Discussion Solutions 1

Some of these questions from Maurizio Caligaris

Problem One: Identity Elements

Recall from lecture that a binary operator ★ has identity element z iff for any a:

$$a \star z = z \star a = a$$

Not all binary operators have identity elements, though many do.

However, it's impossible for a binary operator to have several identity elements. Prove that if ★ is a binary operation with identity elements z_1 and z_2, then $z_1 = z_2$. This is an example of a *uniqueness proof*, in which you show that at most one object with a certain property can exist by proving that if there are two objects with a certain description, they must actually be the same object. All we did was give that object two names.

Theorem: If ★ is a binary operator with an identity element, ★ has exactly one identity element.

Proof: Let ★ be an arbitrary binary operator. Suppose that z_1 and z_2 are identity elements for ★. Since z_1 is an identity element, we know that $z_1 \star a = a$ for any a. In particular, this means that $z_1 \star z_2 = z_2$. Similarly, since z_2 is an identity element, we have that $a \star z_2 = a$ for any a, so in particular we have $z_1 \star z_2 = z_1$. Thus $z_1 = z_1 \star z_2 = z_2$, so $z_1 = z_2$. Therefore, if ★ has an identity element, that identity element is unique. ■

Problem Two: Balls in Bins

Suppose that you have twenty-five balls to place into five different bins. Eleven of the balls are red, while the other fourteen are blue. Prove that no matter how the balls are placed into the bins, there must be at least one bin containing at least three red balls.

Theorem: If eleven red balls and fourteen blue balls are distributed into bins, at least one bin must contain at least three red balls.

Proof: By contradiction; suppose that every bin has at most two red balls. Since there are five bins, this means that there can be a total of at most ten red balls distributed across the bins. This contradicts the fact that there are eleven total red balls. We have reached a contradiction, so our assumption must have been wrong. Thus if eleven red balls and fourteen blue balls are distributed across five bins, some bin must have at least three red balls in it. ■

Note that the blue balls don't feature anywhere in this proof. We can focus purely on the red balls.

Problem Three: Quadratic Equations

A *quadratic equation* is an equation of the form $ax^2 + bx + c = 0$. A *root* of the equation is a real number x satisfying the equation.

Recall from lecture that a rational number is one that can be written as p/q for integers p and q where $q \neq 0$ and p and q have no common divisor other than 1.

i. Prove that mn is odd iff m is odd and n is odd.

Theorem: mn is odd iff m is odd and n is odd.

Proof: We prove both directions of implication.

First, we prove that if m is odd and n is odd, then mn is odd. To do this, consider any odd m and n. Since m and n are odd, there must exist $r, s \in \mathbb{Z}$ such that $m = 2r + 1$ and $n = 2s + 1$. Therefore, we have that $mn = (2r + 1)(2s + 1) = 4rs + 2r + 2s + 1 = 2(2rs + r + s) + 1$. Since $2rs + r + s$ is an integer, this means that mn is odd, as required.

Next, we prove that mn is odd, then m is odd and n is odd. To do this, we proceed by contrapositive and instead prove that if m is even or n is even, then mn is even. Assume without loss of generality that m is even. Since m is even there must be some integer r such that $m = 2r$. Then we have that $mn = 2rn$. Since rn is an integer, this means that mn is even, as required. ■

Note that the second half of this proof uses the statement *without loss of generality* to handle the case where m is even and the case that n is even uniformly. In many cases when you would normally do a proof by cases, you can instead claim that all the cases are essentially the same as one another and proceed from there. Check out Chapter 2 of the online course notes for more details.

ii. Prove, by contradiction, that if a, b, and c are odd numbers, then there are no rational numbers x for which $ax^2 + bx + c = 0$. Be sure to explicitly state what assumption you are attempting to contradict. As a hint, if the rational solution is p / q, consider what happens if both p and q are odd and what happens if exactly one of p and q is odd. (Why can't both p and q be even?)

Theorem: If a, b, and c are odd, there are no rational numbers x for which $ax^2 + bx + c = 0$.

Proof: By contradiction; assume that a, b, and c are odd but that there is a rational number x such that $ax^2 + bx + c = 0$. Since x is rational, it can be written as p / q for integers p and q with $q \neq 0$ and where p and q have no common factors other than ± 1.

Substituting p / q into the equation, we get that

$$a(p / q)^2 + b(p / q) + c = 0$$

$$ap^2 / q^2 + bp / q + c = 0$$

Multiplying both sides by q^2 (since $q \neq 0$), we get

$$ap^2 + bpq + cq^2 = 0$$

Since p and q have no common factors other than ± 1, we know that both p and q cannot be even. We therefore consider three cases:

Case 1: p is odd and q is odd. Then ap^2, bpq, and cq^2 are all odd, because they are the product of three odd numbers. Therefore, $ap^2 + bpq + cq^2$ is odd. But this is impossible, since $ap^2 + bpq + cq^2 = 0$, and 0 is even.

Case 2: p is odd and q is even. Then ap^2 and bpq are even and cq^2 is odd. Therefore, $ap^2 + bpq + cq^2$ is the sum of two even numbers and an odd number, so it is odd. But this is impossible, since this sum is equal to 0 and 0 is even.

Case 3: p is even and q is odd. Then bpq and ap^2 are even and cq^2 is odd. Therefore, $ap^2 + bpq + cq^2$ is the sum of two even numbers and an odd number, so it is odd. But this is impossible, since this sum is equal to 0 and 0 is even.

In all cases, we reach a contradiction. Therefore, our assumption must have been incorrect, so if a, b, and c are odd, then there are no rational numbers x for which $ax^2 + bx + c = 0$. ■

Problem Four: Finding Flaws in Proofs

The following proofs all contain errors that allow them to prove results that are patently false. For each proof, identify at least one flaw in the proof and explain what the problem is, then give a counterexample that demonstrates why the error occurs. In each case, **make sure you understand what logical error is being made**. The mistakes made here are extremely common.

Theorem: If n is even, then n^2 is odd.
Proof: By contradiction; assume that n is odd but that n^2 is even. Since n is odd, $n = 2k + 1$ for some integer k. Thus $n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, which is odd. This contradicts our earlier claim that n^2 is even. We have reached a contradiction, so our initial assumption was wrong. Thus if n is even, n^2 is odd. ■

Flaw: The negation of "If n is even, then n^2 is odd" is not "n is odd but n^2 is even." Instead, it's the statement "n is even, but n^2 is even." Remember that the contradiction of "If P, then Q" is "P and not Q," not "not P and not Q."

Theorem: For all sets A and B, $A \cup B = A$.
Proof: By contradiction; assume that for all sets A and B, $A \cup B \neq A$. So consider $A = \emptyset$ and $B = \emptyset$; then $A \cup B = A$. This contradicts our earlier claim that $A \cup B \neq A$ for all A and B. We have reached a contradiction, so our initial assumption was wrong. Thus for any sets A and B, $A \cup B = A$. ■

Flaw: The negation of "For all sets A and B, $A \cup B = A$" is not "For all sets A and B, $A \cup B \neq A$." instead, it's the statement "There exist sets A and B such that $A \cup B \neq A$." Remember to be careful how negations interact with universally-quantified statements.

Theorem: If $C \subseteq A \cup B$, then $C \subseteq A$.
Proof: By contrapositive. We prove that if C is not a subset of $A \cup B$, then it is not a subset of A. Since C is not a subset of $A \cup B$, there is some $x \in C$ such that $x \notin A \cup B$. Since $x \notin A \cup B$, $x \notin A$ and $x \notin B$. Thus $x \in C$ but $x \notin A$, and so C is not a subset of A. ■

Flaw: The contrapositive statement "If $C \subseteq A \cup B$, then $C \subseteq A$" isn't "If C is not a subset of $A \cup B$, then it is not a subset of A." It should be "If C is not a subset of A, then it isn't a subset of $A \cup B$."