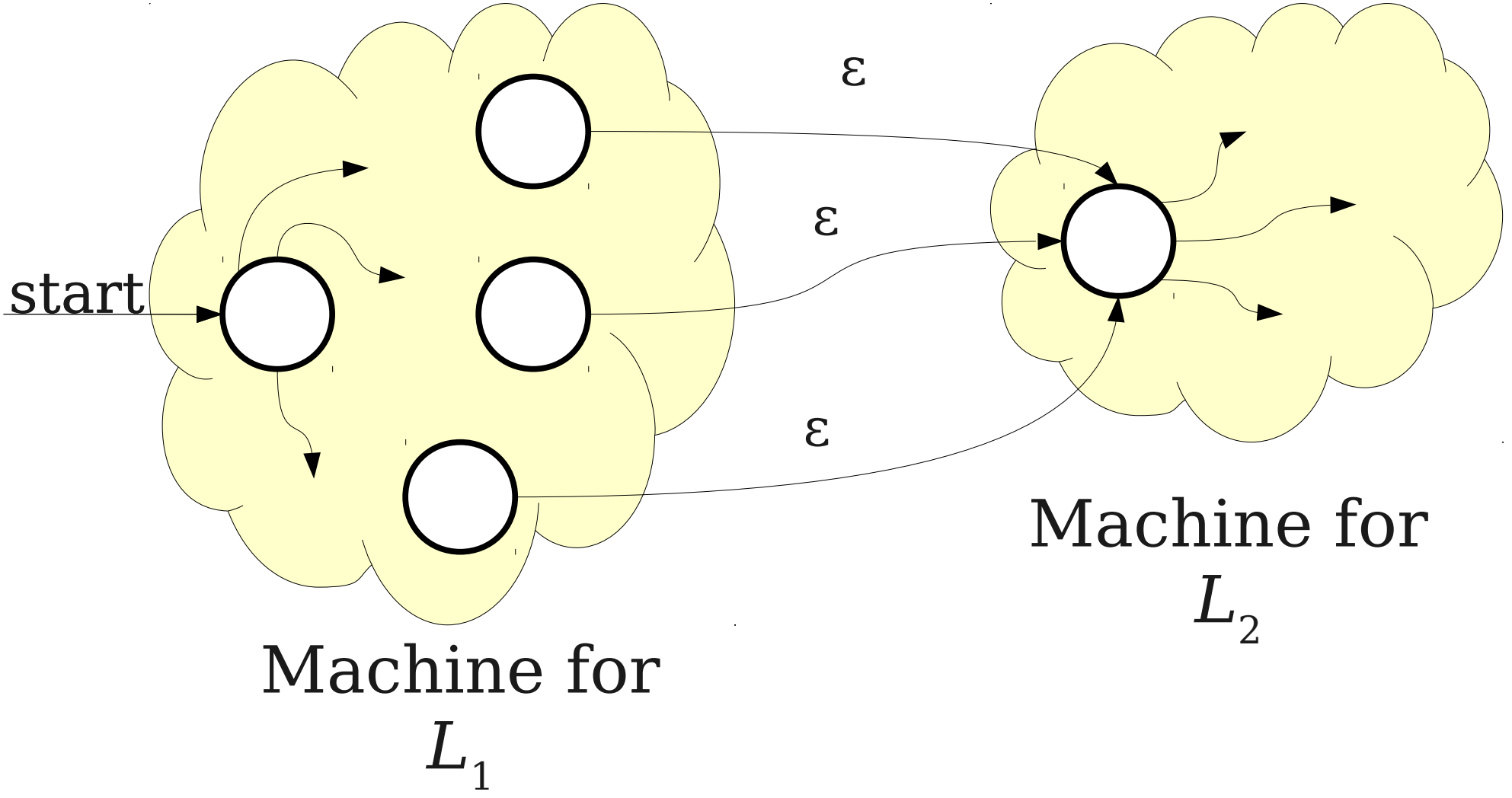
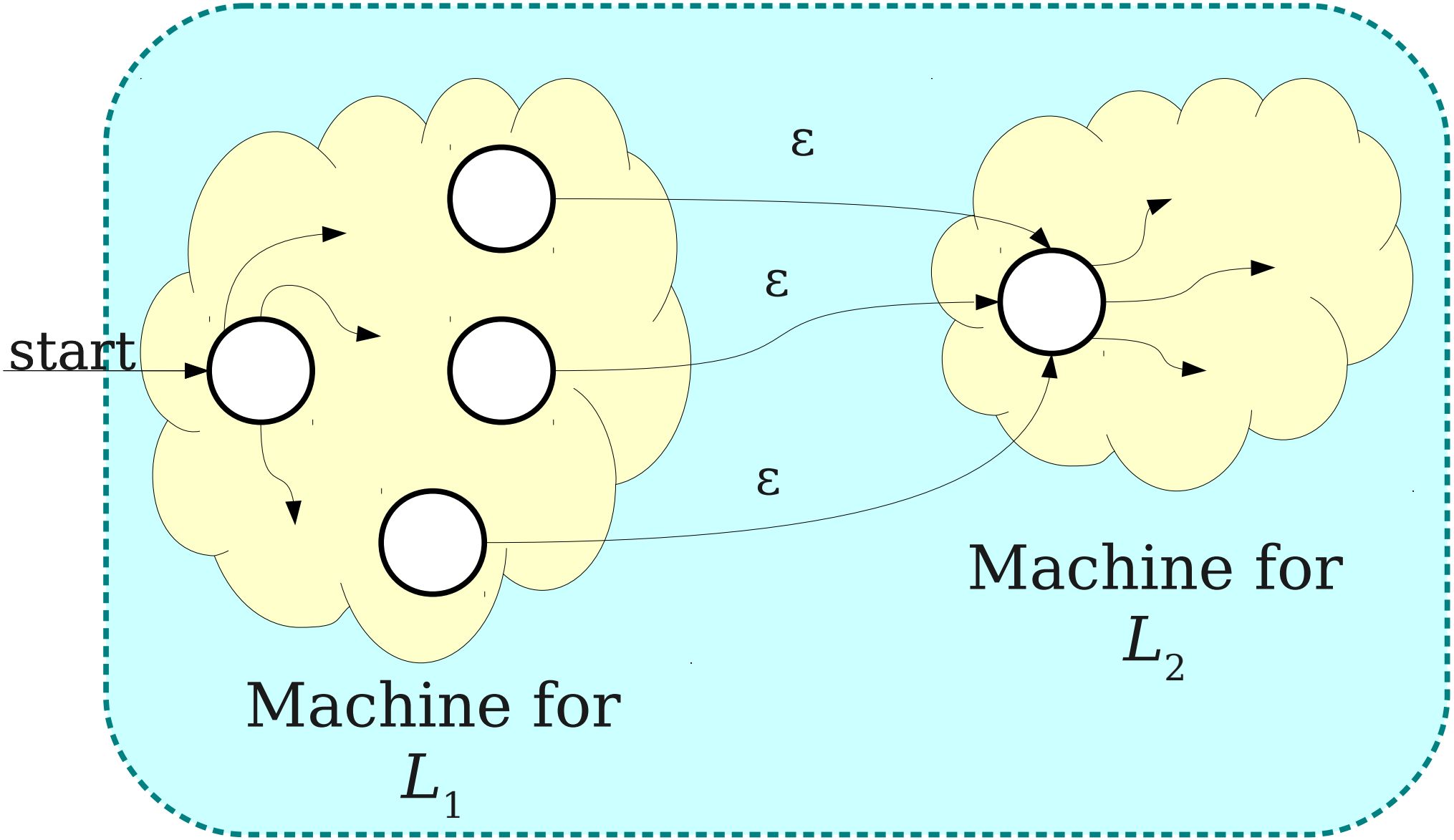


Concatenating Regular Languages



Machine for $L_1 L_2$

Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to "exponentiate" a language as follows:
- $L^0 = \{ \varepsilon \}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.

The Kleene Closure

- An important operation on languages is the **Kleene Closure**, which is defined as

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Mathematically:

$$w \in L^* \quad \textbf{iff} \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, all possible ways of concatenating any number of copies of strings in L together.

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

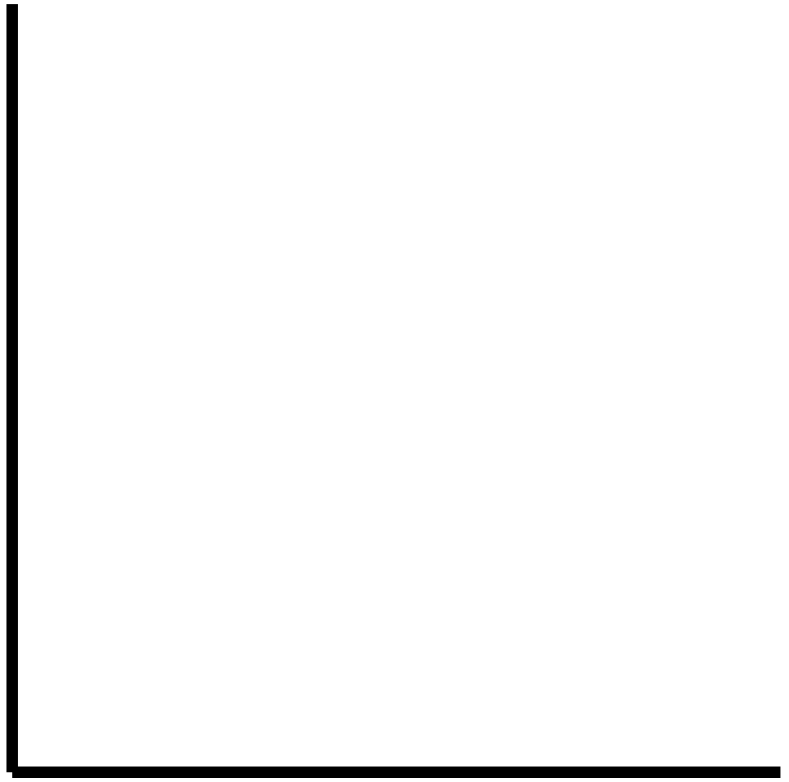
...

}

Reasoning about Infinity

- If L is regular, is L^* necessarily regular?
- **A Bad Line of Reasoning**:
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

Reasoning about Infinity



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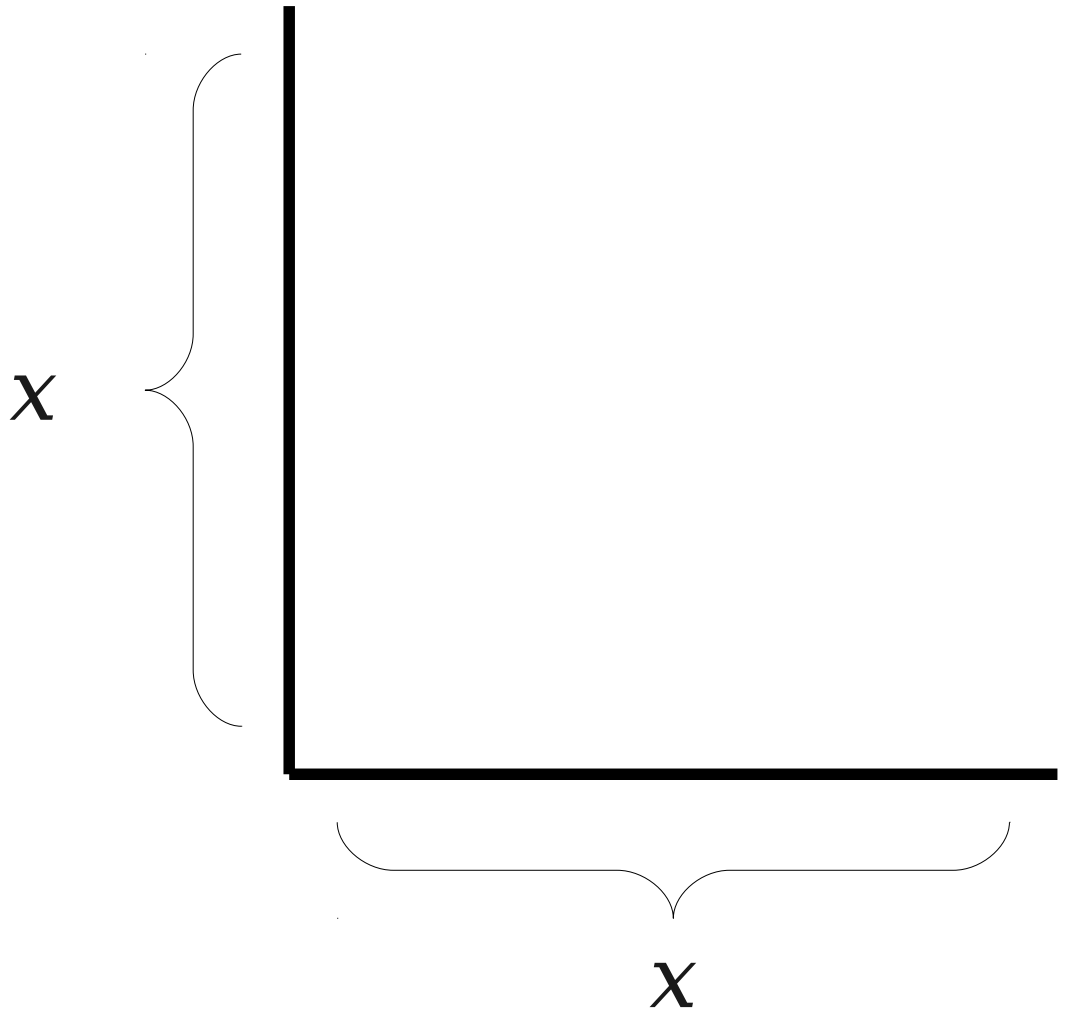

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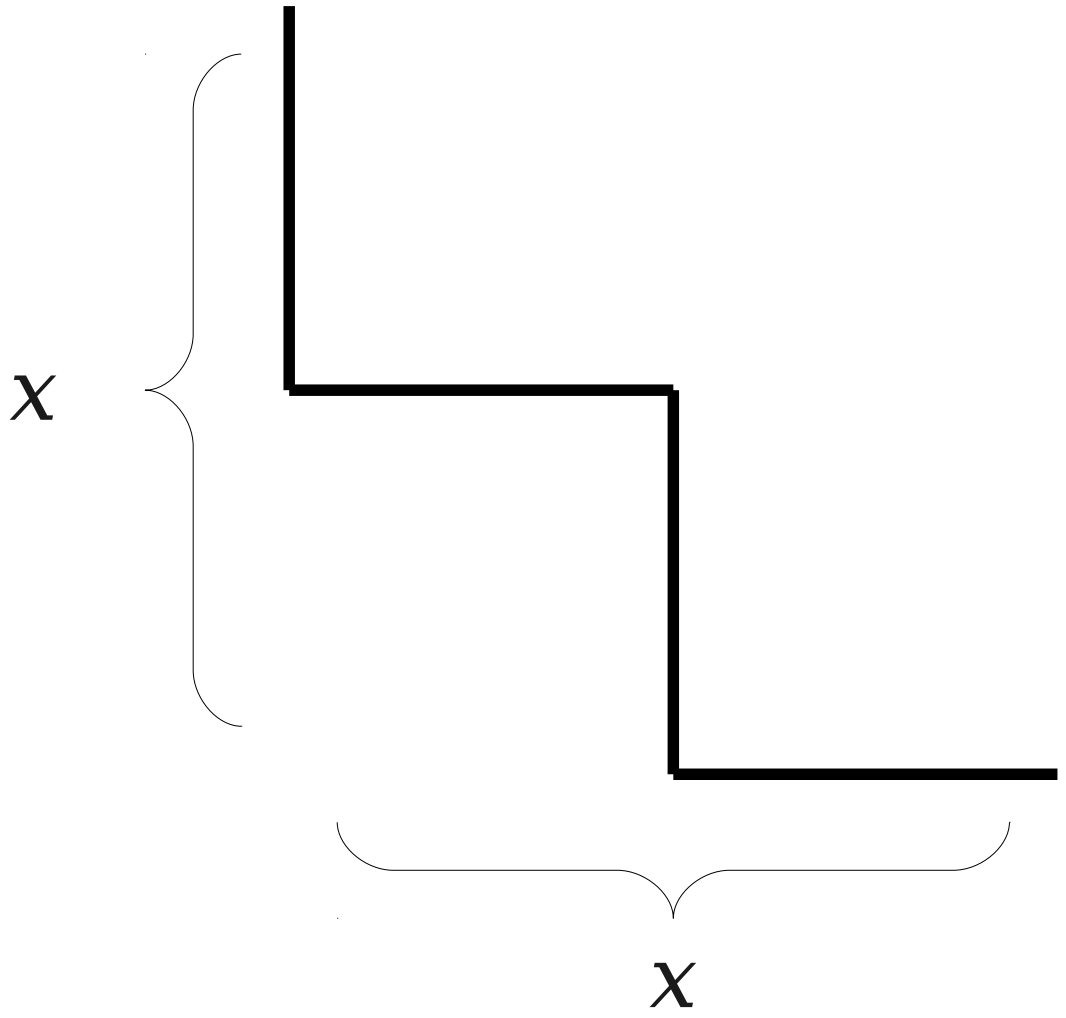

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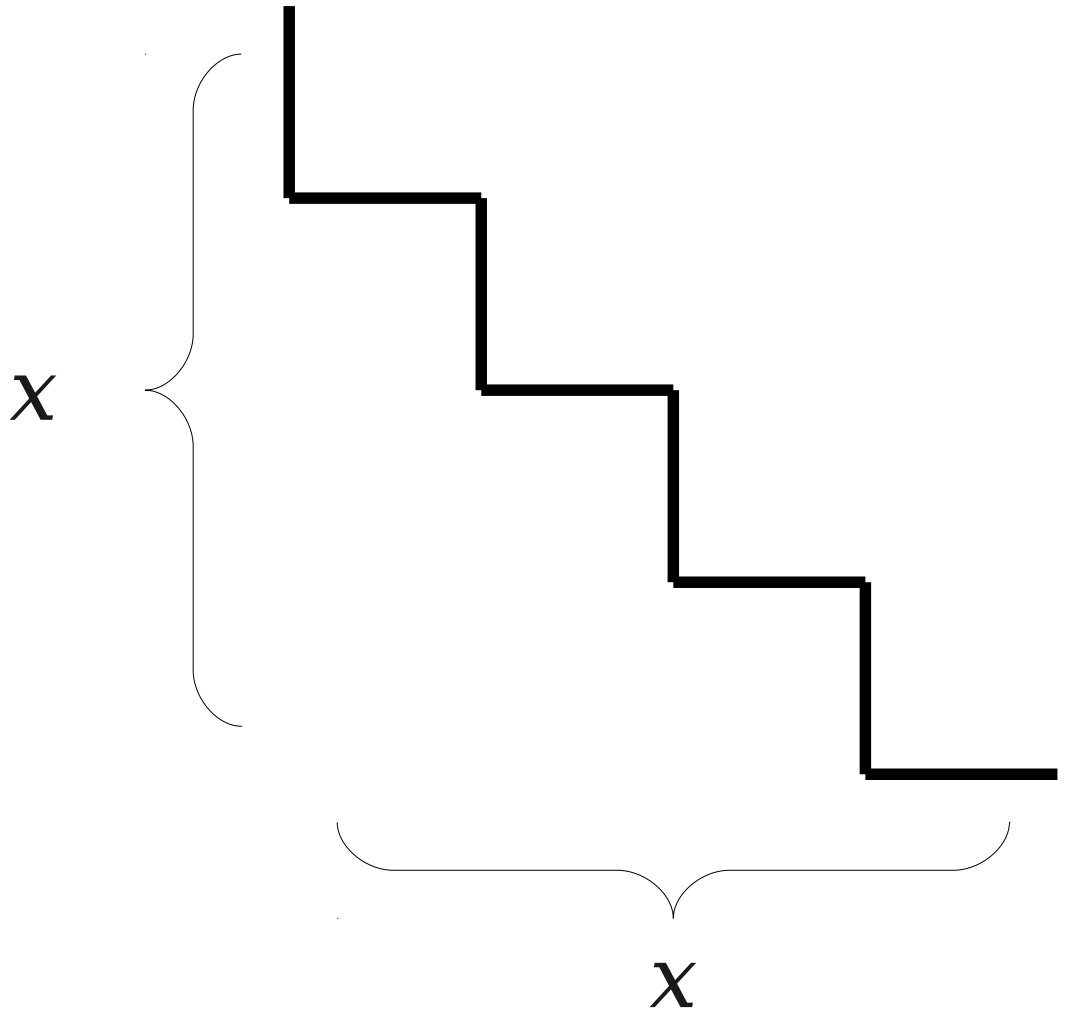

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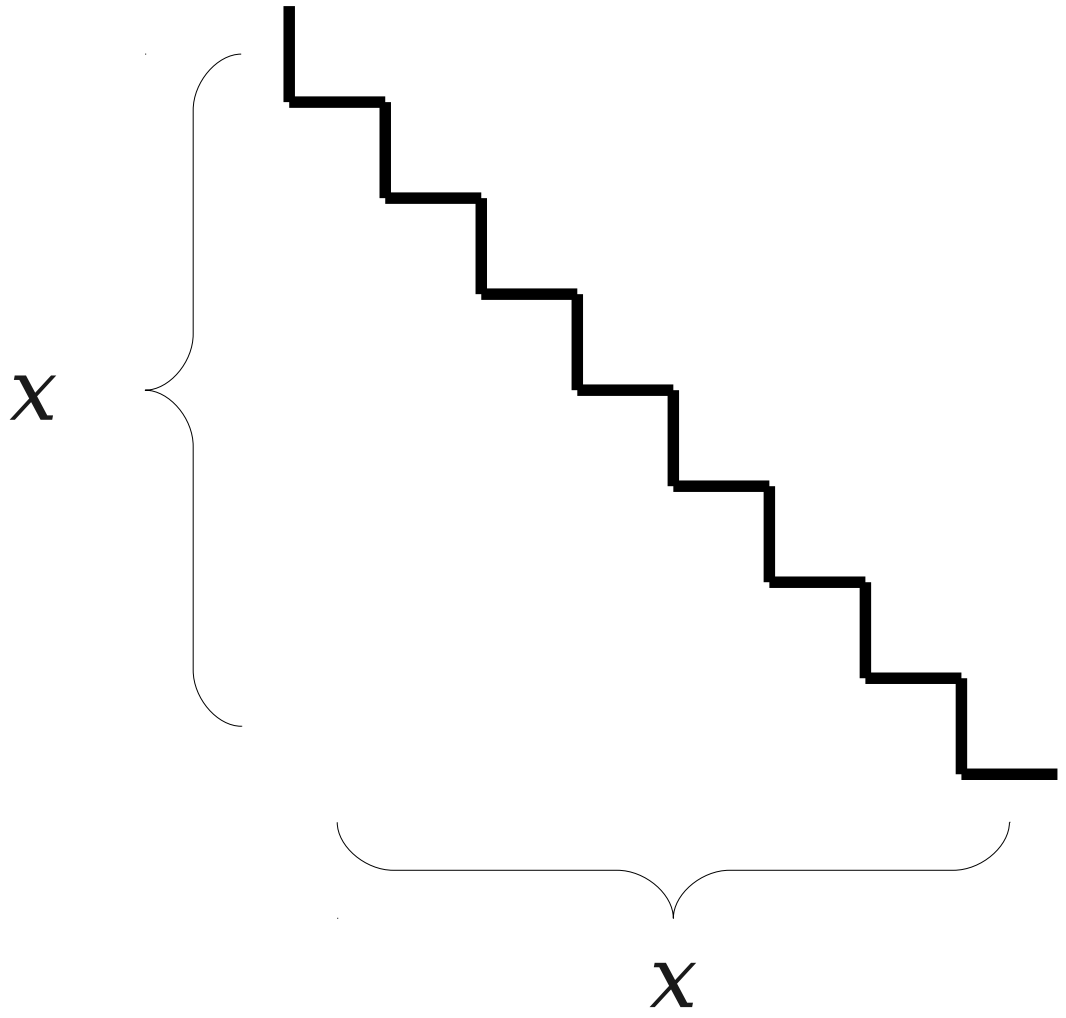
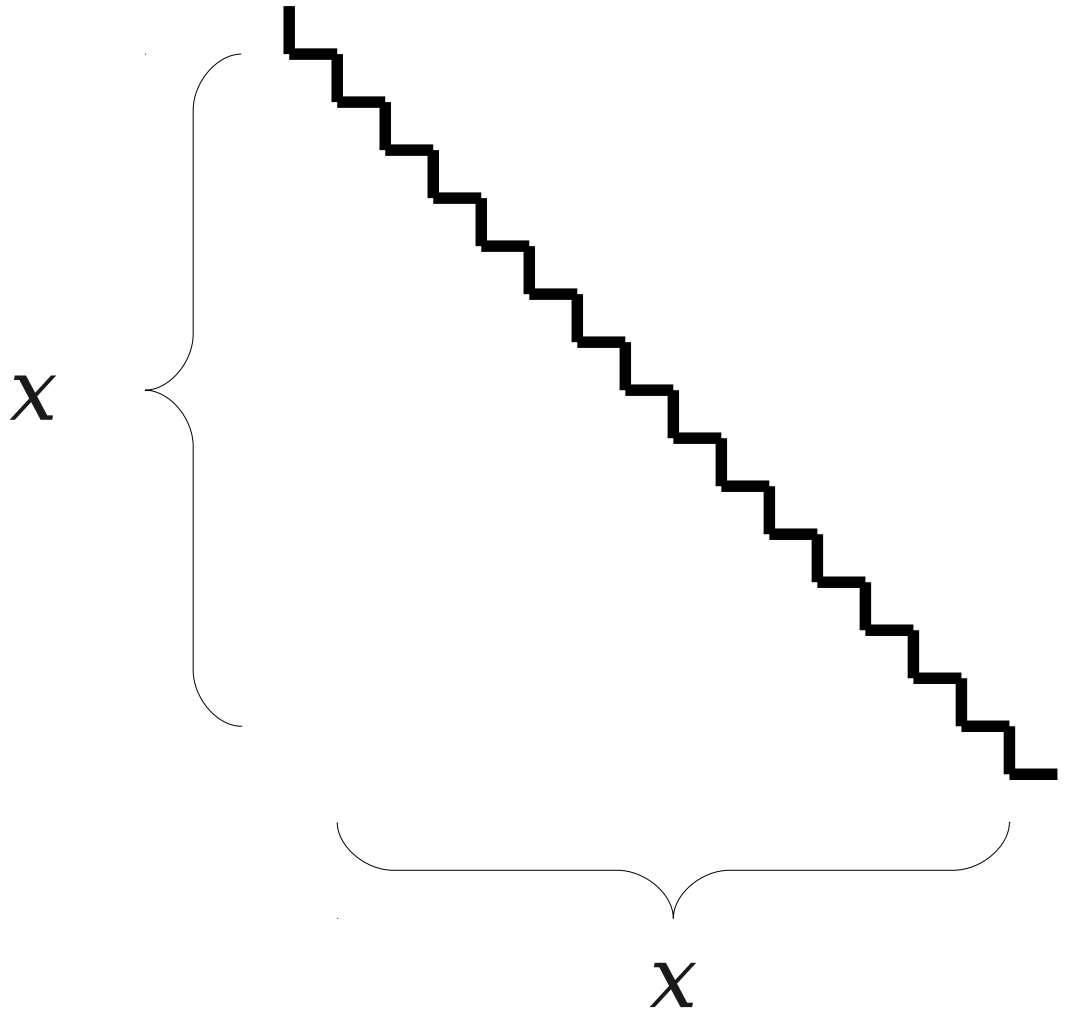
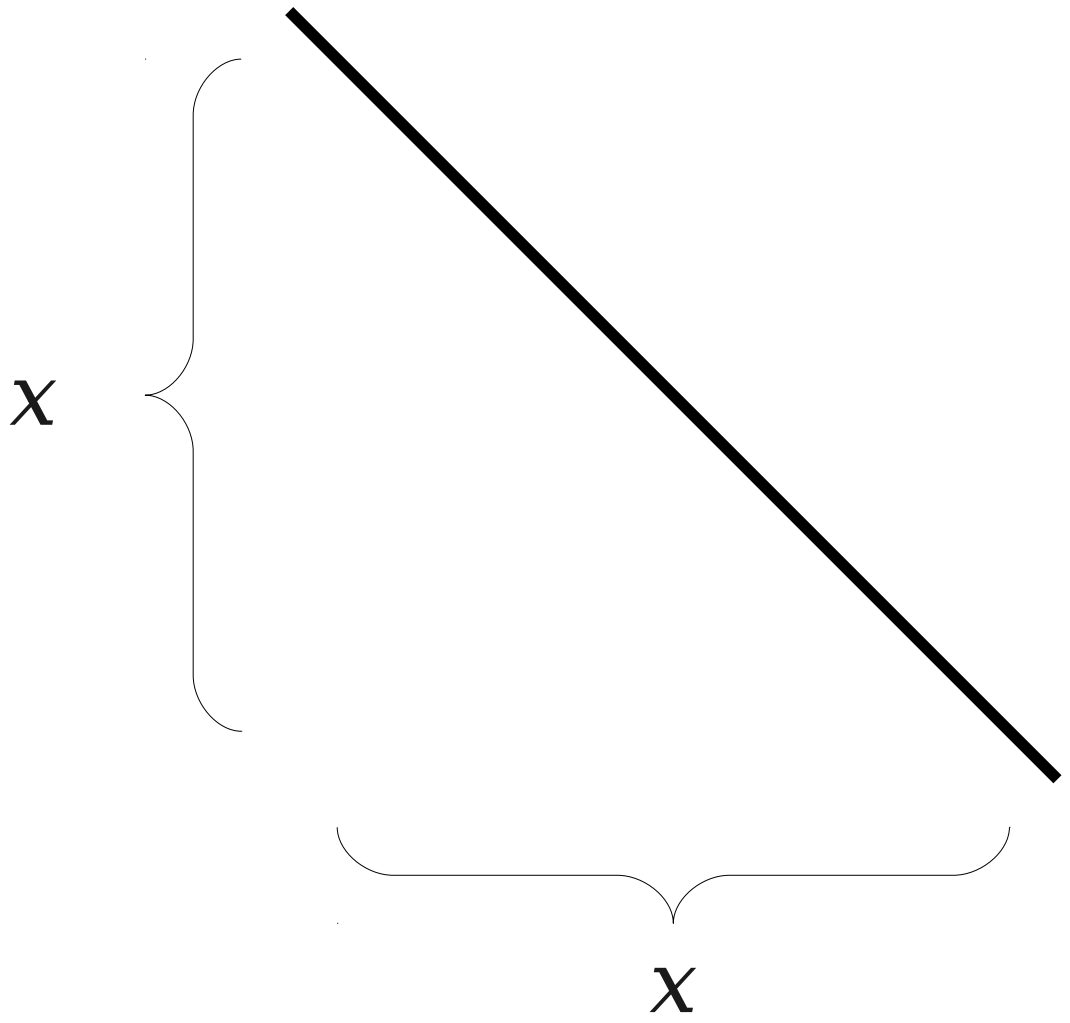


Reasoning about Infinity



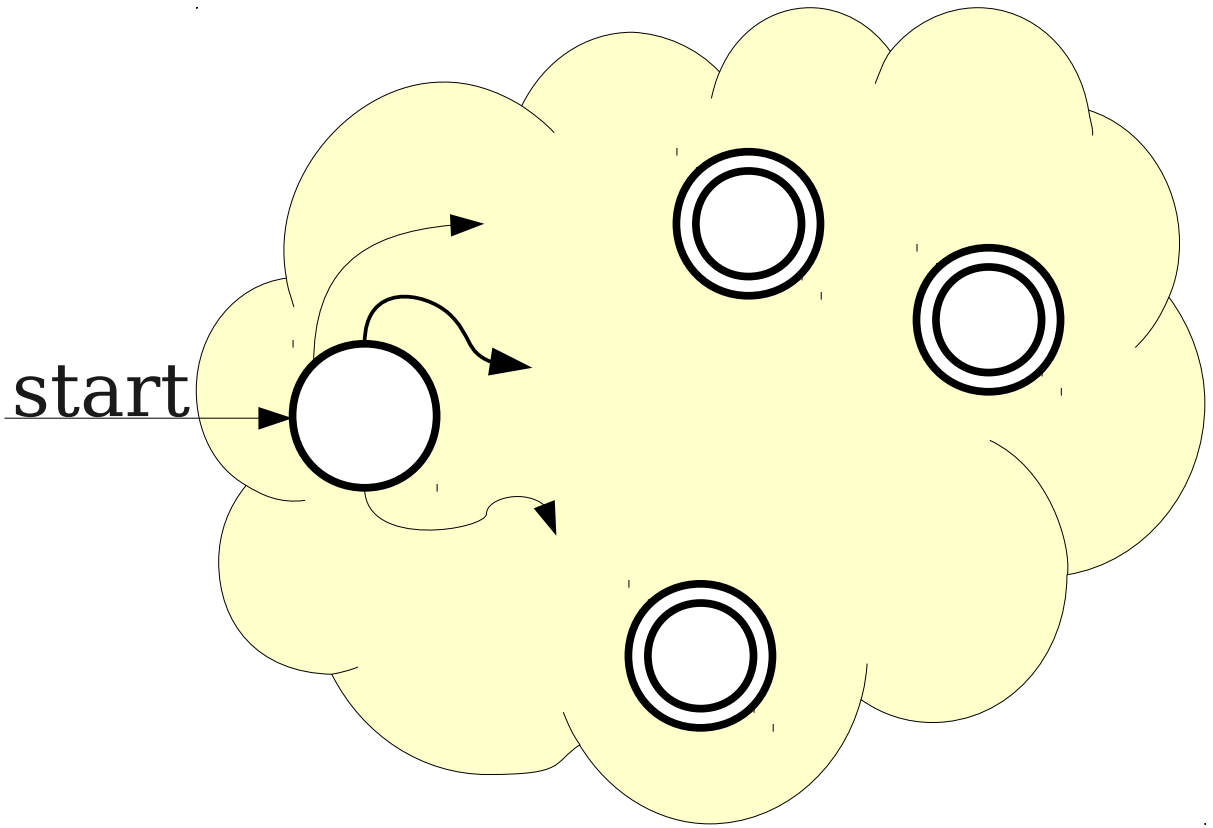
Reasoning about Infinity



Reasoning About the Infinite

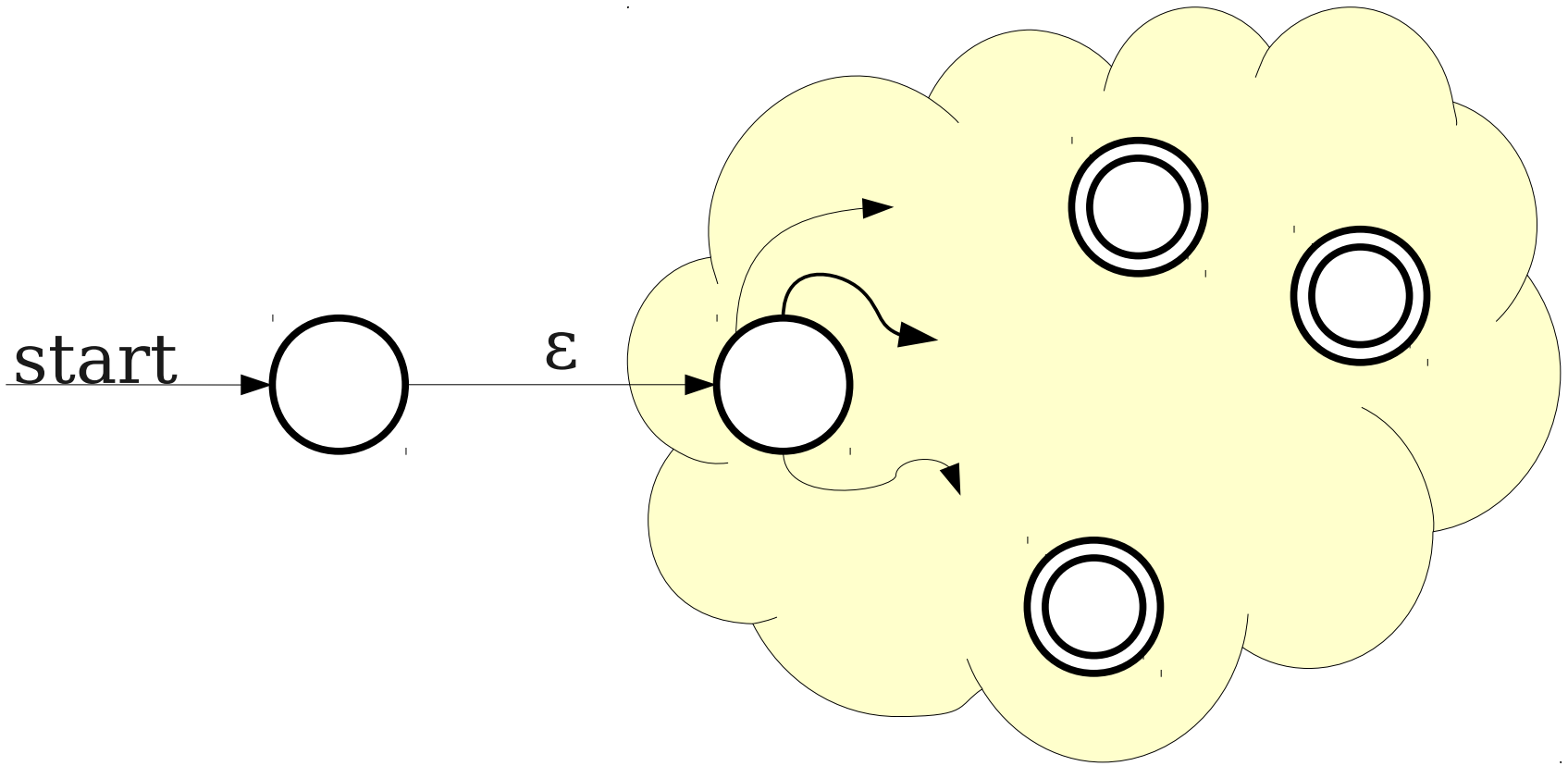
- If a series of finite objects all have some property, their infinite union **does not** necessarily have that property!
  - No matter how many times we zigzag that line, it's never straight.
  - Concluding that it must be equal “in the limit” is not mathematically valid (nor is it correct!)
  - (This is why calculus is interesting).
- **Better idea:** Can we convert an NFA for the language L to an NFA for the language L^*?

The Kleene Star



Machine for L

The Kleene Star



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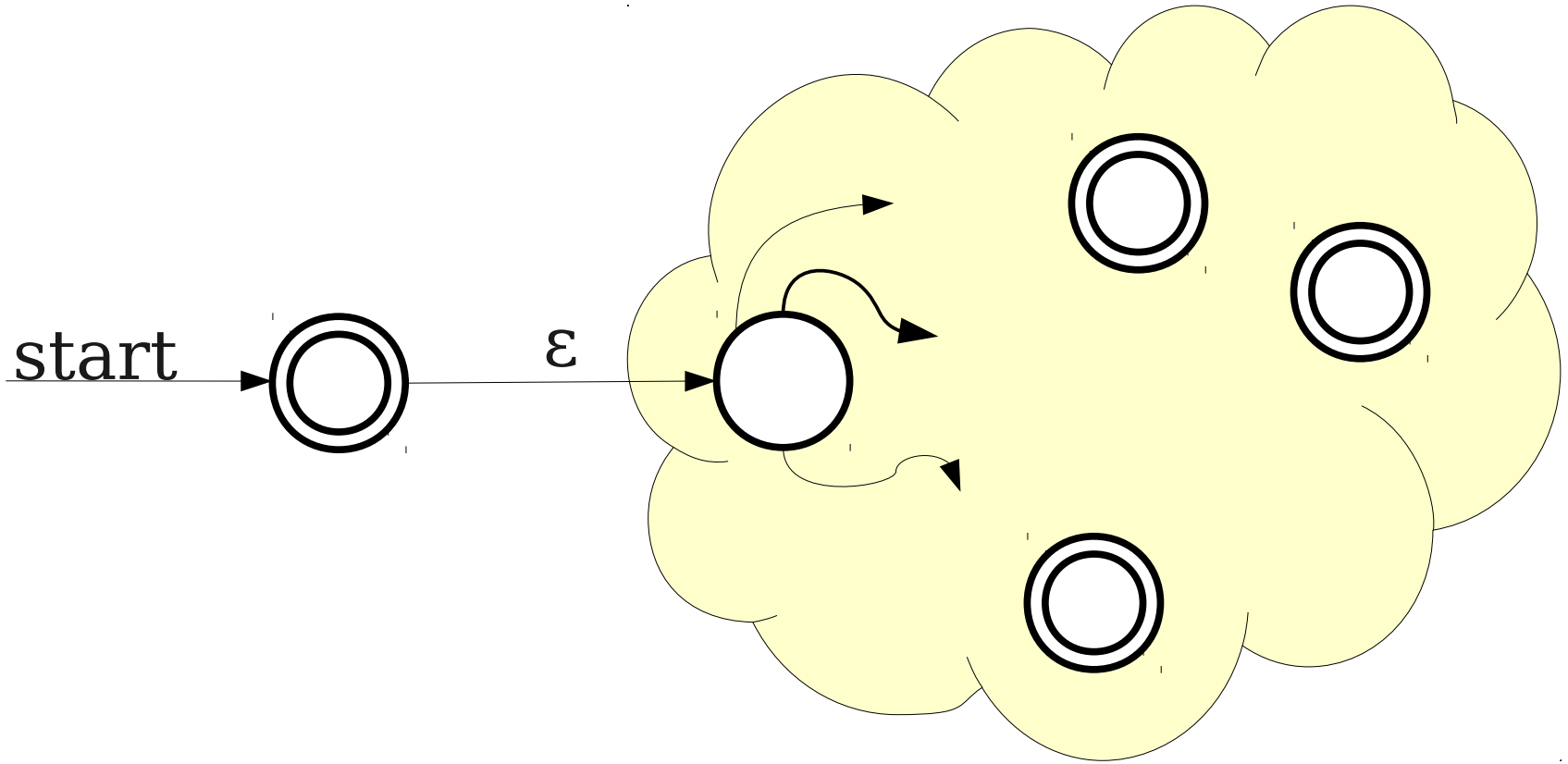

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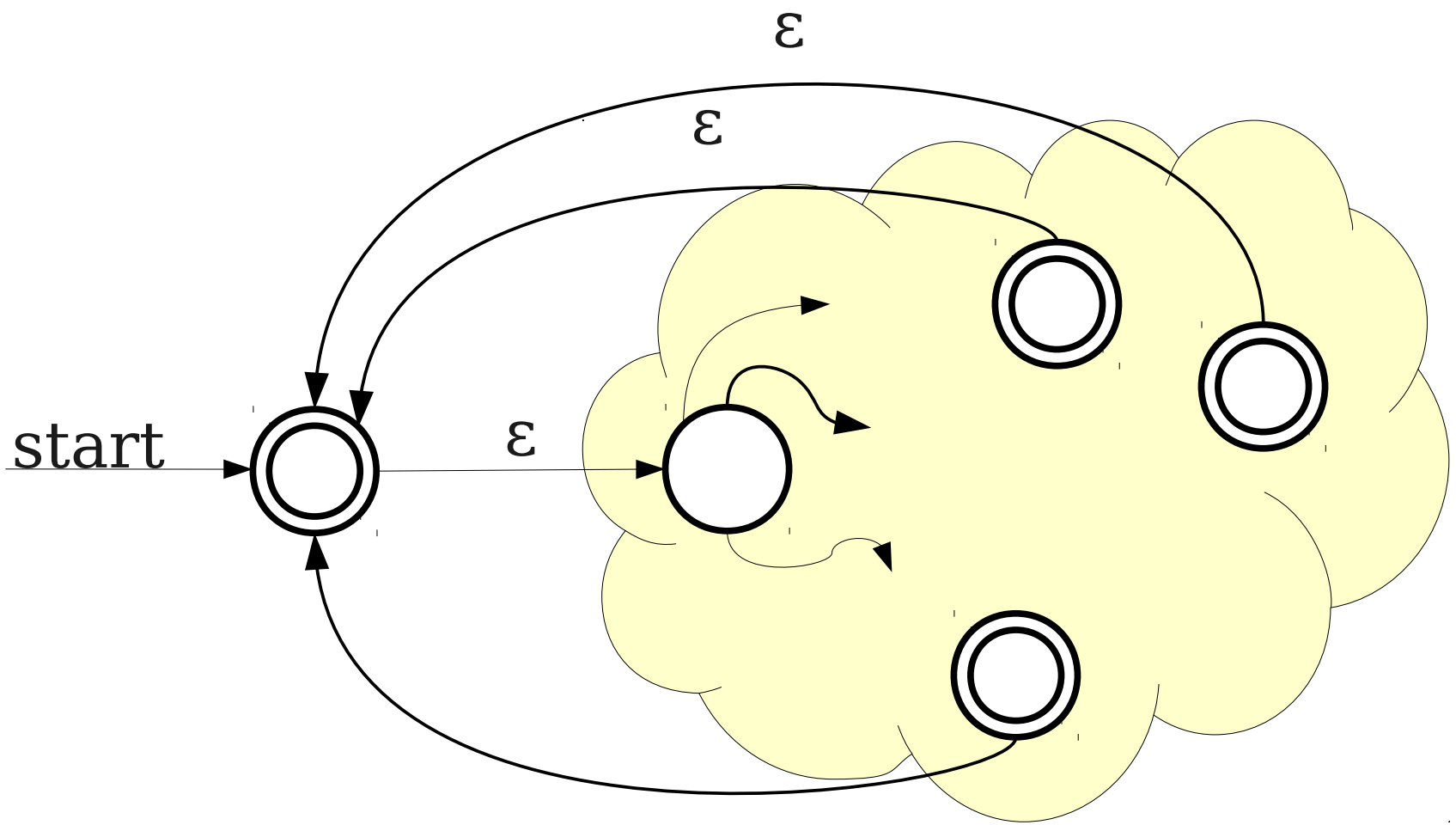

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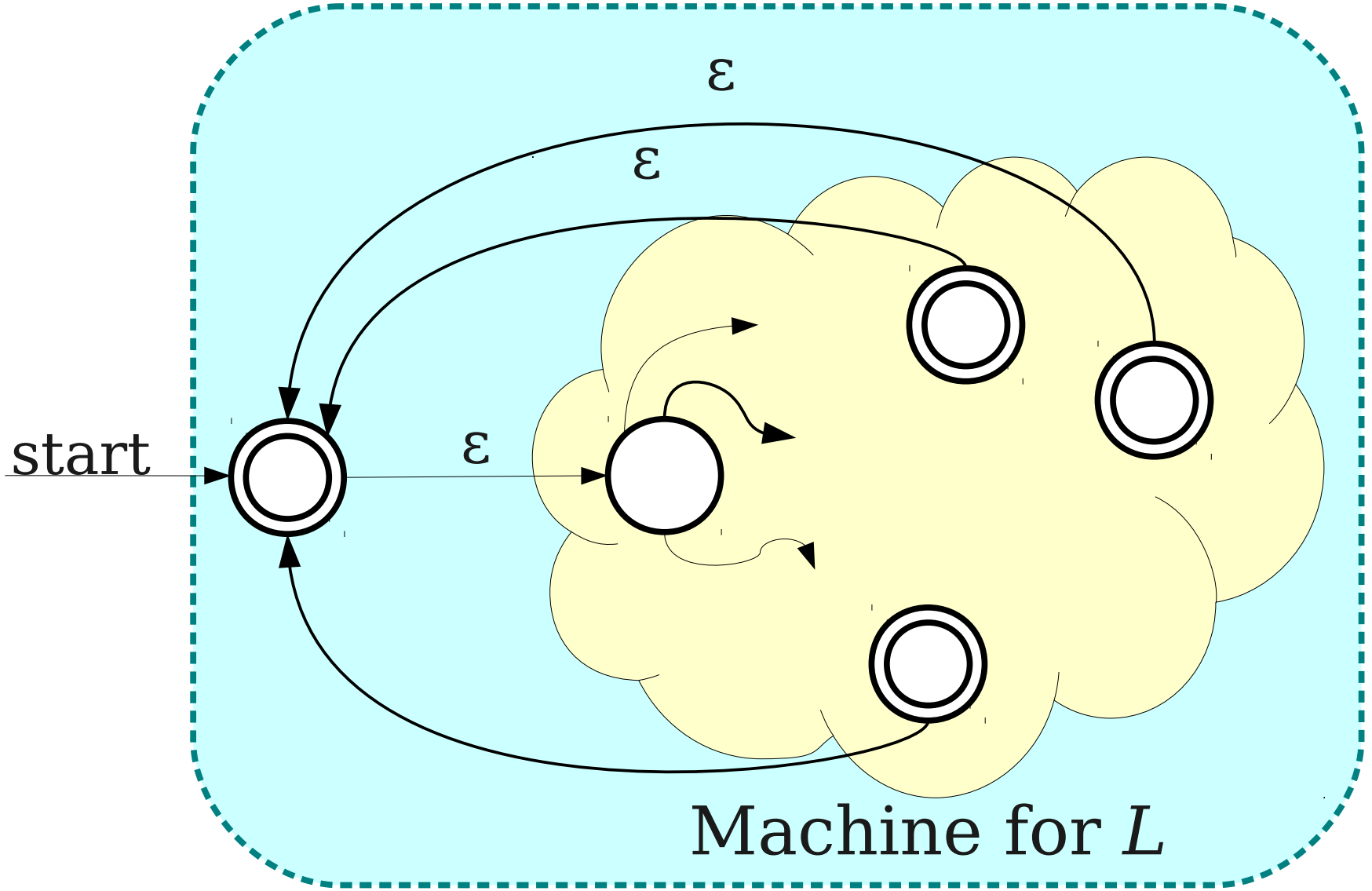
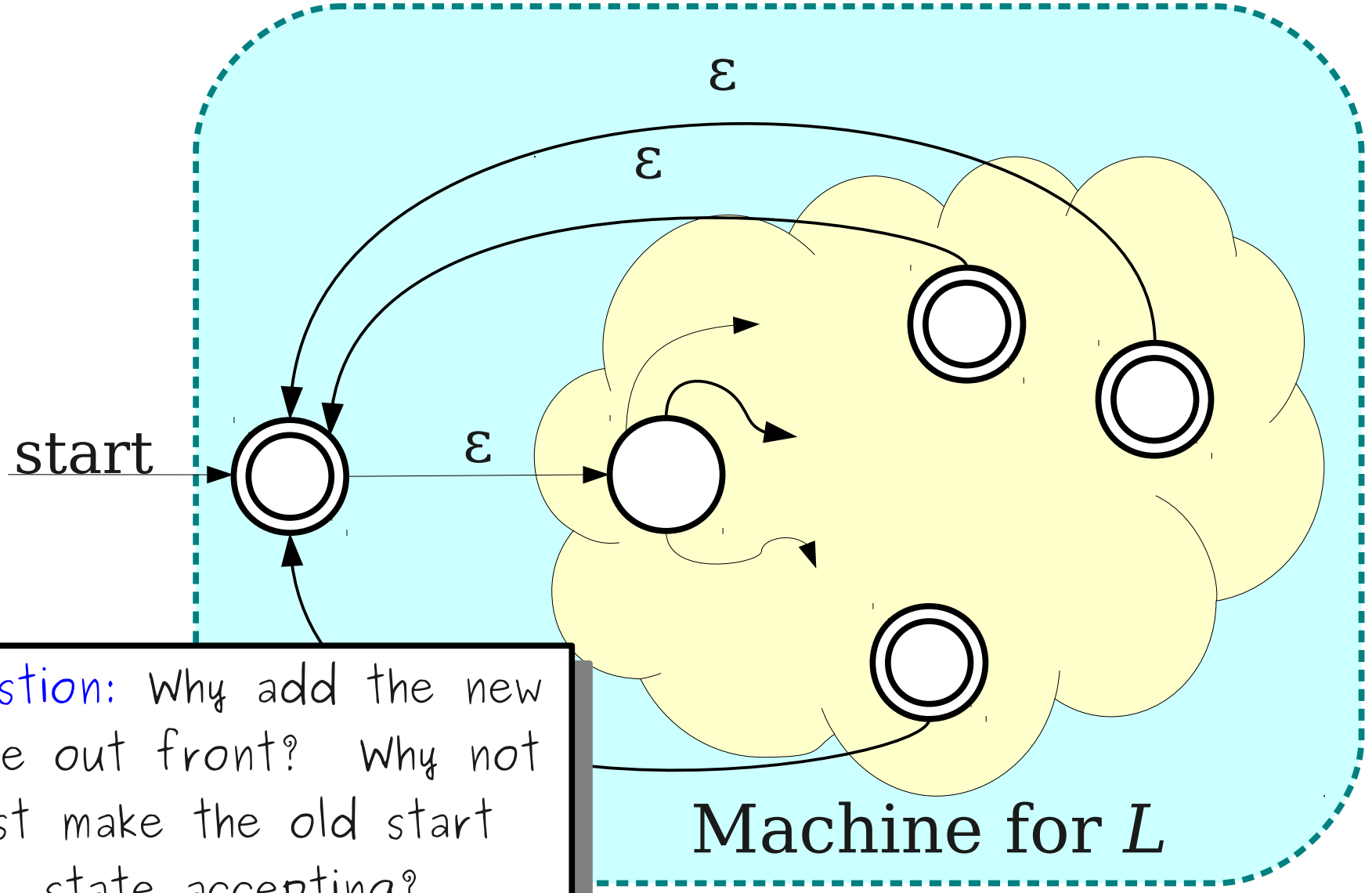


Machine for L^*

The Kleene Star



Summary

- NFAs are a powerful type of automaton that allows for **nondeterministic** choices.
- NFAs can also have **ε-transitions** that move from state to state without consuming any input.
- The **subset construction** shows that NFAs are not more powerful than DFAs, because any NFA can be converted into a DFA that accepts the same language.
- The union, intersection, complement, concatenation, and Kleene closure of regular languages are all regular languages.

Next Time

- **Regular Expressions**
  - Building up the regular languages, one piece at a time.
- **Intuiting Regular Languages**
  - What exactly is a regular language?
  - When would you use them?