Sheet 2

Our Names

May 19, 2025

Exercise 1

1. (a)

$$\mathcal{N}(\mu_k, \Sigma_k) = \frac{1}{\sqrt{2\pi^D \det \Sigma_k}} e^{-\frac{1}{2}(x-\mu_k)^\intercal \Sigma_k^{-1}(x-\mu_k)}$$

$$g_k(x) = -\frac{1}{2}\log\det\Sigma_k - \frac{1}{2}(x-\mu_k)^\intercal \Sigma_k^{-1}(x-\mu_k) + \log\pi_k$$

$$g_a(x) - g_b(x) \stackrel{!}{=} 0$$

$$\longleftrightarrow$$

$$-\frac{1}{2}\log\det\Sigma_A - \frac{1}{2}(x-\mu_A)^\intercal \Sigma_A^{-1}(x-\mu_A) + \log\pi_A$$

$$-\left(-\frac{1}{2}\log\det\Sigma_B - \frac{1}{2}(x-\mu_B)^\intercal \Sigma_B^{-1}(x-\mu_B) + \log\pi_B\right) = 0$$

$$\longleftrightarrow$$

$$\log\det\Sigma_B - \log\det\Sigma_A + (x-\mu_B)^\intercal \Sigma_B^{-1}(x-\mu_B) - (x-\mu_A)^\intercal \Sigma_A^{-1}(x-\mu_A) + 2(\log\pi_A - \log\pi_B)$$

$$\longleftrightarrow$$

$$x^\intercal\ \frac{1}{2}(\Sigma_A^{-1} - \Sigma_B^{-1})\ x + (\Sigma_A^{-1}\mu_A - \Sigma_B^{-1}\mu_B)\ x$$

$$+\frac{1}{2}(\mu_B^\intercal \Sigma_B^{-1}\mu_B - \mu_A^\intercal \Sigma_A^{-1}\mu_A) + \log\pi_A - \log\pi_B + \frac{1}{2}(\log\det\Sigma_B - \log\det\Sigma_A) = 0$$

$$\Lambda = \frac{1}{2}(\Sigma_A^{-1} - \Sigma_B^{-1})$$
$$w = \Sigma_A^{-1}\mu_A - \Sigma_B^{-1}\mu_B$$
$$b = \frac{1}{2}(\mu_B^\intercal \Sigma_B^{-1}\mu_B - \mu_A^\intercal \Sigma_A^{-1}\mu_A) + (\log \pi_A - \log \pi_B) + \frac{1}{2}(\log \det \Sigma_B - \log \det \Sigma_A)$$

A quadratic term indicates non linear decision boundry because quadratic equations are parabolic.

(b)

$$-\frac{1}{2}\log\det\Sigma - \frac{1}{2}(x-\mu_A)^\intercal \Sigma^{-1}(x-\mu_A) + \log \pi_A$$
$$+\frac{1}{2}\log\det\Sigma + \frac{1}{2}(x-\mu_B)^\intercal \Sigma^{-1}(x-\mu_B) - \log \pi_B = 0$$
$$\longleftrightarrow$$
$$-(x-\mu_A)^\intercal \Sigma^{-1}(x-\mu_A) + 2\log \pi_A + (x-\mu_B)^\intercal \Sigma^{-1}(x-\mu_B) - 2\log \pi_B = 0$$

$$(x-\mu_k)^\intercal \Sigma^{-1}(x-\mu_k) = x^\intercal \Sigma^{-1} x - 2\mu_k^\intercal \Sigma^{-1} x + \mu_k^\intercal \Sigma^{-1}\mu_k$$

$$-(x-\mu_A)^\intercal \Sigma^{-1}(x-\mu_A) + (x-\mu_B)^\intercal \Sigma^{-1}(x-\mu_B) + 2(\log \pi_A - \log \pi_B) = 0$$
$$\longleftrightarrow$$
$$2(\mu_A - \mu_B)^\intercal \Sigma^{-1} x \;+\; (\mu_B^\intercal \Sigma^{-1}\mu_B - \mu_A^\intercal \Sigma^{-1}\mu_A) \;+\; 2(\log \pi_A - \log \pi_B) = 0$$
$$\longleftrightarrow$$
$$(\mu_A - \mu_B)^\intercal \Sigma^{-1} x \;+\; \frac{1}{2}(\mu_B^\intercal \Sigma^{-1}\mu_B - \mu_A^\intercal \Sigma^{-1}\mu_A) \;+\; \log \pi_A - \log \pi_B = 0$$

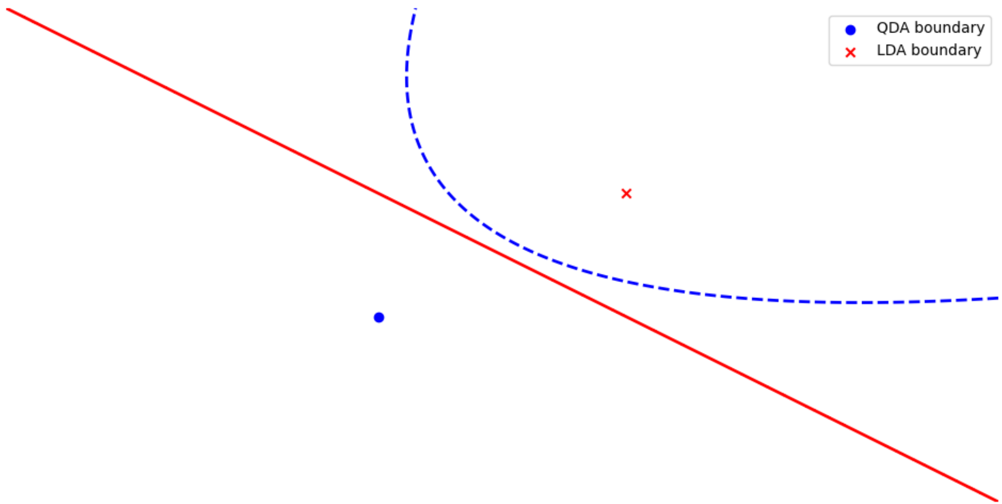
$$w = (\mu_A - \mu_B)^\intercal \Sigma^{-1}$$
$$b = \frac{1}{2}(\mu_B^\intercal \Sigma^{-1}\mu_B - \mu_A^\intercal \Sigma^{-1}\mu_A) \;+\; \log \pi_A - \log \pi_B$$

2. (a) QDA decision boundary:

$$x^\intercal \begin{bmatrix} 0.21008403 & -0.21008403 \\ -0.21008403 & 0.21008403 \end{bmatrix} x + \begin{bmatrix} -1.53846154 \\ -1.53846154 \end{bmatrix} x + 0.4436516$$

LDA decision boundary:

$$\begin{bmatrix} 1.53846154 \\ 1.53846154 \end{bmatrix} x + 0$$



(b)

(c) The linear LDA boundary is preferred when the classes are of the same shape.

Exercise 2