Problem Set 1 Checkpoint Solutions

If n is a multiple of three, n^2 is a multiple of three.

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i. Prove the first of these statements with a direct proof.

Proof: Let n be an arbitrary multiple of three. By definition, this means $n = 3k$ for some integer k. Thus $n^2 = (3k)^2 = 9k^2 = 3(3k)^2$. Since $n^2 = 3(3k^2)$ and $3k^2$ is an integer, the number n^2 is a multiple of three. ■

ii. Prove the second of these statements using the contrapositive. Make sure that you state the contrapositive of the statement explicitly before you attempt to prove it.

Proof: By contrapositive; we show that if n is a not multiple of three, then n^2 is not a multiple of three. If n is not a multiple of three, then either n is congruent to 1 modulo 3 or n is congruent to 2 modulo 3. We consider these cases independently:

Case 1: n is congruent to 1 modulo 3. Then there is an integer k where $n = 3k + 1$, so $n^2 = (3k + 1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$. Since $n^2 = 3(3k^2 + 2k) + 1$ and $(3k^2 + 2k)$ is an integer, this means that n^2 is congruent to 1 modulo 3.

Case 2: n is congruent to 2 modulo 3. Then there is some integer k such that $n = 3k + 2$. Therefore, $n^2 = (3k + 2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$. Since $n^2 = 3(3k^2 + 4k + 1) + 1$ and $(3k^2 + 4k + 1)$ is an integer, this means that n^2 is congruent to 1 modulo 3.

In either case, n^2 is congruent to 1 modulo 3, so n^2 is not a multiple of three. ■

iii. Prove, by contradiction, that $\sqrt{3}$ is irrational. Make sure that you explicitly state what assumption you are making before you derive a contradiction from it. Recall from lecture that a rational number is one that can be written as p / q for integers p and q where $q \neq 0$ and p and q have no common divisor other than ± 1.

Proof: By contradiction; assume that $\sqrt{3}$ is rational. Then there exist integers p and q such that $p / q = \sqrt{3}$, $q \neq 0$, and p and q have no factors in common other than 1 and -1.

Since $p / q = \sqrt{3}$, we have $p = \sqrt{3}\,q$, so $p^2 = 3q^2$. This means p^2 is a multiple of three, so by our above result p is a multiple of three. Thus there exists an integer k such that $p = 3k$.

Since $3q^2 = p^2$ and $p = 3k$, we have $3q^2 = (3k)^2 = 9k^2$, so $q^2 = 3k^2$. This means that q^2 is a multiple of three, so by our above result q is a multiple of three.

But this means that both p and q have 3 as a common divisor, contradicting the fact that p and q have no factors in common other than 1 and -1. We have reached a contradiction, so our assumption must have been wrong. Thus $\sqrt{3}$ is irrational. ■