Discussion Solutions 7

Problem One: Undecidability Reductions

Prove that *ENTERS* = { ⟨*M*, *w*, *q*⟩ | *q* is a state in *M* and *M* enters *q* when run on *w* } is undecidable.

The intuition behind this proof is that *M* accepts *w* precisely when *M* enters its accept state q_{acc} when run on *w*. We can therefore decide whether or not *M* accepts *w* by deciding whether or not *M* enters q_{acc} when run on *w*.

Proof: We show that $A_{TM} \leq_M$ *ENTERS*. since $A_{TM} \notin \mathbf{R}$, this proves that *ENTERS* $\notin \mathbf{R}$.

To see that $A_{TM} \leq_M$ *ENTERS*, we exhibit a mapping reduction from A_{TM} to *ENTERS*. Define *f*(⟨*M*, *w*⟩) = ⟨*M*, *w*, q_{acc}⟩, where q_{acc} is *M*'s accepting state. This function *f* is computable.

We will prove that *f* is a mapping reduction from A_{TM} to *ENTERS* by proving for all TMs *M* and strings *w*, that ⟨*M*, *w*⟩ ∈ A_{TM} iff ⟨*M*, *w*, q_{acc}⟩ ∈ *ENTERS*. To see this, note that by definition of A_{TM}, we have ⟨*M*, *w*⟩ ∈ A_{TM} iff *M* accepts *w*. *M* accepts *w* iff *M* enters q_{acc} when run on *w*. Finally, by the definition of *ENTERS*, *M* enters q_{acc} when run on *w* iff ⟨*M*, *w*, q_{acc}⟩ ∈ *ENTERS.* Combining these statements together, we get that ⟨*M*, *w*⟩ ∈ A_{TM} iff ⟨*M*, *w*, q_{acc}⟩ ∈ *ENTERS*. Therefore, *f* is a mapping reduction from A_{TM} to *ENTERS*, as required. ■

Why we asked this question: On the last set of discussion problems, we asked you to prove that this language was undecidable by showing that, if it were decidable, then A_{TM} would be decidable. This alternate proof, which we hoped would be a lot cleaner, shows that the language is undecidable using the framework of mapping reducibility.

Problem Two: Infinity is Strange

Prove that *INFINITE* = { ⟨*M*⟩ | $\mathscr{L}(M)$ is infinite } is not **RE**. If you have the time, prove that *INFINITE* is not co-**RE** either.

Proof: We will prove that $A_{TM} \le_M$ *INFINITE*. Since $A_{TM} \notin$ co-**RE**, this proves that *INFINITE* ∉ co-**RE**.

To prove that $A_{TM} \le_M$ *INFINITE*, we will find a mapping reduction from A_{TM} to *INFINITE*. For any TM/string pair ⟨*M*, *w*⟩, let *f*(⟨*M*, *w*⟩) = ⟨Amp(*M*, *w*)⟩. By a theorem from lecture, this function is computable.

We now prove that *f* is a mapping reduction from A_{TM} to *INFINITE* by proving that for any TM/string pair ⟨*M*, *w*⟩, that ⟨*M*, *w*⟩ ∈ A_{TM} iff ⟨Amp(*M*, *w*)⟩ ∈ *INFINITE*. By the definition of A_{TM}, we know that ⟨*M*, *w*⟩ ∈ A_{TM} iff *M* accepts *w*.

Now, we claim that *M* accepts *w* iff $\mathscr{L}$(Amp(*M*, *w*)) is infinite. First, note that if *M* accepts *w*, then $\mathscr{L}$(Amp(*M*, *w*)) = Σ*, which is infinite. Next, note that if *M* does not accept *w*, then $\mathscr{L}$(Amp(*M*, *w*)) = Ø, which is not infinite. Therefore, we see that *M* accepts *w* iff $\mathscr{L}$(Amp(*M*, *w*)) is infinite.

Finally, note that by the definition of *INFINITE*, we see that $\mathscr{L}$(Amp(*M*, *w*)) is infinite iff ⟨Amp(*M*, *w*)⟩ ∈ *INFINITE*. Combining the preceding statements together, we see that ⟨*M*, *w*⟩ ∈ A_{TM} iff ⟨Amp(*M*, *w*)⟩ ∈ *INFINITE*. This means that *f* is a mapping reduction from A_{TM} to *INFINITE*, and so $A_{TM} \le_M$ *INFINITE*, as required. ■

Why we asked this question: The “amplifier machine” from lecture is a powerful building block for showing that various languages are not **RE** or co-**RE**. In this proof, we use the amplifier to turn the question of “what does *M* do on *w*?” into the question “does this TM accept every string?” As you'll see in the problem set, it is sometimes useful to build other TMs out of TM/string pairs other than the amplifier machine, and in fact using that other TM you can prove that *INFINITE* ∉ **RE** by reducing $\overline{A_{TM}}$ to *INFINITE*.

Problem Three: Sets and Subsets

The language A_{ALL} is defined as $A_{ALL} = \{ \langle M \rangle \mid \mathscr{L}(M) = \Sigma^* \}$. $A_{ALL} \notin$ **RE** and $A_{ALL} \notin$ co-**RE** (you'll prove this in Problem Set 8).

Prove that the language $SUBSET_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1$ and M_2 are TMs, and $\mathscr{L}(M_1) \subseteq \mathscr{L}(M_2) \}$ is neither **RE** nor co-**RE** by reducing A_{ALL} to $SUBSET_{TM}$.

The idea behind this reduction is to use the fact that A_{ALL} is unrecognizable to show that $SUBSET_{TM}$ is not recognizable either. To do so, we'll use the fact that any language L satisfies $L \subseteq \Sigma^*$. Consequently, if $\Sigma^* \subseteq L$, then $L = \Sigma^*$. In other words, if Σ^* is a subset of some other language, then that language must be Σ^*. Consequently, we can use $SUBSET_{TM}$ as a building block to recognize A_{ALL}, which we know is unrecognizable. The formal proof is given here:

Proof: We will prove that $A_{ALL} \leq_M SUBSET_{TM}$. Since we know $A_{ALL} \notin$ **RE**, this proves that $SUBSET_{TM} \notin$ **RE**.

To prove $A_{ALL} \leq SUBSET_{TM}$, we will give a mapping reduction from A_{ALL} to $SUBSET_{TM}$. For any TM M, let $f(\langle M \rangle) = \langle S, M \rangle$, where S is the TM "On input w, accept." This function is computable.

We further claim that $\langle M \rangle \in A_{ALL}$ iff $\langle S, M \rangle \in SUBSET_{TM}$. To see this, note that by definition of A_{ALL}, $\langle M \rangle \in A_{ALL}$ iff $\mathscr{L}(M) = \Sigma^*$. Now, we claim that $\mathscr{L}(M) = \Sigma^*$ iff $\mathscr{L}(S) \subseteq \mathscr{L}(M)$. To see this, note first that if $\mathscr{L}(M) = \Sigma^*$, then $\mathscr{L}(S) \subseteq \mathscr{L}(M)$ because any language is a subset of Σ^*. Next, note that if $\mathscr{L}(M) \neq \Sigma^*$, then there is some string $w \in \Sigma^*$ such that $w \notin \mathscr{L}(M)$. Then $\mathscr{L}(S)$ is not a subset of $\mathscr{L}(M)$, since $w \in \mathscr{L}(S)$ (recall that $\mathscr{L}(S) = \Sigma^*$) but $w \notin \mathscr{L}(M)$. Therefore, we have $\mathscr{L}(M) = \Sigma^*$ iff $\mathscr{L}(S) \subseteq \mathscr{L}(M)$. Finally, note that by definition of $SUBSET_{TM}$, that $\mathscr{L}(S) \subseteq \mathscr{L}(M)$ iff $\langle S, M \rangle \in SUBSET_{TM}$. Combining these steps together, we see that $\langle M \rangle \in A_{ALL}$ iff $\langle S, M \rangle \in SUBSET_{TM}$. Therefore, we see that f is a mapping reduction from A_{ALL} to $SUBSET_{TM}$, so $A_{ALL} \leq_M SUBSET_{TM}$, as required. ■

Why we asked this question: This question shows that you can prove that a language is neither **RE** nor co-**RE** in one fell swoop by reducing a language already known to be neither **RE** nor co-**RE** to it.