PS8

Problem 4. You are given a system of inequalities with n variables $x_1, x_2, \ldots, x_n$ and m inequalities of the form

$$x_i - x_j \leqslant w_{ij}$$

for integers w_{ij} given in the input. Decide if the system has a solution and print one if it has.

Running time: $O(mn)$.

Algorithm

We need to turn the system of inequalities into a directed graph, with n nodes corresponding to the variables and m edges corresponding to the inequalities.

For an inequality of the form $x_i - x_j \leq w_{ij}$ we create an edge going from node x_j to node x_i with weight w_{ij}.

We run the Bellman-Ford algorithm on this directed graph. If we find negative cycles in the graph, the system of inequalities does not have a solution.

Correctness (proof by contradiction):

Assume a negative cycle C in G exists but the system of inequalities has a solution.

Without loss of generality we can assume that the cycle C is composed of vertices $x_0, x_1, \dots, x_k, x_0$.

The inequalities corresponding to the edges in the cycles are $x_1 - x_0 \leq w_{10}, \dots, x_0 - x_k \leq w_{0k}$

Every variables in the cycles C appears twice, once with positive value (in-edge) and once with negative value (out-edge). Thus we can express the sum of all inequalities as $\sum_{i=0}^{k}(x_i - x_i) \leq \sum w_{ij} \rightarrow 0 \leq \sum w_{ij}$

These inequalities must hold for the system to have a solution but require the sum of edges in the cycle to be greater or equal than 0. Since we have assumed that the cycle is negative, there is a contradiction. With a negative cycle, it is not possible to satisfy the requirement.

Complexity:

Constructing the graph takes O(n + m) to convert each variable into a node and each inequality into an edge.

Running the Bellman-Ford takes O(nm), thus satisfying the complexity requirement