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# Modified boundary integral formulations for the Helmholtz equation

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**Abstract**

In this paper we describe some modified regularized boundary integral equations to solve the exterior boundary value problem for the Helmholtz equation with either Dirichlet or Neumann boundary conditions. We formulate combined boundary integral equations which are uniquely solvable for all wave numbers even for Lipschitz boundaries $\Gamma = \partial\Omega$. This approach extends and unifies existing regularized combined boundary integral formulations.


*Keywords:* Helmholtz equation; Boundary integral equations; Stabilization

## 1. Introduction

We consider the exterior boundary value problem for the Helmholtz equation with Dirichlet boundary conditions,

$$\begin{aligned} &\Delta u(x) + k^2 u(x) = 0 \quad &&\text{for } x \in \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega}, \\ &u(x) = g(x) &&\text{for } x \in \Gamma = \partial\Omega, \end{aligned} \tag{1.1}$$

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where $k \in \mathbb{R}_+$ is the wave number, and where $u$ satisfies in addition the Sommerfeld radiation condition

$$\left| \frac{x}{|x|} \cdot \nabla u(x) - iku(x) \right| = \mathcal{O}\left( \frac{1}{|x|^2} \right) \quad \text{as } |x| \to \infty. \tag{1.2}$$

The unique solution of the exterior boundary value problem (1.1) and (1.2) can be described for $x \in \Omega^c$ by either using a direct approach via the representation formula

$$u(x) = -\frac{1}{4\pi} \int\limits_\Gamma \frac{e^{ik|x-y|}}{|x-y|} \frac{\partial}{\partial n_y} u(y)\, ds_y + \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} g(y)\, ds_y, \tag{1.3}$$

or by using an indirect approach via a single layer potential

$$u(x) = (\widetilde{V}_k w)(x) = \frac{1}{4\pi} \int\limits_\Gamma \frac{e^{ik|x-y|}}{|x-y|} w(y)\, ds_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma, \tag{1.4}$$

or via a double layer potential

$$u(x) = (W_k v)(x) = \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} v(y)\, ds_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma. \tag{1.5}$$

To find either the yet unknown Neumann datum $t(y) = n_y \cdot \nabla u(y)$ for $y \in \Gamma$ or the unknown density functions $w$ and $v$, we have to solve appropriate boundary integral equations. From the representation formula (1.3) we obtain either the first kind boundary integral equation

$$\frac{1}{4\pi} \int\limits_\Gamma \frac{e^{ik|x-y|}}{|x-y|} t(y)\, ds_y = -\frac{1}{2} g(x) + \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} g(y)\, ds_y \tag{1.6}$$

for $x \in \Gamma$, or the second kind boundary integral equation

$$-\frac{1}{2} t(x) - \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_x} \frac{e^{ik|x-y|}}{|x-y|} t(y)\, ds_y = -\frac{1}{4\pi} \frac{\partial}{\partial n_x} \int\limits_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} g(y)\, ds_y \tag{1.7}$$

for $x \in \Gamma$. When using the indirect single layer potential (1.4) we have to solve the first kind boundary integral equation

$$(V_k w)(x) = \frac{1}{4\pi} \int\limits_\Gamma \frac{e^{ik|x-y|}}{|x-y|} w(y)\, ds_y = g(x) \quad \text{for } x \in \Gamma, \tag{1.8}$$

while for the indirect double layer potential (1.5) we have to find the solution of the second kind boundary integral equation

$$\frac{1}{2} v(x) + (K_k v)(x) = \frac{1}{2} v(x) + \frac{1}{4\pi} \int\limits_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} v(y)\, ds_y = g(x) \tag{1.9}$$

for $x \in \Gamma$. Note that we can write the boundary integral equation (1.6) as

$$(V_k t)(x) = -\frac{1}{2} g(x) + (K_k g)(x) \quad \text{for } x \in \Gamma. \tag{1.10}$$

When introducing the adjoint double layer potential

$$(K_k' w)(x) = \frac{1}{4\pi} \int_\Gamma \frac{\partial}{\partial n_x} \frac{e^{ik|x-y|}}{|x-y|} w(y)\, ds_y \quad \text{for } x \in \Gamma$$

and the hypersingular boundary integral operator

$$(D_k v)(x) = -\frac{1}{4\pi} \frac{\partial}{\partial n_x} \int_\Gamma \frac{\partial}{\partial n_y} \frac{e^{ik|x-y|}}{|x-y|} v(y)\, ds_y \quad \text{for } x \in \Gamma,$$

we can write the boundary integral equation (1.7) as

$$\frac{1}{2} t(x) + (K_k' t)(x) = -(D_k g)(x) \quad \text{for } x \in \Gamma. \tag{1.11}$$

Note that the above formulations of boundary integral equations for boundary value problems of the Helmholtz equation are rather standard [11]. The mapping properties of the above introduced boundary integral operators for Lipschitz boundaries $\Gamma = \partial\Omega$ are well known, see, e.g., [14]. In particular, the single layer potential $V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is bounded and satisfies a Gårdings inequality, i.e. the operator $C = V_0 - V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact, see, e.g., [16], and we have

$$\langle V_k w, w \rangle_\Gamma + \langle C w, w \rangle_\Gamma = \langle V_0 w, w \rangle_\Gamma \geqslant c_1^V \|w\|^2_{H^{-1/2}(\Gamma)} \tag{1.12}$$

for all $w \in H^{-1/2}(\Gamma)$. Hence we can apply Fredholm's alternative [12,19] to investigate the unique solvability of the first kind boundary integral equations (1.8) and (1.10). In particular, we need to consider the injectivity of the single layer potential $V_k$.

**Proposition 1.1.** *If $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem*

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x) \quad \text{for } x \in \Omega, \qquad u_\lambda(x) = 0 \quad \text{for } x \in \Gamma, \tag{1.13}$$

*then we have, by using the boundary integral equations of the direct approach, for $x \in \Gamma$*

$$(V_k t_\lambda)(x) = \left(\frac{1}{2} I + K_k\right) u_\lambda(x) = 0, \qquad \left(\frac{1}{2} I - K_k'\right) t_\lambda(x) = (D_k u_\lambda)(x) = 0,$$

*where $t_\lambda = n_x \cdot \nabla u_\lambda(x)$ for $x \in \Gamma$ is the associated normal derivative of $u_\lambda$.*

From Proposition 1.1 we see that the single layer potential $V_k$ is not injective when $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem (1.13). If $k^2$ is not an eigenvalue of the interior Dirichlet eigenvalue problem, then the single layer potential $V_k$ is injective and therefore there exists a unique solution $w \in H^{-1/2}(\Gamma)$ of the boundary integral equation (1.8) as well as a unique solution $t \in H^{-1/2}(\Gamma)$ of the boundary integral equation (1.10).

**Remark 1.1.** When considering the boundary integral equation (1.10) of the direct approach we find a solution $t \in H^{-1/2}(\Gamma)$ even in the case when $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem (1.13). Although the single layer potential $V_k$ is not injective when $k^2 = \lambda$ is an eigenvalue of the interior Dirichlet eigenvalue problem (1.13), we have

$$\left\langle \left(-\frac{1}{2} I + K_k\right) g, t_\lambda \right\rangle_\Gamma = -\left\langle g, \left(\frac{1}{2} I - K_{-k}'\right) t_\lambda \right\rangle_\Gamma = 0$$

and therefore $(-\frac{1}{2}I + K_k)g \in \operatorname{Im} V_k$. In particular, the boundary integral equation (1.10) of the direct approach is solvable, but the solution is not unique. As for the Neumann problem for the Laplace equation [15] one may define a stabilized variational form to obtain a unique solution satisfying some prescribed side condition. Since $(-\frac{1}{2}I + K_k)g \in \operatorname{Im} V_k$ is satisfied, a natural choice would be to require

$$t \in H_\lambda^{-1/2}(\Gamma) = \bigl\{w \in H^{-1/2}(\Gamma)\colon \langle V_0 w, t_\lambda\rangle_\Gamma = 0\bigr\}.$$

Then there exists a unique solution $t \in H_\lambda^{-1/2}(\Gamma)$ of the boundary integral equation (1.10) which can be found as the unique solution $t \in H^{-1/2}(\Gamma)$ satisfying the extended variational problem

$$\langle V_k t, w\rangle_\Gamma + \langle V_0 t, t_\lambda\rangle_\Gamma \langle V_0 w, t_\lambda\rangle_\Gamma = \left\langle \left(-\frac{1}{2}I + K_k\right)g, w\right\rangle_\Gamma$$

for all $w \in H^{-1/2}(\Gamma)$. Since this formulation requires a priori the knowledge of the eigensolution $t_\lambda$ such an approach seems not applicable in general. A similar approach leading to the so-called CHIEF method was proposed in [4,17].

To investigate the unique solvability of the second kind boundary integral equations (1.9) and (1.11) we first note that the operator $C = K_0 - K_k : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is compact and therefore the operator $V_0^{-1}C : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact. Instead of the boundary integral equation (1.9) we then consider the transformed boundary integral equation

$$V_0^{-1}\left(\frac{1}{2}I + K_k\right)v(x) = V_0^{-1}g(x) \quad \text{for } x \in \Gamma. \tag{1.14}$$

Hence we have

$$\left\langle V_0^{-1}\left(\frac{1}{2}I + K_k\right)v, v\right\rangle_\Gamma + \left\langle V_0^{-1}Cv, v\right\rangle_\Gamma = \left\langle V_0^{-1}\left(\frac{1}{2}I + K_0\right)v, v\right\rangle_\Gamma = \langle S_0 v, v\rangle_\Gamma$$

for all $v \in H^{1/2}(\Gamma)$ where $S_0 = V_0^{-1}(\frac{1}{2}I + K_0) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the Steklov–Poincaré operator associated with the Laplace equation. Since the embedding $H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is compact, we conclude that $V_0^{-1}(\frac{1}{2}I + K_k) : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ satisfies a Gårdings inequality. Hence, to ensure the unique solvability of the boundary integral equation (1.14) we need to have the injectivity of $\frac{1}{2}I + K_k$.

**Proposition 1.2.** *If $k^2 = \mu$ is an eigenvalue of the interior Neumann eigenvalue problem*

$$-\Delta u_\mu(x) = \mu u_\mu(x) \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x}u_\mu(x) = 0 \quad \text{for } x \in \Gamma, \tag{1.15}$$

*then we have, by using the boundary integral equations of the direct approach, for $x \in \Gamma$*

$$\left(\frac{1}{2}I + K_k\right)u_\mu(x) = (V_k t_\mu)(x) = 0, \qquad (D_k u_\mu)(x) = \left(\frac{1}{2}I - K_k'\right)t_\mu(x) = 0.$$

From Proposition 1.2 we see that both boundary integral operators $\frac{1}{2}I + K_k$ and $D_k$ are not injective when $k^2 = \mu$ is an eigenvalue of the interior Neumann eigenvalue problem (1.15). If $k^2$ is not an eigenvalue of the interior Neumann eigenvalue problem, the boundary integral operator $\frac{1}{2}I + K_k$ is injective and therefore there exists a unique solution $v \in H^{1/2}(\Gamma)$ of the boundary integral equation (1.9).

## 2. Stabilized boundary integral equations

From Propositions 1.1 and 1.2 we have seen, that either the single layer potential $V_k$ or the double layer potential $\frac{1}{2}I + K_k$ are not injective, when $k^2$ is an eigenvalue of the interior Dirichlet eigenvalue problem (1.13) or of the interior Neumann eigenvalue problem (1.15), respectively.

Following [5], see also [2], we may consider a combined single and double layer potential representation for some positive real parameter $\eta \in \mathbb{R}_+$,

$$u(x) = (\widetilde{V}_k w)(x) - i\eta(W_k w)(x) \quad \text{for } x \in \Omega^c, \tag{2.1}$$

to describe the solution of the exterior Dirichlet boundary value problem (1.1). To find the unknown density function $w \in L_2(\Gamma)$ we have to solve the combined boundary integral equation

$$\left(\frac{1}{2}I + K_k\right)w(x) - i\eta(V_k w)(x) = g(x) \quad \text{for } x \in \Gamma. \tag{2.2}$$

The unique solvability of the combined boundary integral equation (2.2) is again based on Gårdings inequality, but the associated boundary integral operator $\frac{1}{2}I + K_k - i\eta V_k$ is now injective for all $k \in \mathbb{R}_+$. However, the consideration of the boundary integral equation (2.2) in $L_2(\Gamma)$ requires quite strong assumptions on the smoothness of the boundary $\Gamma = \partial\Omega$. In fact, this theory does not apply when $\Omega$ is a Lipschitz domain. Hence, instead of (2.2) one may consider a regularized boundary integral equation to find $w \in H^{-1/2}(\Gamma)$ such that

$$\left(\frac{1}{2}I + K_k\right)Bw(x) - i\eta(V_k w)(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.3}$$

where $B : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is some suitable chosen operator. In particular, for the Laplace–Beltrami operator

$$B = V_0^2 : H^{-1}(\Gamma) \to H^1(\Gamma)$$

we can use the compact imbedding $H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ to prove the unique solvability of the regularized boundary integral equation (2.3), see [7,10].

An alternative regularization of the boundary integral equation (2.2) is to find $v \in H^{1/2}(\Gamma)$ such that

$$\left(\frac{1}{2}I + K_k\right)v(x) - i\eta\big(V_k R^{-1} v\big)(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.4}$$

where $R : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is a suitable given operator. A possible choice is [6]

$$R = \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K_0'\right) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

with a stabilized hypersingular boundary integral operator $\widetilde{D}_0$ for the Laplace equation, see for example [15]. Another possibility is to use a local version of the Dirichlet to Neumann map $S_k : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$, see [3].

Instead of the standard Brakhage–Werner representation (2.1) we may also consider the equivalent representation

$$u(x) = (\widetilde{V}_k w)(x) + i\eta(W_k w)(x) \quad \text{for } x \in \Omega^c, \ \eta \in \mathbb{R}.$$

Then, to find the unknown density function $w \in L_2(\Gamma)$ we have to solve the boundary integral equation

$$(V_k w)(x) + i\eta\left(\frac{1}{2}I + K_k\right)w(x) = g(x) \quad \text{for } x \in \Gamma.$$

Now, an appropriate regularization leads to the regularized combined boundary integral equation

$$(V_k w)(x) + i\eta\left(\frac{1}{2}I + K_k\right)Bw(x) = g(x) \quad \text{for } x \in \Gamma, \tag{2.5}$$

where $B : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$. Again, one may consider the Laplace–Beltrami operator [8]

$$B = V_0^2 : H^{-1}(\Gamma) \to H^1(\Gamma).$$

An alternative choice is

$$(B\varphi)(x) = \int_\Gamma G_\varepsilon(x, y)\varphi(y)\, ds_y \quad \text{for } x \in \Gamma,$$

where $G_\varepsilon(x, y)$ is the fundamental solution of the partial differential operator $(I - \Delta)^{1+\varepsilon}$ for $\varepsilon > 0$, see [8].

## 3. Modified boundary integral equations

In this section we propose an alternative representation of a regularized combined boundary integral equation to find a unique solution of the exterior Dirichlet boundary value problem (1.1). Although this approach combines ideas from the regularized methods as discussed in the previous section, the analysis behind is different.

Considering the regularized combined boundary integral equation (2.5) we may introduce the regularization operator

$$B = \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma), \tag{3.1}$$

where $\widetilde{D}_0 : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is the stabilized hypersingular boundary integral operator for the Laplace equation given by

$$\langle \widetilde{D}_0 u, v\rangle_\Gamma = \langle D_0 u, v\rangle_\Gamma + \langle u, 1\rangle_{L_2(\Gamma)}\langle v, 1\rangle_{L_2(\Gamma)} \quad \text{for all } u, v \in H^{1/2}(\Gamma). \tag{3.2}$$

Note that $\widetilde{D}_0$ is self-adjoint and $H^{1/2}(\Gamma)$-elliptic, see for example [15,18], and therefore invertible.

Using the duality pairing

$$\langle u, w\rangle_\Gamma = \int_\Gamma u(x)\overline{w(x)}\, ds_x$$

for all $u \in H^{1/2}(\Gamma)$ and $w \in H^{-1/2}(\Gamma)$, we then obtain

$$\langle K_k u, w\rangle_\Gamma = \langle u, K'_{-k} w\rangle_\Gamma.$$

With the regularization (3.1) we have to solve a modified regularized combined boundary integral equation to find $w \in H^{-1/2}(\Gamma)$ such that

$$(V_k w)(x) + i\eta\left(\frac{1}{2}I + K_k\right)\widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w(x) = g(x) \quad \text{for } x \in \Gamma. \tag{3.3}$$

The operator

$$A_k = V_k + i\eta\left(\frac{1}{2}I + K_k\right)\widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

satisfies a Gårdings inequality with the compact operator

$$C = V_0 - V_k : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma).$$

In particular,

$$\begin{aligned}
&\mathrm{Re}\big[\langle A_k w, w\rangle_\Gamma + \langle C w, w\rangle_\Gamma\big] \\
&\quad = \mathrm{Re}\left[\langle V_0 w, w\rangle_\Gamma + i\eta\left\langle \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w, \left(\frac{1}{2}I + K'_{-k}\right)w\right\rangle_\Gamma\right] \\
&\quad = \langle V_0 w, w\rangle_\Gamma \geqslant c_1^V \|w\|^2_{H^{-1/2}(\Gamma)}
\end{aligned}$$

for all $w \in H^{-1/2}(\Gamma)$. Hence it is sufficient to prove that $A_k$ is injective, i.e. $A_k w = 0$ implies $w = 0$.

**Lemma 3.1.** *For all $w \in H^{-1/2}(\Gamma)$ we have*

$$\mathrm{Im}\langle V_k w, w\rangle_\Gamma \geqslant 0.$$

**Proof.** For an arbitrary $w \in H^{-1/2}(\Gamma)$ we consider the single layer potential

$$u(x) = (\widetilde{V}_k w)(x) = \frac{1}{4\pi}\int_\Gamma \frac{e^{ik|x-y|}}{|x-y|} w(y)\, ds_y \quad \text{for } x \in \mathbb{R}^3 \setminus \Gamma.$$

The corresponding first Green's formula with respect to the bounded domain $\Omega$ for $u$ and an arbitrary test function $v$ then reads

$$\int_\Omega \nabla u(x)\nabla v(x)\, dx - k^2 \int_\Omega u(x)v(x)\, dx = \int_\Gamma \gamma_1^{\mathrm{int}} u(x)\gamma_0^{\mathrm{int}} v(x)\, ds_x.$$

In particular for $v = \bar{u}$ we obtain

$$\int_\Omega \big|\nabla u(x)\big|^2 dx - k^2 \int_\Omega \big|u(x)\big|^2 dx = \int_\Gamma \gamma_1^{\mathrm{int}} u(x)\gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x,$$

where the Cauchy data on $\Gamma = \partial\Omega$ are

$$\gamma_1^{\mathrm{int}}(\widetilde{V}_k w)(x) = \frac{1}{2}w(x) + (K_k' w)(x), \qquad \gamma_0^{\mathrm{int}}(\widetilde{V}_k w)(x) = (V_k w)(x).$$

Let $x_0 \in \Omega$ be some arbitrary but fixed point, and let $B_R(x_0)$ be the ball of radius $R$ with centre $x_0$ containing $\Omega$. Then we can write Green's first formula with respect to the bounded domain $\Omega_R = B_R(x_0) \setminus \overline{\Omega}$ as

$$\begin{aligned}
&\int_{\Omega_R} \big|\nabla u(x)\big|^2 dx - k^2 \int_{\Omega_R} \big|u(x)\big|\, dx \\
&\quad = \int_{\partial\Omega_R} \gamma_1^{\mathrm{int}} u(x)\gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x = \int_{\partial B_R(x_0)} \gamma_1^{\mathrm{int}} u(x)\gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x - \int_\Gamma \gamma_1^{\mathrm{ext}} u(x)\gamma_0^{\mathrm{ext}} \overline{u(x)}\, ds_x.
\end{aligned}$$

Note that the normal vector on $\Gamma$ is defined with respect to $\Omega$. The Cauchy data on $\Gamma$ are given

$$\gamma_1^{\mathrm{ext}}(\widetilde{V}_k w)(x) = -\frac{1}{2}w(x) + (K_k' w)(x), \qquad \gamma_0^{\mathrm{ext}}(\widetilde{V}_k w)(x) = (V_k w)(x).$$

Hence we have

$$\int_{B_R(x_0)} \left|\nabla u(x)\right|^2 dx - k^2 \int_{B_R(x_0)} \left|u(x)\right|^2 dx = \langle w, V_k w\rangle_\Gamma + \int_{\partial B_R(x_0)} \gamma_1^{\mathrm{int}} u(x) \gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x$$

and therefore

$$\mathrm{Im}\langle w, V_k w\rangle_\Gamma = -\,\mathrm{Im} \int_{\partial B_R(x_0)} \gamma_1^{\mathrm{int}} u(x) \gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x.$$

On the other hand, the Sommerfeld radiation condition (1.2) implies

$$\begin{aligned}
0 &= \lim_{R\to\infty} \int_{\partial B_R(x_0)} \left|\gamma_1^{\mathrm{int}} u(x) - ik\gamma_0^{\mathrm{int}} u(x)\right|^2 ds_x \\
&= \lim_{R\to\infty} \Biggl[ \int_{\partial B_R(x_0)} \left|\gamma_1^{\mathrm{int}} u(x)\right|^2 ds_x + k^2 \int_{\partial B_R(x_0)} \left|\gamma_0^{\mathrm{int}} u(x)\right|^2 ds_x \\
&\qquad - 2k\,\mathrm{Im} \int_{\partial B_R(x_0)} \gamma_1^{\mathrm{int}} u(x) \gamma_0^{\mathrm{int}} \overline{u(x)}\, ds_x \Biggr] \\
&= \lim_{R\to\infty} \Biggl[ \int_{\partial B_R(x_0)} \left|\gamma_1^{\mathrm{int}} u(x)\right|^2 ds_x + k^2 \int_{\partial B_R(x_0)} \left|\gamma_0^{\mathrm{int}} u(x)\right|^2 ds_x + 2k\,\mathrm{Im}\langle w, V_k w\rangle_\Gamma \Biggr]
\end{aligned}$$

and therefore

$$2k\,\mathrm{Im}\langle w, V_k w\rangle_\Gamma = -\lim_{R\to\infty} \Biggl[ \int_{\partial B_R(x_0)} \left|\gamma_1^{\mathrm{int}} u(x)\right|^2 ds_x + k^2 \int_{\partial B_R(x_0)} \left|\gamma_0^{\mathrm{int}} u(x)\right|^2 ds_x \Biggr] \leqslant 0.$$

From this we finally conclude

$$\mathrm{Im}\langle V_k w, w\rangle_\Gamma \geqslant 0. \qquad \square$$

**Theorem 3.1.** *The combined boundary integral operator*

$$A_k = V_k + i\eta\left(\frac{1}{2}I + K_k\right)\widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right) : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

*is injective.*

**Proof.** Let us assume that $w \in H^{-1/2}(\Gamma)$ is a solution of the homogeneous boundary integral equation

$$(A_k w)(x) = 0 \quad \text{for } x \in \Gamma.$$

Then we have

$$0 = \langle A_k w, w\rangle_\Gamma = \langle V_k w, w\rangle_\Gamma + i\eta\left\langle \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w, \left(\frac{1}{2}I + K'_{-k}\right)w\right\rangle_\Gamma$$

which implies

$$\mathrm{Im}\left[\langle V_k w, w\rangle_\Gamma + i\eta\left\langle \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w, \left(\frac{1}{2}I + K'_{-k}\right)w\right\rangle_\Gamma\right] = 0.$$

This gives, by using Lemma 3.1,

$$\eta\left\langle \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w, \left(\frac{1}{2}I + K'_{-k}\right)w\right\rangle_\Gamma = -\operatorname{Im}\langle V_k w, w\rangle_\Gamma \leqslant 0$$

and therefore

$$\left(\frac{1}{2}I + K'_{-k}\right)w = 0.$$

Hence we also have

$$(V_k w)(x) = 0 \quad \text{for } x \in \Gamma.$$

This is satisfied if either $w = 0$ or if $\lambda = k^2$ is an eigenvalue of the interior Dirichlet eigenvalue problem (1.13). For $t_\lambda(x) = n_x \cdot u_\lambda(x)$, $x \in \Gamma$, we then have

$$(V_{\pm\sqrt{\lambda}} t_\lambda)(x) = 0, \quad \left(\frac{1}{2}I - K'_{\pm\sqrt{\lambda}}\right)t_\lambda(x) = 0 \quad \text{for } x \in \Gamma.$$

In particular,

$$\left(\frac{1}{2}I + K'_{-k}\right)w(x) = 0, \qquad \left(\frac{1}{2}I - K'_{-k}\right)w(x) = 0$$

implies $w = 0$ and therefore the injectivity of $A_k$. $\square$

Hence we have unique solvability of the modified regularized combined boundary integral equation (3.3). The associated variational problem is to find $w \in H^{-1/2}(\Gamma)$ such that

$$\langle V_k w, \tau\rangle_\Gamma + i\eta\left\langle \left(\frac{1}{2}I + K_k\right)\widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w, \tau\right\rangle_\Gamma = \langle g, \tau\rangle_\Gamma \tag{3.4}$$

is satisfied for all test functions $\tau \in H^{-1/2}(\Gamma)$. Due to the composite structure of the above bilinear form, a direct Galerkin approximation of the variational problem (3.4) will not be possible in general. Hence we introduce

$$z = \widetilde{D}_0^{-1}\left(\frac{1}{2}I + K'_{-k}\right)w \in H^{1/2}(\Gamma)$$

as the unique solution of the variational problem

$$\langle \widetilde{D}_0 z, v\rangle_\Gamma = \left\langle \left(\frac{1}{2}I + K'_{-k}\right)w, v\right\rangle_\Gamma \quad \text{for all } v \in H^{1/2}(\Gamma).$$

Hence, instead of the variational problem (3.4) we now consider a saddle point problem to find $(w, z) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\begin{aligned} &\langle V_k w, \tau\rangle_\Gamma + i\eta\left\langle \left(\frac{1}{2}I + K_k\right)z, \tau\right\rangle_\Gamma = \langle g, \tau\rangle_\Gamma, \\ &-\left\langle \left(\frac{1}{2}I + K'_{-k}\right)w, v\right\rangle_\Gamma + \langle \widetilde{D}_0 z, v\rangle_\Gamma = 0 \end{aligned} \tag{3.5}$$

is satisfied for all $(\tau, v) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Since the saddle point formulation (3.5) is equivalent to the variational problem (3.4), we obviously have unique solvability of the saddle point formulation (3.5). Moreover, the formulation (3.5) will be more suitable for a numerical approximation since no composite operators are involved.

While the modified regularized combined boundary integral equation (3.3) results from an indirect approach, the density function $w \in H^{-1/2}(\Gamma)$ has in general no physical meaning. However, for the coupling of different physical phenomena, such as a solid acoustic interaction, a stable representation of the Dirichlet to Neumann map is needed. As in the Burton–Miller approach [9] we may combine both boundary integral equations of the direct approach to get a regularized formulation from which we can compute the unknown Neumann data $t = \gamma_1^{\text{ext}} u \in H^{-1/2}(\Gamma)$. Starting from both boundary integral equations of the direct approach, in particular, combining the first kind boundary integral

$$(V_k t)(x) = -\frac{1}{2} g(x) + (K_k) g(x) \quad \text{for } x \in \Gamma$$

and the second kind boundary integral equation

$$\frac{1}{2} t(x) + (K_k' t)(x) = -(D_k g)(x) \quad \text{for } x \in \Gamma,$$

we may consider any linear combination of them. Using the stabilized hypersingular integral operator $\widetilde{D}_0$ of the Laplace equation, we can apply the bounded operator

$$\left(\frac{1}{2} I + K_{-k}\right) \widetilde{D}_0^{-1} : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

to the second equation, multiply the result with $i\eta$, $\eta \in \mathbb{R}_+$, and add this to the first equation to obtain the boundary integral equation

$$\begin{aligned}
&(V_k t)(x) + i\eta \left(\frac{1}{2} I + K_{-k}\right) \widetilde{D}_0^{-1} \left(\frac{1}{2} I + K_k'\right) t(x) \\
&\quad = -\frac{1}{2} g(x) + (K_k g)(x) - i\eta \left(\frac{1}{2} I + K_{-k}\right) \widetilde{D}_0^{-1} (D_k g)(x).
\end{aligned} \tag{3.6}$$

Since the boundary integral operator of the regularized combined boundary integral equation (3.6) is almost the same as in the boundary integral equation (3.3), the unique solvability of (3.6) follows as before. Next we introduce

$$z = \widetilde{D}_0^{-1} \left[ D_k g + \left(\frac{1}{2} I + K_k'\right) t \right] \in H^{1/2}(\Gamma)$$

as unique solution of the variational problem

$$\langle \widetilde{D}_0 z, v \rangle_\Gamma = \langle D_k g, v \rangle_\Gamma + \left\langle \left(\frac{1}{2} I + K_k'\right) t, v \right\rangle_\Gamma$$

for all $v \in H^{1/2}(\Gamma)$. Then, instead of (3.6) we have to find $(t, z) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$ such that

$$\begin{aligned}
&\langle V_k t, \tau \rangle_\Gamma + i\eta \left\langle \left(\frac{1}{2} I + K_{-k}\right) z, \tau \right\rangle_\Gamma = \langle g, \tau \rangle_\Gamma, \\
&-\left\langle \left(\frac{1}{2} I + K_k'\right) t, v \right\rangle_\Gamma + \langle \widetilde{D}_0 z, v \rangle_\Gamma = \langle D_k g, v \rangle_\Gamma
\end{aligned} \tag{3.7}$$

is satisfied for all $(\tau, v) \in H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Again, the unique solvability of the saddle point problem (3.7) follows from the unique solvability of the boundary integral equation (3.6).

We close this section with some comments on a corresponding regularized combined boundary integral equations to solve the exterior Neumann boundary value problem

$$\Delta u(x) + k^2 u(x) = 0 \quad \text{for } x \in \Omega^c = \mathbb{R}^3 \setminus \overline{\Omega},$$
$$\frac{\partial}{\partial n_x} u(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega, \tag{3.8}$$

where we also include the Sommerfeld radiation condition (1.2).

As for the exterior Dirichlet boundary value problem (1.1) and (1.2) we can derive a modified regularized combined boundary integral equation to find $v \in H^{1/2}(\Gamma)$ such that

$$(D_k v)(x) + i\eta\left(\frac{1}{2}I - K_k'\right)V_0^{-1}\left(\frac{1}{2}I - K_{-k}\right)v(x) = g(x) \quad \text{for } x \in \Gamma. \tag{3.9}$$

The unique solvability of the boundary integral equation (3.9) follows as for the boundary integral equation (3.3), we skip the details.

## 4. Conclusions

In this paper we have given alternative regularized combined boundary integral equations to solve the exterior Dirichlet and Neumann boundary value problems for the Helmholtz equation. In particular, we also describe a stable representation of the Dirichlet to Neumann map. The given formulations are suitable for a Galerkin discretization using standard boundary element methods. Open questions concern the optimal choice of the scaling parameter $\eta \in \mathbb{R}_+$ [1,13], the application of fast boundary element methods, preconditioning strategies for the discrete boundary integral operators and efficient algorithms to solve the linear systems resulting from the saddle point formulations (3.5) and (3.7).

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