# Problem Set 8 - Exercise 6

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January 2024

## Problem Description

A country is divided into $k$ provinces, in which in total there are $n$ cities. Some pairs of cities are connected with (bidirectional) roads. In each province we need to select one city to become the capital of the province. For each road at least one of its endpoints must be a capital. Knowing which city is in which province, and which cities are connected by roads, decide if it is possible to choose capitals to satisfy the above condition.

**Running time:** $O(n^2)$.

## Solution

The given problem can be modeled using a 2-CNF (2-Conjunctive Normal Form) satisfiability problem. In this formulation, each city is represented as a boolean variable, where a variable being equal to true means that the corresponding city is selected as capital. The connections between cities, which are bidirectional roads, can be represented as clauses composed of the two variables corresponding to the cities that the road joins.
For each road in the graph that connects cities $x$ and $y$, we introduce a clause of the form

$$(x \vee y)$$

to the 2-CNF formula. This clause ensures that at least one city at the end of each road is selected as a capital.
Additionally, to guarantee that each province has exactly one capital, for each pair $(x, y)$ of cities within the same province, we add a clause

$$(\neg x \vee \neg y)$$

to the 2-CNF formula. This clause enforces that no two cities in the same province can both be capitals, even if there is no direct road between $x$ and $y$.
To ensure that each province has a capital, especially in cases where a province might contain isolated cities or initially appear to have no capitals, a final verification step is necessary. After the primary procedure, we iterate through each province to confirm the presence of a capital. If a province lacks a capital, we attempt to select one by choosing an isolated city or any city whose neighboring cities have not been chosen as capitals. If no such city exists within the province, then it is impossible to satisfy the conditions, and we declare the problem infeasible.
Given $k$ provinces with at most $n$ cities each, the maximum number of city pairs per province is $\frac{n(n-1)}{2}$. With $k$ provinces, the total is

$$k \cdot \frac{n(n-1)}{2}$$

which simplifies to $O(n^2)$ when $k$ is constant. Hence, our 2-CNF formula will have $O(n^2)$ clauses.
The efficiency of solving the 2-CNF formula comes from the polynomial-time complexity of algorithms designed for 2-CNF satisfiability problems. These algorithms, leveraging graph-theoretical approaches, determine satisfiability in linear time relative to the number of variables and clauses, by identifying strongly connected components in a directed graph. Thus, the process adheres to the $O(n^2)$ time complexity, aligning with the problem's requirements.