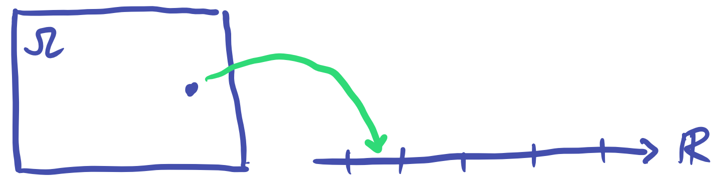
3 Discrete Random Variables



Chapter Preview

In this chapter, we'll introduce random variables, probability distributions, and specifically look at discrete random variables.

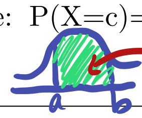
3.1: Random Variables

Def: For a given sample space Ω of some experiment, a random variable (RV) is any rule that associates a number with each outcome in Ω. In mathematical language, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

Def: A discrete random variable is a random variable whose possible values either constitute a finite set or a countably infinite set.

↳ ex) $\{1,2,3,4,5,6\}$ ↳ ex) $\{1,2,3,\dots\}$

Def: A continuous random variable is a random variable whose possible values are uncountably infinite or defined over an interval. Note: P(X=c)=0 for a continuous random variable X and a constant c.



$P(a \le X \le b)$ (for cont. RV)

Example: Roll a fair six-sided die. Let X be the number on the top of the die.

$\Omega = \{1,2,3,4,5,6\}$

finite set → discrete RV

$$p(x) = \begin{cases} 1/6 & x=1 \\ 1/6 & x=2 \\ 1/6 & x=3 \\ 1/6 & x=4 \\ 1/6 & x=5 \\ 1/6 & x=6 \end{cases}$$

probability mass function

support: possible values for the RV

← support: 1, 2, 3, 4, 5, 6

Example: Pick a random person and let Y be the number of keys in their pocket.

$\Omega = \{0,1,2,3,\dots\}$ ← the support

countably infinite → discrete RV

Example: Suppose a student is given 50 minutes to complete an exam. Let Z be the amount of time it takes the student to complete the exam.

$\Omega = [0, 50]$

uncountably infinite → continuous RV

3.2: Probability Distributions for Discrete Random Variables

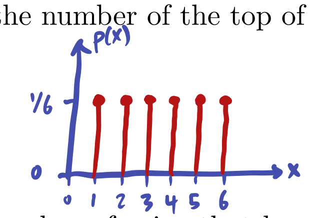
Def: The probability mass function (pmf) or probability distribution of a discrete random variable X is defined for every number x by $p(x) = P(X = x)$. This is also sometimes denoted f(x).

RV specific value

Note: $\sum_{x \in X} p(x) = 1$

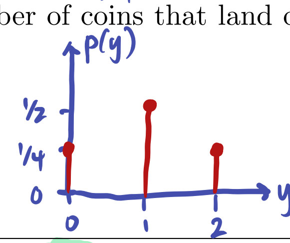
Example: Roll a fair six-sided die. Let X be the number of the top of the die. Find the pmf of X.

$$p(x) = \begin{cases} 1/6 & x = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases}$$



Example: Flip two fair coins. Let Y be the number of coins that land on heads. Find the pmf of Y.

$$p(y) = \begin{cases} 1/4 & y = 0 \ (TT) \\ 1/2 & y = 1 \ (HT, TH) \\ 1/4 & y = 2 \ (HH) \end{cases}$$

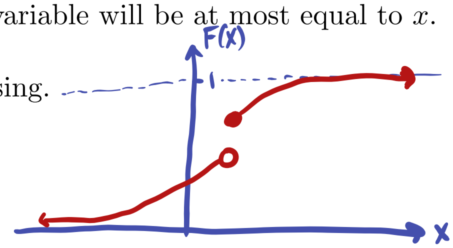


Def: The cumulative distribution function (CDF) $F(x)$ of a discrete random variable X with pmf $p(x)$ is defined for every number x by: $F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y)$

– $F(x)$ gives you the probability that a random variable will be at most equal to x.

– $F(x)$ must be right-continuous and non-decreasing.

– $F(x)$ must satisfy two limit laws:
(i) $\lim_{x \to -\infty} F(x) = 0$
(ii) $\lim_{x \to \infty} F(x) = 1$

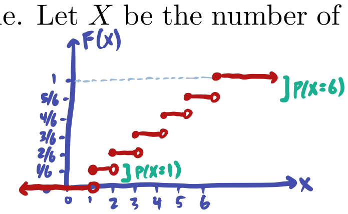


Theorem: For any two numbers a and b with $a \leq b$, $P(a \leq X \leq b) = F(b) - F(a^-)$.

Example: Roll a fair six-sided die. Let X be the number of the top of the die. Find the CDF of X.

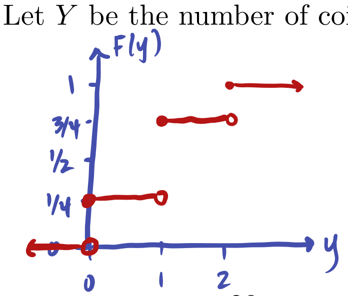
$F(2.5) = P(X \leq 2.5)$
$= p(1) + p(2)$
$= \frac{1}{6} + \frac{1}{6}$
$= \frac{2}{6}$

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & 1 \leq x < 2 \\ 2/6 & 2 \leq x < 3 \\ 3/6 & 3 \leq x < 4 \\ 4/6 & 4 \leq x < 5 \\ 5/6 & 5 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$



Example: Flip two fair coins. Let Y be the number of coins that land on heads. Find the CDF of Y.

$$F(y) = \begin{cases} 0 & y < 0 \\ 1/4 & 0 \leq y < 1 \\ 3/4 & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases}$$



3.3: Expected Values (Means)

Def: Let X be a discrete random variable with set of possible outcomes D and pmf p(x). The expected value or mean value of X, denoted E[X] or μ, is:

"average"

$$\boxed{E[X] = \sum_{x \in D} x \cdot p(x)}$$

Def: Let X have a pmf p(x) and expected value μ. Then the variance of X, denoted by Var(X) or σ^2, is:

$$\boxed{Var(X) = \sigma^2 = \sum_{x \in D} (x-\mu)^2 \cdot p(x) = E[(X-\mu)^2]}$$

Def: The standard deviation, denoted SD or σ is:

$$\boxed{\sigma = \sqrt{Var(X)}}$$

Theorem: $\boxed{Var(X) = E[X^2] - (E[X])^2}$ ★

Example: Roll a fair six-sided die. Let X be the number of the top of the die.
Find $E[X]$ and $Var(X)$.

support of X: {1,2,3,4,5,6}

x	1	2	3	4	5	6	
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6	$\to \sum p(x) = 1$ ✓
$x \cdot p(x)$	1/6	2/6	3/6	4/6	5/6	6/6	$\to \sum x \cdot p(x) = \frac{21}{6} = \boxed{3.5 = E[X]}$
$x^2 \cdot p(x)$	1/6	4/6	9/6	16/6	25/6	36/6	$\to \sum x^2 \cdot p(x) = \frac{91}{6} = E[X^2]$

- $Var(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \boxed{2.917 = Var(X)}$
- $\sigma = \sqrt{Var(X)} = \sqrt{2.917} \to \boxed{\sigma = 1.708}$

Example: Flip two fair coins. Let Y be the number of coins that land on heads.
Find $E[Y]$ and $Var(Y)$.

y	0	1	2	
$p(y)$	1/4	1/2	1/4	$\to \sum p(y) = 1$ ✓
$y \cdot p(y)$	0	1/2	1/2	$\to \sum y \cdot p(y) = \boxed{1 = E[Y]}$
$y^2 \cdot p(y)$	0	1/2	1	$\to \sum y^2 \cdot p(y) = 3/2 = E[Y^2]$

- $Var(Y) = E[Y^2] - (E[Y])^2 = 3/2 - 1^2 = \boxed{\frac{1}{2} = Var(Y)}$

$$E[h(X)] = \sum_x h(x)\cdot p(x) \quad , \quad E[e^{tX}] = \sum_x e^{tx} p(x)$$

Theorem: $E[aX + b] = a \cdot E[X] + b$ ($a, b \in \mathbb{R}$, i.e., constants)

Proof:

$$E[aX+b] = \sum (ax+b)\cdot p(x) = \sum ax\cdot p(x) + \sum b \cdot p(x)$$

$$= a\underbrace{\sum x p(x)}_{E[X]} + b\underbrace{\sum p(x)}_{\sum p(x)=1}$$

$$= aE[X] + b \checkmark$$

Theorem: $Var(aX + b) = a^2 \cdot \sigma^2$

Proof:

$$Var(aX+b) = E[((aX+b)-(a\mu+b))^2] = E[(aX+\cancel{b}-a\mu-\cancel{b})^2]$$

$$= E[(aX-a\mu)^2] = E[a^2(X-\mu)^2]$$

$$= a^2 E[(X-\mu)^2] = a^2 \cdot Var(X) = a^2 \cdot \sigma^2 \checkmark$$

Theorem: $E[h(X)] = \sum f(x) \cdot h(x)$

Example: Consider the following game. You pay \$1 to roll a fair six-sided die and receive a prize based on the outcome. Let X be the outcome of the die roll and the prize is $P = 3 \cdot (X - 3)$. What is expected value and variance of your expected gain, G?

$$p(x) = \begin{cases} 1/6 & x = 1,2,3,4,5,6 \\ 0 & \text{otherwise} \end{cases} \qquad E[X] = 3.5, \quad Var(X) = 2.917$$

$$P = 3(X-3) = 3X - 9$$

$$G = P - 1 = 3X - 10$$

$$E[G] = E[3X-10] = 3E[X] - 10 = 3(3.5) - 10 = \boxed{0.5 = E[G]}$$

$$Var(G) = Var(3X-10) = 3^2 Var(X) = 9 \cdot 2.917 = \boxed{26.253 = Var(G)}$$

"~" = "is distributed as"

$X \sim \text{Bernoulli}(p) \rightarrow$ X is dist. as a Bernoulli RV with parameter p.

3.4: Bernoulli and Binomial Random Variables

Def: A Bernoulli(p) random variable X is a discrete random variable with two possible outcomes (typically, these outcomes are 0 and 1).

ex) $p = \frac{1}{2} \rightarrow$ this models a fair coin flip

The PMF of a Bernoulli(p) is:

p is a parameter (characteristic of the RV/dist.)

$$\boxed{p(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \end{cases}}$$

If $X \sim$ Bernoulli(p) random variable, then $E[X] = p$ and $Var(X) = p(1-p)$

Example: Show that if X is a Bernoulli(p) random variable, then $E[X] = p$ and $Var(X) = p(1-p)$.

$$E[X] = \sum_{x=0}^{1} x \cdot p(x) = 0 \cdot p(0) + 1 \cdot p(1) = 0 \cdot (1-p) + 1 \cdot p = p = E[X] \checkmark$$

$$E[X^2] = \sum_{x=0}^{1} x^2 \cdot p(x) = 0^2 \cdot p(0) + 1^2 \cdot p(1) = 0 \cdot (1-p) + 1 \cdot p = p = E[X^2]$$

$$Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p) = Var(X) \checkmark$$

Def: Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a parameter of the distribution.

Def: The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

two parameters: n, p

Def: X is a Binomial(n,p) random variable if X is a discrete random variable that satisfies the following conditions:

- The experiment consists of n Bernoulli trials, where n is fixed.
- The trials are independent.
- The probability of success, p, is constant from trial to trial.
- X is the number of successes

The PMF of a Binomial(n, p) is:

$$\boxed{p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n}$$

$$\binom{n}{x} = \frac{n!}{x!(n-x)!}$$: "n choose x"

If $X \sim$ Binomial(n, p) random variable, then $E[X] = np$ and $Var(X) = np(1-p)$

Example: Basketball player Lebron James has a career free throw percentage of 73.1% (i.e., there's a 73.1% chance he will make a basket from the free throw line). Suppose Lebron has six free throw attempts in a game and assume all free throw shots are independent.
Answer the following questions.

(a) On average, how many free throws is Lebron expected to make? What is the variance?

X = # of free throws made ($X \sim \text{Binomial}(n=6, p=0.731)$)

$E[X] = np = 6(0.731) = \boxed{4.39 = E[X]}$

$Var(X) = np(1-p) = 6(0.731)(1-0.731) = \boxed{1.18 = Var(X)}$

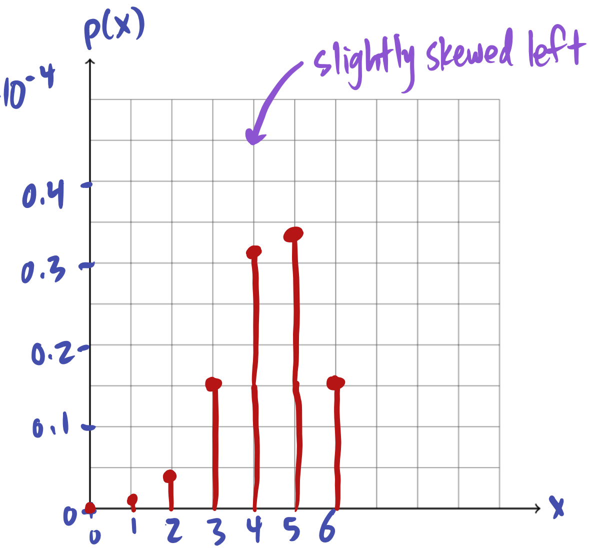
(b) Let X be the number of free throws that Lebron makes. Sketch the pmf of X.

PMF of binomial: $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

x	p(x)
0	3.8×10^{-4}
1	0.006
2	0.042
3	0.152
4	0.310
5	0.337
6	0.153

$= P(0) = \binom{6}{0}(0.731)^0(0.269)^6 = 3.8\times10^{-4}$

$= P(1) = \binom{6}{1}(0.731)^1(0.269)^5 = 0.006$



(c) What is the probability that Lebron makes at least five shots?

$P(X \geq 5) = p(5) + p(6) = 0.337 + 0.153 = \boxed{0.49}$

3.5: Geometric, Discrete Uniform, and Poisson Random Variables

Def: X is a Geometric(p) random variable is X is a discrete random variable with the following properties.

- The experiment consists of independent trials.
- Each trial can result in a success or a failure.
- The probability of success is p and is constant across all trials.
- The experiment continues until a successful trial is observed.
- X is the number of total trials

The PMF of a Geometric(p) is:

$$\boxed{p(x) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots}$$

If $X \sim$ Geometric(p) random variable, then $E[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$.

Example: A fair coin is tossed until a heads occur. Calculate the probability that 3 or more tosses will be required.

↳ geometric

X = # of tosses to get first heads

$X \sim \text{Geometric}(p = \frac{1}{2})$, $P(X \geq 3) = ?$

$P(X \geq 3) = p(3) + p(4) + p(5) + \ldots$ complement rule

$= 1 - P(X < 3)$

$= 1 - [p(1) + p(2)]$

$= 1 - \frac{1}{2} - \frac{1}{4}$

$= \boxed{\frac{1}{4} = P(X \geq 3)}$

$p(1) = \frac{1}{2}(\frac{1}{2})^0 = \frac{1}{2}$

$p(2) = \frac{1}{2}(\frac{1}{2})^1 = \frac{1}{4}$

$a =$ min
$b =$ max

Def: X is a Discrete Uniform(a,b) random variable is X is a discrete random variable such that the outcomes $a, a+1, \ldots, b$ are equally likely. Let $n = b - a + 1$.

ex) fair six-sided die roll
↳ discrete uniform $(a=1, b=6)$

The PMF of a Uniform(a, b) is:

$$\boxed{p(x) = \frac{1}{n}, x = a, a+1, \ldots, b}$$

If $X \sim$ Uniform(a, b) random variable, then $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a+1)^2-1}{12}$.

Example: Calculate the mean and variance of a fair six-sided die roll using the PMF and compare it to the values obtained from the formulas above.

Previously, we found $E[X] = 3.5$, $Var(X) = 2.917$ (found these using the PMF method)

Using the formulas above: $(a=1, b=6)$

$$E[X] = \frac{a+b}{2} = \frac{1+6}{2} = 3.5 \checkmark$$

$$Var(X) = \frac{(b-a+1)^2-1}{12} = \frac{(6-1+1)^2-1}{12} = \frac{35}{12} \approx 2.917 \checkmark$$

models counts

rate parameter (events/time)

Def: A Poisson random variable is a discrete random variable with parameter λ and the following pmf:

$$\boxed{p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots}$$

no upper bound

If $X \sim \text{Poisson}(\lambda)$ random variable, then $E[X] = \lambda$ and $Var(X) = \lambda$.

Note: Poisson random variables are often used to model the number of events that occur in a finite period of time. ex) calls received at a call help center (in an hour)

Example: Mark is going fishing and on average, he catches 2 fish per day. Assume that the time between successive fish caught is independent. What is the probability that Mark catches less than 3 fish?

Poisson : ① models counts
② "interarrival times" are independent

$\lambda = 2 \frac{\text{fish}}{\text{day}}$, $P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}$, $k = 0, 1, 2, \ldots$

$P(X<3) = p(0) + p(1) + p(2)$

$= \frac{e^{-2}\cdot 2^0}{0!} + \frac{e^{-2}\cdot 2^1}{1!} + \frac{e^{-2}2^2}{2!}$

$\rightarrow \boxed{P(X<3) = 0.677}$

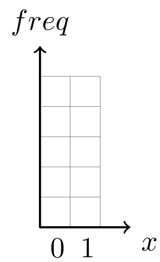
Example: Class exercise. We will collect data based on the following experiments. Plot the sample data as a histogram, determine the random variable, and plot the PMF of the random variable.

(a) Flip your coin one time and let X be the number of heads that you observed.

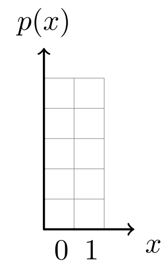
Sample data

x	freq
0	
1	

Histogram



PMF

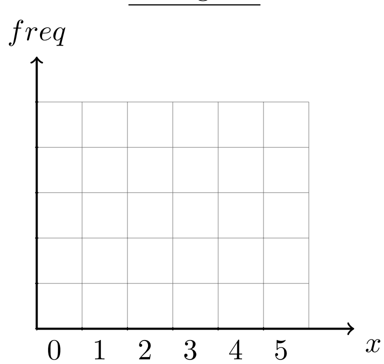


(b) Flip your coin five times and let X be the number of heads that you observed.

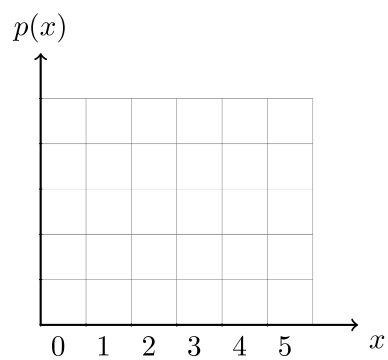
Sample data

x	freq	x	freq
0		3	
1		4	
2		5	

Histogram



PMF

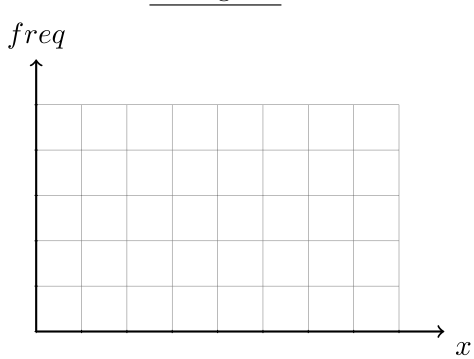


(c) Flip your coin until you flip a heads and let X be the number of coin flips required.

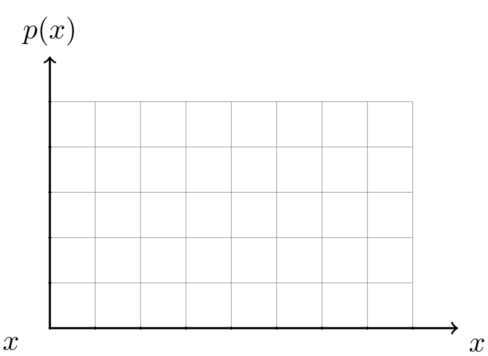
Sample data

x	freq	x	freq
1		5	
2		6	
3		7	
4		8+	

Histogram



PMF



Common Discrete Random Variables summary table

Random Variable	**PMF**	$\mathbf{E}[\mathbf{X}]$	$\mathbf{Var}(\mathbf{X})$
Bernoulli(p)	$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$	p	$p(1-p)$
Binomial(n, p)	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n$	np	$np(1-p)$
Discrete Uniform(a, b)	$p(x) = \frac{1}{b-a+1}, x = a, a+1, \ldots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Geometric(p)	$p(x) = p(1-p)^{x-1}, x = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson(λ)	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, \ldots$	λ	λ