Practice Midterm Exam 2

This exam is open-book and open-note. You may use a computer only to look at notes that you yourself have written, to access the course website and the tools there, and to read an online copy of one of the recommended readings. No other use of the computer is permitted during this exam. You must hand-write all of your solutions on this physical copy of the exam. No electronic submissions will be considered without prior consent of the course staff.

SUNetID: ______________________________

Last Name: ______________________________

First Name: ______________________________

I accept both the letter and the spirit of the honor code. I have not received any unpermitted assistance on this test, nor will I give any. My answers are my own work and are not adapted from any unpermitted sources. I will not use a computer except in the ways specified at the top of the exam.

(signed) ____

You have three hours to complete this midterm. There are 180 total points, and this midterm is worth 15% of your total grade in this course. You may find it useful to read through all the questions to get a sense of what this midterm contains. As a rough sense of the difficulty of each question, there is one point on this exam per minute of testing time.

Question		Points	Grader
(1) First-Order Logic	(25)	/ 25	
(2) Finding Flaws in Proofs	(15)	/ 15	
(3) Induction	(45)	/ 45	
(4) Relations	(50)	/ 50	
(5) The Pigeonhole Principle	(45)	/ 45	
	(180)	**/ 180**	

Note: Normally, there would be a lot of whitespace on this exam so that you would have room to write answers. To conserve paper, we've removed it in this practice exam.

Problem One: First-Order Logic (25 points total)

(i) Directed Acyclic Graphs (10 Points)

Given the predicates

> *DAG*(*G*), which states that *G* is a directed graph with no cycles;
> *NodeIn(v, G)*, which states that *v* is a node in *G*; and
> *EdgeIn(u, v, G)*, which states that in graph *G*, there is an edge from *u* to *v*,

write a statement in first-order logic that says "every directed graph with no cycles has a node with no incoming edges."

(ii) The Logic of Elections (15 Points)

Suppose that it's election night and the two major candidates are examining the vote counts in the US Presidential Election. (To avoid turning this midterm into a political debate, let's call the candidates Candidate *X* and Candidate *Y*.) Candidate *X* argues that more people voted for him than for Candidate *Y* by making the following claim: "For every ballot cast for Candidate *Y*, there were two ballots cast for Candidate *X*." Candidate *X* states this in first-order logic as follows:

$$\forall b. \ (BallotForY(b) \to \exists b_1. \ \exists b_2. \ (BallotForX(b_1) \land BallotForX(b_2) \land b_1 \neq b_2))$$

However, it is possible for the above first-order logic statement to be true even if Candidate *X* didn't get the majority of the votes. Give an example of a set of ballots such that

- Every ballot is cast for exactly one of Candidate *X* and Candidate *Y*,
- The set of ballots obeys the rules described by the above statement in first-order logic, but
- Candidate *Y* gets strictly more votes than Candidate *X*.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement when explaining why your set of ballots obeys it.

Problem Two: Finding Flaws in Proofs (15 points)

In lecture, we sketched a proof that $|\mathbb{N}| = |\mathbb{N}^2|$; that is, that there are the same number of natural numbers as pairs of natural numbers. However, below is a purported proof that $|\mathbb{N}| \neq |\mathbb{N}^2|$.

Theorem: $|\mathbb{N}| \neq |\mathbb{N}^2|$.

Proof: By contradiction; assume that $|\mathbb{N}| = |\mathbb{N}^2|$. Then there exists some bijection $f : \mathbb{N} \to \mathbb{N}^2$, so for any $n \in \mathbb{N}$, $f(n)$ is an ordered pair of natural numbers. For notational simplicity, denote the first value in the ordered pair $f(n)$ by $f_1(n)$ and the second value by $f_2(n)$. Thus $f(n) = (f_1(n), f_2(n))$.

Now, consider the sequence $p_0, p_1, p_2, \ldots$ defined as follows: for any $n \in \mathbb{N}$, let $p_n = (f_1(n) + 1, f_2(n) + 1)$. Note that each p_n is an ordered pair of natural numbers, so $p_n \in \mathbb{N}^2$ for all $n \in \mathbb{N}$.

Since f is a bijection, it is surjective, so for any p_n, there must be some $n \in \mathbb{N}$ such that $f(n) = p_n$. By our definition of p_n, this means that

$$(f_1(n), f_2(n)) = (f_1(n) + 1, f_2(n) + 1)$$

Since two ordered pairs are equal iff their components are equal, this means that $f_1(n) = f_1(n) + 1$ and $f_2(n) = f_2(n) + 1$, which is impossible.

We have reached a contradiction, so our initial assumption must have been wrong. Thus $|\mathbb{N}| \neq |\mathbb{N}^2|$. ■

Of course, this proof is incorrect and contains a fatal flaw. What's wrong with this proof?

Problem Three: Induction (45 points)

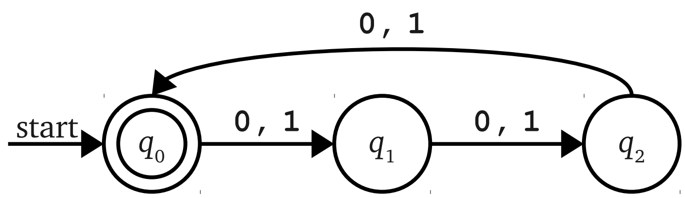
Consider the alphabet $\Sigma = \{\texttt{0}, \texttt{1}\}$. For any string $w \in \Sigma^*$, let's denote the length of w by $|w|$. For example, $|\varepsilon| = 0$, $|\texttt{01}| = 2$, $|\texttt{01000}| = 5$, etc.

Given this definition, consider the following language L_3:

$$L_3 = \{ \ w \in \Sigma^* \mid \ |w| \equiv_3 0 \ \}$$

For example, $\texttt{011} \in L_3$, $\texttt{010101} \in L_3$, and $\varepsilon \in L_3$, but $\texttt{0} \notin L_3$, $\texttt{00} \notin L_3$, and $\texttt{0000} \notin L_3$.

Below is a DFA D for the language L_3:



Prove by induction on $|w|$ that for any string $w \in \Sigma^*$, D accepts w iff $w \in L_3$. This establishes that $\mathscr{L}(D) = L_3$. To make the notation easier, you can denote by wa the string formed by appending the character a to the string w.

Problem Four: Relations (50 Points)

(i) Counting Partial Orders (20 Points)

Let $A = \{a, b, c\}$. How many different binary relations are there over A that are partial orders? You must justify your answer to receive credit, but you don't need to write a formal proof.

A note: two relations R_1 and R_2 over some set X are considered the same iff for any $x, y \in X$, we have that xR_1y iff xR_2y. Two relations are different iff they're not the same.

(Hint: It might be useful to draw some pictures.)

(ii) Relations and Functions (15 Points)

Let A and B be arbitrary sets where $\leq_B$ is a partial order over B. Suppose that we pick an injective function $f : A \to B$. We can then define a relation $\leq_A$ over A as follows: for any $x, y \in A$, we have $x \leq_A y$ iff $f(x) \leq_B f(y)$.

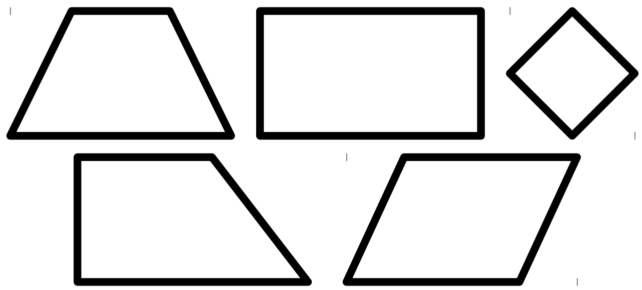
Prove that $\leq_A$ is a partial order over A.

(iii) Overlapping Relations (15 Points)

Are there any binary relations over $\mathbb{N}$ that are both equivalence relations and total orders? If so, give an example of one and prove why it is both an equivalence relation and a total order. If not, prove why not.

Problem Five: The Pigeonhole Principle (45 points total)

Suppose that you color every point in the real plane one of four colors (say, red, green, blue, and yellow). Prove that no matter how you color the plane, there will always be a trapezoid whose corners are all the same color. Recall that a trapezoid is a quadrilateral with at least two parallel sides. For example, all of the following figures are trapezoids:



(Hint: Try placing a specially-constructed object into the real plane such that no matter how that object is colored, the object always contains a trapezoid whose corners are the same color.)