(1Q) Mathematics - MA 203

Assignment - 1

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CSE (B.Tech)

(1Q) $u_{tt}(x,t) = c^2 u_{xx}(x,t)$, $0 < x < \infty$, $t \geq 0$

$$\begin{cases} u(0,t) = 0 & t \geq 0 \\ u(x,0) = f(x) & \\ u_x(x,0) = g(x) & \end{cases} \quad 0 < x < \infty$$

By using sine transformation $F_s[f(x)] = F_s(k)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \sin kx \, f(x) \, dx$$

$$F_s[u_{tt}(x,t)] = c^2 F_s[u_{xx}(x,t)]$$

$$F_s[u_{tt}(x,t)] = \int_0^\infty \sqrt{\frac{2}{\pi}} \sin kx \frac{d^2}{dt^2} u(x,t) \, dx$$

$$= \frac{d^2}{dt^2} \int_0^\infty \sqrt{\frac{2}{\pi}} u(x,t) \sin kx \, dx$$

$$F_s[u_{tt}(x,t)] = \frac{d^2}{dt^2} F_s[u(k,t)] = \frac{d^2}{dt^2} U(k,t)$$

$$F_s[u_{xx}(x,t)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\sin kx}_{u} \underbrace{\frac{d^2 u(x,t)}{dx^2}}_{v} dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\underbrace{\left(\sin kx \, u_x(x,t) \right)_0^\infty}_{=0} - k \int_0^\infty \cos kx \frac{d u(x,t)}{dx} dx \right]$$

$$= -\sqrt{\frac{2}{\pi}} (k) \int_0^\infty \cos kx \frac{d \, u(x,t)}{dx} dx$$

$$= -\sqrt{\frac{2}{\pi}}(k)\left[\left[\cos kx \; u(x,t)\right]_0^\infty + k\int_0^\infty \sin kx \; u(x,t)\,dx\right]$$

$$= -\sqrt{\frac{2}{\pi}}(k)\left[\; 0 + k\underline{F_s\left[u(x,t)\right]}\right]$$

$$= -\cancel{\sqrt{\tfrac{2}{\pi}}}(k^2)\; u(k,t)$$

$$\underbrace{F_s\left[u_{tt}(x,t)\right]}_{\frac{d^2}{dt^2}u(k,t)} = c^2 \; F_s\left[u_{xx}(x,t)\right]$$

$$\frac{d^2}{dt^2} u(k,t) = (c^2)(-k^2)\; u(k,t)$$

$$\boxed{y'' = -k^2c^2 \, y}$$ — single variable (t)

$$\boxed{y'' + k^2c^2 y = 0}$$

$$y = A e^{ickt} + B e^{-ickt}$$

$$u(k,t) = A \boxed{e^{ickt}} + B e^{-ickt}$$

Put $t = 0 \Rightarrow$ $u(k,0) = F(k) = A + B \rightarrow (1)$

Diff $u_t(k,t) = \cancel{g(k)}$ put $t = 0$ $u_t(k,0) = G(k)$

$$ickA - ick(B) = G(k)$$

$$A - B = \frac{-(i)\,G(k)}{(ck)} \rightarrow (2)$$

$$① + ② = F(k) - i\frac{G(k)}{ck} = 2A$$

$$\left(\tfrac{1}{2}\right)\left[F(k) - i\frac{G(k)}{ck}\right]. \cancel{e^{ickx}} = A$$

$$① - ② = F(k) + i\frac{G(k)}{kc} = 2B$$

$$\frac{1}{2}\left[F(k) + \frac{iG(k)}{kc}\right] = B$$

$$\underline{u(k,t)} = \tfrac{1}{2}\left[F(k) - i\frac{G(k)}{ck}\right]e^{ikct} + \tfrac{1}{2}\left[F(k) + i\frac{G(k)}{ck}\right]e^{-ickt}$$

$$F_s[u(x,t)] = F_s(k)\cos(ckt) + \frac{G_s(k)}{ck}\sin ckt$$

$$\boxed{u(x,t) = F_s^{-1}\left[F_s(k)\cos(ckt) + \frac{G_s(k)}{ck}\sin(ckt)\right]}$$

$$\boxed{u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[F_s(k)\cos(ckt) + \frac{G_s(k)}{ck}\sin(ckt)\right]\sin kx \, dk}$$

(2Q) $u_{tt}(x,t) = c^2 U_{xx}(x,t)\ , \quad 0 < x < \infty,\ t \geq 0$

$$\left.\begin{aligned} u(x,0) &= 0 \\ u_t(x,0) &= 0 \end{aligned}\right\} \begin{aligned} 0 < x < \infty \\ t > 0 \end{aligned}$$

$u(0,t) = f(t)\ ;\ t > 0$

$$F_s[u_{tt}(x,t)] = \int_0^\infty \sqrt{2/\pi}\ \frac{d^2 U(x,t)}{dt^2}\ \sin kx\ dx$$

$$= \frac{d^2}{dt^2} \int_0^\infty \sqrt{2/\pi}\ U(x,t) \sin kx\ dx$$

$$= \frac{d^2}{dt^2} U_s(k,t)$$

$$F_s(u_{xx}(x,t)) = \int_0^\infty \sqrt{\frac{2}{\pi}}\ \underbrace{\frac{d^2 u(x,t)}{dx^2}}_{v}\ \underbrace{(\sin kx)}_{u}\ dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\sin kx\ u_x(x,t)\right]_0^\infty - (k) \int_0^\infty \cos kx\ U_x(x,t)\ dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(0 - 0) - (k) \int_0^\infty (\cos kx)\ u_x(x,t)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} (-k) \left[\left[\cos kx\ u(x,t)\right]_0^\infty + (k) \int_0^\infty \sin kx\ u(x,t)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(-k)\left[0 - f(t)\right] + \left[(-k^2)\ U(k,t)\right] \right]$$

$=$

$$= \sqrt{\frac{2}{\pi}}\,(k)\,f(t) + (-k^2)\,U_s(k,t)$$

$$= F_s\left[u_{tt}(x,t)\right] = c^2\,F_s\left[u_{xx}(x,t)\right]$$

$$= \frac{d^2}{dt^2}\cancel{u_s}(k,t) = c^2\left[\sqrt{\frac{2}{\pi}}\,(k)\,f(t) - k^2\,U_s(k,t)\right]$$

$$= \frac{d^2}{dt^2}U_s(k,t) + k^2c^2U_s(k,t) = kc^2\sqrt{\frac{2}{\pi}}\,f(t)$$

$$\boxed{y'' + k^2c^2y = h(t)} \rightarrow \text{of form one variable.}$$

Apply Laplace

$$L(y'') + c^2k^2L(y) = L[h(t)]$$

$$p^2L(y) - \cancel{U_s}(k,0)^{\to 0} - pU_s(k,0)^{\to 0} + k^2c^2L(y) = kc^2\sqrt{\frac{2}{\pi}}\,L(f(t))$$

$$L(y)(p^2+k^2c^2) = c^2\sqrt{\frac{2}{\pi}}\,L(f(t))\cdot(k)$$

$$L(y) = c^2\sqrt{\frac{2}{\pi}}\,L(f(t))\cdot\frac{k}{p^2+k^2c^2}$$

$$y = L^{-1}\left[c^2\sqrt{\frac{2}{\pi}}\,L(f(t))\cdot\frac{k}{p^2+k^2c^2}\right]$$

$$= c\sqrt{\frac{2}{\pi}}\,L^{-1}\left[L(f(t))\cdot\frac{kc}{p^2+k^2c^2}\right]$$

By convolution

$$\boxed{y = c\sqrt{\frac{2}{\pi}}\int_0^t f(t-z)\,g(z)\,dz}$$

$$y = F_s[u(x,t)] = c^{\bullet}\sqrt{\frac{2}{\pi}} \int_0^t f(t-z)\, \underbrace{g(z)} \, dz$$

$$F_s[u(x,t)] = c^2\sqrt{\frac{2}{\pi}} \int_0^t f(t-z)\, \sin(ckz)\, dz$$

$$u(x,t) = F_s^{-1}\left[c^{\bullet}\sqrt{\frac{2}{\pi}} \int_0^t f(t-z)\, \sin(ckz)\, dz \right]$$

$$= c^{\bullet}\sqrt{\frac{2}{\pi}} \left[\sqrt{\frac{2}{\pi}} \int_0^\infty \left(\int_0^t f(t-z)(\sin(ckz))\, dz \right) \sin kx \, dk \right]$$

$$\boxed{u(x,t) = \frac{2}{\pi}(c^{\bullet}) \int_0^\infty \left(\int_0^t f(t-z)\, \sin(ckz)\, dz \right) \sin kx \; dk}$$

(3Q) $u_{xx} + u_{yy} = 0 \quad , 0 < x < \infty \quad , 0 < y < \infty$

$u(x,0) = H(a-x) \quad ; \quad x < a$

$u_x(0,y) = 0 \quad ; \quad x > 0 \, , \, y < \infty$

Apply Forier sine transformation :-

$$F_s[u_{xx}(x,y)] = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\right) \underbrace{(\sin kx)}_{u} \, \underbrace{u_{xx}(x,y)}_{v} \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\sin kx \; u_x(x,y) \right]_0^\infty - (k) \int_0^\infty (\cos kx)\, u_x(x,y)\, dx \right.$$

$$= \sqrt{\frac{2}{\pi}} \left[(0-0) - k \left[\left(\cos kx \; u(x,y) \right)_0^\infty + k \int_0^\infty \sin kx \; u(x,y) \, dx \right] \right.$$

(3a) $U_{xx} + U_{yy} = 0$

$U(x,0) = H(a-x)$

$U_x'(0,y) = 0$ $\quad 0 < y < \infty$

$U_{xx}(x,y) + U_{yy}(x,y) = 0$

Forierer ~~cos~~ cose transformation (taken becaues the conditions satisfy the given ~~rules~~ conditions)

$$F_c[U_{xx}] = \int_0^\infty \left(\sqrt{\frac{2}{\pi}}\right) \underbrace{\frac{d^2 u(x,y)}{dx^2}}_{v} \underbrace{(\cos kx)}_{u} dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[\cos kx \; U_x(x,y)\right]_0^\infty + k\int_0^\infty (\sin kx)\, u_x(x,y)\, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[(k) \int_0^\infty \underbrace{\sin kx}_{u} \; \underbrace{u_x(x,y)}_{v} dx \right.$$

$$= \sqrt{\frac{2}{\pi}} (k) \left[\left[\cancel{\sin kx\; U(x,y)}\right]_0^\infty - k \int_0^\infty \cos kx \; U(x,y)\, dx \right.$$

$$= -k^2 \sqrt{\frac{2}{\pi}} \int_0^\infty \cos kx \; U(x,y)\, dx$$

$$= -k^2 \; F_c\,(U(x,y))$$

$$F_c(U_{xx}(x,y)) = -k^2\, U(k,y)$$

$$F_c(u_{yy}(x,y)) = \int_0^\infty \sqrt{\frac{2}{\pi}}\, u_{yy}(x,y) \cos kx \, dx$$

$$= \frac{d^2}{dy^2} \int_0^\infty \sqrt{\frac{2}{\pi}}\, u(x,y) \cos kx \, dx$$

$$= \frac{d^2}{dy^2} F_c(u(x,y)) = \frac{d^2}{dy^2} U(k,y)$$

$$F_c(u_{yy}(x,y)) + F_c(u_{xx}(x,y)) = 0$$

$$\frac{d^2 U(k,y)}{dy^2} - (k^2)\, U(k,y) = 0$$

$$y'' - k^2 y = 0$$

Apply laplace $\Rightarrow L(y'') - k^2 L(y) = 0$

$$p^2 L(y) - p\, \cancel{y'(0)}^{\,0} - y(0) - k^2 L(y) = 0$$

$$(p^2 - k^2)\, L(y) = y(0)$$

$$y(0) = F_c(u(x,0)) = \int_0^\infty \sqrt{\frac{2}{\pi}} (\cos kx)\, H(a-x)$$

$$= \int_0^a \sqrt{\frac{2}{\pi}} (\cos kx)\,(1) + (0)$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{\sin kx}{k}\right)_0^a$$

$$y(0) = \sqrt{\frac{2}{\pi}}\, \frac{\sin(ka)}{k}$$

$$y'' - k^2 y = 0$$

$$U(k,y) = c_1 e^{ky} + c_2 e^{-ky}$$

① $y = 0 \Rightarrow U(k,0) = \sqrt{\frac{2}{\pi}}\left(\frac{\sin ka}{k}\right) = c_1 + c_2$

f also ② $\lim_{k \to \infty} U(k,y) = 0 \quad \therefore c_1 = 0$

$$c_1 e^{k\infty} + 0 = 0$$

$$\boxed{c_1 = 0}$$

$$c_2 = \sqrt{\frac{2}{\pi}} \sin\left(\frac{ka}{k}\right)$$

$$U(k,y) = \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k} e^{-ky}.$$

$$u(x,y) = \int_0^\infty \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \frac{\sin ka}{k} e^{-ky} \cos kx \, dk$$

$$\boxed{u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\sin ka}{k} (\cos ka) \cdot e^{-ky} \, dk}$$