# **Process Reward Models That Think**

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#### **Abstract**

Step-by-step verifiers—also known as process reward models (PRMs)—are a key ingredient for test-time scaling. PRMs require step-level supervision, making them expensive to train. This work aims to build data-efficient PRMs as verbalized stepwise reward models that verify every step in the solution by generating a verification chain-of-thought (CoT). We propose THINKPRM, a long CoT verifier fine-tuned on orders of magnitude fewer process labels than those required by discriminative PRMs. Our approach capitalizes on the inherent reasoning abilities of long CoT models, and outperforms LLM-as-a-Judge and discriminative verifiers—using only 1% of the process labels in PRM800K—across several challenging benchmarks. Specifically, THINKPRM beats the baselines on ProcessBench, MATH-500, and AIME '24 under best-of-N selection and reward-guided search. In an out-of-domain evaluation on a subset of GPQA-Diamond and LiveCodeBench, our PRM surpasses discriminative verifiers trained on the full PRM800K by 8% and 4.5%, respectively. Lastly, under the same token budget, THINKPRM scales up verification compute more effectively compared to LLM-as-a-Judge, outperforming it by 7.2% on a subset of ProcessBench. Our work highlights the value of generative, long CoT PRMs that can scale test-time compute for verification while requiring minimal supervision for training.1

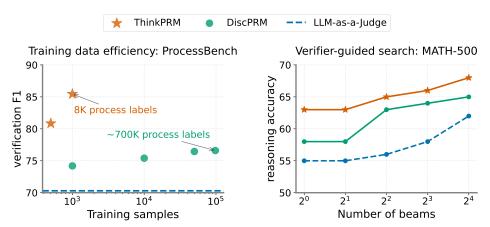


Figure 1: **Left:** Verifier F1-score on ProcessBench (Zheng et al., 2024). THINKPRM-14B, trained on 8K process labels or 1K synthetic examples, outperforms discriminative PRMs trained on about 100x more data. **Right:** Accuracy over MATH-500 using Llama-3.2-3B-Instruct as generator with different verifiers. THINKPRM-1.5B, trained using the same 8K labels, outperforms LLM-as-a-judge and discriminative verifiers in reward-guided search on MATH-500. The LLM-as-a-judge in both figures uses the same base model as THINKPRM.

Our code, data, and models will be released at https://github.com/mukhal/thinkprm.

#### 1 Introduction

Reasoning with large language models (LLMs) can substantially benefit from utilizing more test compute (Jaech et al., 2024; Guo et al., 2025; Akyürek et al., 2024). This typically depends on a high-quality process reward model (PRM)—also known as a process verifier—to select promising paths for search or ranking (Cobbe et al., 2021; Li et al., 2023; Wu et al., 2024; Brown et al., 2024). Given a problem-solution pair, the PRM provides a score that indicates whether the solution or part of it is correct. To date, PRMs have mostly assumed the form of discriminative classifiers, trained to discern correct from incorrect reasoning (Uesato et al., 2022; Zhang et al., 2025). However, training discriminative PRMs requires access to process labels, i.e., step-level annotations, which either require extensive human annotation (Lightman et al., 2023; Zheng et al., 2024), gold step-by-step solutions (Khalifa et al., 2023), or compute-intensive rollouts (Luo et al., 2024; Wang et al., 2023b; Chen et al., 2024a).

LLM-as-a-judge can enable generative verification by treating verification as text generation without any training (Wang et al., 2023a; Liu et al., 2023b; Zheng et al., 2023). Despite the advantages of LLM-as-a-judge such as data-efficiency and interpretability, it has been known to perform quite poorly compared to specialized reward models in the context of complex reasoning (Lambert et al., 2024; Zhang et al., 2024b; Chen et al., 2024c), as uncustomized LLMs frequently fail to recognize incorrect reasoning (Huang et al., 2023; Zhang et al., 2024a; Ye et al., 2024). This raises the question: How can we maintain both the data-efficiency and interpretability advantages and the superior performance of discriminative PRMs?

We approach this question with the following insight: LLM-as-a-judge verification can greatly benefit from scaling up the verifier's inference compute—specifically, by enabling it to *think* through a verification chain of thought (CoT). To this end, we repurpose open-weight large reasoning models (LRMs; Team 2024; Guo et al. 2025; Li et al. 2025) as the foundation for generative PRMs through lightweight tuning. Specifically, we train LRMs using synthetic data (Kim et al., 2023; Wang et al., 2024), utilizing as few as 8K process labels, and obtain THINKPRM—a PRM that not only surpasses LLM-as-a-judge under the same base model, but also outperforms discriminative PRMs trained on two orders of magnitude more data across a variety of test-time scaling scenarios.

To obtain THINKPRM, we train four different reasoning models, namely R1-Distill-Qwen{1.5B,7B,14B} (Guo et al., 2025), and QwQ-32B-Preview (Team, 2024) and our training is extremely efficient, e.g., training our largest

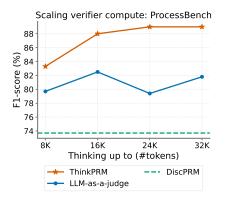


Figure 2: THINKPRM supports scaling verification compute by thinking longer.

PRM takes only 4 hours on a single A100 80GB. We extensively evaluate THINKPRM as a standalone verifier on ProcessBench (Zheng et al., 2024), under different Best-of-N selection and verifier-guided search scenarios—using both in- and out-of-domain tasks.

THINKPRM-14B, based on R1-Distill-Qwen-14B, outperforms a discriminative PRM based on the same base model in terms of accuracy while using far fewer supervision signals as in Fig. 1 left. In addition, THINKPRM-1.5B, based on R1-Distill-Qwen-1.5B, demonstrates strong performance on MATH-500 (Hendrycks et al., 2021) under guided beam search, shown in Fig. 1 right. Lastly, as shown in Fig. 2, THINKPRM can effectively utilize more verification compute than LLM-as-a-judge, by forcing it to think for more tokens. All these results are obtained while training on 1K synthetic examples, utilizing only 8K process labels.

Our work highlights the promise of long CoT verification that verifies reasoning with reasoning, and that can effectively scale up both generator and verifier compute. To summarize, this work makes the following major contributions:

• We introduce a simple recipe to obtain THINKPRM—a generative PRM that verifies solutions via long CoT reasoning. Concretely, THINKPRM can be efficiently trained from off-the-shelf reasoning model via finetuning on synthetic data.

- THINKPRM uses only 8K process labels and outperforms discriminative PRMs, trained on two orders of magnitude more labels by up to 8% Macro F1 points on ProcessBench.
- We further show that THINKPRM outperforms strong baseline PRMs in best-of-N and guided-search setups, on two in-domain tasks: MATH-500, AIME 2024, and two out-of-domain tasks: GPQA-Diamond and LiveCodeBench. THINKPRM also outperforms LLM-as-a-judge baselines, and can enable better scaling of verifier compute by thinking longer.

### 2 Background and Related Work

Process reward models (PRMs) evaluate reasoning solutions step-by-step by taking a problem and a multi-step solution prefix as input, and producing a sequence of scores, each assessing the correctness or quality of a specific step. PRMs can be categorized as discriminative or generative.

**Discriminative PRMs.** Discriminative PRMs are trained as classifiers that directly predict numerical correctness scores for each step, typically relying on extensive step-level annotations (Uesato et al., 2022; Lightman et al., 2023; Zhang et al., 2025). Given a solution prefix, discriminative PRMs encode the solution text and employ a classification head to produce step-level scores, usually optimized with binary cross-entropy. An overall correctness score for a solution is obtained by aggregating these step-level scores (Beeching et al.; Snell et al., 2024; Wu et al., 2024). While discriminative PRMs are effective and straightforward, they do not utilize the language-modeling head of the base language model, making training expensive and labor-intensive (Yuan et al., 2024). Additionally, they offer limited interpretability and require fixed inference-time compute, restricting their dynamic scalability at test-time (Zhang et al., 2024a; Mahan et al., 2024).

Generative PRMs and LLM-as-a-Judge. Generative PRMs (Zheng et al., 2023; Zhu et al., 2023) frame verification as a language-generation task, producing step-level correctness decisions as natural language tokens (e.g., "correct" or "incorrect"), typically accompanied by a verification chain-of-thought (CoT). Generative PRMs rely on the standard language modeling objective, training on verification rationales rather than on binary labels. Step-level correctness scores can be derived from generative PRMs by computing conditional token probabilities, e.g., P("correct"). This approach leverages the strengths of LLMs in text generation, making generative PRMs inherently interpretable and scalable (Zhang et al., 2024a; Mahan et al., 2024; Wang et al., 2023a; Ankner et al., 2024).

Recent work on generative PRMs often utilizes off-the-shelf LLMs prompted to critique solutions—known as LLM-as-a-Judge (Zheng et al., 2024). However, LLM-as-a-Judge can be unreliable, sensitive to prompt phrasing, and prone to invalid outputs, such as infinite looping or excessive overthinking (Bavaresco et al., 2024), issues we further confirm in this work. Prior results with reasoning models like QwQ-32B-Preview (Team, 2024) show promise, but their practical utility in test-time scaling remains limited without additional training (Zheng et al., 2024).

**Test-Time Scaling and PRMs.** Test-time scaling techniques, such as Best-of-N selection (Charniak & Johnson, 2005; Khalifa et al., 2023; Snell et al., 2024) and tree-based search (Wu et al., 2024; Yao et al., 2023; Chen et al., 2024c; Wan et al., 2024), leverage additional inference-time compute to improve reasoning performance. Central to these approaches is the quality of the verifier used to score and select solutions (Beeching et al., Snell et al., 2024). While both discriminative and generative PRMs can guide these processes, generative PRMs uniquely support simultaneous scaling of both generator and verifier compute (Zhang et al., 2024a; Kalra & Tang, 2025). We show that generative PRMs trained based on long CoT models (Jaech et al., 2024; Guo et al., 2025; Muennighoff et al., 2025) enable sequential scaling of verification compute by forcing longer verification CoT.

Motivated by limitations of existing approaches, our work builds upon prior efforts that train reward models using synthetic data (Zhu et al., 2023; Wang et al., 2024), aiming to develop generative PRMs with minimal, carefully filtered synthetic step-level supervision. Specifically, we demonstrate that a generative PRM fine-tuned with as few as 8K synthetic verification chains substantially improves over LLM-as-a-Judge PRMs, and outperforms discriminative PRMs trained on datasets orders of magnitude larger.

#### Example verification chain-of-thought

#### <think>

Alright, I've got this math problem about finding the surface area of a solid after cutting out a small cube from a larger rectangular prism. The original prism has dimensions 4 by 2 by 2, and a 1 by 1 by 1 cube is removed from one corner. The provided solution has four steps, and I need to review each one to see if it's correct.

Let's start with Step 1.

**Step 1:** The solution calculates the surface area of the original rectangular prism. The formula used is  $A_{\text{original}} = 2(lw + lh + wh)$ , which is correct for a rectangular prism. Plugging in the dimensions l = 4, w = 2, and h = 2, the calculation goes like this:

$$A_{\text{original}} = 2(4 \cdot 2 + 4 \cdot 2 + 2 \cdot 2) = 2(8 + 8 + 4) = 2 \cdot 20 = 40$$

This seems right. So, Step 1 is \\boxed{correct}.

**Step 2:** Here, the solution talks about the effect of cutting out the 1 by 1 by 1 cube from one corner. It says that removing the cube takes away three faces but adds three new faces. Each face is 1 by 1, so each has an area of 1 square unit.

Wait a minute, is this accurate? When you cut out a cube from a corner, you are indeed removing material, but how does that affect the surface area?

Let me visualize this. Imagine a rectangular prism and cutting out a small cube from one corner. The cube has three faces exposed to the outside of the prism. When you remove the cube, those three faces are no longer part of the surface area. However, the removal also creates three new internal faces where the cube was taken out, which are now part of the surface area.

...

Alternatively, perhaps I can think about the new solid's dimensions.

But I think the initial approach is correct.

Therefore, Step 2 is  $\setminus \mathbf{boxed}\{\mathbf{correct}\}$ .

...

Step 3: This step calculates the net change in surface area as zero, which follows from Step 2.

Since the net change is zero, the surface area remains the same as the original.

So, Step 3 is \\boxed{correct}.

Step 4: Concludes that the surface area of the new solid is 40, the same as the original.

Therefore, Step 4 is \\boxed{correct}.

Wait, but I recall that sometimes when you make a cut, especially in certain ways, the surface area can change. But in this case, it seems to hold up.

</think>

Figure 3: Example output by THINKPRM-14B, where it verifies and labels every step in the provided prefix via a long verification CoT. We omit the problem and solution for brevity.

#### 3 THINKPRM

Our goal is to obtain a data-efficient, yet powerful, verbalized PRM. Given a problem-solution pair, our generative PRM should verify every step in the solution via an extended chain-of-thought (CoT) as shown in the example in Fig. 3. Generating such CoT has several advantages. First, it opens a window to the reasoning process of the verifier, enabling better interpretability of its decisions. Second, it capitalizes on the capabilities of reasoning models and enables strong verifiers with minimal training. Third, this thinking process enables scaling up the verifier compute either by parallely sampling multiple CoTs and aggregating their decisions, or by allowing the model to revise itself in-context by forcing it to reconsider or double-check its verification.

Throughout the rest of the paper, we aim to answer the following research questions:

- **RQ1:** How well do large reasoning models (LRMs) perform when used as LLM-as-a-judge verifiers for process-level evaluation? Section 3.1
- **RQ2:** Can lightweight finetuning on synthetic verification CoTs improve the reliability and effectiveness of these models as process verifiers? Section 3.2

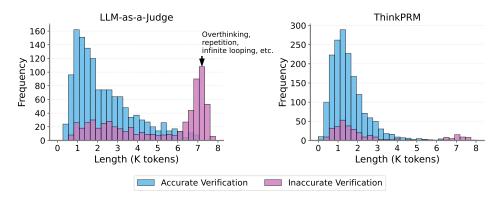


Figure 4: Verifier performance on ProcessBench in light of CoT lengths. On the left, LLM-as-a-judge produces excessively long chains including repetition, infinite looping, and overthinking, leading to worse verifier performance since the output never terminates. Training on collected syntehtic data substantially reduces these issues as shown in the THINKPRM plot on the right.

• **RQ3:** How does a finetuned verbalized PRM (THINKPRM) compare to discriminative PRMs and LLM-as-a-Judge baselines under different test-time scaling scenarios? Section 4

#### 3.1 RQ1: LLM-as-a-judge PRMs are suboptimal

This section highlights limitations we observe when using off-the-shelf reasoning models as process verifiers, suggesting the need for finetuning. For evaluation, we use ProcessBench (Zheng et al., 2024), which includes problem-solution pairs with problems sourced from four different math benchmarks, labeled based on the index of the first incorrect step, if exists. We report the F1-score over the two most challenging subsets of ProcessBench: OlympiadBench (He et al., 2024) and OmniMath (Gao et al., 2024), each comprised of 1K problem-prefix pairs. Table 2 shows LLM-as-a-judge F1 scores and a output generated by QwQ-32B-Preview is displayed in Fig. 25 in Appendix F.

We observe the following issues with LLM-as-a-judge verification. First, the verification quality is *highly sensitive* to the instruction wording, e.g., changing a few words in the instruction could affect the F1-score by up to 3-4 points in some cases. Second, a substantial number of the generated chains include *invalid judgments*, i.e., chains without an extractable overall label as clear in Fig. 6. Such invalid judgments are caused by the following. In some cases, final decision was in the wrong format than instructed e.g., the model tries to *solve* the problem from scratch rather than verify the given solution—a behavior likely stemming from the model training process. In addition, we noted multiple instances of *overthinking* (Chen et al., 2024b; Cuadron et al., 2025), which prevents the model from terminating within the maximum token budget, and *infinite looping/repetitions*, where the model gets stuck trying alternative techniques to verify the solutions.

Fig. 4 (left) shows a histogram of verification CoT lengths generated by R1-Qwen-14B in the LLM-as-a-judge setting. Accurate CoTs tend to be shorter, typically under 3000 tokens, while inaccurate CoTs are more evenly distributed and spike sharply around 7000–8000 tokens, highlighting the prevalence of overthinking and looping in long chains. We show examples of these behaviors in Appendix B. In the next section, we mostly fix these issues via lightweight finetuning over synthetic verification CoTs.

#### 3.2 RQ2: Finetuning on synthetic verification chains boosts LLM-as-a-judge verification

Inspired by recent work on reducing overthinking in long CoT models that by training (Yu et al., 2024; Kang et al., 2024), we aim to improve LLM-as-a-judge performance via finetuning on high-quality verification data. Collecting real data would be expensive, so we rely on filtered synthetic data (Zelikman et al., 2022; Singh et al., 2023; Dong et al., 2023; Zhang et al., 2024b; Wang et al., 2024) also known as rejection sampling finetuning. To keep our approach simple, we refrain from more expensive training techniques such as reinforcement learning or preference-based learning.

#### 1. Sample verification chains

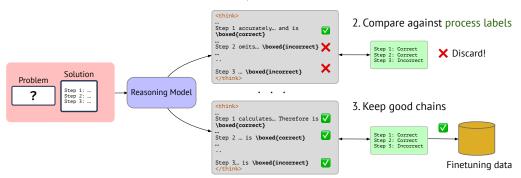


Figure 5: Collecting verification chains for finetuning. First, we prompt a reasoning model, in our case QwQ-32B-Preview to critique a given solution to a problem. Then, we sample multiple verification chains, which we judge against gold process labels from PRM800K, only keeping chains that match the gold process labels.

**Synthetic data collection.** As training data, we sample synthetic verification CoTs from QwQ-32B-Preview, prompting it to verify each step in a solution prefix, using the instruction shown in Fig. 14. The problems and corresponding step-by-step solutions come from the PRM800K dataset (Lightman et al., 2023), which provides both model-generated solutions and human-verified step-level labels.

We sample until we obtain 1K verification CoTs that satisfy the following criteria: (i) they must follow the expected format (i.e., include an extractable decision label for each step inside \boxed{} as shown in Fig. 3), and (ii) these step decisions match the gold step labels from PRM800K, and (iii) are under a certain token length—to avoid the excessive overthinking behaviour we observed in Fig. 4 (left). The filtering process ensure our training data is of sufficient quality while being very cheap to obtain.<sup>2</sup> Data collection is illustrated in Fig. 5. Notably, our filtering relies only on step-level annotations, not on gold verification rationales or CoTs—making this data pipeline scalable and low-overhead. For instance, in the absence of gold step-level annotations, one can obtain silver labels via Monte Carlo rollouts (Wang et al., 2023b; Chen et al., 2024a). Statistics of the dataset are in Appendix A.1 and a training example is in Fig. 13. We note that this data filtering process is crucial for the performance of the resulting PRM as we show in Section 5.2.

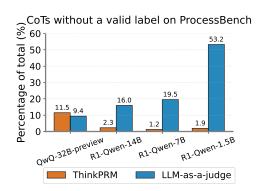
While we train only on math data, the resulting PRM remains robust under other domains such as Physics and code generation as we show in Section 4.2. We then proceed to train our models on the 1K collected chains. Our training is very lightweight; finetuning QwQ-32B-Preview takes only 4.5 hours on a single A100 80GB GPU. Refer to Appendix C.1 for training details.

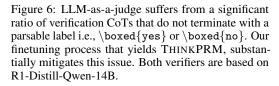
Finetuning on synthetic verification CoTs substantially improves the verifier. THINKPRM trains on the 1K chains and is evaluated on ProcessBench and compared to LLM-as-a-judge under the same base model. Fig. 7 shows verifier accuracy of different models before and after our finetuning. We note a substantial boost in F1 across all models, with the 1.5B model gaining most improvement by over 70 F1 points, and the 14B model performing best. Looking at the ratio of invalid judgements in Fig. 6, we also note a significant reduction in invalid labels with all models, except for QwQ, where it slightly increases. Lastly, the reduction in overthinking and infinite looping behavior discussed in the last section is evident, as in Fig. 4 (right), where THINKPRM generations maintain a reasonable length (1000-5000) tokens while being substantially more accurate.

#### 4 Main Results

This section aims to answer RQ3 introduced in section 3 by comparing THINKPRM to baselines under different scaling scenarios. We will study how THINKPRM performs under different generation budgets (i) best-of-N selection (Wu et al., 2024; Brown et al., 2020) and (ii) guided beam search (Snell et al., 2024; Beeching et al.). We will also explore how THINKPRM performs when verifier compute is scaled either in parallel i.e., by aggregating decisions over multiple verification CoTs or

<sup>&</sup>lt;sup>2</sup>Spurious chains may still exist.





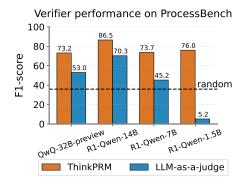


Figure 7: Verification accuracy on 2K questionsolution pairs from two most challenging subsets of ProcessBench: OlympiadBench and OmniMath. THINKPRM obtained by finetuning the correponding model over only 1K verification chains performs better.

sequentially by extending verification tokens by forcing the model to double-check or self-correct its verification process.

#### 4.1 Experimental Setup

In the remainder of the the paper, we will mainly use our finetuned verifiers based on R1-Distill-Qwen-1.5B and R1-Distill-Qwen-14B as these provide the best tradeoff between size and performance. We will refer to these as THINKPRM-1.5B and THINKPRM-14B, respectively.

**Baselines.** We compare THINKPRM to **DiscPRM**, which is the same base model as THINKPRM, finetuned with binary cross-entropy on the *entire* PRM800K dataset, totalling 712K process labels from 98K problem-solution pairs. Details on finetuning DiscPRMs are in Appendix C.2. We also compare to **unweighted majority voting**, which merely selects the most frequent answer across the samples (Wang et al., 2022). We also compare to LLM-as-a-Judge using the same base model as THINKPRM, prompted as in Section 3.1.

**Final Answer Selection.** For our best-of-N experiments except on AIME '24, we use verifier-weighted majority select the best answer, which scores final answers based on the sum of the verifier scores of their solutions (Uesato et al., 2022; Wu et al., 2024). For guided beam search, we select the most common final answer from all beams.

**Tasks and Models.** We show results on three math reasoning tasks, namely 100 problems from MATH-500 (Hendrycks et al., 2021; Lightman et al., 2023) covering all difficulty levels (see Appendix E.5 for more details), and American Invitational Mathematics Examination (AIME) problems for 2024. Since THINKPRM was finetuned only on math data, we study the out-of-domain generalization on two tasks: science QA and code generation. For science QA, we use the physics subset of GPQA-Diamond (Rein et al., 2024), consisting of 86 PhD-level multiple choice questions. For code generation, we use 200-problem subset from the v5 release of LiveCodeBench (Jain et al., 2024).

Over MATH-500, we show results with THINKPRM-1.5B and THINKPRM-14B on two different generator models: Qwen-2.5-14B and Llama-3.2-3B-Instruct. The former model is used for best-of-N and the latter for beam search as search can be quite slow. Showing results with different generators guarantees that our conclusions are not specific to a certain model family or size. For the more challenging tasks, namely AIME24, and GPQA, we use a more capable model, namely Qwen-2.5-32B-Instruct. For code generation, we use Qwen-2.5-Coder-7B (Hui et al., 2024). Hyperparameter details on best-of-N and beam search are in Appendix E.

**Scaling verifier compute.** Compared to DiscPRMs, generative reward models enable an extra dimension of scaling to squeeze more performance: scaling the verifier compute. THINKPRM allows

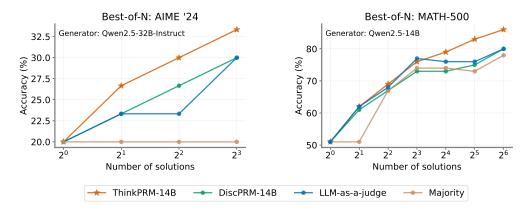


Figure 8: Best-of-N on AIME '24 and MATH-500. Compared to LLM-as-a-judge, DiscPRM, and (unweighted) majority vote, THINKPRM-14B exhibits best accuracy scaling curve.

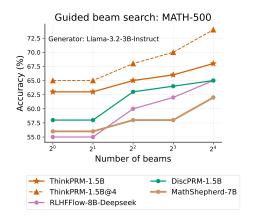


Figure 9: **Left:** Comparison to Off-the-shelf PRMs trained on substantially more process labels than THINKPRM.

Figure 10: **Right:** Ablating the data filtering mechanism, where our process-based filtering yields better PRMs. LLM-as-a-judge is shown with number of beams = 16.

for two types of scaling. First, we use *parallel scaling* (Mahan et al., 2024; Brown et al., 2024), by sampling K independent verification CoT and averaging their performance. We will refer to this scaling using "@K" throughout the rest of the paper. Second, more specific to long reasoning models, we use *sequential scaling* e.g., by enabling the model to self-correct its initial verification (Xiong et al., 2025; Kumar et al., 2024; Ye et al., 2024). Inspired by Muennighoff et al. (2025), we use a trigger phrase such as "Let's verify again" to prime the model to double check its earlier verification for potential errors. See Appendix E.4 for more details.

#### 4.2 Results

THINKPRM outperforms DiscPRM and LLM-as-a-Judge. On best-of-N selection with MATH-500 shown in Fig. 8 (right), THINKPRM leads to higher or comparable reasoning accuracy to DiscPRM under all sampling budgets. The trend holds on the more challenging AIME24 as shown in Fig. 8 left. Under verifier-guided search, Fig. 1 (right) shows beam search results on MATH-500, with THINKPRM 1.5B clearly outperforming discPRM by about 5% points. It also outperforms LLM-as-a-judge with the same base model, i.e., R1-Qwen-1.5B.

THINKPRM surpasses off-the-shelf PRMs. We compare THINKPRM-1.5B to two strong off-the-shelf PRMS, namely RLHFFlow-Deepseek-PRM (Xiong et al., 2024) and MATH-Shepherd-PRM (Wang et al., 2023b). These PRMs are trained on orders of magnitude more data and have more parameters than THINKPRM-1.5B. We show results under verifier-guided search on MATH-500 in

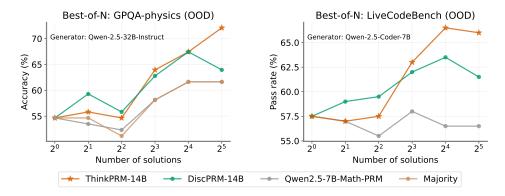


Figure 11: Best-of-N on two out-of-domain tasks: science QA (GPQA-Physics) and code generation (Live-CodeBench). Although THINKPRM was only finetuned on math, it exhibits superior OOD performance than the baselines, especially at larger sampling budgets, where the baselines performance falls short. Discriminative PRMs struggle despite being trained on order of magnitude more process labels.

Fig. 9, with THINKPRM-1.5B's scaling curve surpassing all baselines and outperforming RLHFFlow-Deepseek-PRM, the best off-the-shelf PRM among the ones we tested, by more than 7% at 16 beams.

THINKPRM excels on out-of-domain tasks. As for OOD performance on GPQA-physics (Fig. 11 left), THINKPRM scales better than DiscPRM—which drops substantially at N=32—outperforming it by 8%. On LiveCodeBench (Fig. 11 right), THINKPRM also outperforms DiscPRM by 4.5%. On LiveCodeBench, Qwen2.5-7B-Math-PRM (Zhang et al., 2025)—a discriminative PRM trained on subtantial amount of process labels obtained from LLM-as-a-judge data and Monte Carlo rollouts—struggles when applied out-of-domain. Our results shed light on the fragility of discriminative PRMs under domain shifts in contrast with generative PRMs. These results point at the strong OOD generalization abilities of ThinkPRM compared to discriminative models.

Scaling THINKPRM compute boosts performance. Under beam search (shown in Fig. 9) parallel scaling with THINKPRM-1.5B@4, boosts the verifier performance by more than 5% points, and yields the best accuracy on MATH-500. In addition, parallel scaling with THINKPRM-14B@4 and THINKPRM-14B@8 boosts best-of-N performance on MATH-500 as shown in Fig. 20 in Appendix E.6. Now we move to sequential scaling of verifier compute by forcing THINKPRM to recheck its own verification. Since this can be compute-intensive, we only run this on 200 problems from OmniMath subset of ProcessBench, we compare how verification accuracy improves as we force the model to think for longer as shown in Fig. 2. THINKPRM exhibits better scaling behaviour compared to LLM-as-a-judge, which drops after 16K tokens, and outperforms DiscPRM-14B by 15 F1 points. In summary, THINKPRM is consistently better than LLM-as-a-Judge under parallel and sequential scaling.

**Parallel scaling vs. sequential scaling.** Is it preferable to scale verifier compute in parallel or sequentially? We answer this question by comparing the two modes of scaling under the same token budget. Fig. 21 in Appendix E.6 shows performance of best-of-N with Qwen-2.5-14B under parallel and sequential scaling with K=2,4 under both parallel scaling and sequential scaling. Overall, the performance of both methods is fairly comparable, but we observe a slight advantage to parallel scaling in some cases.

## 5 Analysis

In this section, we study the training data efficiency of THINKPRM compared to discriminative PRMs (Section 5.1). Next, we validate the effect of process-based data filtering strategy by comparing it against outcome-based filtering (Section 5.2). We then examine how THINKPRM's performance varies with problem difficulty, highlighting its effectiveness on more challenging reasoning tasks

(Section 5.3). Finally, we discuss some inherent limitations of the current generative PRM approach (Section 5.4).

## 5.1 Training data efficiency

A major strength of THINKPRM is training data efficiency compared to discriminative versions. We study the training scaling behavior by training THINKPRM-14B using 500 and 1K examples in total collected using the process in Section 3.2, which roughly corresponds to 4K and 8K process labels from PRM800K in total. We compare that to DiscPRM-14B trained with 1K, 10K, 50K and 98K examples in total, where 98K corresponds to training on the full PRM800K train set, which includes about 712K step labels. Fig. 1 (Left) contrasts the training data scaling behavior of THINKPRM-14B with that of DiscPRM-14B, where THINKPRM-14B's performance scales substantially better with two orders of magnitude fewer process labels. This primarily stems from THINKPRM's utilization of text generation and reasoning abilities of the underlying models.

While we train THINKPRM using only 1K data points, we investigate whether it will benefit from training on more data. Using the same data collection process described in Section 3.2, we collect and filter additional verification CoTs and obtain a total of 65K chains. We then finetune R1-Distill-Qwen-1.5B and R1-Distill-Qwen-14B on these for a single epoch while keeping all other training hyperparameters fixed. We then compare the resulting models to the 1K-trained version of THINKPRM under best-of-N selection on MATH-500. Figs. 22 and 23 in Appendix E.7 show clear performance boost from training on the 65K examples, showing that THINKPRM can further benefit from training on more data.

#### 5.2 Filtering based on process vs. outcome labels

Section 3.2 explains how we filter verification CoTs based on the match between the generated step-level decisions with the gold process labels. To validate this choice in yielding high-quality synthetic CoTs for training THINKPRM, we compare this process-based filtering to a version using final answer correctness, as done in GenRM (Zheng et al., 2024), where we ignore step-level labels, and only consider the matching between the final step decision and the solution correctness. Precisely, this version only keeps chains if the solution final answer is correct and the final step label is \boxed{correct} or the final answer is incorrect and the final step label is \boxed{incorrect}, thereby ignoring intermediate step labels when filtering the generated data.

In total, we got 65K and 128K chains from process- and outcome-based filtering, respectively. Fig. 10 shows beam search performance of the resulting PRM from finetuning R1-Distill-Qwen-1.5B under the two filtering strategies. Clearly, data filtered with outcome only yields substantially worse verifier compared to data filtered based on process labels, even when containing almost 2x more examples. This emphasizes the quality of the synthetic data needed to train generative PRMs and the benefit of process labels in the filtering of data.

#### 5.3 THINKPRM helps with difficult reasoning problems

The hypothesis is that THINKPRM's reasoning process enables it to tackle verification of harder problems. To check if this is the case, we analyze performance of THINKPRM vs. DiscPRM in light of problem difficulty over MATH-500 and GPQA-physics (details on how we estimate problem difficulty for GPQA-Physics are in Appendix E.8), shown in Fig. 12. The generators here are Qwen-2.5-14B for MATH-500 and Qwen-2.5-32B-Instruct for GPQA-Physics. Primarily, THINKPRM improves reasoning on the *difficult* (levels 3, 4, 5 in MATH-500 and 2, 3, 4 in GPQA-Physics), with which DiscPRM struggles. This is likely due to the reasoning power of THINKPRM, which allows it to reason more effectively about the harder problems and their solutions. This could potentially hint at a compute-optimal verifier scaling strategy, where verification compute is allocated based on the query difficulty.

#### 5.4 Limitations of Generative PRMs

We note a few limitations with generative PRMs that may impact their usage in different reasoning setups. First, overconfidence is a known issue in LLMs (Liu et al., 2023a; Stechly et al., 2023; Zhou et al., 2024) and, in the case of PRMs, it can cause the predicted PRM scores to cluster near extremes:

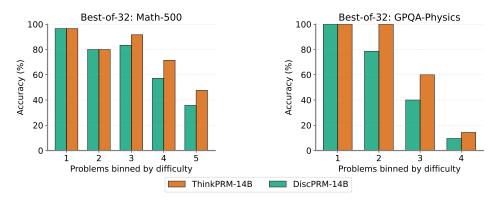


Figure 12: THINKPRM helps with challenging reasoning problems compared to DiscPRM. The generator model here is Qwen-2.5-14B for MATH-500 and Qwen-2.5-32B-Instruct for GPQA.

close to either 0 or 1. One reason is that we are using probabilities of certain tokens such as "yes" or "no", which by nature will be either very high or very low. Future work should explore better ways to extract better calibrated scores from generative reward models. Another limitation is a result of the autoregressive nature of their reasoning, which can lead them to prematurely commit to an earlier judgment. For example, we observe a phenomenon we term *step label interference*, where verification errors in earlier solution steps could impact verification of later steps. For example, we noted that if the PRM judges a particular step as incorrect, it becomes more likely to label subsequent steps as incorrect. Lastly, generating a verification chains-of-thought introduces additional computational overhead compared to discriminative PRMs. However, the performance gains offered by generative PRMs compared to the baselines justifies this extra cost.

#### Conclusion

This work introduced THINKPRM, a generative process reward model trained with minimal supervision on synthetic data, enabling efficient and scalable verification of step-by-step reasoning. Through extensive evaluation, we demonstrated that lightweight fine-tuning of generative PRMs on as few as 8K process labels can substantially improve upon zero-shot LLM-as-a-judge baselines. Importantly, our approach also surpasses discriminative PRMs trained with orders of magnitude more process labels, highlighting the advantages of utilizing generative language-modeling objectives for interpretability, scalability, and data efficiency. Our results underscore the potential of generative PRMs to effectively scale verification compute at test-time, particularly benefiting challenging domains such as mathematical and scientific reasoning.

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## A Training data

#### A.1 Sampling

Fig. 5 shows the prompt used with QwQ-32B-Preview to sample verification chains for training. We use the problem-prefix pairs from PRM800K train split (Lightman et al., 2023), which is based on MATH (Hendrycks et al., 2021). We sample 4 verification chains for each prefix in the dataset with temperature T=0.1, until we are left with 1K verification CoTs that (1) follow the desired format i.e., the model generated an extractable label for every step in the prefix and (2) the verbalized step labels match the gold labels in the PRM800K dataset annotation, (3) at most 4096 tokens long, and are (4) roughly balanced with respect to final answer correctness. Table 1 shows the statistics of the final dataset. We also show an example of 1K training CoTs in Fig. 13.

#### A.2 Preprocessing

From the sampled verification CoTs, we extract boxed predictions (e.g., \boxed{correct}) or \boxed{incorrect}) for each reasoning step. Then we filter data as follows:

- CoTs with missing or malformed verification decisions are discarded.
- CoTs where verification decisions don't match ground truth labels are filtered out. Precisely, we require all step-by-step verification decisions to match the ground truth labels.

Then we clean the data as follows:

- We standardize boxed notation formats.
- Content after the last verification decision is removed.
- Special tokens (e.g., <think> and </think>) are added to designate verification reasoning.

<b>Correctness Distribution</b>	
# correct	486 (48.6%)
# incorrect	514 (51.4%)
Step label distribution	
# correct	7474 (92.3%)
# Incorrect	625 (7.7%)
Prefix statistics	
# unique questions	869
average steps per prefix	9.5
min # steps	1
max # steps	53
Verification Chain Statistics	
Average verification chain length (tokens)	1037.0
Min verification chain length (tokens)	207
Max verification chain length (tokens)	3669

Table 1: Statistics of the 1K finetuning dataset.

```
Synthetic training sample
Problem: Find the area in square feet of a square with a perimeter of 32ft.
Solution:
Step 1: A square has four equal sides, so the perimeter is four times the length of one side.
Step 2: I can write an equation to relate the perimeter and the side length: 4s = 32, where s is the side
length in feet.
Step 3: To solve for s, I can divide both sides by 4: s = 32/4 = 8.
Step 4: Now I know the side length of the square is 8 feet.
Step 5: The area of a square is the side length squared, so the area is s^2 = 8^2 = 64.
Step 6: The area is in square feet, so the units are ft<sup>2</sup>.
Step 7: The answer is 64 ft<sup>2</sup>
Alright, let's verify step by step the solution provided for finding the area of a square with a perimeter
of 32 feet.
Step 1: "A square has four equal sides, so the perimeter is four times the length of one side."
Critique: This statement is accurate. A square indeed has four sides of equal length, and the perimeter
is the sum of all sides. Therefore, perimeter (P) equals 4 times the side length (s), or P = 4s.
The step is \\boxed{correct}
Step 2: "I can write an equation to relate the perimeter and the side length: 4s = 32, where s is the side
length in feet."
Critique: Given that the perimeter is 32 feet, setting up the equation 4s = 32 is correct. It properly
represents the relationship between the perimeter and the side length of the square.
The step is \\boxed{correct}
Step 3: "To solve for s, I can divide both sides by 4: s = 32/4 = 8."
Critique: This step correctly solves for s by dividing both sides of the equation by 4. The calculation 32
divided by 4 equals 8 is accurate.
The step is \\boxed{correct}
Step 4: "Now I know the side length of the square is 8 feet."
Critique: Based on the previous step, s = 8 feet, which is correctly stated here.
The step is \\boxed{correct}
Step 5: "The area of a square is the side length squared, so the area is s^2 = 8^2 = 64."
Critique: The formula for the area of a square is indeed s squared. Calculating 8 squared equals 64 is
correct.
The step is \\boxed{correct}
Step 6: "The area is in square feet, so the units are ft<sup>2</sup>."
Critique: Since the side length is in feet, the area will be in square feet (ft²). This is the correct unit for
area.
The step is \\boxed{correct}
Step 7: "The answer is 64 ft2"
Critique: This is the final answer, correctly stating the area as 64 square feet.
```

Figure 13: Training example from our sampled 1K synthetic chains.

**The step is** \\boxed{correct}

Is the solution correct? Yes

</think>

#### Data generation prompt

You are given a math problem and a proposed multiple-step solution (with a step on each line):

```
[Math Problem] {problem}
```

#### [Solution]

{solution}

Review and critique the proposed solution steps and determine whether each step is correct. If the solution is incomplete, only critique the steps that are provided. Your output must be in the following format:

```
Let's verify step by step:

Step 1: <critique>...The step is \boxed{correct/incorrect}

Step 2: <critique>...The step is \boxed{correct/incorrect}

...

Step n: <critique>...The step is \boxed{correct/incorrect}
```

Once you find an incorrect step, you should stop since you don't need to analyze the remaining steps.

Figure 14: Prompt used to generate synthetic verification chains as described in Section 3.2.

### **B** LLM-as-a-Judge Limitations

We show examples of failures from QwQ-32B-Preview and Deepseek-R1-Distill-Qwen-14B when used for LLM-as-judge verification in Figs. 16 and 18.

## C Training details

#### C.1 Training THINKPRM

We train our models on the filtered 1K chains using LoRA (Hu et al., 2022) with rank r=32 and  $\alpha=16$ . Specifically, we finetune Qwen-14B-Instruct and R1-Deepseek (14B) for 3 epochs, and QwQ-32B-Preview for 5 epochs using LoRA. We use an effective batch size of 16 and a fixed learning rate of  $4\times10^{-4}$  without warmup is used. Training took 1.5 hours for the 14B models and 4.5 hours for QwQ-32B-Preview on a single A100 GPU. Without particularly found QwQ to be hard to train with LoRA and still generates a relatively high percentage of invalid judgments after training. Full training of the model will likely resolve these issues but that would require more compute than we have.

The R1-Distill-Qwen $\{1.5B,7B\}$  models use *full* training with the following parameters. The 1.5B model We trained for 3 epochs with an effective batch size of 32, using a constant learning rate of  $6\times10^{-5}$  without decay or warmup. We train both models using four RTX A6000 GPU using data parallel. Training the 1.5B model on the 1K chains took about half an hour and the 7B model about two hours.

### **C.2** Training Discriminative Verifiers

We train R1-Qwen-14B for 1 epoch over the entire PRM800K dataset using two A100 80GB GPUs with a batch size of 8 and a learning rate of  $6 \times 10^{-5}$ . We use a constant learning rate scheduler with no warmup. Following prior work (Wang et al., 2023b; Zhang et al., 2025) We train the model using binary cross-entropy to maximize the probability of the tokens "+" and "-" for correct and incorrect steps, respectively. The R1-Qwen-1.5B model is trained with the same infrastructure with a batch size of 64 and a learning rate of  $1 \times 10^{-4}$  with a warm up ratio of 10%.

```
QwQ-32B-Preview stuck in a loop
Okay, so I need to analyze each step of the proposed solution to the math problem. The problem is: If f(x) = \frac{3x-2}{x-2}, what is the
value of f(-2) + f(-1) + f(0)? The proposed solution is given, and I have to check if each step is correct.
First, let me read through the problem and the solution carefully.
The function is f(x) = \frac{3x-2}{x-2}. They want the sum of f(-2) + f(-1) + f(0).
Looking at the solution steps:
Step 0: They plug in x=-2 into the function and calculate \frac{3(-2)-2}{-2-2}=\frac{-8}{-4}=2.
Step 1: They mention that plugging x = -2 into the denominator gives zero, which would be division by zero.
Step 2: They generalize that we should avoid plugging in values that make the denominator zero.
Step 3: Similarly, plugging x = -1 into the denominator also gives zero.
Step 4: Plugging x = 0 into the denominator also results in division by zero.
Wait no! The original function is given as f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
No! The problem says: If f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
No! The problem says: If f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
No! The problem says: If f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
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No! The problem says: If f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
No! The problem says: If f(x) = \frac{\text{numerator}}{\text{denominator}} = ?
```

Figure 15: Degeneration behavior from QwQ-32B-Preview with LLM-as-judge verification. Problem and solution are omitted for brevity.

### D Results on ProcessBench before and after finetuning

Table 2 shows the performance numbers of LLM-as-a-Judge and THINKPRM on ProcessBench.

### E Evaluation details

This section includes exact details on the test-time scaling shown in Section 4.2

#### **E.1** Predicting verification labels

Following prior work (Snell et al., 2024; Beeching et al.), we aggregate scores from DiscPRM by using the score of the *last* step. For THINKPRM, we first prompt the model to generate the verification chain up to a maximum of 8192 tokens, then we force decode the string "Is the solution correct?" and use  $\frac{P(\text{"yes"})}{P(\text{"yes"})+P(\text{"no"})}$  as the solution score.

#### E.2 Best-of-N selection

We sample solutions using a temperature of T=0.8 for Llama-3.2-3B-Instruct and T=0.4 for Qwen-2.5-14B. We instruct all models to think step by step and put the final answer in \boxed{}. All

```
Deepseek-R1-Distill-Qwen-14B stuck in a loop
Okay, so I need to review and critique the solution step by step for the given math problem. The problem is to find the exact value of
the continued fraction expression:
1 + 1/(1 + 2/(1 + 1/(1 + 2/(1 + ...))))
The solution provided is split into 16 steps, each enclosed in a tag. I need to go through each step, analyze it, and see if there are any
errors or if the logic holds. After that, I'll determine if the solution is correct by outputting "yes" or "no" in a box.
Let me start by understanding the problem. It's a continued fraction that alternates between 1 and 2 in the numerators. The expression
So, the pattern repeats every two steps: 1, then 2, then 1, then 2, etc. The solution starts by denoting the expression as x and then tries
to find a relationship by substituting the inner expression back into x.
Wait, the original expression is x = 1 + 1/(1 + 2/(1 + 1/(1 + 2/(1 + ...)))). So, the part after the first 1 is 1 + 2/(1 + 1/(1 + 2/(1 + ...))). But that part isn't x, because x starts with 1 + 1/(...). So, maybe the substitution isn't directly x. Let me think.
Wait, perhaps the inner part is similar but not exactly x. Let me denote the inner part as y. So, y = 1 + 2/(1 + 1/(1 + 2/(1 + ...))). But
then, the expression for y would involve another similar structure. Hmm, maybe the solution is oversimplifying here.
Wait, actually, when you multiply both sides by (1 + 2/x), the left side is x*(1 + 2/x) = x + 2.
The right side is [1 + 1/(1 + 2/x)]^*(1 + 2/x) = 1^*(1 + 2/x) + [1/(1 + 2/x)]^*(1 + 2/x) = (1 + 2/x) + 1 = 2 + 2/x.
So, the correct equation after multiplying both sides by (1 + 2/x) is:
x + 2 = 2 + 2/x.
But in step 6, they have:
x + 2 = 1 + 1/(1 + 2/x) + 2/x + 4/x^2
Which seems to be incorrect. They seem to have distributed the multiplication incorrectly.
Wait, let me see. If you have x*(1 + 2/x) = [1 + 1/(1 + 2/x)]*(1 + 2/x) + [2/x*(1 + 2/x)]
But that doesn't make sense because when you multiply both sides by (1 + 2/x), you should only have:
Left side: x*(1 + 2/x)
Right side: [1 + 1/(1 + 2/x)]*(1 + 2/x)
Which simplifies to x + 2 = 2 + 2/x.
But in the solution, step 6 says:
x*(1 + 2/x) = 1 + 1/(1 + 2/x) + 2/x*(1 + 2/x)
Which seems to be adding an extra term, 2/x*(1 + 2/x), which is not correct. So, step 6 is incorrect.
Therefore, the solution has an error in step 6. Instead of correctly multiplying both sides, they incorrectly expanded the right-hand
side, leading to an incorrect equation.
Because of this mistake, the subsequent steps are likely incorrect as well. So, the entire solution is flawed from step 6 onwards.
Wait, let me compute the continued fraction numerically to check.
Let me compute the continued fraction up to a certain depth.
Let me compute it step by step:
Start with the innermost part and work outwards.
So, it seems to converge to approximately 2.414, which is 1 + \text{sqrt}(2) \approx 2.4142.
Wait, but according to my earlier solving, x should be sqrt(2) \approx 1.414, but the numerical computation suggests it converges to
approximately 2.414, which is 1 + \text{sqrt}(2). So, which one is correct?
Wait, perhaps I made a mistake in setting up the equation.
Wait, let's re-examine the original expression:
But when I tried to solve it earlier, I set x = 1 + 1/(1 + 2/x), which led to x = \text{sqrt}(2). But the numerical computation suggests it
converges to 1 + sqrt(2). So, perhaps my initial substitution was incorrect.
Wait, perhaps I need to define it differently. Let me try to define the continued fraction properly.
Wait, let's solve the system:
From x = 1 + 1/y and y = 1 + 2/x.
Substitute y = 1 + 2/x into x = 1 + 1/y:
x = 1 + 1/(1 + 2/x)
Multiply both sides by (1 + 2/x):
Alternatively, perhaps I need to define y differently. Let me try to define y as the expression starting with 1 + 1/(...):
y = 1 + 1/(1 + 2/(1 + 1/(1 + 2/(1 + ...))))
But then, y = x, which brings us back to the original substitution.
Which is the same equation as before, leading to x = sqrt(2). But the numerical computation suggests it's approaching 1 + sqrt(2). So,
perhaps my initial substitution is incorrect.
Alternatively, perhaps the continued fraction is structured such that after the first 1 + 1/(...), the denominator is another continued
fraction similar to x but shifted.
```

Figure 16: Degeneration behavior from Deepseek-R1-Distill-Qwen-14B with LLM-as-judge verification. Problem and solution are omitted for brevity.

**Problem:** Let a,b be positive reals with  $a>b>\frac{1}{2}a$ . Place two squares of side lengths a,b next to each other, such that the larger square has lower left corner at (0,0) and the smaller square has lower left corner at (a,0). Draw the line passing through (0, a) and (a + b, 0). The region in the two squares lying above the line has area 2013. If (a, b) is the unique pair maximizing

#### Solution:

We begin by finding the equation of the line through (0, a) and (a + b, 0). The slope is:

$$m = \frac{0-a}{(a+b)-0} = -\frac{a}{a+b}$$

Using point-slope form from (0, a):

$$y = a - \frac{a}{a+b}x$$

Area above the line in the larger square:

At x = a, the height is:

$$y = a - \frac{a^2}{a+b} = \frac{ab}{a+b}$$

So the area is a trapezoid:

$$\operatorname{Area}_1 = \frac{1}{2} \left( a + \frac{ab}{a+b} \right) a = \frac{a^3 + 2a^2b}{2(a+b)}$$

Area above the line in the smaller square: This is a triangle with height  $\frac{ab}{a+b}$  and base b:

$$Area_2 = \frac{1}{2} \cdot \frac{ab}{a+b} \cdot b = \frac{ab^2}{2(a+b)}$$

Total Area:

$$\text{Total} = \text{Area}_1 + \text{Area}_2 = \frac{a^3 + 2a^2b + ab^2}{2(a+b)} = \frac{a(a^2 + 2ab + b^2)}{2(a+b)} = \frac{a(a+b)^2}{2(a+b)} = \frac{a(a+b)^2}{2}$$

We are given:

$$\frac{a(a+b)}{2} = 2013 \Rightarrow a(a+b) = 4026$$

Express b in terms of a:

$$b = \frac{4026}{a} - a$$

Apply the constraints  $a > b > \frac{1}{2}a$ :

• 
$$a > b \Rightarrow a > \frac{4026}{a} - a \Rightarrow a^2 > 2013 \Rightarrow a > \sqrt{2013}$$

• 
$$b > \frac{1}{2}a \Rightarrow \frac{4026}{a} - a > \frac{1}{2}a \Rightarrow a^2 < 2684 \Rightarrow a < \sqrt{2684}$$

So  $\sqrt{2013} < a < \sqrt{2684}$ . To maximize  $a+b=\frac{4026}{a}$ , we minimize a, so let  $a=\sqrt{2013}$ . Then  $b=\sqrt{2013}$ , satisfying the

$$\frac{a}{b} = \boxed{2}$$

Figure 17: An example of problem and solution from OmniMath subset of ProcessBench.

#### Incorrect verification

Okay, so I need to review and critique the given solution step by step. Let me go through each step carefully.

Step 0: The solution starts by identifying key points and properties. It mentions that J is the excenter opposite to A, which is correct.  $A_1$  is where the excircle touches BC, and  $B_1$ ,  $C_1$  are where it touches the extensions of AC and AB. Then, it says  $A_1B_1$  is perpendicular to AB and intersects at D. E is the foot of the perpendicular from  $C_1$  to DJ.

Hmm, this seems okay, but I wonder if they've correctly identified all the properties. Maybe they should have mentioned more about the excenter's properties, like the fact that it's the intersection of external angle bisectors. But maybe that's assumed knowledge.

**Step 1:** They note that since  $A_1B_1$  is perpendicular to AB, D is the projection of  $A_1$  onto AB. So, D lies on AB and  $A_1D$  is perpendicular to AB.

Wait, is that necessarily true? If  $A_1B_1$  is perpendicular to AB, and they intersect at D, then yes, D is the foot of the perpendicular from  $A_1$  to AB. That seems correct.

Step 2: They examine the excircle and the triangle, mentioning that the excircle is tangent to BC at  $A_1$ , and the tangents from  $A_1$  to  $B_1$  and  $C_1$  form right angles with the radii. Then, DJ is a line through the excenter and the intersection point of the perpendiculars from  $A_1$  and  $B_1$  to AB.

Hmm, I'm not sure about this part. The line DJ is through the excenter J and point D, which is the foot from  $A_1$ . But does it involve perpendiculars from  $A_1$  and  $B_1$ ? I think the solution might be conflating different properties here. The excenter is equidistant from all sides, but DJ is just a line from D to J, not necessarily involving perpendiculars from both  $A_1$  and  $B_1$ . Maybe this is a misstatement.

Step 3: They state that since E is the foot of the perpendicular from  $C_1$  to DJ, angle  $C_1ED$  is 90 degrees. Then, they argue that angles  $BEA_1$  and  $AEB_1$  are equal due to symmetry and properties of the cyclic quadrilateral.

Wait, why is E symmetrically placed? The solution says it's because of the properties of the excircle and right angles, but I'm not entirely convinced. Just because E is a foot of a perpendicular doesn't automatically make it symmetric with respect to  $A_1B_1$ . Maybe there's a better reason, like cyclic quadrilaterals or similar triangles, but this step feels a bit hand-wavy.

Step 4: They conclude both angles are 45 degrees because of symmetry and right angles, forming a cyclic quadrilateral. But I'm not sure if the cyclic quadrilateral part is clearly established. Also, 45 degrees seems specific—does the problem necessarily lead to that? Wait, maybe I should draw a diagram to visualize. If  $A_1B_1$  is perpendicular to AB, and E is the foot from  $C_1$  to DJ, which is connected to the excenter. Hmm, maybe the angles are indeed 45 degrees because of the right angles and the properties of the excentral triangle, but I'm not entirely certain.

Overall Thoughts: The solution seems to have some correct steps but might be making assumptions without sufficient justification. For example, in Step 3, the symmetry argument isn't fully explained. Also, in Step 2, the description of DJ might not be accurate. The conclusion of 45 degrees might be correct, but the reasoning feels a bit rushed.

**Final Answer** After considering the steps, I think the solution might have some gaps in justification but arrives at the correct answer. So, I'll say yes, but with some reservations.

yes

Figure 18: Incorrect verification by Deepseek-R1-Distill-Qwen-14B for the problem-solution pair in Fig. 17.

Model	LLM-as-a-Judge		THINKPRM	
	OlympiadBench	OmniMath	OlympiadBench	OmniMath
Random baseline	39.1	32.7	39.1	32.7
R1-Qwen-1.5B R1-Qwen-7B R1-Qwen-14B QwQ-32B-preview	5.0 (51.4 %) 44.8 (18.2 %) <b>72.8</b> (13.3 %) 50.6 (7.9 %)	5.4 (55.1 %) 45.7 (20.9 %) <b>67.8</b> (18.6 %) 55.5 (10.9 %)	76.3 (1.4 %) 73.4 (1.1 %) <b>87.3</b> (2.3 %) 73.1 (15.1 %)	75.7 (2.4 %) 74.0 (1.4 %) <b>85.7</b> (2.3 %) 73.2 (7.9 %)

Table 2: Average F1-score on OlympiadBench and OmniMath subsets of ProcessBench (Zheng et al., 2024) comparing LLM-as-a-Judge to THINKPRM finetuned on 1K examples. Random baseline for OlympiadBench is 39.1% and for OmniMath is 32.7%. Percentage of bad outputs (repetitions, invalid label formatting, overthinking, etc.) are shown in red. LLM-as-a-judge with reasoning models suffer from issues that limits their utility as generative verifiers.

our Best-of-N experiments use weighted majority voting (Sun et al., 2024) except for our experiments on AIME '24, where we use the verifier score directly to rank the solution, as we found this to perform better for all verifiers.

#### E.3 Verifier-guided beam search

Under verifier-guidedbeam search, we sample candidate next steps and score them with the process verifier, then selects top-K out of these to further expand and so on. Our implementation is based on (Snell et al., 2024; Beeching et al.), which maintains N beams in total, and samples M candidate next steps per beam. We set M=4 for all experiments and run search for a maximum of 20 steps per beam. To sample next steps, we use T=0.6 and use double newlines as the step delimiter.

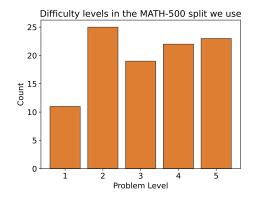
#### **E.4** Sequential scaling of verifier compute

We achieve budget forcing (Muennighoff et al., 2025) by triggering the model to think again for R rounds, where each round uses a unique trigger phrase that incites the model to revisit or double-check its earlier verification. We use different trigger phrases for each round since we found that using the same phrase causes the model to repeat what it did in the last round.

We do a maximum of R=4 thinking rounds, and use the phrases "Let me double check", "Let's verify again", and "Did I miss something?", for rounds 2, 3, and 4 respectively. We do not investigate deeply into optimizing the trigger phrase, but we note that performance may depend on these and we use the same phrases for both THINKPRM and LLM-as-a-judge to ensure fair comparison.

#### E.5 MATH-500 test set

As running on all 500 examples from MATH-500 will require a lot of compute, we run all our experiments on 100 randomly sampled subsets from MATH-500 (Hendrycks et al., 2021). We pick the 100 problems such that they cover different difficulty levels, as shown in Fig. 19.



Parallel scaling of verifier compute: MATH-500

85
80
75
75
60
60
55
0
20
21
22
23
24
25
Number of solutions

ThinkPRM-14B
— ThinkPRM-14B@4
— ThinkPRM-14B@8

Figure 19: Histogram of difficulty levels in our 100-problem subset from MATH-500.

Figure 20: Scaling of verifier compute by parallel sampling of multiple verification CoTs and averaging their scores. Parallel scaling (THINKPRM-14B@4 and THINKPRM-14B@8) further boosts performance curve compared to scoring based on a single CoT (THINKPRM-14B).

#### E.6 Additional results on scaling verifier compute

Fig. 20 shows results of THINKPRM-14B when parallely scaling verifier compute by sampling K=4 and K=8 CoTs with temperature T=0.6 and aggregating their scores. Parallel scaling indeed lifts up the accuracy curve of THINKPRM-14B compared to standard K=1 with greedy decoding. However, performance plateaus rather quickly and K=8 remains comparable to K=4, while slightly better at smaller sampling budgets. Fig. 21 compares parallel to sequential scaling under the same token budget. While there is no clear winner, parallel scaling seems to perform slightly better at best-of-8.

#### E.7 Scaling training data of THINKPRM

Here, we show results when training THINKPRM-14B and THINKPRM-1.5B using synthetic data from all PRM800K. The goal is to show that THINKPRM can still benefit from training on more synthetic data. Here, we train both R1-Distill-Qwen-1.5B and R1-Distill-Qwen-14B on a total of 65K verification CoTs we obtained by sampling and filtering as explained in Section 3.2. Figs. 22 and 23 show best-of-N performance with THINKPRM-1.5B and THINKPRM-14B respectively, but when trained on 65K and compares it to training on 1K examples—this is the version we use for all other results in the paper. Interstingly, THINKPRM benefits from additional training, and can further improves the accuracy curve compared to the 1K-trained version on MATH-500. We note,

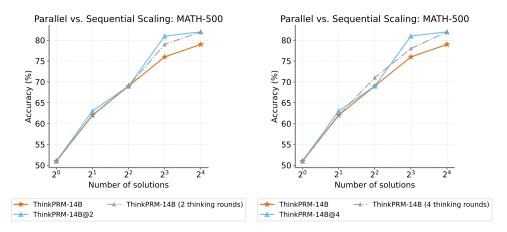


Figure 21: Parallel vs. sequential scaling of THINKPRM compute under the same generation budget with Qwen-2.5-14B generator. Parallel scaling (model@K) is done by independently sampling K verification CoTs and aggregating their scores. Sequential scaling is done by prompting the model K times to revise its own verification for K thinking rounds. Both setups generate up until 8192 tokens per generation. We do not observe a clear winner although parallel scaling seems slightly better especially at larger sampling budgets.

however, that while training on more math data boosts performance on MATH-500, we observe some performance drop on out-of-domain tasks due to the distribution shift.

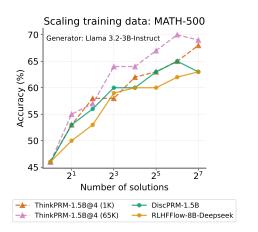


Figure 22: Best-of-N results with THINKPRM-1.5B comparing the version trained on 1K examples (used throughout the paper) and a version trained on 65K examples. THINKPRM benefits from training on more synthetic data as the performance can further improve with more training.

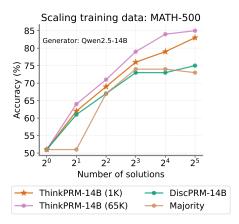


Figure 23: Best-of-N results with THINKPRM-14B comparing the version trained on 1K examples (used throughout the paper) and a version trained on 65K examples. THINKPRM benefits from training on more synthetic data as the performance can further improve with more training.

### E.8 Verifier performance in terms of problem difficulty

We the difficulty We do not estimate the difficulty over MATH problem since each problem in MATH is annotated based on 1 of 5 difficulty levels. For GPQA-Physics problems, we first compute the pass@1 rate of Qwen2.5-32B-Instruct for every problem by sampling 32 solutions for each problem. Then, we bin the pass@1 rates into four different bins and use the bin assignment for each problem as the difficulty level. We discuss how the verifier accuracy changes with problem difficulty in Section 5.3.

	Prompt	OlympiadBench	OmniMath
	Single Yes/No	50.6 (7.9 %)	55.5 (3.2 %)
QwQ-32B	Bad-step-index	65.4 (18.3 %)	69.0 (16.8 %)
	Decision-for-each-step	40.1 (6.4 %)	36.6 (9.2 %)
R1-Qwen-14B	Single Yes/No	72.8 (13.3 %)	67.8 (18.6 %)
	Bad step index	77.7 (17.0 %)	76.6 (17.9 %)
	Decision-for-each-step	73.3 (14.6 %)	71.4 (14.7 %)

Table 3: F1-score on OlympiadBench and OmniMath subsets from ProcessBench with different prompts. Ratio of invalid judgments is shown in red.

## F More details on LLM-as-judge verification

The instruction used for LLM-as-judge verification in Section 3.1 is shown in Fig. 26, which we refer to as Single-label. Table 3 shows performance of QwQ-32B-Preview and Qwen-Deepseek-R1-14B with two more instructions. **Bad-step-index** refers to instructing the model to find the index of the first incorrect step and then if a step is found generate a single yes/no label regarding the solution correctness. **Decision-for-each-step** instructs the model to critique and produce a boxed label for each step. This instruction is shown in Fig. 14. We observe high ha variance in verification performance and invalid judgment ratio depending on the prompt.

We also an example from ProcessBench in Fig. 24 and the generated verification chain from QwQ-32B-Preview in Fig. 25.

#### Example problem-solution pair from ProcessBench

#### Problem:

In quadrilateral ABCD,  $m \angle B + m \angle D = 270^{\circ}$ . The circumcircle of  $\triangle ABD$  intersects  $\overline{CD}$  at point E, distinct from D. Given that BC = 4, CE = 5, and DE = 7, compute the diameter of the circumcircle of  $\triangle ABD$ .

#### Solution

Step 0: To solve this problem, we will use several properties of geometry, particularly related to circles and quadrilaterals. Let's break down the steps: First, we understand the given information. We have a quadrilateral ABCD where  $\angle B + \angle D = 270^{\circ}$ . The circumcircle of  $\triangle ABD$  intersects  $\overline{CD}$  at point E, distinct from D. Additionally, BC = 4, CE = 5, and DE = 7.

Step 1: Next, we apply properties of cyclic quadrilaterals and circles. Since E lies on the circumcircle of  $\triangle ABD$ , E is also part of the circumcircle of  $\triangle ABCD$  (assuming ABCD is cyclic, which will be justified later). In a cyclic quadrilateral, opposite angles sum up to  $180^\circ$ . Given  $\angle B + \angle D = 270^\circ$ , it suggests an error in our initial understanding or misinterpretation, since angles cannot sum to more than  $360^\circ$  in a quadrilateral. However, if we consider the context, let's assume a correction was intended for the sake of explanation: Correct interpretation: If we adjust for logical consistency, perhaps the intention was to imply that one of the angles at B or D is  $90^\circ$  (since this would fit the property of cyclic quadrilaterals better, aligning with the geometry principles involved). For this explanation, let's proceed with an assumption that either  $\angle B$  or  $\angle D$  is  $90^\circ$ , aiming for a coherent solution path.

Step 2: We then use the Power of a Point Theorem. The Power of a Point Theorem states that for a point P outside a circle, the product of the lengths of the two segments from P to the circle equals the product of the lengths of the other two segments when considering tangents or secants from that point. However, in this case, E is not outside but on the circumference of the circle, and we apply it differently. Given BC=4 and CE=5, DE=7, and knowing E is part of the circumcircle, we can apply the property of intersecting chords in a circle, which states that the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord. Thus, in this scenario, we consider BC and CD as intersecting chords, where CD=BC+CE=4+5=9.

$$BC \cdot CD = CE \cdot DE$$

Substituting the known values:

$$4 \cdot 9 = 5 \cdot DE$$

$$36 = 5 \cdot DE$$

Thus,

$$DE = \frac{36}{5} = 7.2.$$

**Step 3:** This discrepancy suggests a reevaluation of the given values or a misunderstanding in their application within the context provided. However, for the purpose of demonstrating the method:

Step 4: Finally, we find the diameter of the circumcircle of  $\triangle ABD$ . The key to finding the diameter involves recognizing that the circumcircle's radius or diameter is crucial for calculating distances involving points A, B, and D. However, without specific details about how E relates to these points geometrically beyond being on the circle, or additional information such as the length of another segment or angle measures, directly calculating the diameter becomes complex.

Step 5: The question's focus on E and its relation to BC, CE, and DE might suggest looking at the properties of chords in a circle or applying trigonometric relationships based on the angles given initially. However, due to the discrepancy noted and the complexity introduced by the misinterpretation of the angle condition, a direct calculation for the diameter of the circumcircle of  $\triangle ABD$  isn't straightforward with the information provided.

Step 6: Given the complexities and discrepancies in the original problem statement, particularly with the angles summing to  $270^{\circ}$  in a quadrilateral, the solution path becomes less clear. The explanation provided aimed to creatively interpret and apply geometric principles under the constraints of the given information. However, the direct calculation for the diameter of the circumcircle of  $\triangle ABD$  requires additional, precise information about the positions and relationships of the points A, B, C, D, and E within the quadrilateral and its circumcircle.

Figure 24: An example problem and solution from the OlympiadBench (He et al., 2024) in ProcessBench (Zheng et al., 2024).

#### Example from QwQ-32B-Preview

Let's verify step by step:

Step 0: The initial step seems to correctly identify the given information in the problem. It mentions the quadrilateral ABCD with angles at B and D summing to  $270^{\circ}$ , and the circumcircle of  $\triangle ABD$  intersecting  $\overline{CD}$  at E, distinct from D. The lengths BC=4, CE=5, and DE=7 are also noted. This step is accurate in summarizing the problem statement.

Step 1: This step attempts to apply properties of cyclic quadrilaterals but seems to get confused about the sum of angles in quadrilateral ABCD. It is mentioned that in a cyclic quadrilateral, opposite angles sum to  $180^{\circ}$ , but the given sum of angles  $\angle B$  and  $\angle D$  is  $270^{\circ}$ , which indeed seems inconsistent. However, the step then suggests assuming that one of the angles is  $90^{\circ}$  for the sake of explanation, which is an adjustment to proceed with the solution. This part is a bit problematic because it alters the given information, but perhaps there was a misinterpretation in the problem statement.

Step 2: Here, the Power of a Point Theorem is applied to points related to the circle. However, there is confusion in calculating the length of  $\overline{CD}$ . It states that CD=BC+CE=4+5=9, but since E is a point on  $\overline{CD}$  distinct from D, and DE is given as 7, this suggests that CD=CE+DE=5+7=12, not 9. This is an error in calculation. Furthermore, the application of the Power of a Point Theorem seems misapplied because E is on the circle, not outside it, so the theorem should be applied differently, possibly considering intersecting chords.

Step 3: Recognizing the discrepancy in the previous step, the solution acknowledges the need for reevaluation, which is a good practice. However, without correcting the earlier mistake in calculating  $\overline{CD}$ , the subsequent steps may be built on incorrect foundations.

Step 4: This step aims to find the diameter of the circumcircle of  $\triangle ABD$ , which is the main goal of the problem. It notes that the diameter is related to the circumradius, but without specific details about the triangle's sides or angles, it's challenging to compute directly. The solution hints at the need for more information, which is accurate.

Step 5: The insight here is that the relationship between E and the other points might provide the necessary information through chord properties or trigonometric relationships. However, due to the earlier discrepancies, this step does not lead to a concrete calculation.

**Step 6:** The conclusion reflects on the complexities and discrepancies in the problem and the solution approach. It acknowledges that the direct calculation for the diameter is not straightforward with the given information, which is a fair assessment.

Overall, while the solution attempt shows an understanding of relevant geometric principles, there are significant errors in calculation and application of theorems, particularly in Step 2. These errors propagate through the subsequent steps, leading to an inconclusive solution. Therefore, the solution is not correct.

\boxed{no}

Figure 25: Example verification chain generated by QwQ-32B-Preview in response to the problem-solution pair in Fig. 24.

## LLM-as-a-judge prompt

The following is a math problem and a solution (split into steps, enclosed with tags and indexed from 0):

#### [Math Problem]

{problem}

### [Solution]

{prefix}

Your task is to review and critique the solution step by step. Your output must be in the following format:

Step 0: <critique>
Step 1: <critique>
...
Step n: <critique>

After your analysis, return yes/no depending on whether the solution is correct.

Please put your final answer (i.e., yes/no) in \boxed{}

Figure 26: Prompt used for LLM-as-a-judge verification in Section 3.1