Tutorial 5 and 6

1/a $\quad f(x) = x \exp(-a|x|) \quad , \quad a>0$

$$f(x) = x \tilde{f}(x) \quad \text{(let)}$$

$$\tilde{f}(x) = \begin{cases} e^{-ax} , & x>0 \\ e^{ax} , & x<0 \end{cases}$$

$$\mathcal{F}\{\tilde{f}(x)\} = \int_{-\infty}^{0} e^{-i\omega x} e^{ax}\, dx + \int_{0}^{\infty} e^{-i\omega x} e^{-ax} dx$$

$$= \int_{-\infty}^{0} e^{(a-i\omega)x} dx + \int_{0}^{\infty} e^{-(a+i\omega)x} dx$$

$$= \left[\frac{e^{(a-i\omega)x}}{a-i\omega} \right]_{-\infty}^{0} + \left[\frac{e^{-(a+i\omega)x}}{-(a+i\omega)} \right]_{0}^{\infty}$$

$$= \left(\frac{1}{a-i\omega} - 0 \right) + \left(0 + \frac{1}{a+i\omega} \right)$$

$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega}$$

$$\mathcal{F}\{\tilde{f}(x)\} = \frac{2a}{a^2+\omega^2}$$

$$\boxed{\mathcal{F}\{x^n f(x)\} = (-i)^n \frac{d^n}{d\omega^n} F(\omega)}$$

Now $\mathcal{F}\left\{x e^{-a|x|}\right\} = -i \dfrac{d}{d\omega} \left(\dfrac{2a}{a^2+\omega^2} \right)$

$$= -i\left(\frac{-4a\omega}{(a^2+\omega^2)^2}\right)$$

$$\boxed{\mathcal{F}\left\{x e^{-a|x|}\right\} = \frac{4a\omega i}{(a^2+\omega^2)^2}}$$

1/b $f(x) = x\exp(-ax^2)$, $a>0$

$$\mathcal{F}\{f(x)\} = \mathcal{F}\{x\exp(-ax^2)\}$$

$$= -i\frac{d}{d\omega}\mathcal{F}\{e(-ax^2)\}$$

We know that,

$$\boxed{\mathcal{F}\left\{e^{-a^2x^2}\right\} = \frac{\sqrt{\pi}\, e^{-\omega^2/4a^2}}{a}}$$

Hence, $\mathcal{F}\{f(x)\} = -i\dfrac{d}{d\omega}\left(\dfrac{\sqrt{\pi}\; e^{-\omega^2/4a}}{\sqrt{a}}\right)$

$$= -i\frac{d}{d\omega}\left(\sqrt{\frac{\pi}{a}}\; e^{-\omega^2/4a}\right)$$

$$= -i\sqrt{\frac{\pi}{a}}\left(e^{-\omega^2/4a}\times\left(-\frac{2\omega}{4a}\right)\right)$$

$$\boxed{\mathcal{F}\{f(x)\} = \frac{i\omega}{2a}\sqrt{\frac{\pi}{a}}\; e^{-\omega^2/4a}}$$

1/c $f(x) = u(x)$

Unit step function

$$u(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}$$

One of the condition of existence of fourier transform

(i) Signal should be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

But $\int_{-\infty}^{\infty} |u(t)|\,dt = \infty$

Hence, we cannot apply the definition of Fourier transform.

$$u(x) = \lim_{\alpha \to 0} \cos\alpha x \, u(x)$$

$$F\{\cos\alpha x\, u(x)\} = \frac{\pi}{2}\left[\delta(\omega+\alpha) + \delta(\omega-\alpha)\right] + \frac{i\omega}{\alpha^2-\omega^2}$$

$$F\{\sin\alpha x\, u(x)\} = \frac{\pi}{2i}\left[\delta(\omega-\alpha) - \delta(\omega+\alpha)\right] + \frac{\omega}{\alpha^2-\omega^2}$$

Hence, $F\{u(x)\} = \lim_{\alpha \to 0} \frac{\pi}{2}\left[\delta(\omega+\alpha) + \delta(\omega-\alpha)\right] + \frac{i\omega}{\alpha^2-\omega^2}$

$$\boxed{F\{u(x)\} = \pi\delta(\omega) - \frac{i}{\omega}}$$

1/d $f(x) = K$, K is constant

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Duality Principle

If $\mathcal{F}\{f(x)\} = \tilde{f}(\omega)$

$$\mathcal{F}\{\tilde{f}(x)\} = 2\pi f(-\omega)$$

Dirac delta function

$$\boxed{\begin{gathered} \delta(t) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \\ \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \\ \delta(-x) = \delta(x) \\ \int_{-\infty}^{\infty} G(t)\, \delta(t - t_0)\, dt = G(t_0) \end{gathered}}$$

$$\mathcal{F}\{\delta(x)\} = \int_{-\infty}^{\infty} \delta(x)\, e^{-i\omega x}\, dx$$

$$\mathcal{F}\{\delta(x)\} = 1$$

$$\mathcal{F}\{1\} = 2\pi\, \delta(-x) = 2\pi\, \delta(x)$$

$$\Rightarrow \boxed{\mathcal{F}\{K\} = 2\pi K \delta(x)}$$

(2) $\mathcal{F}\{e^{-|x|}\}$

$$F_s\left\{e^{-|x|}\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\mathcal{L}\left\{\sin\omega x\right\}$$

$$F_s\left\{e^{-|x|}\right\} = \sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+1} = \tilde{f}(\omega)$$

$$e^{-|x|} = F_s^{-1}\left\{\tilde{f}(\omega)\right\}$$

$$e^{-|x|} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\sqrt{\frac{2}{\pi}}\frac{\omega}{\omega^2+1}\sin\omega x\,d\omega$$

$$e^{-|x|} = \frac{2}{\pi}\int_0^{\infty}\frac{\omega}{\omega^2+1}\sin x\omega\,d\omega$$

Hence, $\frac{2}{\pi}\int_0^{\infty}\frac{x}{x^2+1}\sin mx\,dx = e^{-m}$

$$\Rightarrow \boxed{\int_0^{\infty}\frac{x}{x^2+1}\sin mx\,dx = \frac{\pi}{2}e^{-m}}$$

③ $F_s\left\{\frac{x}{1+x^2}\right\}$

$$\Rightarrow F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{x}{1+x^2}\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\times\frac{\pi}{2}e^{-\omega}$$

$$\boxed{F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-\omega}}$$

④ $F_s\left\{f(x)\right\} = \frac{e^{-as}}{s}$

$$f(x) = F_s^{-1}\left\{\frac{e^{-as}}{s}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-as}}{s}\sin sx \, ds$$

$$\mathcal{L}\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(u)\,du$$

$$= \sqrt{\frac{2}{\pi}}\,\mathcal{L}\left\{\frac{\sin sx}{s}\right\}$$

$$= \sqrt{\frac{2}{\pi}}\int_a^\infty \frac{x}{u^2+x^2}\,du$$

$$= \sqrt{\frac{2}{\pi}}\left[\tan^{-1}\left(\frac{u}{x}\right)\right]_a^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{x}\right)\right) = \sqrt{\frac{2}{\pi}}\cot^{-1}\left(\frac{a}{x}\right)$$

$$\boxed{f(x) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)}$$

We have $F_s^{-1}\left(\frac{e^{-as}}{s}\right) = \sqrt{\frac{2}{\pi}}\tan^{-1}\left(\frac{x}{a}\right)$

Put $a=0$

$$\boxed{F_s^{-1}\left(\frac{1}{s}\right) = \sqrt{\frac{2}{\pi}} \times \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}}$$

(5) $F_c\{f(x)\} = \frac{1}{2}\tan^{-1}\frac{2}{s^2}$

$$\tan^{-1}\frac{2}{s^2} = \tan^{-1}\left(\frac{1}{s-1}\right) - \tan^{-1}\left(\frac{1}{s+1}\right)$$

$$F_c\{f(x)\} = \frac{1}{2}\left(\tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}\right)$$

$$f(x) = \frac{1}{2} F_c^{-1}\left(\tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}\right)$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \tan^{-1}\frac{1}{s-1}\cos sx\,ds - \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty \tan^{-1}\frac{1}{s+1}\cos sx\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(\tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}\right)\cos sx\,ds$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \tilde{f}(s)\cos sx\,ds, \quad \tilde{f}(s) = \tan^{-1}\frac{1}{s-1} - \tan^{-1}\frac{1}{s+1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{\tilde{f}(s)\frac{\sin sx}{x}\right\}_0^\infty - \int_0^\infty \frac{d\tilde{f}}{ds}\frac{\sin xs}{x}\,ds\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[0 - \int_0^\infty \frac{d\tilde{f}}{ds}\frac{\sin xs}{x}\,ds\right]$$

$$f(x) = \frac{-1}{\sqrt{2\pi}}\int_0^\infty \frac{d\tilde{f}}{ds}\frac{\sin xs}{x}\,ds$$

Now $$\frac{d\tilde{f}}{ds} = \frac{1}{1+\frac{1}{(s-1)^2}}\left(\frac{-1}{(s-1)^2}\right) - \frac{1}{1+\frac{1}{(s+1)^2}}\left(\frac{-1}{(s+1)^2}\right)$$

$$= \frac{-1}{(s-1)^2+1} + \frac{1}{(s+1)^2+1}$$

$$\frac{d\tilde{f}}{ds} = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}$$

Hence, $$f(x) = \frac{1}{\sqrt{2\pi}\,x}\int_0^\infty \left(\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$\int_{-\infty}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$= \int_{-\infty}^{0}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$+ \int_{0}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$t = -s \Rightarrow ds = -dt$$

$$= \int_{\infty}^{0}\left(\frac{1}{(t+1)^2+1}-\frac{1}{(t-1)^2+1}\right)\sin xt\,dt$$

$$+ \int_{0}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$= -\int_{\infty}^{0}\left(\frac{1}{(t-1)^2+1}-\frac{1}{(t+1)^2+1}\right)\sin xt\,dt$$

$$+ \int_{0}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$= \int_{0}^{\infty}\left(\frac{1}{(t-1)^2+1}-\frac{1}{(t+1)^2+1}\right)\sin xt\,dt$$

$$+ \int_{0}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds$$

$$\boxed{\int_{-\infty}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds = 2\int_{0}^{\infty}\left(\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right)\sin xs\,ds}$$

Hence, $f(x) = \frac{1}{2\sqrt{2\pi}x}\int_{-\infty}^{\infty}\left(\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right)\sin xs\, ds$

Using formula

$$\int_{-\infty}^{\infty}\frac{\sin px}{(x-b)^2+a^2}dx = \frac{\pi}{a}e^{-ap}\sin bp$$

$\therefore \quad f(x) = \frac{1}{2\sqrt{2\pi}x}\left[\frac{\pi}{1}e^{-x}\sin x - \frac{\pi}{1}e^{-x}\sin(-1)x\right]$

$$\boxed{f(x) = \sqrt{\frac{\pi}{2}}\,\frac{e^{-x}\sin x}{x}}$$

(6) $f(x) = \begin{cases} 1-|x| & , |x|<1 \\ 0 & , |x|>1 \end{cases}$

$$f(x) = \begin{cases} 1-x & , 0<x<1 \\ 1 & , x=0 \\ 1+x & , -1<x<0 \\ 0 & , |x|>1 \end{cases}$$

$$F\{f(x)\} = \int_{-1}^{0}(1+x)e^{-i\omega x}dx + \int_{0}^{1}(1-x)e^{-i\omega x}dx$$

$$= \left[\frac{(1+x)e^{-i\omega x}}{-i\omega}\right]_{-1}^{0} + \frac{1}{i\omega}\int_{-1}^{0}e^{-i\omega x}dx$$

$$+ \left[\frac{(1-x)e^{-i\omega x}}{-i\omega}\right]_{0}^{1} - \frac{1}{i\omega}\int_{0}^{1}e^{-i\omega x}dx$$

$$= \left[\frac{1}{-i\omega}\right] - \frac{1}{(i\omega)^2}\left[e^{-i\omega x}\right]_{-1}^{0}$$

$$+ \left(\frac{1}{i\omega}\right) + \frac{1}{(i\omega)^2}\left[e^{-i\omega x}\right]_{0}^{1}$$

$$= \frac{1}{\omega^2}\left(1 - e^{i\omega}\right) - \frac{1}{\omega^2}\left(e^{-i\omega} - 1\right)$$

$$= \frac{1}{\omega^2}\left(2 - \left(e^{i\omega} + e^{-i\omega}\right)\right)$$

$$= \frac{1}{\omega^2}\left(2 - 2\cos\omega\right)$$

$$\boxed{F\{f(x)\} = \frac{2(1-\cos\omega)}{\omega^2}}$$

Parseval's identity for fourier transforms

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

$$f(\omega) = \frac{2(1-\cos\omega)}{\omega^2} = \frac{2(\cancel{1}-\cancel{1}+2\sin^2\omega/2)}{\omega^2}$$

$$= \frac{4\sin^2\omega/2}{\omega^2}$$

$$\frac{16}{2\pi}\int_{-\infty}^{\infty}\frac{\sin^4\omega/2}{\omega^4}\,d\omega = \int_{-1}^{0}(1+x)^2dx + \int_{0}^{1}(1-x)^2dx$$

$$\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4 \omega/2}{\omega^4}\,d\omega = \frac{1}{3}+\frac{1}{3} = \frac{2}{3}$$

Let $t = \frac{\omega}{2}$ $\quad dt = \frac{d\omega}{2}$

$$\frac{16}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4 t\,dt}{16\,t^4} = \frac{2}{3}$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{2}{3}$$

$$\Rightarrow\quad 2\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{2\pi}{3} \quad\Rightarrow\quad \boxed{\int_{0}^{\infty}\frac{\sin^4 t}{t^4}\,dt = \frac{\pi}{3}}$$

Note: If $u(0,t)$ is given, use fourier sine transform

If $\frac{\partial u}{\partial x}(0,t)$ is given, use fourier cosine transform

⑦ $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 \le x < \infty,\ t > 0$$

Conditions i) $u(x,0) = 0$, for $x \ge 0$

ii) $\frac{\partial u}{\partial x}(0,t) = -a$

iii) $u(x,t)$ is bounded

We know that

$$F\{f^n(x)\} = (-i\omega)^n F(\omega)$$

$$F_c\left\{u''(x)\right\} = -\omega^2 \bar{u} - \sqrt{\frac{2}{\pi}}\, u'(0)$$

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = k\, F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{\partial}{\partial t}\int_0^\infty u \cos\omega x\, dx = k\left\{-\omega^2\bar{u} - \sqrt{\frac{2}{\pi}}u'(0)\right\}, \quad \bar{u} = F_c\{u\}$$

$$\frac{\partial \bar{u}}{\partial t} = k\left\{-\omega^2\bar{u} + a\sqrt{\frac{2}{\pi}}\right\}$$

$$\Rightarrow \quad \frac{\partial \bar{u}}{\partial t} + k\omega^2\bar{u} = ka\sqrt{\frac{2}{\pi}}$$

$$\bar{u} = c_1 e^{-k\omega^2 t} + \frac{a}{\omega^2}\sqrt{\frac{2}{\pi}}$$

$$t=0, \quad u=0 \Rightarrow t=0, \quad \bar{u}=0$$

$$0 = c_1 + \frac{a}{\omega^2}\sqrt{\frac{2}{\pi}} \Rightarrow c_1 = -\frac{a}{\omega^2}\sqrt{\frac{2}{\pi}}$$

Hence, $\bar{u} = \sqrt{\frac{2}{\pi}}\frac{a}{\omega^2}\left(1 - e^{-k\omega^2 t}\right)$

$$F_c^{-1}(\bar{u}) = a\sqrt{\frac{2}{\pi}}\, F_c^{-1}\left\{\frac{\left(1-e^{-k\omega^2 t}\right)}{\omega^2}\right\}$$

$$\boxed{u = \frac{2a}{\pi}\int_0^\infty \frac{\left(1-e^{-k\omega^2 t}\right)}{\omega^2}\cos\omega x\, d\omega}$$

8 $\quad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$

Conditions i) $u(0,t)=0$,

ii) $u(x,0) = e^{-x}$

iii) $u(x,t)$ is bounded.

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = 2\,F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\boxed{F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\omega^2\bar{u} + \omega u(0)\sqrt{\frac{2}{\pi}}}$$

$$\frac{\partial}{\partial t}\int_0^\infty u\sin\omega x\,dx = 2\left\{-\omega^2\bar{u} + \omega u(0)\sqrt{\frac{2}{\pi}}\right\}$$

$$\frac{\partial \bar{u}}{\partial t} = -2\omega^2\bar{u}$$

$$\frac{\partial \bar{u}}{\partial t} + 2\omega^2\bar{u} = 0$$

$$\bar{u} = c\,e^{-2\omega^2 t}$$

When $t=0$, $u = e^{-x}$, $\bar{u} = F_s\{e^{-x}\}$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\omega x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\,\mathcal{L}\{\sin\omega x\}$$

$$\bar{u} = \sqrt{\frac{2}{\pi}}\,\frac{\omega}{\omega^2+1}$$

$$\Rightarrow \boxed{\sqrt{\frac{2}{\pi}}\,\frac{\omega}{\omega^2+1} = c}$$

$$\bar{u} = \sqrt{\frac{2}{\pi}}\,\frac{\omega}{\omega^2+1}\,e^{-2\omega^2 t}$$

$$F_s^{-1}\{\bar{u}\} = \sqrt{\frac{2}{\pi}}\,F^{-1}\left\{\frac{\omega}{\omega^2+1}\,e^{-2\omega^2 t}\right\}$$

$$\boxed{u = \frac{2}{\pi}\int^\infty \frac{\omega}{\omega^2+1}\,e^{-2\omega^2 t}\,\sin\omega x\,d\omega}$$

$\int_0$

(9) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$; $x>0$, $t>0$

conditions

i) $u_x(0,t) = 0$

ii) $u(x,0) = \begin{cases} x & , 0 \le x \le 1 \\ 0 & x>1 \end{cases}$

iii) $u(x,t)$ is bounded

#

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\frac{\partial \bar{u}}{\partial t} = -\omega^2 \bar{u} - u'(0)\sqrt{\frac{2}{\pi}}$$

$$\Rightarrow \frac{\partial \bar{u}}{\partial t} + \omega^2 \bar{u} = 0$$

$$\Rightarrow \bar{u} = c e^{-\omega^2 t}$$

When $t=0$, $u = \begin{cases} x & , 0 \le x \le 1 \\ 0 & x>1 \end{cases}$

$$\bar{u} = \sqrt{\frac{2}{\pi}}\int_0^1 x \cos \omega x \, dx$$

$$\bar{u} = \sqrt{\frac{2}{\pi}} \frac{\omega \sin\omega + \cos\omega - 1}{\omega^2}$$

$$\Rightarrow c = \sqrt{\frac{2}{\pi}} \frac{\omega \sin\omega + \cos\omega - 1}{\omega^2}$$

$$\Rightarrow \bar{u} = \sqrt{\frac{2}{\pi}} \frac{\omega \sin\omega + \cos\omega - 1}{\omega^2} e^{-\omega^2 t}$$

$$\boxed{u = \frac{2}{\pi}\int_0^\infty \frac{\omega \sin\omega + \cos\omega - 1}{\omega^2} e^{-\omega^2 t} \cos \omega x \, d\omega}$$