Problem Set 2 Solutions

Problem One: Recurrence Relations (16 Points)

$$a_0 = 1$$
$$a_{n+1} = 2a_n$$

The first few terms of this sequence are 1, 2, 4, 8, 16, 32, …, which happen to be powers of two.

i. Prove by induction that for any $n \in \mathbb{N}$, we have $a_n = 2^n$.

Theorem: For any $n \in \mathbb{N}$, we have $a_n = 2^n$.

Proof: By induction. Let $P(n)$ be "$a_n = 2^n$." We prove $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that $a_0 = 2^0 = 1$. This is true by the definition of a_0.

For the inductive step, assume that for some $n \in \mathbb{N}$ we have $P(n)$, meaning that $a_n = 2^n$. We prove $P(n + 1)$, that $a_{n+1} = 2^{n+1}$. To see this, note that $a_{n+1} = 2a_n = 2(2^n) = 2^{n+1}$. Thus $a_{n+1} = 2^{n+1}$, so $P(n + 1)$ holds, completing the induction. ■

$$b_0 = 1$$
$$b_{n+1} = 2b_n – 1$$

ii. Find non-recursive definitions for b_n and c_n, then prove by induction that your definitions are correct.

Theorem: For any $n \in \mathbb{N}$, we have $b_n = 1$.

Proof: By induction. Let $P(n)$ be "$b_n = 1$." We prove $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that $b_0 = 1$. This is true by the definition of b_0.

For the inductive step, assume that for some $n \in \mathbb{N}$ we have $P(n)$, meaning that $b_n = 1$. We prove $P(n + 1)$, that $b_{n+1} = 1$. To see this, note that $b_{n+1} = 2b_n – 1= 2 – 1 = 1$. Thus we have $b_{n+1} = 1$, so $P(n + 1)$ holds, completing the induction. ■

Theorem: For any $n \in \mathbb{N}$, we have $c_n = 2^{n+1} – 1$.

Proof: By induction. Let $P(n)$ be "$c_n = 1$." We prove $P(n)$ holds for all $n \in \mathbb{N}$.

As our base case, we prove $P(0)$, that $c_0 = 2^1 – 1 = 1$. This is true by the definition of c_0.

For the inductive step, assume that for some $n \in \mathbb{N}$ we have $P(n)$, meaning $c_n = 2^{n+1} – 1$. We prove $P(n + 1)$, that $c_{n+1} = 2^{n+2} – 1$. To see this, note that

$$c_{n+1} = 2a_n + 1= 2(2^{n+1} – 1) + 1 = 2^{n+2} – 2 + 1 = 2^{n+2} – 1$$

Therefore, $c_{n+1} = 2^{n+2} – 1$ holds, completing the induction. ■

Why we asked this question: Recurrence relations arise frequently in computer science, especially in the design and analysis of algorithms. Induction is a great way to confirm that a guess about the recurrence is actually correct.

Problem Two: Nim (20 points)

Theorem: If both piles have n stones, the second player can win if she plays correctly.

Proof: By complete induction. Let P(n) be "In a game of Nim, if both piles of stones have n stones in them and it's the first player's turn, the second player can always win if she plays correctly." We prove $P(n)$ is true for all $n \in \mathbb{N}$.

For the base case, we prove $P(0)$, that in a game of Nim, if both piles of stones have 0 stones in them and it's the first player's turn, the second player can always win if she plays correctly. This is true because the first player immediately loses the game, so the second player always wins.

For the inductive step, assume that for some $n \in \mathbb{N}$ that $P(0)$, $P(1)$, $P(2)$, …, $P(n)$ are all true. That is, for any $n' \in \mathbb{N}$ where $0 \le n' \le n$, in any game of Nim where there are n' stones in each pile and it's the first player's turn, the second player can always win if she plays correctly. We prove $P(n + 1)$, that if there are $n + 1$ stones in each pile and it's the first player's turn, then the second player can always win if she plays correctly.

Suppose the first player removes k stones from some pile (where $1 \le k \le n + 1$); let that pile be A and the other be B. This leaves $n + 1 - k$ stones in pile A. If the second player then removes k stones from pile B, there will be $n + 1 - k$ stones in pile B. Because $1 \le k \le n + 1$, we see that $n \ge n + 1 - k \ge 0$, so each pile has between 0 and n stones in it. It is now the first player's turn. Thus by the inductive hypothesis, the second player can always win in this situation if she plays correctly. Using this strategy, the second player can always win given a game with two piles with $n + 1$ stones in each pile, so $P(n + 1)$ is true, completing the induction. ■

Why we asked this question: This question relies on complete induction to establish the main result. We know that the number of stones in each pile will decrease, but we're not sure *how much* they'll decrase. Accordingly, we use complete induction to assume that if the two piles have the same size for any size up to an including n, the second player has a winning strategy.

Problem Three: Contract Rummy (16 points)

Theorem: For any $n \in \mathbb{N}$ with $n \ge 6$, there exists natural numbers r and s such that $n = 3r + 4s$.

Proof: By induction on n. Let $P(n)$ be "there exists natural numbers r and s such that $n = 3r + 4$s." We prove that $P(n)$ is true for all $n \in \mathbb{N}$ where $n \ge 6$.

As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$. To see this, note that

$$6 = 3\cdot 2 + 4\cdot 0 \qquad 7 = 3\cdot 1 + 4\cdot 1 \qquad 8 = 3\cdot 0 + 4\cdot 2$$

For the inductive step, assume that for some $n \in \mathbb{N}$ where $n \ge 6$ that $P(n)$ holds and that n can be written as $3r + 4s$ for some natural numbers r and s. We prove $P(n + 3)$, that $n + 3$ can be written as $3r' + 4s'$ for some natural numbers r' and s'. Note that

$$n + 3 = 3r + 4s + 3 = 3r + 3 + 4s = 3(r + 1) + 4s$$

So there exist natural numbers r' and s' (namely, $r + 1$ and s) such that $n + 3 = 3r' + 4s'$, so $P(n + 3)$ holds, completing the induction.

Why we asked this question: This problem is an example of using multiple base cases and taking steps of larger sizes. Although this isn't the only way to prove this result (you can also use normal induction starting at 6 or complete induction if you'd like), it's one of the most straightforward.

Problem Four: Colored Cubes (24 Points)

i. Prove that if $n \geq 1$, then there must be some color such that there are at least k cubes of that color and some color such that there are at most k cubes of that color.

Theorem: If there are n different colors of cubes and nk cubes, then there must be some color such that there are at least k cubes of that color and a color such that there are at most k cubes of that color.

Proof: We prove each part in turn. To show that there is a color for which there are at least k cubes of that color, assume for the sake of contradiction that this is not the case and that there are fewer than k cubes of each color. This means that there are at most $k - 1$ cubes of each color. Since there are n colors, this means that there are at most $n(k - 1) = nk - n < nk$ total cubes, contradicting that there are exactly nk total cubes. Thus our assumption is wrong and there must be some color for which at least k cubes are that color.

To show that there is a color for which there are at most k cubes of that color, assume for the sake of contradiction that this is not the case and that there are greater than k cubes of each color. This means that there are at least $k + 1$ cubes of each color. Since there are n colors, this means that there are at least $n(k + 1) = nk + n > nk$ total cubes, contradicting that there are exactly nk total cubes. Thus our assumption is wrong and there must be some color for which at most k cubes are that color. ■

ii. Prove by induction on n that it is always possible to distribute the cubes into n boxes such that each box contains exactly k cubes and each box contains cubes of at most two different colors.

Theorem: If there are cubes of n colors and nk total cubes, then they can be put into n boxes such that each box contains exactly k cubes and cubes of at most two different colors.

Proof: By induction. Let $P(n)$ be "If there are n colors and nk total cubes, the cubes can be put into n boxes such that each box has exactly k cubes and cubes of at most two different colors." We prove $P(n)$ is true for all $n \in \mathbb{N}$.

For our base case, we prove $P(0)$, that given cubes of 0 colors and 0 total cubes, the cubes can be put into 0 boxes such that each box has exactly k cubes and cubes of at most two different colors. Since there are no cubes to distribute, this is possible by distributing 0 cubes into each box.

For our inductive step, assume that $P(n)$ is true for some $n \in \mathbb{N}$, meaning that given cubes of n colors and kn total cubes, the cubes can be put into n boxes such that each box has exactly k cubes and cubes of at most two different colors. We prove $P(n + 1)$, that given $n + 1$ colors of cubes and $k(n + 1)$ total cubes, the cubes can be put into $n + 1$ boxes such that each box has exactly k cubes and cubes of at most two different colors.

By our result from part (i), there must be a color c_1 where there are at most k cubes of that color and a color c_2 (possibly the same as c_1) where there are at least k cubes of that color. Put all cubes of color c_1 into one box and then fill the remainder with cubes of color c_2. (If c_1 and c_2 are the same, this is not a problem, since there must be exactly k cubes of that color). This uses all cubes of color c_1, so there are now n remaining colors. We also used exactly k cubes, so there are now $k(n + 1) - k = kn$ total cubes. Thus by the inductive hypothesis, it is possible to distribute the remaining cubes into n boxes where each box has k total cubes and no more than two colors of cubes in each box. Collectively, this proves $P(n + 1)$, completing the induction. ■

Why we asked this question: This is an example of an induction where two different variables exist (here, n and k), but the induction only uses one of them. We wanted to give you experience working with induction in which multiple variables come into play. We also wanted to give you a chance to translate a mechanical procedure (here, for filling the boxes) into a mathematical proof.

Problem Five: Binomial Trees (16 Points)

Theorem: A binomial tree of order n contains exactly 2^n nodes.

Proof: By complete induction. Let $P(n)$ be "A binomial tree of order n contains exactly 2^n nodes." We prove that $P(n)$ holds for all $n \in \mathbb{N}$.

For our base case, we prove $P(0)$, that a binomial tree of order 0 contains exactly $2^0 = 1$ nodes. To see this, note that by definition, a binomial tree of order 0 is a single node with no children, so it contains exactly one node.

For our inductive step, assume that for some $n \in \mathbb{N}$ that $P(0)$, $P(1)$, $P(2)$, …, $P(n)$ are true, meaning that for any $n' \in \mathbb{N}$ with $0 \leq n' \leq 0$, any binomial tree of order n' contains exactly $2^{n'}$ nodes. We prove $P(n + 1)$, that a binomial tree of order $n + 1$ contains exactly 2^{n+1} nodes.

To see this, note that the number of nodes in an order-(n+1) binomial tree is given by

$$
\begin{aligned}
&1+\sum_{i=0}^{n}(\text{\# of nodes in a binomial tree of order i}) \\
&=1+\sum_{i=0}^{n}2^i && \textit{(by the inductive hypothesis)} \\
&=1+2^{n+1}-1 && \textit{(using our result from lecture)} \\
&=2^{n+1}
\end{aligned}
$$

Thus $P(n)$ holds, completing the induction. ■

Why we asked this question: This problem asks you to write an inductive proof about a recursively-defined structure (namely, the binomial tree). This sort of induction is common in computer science and we hoped that the binomial tree, which has a rigid structure, would serve as a good introduction. Additionally, it gives practice using complete induction, which is tested only once elsewhere on this problem set.

Problem Six: Tournament Graphs (28 points)

Theorem: Every tournament graph for $n \geq 1$ players has a tournament winner.

Proof: By induction. Let $P(n)$ be "any tournament graph of n players has a tournament winner." We prove that $P(n)$ is true for all $n \in \mathbb{N}$ with $n \geq 1$ by induction on n.

For the base case, we prove $P(1)$, that any tournament graph of 1 players has a tournament winner. In any tournament with just one player, that player must be a winner because the statement "the one player beat all other players" is vacuously true.

For the inductive step, assume that for some $n \in \mathbb{N}$ with $n \geq 1$ that $P(n)$ holds and any tournament graph with n players has a tournament winner. We will prove $P(n + 1)$, that any tournament graph with $n + 1$ players has a tournament winner.

Consider any tournament graph G with $n + 1$ players and choose an arbitrary player x. Consider the tournament graph G' formed by removing x from G. This tournament graph has n players, so by the inductive hypothesis it has a tournament winner w. Now, if w is a tournament winner in G, then we're done. Otherwise, w is not a winner. Therefore, w lost to x, and each player that w beat also lost to x. Using these facts, we will prove that x is a winner in G. To see this, consider any player p in G other than x. We consider three cases:

Case 1: $p = w$. Since w lost to x, we know that x beat w, so x beat p.

Case 2: w beat p. Since every player w beat lost to x, we know that x beat p.

Case 3: w lost to p. Since w is a winner, there must be some player p' where w beat p' and p' beat p. Since w beat p', we know that x beat p'. Therefore, x beat p' and p' beat p.

In all three cases, we see that x either beat p or beat some player p' who in turn beat p. Therefore, x is a tournament winner in G. Thus $P(n + 1)$ holds, completing the induction. ■

Why we asked this question: This is one of the more surprising results on the problem set: it doesn't at all seem like this should be true! We asked this problem because we wanted to give you a chance to tackle a more challenging induction problem where the inductive step was not immediately clear. Additionally, most proofs of this result work by cases, and we wanted you to write a proof that combined a proof by induction and a proof by cases.

It is possible to prove this result without using induction, and we actually invite you to try to do this. As a hint: choose a player who won the most games or was tied for the most wins. Can you show that they have to be a tournament winner?