Problem Set 2 Checkpoint Solutions

Prove that any $2^n \times 2^n$ grid with one square removed can be tiled by right triominoes.

Proof: By induction; let $P(n)$ be "Any $2^n \times 2^n$ grid with one square removed can be tiled with right triominoes." We prove that $P(n)$ is true for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that a $2^0 \times 2^0$ grid with any one square removed can be tiled with right triominoes. This is a 1×1 grid with one square removed, so it's vacuously true that all of the uncovered squares can be tiled with right triominoes.

For the inductive step, assume that $P(n)$ is true and any $2^n \times 2^n$ grid with one square removed can be tiled with right triominoes. We prove that $P(n + 1)$ is true, namely, that any $2^{n+1} \times 2^{n+1}$ grid with one square removed can be tiled with right triominoes. Consider any $2^{n+1} \times 2^{n+1}$ grid with one square removed. Split the grid into four $2^n \times 2^n$ grids and consider the subgrid with the missing square; without loss of generality, assume that the missing square is in the upper-left subgrid. By the inductive hypothesis, we can tile this $2^n \times 2^n$ grid with right triominoes. Next, place a right triomino so that it covers the upper-right corner of the lower-left grid, the upper-left corner of the lower-right grid, and the lower-left corner of the upper-right grid. Each of these three $2^n \times 2^n$ subgrids now has exactly one square covered, so by the inductive hypothesis we can cover the uncovered portions of these subgrids using right triominoes. This covers every square of the original grid, except for the original missing square, with right triominoes. Thus $P(n + 1)$ holds, completing the induction. ■

Why we asked this question: This is a classic induction problem that, at first glance, might not even seem like it's true. We chose this problem for a few reasons. First, this problem tests your understanding of what the base case ought to be. Although $P(1)$ seems like a reasonable base case, it's possible to simplify even further and use $P(0)$ as the base case instead. Additionally, this problem has the interesting property that, at least in our solution, the inductive hypothesis is used four times to cover the four smaller squares.

On top of the nice inductive structure, this problem has other nice properties that make it useful as a checkpoint problem. On the last problem set, you proved that it's impossible to tile an $n \times n$ chessboard missing two opposite corners by showing that $n^2 - 2$, the number of squares on the board, isn't a multiple of three. Although it's true that $4^n - 1$ is always a multiple of three, that in of itself isn't sufficient to prove that you can tile a $2^n \times 2^n$ grid missing a square. Many grids have a number of squares that's a multiple of three but can't be tiled with right triominoes (take, for example, a 1×3 grid or a 3×3 grid).

Additionally, the main proof of this result requires you to show that you can tile a $2^n \times 2^n$ grid missing *any* square with right triominoes. A common mistake on this problem was to show that for certain choices of squares the result is true without proving the overall, more general result.

There are many other solutions to this problem as well. Some solutions work by building larger L-shaped tiles out of smaller tiles, then using those larger tiles to tile the overall grid. If you're curious, try approaching the problem this way and see what happens!