Discussion Solutions 4

Problem One: Infinity Plus One

Let ★ be any value that isn't a natural number. Prove that $|\mathbb{N} \cup \{\star\}| = |\mathbb{N}|$.

Proof: Consider the function $f : \mathbb{N} \cup \{\star\} \to \mathbb{N}$ defined as follows:

$$f(n) = \begin{cases} n+1 & \text{if } n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

We will prove that f is a bijection by showing it is injective and surjective, from which we can conclude that $|\mathbb{N} \cup \{\star\}| = |\mathbb{N}|$.

To see that f is injective, consider any $n_1, n_2 \in \mathbb{N} \cup \{\star\}$ $n_1 \neq n_2$. We will prove $f(n_1) \neq f(n_2)$. Since $n_1 \neq n_2$, at most one of n_1 and n_2 can be ★. We consider two cases:

Case 1: Neither n_1 nor $n_2 = \star$. Then $f(n_1) = n_1 + 1$ and $f(n_2) = n_2 + 1$, and since $n_1 \neq n_2$, we see $n_1 + 1 \neq n_2 + 1$. Thus $f(n_1) \neq f(n_2)$, as required.

Case 2: At least one of n_1 and n_2 is ★. Without loss of generality, assume that $n_1 = \star$ and that $n_2 \neq \star$. Then $f(n_1) = 0$ and $f(n_2) = n_2 + 1$. Since $n_2 \in \mathbb{N}$ and $f(n_2) = n_2 + 1$, we can see that $f(n_2) = n_2 + 1 \geq 1 > 0 = f(n_1)$, and so $f(n_1) \neq f(n_2)$, as required.

In both cases we see $f(n_1) \neq f(n_2)$, so f is injective, as required.

To see that f is surjective, consider any $n \in \mathbb{N}$. We will prove there is an $m \in \mathbb{N} \cup \{\star\}$ such that $f(m) = n$. Now, if $n = 0$, then we can take $m = \star$, since $f(m) = f(\star) = 0 = n$. Otherwise, we know $n > 0$. This means $n \geq 1$, and so $n - 1$ is also a natural number. Taking $m = n - 1$ then guarantees that $f(m) = f(n - 1) = (n - 1) + 1 = n$, as required. Thus f is surjective. ■

Why we asked this question: There are two equivalent definitions of injectivity:

$f : A \to B$ is injective iff for any $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$, then $a_1 = a_2$.

$f : A \to B$ is injective iff for any $a_1, a_2 \in A$, if $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

You can always prove injectivity by using either of these definitions, but in some cases it's easier to use one of the definitions rather than the other. In this case, the second definition ends up being a bit easier to work with.

Additionally, we've gotten lots of questions about how to use the phrase "without loss of generality" correctly in a proof. The above proof includes this phrase to ignore the case where $x_1 \neq \star$ and $x_2 = \star$ by bundling it together with the case where $x_1 = \star$ and $x_2 \neq \star$ because the logic in both of the cases is the same, just with x_1 and x_2 swapped.

Finally, we asked this because it gives a formal way of defining what infinity plus one is. If you have a collection of size $\aleph_0$ and add in one more element, you get back a collection of size $\aleph_0$. You now know what infinity plus one is!

Problem Two: Propositional Equivalences

Prove that the formula $p \land q \to r$ is equivalent to the formula $p \land \neg r \to \neg q$ by using a truth table.

p	q	r	$p \land q$	$\to r$
F	F	F	F	T
F	F	T	F	T
F	T	F	F	T
F	T	T	F	T
T	F	F	F	T
T	F	T	F	T
T	T	F	T	F
T	T	T	T	T

p	q	r	$p \land$	$\neg r$	$\to$	$\neg q$
F	F	F	F	T	T	T
F	F	T	F	F	T	T
F	T	F	F	T	T	F
F	T	T	F	F	T	F
T	F	F	T	T	T	T
T	F	T	F	F	T	T
T	T	F	T	T	F	F
T	T	T	F	F	T	F

Why we asked this question: One of the major reasons for studying logic in the first place is to make sure we have a firm logical basis for manipulating terms and definitions. Here, we're using logic to check out why two definitions that at face value don't appear similar must actually be the same as one another.

Problem Three: Translating into Logic

i. Given the predicate *Person(x)*, which states that *x* is a person, and *Muggle(x)*, which states that *x* is a muggle, write a statement in first-order logic that says "some (but not all) people are muggles."

One possible option is the following:

$$\exists x.\ (Person(x) \land Muggle(x)) \land \exists x.\ (Person(x) \land \neg Muggle(x)).$$

The first part of this statement asserts that someone is a muggle, and the second part states that someone is not a muggle. Therefore, there is at least one muggle, but it's not possible for everyone to be a muggle.

We could also have written that second part as $\neg\forall x.\ (Person(x) \to Muggle(x))$, which is logically equivalent.

ii. Given the predicate *Person(x)*, which states that *x* is a person, and *Commoner(x)*, which states that *x* is a commoner, write a statement in first-order logic that says "there are either zero or one people who are not commoners."

One way to do this is as follows:

$$\forall x.\ (Person(x) \to Commoner(x)) \lor$$
$$\exists x.\ (Person(x) \land \neg Commoner(x) \land \forall y.\ (Person(y) \land \neg Commoner(y) \to x = y))$$

This first statement states that everyone is a commoner, which is one option. The second statement says that there is someone who is not a commoner, and that any person who is not a commoner must be this person x. Note how we express uniqueness by using a combination of the $\exists$ and $\forall$ quantifiers along the lines of what was suggested in lecture.

Why we asked this question: Both parts of the problem involve statements that are not qualified at the top level. We asked part (i) to get you comfortable placing $\land$ with $\exists$ and placing $\to$ with $\forall$ and seeing how to filter down every possible input to a set of values that you actually care about. We also wanted you to be comfortable seeing how to write statements that assert that some property is true some but not all of the time.

Part (ii) of this question was designed to get you comfortable counting with first-order logic. First-order logic doesn't have numbers in it, so talking about "exactly zero" or "exactly one" objects of some type requires some clever thinking. The "exactly one" case in particular uses the uniqueness construction, which we thought would be important to review,