P1)

$$T(n) = \max_{i \in \{1 \dots n-1\}} \{\min(i, n-i) + T(i) + T(n-i)\} = O(n \log n) \qquad T(1)=1$$

Proof: • $T(n)$ is $O(n \log n)$ if $\exists\, c>0,\ \exists\, n_0$ s.t. $T(n) \leq c\, n \log n \quad \forall n \geq n_0$

• $n_0 := 2,\ c := 3$

$\downarrow$ I prove this by by induction

* Base case

$$T(2) = 1 + T(1) + T(2-1) = 3 < 3 \cdot 2 \ln 2 = 4.159$$

* Inductive Step

$$T(n) = \max_{x \in \{1,\dots,n-1\}} \{\min(x, n-x) + T(x) + T(n-x)\}$$

$$\leq \max_{x \in \{1,\dots,n-1\}} \{\min(x, n-x) + 3x \log(x) + 3(n-x) \log(n-x)\} + 2T(1)$$

(*) edge case

$T(1) = 1 \neq 1 \log(1)$

$1 + 1 = 2$

$$\leq \max_{x \in [1, n-1]} \{\overbrace{\min(x, n-x) + 3x \log(x) + 3(n-x) \log(n-x)}^{f_n(x) :=}\} + \overset{2}{\overset{\parallel}{2T(1)}}$$

$$\overset{?}{\leq} 3n \log n$$

$$f_n(x) := \begin{cases} \overbrace{x + 3x \log(x) + 3(n-x) \log(n-x) + 2}^{g_n(x)}, & 1 \leq x \leq \frac{n}{2} \\ \underbrace{n - x + 3x \log(x) + 3(n-x) \log(n-x) + 2}_{h_n(x)}, & \frac{n}{2} \leq x \leq n-1 \end{cases}$$

Observations:

① h_n, g_n are sym. w.r.t. $\frac{n}{2}$ (i.e. $g_n(x) = h_n(-x+n)$)

$g_n(x) = x + 3x \ln x + 3(n-x) \ln(n-x) + 2$

$h_n(-x+n) = x + 3(n-x) \log(n-x) + 3x \log(x) + 2$

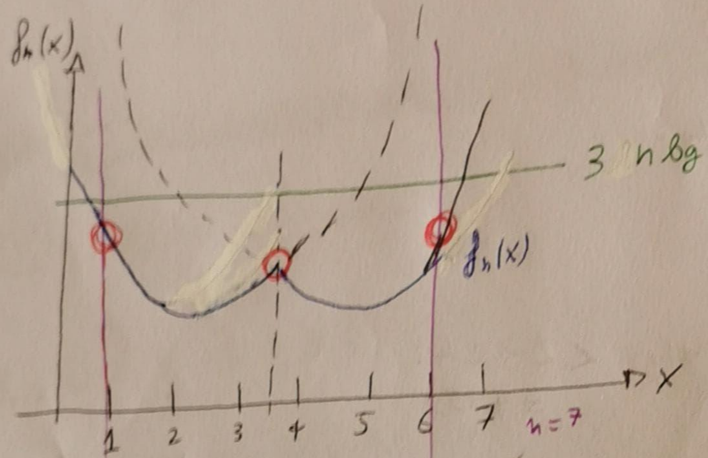
• $T(K) \overset{?}{\le} \max_i \{c_i + T(i) + T(K-1)\}$

② $f_n'(x) = 1 + 3\log(x) + 3 + 3\cancel{(n-x)}\frac{1}{\cancel{(n-x)}}(-1) + 3\log(n-x)(-1)$

$= 1 + 3\log(x) - 3 - 3\log(n-x) = 1 + 3\log\left(\frac{x}{n-x}\right)$

$f_n''(x) = \frac{3}{x} + \frac{3}{n-x} = \frac{3(n-x)+3}{x(n-x)} = \frac{3n}{x(n-x)} > 0$

$x < n$ ✓



• $f_n(1) = \cancel{1} + \cancel{3}(n-1)\log(n-1) + \cancel{2} \overset{?}{\le} \cancel{3}n\log n$

$1 + n\log(n-1) - \log(n-1) \le n\log(n)$

$1 + n\log\left(\frac{n-1}{n}\right) \le \log(n-1)$

$\log\left[\left(\frac{n-1}{n}\right)^n\right] \le \log(n-1) - \log(e)$

$= \log\left(\frac{n-1}{e}\right)$

$\forall n \ge 2$:

$\left(\frac{n-1}{n}\right)^n \le \frac{(n-1)^2}{e} \le \frac{n-1}{2}$

$r(x) = \left(\frac{x-1}{x}\right)^x - \frac{(x-1)^2}{e}$

$r'(x) = \left(\frac{x-1}{x}\right)^x \ln\left(\frac{x-1}{x}\right) \cdot \frac{x}{x-1} \cdot \frac{x - (x-1)\,1}{x^2} - 2\frac{(x-1)}{e}$

$$\left(\frac{\;}{n}\right) \le \left(\frac{\;}{e}\right)$$

$$\frac{(n-1)^{n-1}}{n^n} \le \frac{1}{e} = 0.367$$

$n=2$ $\qquad \frac{1}{4} = 0.25$

$n=3$ $\qquad = 0.148$

$$z(x) = x\ln(x) - 1 - x\ln(x-1) + \ln(x-1)$$

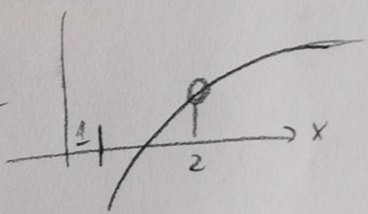
$$z'(x) = 1 + \ln(x) - \ln(x-1) - \frac{x}{x-1} + \frac{1}{x-1} = \cancel{1} + \ln\left(\frac{x}{x-1}\right) + \cancel{\frac{1-x}{x-1}}$$

$$= \ln\left(\frac{x}{x-1}\right) > 0$$

fr $x > 2$

$$z(2) = 2\ln(2) - 1 - 2\ln(2-1) + \ln(2-1)$$

$$= 0.386 > 0$$



$\Rightarrow 3n\log n \ge f_n(1) \quad \forall n \ge 2$

$$f_n\left(\frac{n}{2}\right) = \frac{n}{2} + 3\frac{n}{2}\ln\left(\frac{n}{2}\right) + 3\left(\frac{n}{2}\right)\ln\left(\frac{n}{2}\right) + 2$$

$$= \frac{n}{2} + 3\cdot n\ln\left(\frac{n}{2}\right) + 2 \overset{?}{\le} 3n\ln(n)$$

$$\frac{n}{2} + 3n\ln\left(\frac{n}{2n}\right) + 2 \overset{?}{\le} 0$$

$$\frac{n}{2} + 3n\ln\left(\frac{1}{2}\right) + 2 < \underbrace{\left(\frac{1}{2} + 3\ln\left(\frac{1}{2}\right) + 1\right)}_{=-0.57} n \le 0 \qquad n \ge 2$$

$\Rightarrow 3n\log n \ge f_n\left(\frac{n}{2}\right) \quad \forall n \ge 2 \qquad \Rightarrow T(n) \le 3n\log n$ $\square$