Problem Set 3 Checkpoint Solutions

i. What three properties must a relation have to be an equivalence relation?

An equivalence relation is a binary relation that is reflexive, symmetric, and transitive.

ii. Define the relation $=_A$ over the set of all polygons as follows: if x and y are polygons, then $x =_A y$ iff $A(x) = A(y)$. Is $=_A$ an equivalence relation? If so, prove it. If not, prove why not.

This statement is true.

Proof: We will show that $=_A$ is reflexive, symmetric, and transitive.

To show the relation $=_A$ is reflexive, note for any polygon p that $A(p) = A(p)$, so by definition $p =_A p$.

To show $=_A$ is symmetric, consider any polygons x and y where $x =_A y$. We will prove $y =_A x$. Since $x =_A y$, we see $A(x) = A(y)$. Thus $A(y) = A(x)$, so by definition of $=_A$ we have $y =_A x$, as required.

To show the relation $=_A$ is transitive, consider any polygons x, y, and z where $x =_A y$ and $y =_A z$. We will show that $x =_A z$. Since $x =_A y$, we have $A(x) = A(y)$, and since $y =_A z$, we have $A(y) = A(z)$. Since $A(x) = A(y)$ and $A(y) = A(z)$, we have $A(x) = A(z)$. Thus $x =_A z$, as required. ■

A note: We saw many proofs of symmetry or transitivity that had the following structure:

To show that the relation $=_A$ is symmetric, consider any polygons x and y where $A(x) = A(y)$. Since $A(x) = A(y)$, we see $A(y) = A(x)$. Therefore, $=_A$ is symmetric.

Although everything stated here is true, this structure of this proof is incorrect. Remember that a binary relation R over a set A is symmetric iff the following is true:

For any $x, y \in A$, if xRy, then yRx.

In the context of this question, this means that

For any polygons x and y, if $x =_A y$, then $y =_A x$.

You should try to prove this statement just as you would any normal implication: start off by assuming that x and y are polygons where $x =_A y$, then prove that $y =_A x$. Although $x =_A y$ and $A(x) = A(y)$ are equivalent to one another, the ultimate goal of the proof is to reason about the $=_A$ relation, and so the proof should start off by assuming x $=_A$ y rather than $A(x) = A(y)$. The most correct way to do this would be the following:

1. Assume that $x =_A y$ for some polygons x and y.
2. Claim by the definition of $=_A$ that since $x =_A y$, we must have $A(x) = A(y)$.
3. Show that $A(y) = A(x)$.
4. Claim by the definition of $=_A$ that $y =_A x$.

The structure of this proof mirrors the structure of what needs to be shown. More generally, to prove symmetry, you should start by assuming xRy for your relation R and only then apply the definition of R. Similarly, you ultimately should prove that yRx for your relation R. You should also treat proofs of antisymmetry and transitivity the same way.

iii. What three properties must a relation have to be a partial order?

A partial order is a binary relation that is reflexive, antisymmetric, and transitive.

iv. Define the relation $\leq_A$ over the set of all polygons as follows: if x and y are polygons, then $x \leq_A y$ iff $A(x) \leq A(y)$. Is $\leq_A$ a partial order? If so, prove it. If not, prove why not.

This is not a partial order. Consider a 1×4 rectangle (call it a) and a 2×2 rectangle (call it b). These polygons have the same areas, so $a \leq_A b$ and $b \leq_A a$. However, $a \neq b$. Thus $\leq_A$ is not antisymmetric, so $\leq_A$ is not a partial order.

A note: The relation $\leq_A$ is both reflexive and transitive, but it's not antisymmetric. Many of the proofs we received did try to show that $\leq_A$ is antisymmetric by using the following line of reasoning:

> To prove $\leq_A$ is antisymmetric, consider any two polygons x and y where $x \leq_A y$ and $y \leq_A x$.
> This means that $A(x) \leq A(y)$ and $A(y) \leq A(x)$. Thus, $A(x) = A(y)$, so $\leq_A$ is antisymmetric.

This proof is incorrect. Remember that a binary relation R over a set A is antisymmetric iff this statement is true:

$$\text{For any } x, y \in A, \text{ if } xRy \text{ and } yRx, \text{ then } x = y.$$

In our case, since R is a binary relation over polygons, this is the statement

$$\text{For any polygons } x \text{ and } y, \text{ if } x \leq_A y \text{ and } y \leq_A x, \text{ then } x = y.$$

Notice that the conclusion of this statement is "$x = y$," meaning that x and y must be the same object. This is *not* the same as the statement "$A(x) = A(y)$," which means x and y have the same area. This is a subtle but extremely important point about antisymmetry, and in fact it's the reason we asked you to work through this problem.

When proving antisymmetry, make sure you prove that if xRy and yRx, then x and y are *identically* the same object as one another. It's not enough to show that they have some property in common with one another.