3 Discrete Random Variables

Chapter Preview

In this chapter, we'll introduce random variables, probability distributions, and specifically look at discrete random variables.

3.1: Random Variables

Def: For a given sample space Ω of some experiment, a random variable (RV) is any rule that associates a number with each outcome in Ω. In mathematical language, a random variable is a function whose domain is the sample space and whose range is the set of real numbers.

Def: A discrete random variable is a random variable whose possible values either constitute a finite set or a countably infinite set.

Def: A continuous random variable is a random variable whose possible values are uncountably infinite or defined over an interval. Note: P(X=c)=0 for a continuous random variable X and a constant c.

Example: Roll a fair six-sided die. Let X be the number on the top of the die.

Example: Pick a random person and let Y be the number of keys in their pocket.

Example: Suppose a student is given 50 minutes to complete an exam. Let Z be the amount of time it takes the student to complete the exam.

3.2: Probability Distributions for Discrete Random Variables

Def: The probability mass function (pmf) or probability distribution of a discrete random variable X is defined for every number x by $p(x) = P(X = x)$. This is also sometimes denoted f(x).

Note: $\sum_{x \in X} p(x) = 1$

Example: Roll a fair six-sided die. Let X be the number of the top of the die. Find the pmf of X.

Example: Flip two fair coins. Let Y be the number of coins that land on heads. Find the pmf of Y.

Def: The cumulative distribution function (CDF) $F(x)$ of a discrete random variable X with pmf $p(x)$ is defined for every number x by: $F(x) = P(X \leq x) = \sum_{y: y \leq x} p(y)$

– $F(x)$ gives you the probability that a random variable will be at most equal to x.

– $F(x)$ must be right-continuous and non-decreasing.

– $F(x)$ must satisfy two limit laws:
(i) $\lim_{x \to -\infty} F(x) = 0$
(ii) $\lim_{x \to \infty} F(x) = 1$

Theorem: For any two numbers a and b with $a \leq b$, $P(a \leq X \leq b) = F(b) - F(a^-)$.

Example: Roll a fair six-sided die. Let X be the number of the top of the die. Find the CDF of X.

Example: Flip two fair coins. Let Y be the number of coins that land on heads. Find the CDF of Y.

3.3: Expected Values (Means)

<u>Def:</u> Let X be a discrete random variable with set of possible outcomes D and pmf p(x).
The <u>expected value</u> or <u>mean value</u> of X, denoted E[X] or μ, is:

$$E[X] = \sum_{x \in D} x \cdot p(x)$$

<u>Def:</u> Let X have a pmf p(x) and expected value μ. Then the <u>variance</u> of X, denoted by Var(X) or σ^2, is:

$$Var(X) = \sigma^2 = \sum_{x \in D} (x-\mu)^2 \cdot p(x) = E[(X-\mu)^2]$$

<u>Def:</u> The <u>standard deviation</u>, denoted SD or σ is:

$$\sigma = \sqrt{Var(X)}$$

<u>Theorem:</u> $Var(X) = E[X^2] - (E[X])^2$

<u>Example</u>: Roll a fair six-sided die. Let X be the number of the top of the die.
Find $E[X]$ and $Var(X)$.

<u>Example</u>: Flip two fair coins. Let Y be the number of coins that land on heads.
Find $E[Y]$ and $Var(Y)$.

Theorem: $E[aX + b] = a \cdot E[X] + b$
Proof:

Theorem: $Var(aX + b) = a^2 \cdot \sigma^2$
Proof:

Theorem: $E[h(X)] = \sum f(x) \cdot h(x)$

Example: Consider the following game. You pay \$1 to roll a fair six-sided die and receive a prize based on the outcome. Let X be the outcome of the die roll and the prize is $P = 3 \cdot (X - 3)$. What is expected value and variance of your expected gain, G?

3.4: Bernoulli and Binomial Random Variables

Def: A Bernoulli(p) random variable X is a discrete random variable with two possible outcomes (typically, these outcomes are 0 and 1).

The PMF of a Bernoulli(p) is:

$$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$$

If $X \sim$ Bernoulli(p) random variable, then $E[X] = p$ and $Var(X) = p(1-p)$

Example: Show that if X is a Bernoulli(p) random variable, then $E[X] = p$ and $Var(X) = p(1-p)$.

Def: Suppose $p(x)$ depends on a quantity that can be assigned any one of a number of possible values, with each different value determining a different probability distribution. Such a quantity is called a parameter of the distribution.

Def: The collection of all probability distributions for different values of the parameter is called a family of probability distributions.

Def: X is a Binomial(n,p) random variable if X is a discrete random variable that satisfies the following conditions:

- The experiment consists of n Bernoulli trials, where n is fixed.
- The trials are independent.
- The probability of success, p, is constant from trial to trial.
- X is the number of successes

The PMF of a Binomial(n, p) is:

$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, 2, \ldots, n$

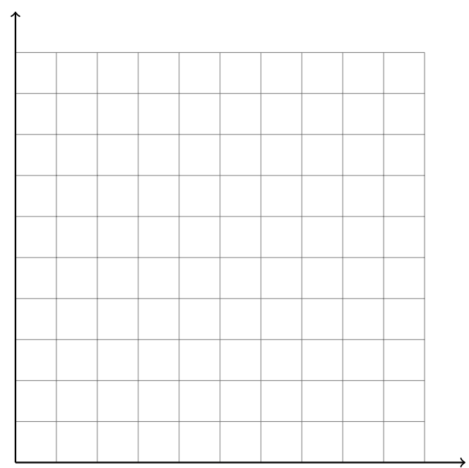
If $X \sim$ Binomial(n, p) random variable, then $E[X] = np$ and $Var(X) = np(1-p)$

Example: Basketball player Lebron James has a career free throw percentage of 73.1% (i.e., there's a 73.1% chance he will make a basket from the free throw line). Suppose Lebron has six free throw attempts in a game and assume all free throw shots are independent.
Answer the following questions.

(a) On average, how many free throws is Lebron expected to make? What is the variance?

(b) Let X be the number of free throws that Lebron makes. Sketch the pmf of X.

x	p(x)
0	
1	
2	
3	
4	
5	
6	



(c) What is the probability that Lebron makes at least five shots?

3.5: Geometric, Discrete Uniform, and Poisson Random Variables

Def: X is a Geometric(p) random variable is X is a discrete random variable with the following properties.

- The experiment consists of independent trials.
- Each trial can result in a success or a failure.
- The probability of success is p and is constant across all trials.
- The experiment continues until a successful trial is observed.
- X is the number of total trials

The PMF of a Geometric(p) is:

$p(x) = p(1-p)^{x-1}, x = 1, 2, 3, \ldots$

If $X \sim$ Geometric(p) random variable, then $E[X] = \frac{1}{p}$ and $Var(X) = \frac{1-p}{p^2}$.

Example: A fair coin is tossed until a heads occur. Calculate the probability that 3 or more tosses will be required.

Def: X is a Discrete Uniform(a,b) random variable is X is a discrete random variable such that the outcomes $a, a+1, \ldots, b$ are equally likely. Let $n = b - a + 1$.

The PMF of a Uniform(a, b) is:

$p(x) = \frac{1}{n}, x = a, a+1, \ldots, b$

If $X \sim$ Uniform(a, b) random variable, then $E[X] = \frac{a+b}{2}$ and $Var(X) = \frac{(b-a+1)^2-1}{12}$.

Example: Calculate the mean and variance of a fair six-sided die roll using the PMF and compare it to the values obtained from the formulas above.

Def: A Poisson random variable is a discrete random variable with parameter λ and the following pmf:

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots$$

If $X \sim$ Poisson(λ) random variable, then $E[X] = \lambda$ and $Var(X) = \lambda$.

Note: Poisson random variables are often used to model the number of events that occur in a finite period of time.

Example: Mark is going fishing and on average, he catches 2 fish per day. Assume that the time between successive fish caught is independent. What is the probability that Mark catches less than 3 fish?

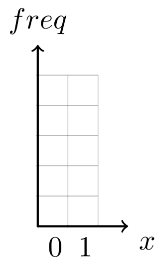
Example: Class exercise. We will collect data based on the following experiments. Plot the sample data as a histogram, determine the random variable, and plot the PMF of the random variable.

(a) Flip your coin one time and let X be the number of heads that you observed.

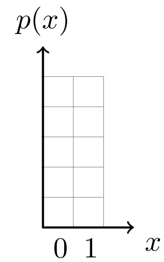
Sample data

x	freq
0	
1	

Histogram



PMF

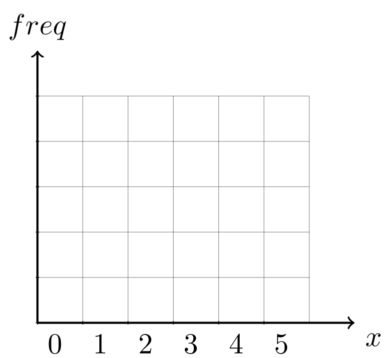


(b) Flip your coin five times and let X be the number of heads that you observed.

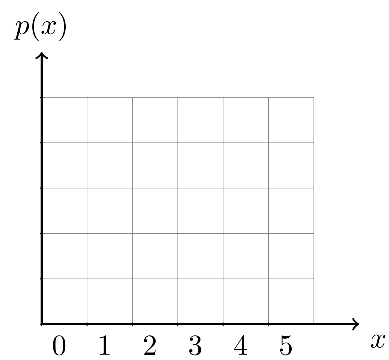
Sample data

x	freq	x	freq
0		3	
1		4	
2		5	

Histogram



PMF

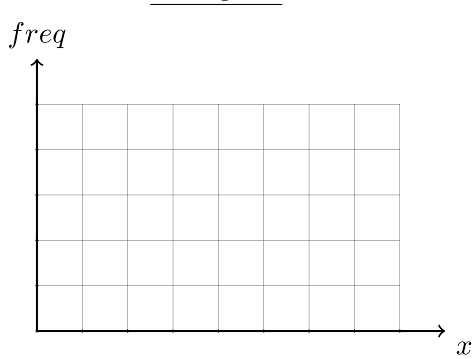


(c) Flip your coin until you flip a heads and let X be the number of coin flips required.

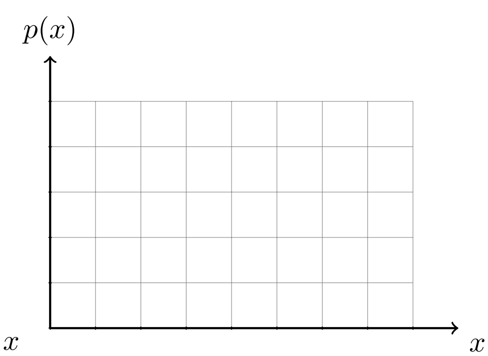
Sample data

x	freq	x	freq
1		5	
2		6	
3		7	
4		8+	

Histogram



PMF



Common Discrete Random Variables summary table

Random Variable	**PMF**	$\mathbf{E}[\mathbf{X}]$	$\mathbf{Var}(\mathbf{X})$
Bernoulli(p)	$p(x) = \begin{cases} 1-p & x=0 \\ p & x=1 \end{cases}$	p	$p(1-p)$
Binomial(n, p)	$p(x) = \binom{n}{x} p^x (1-p)^{n-x}, x = 0, 1, \ldots, n$	np	$np(1-p)$
Discrete Uniform(a, b)	$p(x) = \frac{1}{b-a+1}, x = a, a+1, \ldots, b$	$\frac{a+b}{2}$	$\frac{(b-a+1)^2-1}{12}$
Geometric(p)	$p(x) = p(1-p)^{x-1}, x = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson(λ)	$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, \ldots$	λ	λ