Practice Midterm 2 Exam Solutions

Problem One: First-Order Logic (25 points total)

(i) Directed Acyclic Graphs (10 Points)

Given the predicates

> *DAG*(*G*), which states that *G* is a directed graph with no cycles;
> *NodeIn(v, G)*, which states that *v* is a node in *G*; and
> *EdgeIn(u, v, G)*, which states that in graph *G*, there is an edge from *u* to *v*,

write a statement in first-order logic that says "every directed graph with no cycles has a node with no incoming edges."

One option is

$$\boldsymbol{\forall G. (DAG(G) \to \exists v. (NodeIn(v, G) \land \forall u. (NodeIn(u, G) \to \neg EdgeIn(u, v, G))}$$

Common mistakes included mixing up the ∧ and → connectives, or factoring out quantifiers in a way that changed the meaning of the first-order statement. For example, the statement

$$\exists v. (NodeIn(u, G) \land \forall u. (NodeIn(u, G) \to \neg EdgeIn(u, v, G))$$

Is not the same as the following statement, since the parentheses are significant:

$$\exists v. \forall u. (NodeIn(u, G) \land NodeIn(u, G) \to \neg EdgeIn(u, v, G)$$

(ii) The Logic of Elections (15 Points)

Suppose that it's next Tuesday night and the two major candidates are examining the vote counts in the 2012 US Presidential Election. (To avoid turning this midterm into a political debate, let's call the candidates Candidate *X* and Candidate *Y*.) Candidate *X* argues that more people voted for him than for Candidate *Y* by making the following claim: "For every ballot cast for Candidate *Y*, there were two ballots cast for Candidate *X*." Candidate *X* states this in first-order logic as follows:

$$\forall b.\ (BallotForY(b) \to \exists b_1.\ \exists b_2.\ (BallotForX(b_1) \land BallotForX(b_2) \land b_1 \neq b_2))$$

However, it is possible for the above first-order logic statement to be true even if Candidate *X* didn't get the majority of the votes. Give an example of a set of ballots such that

- Every ballot is cast for exactly one of Candidate *X* and Candidate *Y*,
- The set of ballots obeys the rules described by the above statement in first-order logic, but
- Candidate *Y* gets strictly more votes than Candidate *X*.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement when explaining why your set of ballots obeys it.

Take any collection of ballots where there are at least three ballots for Candidate *Y* and exactly two ballots for Candidate *X*. Thus every ballot is cast for exactly one of Candidate *X* and Candidate *Y*, and Candidate *Y* gets strictly more votes than Candidate *X*. Additionally, the above first-order statement logic is true: for every choice of b you can make that is a ballot for Candidate *Y*, there are choices b_1 and b_2 of ballots for Candidate *X* you can make. They just happen to be the same ballots every time.

The most common mistake we saw was arguing that the requirement for b_1 and b_2 could be vacuously true if there are no ballots for candidate *X*. Vacuous truth can only apply to implications involving the universal quantifier, rather than the existential quantifier.

Problem Two: Finding Flaws in Proofs (15 points)

In lecture, we sketched a proof that $|\mathbb{N}| = |\mathbb{N}^2|$; that is, that there are the same number of natural numbers as pairs of natural numbers. However, below is a purported proof that $|\mathbb{N}| \neq |\mathbb{N}^2|$.

Theorem: $|\mathbb{N}| \neq |\mathbb{N}^2|$.

Proof: By contradiction; assume that $|\mathbb{N}| = |\mathbb{N}^2|$. Then there exists some bijection $f : \mathbb{N} \to \mathbb{N}^2$, so for any $n \in \mathbb{N}$, $f(n)$ is an ordered pair of natural numbers. For notational simplicity, denote the first value in the ordered pair $f(n)$ by $f_1(n)$ and the second value by $f_2(n)$. Thus $f(n) = (f_1(n), f_2(n))$.

Now, consider the sequence $p_0, p_1, p_2, \ldots$ defined as follows: for any $n \in \mathbb{N}$, let $p_n = (f_1(n) + 1, f_2(n) + 1)$. Note that each p_n is an ordered pair of natural numbers, so $p_n \in \mathbb{N}^2$ for all $n \in \mathbb{N}$.

Since f is a bijection, it is surjective, so for any p_n, there must be some $n \in \mathbb{N}$ such that $f(n) = p_n$. By our definition of p_n, this means that

$$(f_1(n), f_2(n)) = (f_1(n) + 1, f_2(n) + 1)$$

Since two ordered pairs are equal iff their components are equal, this means that $f_1(n) = f_1(n) + 1$ and $f_2(n) = f_2(n) + 1$, which is impossible.

We have reached a contradiction, so our initial assumption must have been wrong. Thus $|\mathbb{N}| \neq |\mathbb{N}^2|$. ■

Of course, this proof is incorrect and contains a fatal flaw. What's wrong with this proof?

The issue here is that while it's true that there must be some natural number that maps to p_n, it doesn't have to be the natural number n. The proof should read "for any p_n, there must be some $m \in \mathbb{N}$ such that $f(m) = p_n$." This prevents the contradiction derived in the next paragraph from occurring.

Common mistakes included noting that the series p_n didn't cover all of $\mathbb{N}^2$ (which is true, but not argued by the proof); arguing that the definition of p_n didn't legally define elements of $\mathbb{N}^2$; and mentioning that the proof needed to show that all choices of f didn't work, not just one (though the proof indeed tries to argue that no choice of f would work).

Problem Three: Induction (45 points)

There are *many* ways to prove this result. Here's one of them:

Proof: By induction. Let $P(n) \equiv$ "For any string $w \in \Sigma^*$ with $|w| = n$, running D on w takes w into state $q_{n \bmod 3}$." We will prove $P(n)$ is true for all $n \in \mathbb{N}$. From this, we can conclude that for any string w, that running D on w takes w into state q_0 iff $|w| \equiv_3 0$. This in turn implies that D accepts w iff $w \in L_3$, as required.

For our base case, we prove $P(0)$, that for any string $w \in \Sigma^*$ with $|w| = 0$, running D on w takes w into state q_0. There is only one string $w \in \Sigma^*$ such that $|w| = 0$, namely ε, and if we run D on ε, it will end in state q_0, as required.

For our inductive step, assume $P(n)$ is true, that for any string $w \in \Sigma^*$ with $|w| = n$, running D on w takes w into state $q_{n \bmod 3}$. We prove $P(n + 1)$, that for any string $w \in \Sigma^*$ with $|w| = n + 1$, running D on w takes w into state $q_{(n+1) \bmod 3}$. To see this, take any string $w \in \Sigma^*$ with $|w| = n + 1$. Then $w = xa$ for some string $x \in \Sigma^*$ with $|x| = n$. When we run D on w, it is equivalent to running D on x, then taking the transition for a. By the inductive hypothesis, running D on x takes the machine to state $q_{n \bmod 3}$. When we follow the transition for a, we consider three cases:

Case 1: $n \bmod 3 = 0$. Then the transition is from q_0 to q_1, and $(n + 1) \bmod 3 = 1$.

Case 2: $n \bmod 3 = 1$. Then the transition is from q_1 to q_2, and $(n + 1) \bmod 3 = 2$.

Case 3: $n \bmod 3 = 2$. Then the transition is from q_2 to q_0, and $(n + 1) \bmod 3 = 0$.

Thus in each case running D on x and taking the transition for a ends in state $q_{(n+1) \bmod 3}$. Since this is equivalent to running D on w, this means that running D on w ends in state $q_{(n+1) \bmod 3}$. Thus $P(n + 1)$ holds, completing the induction. ■

Other ways to prove the result include the following:

- Using strong induction on n to show that any string w can be written as wx, where $|w| \equiv_3 0$ and $|x| \leq 2$.
- Proving $P(0)$, $P(1)$, and $P(2)$, then proving $P(n) \rightarrow P(n + 3)$.
- Proving $P(0)$, then proving that $P(3n) \rightarrow P(3n + 1) \land P(3n + 2) \land P(3n + 3)$.

Common mistakes included missing one direction of the biconditional; assuming that if $|w| \equiv_3 1$, then w ends in state q_1 without proving it; only proving that the result holds for strings whose lengths were multiples of three; and only proving one direction of implication in the base case.

Problem Four: Relations (50 Points)

(i) Counting Partial Orders (20 Points)

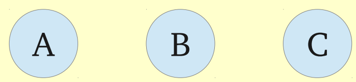
Let $A = \{a, b, c\}$. How many different binary relations are there over A that are partial orders? You must justify your answer to receive credit, but you don't need to write a formal proof.

A note: two relations R_1 and R_2 over some set X are considered the same iff for any $x, y \in X$, we have that xR_1y iff xR_2y. Two relations are different iff they're not the same.

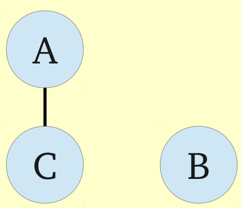
(Hint: It might be useful to draw some pictures.)

There are 19 partial orders over this set. They fall into one of five different "families":

(1x) All elements are incomparable:

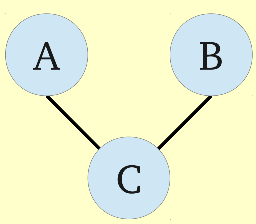


(6x) Exactly one pair is comparable:

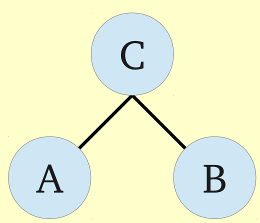


(plus five others of the same shape)

(6x) Exactly two pairs are comparable:

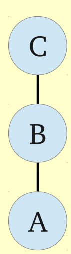


and



(plus two copies of each)

(6x) All three elements are comparable:



(plus five other copies)

There are many different ways to approach this problem. One option would be to draw out all possible Hasse diagrams and figure out how many copies of each there are (this is the approach taken above). Another option would be to draw out the graphical representation of these relations, which must have at most one edge between any pair of nodes, then to count how many of these give partial orders. An alternative approach would be to start off with all of the possible reflexive and antisymmetric relations, then to filter out those that aren't transitive.

Common mistakes included only listing total orders (of which there are six); or claiming that there are infinitely many partial orders (while there are infinitely many *descriptions* of partial orders, all but 19 of them are duplicates); or claiming that within any partial order there are at most six pairs of comparable elements (which is true, but not what we were asking for); only listing "meaningful" relations like ≤, ≥, and =; or saying that two partial orders were the same if any single pair of elements were related together the same way.

A note from Keith: In retrospect, this question was far too difficult to put on a midterm exam. We saw very few correct solutions. I've left the question on the practice exam because I believe it's a very interesting question that exercises the intuition behind partial orders. Please don't panic if you couldn't figure this one out – this question is much harder than the sorts of questions that are likely to appear on the actual midterm!

(ii) Relations and Functions (15 Points)

Let A and B be arbitrary sets where $\leq_B$ is a partial order over B. Suppose that we pick an injective function $f : A \to B$. We can then define a relation $\leq_A$ over A as follows: for any $x, y \in A$, we have $x \leq_A y$ iff $f(x) \leq_B f(y)$.

Prove that $\leq_A$ is a partial order over A.

Proof: We'll show $\leq_A$ is a partial order by showing it is reflexive, antisymmetric, and transitive.

To see that $\leq_A$ is reflexive, consider any $x \in A$. We will prove that $x \leq_A x$. To see this, note that $f(x) \leq_B f(x)$, since $\leq_B$ is reflexive. Thus $x \leq_A x$.

To see that $\leq_A$ is antisymmetric, consider any $x, y \in A$ where $x \leq_A y$ and $y \leq_A x$. We will prove $x = y$. Since $x \leq_A y$ and $y \leq_A x$, we have that $f(x) \leq_B f(y)$ and $f(y) \leq_B f(x)$. Since $\leq_B$ is antisymmetric, this means that $f(x) = f(y)$. Since f is injective and $f(x) = f(y)$, we know that $x = y$, as required.

To see that $\leq_B$ is transitive, consider any $x, y, z \in A$ where $x \leq_A y$ and $y \leq_A z$. We will prove that $x \leq_A z$. To see this, note that because $x \leq_A y$ and $y \leq_A z$, we have that $f(x) \leq_B f(y)$ and $f(y) \leq_B f(z)$. Since $\leq_B$ is transitive, we have $f(x) \leq_B f(z)$. Thus $x \leq_A z$, as required. ■

The most common mistake we saw was using the wrong definition of antisymmetry. Another common error was to prove a true statement that wasn't equivalent to what needed to be shown.

(iii) Overlapping Relations **(15 Points)**

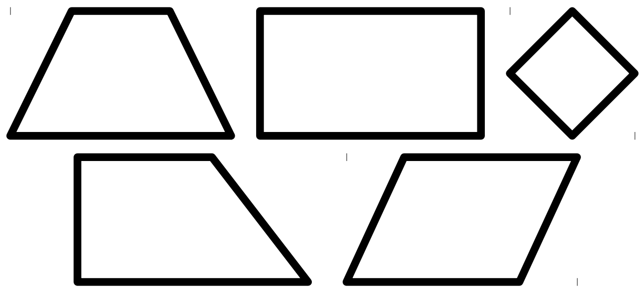
Are there any binary relations over $\mathbb{N}$ that are both equivalence relations and total orders? If so, give an example of one and prove why it is both an equivalence relation and a total order. If not, prove why not.

This is impossible. To see this, suppose for contradiction that there is a binary relation R over $\mathbb{N}$ such that R is a total order and equivalence relation. Since R is total, either $0R1$ or $1R0$ (or both). If $0R1$, by symmetry we have that $1R0$, so $0R1$ and $1R0$. Otherwise, if $1R0$, by symmetry we have that $0R1$, so $0R1$ and $1R0$. Either way, we get $0R1$ and $1R0$, so by antisymmetry we have that $0 = 1$, which is impossible. We have reached a contradiction, so our assumption must have been wrong. Thus there is no binary relation R over $\mathbb{N}$ such that R is a total order and an equivalence relation.

Common mistakes included confusing total *relations* with total *orders* (a total order is a partial order that is also a total relation); proving that a relation like = was a total order or that ≤ was an equivalence relation; or claiming that it is impossible for a relation to be simultaneously symmetric and antisymmetric.

Problem Five: The Pigeonhole Principle (45 points total)

Suppose that you color every point in the real plane one of four colors (say, red, green, blue, and yellow). Prove that no matter how you color the plane, there will always be a trapezoid whose corners are all the same color. Recall that a trapezoid is a quadrilateral with at least two parallel sides. For example, all of the following figures are trapezoids:



(Hint: Try placing a specially-constructed object into the real plane such that no matter how that object is colored, the object always contains a trapezoid whose corners are the same color.)

We saw two major strategies for proving this result. One involving placing a 5 × 5 grid of points into the plane and finding a trapezoid in it:

Proof: Consider a 5 × 5 rectangular grid of points. Place this grid anywhere in the plane and look at the colors assigned to the points. Note that in each row, there are 5 points and 4 possible colors, so there must be at least two points of the same color.

For each row, define the dominant color of that row to be some color (chosen arbitrarily) such that at least two points in that row are assigned that color. Since there are 5 rows and 4 possible dominant colors, two rows must have the same dominant colors.

We can then form a trapezoid as follows: connect two points in each of the two rows that have the dominant color, then connect the leftmost point in the first row to the leftmost point in the second row and the rightmost point in the first row to the rightmost point in the second row. Since the rows are parallel, this forms a quadrilateral with at least two parallel sides. We thus have a trapezoid whose corners are all the same color, as required. ■

The other common strategy was to place a 5 × 1025 grid into the plane and to find a rectangle in it. This is a variation on the proof that a 3 × 9 grid colored red and blue always has a rectangle whose corners are the same color:

Proof: Consider a 5 × 1025 rectangular grid of points. Place this grid anywhere in the plane and look at the colors assigned to the points. First, we will prove that two of the columns in the grid must have identical colorings. Second, we will prove that if there are two identically-colored columns in the grid, then there must be a rectangle whose corners have the same color.

To see that two columns in the grid must have the same color, note that there are only 1024 possible ways to assign colors to one column, since there are five points and four possible colors each (here, $4^5 = 1024$). Since there are 1025 columns and 1024 possible colorings, by the pigeonhole principle two columns must have exactly the same coloring.

Now, we will show that this means that there must be a rectangle whose corners are all the same color. Take the two columns with the same coloring. Since there are only four colors and five points in each column, two of the points in each column must have the same color. Using this, we can form a rectangle as follows: pick any two points of the same color in one of the repeated columns, then pick the same points in the duplicated column. This ensures all four points have the same color. This picks two points out of one column and the same two points out of another column, so those points form a rectangle. Since all rectangles are trapezoids, we have found a trapezoid whose corners are the same color. ■

Common mistakes included finding ways to get four points of the same color, but which didn't form a trapezoid (for example, picking 13 arbitrary points in space).