CS103 Handout 14S
Fall 2013 October 23, 2013

Practice Midterm Exam Solutions 1

Problem One: Mathematical Logic (20 points total)

(i) Set Theory (10 Points)

Given the predicates

$Set(S)$, which states that S is a set, and

$x \in S$, which states that x is an element of S,

Write a statement in first-order logic that states "for any x and y, there is a set containing just the elements x and y." This is called the ***axiom of pairing***. Your formula can use any constructs of first-order logic (quantifiers, connectives, equality, etc.), but you should not use any functions or constants and should only use the predicates given above.

One option is

$$\forall x.\ \forall y.\ \exists S.\ (Set(S) \land \forall z.\ (z \in S \leftrightarrow x = z \lor y = z))$$

This says that for any x and y, there's a set S containing x and y and nothing else.

A common mistake was writing a first-order sentence that said that a set S existed such that *if* elements were in the set, then they must be either x or y, but not actually asserting that S contains x and y. Another common mistake was reordering the quantifiers so that the statement says "there is a set S such that for any x and y, any element of S is either x or y."

(ii) Set Theory, Part Two (10 Points)

For any sets A and B, consider the set S defined below:

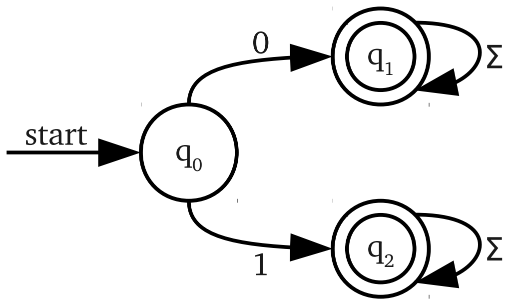
$$S = \{\ x \mid \neg(x \in A \to x \in B)\ \}$$

Write an expression for S in terms of A and B using the standard set operators (union, intersection, etc.) and briefly justify why your answer is correct.

$S = \{\ x \mid \neg(x \in A \to x \in B)\ \}$ is equivalent to $S = \{\ x \mid x \in A \land x \notin B\ \}$, which in turn is equivalent to $A - B$.

Problem Two: Finite Automata (15 Points)

Consider the following DFA:



(i) Identifying the Alphabet (5 Points)

What is the alphabet of *D*? Briefly justify your answer (a sentence or two should be sufficient).

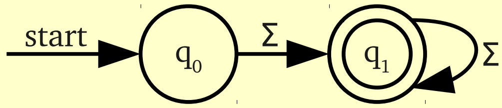
Since all states in a DFA must have a transition defined for all symbols and there are only two transitions from q_0, namely 0 and 1, the alphabet must be $\Sigma = \{ 0, 1 \}$.

Common mistakes included giving the *language* of *D* rather than the *alphabet* of *D*, including Σ as a symbol in the alphabet of *D*, or claiming that it is impossible to determine *D*'s alphabet from the picture.

(ii) DFA Compression (10 Points)

Construct a DFA with the same language as *D*, but which has fewer states.

The language is the set of all nonempty strings. After reading any first character, the automaton transitions into state q_1 or q_2, at which point all remaining characters are consumed and the machine accepts. Given this, we don't need both q_1 and q_2; we just need some state other than the start state:



Common mistakes included building a DFA for Σ* (which accepts the empty string) or giving automata that were not DFAs.

Problem Three: Induction (45 points)

(i) Counting Nodes (15 Points)

Prove that for any $n \in \mathbb{N}$, a perfect binary tree of order n has $2^{n+1} - 1$ nodes.

Proof: By induction. Let $P(n)$ be "a perfect binary tree of order n has 2^{n+1} nodes." We prove that $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that a perfect binary tree of order 0 has $2^1 - 1 = 1$ nodes. This is true, since a perfect binary tree of order 0 is just a single node.

For our inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that a perfect binary tree of order n has $2^{n+1} - 1$ nodes. We prove $P(n + 1)$, that a perfect binary tree of order $n + 1$ has $2^{n+2} - 1$ nodes. To see this, note that a perfect binary tree of order $n + 1$ is a root node with two subtrees, each of which is a perfect binary tree of order n. This means that the total number of nodes is given by twice the number of nodes in a perfect binary tree of order n, plus one for the new root node. By our inductive hypothesis, this number is given by $2(2^{n+1} - 1) + 1 = 2^{n+2} - 2 + 1 = 2^{n+2} - 1$. Thus $P(n + 1)$ holds, completing the induction. ■

This could also be done by recognizing that we've seen something very similar before on the problem set:

Proof: We can set up a recurrence relation that defines the number of nodes in a perfect binary tree of order n. When $n = 0$, there is just one node, so we set $a_0 = 1$. The number of nodes in an order-$(n + 1)$ perfect binary tree is equal to twice the number of nodes in a perfect binary tree of order n, plus one for the root. Thus $a_{n+1} = 2a_n + 1$. As proven in Problem Set 2, Question 1, this recurrence solves to $a_n = 2^{n+1} - 1$. Therefore, there are $2^{n+1} - 1$ nodes in a perfect binary tree of order n. ■

Many solutions worked by proving that the number of nodes in a perfect binary tree is $2^{n+1} - 1$ by summing up over the number of nodes in each level of the tree (2^k nodes in level k). While this statement is true, this does not immediately follow from the definition of a perfect binary tree. You would need to write an auxiliary proof that shows this to be the case in order to use this proof.

(ii) Grouping Nodes **(30 Points)**

Prove that for any $n \in \mathbb{N}$, any set of n nodes has a skew binary decomposition.

Proof: By induction. Let $P(n)$ be "any set of n nodes has a skew binary decomposition." We prove $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that any 0 nodes has a skew binary decomposition. The empty collection of perfect binary trees is a skew binary decomposition, since (1) holds because there are 0 trees of each order and (2) holds because it is vacuously true.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds and any n nodes has a skew binary decomposition. We prove $P(n + 1)$, that any collection of $n + 1$ nodes has a skew binary decomposition.

To see this, consider any $n + 1$ nodes and remove any single node v. This leaves n nodes remaining, so by the inductive hypothesis these nodes have a skew binary decomposition. We now consider two cases:

Case 1: This skew binary decomposition does not contain any two trees of the same order. Then we can add v to this collection as a perfect binary tree of order 0. In this case, claim (1) holds because for any order other than 0, there are the same number of trees of that order as before, and since there were either 0 or 1 trees of order 0 to begin with, there are now either 1 or 2 trees of order 0. Claim (2) also holds, since the only possible order for which there could be two trees is order 0, in which case there cannot be any trees of smaller orders anyway because there are no trees of order less than 0.

Case 2: This skew binary decomposition contains two trees of the same order (call that order n). Then we can insert v into the collection as follows: make v the root of an order-(n+1) perfect binary tree, whose left and right children are the two trees of order n in the collection. We claim that this forms a skew binary decomposition. To see this, note that for any order greater than $n + 1$, there are the same number of trees of this order as there were before (either 0 or 1). There is now one more tree of order $n + 1$ than of before, so there are either 1 or 2 trees of this order. There are now 0 trees of order n, and since the original decomposition had two trees of order n, we know there are 0 trees of any order smaller than n. Thus claims (1) and (2) holds, so this is a valid skew binary decomposition.

In either case, we have that $P(n + 1)$ holds, completing the induction. ■

There is another possible proof that uses strong induction. Given n nodes, find the largest perfect binary tree that can be formed from those nodes. Using the inductive hypothesis, group the remaining nodes into a skew binary decomposition. You can prove that this overall collection of trees forms a skew binary decomposition, though there are more cases to check.

Problem Four: Relations and Functions (55 Points)

(i) Subsets and Cardinality (10 Points)

Prove, for all sets A and B, that if $A \subseteq B$, then $|A| \leq |B|$.

Proof: Consider the function $f : A \to B$ defined by $f(x) = x$. We will show that this is a legal function from A to B, then that it is injective. Consequently, by definition we have $|A| \leq |B|$.

To see that f is a legal function, note that for any $x \in A$, since $A \subseteq B$, we have $x \in B$ as well. Therefore, for any $x \in A$, we have $f(x) = x \in B$, so f is a legal function from A to B.

To see that f is injective, consider any $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$. We will prove that $x_1 = x_2$. To see this, note that $x_1 = f(x_1) = f(x_2) = x_2$, so $x_1 = x_2$, as required. ■

The most common mistake we saw was giving an intuitive explanation of why the result was true rather than defining an injection from A to B.

(ii) The Set of all Sets (20 Points)

Using your result from part (i) and Cantor's Theorem, prove that $U = \{ S \mid S \text{ is a set} \}$ does not exist.

Proof: By contradiction; assume that U exists. Then consider the set $\wp(U)$. By Cantor's theorem, we have that $|U| < |\wp(U)|$.

We claim that $\wp(U) \subseteq U$. To see this, note that every $S \in \wp(U)$ satisfies $S \subseteq U$. Consequently, every $S \in \wp(U)$ must be a set. Since U is the set of all sets, we have that any $S \in \wp(U)$ satisfies $S \in U$, so $\wp(U) \subseteq U$. Using our result from part (i), this means that $|\wp(U)| \leq |U|$. But this is impossible, since by Cantor's theorem we know $|\wp(U)| > |U|$.

We have reached a contradiction, so our assumption must have been false. Thus U does not exist. ■

There were two common mistakes on this question. Many solutions correctly claimed that since $\wp(U)$ is a set, then $\wp(U) \in U$. Although this is true, this is *not* the same thing as showing that $\wp(U) \subseteq U$, which is necessary to use the result from part (i).

Many proofs claimed that since $|U| < |\wp(U)|$, there must be some $x \in \wp(U)$ where $x \notin U$, thus contradicting that U is the set of all sets. This is correct. However, it is nontrivial to prove that if $|A| < |B|$, then there is an element $x \in B$ where $x \notin A$ (this is a great exercise, by the way!) Accordingly, we deducted points for solutions that used this reasoning unless they justified why this result was true.

Consider an arbitrary set A. There may be many different partial order relations over this set A. We'll denote by PO the set of all partial order relations over the set A.

Let $\leq_1$ and $\leq_2$ be partial orders over the set A. We define a new relation R over PO as follows:

$$\leq_1 R \leq_2 \quad \text{iff} \quad \text{for every } x, y \in A, \text{ if } x \leq_1 y, \text{ then } x \leq_2 y$$

(iii) Meta Partial Orders **(25 Points)**

Prove that R is a partial order over PO. (A note: two relations R_1 and R_2 over some set A are equal iff for any $x, y \in A$, we have that xR_1y iff xR_2y.)

Proof: We will show that R is reflexive, antisymmetric, and transitive.

To see that R is reflexive, consider any partial order $\leq_1 \in PO$. We will prove that $\leq_1 R \leq_1$. To see this, note that for any $x, y \in A$, we trivially have $x \leq_1 y \rightarrow x \leq_1 y$, since any statement implies itself. Consequently, $\leq_1 R \leq_1$ holds, so R is reflexive.

To see that R is transitive, consider any $\leq_1, \leq_2, \leq_3 \in PO$ such that $\leq_1 R \leq_2$ and $\leq_2 R \leq_3$. We will prove that $\leq_1 R \leq_3$. To see this, note that since $\leq_1 R \leq_2$, we have for any $x, y \in A$, that if $x \leq_1 y$, then $x \leq_2 y$. Similarly, since $\leq_2 R \leq_3$, we have that for any $x, y \in A$, that if $x \leq_2 y$, then $x \leq_3 y$. Combining these together, we get that for any $x, y \in A$, that if $x \leq_1 y$, then $x \leq_3 y$. Therefore, by definition, $\leq_1 R \leq_3$.

To see that R is antisymmetric, consider any partial orders $\leq_1, \leq_2 \in PO$ such that both $\leq_1 R \leq_2$ and $\leq_2 R \leq_1$. We prove that $\leq_1 = \leq_2$. Since $\leq_1 R \leq_2$, we have that for any $x, y \in A$, that if $x \leq_1 y$, then $x \leq_2 y$. Similarly, since $\leq_2 R \leq_1$, we have that for any $x, y \in A$, that if $x \leq_2 y$, then $x \leq_1 y$. Combining these together, we have that for any $x, y \in A$, that $x \leq_1 y$ iff $x \leq_2 y$. Thus by definition, we have $\leq_1 = \leq_2$, as required.

Since R is reflexive, antisymmetric, and transitive, it is a partial order. ■

The most common mistake we saw in this problem was mixing up what results needed to be proven. For example, to show that R is reflexive, you need to show that $\leq_1 R \leq_1$ for any $\leq_1 \in PO$. This means that you need to show that whenever $x \leq_1 y$, it's also true that $x \leq_1 y$. Showing that $x \leq_1 x$ for any $x \in A$ does not establish this result. In fact, none of the proofs in this problem actually rely on the fact that the relation R is a relation over partial orders. The relation R, more generally, is a partial order over arbitrary binary relations over A.

Problem Five: The Pigeonhole Principle (45 points total)

(i) Measuring Subsequences (5 Points)

Let k be an arbitrary natural number where $1 \leq k \leq rs + 1$. Prove that $I_k \geq 1$ and $D_k \geq 1$.

Proof: Given any element x_k from the sequence, the sequence $\langle x_k \rangle$ is an increasing subsequence of length 1 and a decreasing subsequence of length 1. Consequently, the largest possible increasing subsequence ending at k has length at least 1 and the largest possible decreasing subsequence ending at k has length at least 1. Therefore, $I_k \geq 1$ and $D_k \geq 1$, as required. ■

Common mistakes included claiming that $I_k = 1$ or $D_k = 1$ because $\langle x_k \rangle$ is a possible subsequence (we just know that they're at least one), or proceeding by contradiction and claiming that the negation of the statement is that some k satisfies $I_k < 1$ ***and*** $D_k < 1$, rather than $I_k < 1$ ***or*** $D_k < 1$.

(ii) Distinguishing Pairs (10 Points)

Let j and k be arbitrary natural numbers where $1 \leq j \leq rs + 1$ and $1 \leq k \leq rs + 1$. Prove that if $j \neq k$, then $(I_j, D_j) \neq (I_k, D_k)$. To keep your proof short, we recommend assuming without loss of generality that $j < k$.

Proof: Consider any j and k such that $j < k$. We consider two cases:

Case 1: $x_j < x_k$. In that case, the longest increasing subsequence ending at position k has to have length at least $I_j + 1$, since we can always take the longest increasing subsequence ending that position j and extend its length by 1. Consequently, $I_k > I_j$, so $(I_j, D_j) \neq (I_k, D_k)$.

Case 2: $x_j > x_k$. In that case, the longest decreasing subsequence ending at position k has to have length at least $D_j + 1$, since we can always take the longest decreasing subsequence ending that position j and extend its length by 1. Consequently, $D_k > D_j$, so $(I_j, D_j) \neq (I_k, D_k)$.

In both cases, we find that $(I_j, D_j) \neq (I_k, D_k)$, as required. ■

Common mistakes included assuming that the longest increasing or decreasing subsequence at position k *must* be formed by extending the appropriate subsequence ending at position j by one element. This is just one possible option; it's not required to be the best possible option.

(iii) Putting Everything Together **(30 Points)**

Using your results from parts (i) and (ii), prove that any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$. *(Hint: Proceed by contradiction. If the sequence does not have an ascending subsequence of length* $r + 1$ *or a decreasing subsequence of length* $s + 1$*, what do you know about the values of all the (I, D) pairs?)*

Proof: By contradiction; assume there is a sequence of distinct elements of length $rs + 1$ such that there is no ascending subsequence of length $r + 1$ and no descending subsequence of length $s + 1$. This means that for any k in the range $1 \le k \le rs + 1$, that $I_k \le r$ and $D_k \le s$. By our result from part (i), we also know that $I_k \ge 1$ and $D_k \ge 1$. Therefore, there are a total of rs different possible pairs of the form (I_k, D_k) that are possible. Since there are $rs + 1$ elements of the sequence, each of which has an (I, D) pair, by the pigeonhole principle there must be two different elements of the sequence (call them j and k) such that $(I_j, D_j) = (I_k, D_k)$. This contradicts our result from part (ii).

We have reached a contradiction, so our assumption must have been wrong. Thus in any sequence of $rs + 1$ distinct real numbers, there must be an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$. ■

Common mistakes included correctly noting that there are at most rs possible values for each of I and D (since I can't be $r + 1$ and D can't be $s + 1$) and claiming some I value must be repeated (since there are $rs + 1$ of them and rs possible values) and some D value must be repeated, but incorrectly claiming that this means that some (I, D) pair must be repeated.