Solutiosn to Extra Review Problems

Problem One: Translating into Logic

i. Given the predicates

Person(p), which states that p is a person, and

ParentOf(p_1, p_2), which states that p_1 is the parent of p_2,

write a statement in first-order logic that says "someone is their own grandparent." (Paraphrased from an old novelty song.)

One possible answer is

$$\exists p_1.\ (Person(p_1) \land \exists p_2.\ (Person(p_2) \land\ ParentOf(p_1, p_2) \land ParentOf(p_2, p_1)))$$

In other words, p_1 is p_2's parent, and p_2 is p_1's parent. Thus p_1 is their own grandparent.

ii. Given the predicates

Natural(n), which states that n is a natural number, and

Integer(n), which states that n is an integer,

along with the function symbol $f(n)$, which represents some particular function f, write a statement in first-order logic that says "$f : \mathbb{N} \to \mathbb{Z}$ is a bijection."

One possible answer is

$$\forall x.\ (Natural(x) \to Integer(f(x))) \land$$
$$\forall x_1.\ \forall x_2.\ (Natural(x_1) \land Natural(x_2) \land x_1 \neq x_2 \to f(x_1) \neq f(x_2)) \land$$
$$\forall y.\ (Integer(x) \to \exists x.\ (Natural(x) \land f(x) = y))$$

Line 1 says that f is a function from naturals to integers: any input to f that is a natural number produces an integer as output. Line 2 says that f is injective, and Line 3 says that f is surjective.

Problem Two: Translating Out Of Logic

For each first-order statement below, write an English sentence that describes what that sentence says. Then, determine whether the statement is true or false. No proofs are necessary.

- $\exists S. (Set(S) \land \forall x. x \notin S)$

This says that an empty set exists (that is, there's a set with no elements.) It's true.

- $\forall x. \exists S. (Set(S) \land x \notin S)$

This says that every object x doesn't belong to at least one set S. It's true – nothing belongs to the empty set.

- $\forall S. (Set(S) \to \exists x. x \notin S)$

This says that the universal set doesn't exist: every set has some element it doesn't contain. This is true; you proved it in the problem set!

- $\forall S. (Set(S) \land \exists x. x \notin S)$

This says that everything that exists is a set that doesn't contain everything. This is false – I for one resent being called a set! ☺

- $\exists S. (Set(S) \land \exists x. x \notin S)$

This says that there's a set that doesn't contain everything. It's true; try Ø

- $\exists S. (Set(S) \to \forall x. x \in S)$

This says that there is some object S where if S is a set, then S is the universal set. It's true: choose a puppy for S. The puppy isn't a set, so the statement "if the puppy is a set, it's the universal set" is vacuously true!

- $\exists S. (Set(S) \land \forall x. x \notin S \land \forall T. (Set(T) \land S \neq T \to \exists x. x \in T))$

This says that there is a unique empty set: there's an empty set, and any set other than it can't be empty. This is also true, since any two sets with no elements are the same.

- $\exists S.\ (Set(S) \land \forall x.\ x \notin S \land \exists \mathrm{T}.\ (Set(T) \land \forall x.\ x \notin T \land S \neq T))$

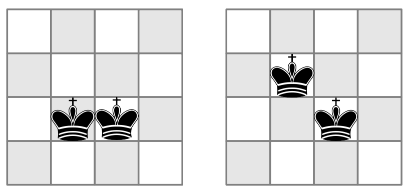
This says that there are at least two different empty sets, which is false.

- $\exists S.\ (Set(S) \land \forall x.\ x \notin S) \land \exists \mathrm{T}.\ (Set(T) \land \forall x.\ x \notin T))$

This says that an empty set exists and that an empty set exists. This is true.

Problem Three: A Clash of Kings

Chess is a game played on an 8 × 8 grid with a variety of pieces. In chess, no two king pieces can ever occupy two squares that are immediately adjacent to one another horizontally, vertically, or diagonally. For example, the following positions are illegal:



Prove that it is impossible to legally place 17 kings onto a chessboard.

Proof: Take an 8 × 8 chessboard and subdivide it into 16 blocks of size 2 × 2. By the pigeonhole principle, since there are 17 kings and 16 blocks, some block must contain at least two kings. Those two kings must be adjacent, so the position would be illegal. Therefore, any arrangement of 17 kings would cause two kings to attack one another, so it's impossible to legally place 17 kings on a chessboard.

Problem Four: Coloring a Grid, Take Two

Suppose every point in a 3×7 grid is colored either red or blue. Prove that there must be four points of the same color that form a rectangle.

Proof: Since each column consists of three points and there are two possible colors, by the pigeonhole principle each column must have at least two points of the same color. For each column, let its dominant color be the color that's repeated. Since there are two colors and there are seven columns, by the pigeonhole principle there must be at least $\lceil 7/2 \rceil = 4$ columns of the same dominant color. Let this dominant color be c_1 and the other color be c_2.

There are only four possible arrangements of colors in any column of the dominant color c_1: (c_1, c_1, c_1), (c_1, c_1, c_2), (c_1, c_2, c_1), and (c_2, c_1, c_1). If any of the columns with the dominant color are of this first type, then that column paired with any other column will form a rectangle whose corners are of color c_1: choose any two points from the second column of color c_1 and connect them with their corresponding points in the column colored (c_1, c_1, c_1). Otherwise, if there is no column of this type, then since there are three possible colorings of the colors and four total columns, by the pigeonhole principle two of these columns must be identical, at which point we can form a rectangle whose corners have the same color by joining together the two points of color c_1 in each color with one another. ■

Problem Five: Repeated Squaring

Prove that for any $x \in \mathbb{R}$ and any $y \in \mathbb{N}$, that $RS(x, y) = x^y$.

Proof: By complete induction. Let $P(y)$ be "for any $x \in \mathbb{R}$, we have $RS(x, y) = x^y$." We will prove that $P(y)$ holds for all $y \in \mathbb{N}$.

As our base case, we need to show $P(0)$, that for any $x \in \mathbb{R}$, we have $RS(x, 0) = x^0$. By definition, $RS(x, 0) = 1 = x^0$, so $P(0)$ holds.

For the inductive step, assume that $P(0), P(1), P(2), \ldots, P(y)$ hold. In other words, we assume that $RS(x, y') = x^{y'}$ for any natural number in the range $0 \leq y' \leq y$. We will prove $P(y + 1)$, that for any $x \in \mathbb{R}$ we have $RS(x, y + 1) = x^{y+1}$.

Consider any $x \in \mathbb{R}$. We consider two cases:

Case 1: $y + 1$ is even. Then $RS(x, y + 1) = RS(x, (y + 1) / 2)^2$. By our inductive hypothesis, we know that $RS(x, (y + 1) / 2) = x^{(y+1)/2}$. Therefore, $RS(x, y + 1) = RS(x, (y + 1) / 2)^2 = x^{y+1}$.

Case 2: $y + 1$ is odd. Then $RS(x, y) = x \cdot RS(x, y / 2)^2$. By our inductive hypothesis, we know that $RS(x, y / 2) = x^{y/2}$, so $RS(x, y) = x \cdot RS(x, y / 2)^2 = x \cdot (x^{y/2})^2 = x \cdot x^y = x^{y+1}$.

Thus in either case $RS(x, y + 1) = x^{y+1}$, so P(y+1) holds, completing the induction. ■

Problem Six: The Harmonic Series

Prove, by induction on n, that

$$H_{2^n} \geq \frac{n}{2} + 1$$

Proof: By induction. Let $P(n)$ be defined as follows:

$$P(n) \equiv \sum_{k=1}^{2^n} \frac{1}{k} \geq \frac{n}{2} + 1$$

We prove by induction that $P(n)$ holds for all n ∈ ℕ. For our base case, we prove $P(0)$, that

$$\sum_{k=1}^{1} \frac{1}{k} \geq \frac{0}{2} + 1$$

The left-hand side of this summation is 1, and the right-hand side is 1. Since $1 \geq 1$, $P(0)$ holds.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(n)$ holds, meaning that

$$\sum_{k=1}^{2^n} \frac{1}{k} \geq \frac{n}{2} + 1$$

We will prove $P(n + 1)$, that

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} \geq \frac{n+1}{2} + 1$$

To do so, first note that

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k}$$

Note that for any $k \in \mathbb{N}$ where $k \leq 2^{n+1}$, that $1 / k \geq 1 / 2^{n+1}$. Therefore:

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}}$$

Since this second sum is 2^n copies of $1 / 2^{n+1}$, the second sum is equal to $2^n (1 / 2^{n+1}) = 1 / 2$. Combined with our inductive hypothesis, we have that

$$\sum_{k=1}^{2^{n+1}} \frac{1}{k} = \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \geq \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{2^{n+1}} = \sum_{k=1}^{2^n} \frac{1}{k} + \frac{1}{2} \geq \frac{n}{2} + 1 + \frac{1}{2} = \frac{n+1}{2} + 1$$

Thus $P(n + 1)$ holds, completing the induction. ■

Problem Eight: Increasing Functions

A function $f : \mathbb{R} \to \mathbb{R}$ is called monotonically increasing iff for every $x, y \in \mathbb{R}$, if $x < y$, then $f(x) < f(y)$. Prove that any monotonically-increasing function must be injective.

Proof: Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically-increasing function. Consider any $x, y \in \mathbb{R}$. We will prove that if $x \neq y$, then $f(x) \neq f(y)$. To do this, note that since $x \neq y$, we know that either $x < y$ or $y < x$. Without loss of generality, assume $x < y$. Then since f is monotonically-increasing, we know that $f(x) < f(y)$, and so $f(x) \neq f(y)$. Thus f is injective. ■

Problem Nine: Infinity is Strange!

For any $n \in \mathbb{N}$, let $S_n = \{ k \in \mathbb{N} \mid k \geq n \}$. Prove for any $n \in \mathbb{N}$ that $|\mathbb{N}| = |S_n|$.

Proof: Let k be an arbitrary natural number. Define the function $f : S_k \to \mathbb{N}$ as $f(n) = n - k$. This function is a legal function from S_k to $\mathbb{N}$, since any $n \in S_k$ satisfies $n \geq k$ and so $n - k \geq 0$. We will prove that this function is a bijection, from which we can conclude $|\mathbb{N}| = |S_k|$.

To show that f is a bijection, we will prove it is injective and surjective. To prove injectivity, let's consider any $n_0, n_1 \in S_k$ where $n_0 \neq n_1$. We will prove that $f(n_0) \neq f(n_1)$. To see this, note that since we know $n_0 \neq n_1$, we also know $n_0 - k \neq n_1 - k$. Therefore, $f(n_0) \neq f(n_1)$, as required.

To prove surjective, let's consider any $n \in \mathbb{N}$. We will show that there is an $m \in S_k$ such that $f(m) = n$. To do this, let $m = n + k$. Since $n \in \mathbb{N}$, we know $n \geq 0$, and so $n + k \geq k$. Therefore, we know that $m \in S_k$. Moreover, $f(m) = f(n + k) = n + k - k = n$, as required. ■