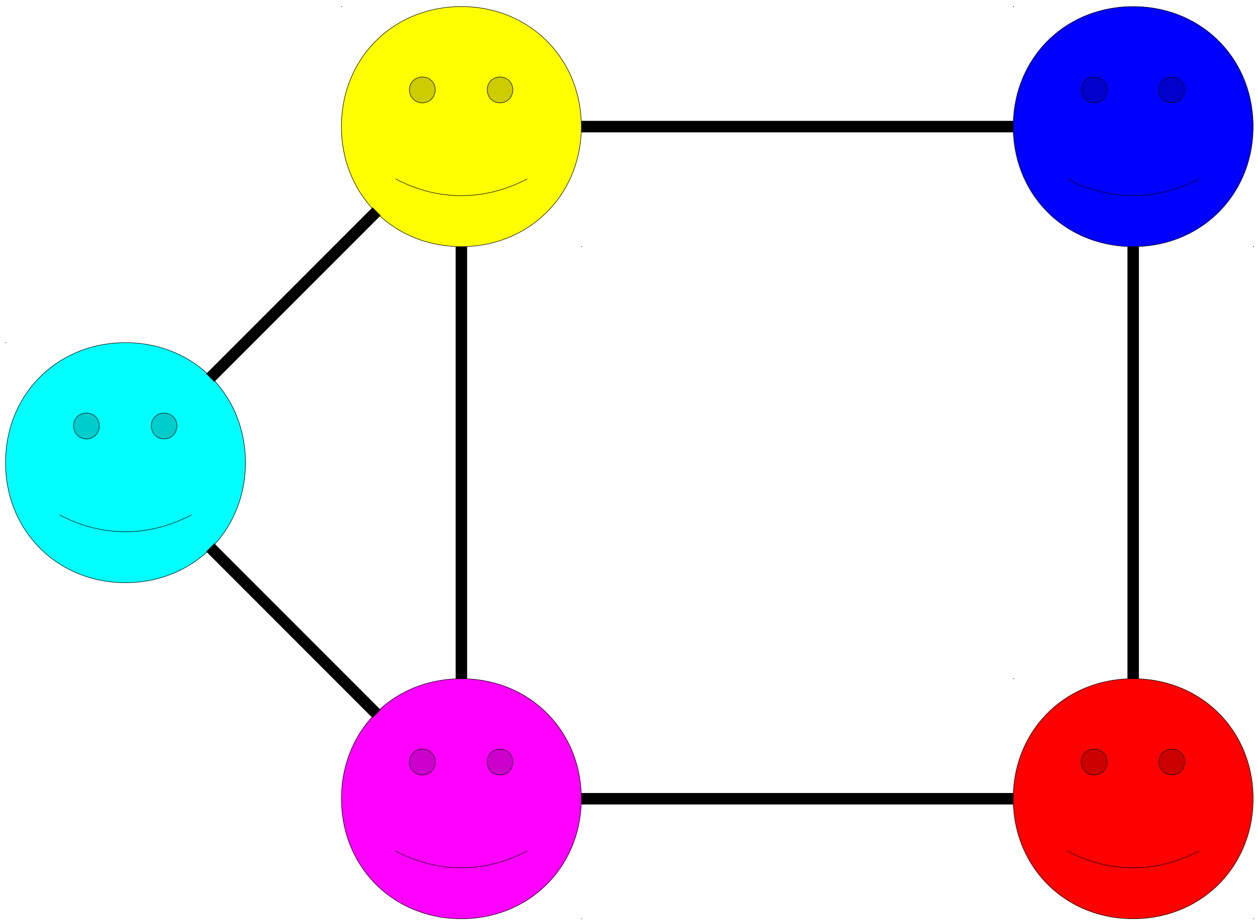
Graphs II

Problem Set Two checkpoint problem due in the box up front. Problem Set One due in the box up front if you're using a late period.

Quick Announcements

- *Sorry about the fire alarm!*
- We're going to be offset by about half a lecture for a few days.
- No deadlines will be adjusted. We're still on track!

A **graph** is a mathematical structure for representing relationships.



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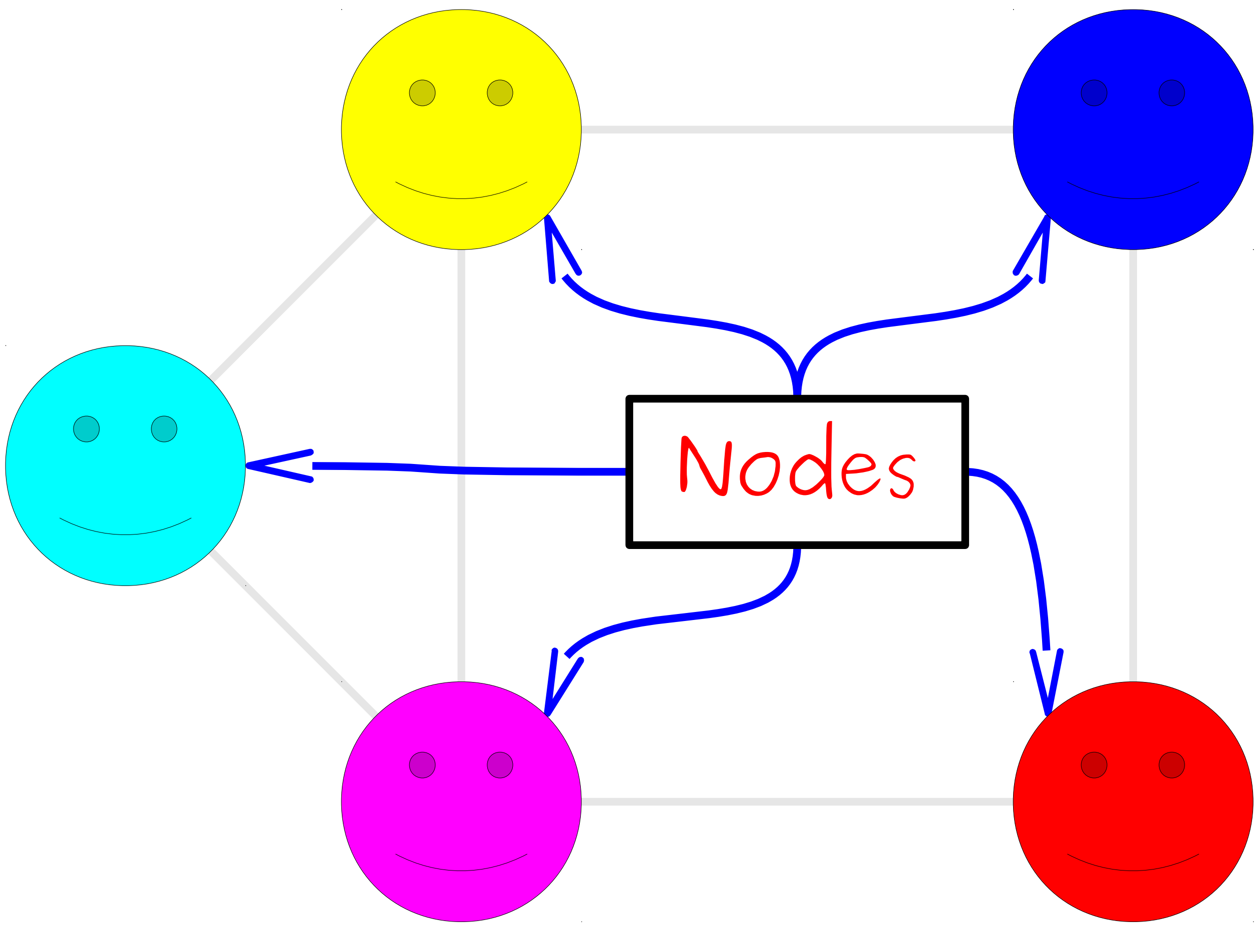

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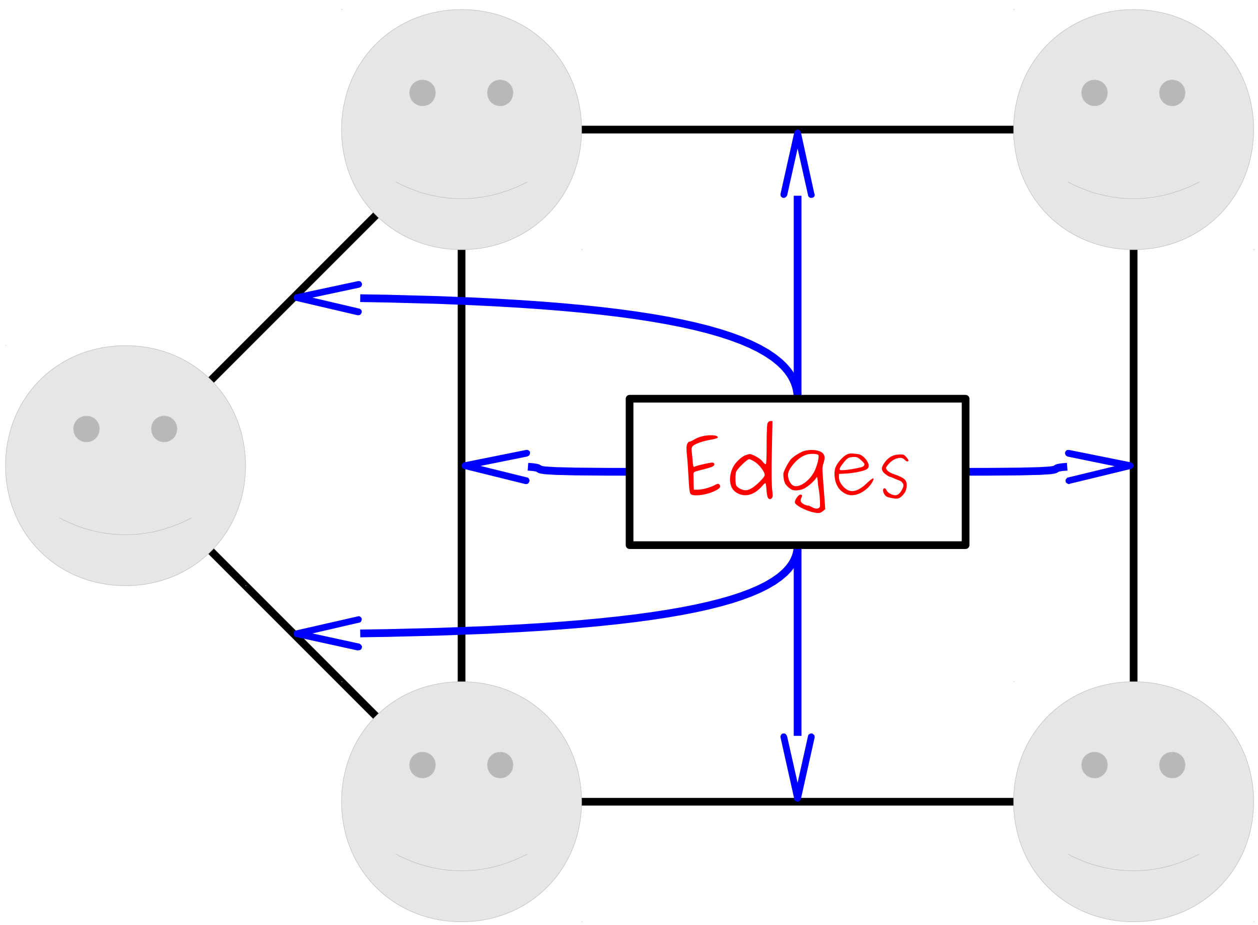

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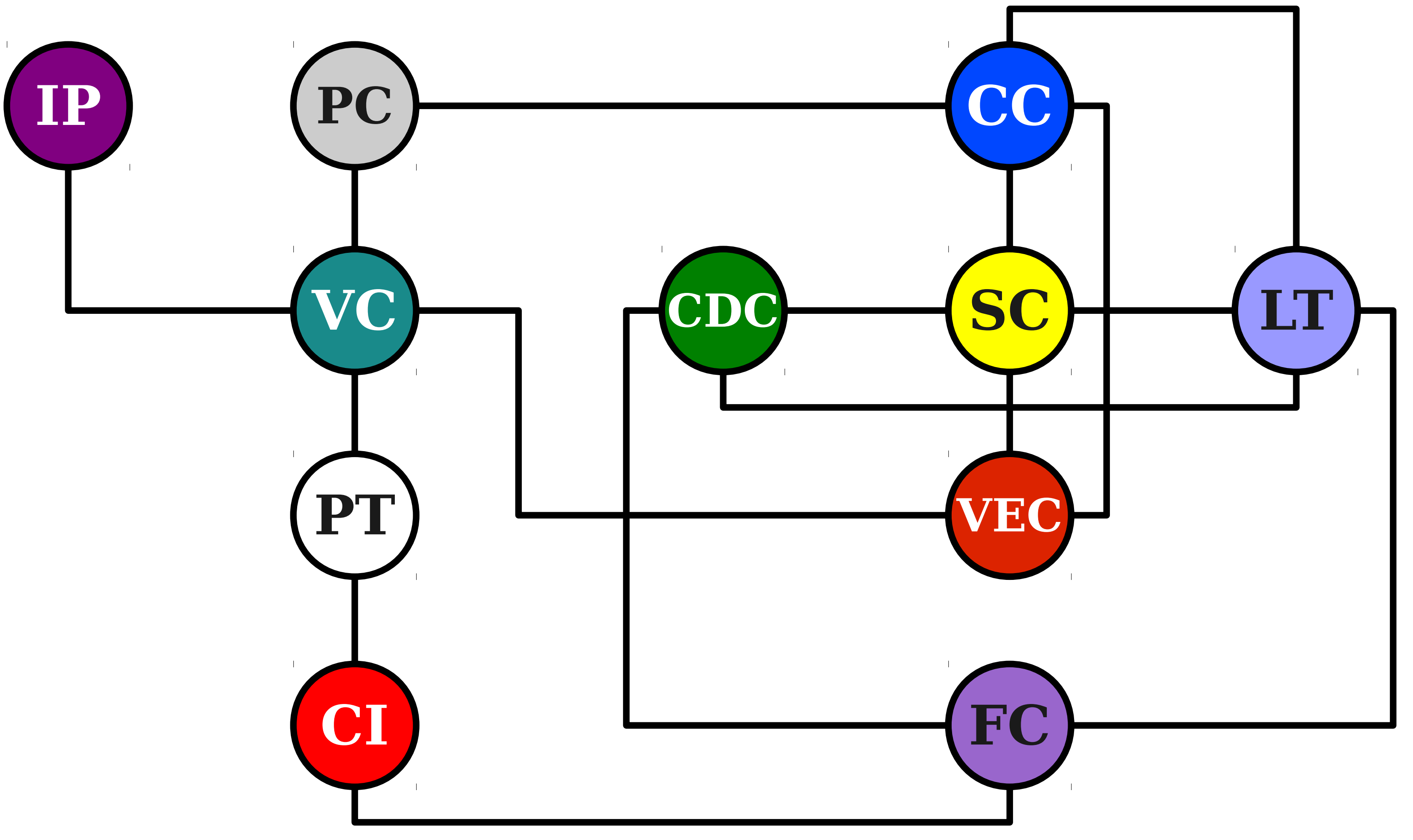
Formalizing Graphs

- Formally, a **graph** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes.
  - E is a set of edges, which are either ordered pairs or unordered pairs of elements from V.

Undirected Connectivity

Navigating a Graph



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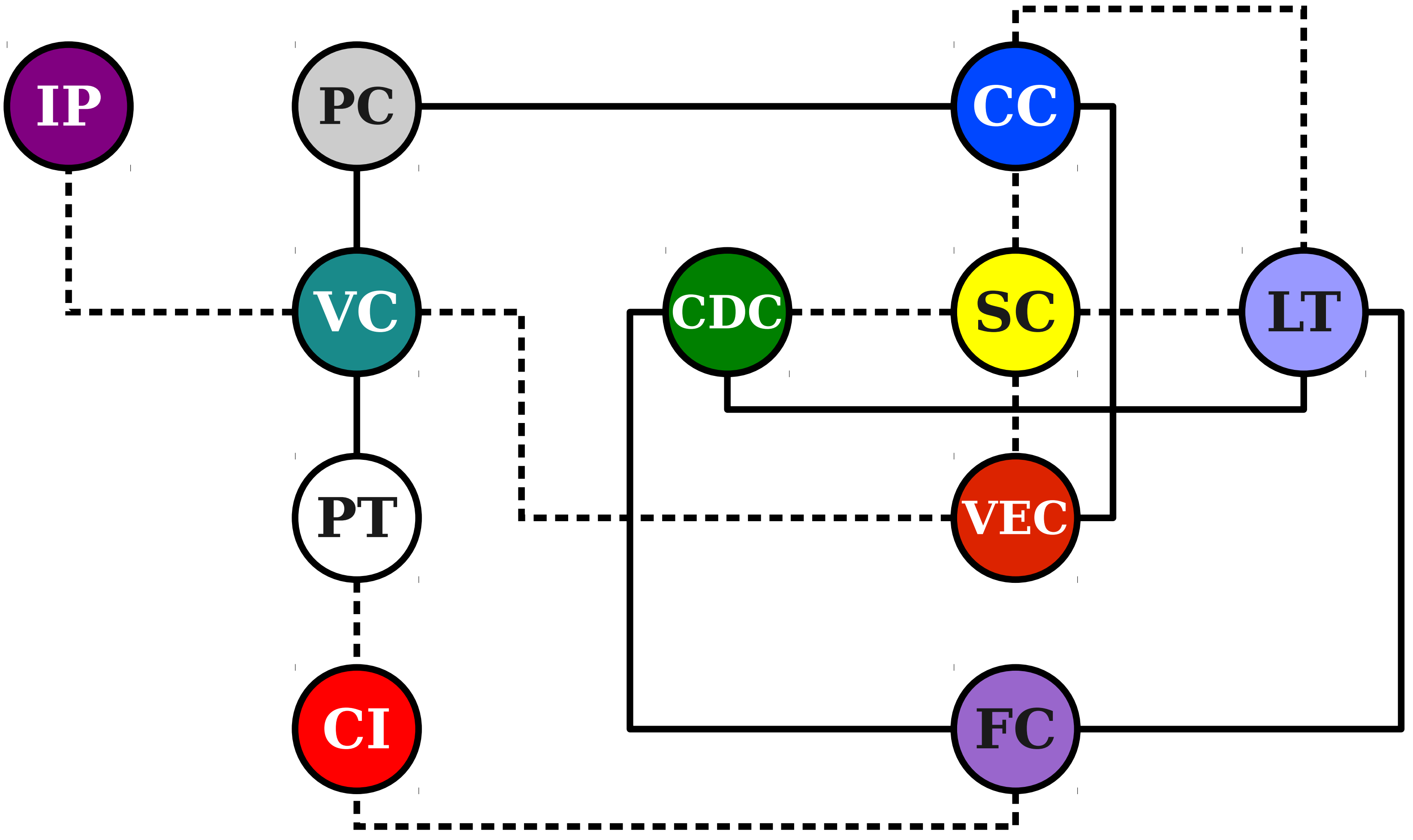

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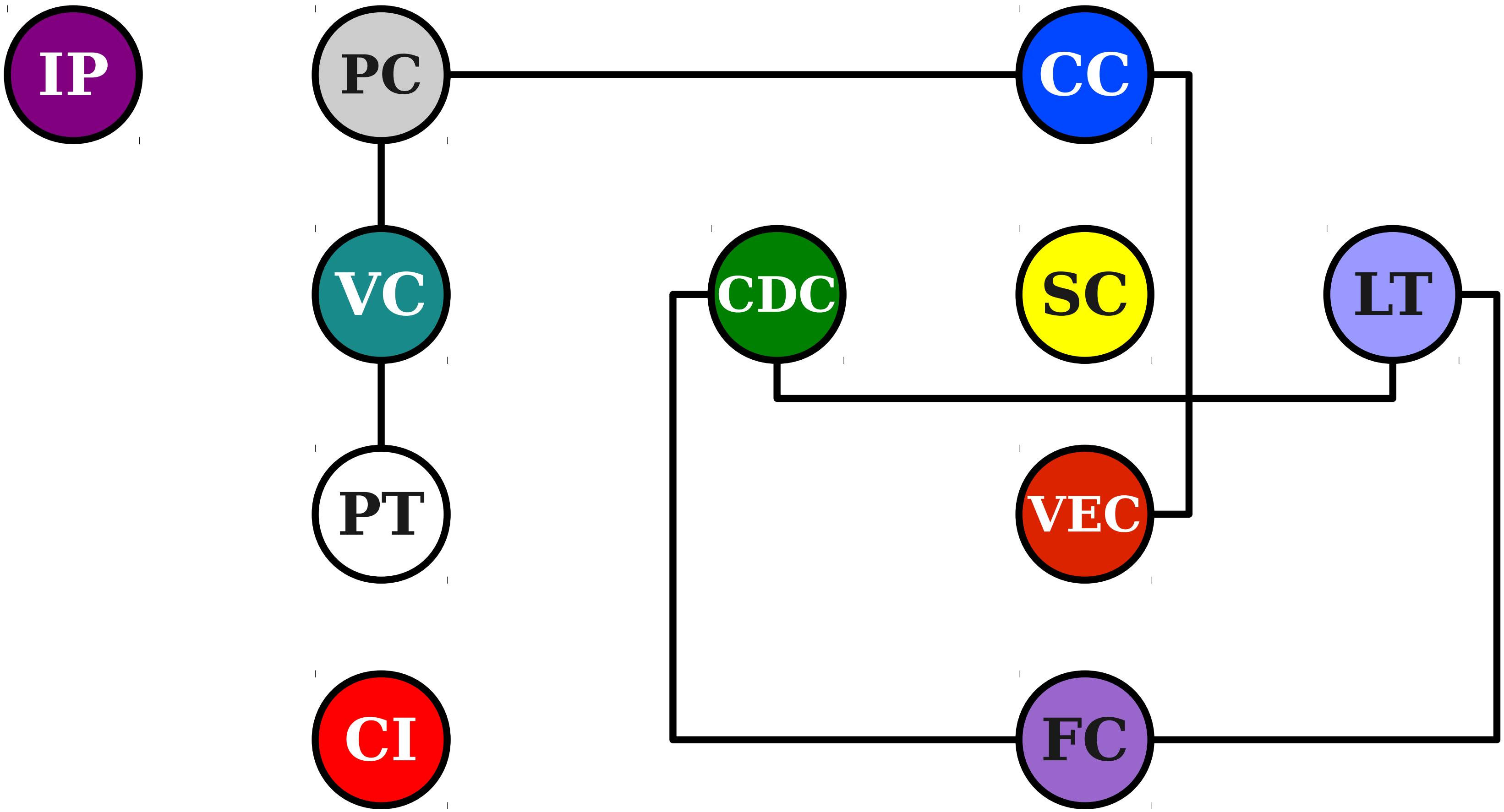

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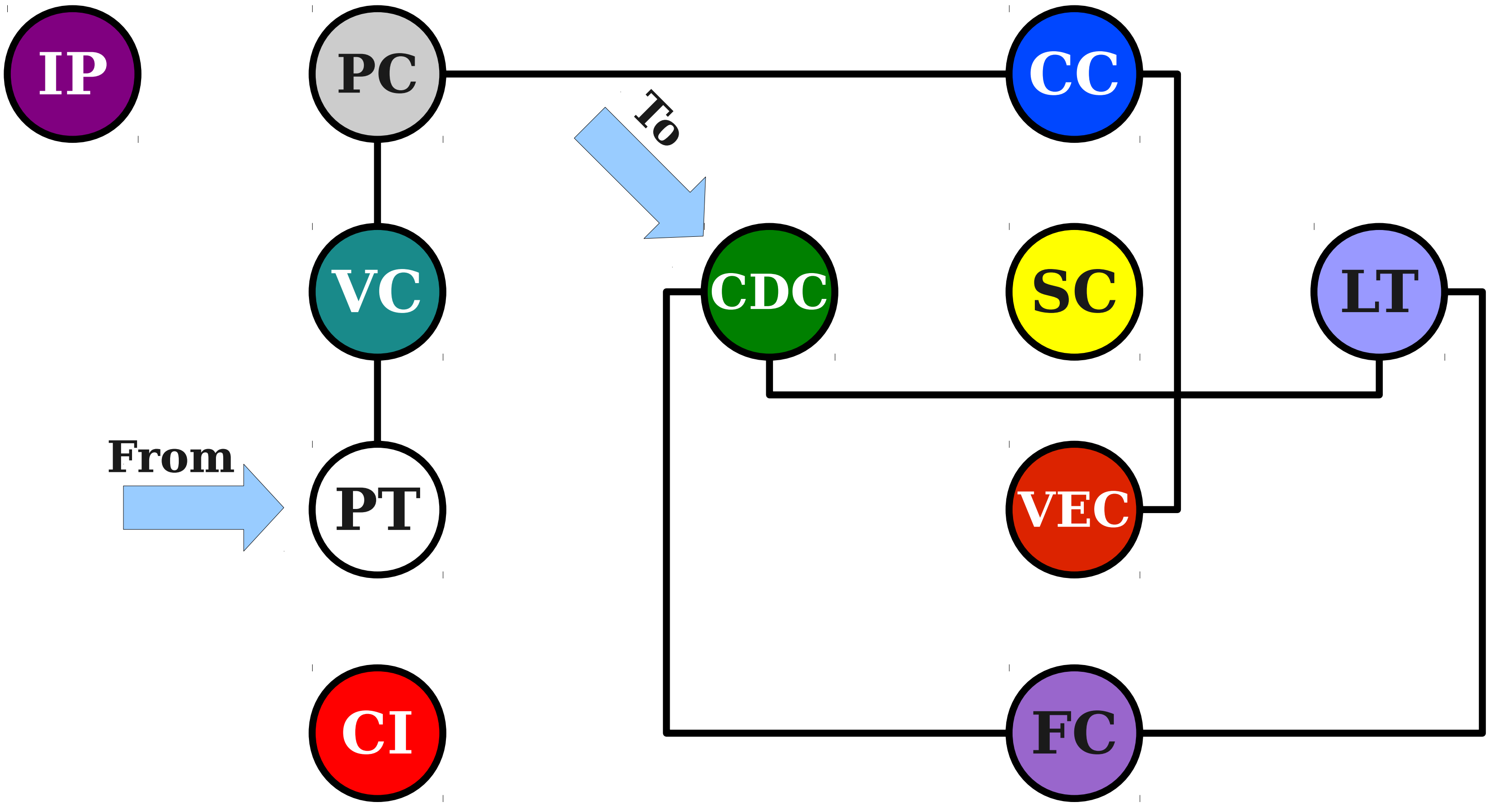

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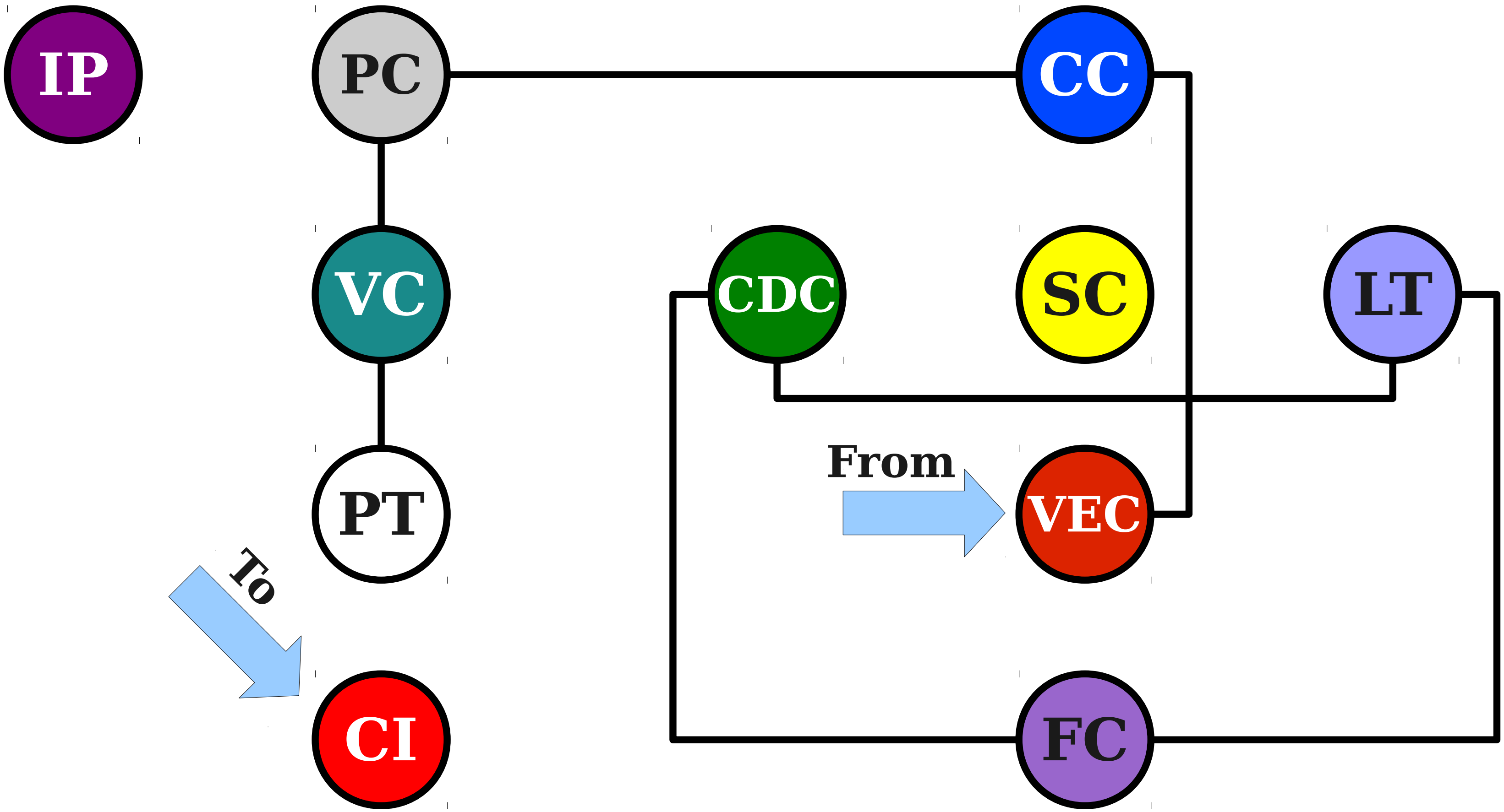


In an undirected graph, two nodes u and v are called **connected** iff there is a path from u to v.

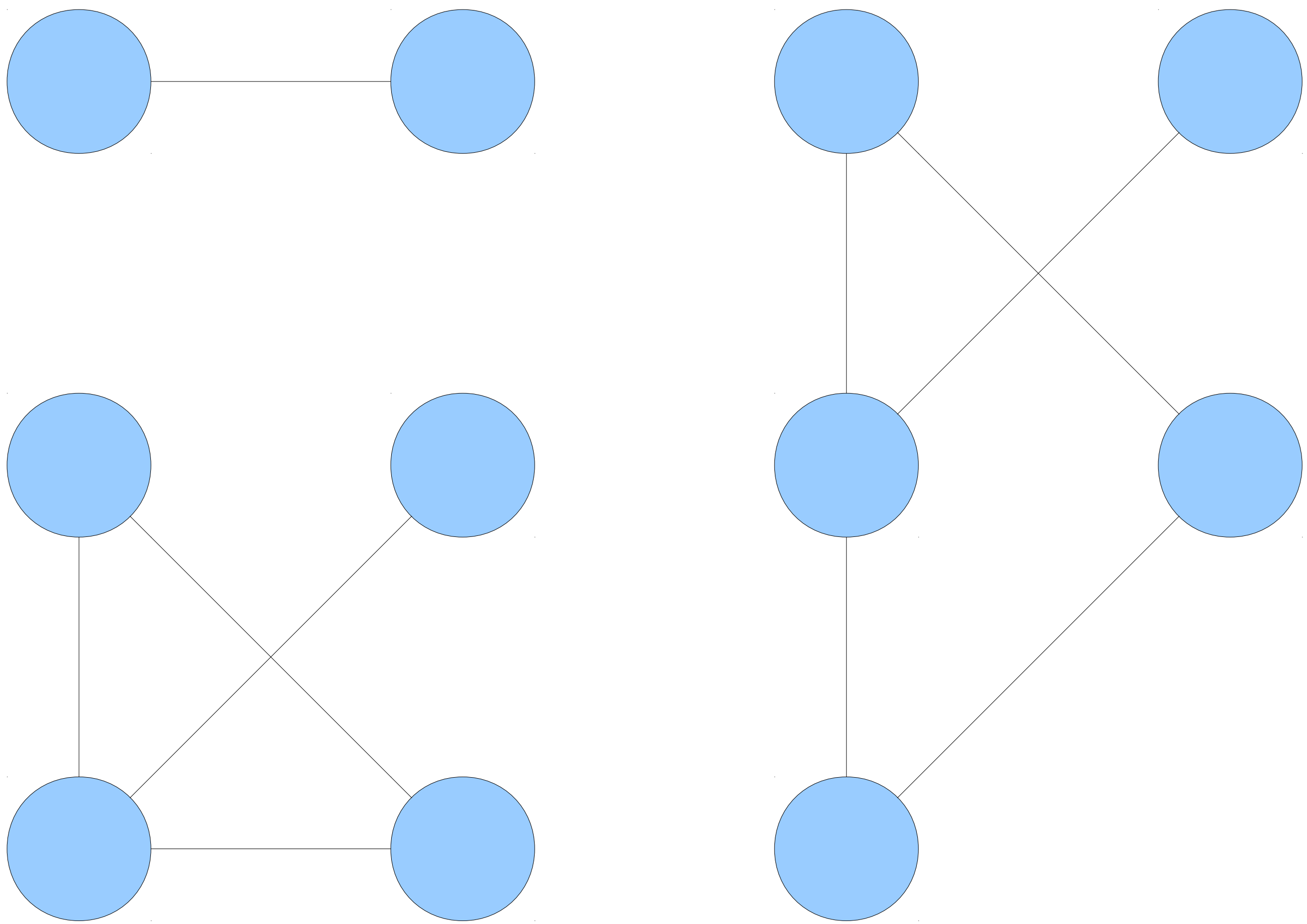
We denote this as $\boldsymbol{u \leftrightarrow v}$.

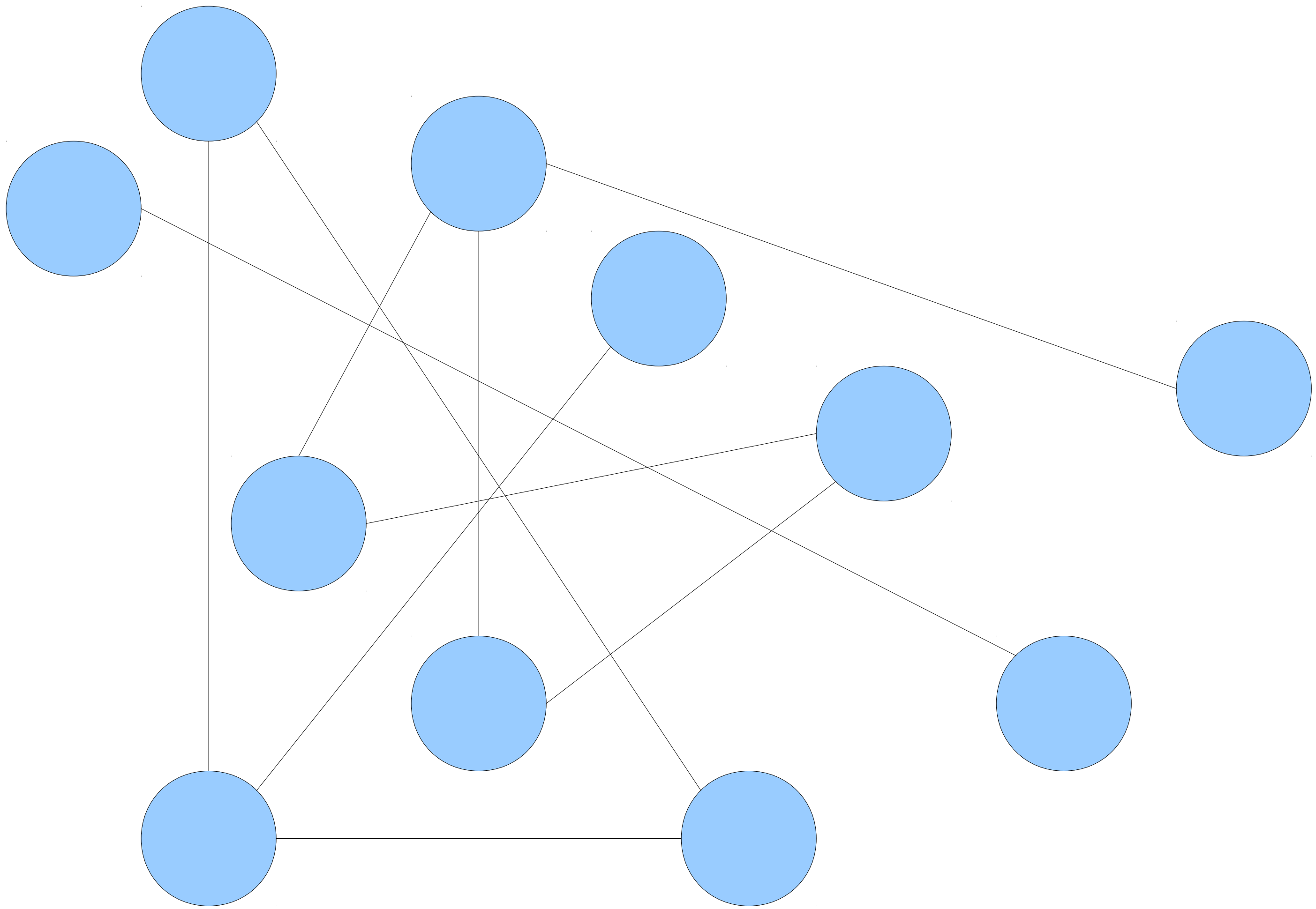
If u is not connected to v, we write $\boldsymbol{u \nleftrightarrow v}$.

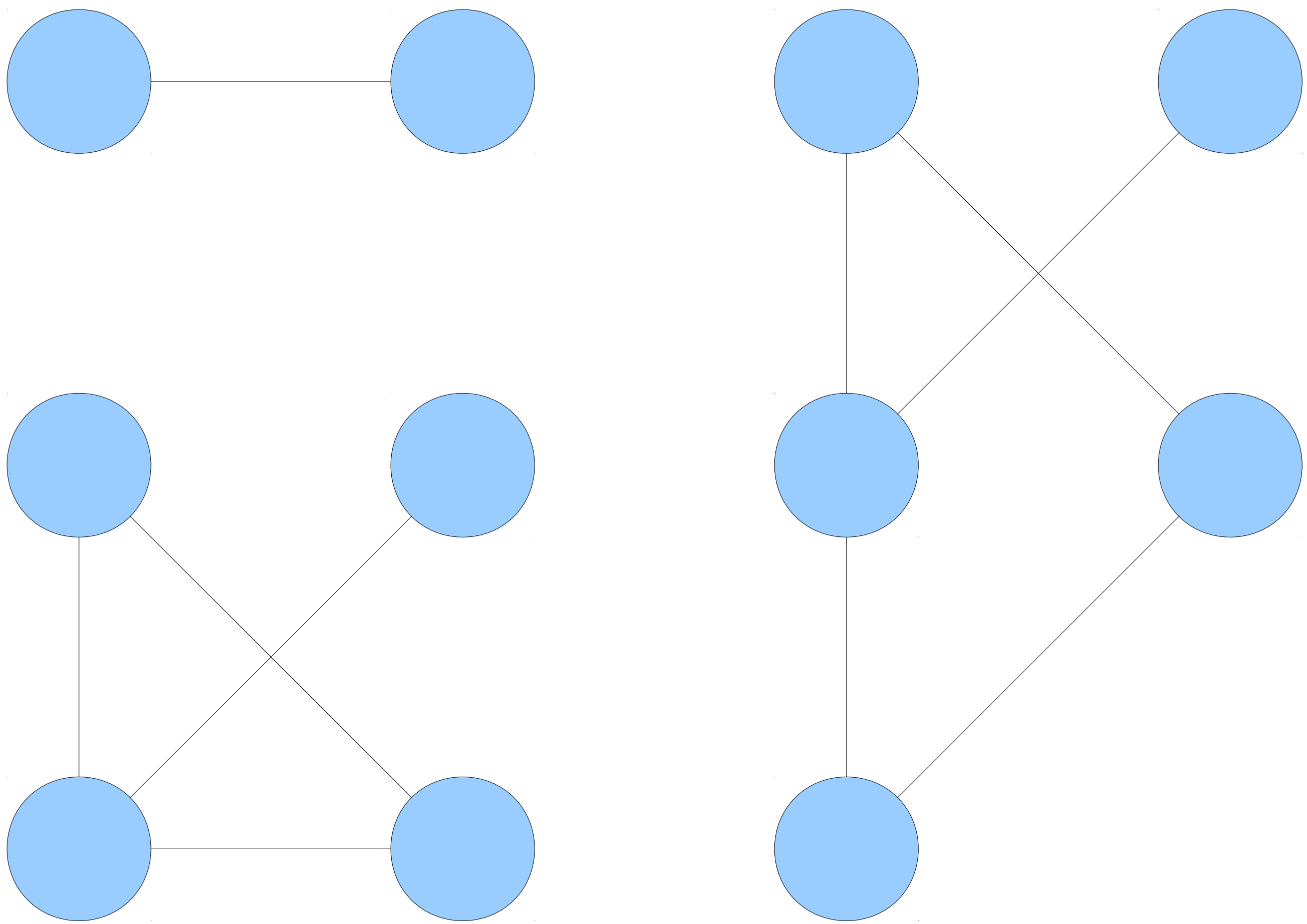
Properties of Connectivity

- ***Theorem:*** The following properties hold for the connectivity relation $\leftrightarrow$:
  - For any node $v \in V$, we have $v \leftrightarrow v$.
  - For any nodes $u, v \in V$, if $u \leftrightarrow v$, then $v \leftrightarrow u$.
  - For any nodes $u, v, w \in V$, if $u \leftrightarrow v$ and $v \leftrightarrow w$, then $u \leftrightarrow w$.
- Can prove by thinking about the paths that are implied by each.

Connected Components



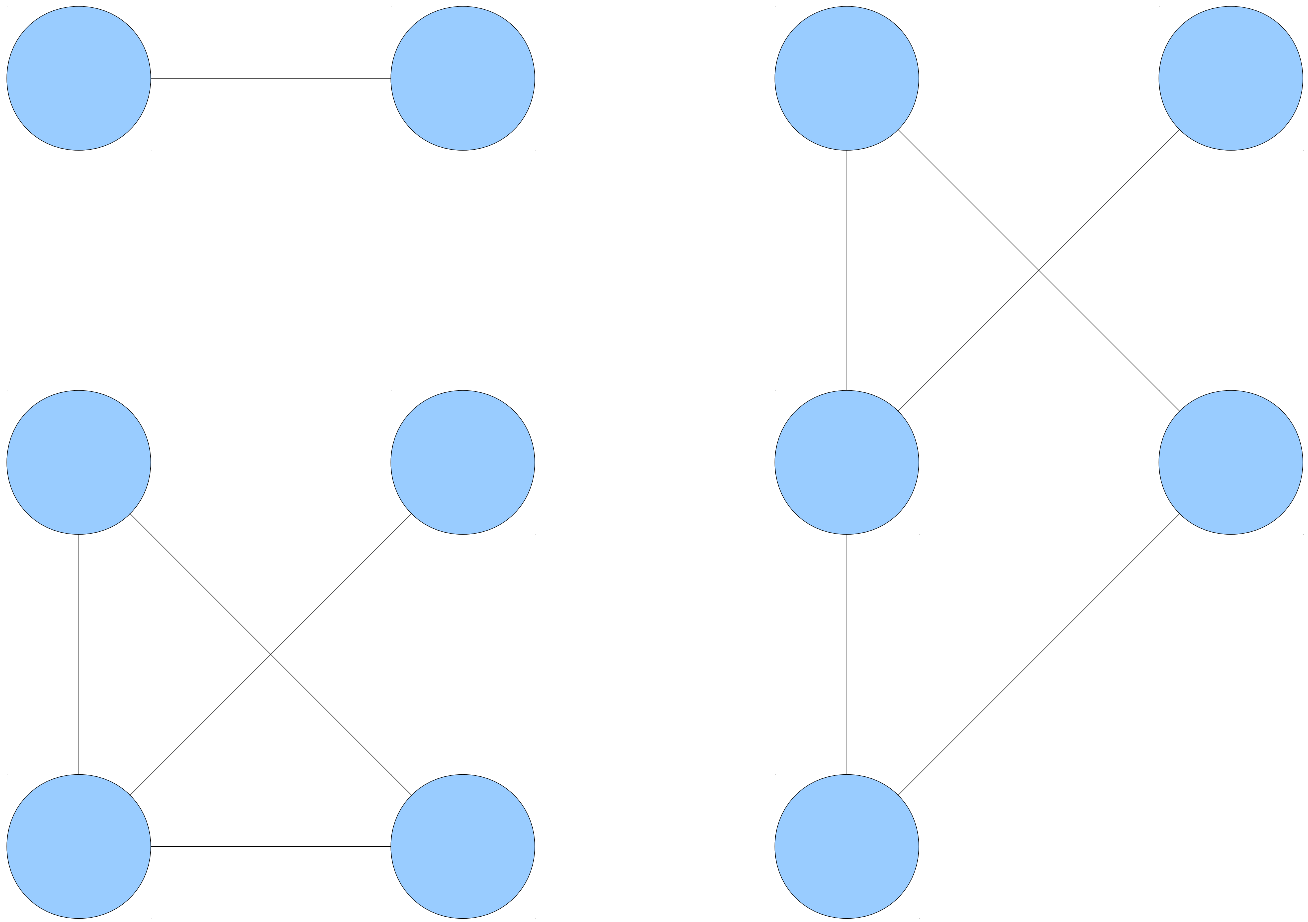


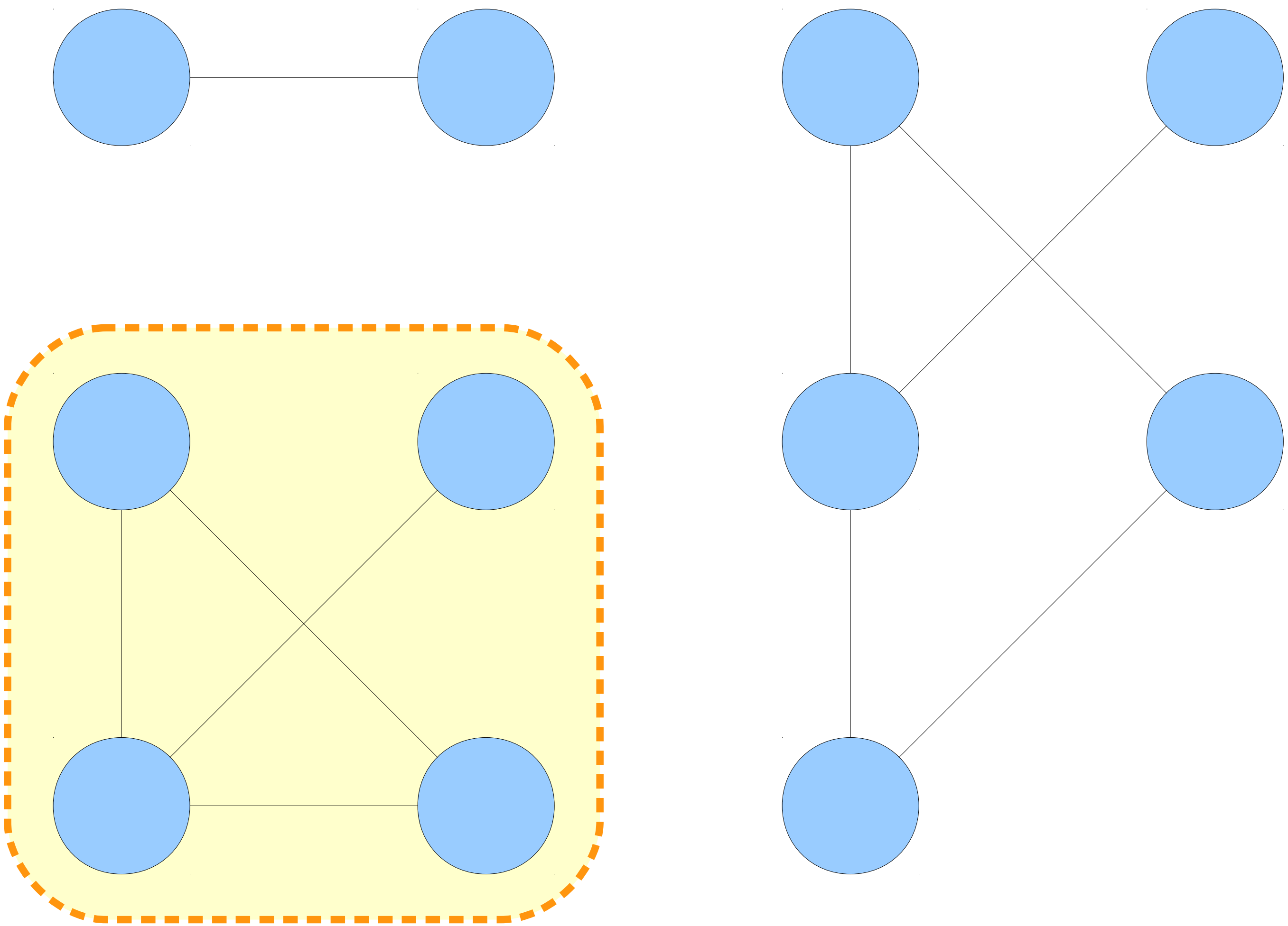


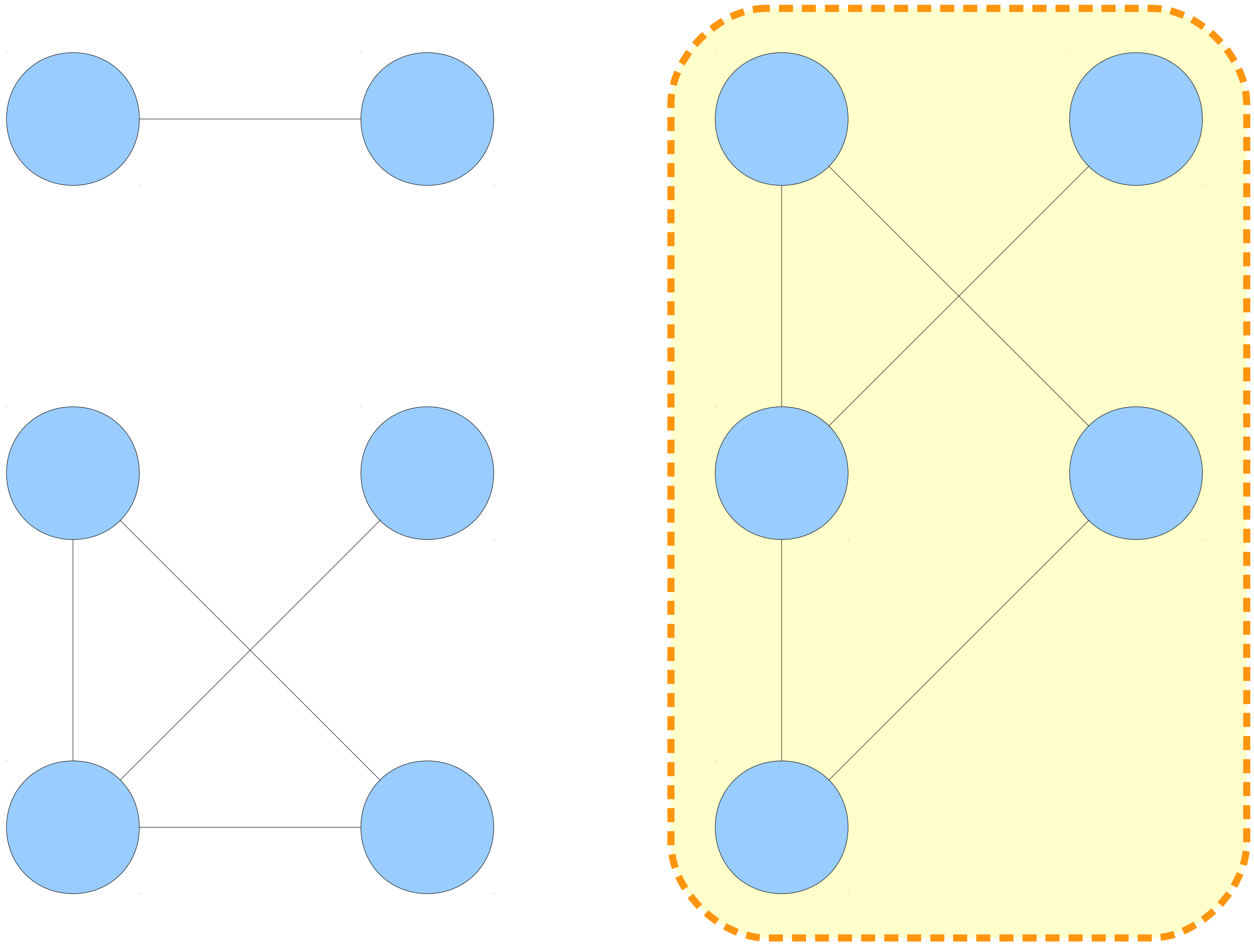
An Initial Definition

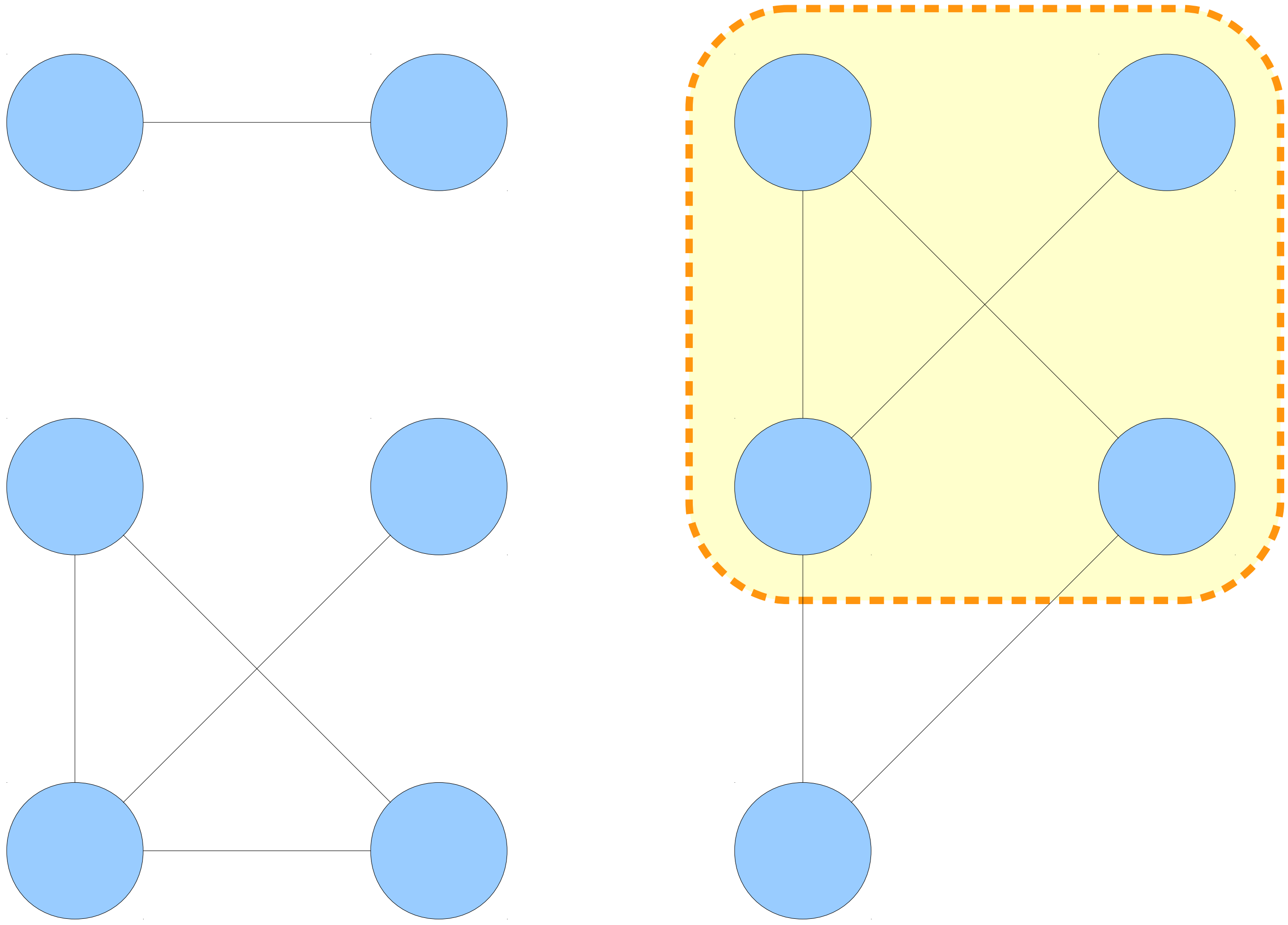
- **Attempted Definition #1:** A *piece* of an undirected graph $G = (V, E)$ is a set $C \subseteq V$ such that for any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
- Intuition: a piece of a graph is a set of nodes that are all connected to one another.

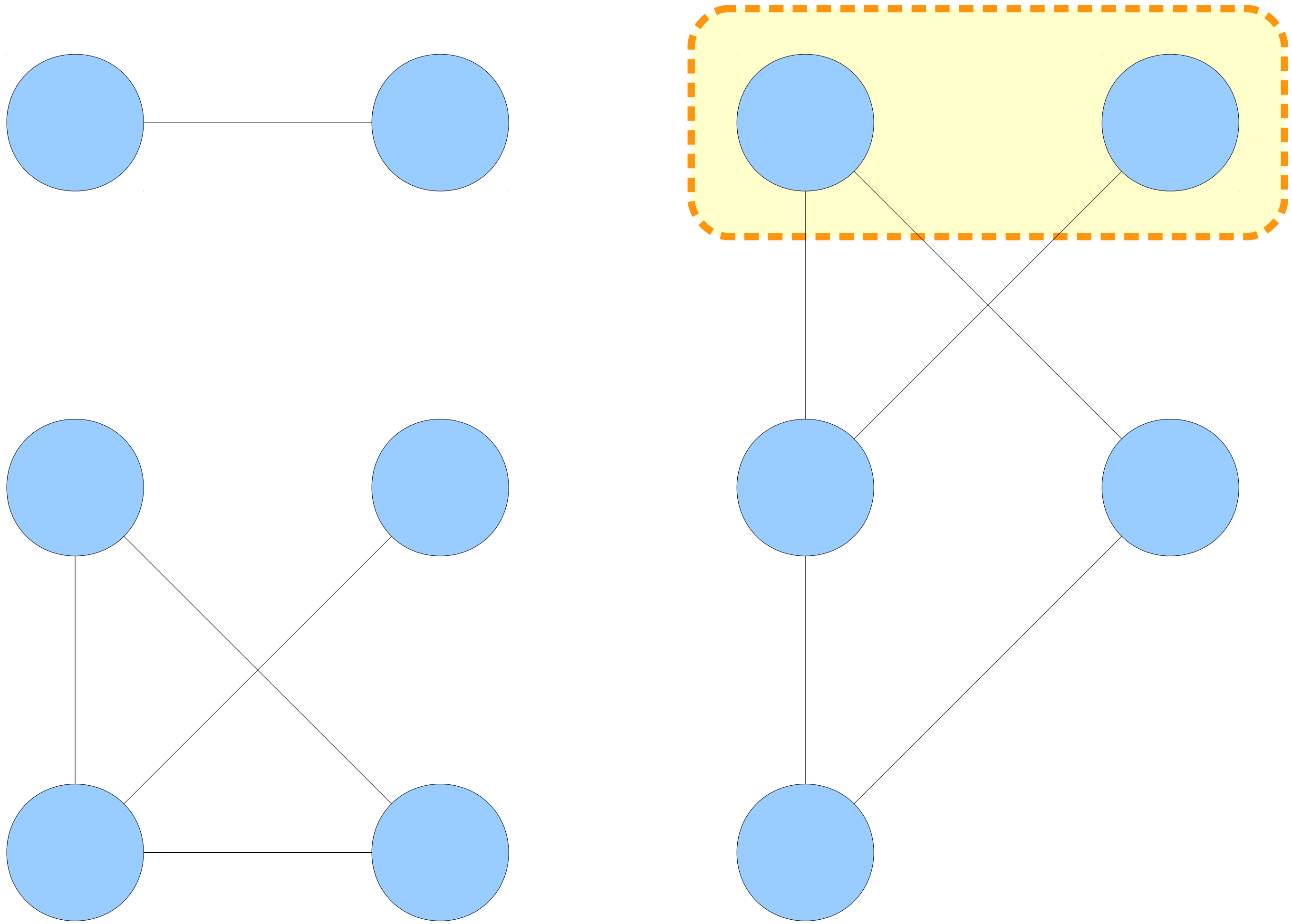
This definition has some problems; please don't use it as a reference.

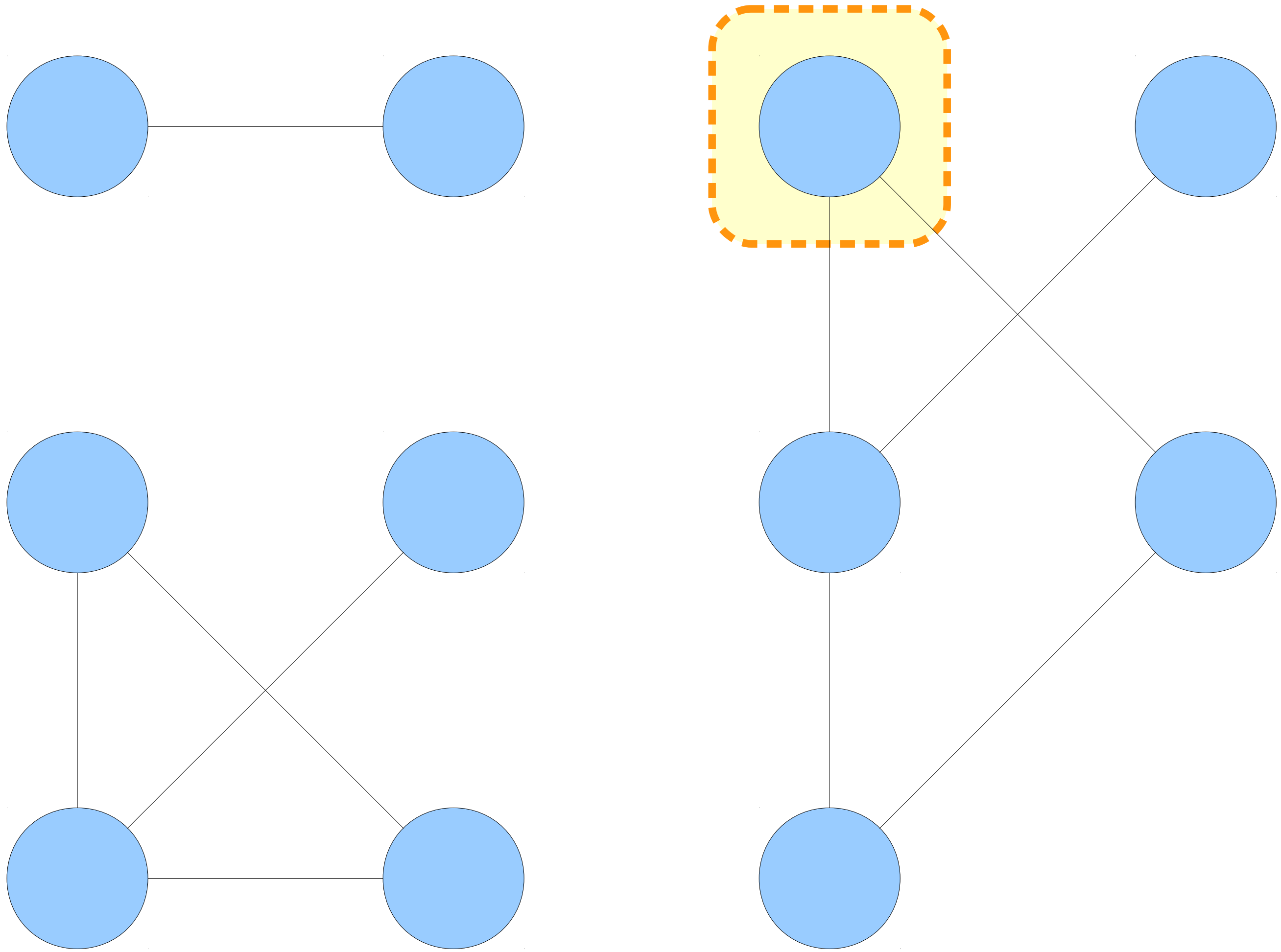








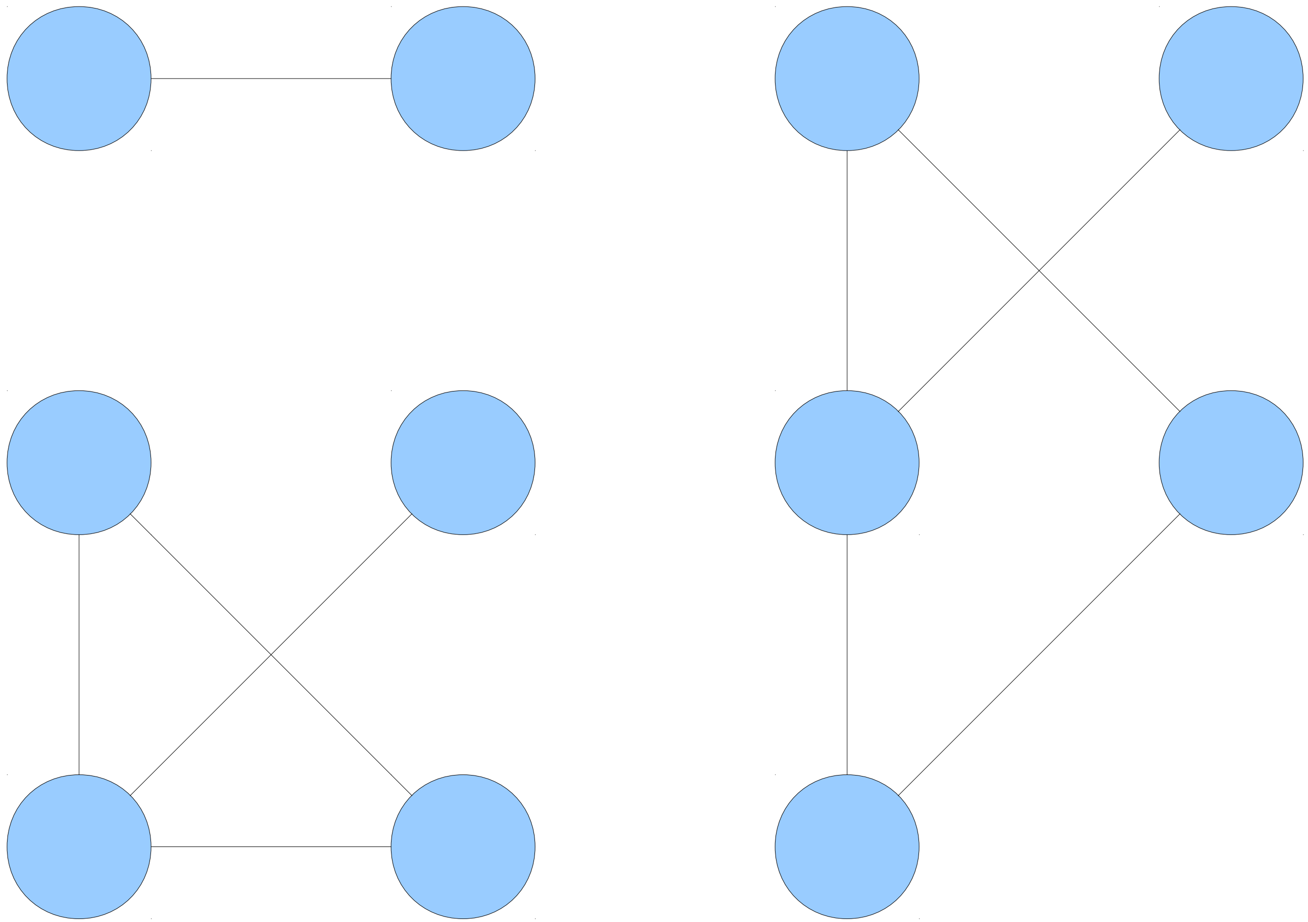


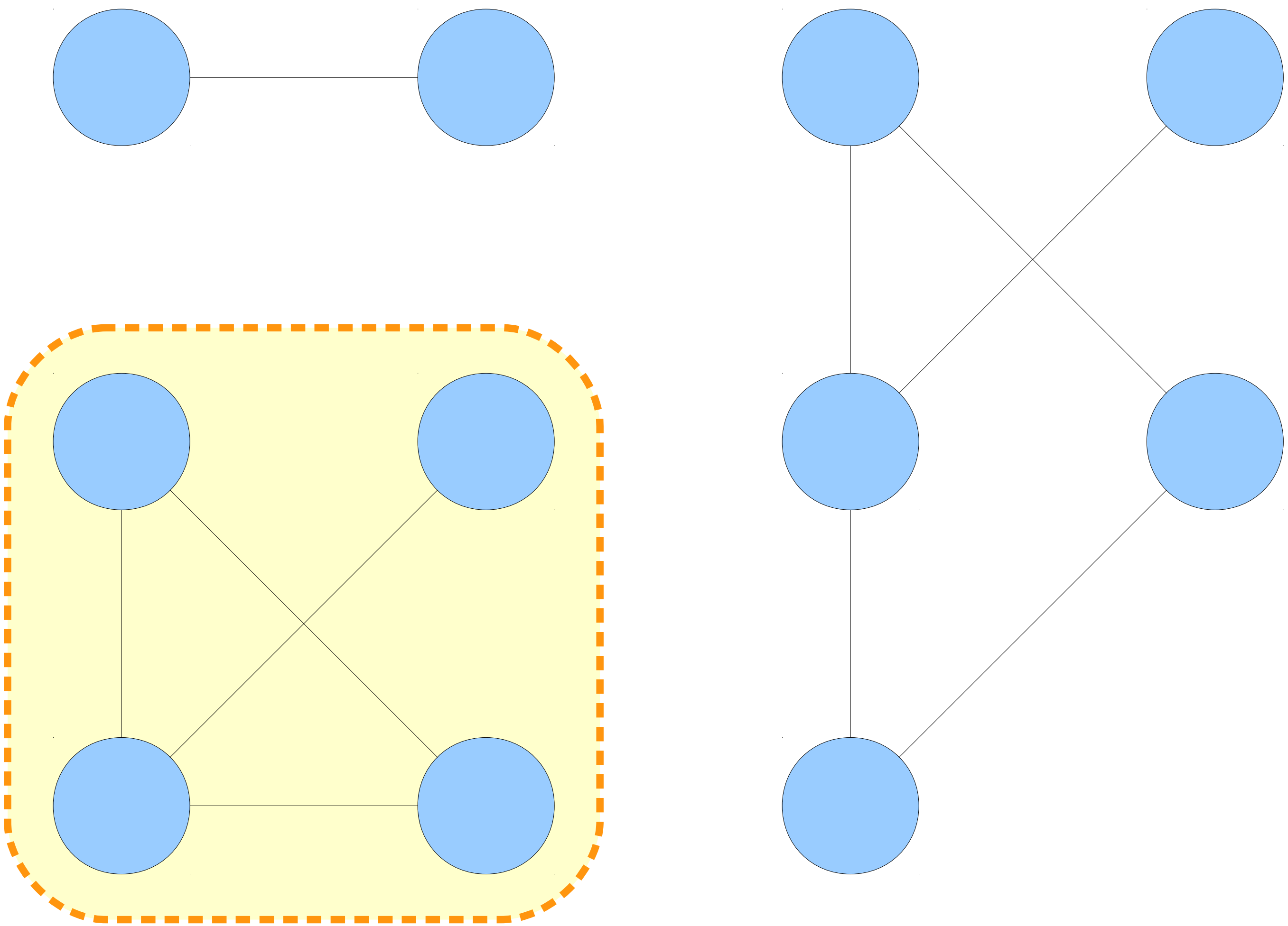


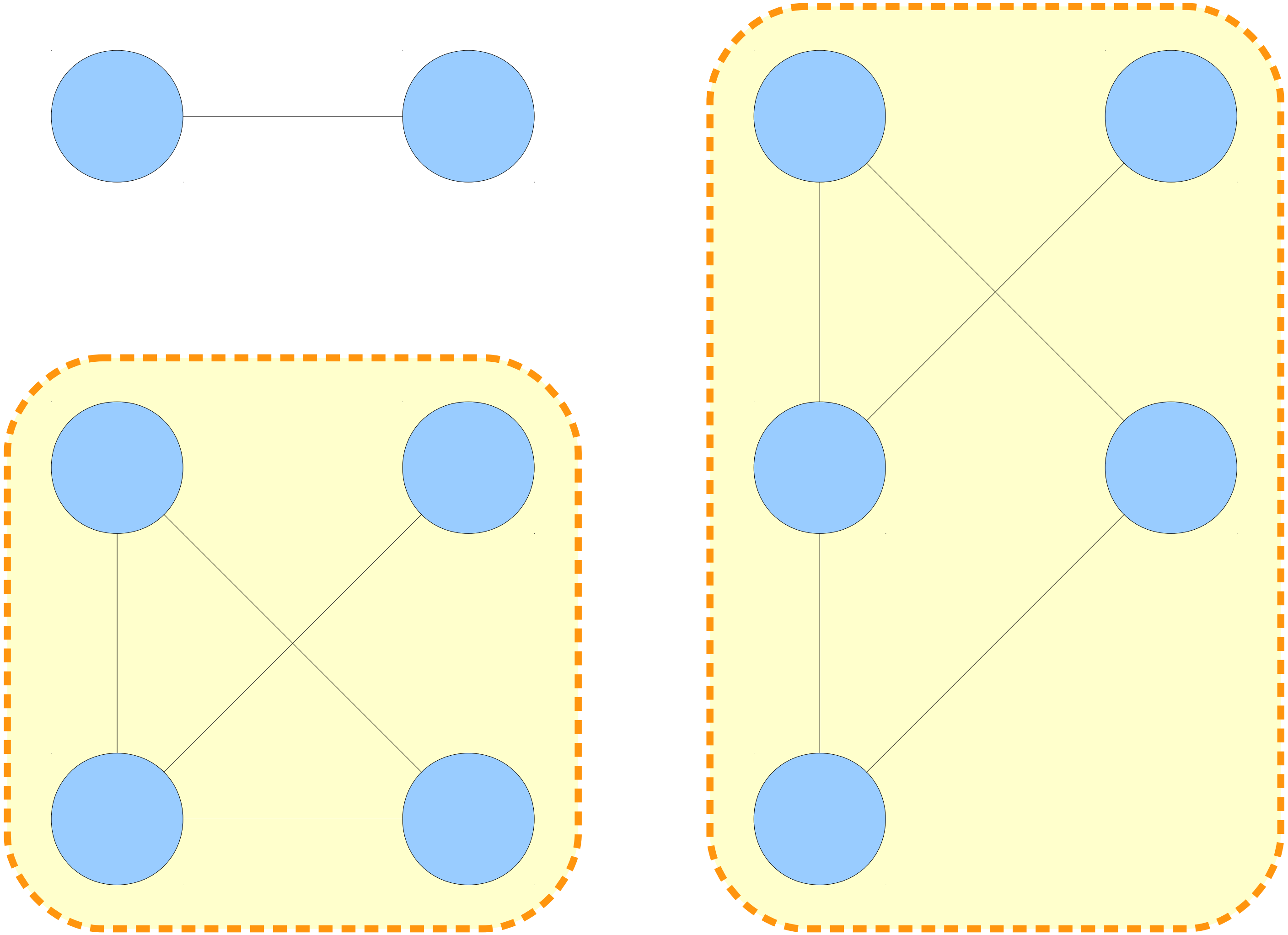
An Updated Definition

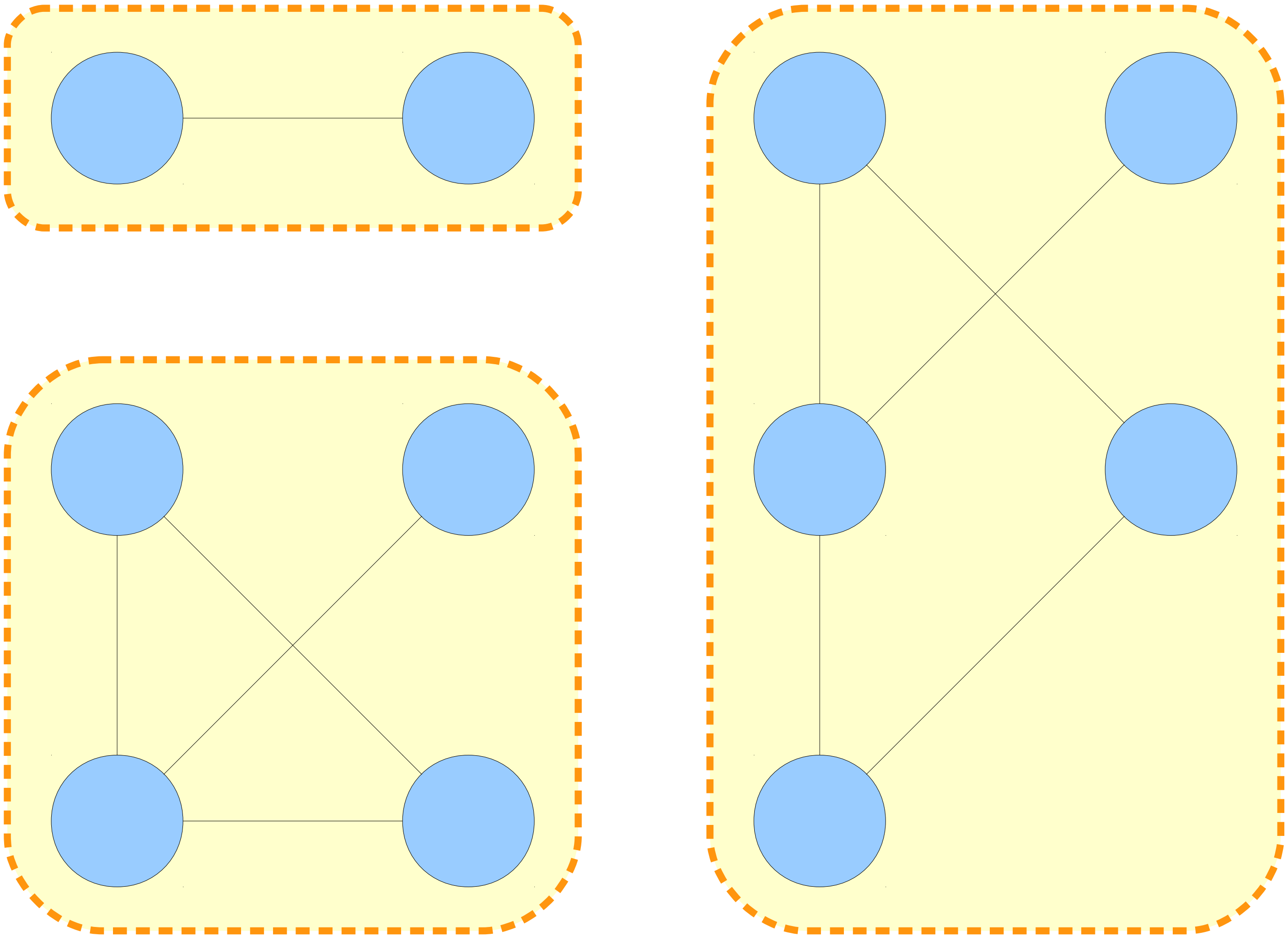
- **Attempted Definition #2:** A *piece* of an undirected graph $G = (V, E)$ is a set $C \subseteq V$ where
  - For any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
  - For any nodes $u \in C$ and $v \in V - C$, the relation $u \nleftrightarrow v$ holds.
- Intuition: a piece of a graph is a set of nodes that are all connected to one another that doesn't “miss” any nodes.

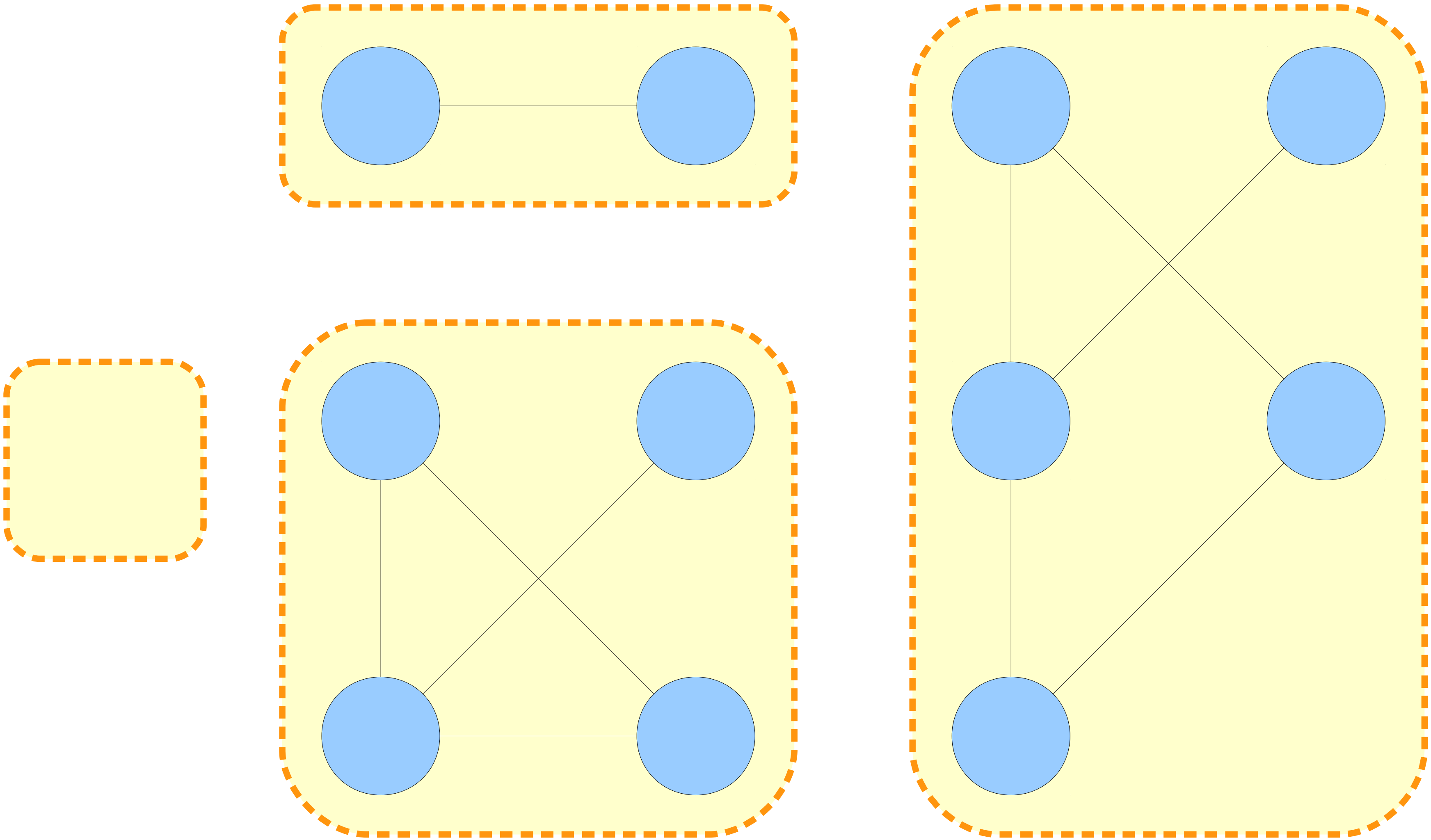
This definition still has problems; please don't use it as a reference.

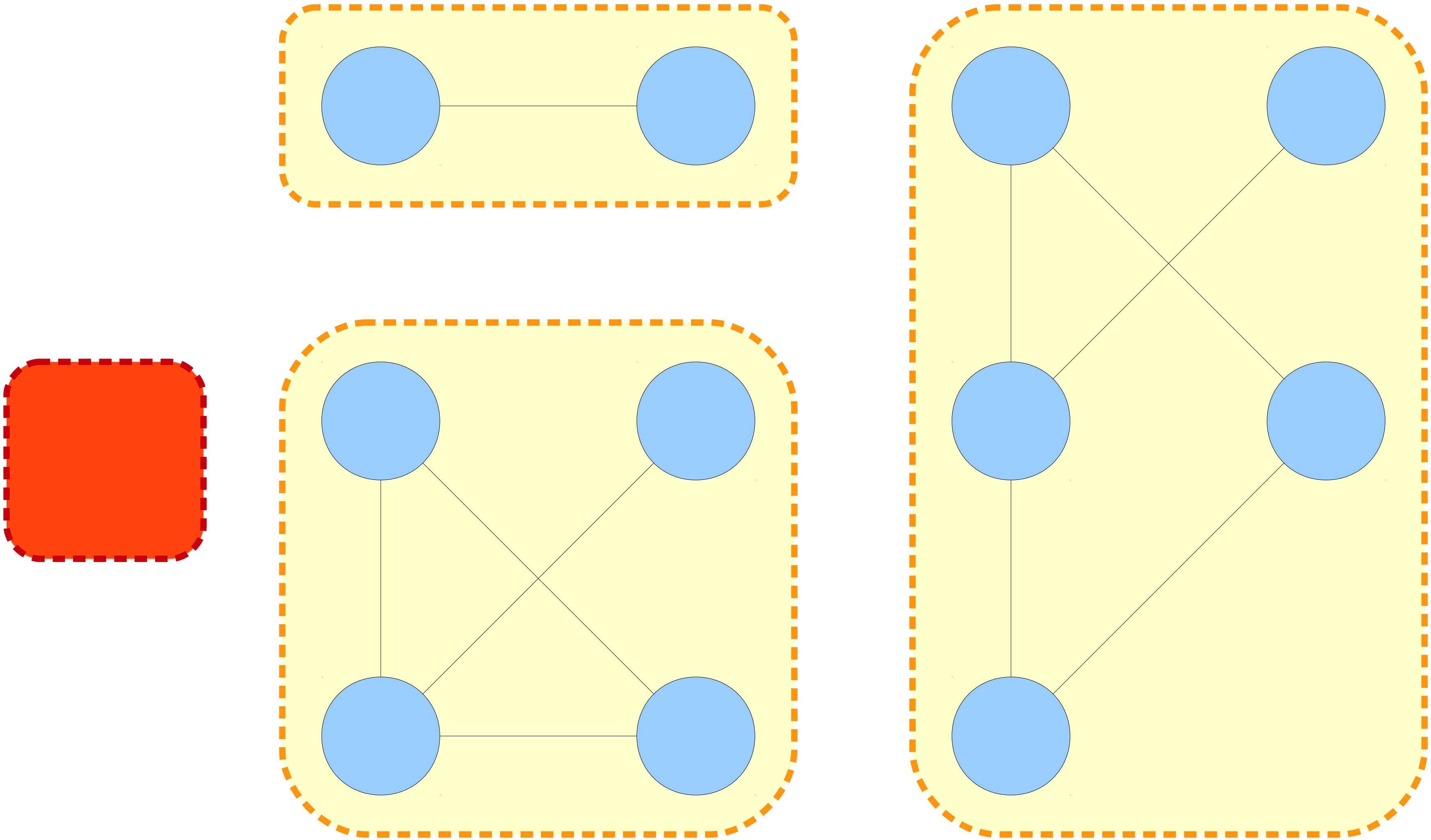












A Final Definition

- **Definition:** A **connected component** of an undirected graph $G = (V, E)$ is a nonempty set $C \subseteq V$ where
  - For any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
  - For any nodes $u \in C$ and $v \in V - C$, the relation $u \not\leftrightarrow v$ holds.
- Intuition: a connected component is a nonempty set of nodes that are all connected to one another that includes as many nodes as possible.

Some Announcements

Announcements

- Problem Set 1 solutions released at end of today's lecture.
  - Aiming to return problem sets no later than Thursday.
- Problem Set 2 out, due Friday at the start of lecture.
  - Checkpoints should be returned by Wednesday.

Announcements

- Two new TAs:
  - Je-ok Choi
  - Bertrand Decoster
- Welcome!

Casual CS Dinner

- Casual dinner for women studying computer science tomorrow.
  - 5:30PM – 8:00PM in Gates 519 (the newly renovated fifth floor!)
  - RSVP at **http://bit.ly/cscasualdinners**.
- *Highly recommended!*

Your Questions

</announcements>

Manipulating our Definition

Proving the Obvious

- ***Theorem:*** If $G = (V, E)$ is a graph, then every node $v \in V$ belongs to exactly one connected component.
- How exactly would we prove a statement like this one?
- Use an **existence and uniqueness proof**:
  - Prove there is *at least* one object of that type.
  - Prove there is *at most* one object of that type.
- These are usually separate proofs.

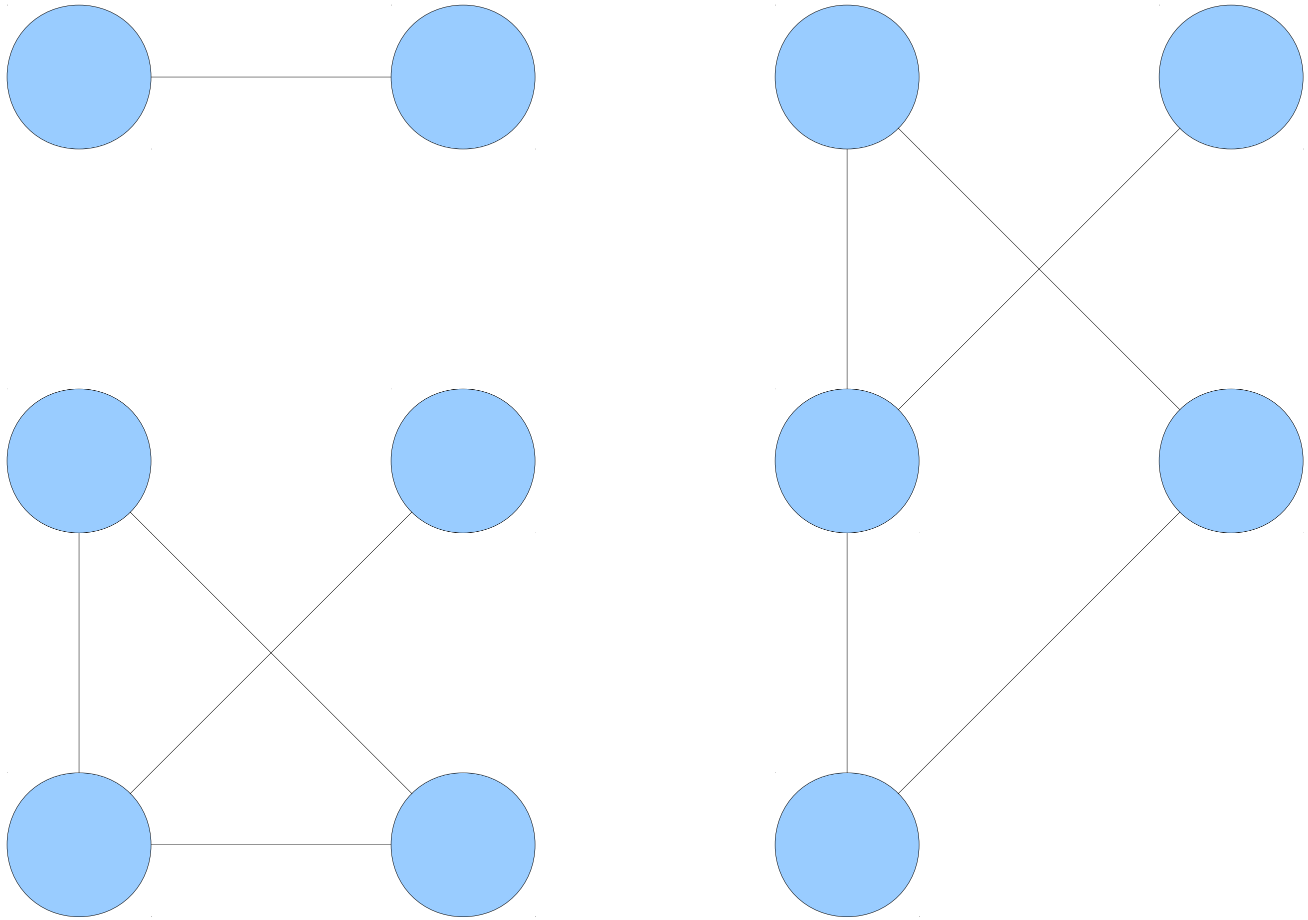
Part 1: **Every node belongs to at least one connected component.**

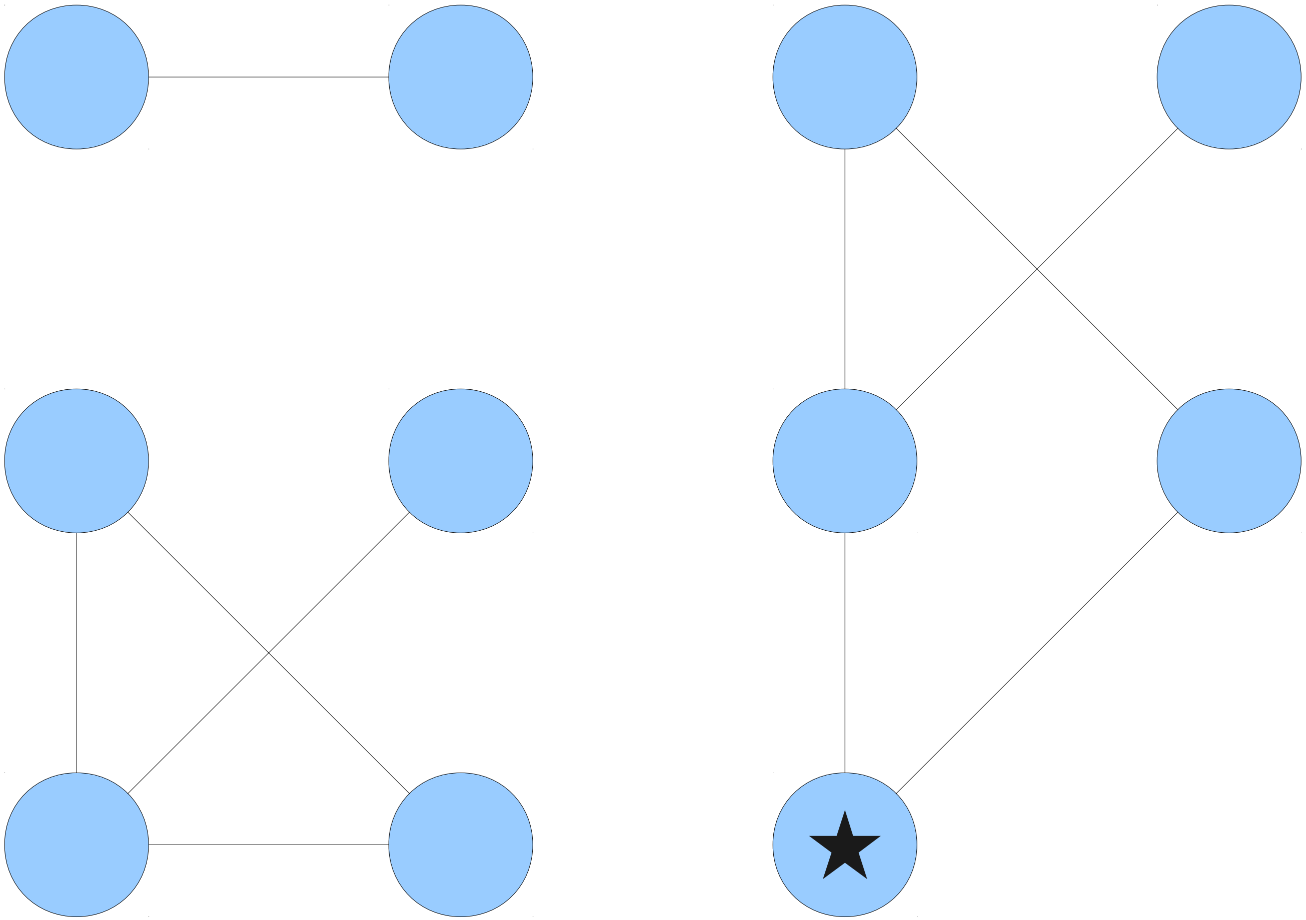
Proving Existence

- Given an arbitrary graph $G = (V, E)$ and an arbitrary node $v \in V$, we need to show that there exists some connected component C where $v \in C$.
- The key part of this is the existential statement

  There exists a connected component C such that $v \in C$.

- The challenge: how can we find the connected component that v belongs to given that v is an arbitrary node in an arbitrary graph?





The Conjecture

- **Conjecture:** Let $G = (V, E)$ be an undirected graph. Then for any node $v \in V$, the set $\{ x \in V \mid v \leftrightarrow x \}$ is a connected component and it contains v.
- If we can prove this, we have shown *existence*: at least one connected component contains v.

Lemma 1: Let $G = (V, E)$ be an undirected graph. For any node $v \in V$, the set $C = \{ x \in V \mid v \leftrightarrow x \}$ contains v.

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The Tricky Part

- We need to show for any $v \in V$ that the set $C = \{ x \in V \mid v \leftrightarrow x \}$ is a connected component.
- Therefore, we need to show
  - $C \neq \emptyset$;
  - for any $x, y \in C$, the relation $x \leftrightarrow y$ holds; and
  - for any $x \in C$ and $y \notin C$, the relation $x \nleftrightarrow y$ holds.

Lemma 2: Let $G = (V, E)$ be an undirected graph. Choose some node $v \in V$ and let $C = \{ x \in V \mid v \leftrightarrow x \}$. Then for any nodes $x, y \in C$, we have $x \leftrightarrow y$.

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Proof: By contradiction; assume $x \in C$ and $y \in V - C$, but that $x \leftrightarrow y$.

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Part 2: **Every node belongs to at most one connected component.**

Uniqueness Proofs

- To show there is at most one object with some property P, show the following:

If x has property P and y has property P, then $x = y$.

- Rationale: x and y are just different names for the same thing; at most one object of the type can exist.

Uniqueness Proofs

- Suppose that C_1 and C_2 are connected components containing v.
- We need to prove that $C_1 = C_2$.
- Idea: C_1 and C_2 are sets, so we can try to show that $C_1 \subseteq C_2$ and that $C_2 \subseteq C_1$.
  - Just because we're working at a higher level of abstraction doesn't mean our existing techniques aren't useful!

Lemma: Let C be a connected component of an undirected graph $G = (V, E)$ and $v \in V$ a node contained in C. Then for any $x \in V$, we have $x \in C$ iff $v \leftrightarrow x$.

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(⇒) First, we prove that if $x \in C$, then $v \leftrightarrow x$.

(⇐) Next, we prove that if $v \leftrightarrow x$, then $x \in C$.

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When proving a biconditional, it is common to split the proof apart into two directions. The symbols ($\Rightarrow$) and ($\Leftarrow$) denote where in the proof the two directions can be found.

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By using a similar line of reasoning and interchanging the roles of C_2 and C_1, we also see that $C_2 \subseteq C_1$.

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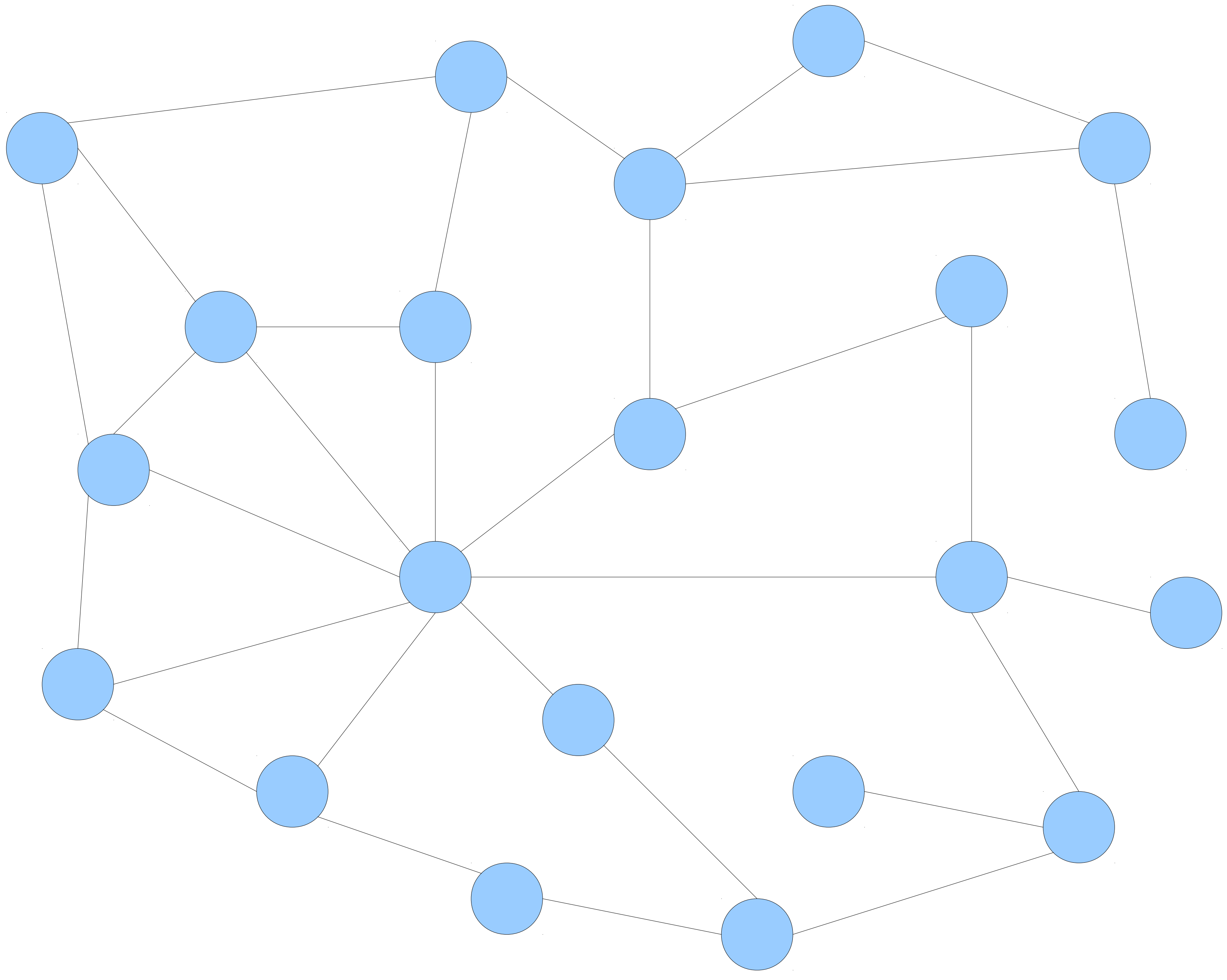
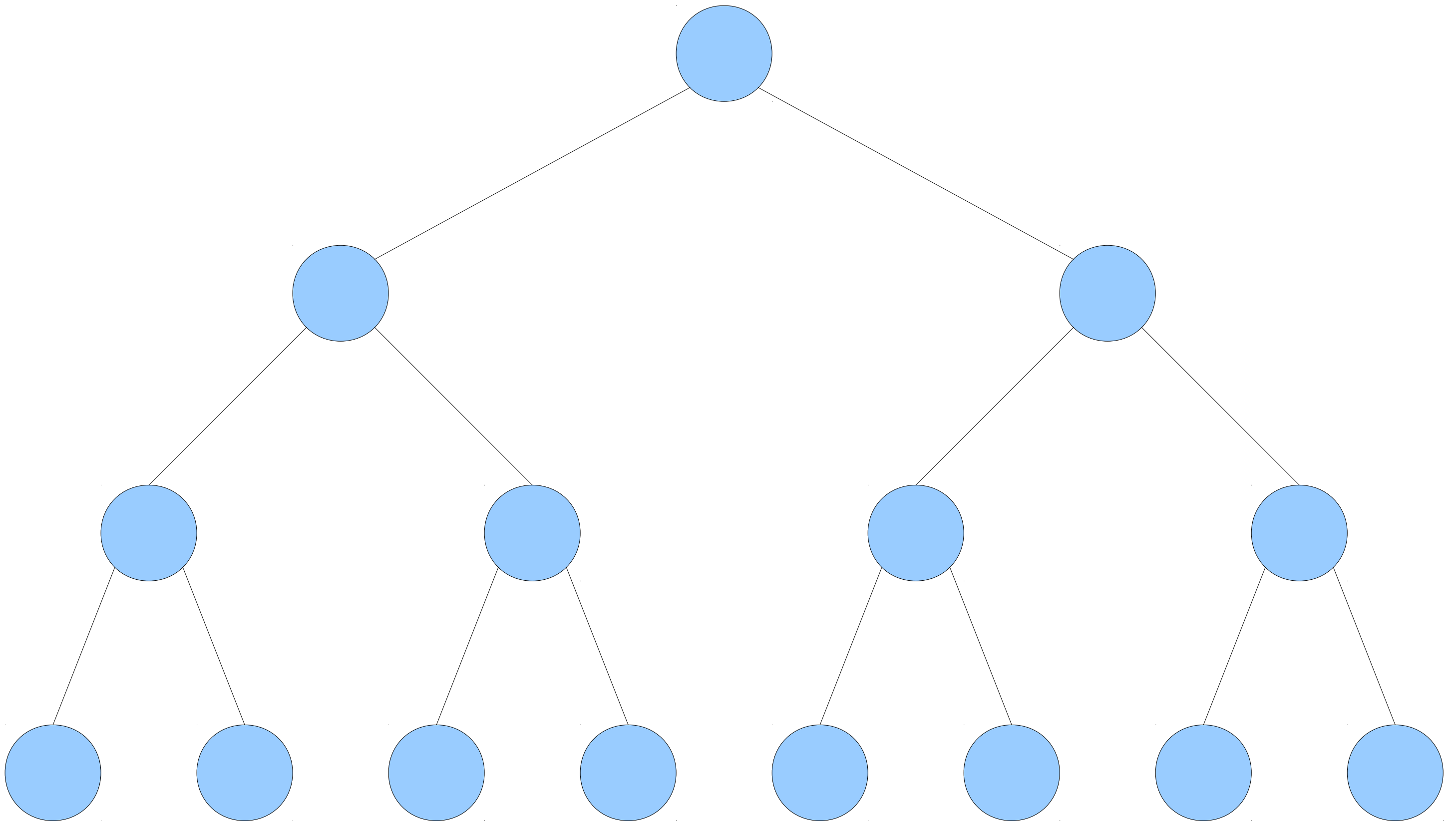
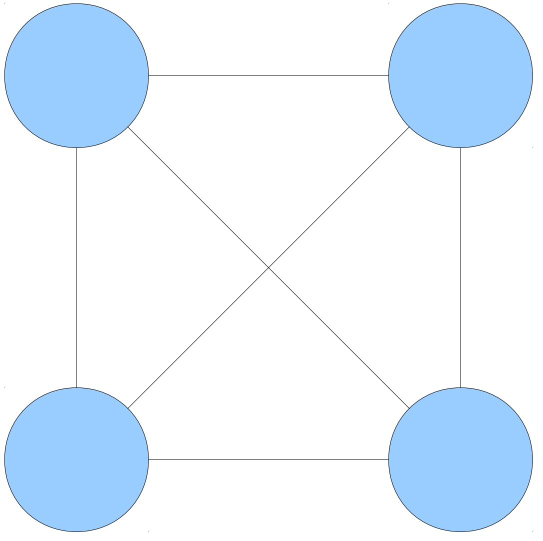
Why All This Matters

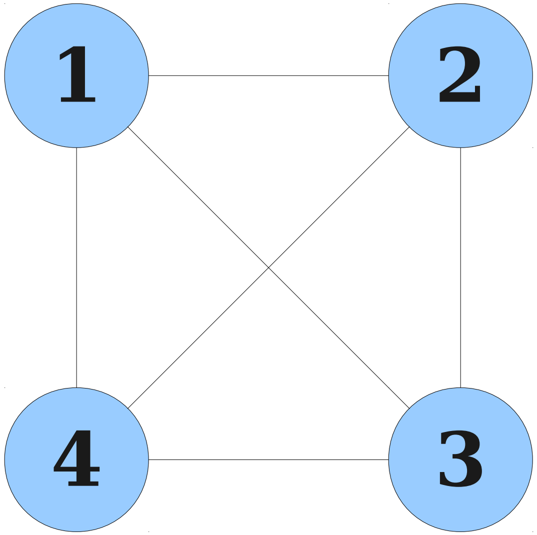
- I chose the example of connected components to
  - describe how to come up with a precise definition for intuitive terms;
  - see how to manipulate a definition once we've come up with one;
  - explore existence and uniqueness proofs, which we'll see more of later on; and
  - explore multipart proofs with several different lemmas.

Planar Graphs





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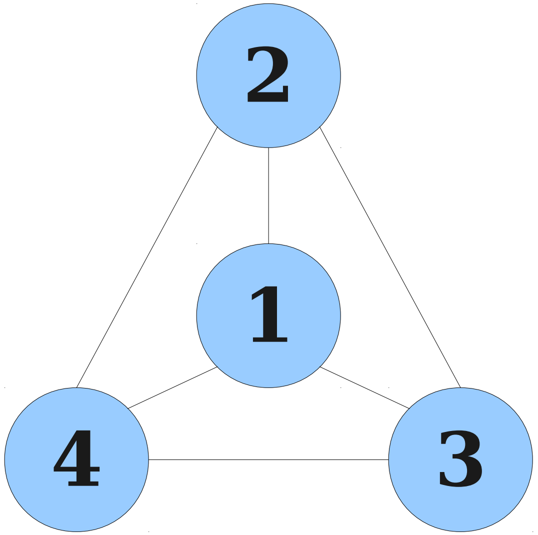
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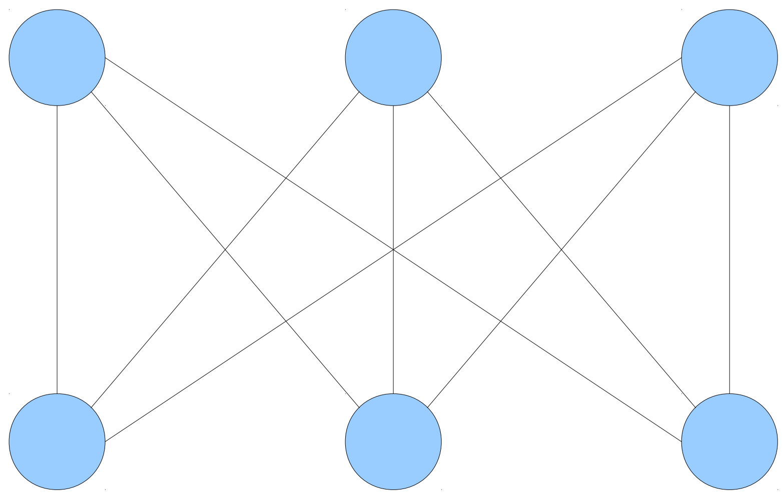
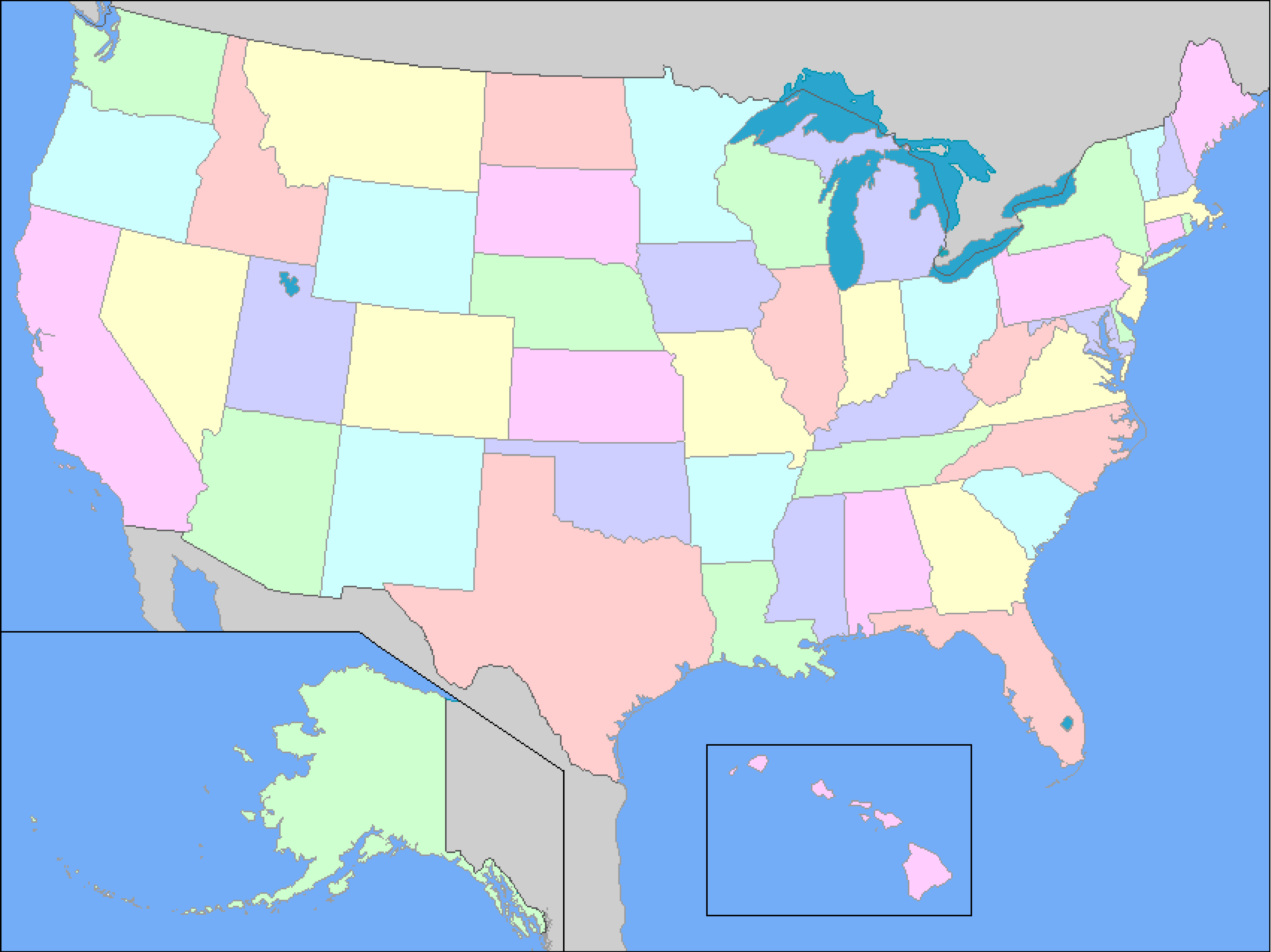
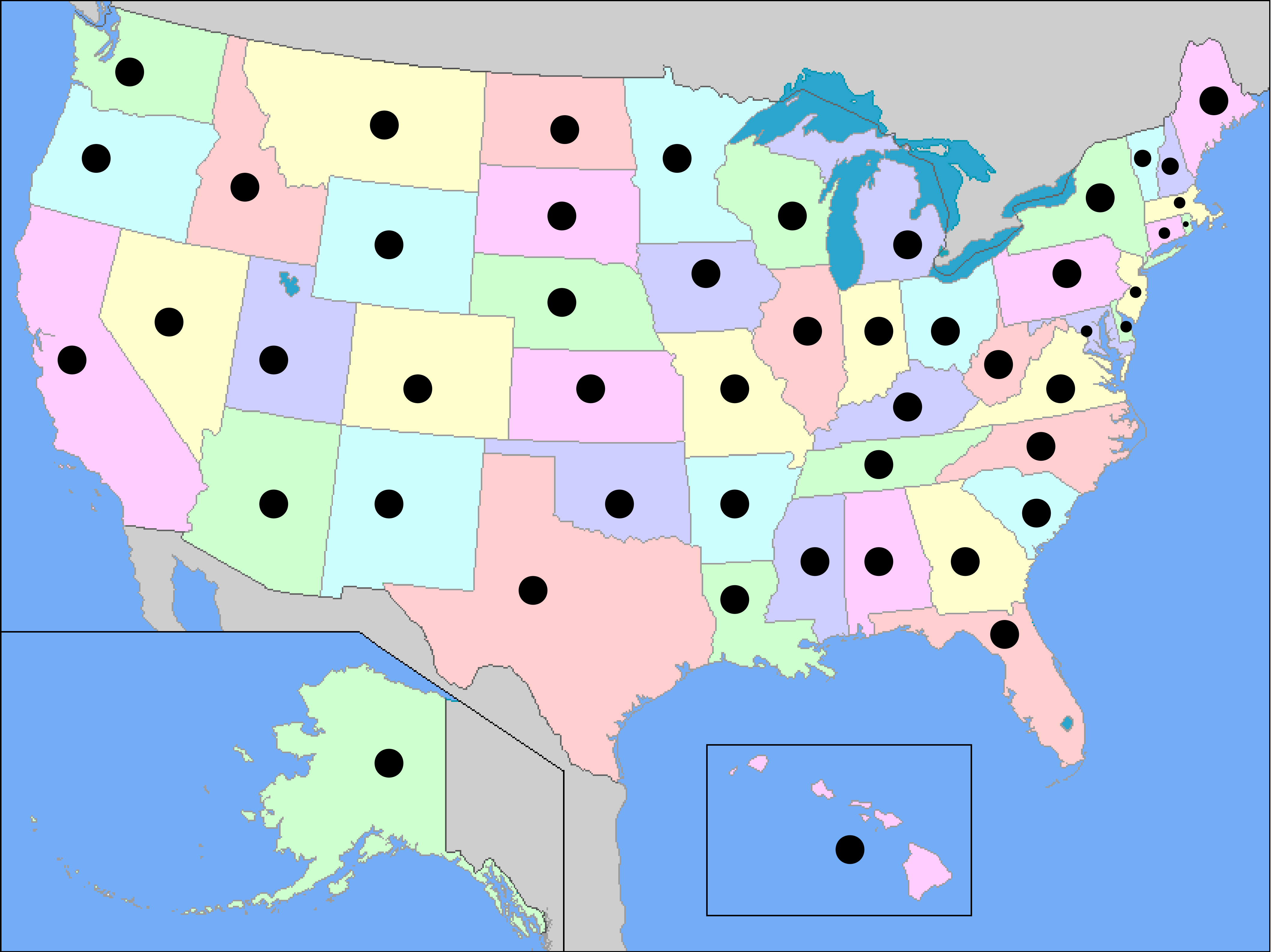
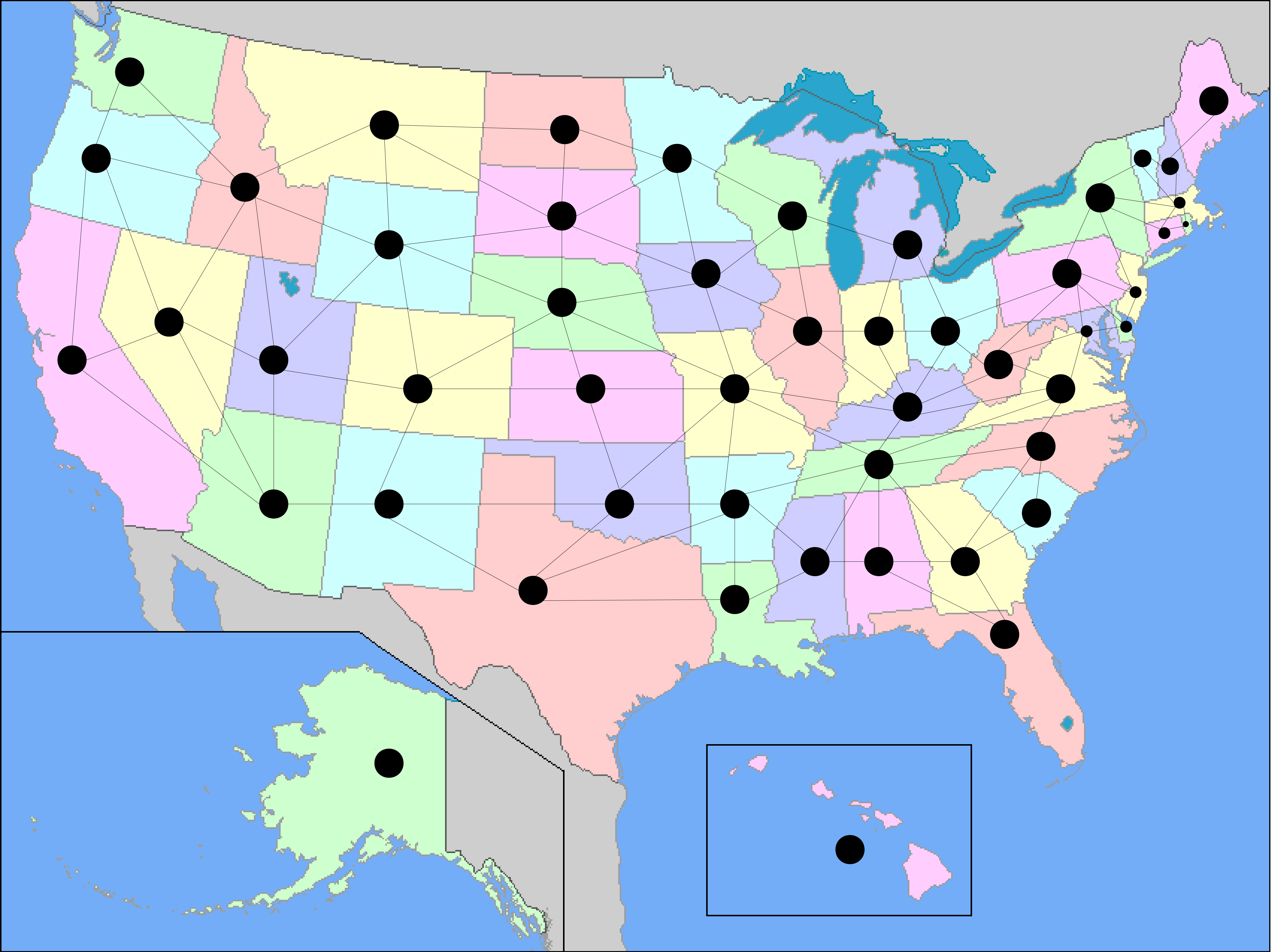


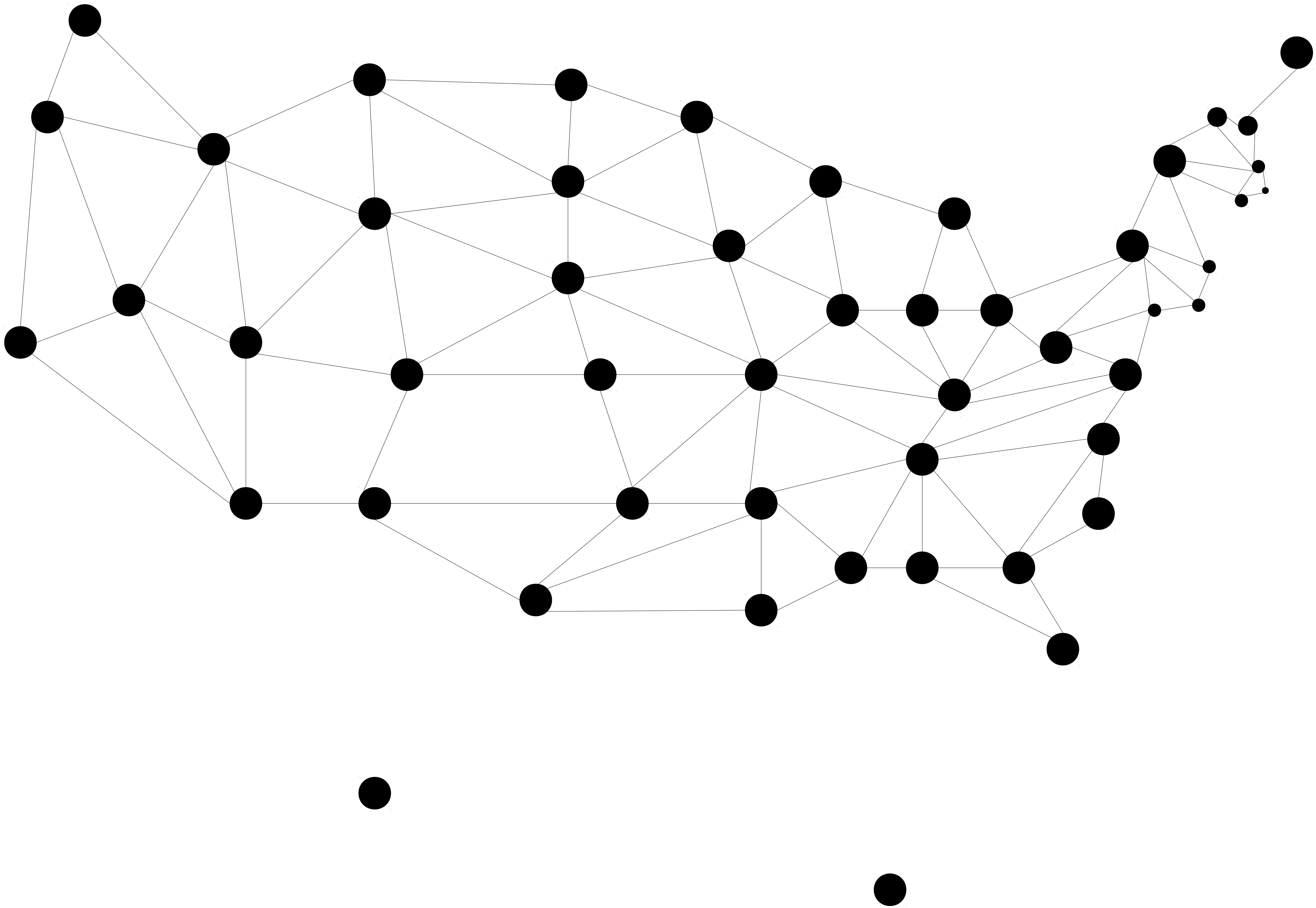
This graph is sometimes called the **utility graph**.

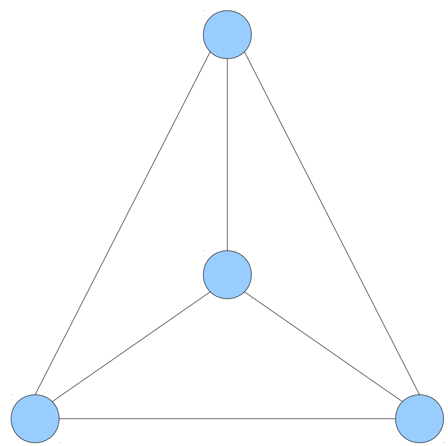
A graph is called a **planar graph** iff there is some way to draw it in a 2D plane without any of the edges crossing.

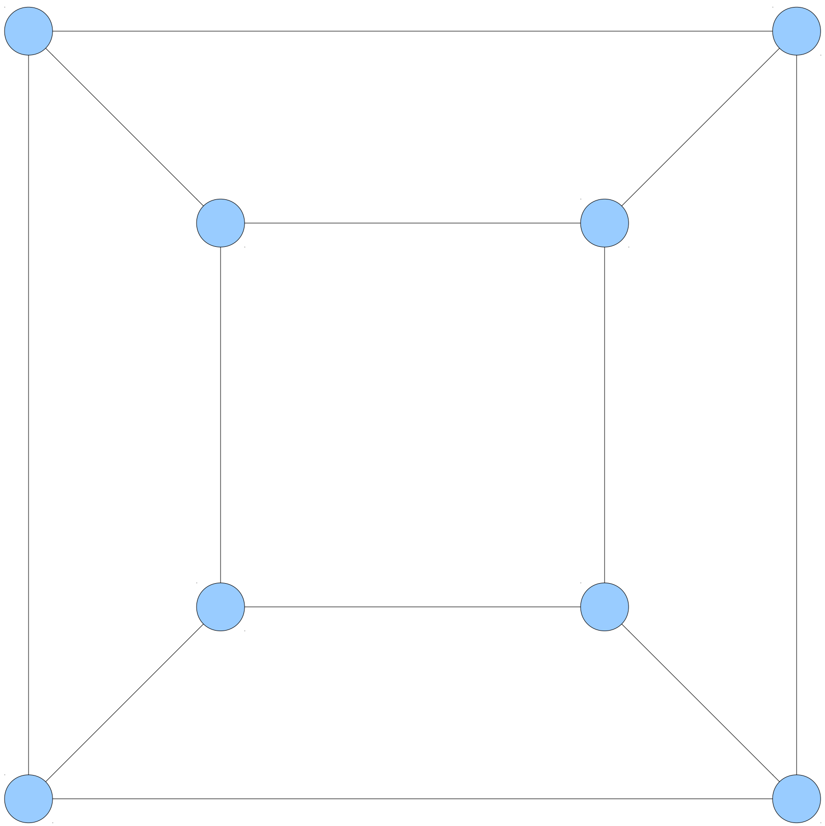


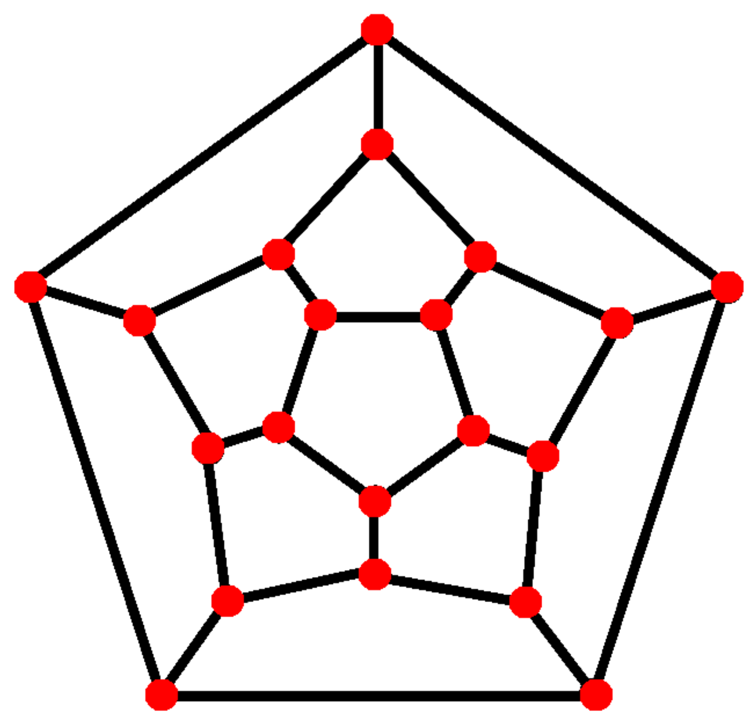




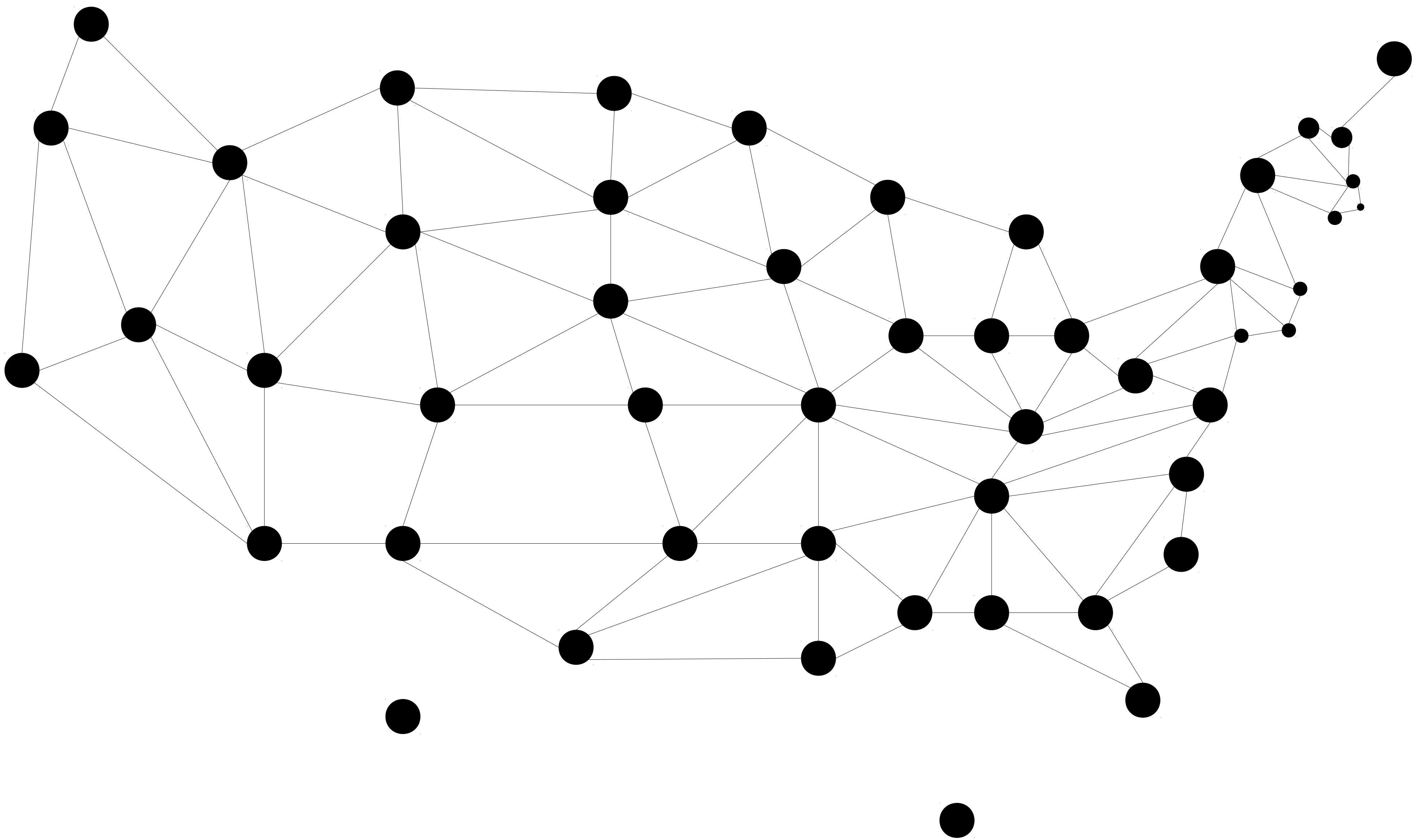








Graph Coloring



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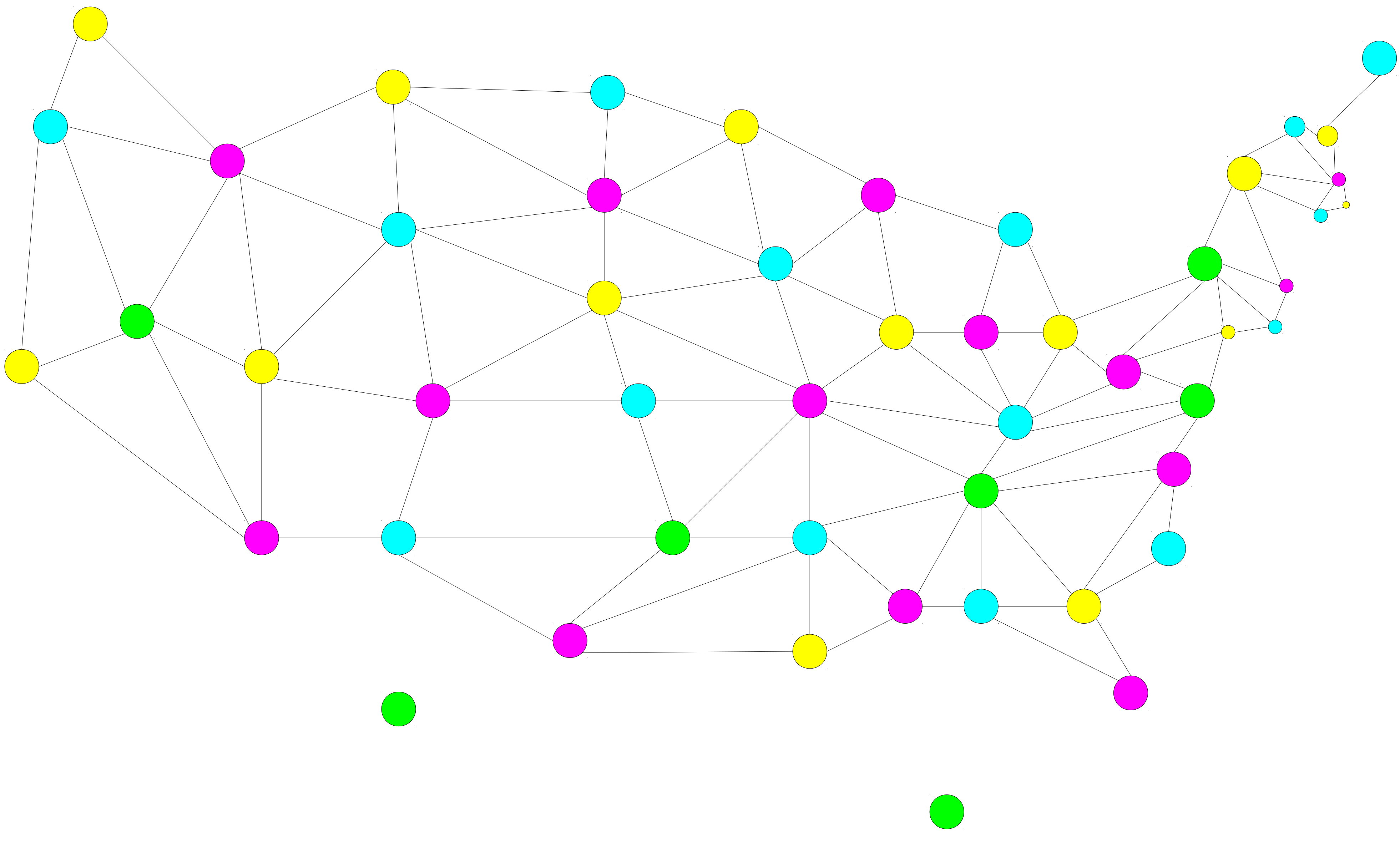

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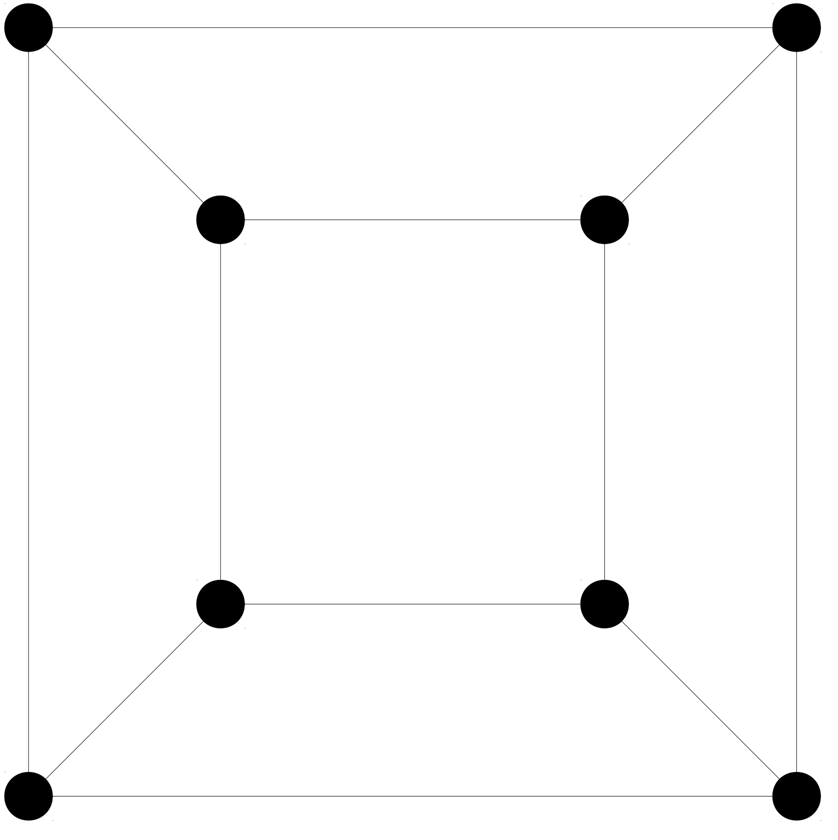

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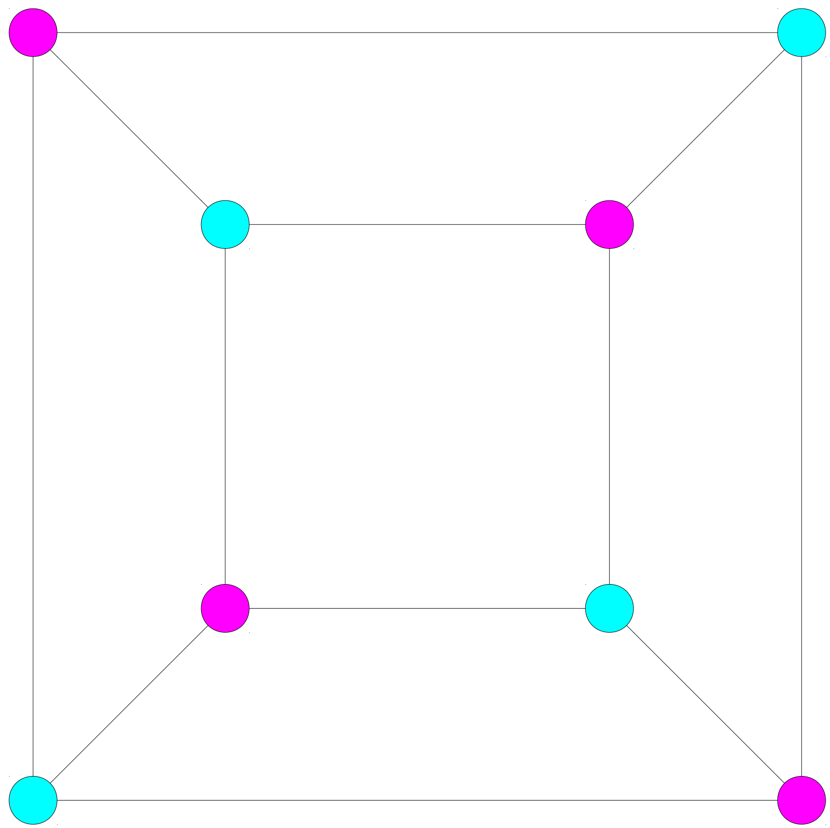

Graph Coloring

- An undirected graph $G = (V, E)$ with no self-loops (edges from a node to itself) is called ***k*-colorable** iff the nodes in V can be assigned one of k different colors such that no two nodes of the same color are joined by an edge.
- The minimum number of colors needed to color a graph is called that graph's **chromatic number**.

Theorem (Four-Color Theorem): Every planar graph is 4-colorable.

- **1850s:** Four-Color Conjecture posed.
- **1879:** Kempe proves the Four-Color Theorem.
- **1890:** Heawood finds a flaw in Kempe's proof.
- **1976:** Appel and Haken design a computer program that proves the Four-Color Theorem. The program checked 1,936 specific cases that are “minimal counterexamples;” any counterexample to the theorem must contain one of the 1,936 specific cases.
- **1980s:** Doubts rise about the validity of the proof due to errors in the software.
- **1989:** Appel and Haken revise their proof and show it is indeed correct. They publish a book including a 400-page appendix of all the cases to check.
- **1996:** Roberts, Sanders, Seymour, and Thomas reduce the number of cases to check down to 633.
- **2005:** Werner and Gonthier repeat the proof using an established automatic theorem prover (Coq), improving confidence in the truth of the theorem.

Next Time

- **Binary Relations**
  - Another way of studying connectivity.
- **The Pigeonhole Principle**
  - Proof by counting?