Problem Set 3 Solutions

Problem One: Meet Semilattices (20 points)

i. Prove that $\leq$ is a partial order.

Proof: Let $(\wedge, D)$ be a meet semilattice and let $x \leq_S y$ iff $x \wedge y = x$. We will prove that $\leq_S$ is a partial order over D. To do so, we'll prove $\leq_S$ is reflexive, antisymmetric, and transitive.

To show that $\leq_S$ is reflexive, we need to prove that $x \leq_S x$ for all $x \in D$. Note that since $\wedge$ is idempotent, we have that $x \wedge x = x$ for all $x \in D$. Thus by definition, $x \leq_S x$, as required.

To show that $\leq_S$ is antisymmetric, suppose that for some $x, y \in D$ that $x \leq y$ and $y \leq x$. We will prove that $x = y$. Since $x \leq_S y$, we have $x \wedge y = x$. Since $y \leq x$, we have $y \wedge x = y$. Since $\wedge$ is commutative, from $y \wedge x = y$ we see that $x \wedge y = y$. Thus $x = x \wedge y = y \wedge x = y$, so $x = y$, as required.

To show that $\leq_S$ is transitive, suppose that for some $x, y, z \in D$ that $x \leq_S y$ and $y \leq_S z$. We will show that $x \leq_S z$. Since $x \leq_S y$, we know $x \wedge y = x$. Since $y \leq_S z$, we know $y \wedge z = y$. Since $\wedge$ is associative, we thus have that

$$x = x \wedge y = x \wedge (y \wedge z) = (x \wedge y) \wedge z = x \wedge z$$

So $x \wedge z = x$, and therefore $x \leq_S z$, as required. ■

ii. Prove that for any $x, y \in D$, that $x \wedge y \leq_S x$ and $x \wedge y \leq_S y$. This proves that $x \wedge y$ is a *lower bound* of x and y.

Proof: Consider any $x, y \in D$. We will show that $x \wedge y \leq_S x$ and $x \wedge y \leq_S y$.

To show that $x \wedge y \leq_S x$, note that

$$(x \wedge y) \wedge x = x \wedge (y \wedge x) = x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y$$

So $(x \wedge y) \wedge x = x \wedge y$. Therefore, $x \wedge y \leq_S x$.

To show that $x \wedge y \leq_S y$, note that

$$(x \wedge y) \wedge y = x \wedge (y \wedge y) = x \wedge y$$

So $(x \wedge y) \wedge y = x \wedge y$. Therefore, $x \wedge y \leq_S y$. ■

iii. Prove that for any $x, y \in D$, that if $z \leq_S x$ and $z \leq_S y$, that $z \leq_S x \wedge y$. This proves that $x \wedge y$ is a *greatest lower bound* of x and y.

Proof: Suppose that $z \leq_S x$ and $z \leq_S y$. We will prove that $z \leq_S x \wedge y$.

Since $z \leq_S x$, we know $z \wedge x = z$. Similarly, since $z \leq_S y$, we know $z \wedge y = z$. Thus

$$z \wedge (x \wedge y) = (z \wedge x) \wedge y = z \wedge y = z$$

So $z \wedge (x \wedge y) = z$, and therefore $z \leq_S x \wedge y$. ■

Why we asked this question: We asked this question for several reasons. First, this problem introduced the terms *lower bound* and *greatest lower bound*, which arise frequently in discrete mathematics and computer science. Second, this problem tests your ability to reason about one abstract set of definitions (the properties of a partial order) by using another abstract set of definitions (the properties of a meet operator). Finally, as mentioned at the bottom of the problem, this problem calls forward to material covered in other CS classes and helps explain why the material we are covering is relevant in computer science.

Problem Two: The Six-Color Theorem (20 Points)

Prove the six-color theorem.

Proof: By induction. Let $P(n)$ be "any planar graph with n nodes is 6-colorable." We will prove that $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that any planar graph with 0 nodes is 6-colorable. Such a graph is vacuously six-colorable, since coloring all of its 0 nodes satisfies the conditions of a 6-coloring.

For our inductive step, assume that $P(n)$ holds for some $n \in \mathbb{N}$ and that any planar graph with n nodes is 6-colorable. We will prove $P(n + 1)$, that any planar graph with $n + 1$ is 6-colorable.

Consider any planar graph $G = (V, E)$ with $n + 1$ nodes. Consider any node $v \in V$ with degree five or less; such a node must exist. Now, consider the graph G' formed by deleting v and all edges incident to it from G. This new graph is planar, because if we draw G without any edges crossing and then remove v and all its edges, the resulting graph also has no crossing edges. Therefore, by our inductive hypothesis, the graph G' is 6-colorable. Color the nodes in G using this 6-coloring. Since v has degree five or less, there are at most five colors adjacent to v, so we can color v using one of the unused six colors. Therefore, G is 6-colorable, completing the induction. ■

Why we asked this problem: This problem shows another way of writing proofs about graphs: induction. Here, the induction works on the number of nodes in the graph, though it's also possible to use induction to prove results about graphs by inducting on the number of edges, the degree of the nodes, etc.

Another reason we asked this problem was to make sure that you structured the induction correctly. An *extremely common* mistake on this problem was to start with a planar graph with n nodes, add in a new node with degree five or less, and conclude that the resulting planar graph with $n + 1$ nodes is therefore 6-colorable. *This is not a sound line of reasoning*. Remember, the goal is to prove that any planar graph with n+1 nodes is 6-colorable. The preceding line of reasoning only shows that any (n+1)-node planar graph formed by adding a node of degree five or less to a planar graph of n nodes is 6-colorable. Unless you can prove that every (n+1)-node planar graph can be formed this way, the proof is invalid.

Think of the proof this way: to prove $P(n) \rightarrow P(n+1)$, you should assume $P(n)$ and prove $P(n+1)$. $P(n+1)$ says that any planar graph with n+1 nodes is 6-colorable, so the proper way to start this proof off would be to consider an arbitrary (n+1)-node planar graph G and prove it's 6-colorable.

Problem Three: Tournament Cycles (20 Points)

Prove that if a tournament graph contains a cycle, then its girth is three.

Proof: Let $G = (V, E)$ be any tournament graph that contains at least one cycle and let C be the smallest cycle in G. C can't have length one, because there are no self-loops in a tournament graph. C also can't have length two, because if $(u, v) \in E$, then $(v, u) \notin E$ because tournament graphs only have one pair of edges between nodes.

We claim that C has to have length three and proceed by contradiction; suppose its length is greater than three. Let n denote the length of C (by assumption, $n \geq 4$) and let the nodes in C be $v_1, v_2, v_3, \ldots, v_n, v_1$. Consider the edge between nodes v_2 and v_n. If the edge is of the form (v_2, v_n), then v_1, v_2, v_n, v_1 is a cycle of length three. This is impossible, because cycle C is the smallest cycle in G and its length is at least four. Therefore, the edge must be (v_n, v_2). Then $v_2, v_3, \ldots, v_n, v_2$ is a cycle in G with length $n-1$, contradicting that C, the smallest cycle in G, has length n. In either case we reach a contradiction, so our assumption was wrong. Therefore, C must have length three. ■

Why we asked this question: This question was designed to introduce the idea of proofs by extremal case. Notice how the proof works here – if we assume C is the smallest cycle and C has length greater than three, we can get a contradiction by showing that a shorter cycle must exist. Proofs of the form "the largest X is Y" or "the smallest X is Y" can often be structured this way.

Here is an alternative proof using tournament winners and induction:

Proof: First, note that no tournament graph has a cycle of length 1 (because there are no self-loops) or length 2 (because any pair of nodes has only one edge between them). Our proof proceeds by induction. Let $P(n)$ be "any n-player tournament containing a cycle has girth three." We prove $P(n)$ holds for all natural numbers $n \geq 1$ by induction on n. As our base case, we prove $P(1)$, that any 1-player tournament containing a cycle has girth three. This is vacuously true, as any tournament with one node can't contain any cycles because there are no edges.

For the inductive step, assume for some $n \geq 1$ that $P(n)$ holds and any n-player tournament containing a cycle has girth three. We prove $P(n+1)$, that any $(n+1)$-player tournament containing a cycle has girth three. Let $G = (V, E)$ be any $(n+1)$-player tournament containing a cycle. This tournament has at least one winner; choose one and call her w. Let X be the set of players w lost to. If $X \neq \emptyset$, there is at least one player that w lost to (call this player x). Since w is a tournament winner, there must be some player y such that w beat y and y beat z. This creates a cycle w, y, z, w of length three, and since there aren't any shorter cycles possible, this must be the shortest cycle in the graph.

Otherwise, $X = \emptyset$, so w beat all other players. Then w cannot be a part of the cycle in the tournament, since there are no edges entering w. Therefore, any cycles in G must be cycles in the subtournament G' of all n players except w. By our inductive hypothesis, we know G' has girth three. Therefore, the overall graph has girth three as well.

In both cases we see $P(n+1)$ holds, completing the induction. ■

Problem Four: Complements and Connectivity (20 Points)

Prove that for every undirected graph G, at least one of G and G^c must be connected.

Proof: Let $G = (V, E)$ be an undirected graph. If G is connected then we are done. Otherwise, G is not connected, so it consists of two or more connected components.

Consider any nodes $u, v \in V$. If u and v belong to different connected components of G, then the edge $\{u, v\} \notin E$; otherwise, u and v would be connected and would belong to the same connected component. Therefore, $\{u, v\}$ must be an edge in G^c, and so u and v are connected in G^c.

Otherwise, nodes u and v belong to the same connected component. Consider any node $x \in V$ that belongs to a different connected component than u and v. Then $\{u, x\}$ and $\{x, v\}$ are not edges in G, so they must be edges in G^c. Therefore, u is connected to v in G^c because we can follow the path u, x, v.

Since our choice of nodes was arbitrary, this establishes that any two nodes in G^c are connected, so G^c is connected, as required. ■

Why we asked this question: This problem is closely connected to the connectivity relation we explored in our lecture on graphs and on connected components and their properties. We hoped that you would be able to make the connections between graph connectivity, connected components, and the edges in the graph.

Problem Five: Pigeonhole Party! (20 Points)

i. Prove that at a party with at least two people present, there are at least two people with the same number of acquaintances at that party.

Proof: Suppose that there are n people at the party. We consider two cases.

Case 1: Everyone at the party knows someone else. Then of the n people at the party, those people can each know 1, 2, 3, …, or $n - 1$ other people in the room. This means that there are n different people at the party and $n - 1$ different possible values for the number of people they can know. By the pigeonhole principle, this means that two people must know the same number of people at the party.

Case 2: Someone at the party knows no one else. Then no one at the party can know everyone else, so people at the party can know either 0, 1, 2, …, or $n - 2$ other people in the room. Since there are n people at the party and only $n - 1$ different possible numbers of people they can know in this case, by the pigeonhole principle two people must know the same number of people at the party. ■

ii. Using the generalized pigeonhole principle, prove that in any group of six people that there are at least three mutual acquaintances or at least three mutual strangers.

Proof: Consider any one person p at the party. There are five other people at the party, each of which either knows person p or does not. Since there are five other people and two possible options for how each of those people relate to p, by the generalized pigeonhole principle there must be a group of at least $\lceil 5 / 2 \rceil = 3$ acquaintances of p or $\lceil 5 / 2 \rceil = 3$ strangers to p. Without loss of generality, assume that there are 3 acquaintances to p; the case where there are three strangers to p is analogous but with the roles "acquaintance" and "stranger" reversed.

Now, if any two of p's acquaintances know each other, then there is a group of three mutual acquaintances at the party, namely p and the two acquaintances who know one another. Otherwise, none of p's acquaintances know each other, so there is a group of three mutual strangers at the party. ■

Why we asked this question: These questions are classic results that can be proven with the pigeonhole principle. The first problem is interesting in that, initially, the number of possible options for numbers of acquaintances is too large to apply the pigeonhole principle, but can be reduced by use of a case analysis. The second problem is an example of *Ramsey theory*, which states that for sufficiently large graphs, there will either be a sufficiently large cluster of mutually connected nodes or a sufficiently large cluster of disconnected nodes. We chose to put this problem on the problem set because the pigeonhole principle by itself doesn't suffice to show this result; you need to take it a step further and consider the triangle of people found after the first step.

Problem Six: Coloring a Grid (20 Points)

Suppose that you color each point in a 3×9 grid either red or blue. Prove that no matter how you color those points, you can always find four points of the same color that form the corners of a rectangle.

Proof: We will prove this result in two parts. First, we will prove that two of the columns in the grid must have identical colorings. Second, we will prove that if there are two identically-colored columns in the grid, then there must be a rectangle whose corners have the same color.

To see that two columns in the grid must have the same color, note that there are only eight possible ways to assign two colors to three dots: (r, r, r), (r, r, b), (r, b, r), (r, b, b), (b, r, r), (b, r, b), (b, b, r), and (b, b, b). Since there are nine columns and eight possible colorings, by the pigeonhole principle two columns must have exactly the same coloring.

Now, we will show that this means that there must be a rectangle whose corners are all the same color. Take the two columns with the same coloring. Since there are two possible colors and three points in each column, two of the points in each column must have the same color. Using this, we can form a rectangle as follows: pick any two points of the same color in one of the repeated columns, then pick the same points in the duplicated column. This ensures all four points have the same color. This picks two points out of one column and the same two points out of another column, so those points form a rectangle. ■

Why we asked this question: This question uses the pigeonhole principle *twice* – once to get a duplicated column, and once to find the duplicated color within the column. There are many solutions to this problem; in fact, it's possible to show that any 3×7 grid also has a rectangle in it by using the generalized pigeonhole principle rather than the normal pigeonhole principle. Can you see how?