2 Probability

Chapter Preview

In this chapter, we'll introduce the study of Probability which refers to the study of randomness or uncertainty.

2.1: Sample Space and Events

Def: An experiment is any activity or process whose outcome is subject to uncertainty.

Def: The sample space of an experiment, denoted by Ω or $\mathcal{S}$, is the set of all possible outcomes of that experiment.

Def: An event is any collection (subset) of outcomes contained in the sample space Ω.

Example: For the following examples, define the sample space and give an example of an event. Code for running the experiment in R is given.

Experiment 1: Roll a fair six-sided die.

```r
# roll one fair six-sided die in R
sample(x=1:6,size=1)

## [1] 5
```

Experiment 2: Roll two fair six-sided dice.

```r
# roll two fair six-sided dice in R
sample(x=1:6,size=2)

## [1] 6 4
```

Experiment 3: Randomly select a real number in the closed interval from 0 to 1.

```r
# generate one random number between 0 and 1
runif(n=1)

## [1] 0.4507613

# generate ten random numbers between 0 and 1
runif(n=10)

##  [1] 0.69412641 0.20747026 0.41271062 0.84760718 0.33069935 0.69816595
##  [7] 0.64086992 0.79723748 0.03773936 0.53042254
```

Set Theory

Note: For the following examples, suppose a fair six-sided die is rolled, so $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Define the following events: $A =$ roll an even, $B =$ roll a 3 or higher, $C =$ roll a prime number.

Def: The complement of an event A, denoted by A^c or A', is the set of all outcomes in Ω that are not contained in A.

Illustration:

Def: The union of two events A and B, denoted by $A \cup B$ and read "A or B," is the event consisting of all outcomes that are either in A or B or in both.

Illustration:

Def: The intersection of two events A and B, denoted by $A \cap B$ and read "A and B," is the event consisting of all outcomes that are in both A and B.

Illustration:

Def: The difference of two events A and B, denoted by A\B and read "difference of A and B," is the event consisting of all outcomes that are in A but not in B.

Illustration:

As a reminder, define the following events:
$A =$ roll an even, $B =$ roll a 3 or higher, $C =$ roll a prime number.

Example: Draw and determine $A \cup B^c$.

Example: Draw and determine $A \cap B \cap C^C$.

Example: Draw and determine $[A \cup B] \cap C$.

Example: Draw and determine $A \setminus (B \cap C)$.

2.2: Axioms and Properties of Probability

Axioms of Probability

Axiom 1: For any event A, $P(A) \geq 0$.

Axiom 2: $P(\Omega) = 1$.

Axiom 3: If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P\big(\cup_{i=1}^{\infty} A_i\big) = P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$$

Some Basic Properties of Probability

Def: The null or empty set is the set of no events (or elements), denoted $\emptyset$.

The following can be derived from the Axioms of Probability.

Theorem: $P(\emptyset) = 0$

Proof:

Theorem: For any event A, $P(A) + P(A^c) = 1$.

Proof:

Corollary: $P(A^c) = 1 - P(A)$

Proof:

Theorem: For any event A, $0 \leq P(A) \leq 1$.

Theorem: For any two events A and B,
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Theorem: For any three events A, B, and C,
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$.

Theorem: (DeMorgan's Laws)

1. $[A \cup B]^C = A^C \cap B^C$
2. $[A \cap B]^C = A^C \cup B^C$

Example: Suppose our class has the following breakdown:

70% Math students
20% seniors
30% Denver natives
10% Math students and seniors
15% Math students and Denver natives
10% Seniors and Denver natives
5% Math students, seniors, and Denver natives

Define events as follows: A = Math student, B = senior, C = Denver native.
Suppose a random student from the class is selected. Calculate the following probabilities:

(a) $P(A \cup B)$

(b) $P(C \setminus B)$

(c) $P([A \cap C]^c)$

(d) $P(A \cup B \cup C)$

2.3: Counting Techniques

Equally Likely Outcomes

If $\Omega = \{x_1, x_2, ...x_N\}$ and all outcomes are equally likely, then $P(A) = |A|/N$, where $|A|$ is the number of outcomes contained in the event A.

Example: You roll a fair six-sided die. Calculate $P(even)$.

Product Rule

Theorem: (Product Rule for Ordered Pairs) If the first element of an ordered pair can be selected in n_1 ways and for each of the these n_1 ways the second element of the pair can be selected in n_2 ways, then the number of pairs is $n_1 \cdot n_2$.

Example: You flip three fair coins. Calculate the probability that all are heads or all are tails.

Theorem: (Generalized Product Rule) Suppose a set consists of k elements (k-tuples) and that there are n_1 possible choices for the first element, n_2 possible choices for the second element, ... , and n_k possible choices for the k^{th} element, then there are $n_1 \cdot n_2 \cdots n_k$ possible k-tuples.

Example: How many license plates can be chosen if the first three symbols are letters and the last three symbols are numbers?

The factorial function — $n!$ — is important in counting methods and probability. It is the product of all positive integers less than or equal to n, that is, $n! = n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 2\cdot 1$ for $n \in \mathbb{Z}^+$.

Note: $0! = 1$ and $1! = 1$

Def: An ordered subset is called a permutation. The number of permutations of size k that can be formed from the n elements in a set will be denoted by $P_{n,k}$.

Note: $P_{n,k} = \frac{n!}{(n-k)!}$

Def: An unordered subset is called a combination. The number of combinations of size k that can be formed from the n elements in a set will be denoted by $\binom{n}{k}$ or $C_{n,k}$.

Note: $\binom{n}{k} = C_{n,k} = \frac{n!}{k!\cdot(n-k)!}$

Example: Suppose there are five competitors in an Olympic competition. A gold, a silver, and a bronze medal are awarded for the top three competitors. Are combinations or permutations more appropriate for this situation? How many possible ways can these three awards be given out?

Example: Suppose there are five competitors in an Olympic competition and only three can advance to the final competition. Are combinations or permutations more appropriate for this situation? How many possible ways can you choose three people from a group of five?

Example: Powerball is an American lottery game where participants pay \$2 for a chance of winning the grand prize, typically hundreds of millions of dollars. Participants choose five numbers between 1 and 69 and one additional number between 1 and 26. If they choose all the numbers correctly, they win the grand prize. What is the probability of matching all six numbers?

Example: Suppose you are dealt three cards from a standard 52-card playing deck. What is the probability that you get a heart, a diamond, and a club in that order? What if order doesn't matter?

2.4: Conditional Probability

Conditional Probabilities and Formulas

Def: For any two events A and B with $P(B) > 0$, the conditional probability of A given B is defined by:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example: Suppose two fair dice are rolled and their sum is 7. What is the probability that one of the dice is a 6?

Example: The following is a contingency table from a survey on presidential approval.

Political Affiliation	Approves of President	Disapproves of President	Total
Republican	100	20	120
Democrat	20	80	100
Independent	40	40	80
Total	160	140	300

(a) What is the probability that a randomly selected individual from the survey approves of the president?

(b) What is the probability that a randomly selected individual approves of president given that they are Democrat?

(c) Given that the randomly selected person approves of the president, what is the probability that they are Republican?

Multiplication Rule

Theorem: $P(A \cap B) = P(A|B) \cdot P(B)$

Example: Suppose that people with a genetic defect have a 50% chance of getting a rare disease. The probability of having the genetic defect is 1%. What is the probability that a randomly chosen person has the genetic defect and also has the disease?

Law of Total Probability

Theorem: Let $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events (i.e., the events form a partition). Then for any event B,

$$P(B) = P(B|A_1) \cdot P(A_1) + P(B|A_2) \cdot P(A_2) + \ldots P(B|A_k) \cdot P(A_k)$$

Illustration of a Partition:

Note: In the example at the top of the page, we can partition the population into those with the disease and those who do not have the disease.

Example: Box #1 has ten computer chips (five working and five broken). Box #2 has five (one working and four broken). Suppose we randomly select one of the two boxes and then randomly select a chip. What is the probability that the chip is broken?

Bayes' Theorem

Theorem: Suppose that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \Omega$, then if $P(B) > 0$,

$$P(A_1|B) = \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)}$$

Theorem: Let $A_1, A_2, \ldots, A_k$ be a collection of mutually exclusive and exhaustive events, then if $P(B) > 0$,

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j) \cdot P(A_j)}{\sum_{i=1}^{k} P(B|A_i) \cdot P(A_i)}$$

Example: Suppose a test to detect a rare disease is positive 95% of the time for a person with a disease and is positive 1% of the time for a person without the disease. Further, assume that 0.5% of the population has the disease. Given that a person tests positive for a disease, what is the probability that he actually has the disease?

2.5: Independence

Def: When $A \cap B = \emptyset$, A and B are said to be mutually exclusive (or disjoint) events.

Def: Two events A and B are independent if $P(A|B) = P(A)$ and dependent otherwise.

Theorem: A and B are independent if and only if $P(A \cap B) = P(A) \cdot P(B)$.

Example: Suppose we roll two fair six-sided dice. Let A be the event that the first die is a 1, let B be the event that the sum of the dice is 7, and let C be the event that the sum of the dice is 8.

(a) Are A and B mutually exclusive and/or independent?

(b) Are A and C mutually exclusive and/or independent?

Objective vs. Subjective Probability

There are two different interpretations of Probability. Both follow the Axioms of Probability.

Def: Objective Probability refers to probabilities associated with limits of relative frequencies.

Example: I flip a coin 100 times and it lands on head 53 times. I estimate the probability of flipping a heads to be $P_{100}(head) = \frac{53}{100} = 0.53$. As we flip the coin more and more, we would expect that $P_n(head) \to 0.5$ as $n \to \infty$ if the coin is actually a fair coin.

Def: Subjective Probability refers to probabilities with measures of belief. One popular version is Bayesian probability.

Example: I am given a coin which I have no reason to believe it is unfair (i.e, $P(head) = 0.5$). I flip the coin 100 times and 3 land on heads. Since this is very unlikely to happen if the coin is fair, I update my estimate to be $P(head) = 0.1$ (i.e., somewhere in between $3/100 = 0.03$ and $1/2 = 0.5$).