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Civil Engineering (B.Tech Sem IV)
MA203 Assignment 1

Solution ①: on Multiplying Both side By $\sqrt{\frac{2}{\pi}}$

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\xi x)\,dx = \begin{cases}\sqrt{\frac{2}{\pi}}(1-\xi) & 0\le\xi\le1\\ 0 & \xi\ge1\end{cases}$$

we have,

$$\phi(\xi) = F_c(f(x)) = \begin{cases}\sqrt{\frac{2}{\pi}}(1-\xi) & 0\le\xi\le1\\ 0 & \xi\ge1\end{cases}$$

To find f(x) we have to Do Inverse Fourier Cosine Transform on $\phi(\xi)$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \phi(\xi)\cos\xi x\,d\xi$$

$$= \sqrt{\frac{2}{\pi}}\int_0^1 \sqrt{\frac{2}{\pi}}(1-\xi)\cos\xi x\,d\xi + \sqrt{\frac{2}{\pi}}\int_1^\infty 0\cdot\cos\xi x\,d\xi$$

$$= \frac{2}{\pi}\int_0^1 (1-\xi)\cos\xi x\,d\xi$$

$$= \frac{2}{\pi}\left[\int_0^1\cos\xi x\,d\xi - \left(\int_0^1 \underset{I}{\xi}\,\underset{II}{\cos\xi x}\,d\xi\right)\right]$$

$$-\left(\xi\frac{\sin\xi x}{x} - \int\frac{\sin\xi x}{x}\right)$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin\xi x}{x}\right)_0^1 - \left(\xi\frac{\sin\xi x}{x}\right)_0^1 + \left(\frac{-\cos\xi x}{x^2}\right)_0^1\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\sin x}{x}} - \cancel{\frac{\sin x}{x}} - \frac{\cos x}{x^2} + \frac{1}{x^2}\right]$$

$$f(x) = \frac{2}{\pi}\left(\frac{1-\cos x}{x^2}\right)$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\left[\frac{2\sin^2 x/2}{x^2}\right]$$

put in given, integral eqn

$$\int_0^\infty \frac{4}{\pi}\frac{\sin^2 x/2}{x^2}\cos(\xi x)\,dx = \begin{cases} 1-\xi & 0 \le \xi \le 1 \\ 0 & \xi \ge 1 \end{cases}$$

put $\boxed{\xi = 0}$

$$\int_0^\infty \frac{4}{\pi}\frac{\sin^2 x/2}{x^2}\,dx = 1$$

let $\frac{x}{2} = t \qquad dx = 2dt$

$$\int_0^\infty \frac{\cancel{4}}{\pi}\frac{\sin^2 t}{\cancel{4}t^2}\cdot 2dt = 1$$

$$\boxed{\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}}$$

Solution ②: a) $F_c\left[e^{-x^2}\right] = ?$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\xi x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\left(\frac{e^{i\xi x}+e^{-i\xi x}}{2}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[e^{-(x^2 - i\xi x)} + e^{-(x^2 + i\xi x)} \right] dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \left[e^{-\left(x - \frac{i\xi}{2}\right)^2} + e^{-\left(x + \frac{i\xi}{2}\right)^2} \right] e^{-\xi^2/4} \, dx$$

$$= \frac{e^{-\xi^2/4}}{\sqrt{2\pi}} \int_0^\infty \left[e^{-\left(x - \frac{i\xi}{2}\right)^2} + e^{-\left(x + \frac{i\xi}{2}\right)^2} \right] dx$$

$$= \frac{e^{-\xi^2/4}}{\sqrt{2\pi}} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \right]$$

$$= \frac{1}{\sqrt{2}} e^{-\xi^2/4} \quad \underline{\underline{Ans}}$$

(b) fourier transform of f(x) -

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \, e^{i\xi x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin ax}{x} \left(\cos \xi x + i \sin \xi x \right) dx$$

$$= \frac{1}{2\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} \frac{2 \sin ax \cdot \cos \xi x}{x} dx}_{\text{even function}} + i \underbrace{\int_{-\infty}^{\infty} \frac{2 \sin ax \sin \xi x}{x} dx}_{\text{odd function so 'zero'}} \right]$$

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{2 \sin ax \cos \xi x}{x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{(\sin x(a+\xi) + \sin x(a-\xi))}{x} \, dx$$

Case I) $a+\xi > 0$ & $a-\xi > 0$

$\xi > -a$ & $\xi < a$ $\Rightarrow |\xi| < a$

$$\phi(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$\boxed{\phi(\xi) = \sqrt{\frac{\pi}{2}}}$$

Case II) $|\xi| > a$

then, $\displaystyle\int_0^\infty \frac{\sin x(a-\xi)}{x} \, dx = -\frac{\pi}{2}$

so, $\phi(\xi) = \frac{1}{\sqrt{2\pi}} \left(\frac{\pi}{2} - \frac{\pi}{2} \right)$

$$\boxed{\phi(\xi) = 0}$$

$$\Rightarrow \phi(\xi) = \begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |\xi| < a \\ 0 & \text{if } |\xi| > a \end{cases}$$

Solution (3)

given, steady-state temperature distribution in a semi-infinite solid –

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \qquad -\infty < x < \infty, \; y > 0$$

Conditions, $\theta(x,0) = \begin{cases} 1 & , |x| < a \\ 0 & , |x| > a \end{cases}$

on doing fourier-cosine transform,

$$\underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 \theta}{\partial x^2} \cos \xi x \, dx}_{I} + \underbrace{\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 \theta}{\partial y^2} \cos \xi x \, dx}_{II} = 0$$

Apply By Parts in (I)

$$\left(\cos \xi x \frac{\partial \theta}{\partial x} \right)_0^\infty + \int \frac{\partial \theta}{\partial x} \xi (\sin \xi x) \, dx$$

since 'θ' is Bounded so, $\frac{d\theta}{dx} \to 0$ (as $x \to \infty$)

& since, for $-a \leq x \leq a$ $\theta(x,y) = 1$ so, $\frac{\partial \theta}{\partial x} = 0$

so, we have only.

$$\int \frac{\partial \theta}{\partial x} \xi (\sin \xi x) \, dx$$

$$\left| \xi (\sin \xi x) \, \theta(x,y) \right|_0^\infty - \xi^2 \int \cos \xi x \, (\theta(x,y)) \, dx$$

since θ is Bounded so $\theta(x,y) = 0$ as $x \to \infty$

& at $x = 0$ $\sin \xi x = 0$

so, only we have. $-\xi^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \theta(x,y) \cos \xi x \, dx = -\xi^2 \bar{\theta}(\xi$ (say)

In II nd part

Apply Leibnitz,

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 \theta}{\partial y^2} \cos \xi x \, dx = \sqrt{\frac{2}{\pi}} \frac{\partial^2}{\partial y^2} \int_0^\infty \theta(x,y) \cos \xi x \, dx$$

$$= \frac{\partial^2}{\partial y^2} \bar{\theta}(\xi, y)$$

So we have Now,

$$-\xi^2 \bar{\theta}(\xi, y) + \frac{\partial^2}{\partial y^2} \bar{\theta}(\xi, y) = 0 \quad \text{--- (1)}$$

Where,
$$\bar{\theta}(\xi, y) = F\left[\theta(x,y) : x \to \xi\right]$$

Solution of eqn (1) is given by

$$\bar{\theta}(\xi, y) = A(\xi) e^{\xi y} + B(\xi) e^{-\xi y}$$

Since $\theta(x,y)$ is Bounded so, $\bar{\theta}(\xi, y)$ should be Bounded so, $\bar{\theta}(\xi, y) \to 0$ as $\xi \to \infty$

possible only if $A(\xi) = 0$ So,

$$\boxed{\bar{\theta}(\xi, y) = B(\xi) e^{-\xi y}}$$

Now find $B(\xi)$ By using given Conditions.

$$B(\xi) = \bar{\theta}(\xi, 0)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \theta(x, 0) \cos \xi x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^a 1 \cos \xi x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left| \frac{\sin \xi x}{\xi} \right|_0^a$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin a\xi}{\xi} \right)$$

So, $$\boxed{\bar{\theta}(\xi, y) = \sqrt{\frac{2}{\pi}} \frac{\sin a\xi}{\xi} e^{-y\xi}}$$

Now By inverse fourier Transform,

$$\theta(x, y) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{\theta}(\xi, y) \cos \xi x \, d\xi$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \frac{\sin a\xi}{\xi} e^{-y\xi} \cos \xi x \, d\xi$$

$$= \frac{2}{2 \times \pi} \int_0^\infty \frac{e^{-y\xi}}{\xi} (2 \sin a\xi \cos \xi x) \, d\xi$$

$$= \frac{1}{\pi} \int_0^\infty \frac{e^{-y\xi}}{\xi} [\sin(a+x)\xi + \sin(a-x)\xi] \, d\xi$$

$$= \frac{1}{\pi} \left[L \left(\frac{\sin(a+x)\xi}{\xi} \right) + L \left(\frac{\sin(a-x)\xi}{\xi} \right) \right]$$

$$= \frac{1}{\pi}\left[\int_y^\infty \frac{(a+x)}{y^2+(a+x)^2}\,dy + \int_y^\infty \frac{(a-x)}{y^2+(a-x)^2}\,dy\right]$$

$$= \frac{1}{\pi}\left[\left(\tan^{-1}\left(\frac{y}{a+x}\right)\right]_y^\infty + \left[\tan^{-1}\left(\frac{y}{a-x}\right)\right]_y^\infty\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{y}{a+x}\right)\right) + \left[\frac{\pi}{2} - \tan^{-1}\left(\frac{y}{a-x}\right)\right]\right]$$

$$= \frac{1}{\pi}\left[\cot^{-1}\left(\frac{y}{a+x}\right) + \cot^{-1}\left(\frac{y}{a-x}\right)\right]$$

$$\boxed{\theta(x,y) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x+a}{y}\right) + \tan^{-1}\left(\frac{a-x}{y}\right)\right]}$$ for $\frac{a+x}{y}$ & $\frac{a-x}{y} > 0$

Solution ④ $f(x) = e^{-ax}$ $\quad g(x) = u(a - x)$

$u(a)$ is unitstep function

$$g(x) = \begin{cases} 1 & 0 < x < a \\ 0 & \text{elsewhere} \end{cases}$$

Cosine Fourier Transform of $f(x)$

$$\phi(\xi) = F_c[f(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-ax}\cos\xi x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-ax}}{\xi^2+a^2}(-a\cos\xi x + \xi\sin\xi x)\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}$$

Similarly fourier cosine transform of $g(x)$

$$\psi(\xi) = F_c[g(x)] = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\xi x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^a 1\cdot\cos\xi x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{\sin a\xi}{\xi}\right)$$

Using Parseval Identity,

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty \phi(\xi)\, \psi(\xi)\, d\xi$$

$$\int_0^\infty e^{-ax} \cdot 1\, dx = \int_0^\infty \frac{2}{\pi} \left(\frac{a}{a^2+\xi^2}\right) \frac{\sin a\xi}{\xi}\, d\xi$$

$$\boxed{\int_0^\infty \frac{\sin(a\xi)}{\xi(a^2+\xi^2)}\, d\xi = \frac{\pi}{2a^2}\left(1-e^{-a^2}\right)}$$

hence Proved

Solution ⑤ A) $L[g(t)] = ?$

given, $g(t) = \int_0^t e^{-2t}\, t \sin^3 t\, dt$

we know that

if $L[f(t)] = F(s)$

then $L\left[\int_0^t f(t)\right] = \frac{F(s)}{s}$

So, $L[\sin^3 t] = L\left[\frac{3\sin t - \sin 3t}{4}\right]$

$= \frac{3}{4} L[\sin t] - \frac{1}{4} L[\sin 3t]$

$= \frac{3}{4}\left[\frac{1}{s^2+1}\right] - \frac{1}{4}\left[\frac{3}{s^2+3^2}\right]$

$$L[\sin^3 t] = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+3^2}\right]$$

$$L[t\sin^3 t] = \frac{3}{4}\left[\frac{-2s}{(s^2+1)^2} + \frac{2s}{(s^2+3^2)^2}\right]$$

using second shifting theorem,

$$L[e^{-2t} t\sin^3 t] = \frac{3}{4}\left[\frac{-2(s+2)}{[(s+2)^2+1]^2} + \frac{2(s+2)}{[(s+2)^2+3^2]^2}\right]$$

then,

$$L\left[\int_0^t e^{-2t} t\sin^3 t\right]$$

$$= \frac{3}{4s}\left[\frac{-2(s+2)}{[(s+2)^2+1]^2} + \frac{2(s+2)}{[(s+2)^2+3^2]^2}\right]$$

$$= \frac{3(s+2)}{2}\left[\frac{1}{[(s+2)^2+3^2]} - \frac{1}{[(s+2)^2+1^2]}\right]$$

Ans

B) Solve

$y'' + t y' - y = 0$, $y(0) = 0$; $y'(0) = 0$

Apply laplace transform Both sides,

$L[y''] + L[t y'] - L[y] = 0$

$[s^2 F(s) - s\cancel{y(0)}^{0} - \cancel{y'(0)}^{0}] - \frac{d}{ds}[sF(s) - \cancel{y(0)}^{0}] - F(s) = 0$

$s^2 F(s) - s\frac{d}{ds}F(s) - F(s) - F(s) = 0$

$F(s)[s^2 - 2] = s\frac{d}{ds}F(s)$

$\frac{d\,F(s)}{F(s)} = (s - \frac{2}{s})\,ds$

on integrating Both sides we get

$\ln[F(s)] = \frac{s^2}{2} - 2\ln s + c$

$\ln[F(s) \cdot s^2] = \frac{s^2}{2} + c$

$$\boxed{F(s) = \frac{e^{s^2/2}\,c'}{s^2}}$$

Now to find $f(t)$ we have to find inverse laplace of this,

$$f(t) = \int_0^\infty f(s)\, e^{st}\, ds$$

$$= \int_0^\infty c' \frac{e^{s^2/2}}{s^2}\, e^{st}\, ds$$

we can see, as $s \to \infty$ $f(s) \to \infty$ because of $e^{s^2/2}$

so, the possible solution will ~~possible~~ only if

$$\boxed{c' = 0}$$

so,

$$\boxed{f(t) = 0}$$