Modelling and Networked Control of Water Distribution Networks

January 21, 2022

CA733

Control and Automation, Group 733
Aalborg University
Denmark

AALBORG UNIVERSITY
DENMARK

Agenda

Introduction

Modelling of WDN
- Fast Dynamics and Graph theory
- Slow Dynamics and Linearisation
  - Slow Dynamics
  - Linearisation

Control Structure and Root Locus

Optimal Control

Disturbance Estimator

Network Effects

Results
- Best-Case Results
- Pathological Synchronisation

References

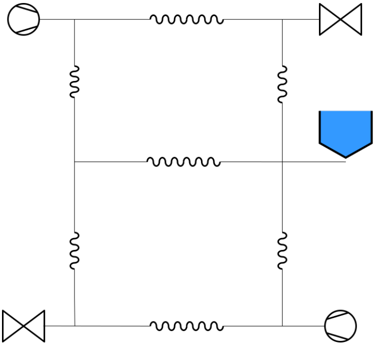
Modelling and Networked Control of Water Distribution Networks
CA733
Introduction 2
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Critical societal infrastructure, responsible for the provision of water to both domestic and industrial consumers.
- Network pressure must be controlled.
  - Underpressure $\rightarrow$ insufficient service pressure $\rightarrow$ dissatisfied end users.
  - Overpressure $\rightarrow$ component failure $\rightarrow$ repair costs, supply intermittency, etc.
- Network pressure tied to level in Elevated Water Reservoir(s) (EWR) $\rightarrow$ level control is key!
- Other key components are pumps, valves, and pipes.

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction 3
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

We analyse a small-scale test WDN with the following layout:



Larger WDNs are typically not amenable to first-principles modelling.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- General principles for modelling fast regime very similar to circuit analysis.
  - Water flows = currents.
  - Pressures = voltages.
  - Network components behave similarly to circuit components, but resistors generally non-linear.
- Graph theory is combined with physics-induced boundary conditions to yield model of fast regime.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory 5
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Graph theory describes a system as a directed graph.

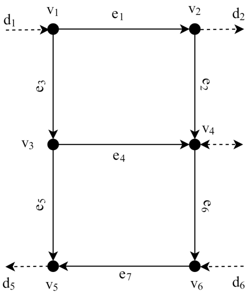


Figure: Graph of simplified WDN network. From [1].

- Spanning tree
- Chords

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Auxiliary matrices can be defined to mathematically describe the network:

$$H_{i,j} = \begin{cases} 1 & \text{If } j\text{th edge leaves } i\text{th node} \\ -1 & \text{If } j\text{th edge enters } i\text{th node} \\ 0 & \text{If } j\text{th edge and } i\text{th node unconnected} \end{cases}$$

$$B_{i,j} = \begin{cases} 1 & \text{If direction of } i\text{th loop and } j\text{th edge agree} \\ -1 & \text{If direction of } i\text{th loop and } j\text{th edge disagree} \\ 0 & \text{If } i\text{th loop excludes } j\text{th edge} \end{cases}$$

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Example:

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{1}$$

$$\bar{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} \tag{3}$$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Whether a node is open to atmosphere or not, or connected to a tank, can be described with matrices F and G respectively.

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The edges of the WDN can be described with a pressure and flow relationship, analogous to the current and voltage of an electrical component.

This project uses pipes, valves and pumps as edge components.

The general relationship between pressure and flow can be defined as:

$$\Delta p = \mathcal{J}\dot{q} + \lambda(q) + \mu(q, \Theta) + \alpha(q, \omega) - \Delta h \tag{5}$$

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The pressure drop across a pipe is defined as:

$$\Delta p_k = \mathcal{J}\dot{q} + \lambda(q) + \Delta z \tag{6}$$

where:

$$\lambda_k(q_k) = \left(f \cdot \frac{8 \cdot L \cdot q^2}{\pi^2 \cdot g \cdot D^5} + k_f \cdot \frac{8 \cdot q^2}{\pi^2 \cdot g \cdot D^4}\right) \cdot g \cdot \rho \tag{7}$$

$$\mathcal{J} = \frac{L \cdot \rho}{A} \tag{8}$$

$$\Delta z_k = \rho \cdot g \cdot \Delta h_k \tag{9}$$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The pressure drop across a valve is defined as:

$$\Delta p_k = \mu(q, \Theta) = \frac{1}{K_{valve}(\Theta)^2} \cdot |q| \cdot q \tag{10}$$

where Θ is the opening degree of the valve.
Derivation:

$$\frac{\Delta p_1}{q_1^2} = \frac{\Delta p_2}{q_2^2} \Leftrightarrow q_1 = q_2 \cdot \sqrt{\frac{\Delta p_1}{\Delta p_2}} \tag{11}$$

$$q = q_n(\Theta) \cdot \sqrt{\frac{\Delta p_1}{1}} = K_{valve}(\Theta) \cdot \sqrt{\Delta p_1} \tag{12}$$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory 12
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The pressure drop across a pump is defined as:

$$\Delta p_k = a_0 \cdot \omega^2 + a_1 \cdot \omega \cdot q - a_2 \cdot |q| \cdot q \tag{13}$$

where $[a_0, a_1, a_2]$ is a tuple of coefficients that describe the pump's characteristic curve, q is the flow rate through the pump, and ω is the rotational velocity of the pump.

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Kirchhoff's node and mesh law:

$$Hq = d \tag{14}$$

$$B\Delta p = BH^T p = 0 \wedge B\Delta h = BH^T h = 0 \tag{15}$$

Mass conservation:

$$d_n = -\sum_{i=1}^{n-1} d_i \tag{16}$$

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Lemma 4.1:

$$H_T \bar{H}_T^{-1} = \begin{bmatrix} I_{n-1} \\ -\mathbf{1}^T \end{bmatrix} \tag{17}$$

where $\mathbf{1}$ is a vector of ones and $I_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ is an identity matrix.

Lemma 4.2:

$$q = B^T q_C + \begin{bmatrix} 0_{C \times n-1} \\ \bar{H}_T^{-1} \end{bmatrix} \bar{d} \tag{18}$$

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Non linear differential equation:

$$\Phi \mathcal{J} \Phi^T \dot{q} = -\Phi\Big(\lambda(q_n) + \mu(q_n) + \alpha(q_n)\Big) + \Psi(\bar{h} - \mathbf{1} h_0) + \mathcal{I}(p_\tau - \mathbf{1} p_0) \quad (19)$$

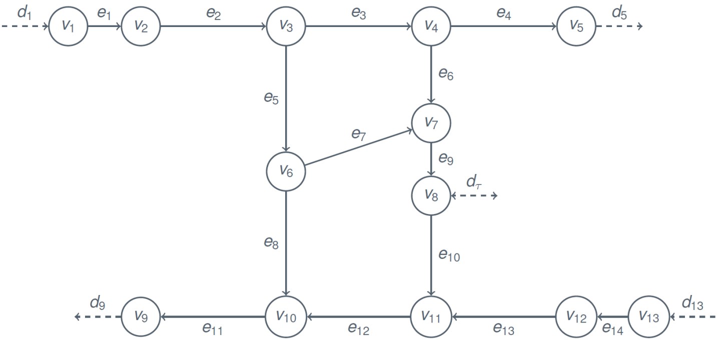
Where the matrices $\Phi, \Psi, \mathcal{I}$ are defined as:

$$\Phi \triangleq \begin{bmatrix} I & -\bar{H}_C^T \bar{H}_T^{-T} \\ 0 & \bar{F}^T \bar{H}_T^{-T} \\ 0 & \bar{G}^T \bar{H}_T^{-T} \end{bmatrix}, \qquad \Psi \triangleq \begin{bmatrix} 0 \\ \bar{F}^T \\ \bar{G}^T \end{bmatrix}, \qquad \mathcal{I} \triangleq \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad (20)$$

$$\mathcal{P} : (\Phi \mathcal{J} \Phi^T)^{-1} \quad (21)$$

$$\dot{q}_n = -\mathcal{P}\Phi\Big(\lambda(q_n) + \mu(q_n) + \alpha(q_n)\Big) + \mathcal{P}\Big(\Psi(\bar{h} - \mathbf{1} h_0) + \mathcal{I}(p_\tau - \mathbf{1} p_0) \quad (22)$$

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory 16
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References



Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Fundamentals

$$p \propto h$$

$$\dot{V} = q$$

- Assume constant cross sectional area A

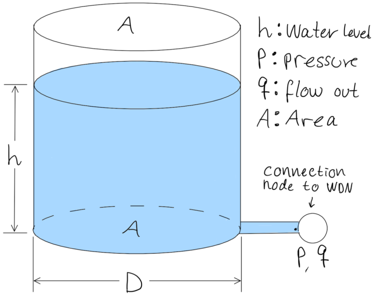
$$V \propto h \implies V \propto p$$

$$\dot{p} \propto \dot{V} \wedge \dot{V} = q \implies \dot{p} \propto q$$

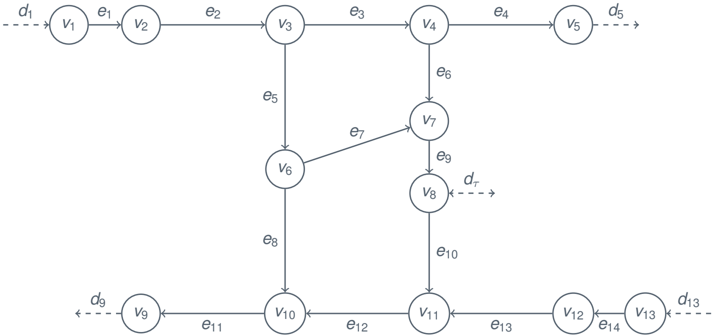
- We arrive at

$$\dot{p} = -\tau q, \text{ where}$$

$$\tau = \rho g \frac{1}{A}$$



Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References



In context of WDN we now consider flows to and from network as external demands d_i

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Mass conservation holds, and such

$$d_n = -\sum_{i=1}^{n-1} d_i \implies d_\tau = -(d_p + d_c)$$

$$\dot{p} = -\tau d_\tau = \tau(d_p + d_c)$$

When discretised by forward Euler:

$$p_\tau(k+1) = p_\tau(k) - \tau d_\tau(k) t_s = p_\tau(k) + \tau(d_p(k) + d_c(k)) t_s$$

Which corresponds to a discrete, linear state-space model:

$$p_\tau(k+1) = A p_\tau(k) + B_p d_p(k) + B_c d_c(k) \tag{23}$$

$$d_p = \begin{bmatrix} d_1 \\ d_{13} \end{bmatrix}, d_c = \begin{bmatrix} d_5 \\ d_9 \end{bmatrix}, B_p = B_c = t_s \begin{bmatrix} \tau & \tau \end{bmatrix}, A = 1$$

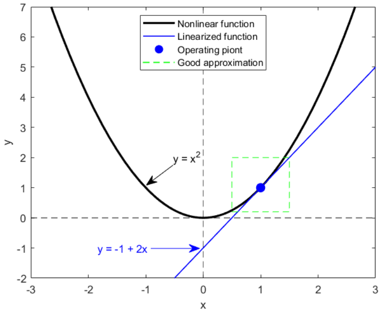
Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Fast dynamics are non-linear
- Linearisation required

In near vicinity of linearisation point x_0,

$$\dot{x} \approx f(x_0) + \nabla f\Big|_{x_0} (x - x_0)$$

Linearising around equilibrium point preferred



Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Recalling the fast dynamics differential equation is given as

$$\dot{q}_n = -\mathcal{P}\Phi\Big(\lambda(q_n) + \mu(q_n, \Theta) + \alpha(q_n, \omega)\Big) + \mathcal{P}\Big(\Psi(\bar{h} - \mathbf{1}h_0) + \mathcal{I}(p_\tau - \mathbf{1}p_0)\Big) \tag{24}$$

The linear model such becomes

$$\dot{q}_n \approx f(x_0) + \left.\frac{\partial f}{\partial q_n}\right|_{x_0} \tilde{q}_n + \left.\frac{\partial f}{\partial \Theta}\right|_{x_0} \tilde{\Theta} + \left.\frac{\partial f}{\partial \omega}\right|_{x_0} \tilde{\omega} + \left.\frac{\partial f}{\partial p_\tau}\right|_{x_0} \tilde{p}_\tau \tag{25}$$

where $x_0 = \{q_0, \Theta_0, \omega_0, p_{\tau_0}\}$, $\tilde{q} = q - q_0$, likewise for $\tilde{\Theta}$, $\tilde{\omega}$, $\tilde{p}_\tau$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation 22
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The full linearised model is then obtained as

$$
\begin{aligned}
\dot{q}_n \approx f(x_0) - \mathcal{P}\Phi\Bigg(& a_1\omega_0 + \Big(|q_0| + \mathrm{sign}(q_0)q_0\Big) \\
& \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\Bigg) \\
-\mathcal{P}\Phi & \left(\left(-|q_0|q_0\frac{2}{K_v^2\Theta_0^3}\right)\tilde{\Theta}\right) \\
-\mathcal{P}\Phi & \Big(\Big(a_1 q_0 + 2a_0\omega_0\Big)\tilde{\omega}\Big) \\
+\mathcal{P}\mathcal{I}\tilde{p}_\tau &
\end{aligned}
\tag{26}
$$

The equation can be simplified further making some assumptions

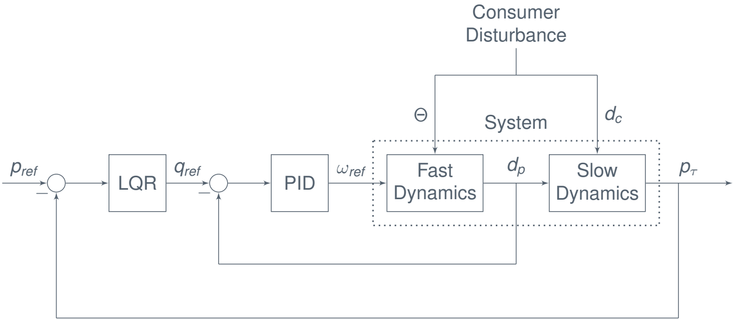
Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation 23
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Equilibrium disappears, valve and tank dynamics assumed to be constant disturbances

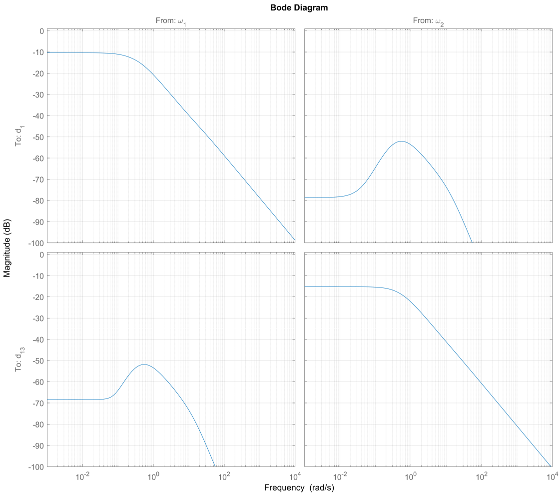
$$\begin{aligned}\dot{q}_n \approx -\mathcal{P}\Phi\Bigg(& a_1\omega_0 + \Big(|q_0| + \operatorname{sign}(q_0)q_0\Big) \\ & \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\Bigg) \\ & -\mathcal{P}\Phi\Bigg(\Big(a_1 q_0 + 2a_0\omega_0\Big)\tilde{\omega}\Bigg)\end{aligned} \tag{27}$$

Control Structure

We elect for a cascaded control structure with optimal central control and local PI control.

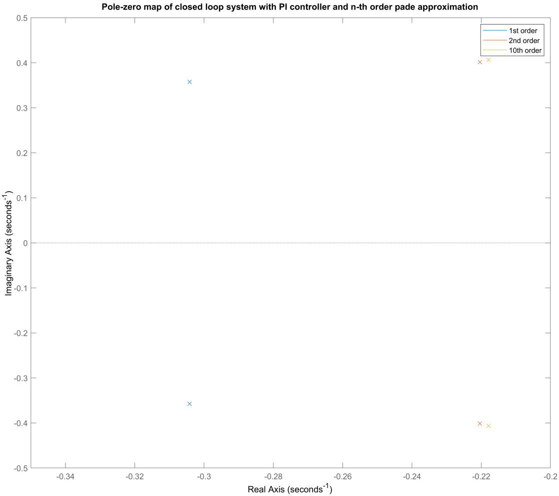


Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References



Modelling and Networked Control of Water Distribution Networks
CA733

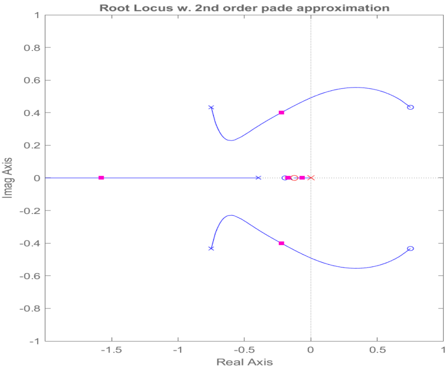
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus 26
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References



Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus 27
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

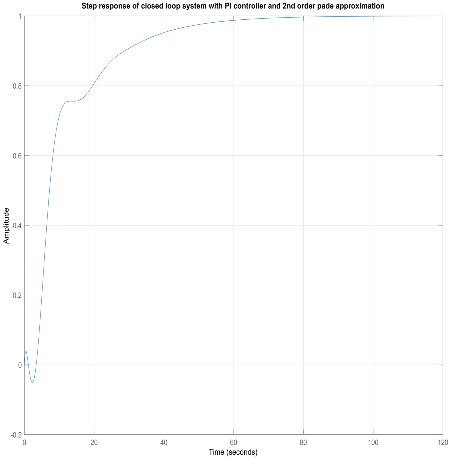
- Faster than outer loop.
- No steady state error.
- No overshoot, and low amount of oscillation.
- Stability.



Results in controller transfer function:

$$C(s) = \frac{K_p s + K_i}{s} = \frac{1.8s + 0.225}{s} \quad (28)$$

Control Structure
Step Response



Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus 28
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

The basic structure of an optimal control problem is sketched in the language of calculus of variations.

$$\dot{x} = f(x, u, t) \tag{29}$$

$$x(t_0) = x_0,\ x \in \mathbb{R}^n,\ \ u \in U \subseteq \mathbb{R}^m \tag{30}$$

where $t \in \mathbb{R}$ is the time and x, u are functions of t, with U the set of admissible controls.

Cost functional:

$$J = \mathcal{M}(x(T)) + \int_0^T \mathcal{L}(x(t), u(t))dt \tag{31}$$

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Infinite-Horizon Linear-Quadratic Regulator

The LQR time-invariant case with no terminal cost.
System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{32}$$

Cost functional:

$$J = \int_{t_0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \tag{33}$$

State feedback control law:

$$u^*(t) = -R^{-1}B^T P x^*(t) \tag{34}$$

P is time-invariant and fullfills the *algebraic Riccati equation*:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \tag{35}$$

Modelling and Networked Control of Water Distribution Networks

CA733

Introduction

Modelling of WDN

Fast Dynamics and Graph theory

Slow Dynamics and Linearisation

Slow Dynamics

Linearisation

Control Structure and Root Locus

Optimal Control

Disturbance Estimator

Network Effects

Results

Best-Case Results

Pathological Synchronisation

References

Let $\hat{x} = x - x_r$ and $\hat{u} = u - u_r$. Shifted coordinate system cost functional framed as an output tracking problem:

$$J = \int_{t_0}^{\infty} \left((C^T \hat{x}^T) Q_y (C\hat{x}) + \hat{u}^T R \hat{u} \right) dt = \int_{t_0}^{\infty} \left(\hat{y}^T Q_y \hat{y} + \hat{u}^T R \hat{u} \right) dt \tag{36}$$

- If linearised around some equilibrium point, these can be used as reference.

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

States are extended with integral state x_i

$$u = -\bar{K}\bar{x} \tag{37}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + B_r r \tag{38}$$

$$y = \bar{C}\bar{x} \tag{39}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{40}$$

$$\bar{K} = -\begin{bmatrix} K & -K_i \end{bmatrix} \tag{41}$$

K_i can be an awkward weight to choose.

Velocity-Form LQR

Deviation variables:

$$\Delta x_k = x_k - x_{k-1}, \; \Delta y_k = y_k - r_k, \; \Delta u_k = u_k - u_{k-1} \tag{42}$$

Extended vectors and matrices:

$$\begin{gathered} \tilde{\zeta}_k = \begin{bmatrix} \Delta x_k \\ \Delta y_k \end{bmatrix}, \; \tilde{u}_k = \Delta u_k, \\ \tilde{A} = \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}, \; \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix}, \; \tilde{C} = \begin{bmatrix} 0 & I \end{bmatrix} \end{gathered} \tag{43}$$

Cost functional:

$$J = \sum_{k_0}^{\infty} \left(\tilde{\zeta}^T Q \tilde{\zeta} + \tilde{u}^T R \tilde{u} \right) \tag{44}$$

Origin regulation! If $\tilde{\zeta} \to 0 \Rightarrow \Delta y \to 0 \Rightarrow y \to r$.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Control input applied at time k is:

$$u^*(k) = \sum_{i=1}^{k} \Delta u^*(i) = \sum_{i=1}^{k} \tilde{K}\zeta^*(i) \tag{45}$$

with

$$\tilde{K} = (\tilde{B}^T P \tilde{B} - R)^{-1}(\tilde{B}^T P \tilde{A}) \tag{46}$$

See e.g. Pannocchia et al. [2] for details.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Standard LQR does not accommodate exogenous inputs (such as the the model of the consumer demand flows) but can be modified to do so [3]:

$$u(k) = \sum_{i=1}^{k} \Delta u^*(i) - B^\dagger \mathcal{B}\delta(i) \quad (47)$$

where $B^\dagger$ is the Moore-Penrose pseudoinverse of B and $\mathcal{B}$ is the disturbance input matrix.

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Disturbance Estimation

Objectives with the use of disturbance estimation

- VF-LQR controller needs an estimate of the consumer demand.
- The optimal estimator is the **Kalman Filter**.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator 37
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Need a linear model of consumer behaviour for the Kalman Filter
- Under normal circumstances the Kalman gain is found recursively.
- In the case of LTI system the Kalman filter itself becomes time invariant $\rightarrow$ constant Kalman gain.
  ARE:

$$\begin{aligned} \Pi &= A(\Pi^{-1} + C^T R^{-1} C)^{-1} A^T + Q \\ K &= \Pi C^T (C \Pi C^T + R)^{-1} \end{aligned} \tag{48}$$

- Kalman filter is also very interesting from a leakage detection POV.

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator 38
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Data of consumption pattern over a 35 day period courtesy of CSK and Grundfos.

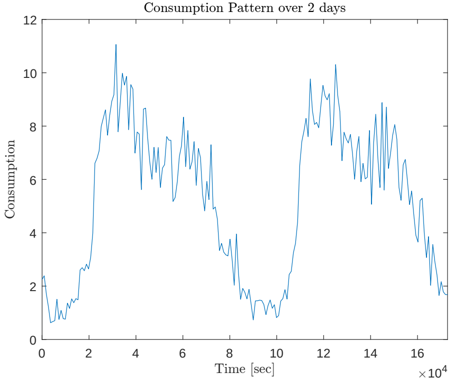


Figure: Consumption pattern over two days

- Frequency analysis of the data.

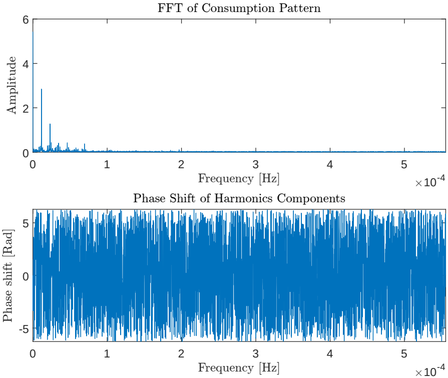


Figure: Amplitude and phase content of full consumption pattern

We want a sparse representation. Largest frequency components are: DC, 24.05hr, 12.03hr, 8.02hr and 5.99hr.

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Fourth-order Fourier approximation of consumer demand:

$$\begin{aligned} d_c(t) \approx\, & k_0 + k_1 cos(\omega_1 t + \phi_1) + k_2 cos(\omega_2 t + \phi_2) \\ & + k_3 cos(3\omega_3 t + \phi_3) + k_4 cos(4\omega_4 t + \phi_4) \end{aligned} \tag{49}$$

- We wish to model equation 49 as a state-space model:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

Modelling and Networked Control of Water Distribution Networks
CA733

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator 41
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Need to represent the evolution of the "states" in the approximation as a linear combination of states.
- This can not be achieved using the current states.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & \omega_2 & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 cos(\omega_1 t) \\ k_1 sin(\omega_1 t) \\ k_2 cos(\omega_2 t) \\ k_2 sin(\omega_2 t) \end{bmatrix} \tag{50}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} 0 \\ -k_1\omega_1 sin(\omega_1 t) \\ k_1\omega_1 cos(\omega_1 t) \\ -k_2\omega_2 sin(\omega_2 t) \\ k_2\omega_2 cos(\omega_2 t) \end{bmatrix} \tag{51}$$

- Fourth order doesn't fit the page!

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator 42
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- The model compared to the real data - shown over 2 days.

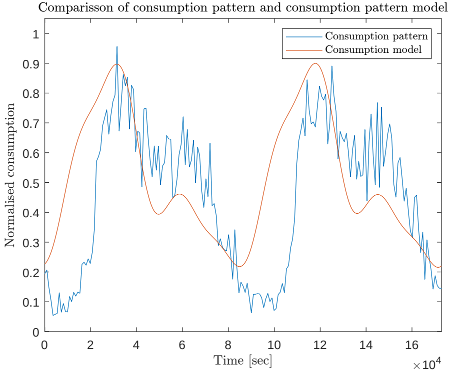


Figure: Comparison of raw historical data, and model

- The model follows the visual trend in real data.

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

Considerations when designing the KF.

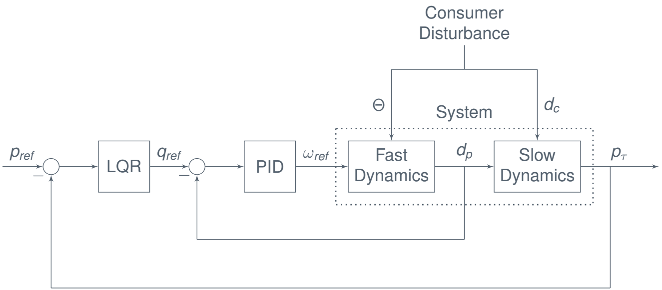
- In practice the Kalman gain is found using lqr in matlab.
- Stiffness of the filter is decided by Q-R ratio.
- Big R means uncertainty concerning observation and that we trust model much $\rightarrow$ stiff filter.

Control and Automation
Aalborg University
Denmark

Network Effects

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects 44
Results
Best-Case Results
Pathological Synchronisation
References

- Outer loop
  - Central control unit
  - Transmit data
- Urban environment
  - Packet loss scaling with distance
  - 60% or more expected at 20 km [4]
- Try-Once-Discard protocol used
  - Packet loss → assume states = 0



Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References

- Packet loss is stochastic, must think in terms of mean-square stability:

$$\lim_{k\to\infty} \mathrm{E}[x(k)^2] = 0 \tag{52}$$

- Assuming 60% as upper bound for loss, α, the condition for mean-square stability is [5]:

$$\mathcal{S}\Big(\alpha A \otimes A + (1-\alpha)(A - BK) \otimes (A - BK)\Big) < 1 \tag{53}$$

- This is because propagation matrix of the state covariance is:

$$\alpha A \otimes A + (1-\alpha)(A - BK) \otimes (A - BK) \tag{54}$$

- We get $\mathcal{S} = 0.9453$

Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects 46
Results
Best-Case Results
Pathological Synchronisation
References

- Upper bound on packet loss Ξ. All values $\alpha < \Xi$ will also be stable.

$$\Xi = \frac{1}{\|\sigma_+(V)\|_\infty}$$

$$V = \begin{bmatrix} (S \otimes \hat{S} + \hat{S} \otimes S)(I - S \otimes S)^{-1} & \hat{S} \otimes \hat{S} \\ (I - S \otimes S)^{-1} & 0 \end{bmatrix} \tag{55}$$

$$S = (A - BK) \otimes (A - BK)\,, \quad \hat{S} = A \otimes A - S$$

- In the limit of zero control:

$$\begin{aligned} \lim_{\tilde{K} \to 0} & \left(\alpha \tilde{A} \otimes \tilde{A} + (1-\alpha)(\tilde{A} - \tilde{B}\tilde{K}) \otimes (\tilde{A} - \tilde{B}\tilde{K})\right) \\ &= \left(\alpha \tilde{A} \otimes \tilde{A} + (1-\alpha)\tilde{A} \otimes \tilde{A}\right) \end{aligned} \tag{56}$$

Introduction

Modelling of WDN

Fast Dynamics and Graph theory

Slow Dynamics and Linearisation

Slow Dynamics

Linearisation

Control Structure and Root Locus

Optimal Control

Disturbance Estimator

Network Effects

Results

Best-Case Results

Pathological Synchronisation

References

Network Effects

Upper Loss Bound - Continued

$$\forall \alpha \in \{0 \ldots 1\} : \quad \left(\alpha \tilde{A} \otimes \tilde{A} + (1-\alpha) \tilde{A} \otimes \tilde{A}\right) \\ = \tilde{A} \otimes \tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{57}$$

Thus *any* EWR is marginally stable in the zero limit of control for any packet loss as $\mathcal{S}(1,1,1,1) = 1$.

Modelling and Networked Control of Water Distribution Networks

CA733

Introduction

Modelling of WDN

Fast Dynamics and Graph theory

Slow Dynamics and Linearisation

Slow Dynamics

Linearisation

Control Structure and Root Locus

Optimal Control

Disturbance Estimator

Network Effects

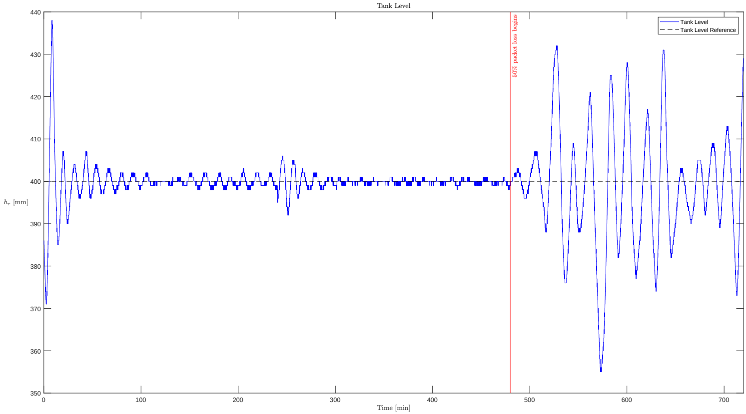
Results 48

Best-Case Results

Pathological Synchronisation

References

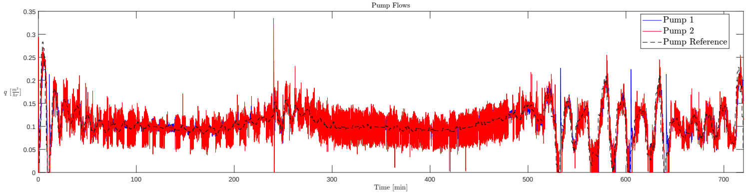
Results
Level Control

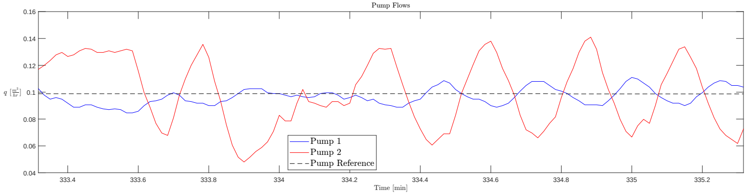


Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results 49
Pathological Synchronisation
References

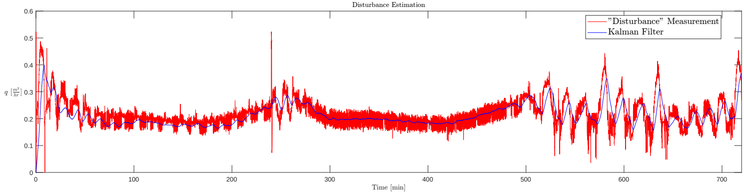
Results
Flow Control

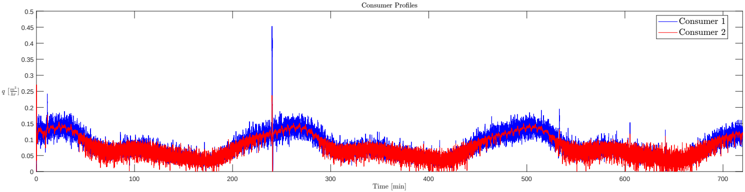
Modelling and Networked Control of Water Distribution Networks
CA733
Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results 50
Best-Case Results
Pathological Synchronisation
References





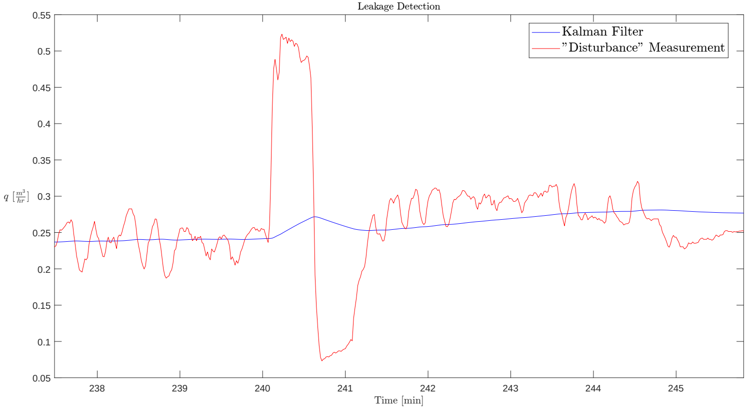
Results
Disturbance Estimation





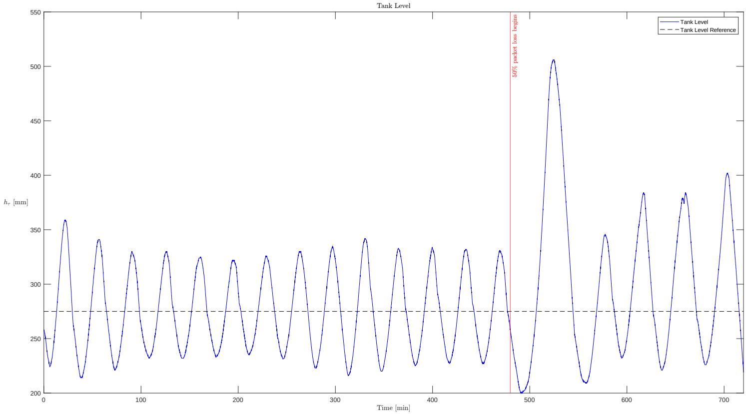
Results
Leakage Detection

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results 52
Pathological Synchronisation
References



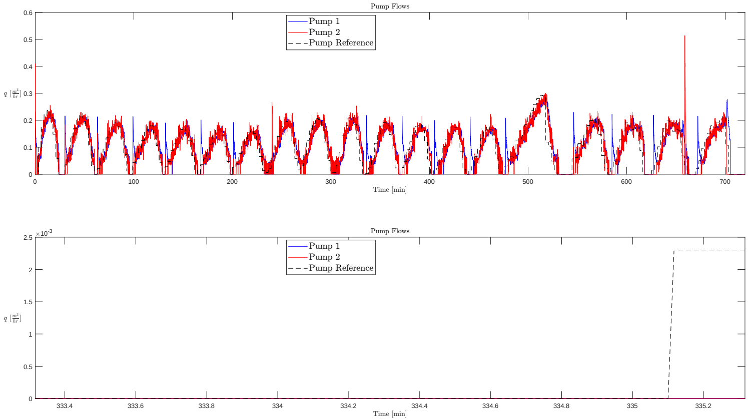
Results

Level Control - Pathological Behaviour



Results

Flow Control - Pathological Behaviour



Introduction

Modelling of WDN

Fast Dynamics and Graph theory

Slow Dynamics and Linearisation

Slow Dynamics

Linearisation

Control Structure and Root Locus

Optimal Control

Disturbance Estimator

Network Effects

Results

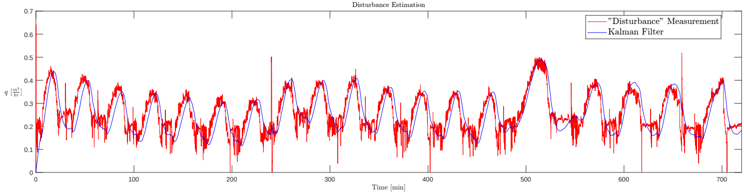
Best-Case Results

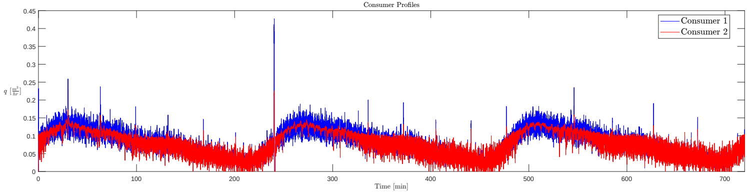
Pathological Synchronisation 54

References

Results

Disturbance Estimation - Pathological Behaviour





References

Introduction
Modelling of WDN
Fast Dynamics and Graph theory
Slow Dynamics and Linearisation
Slow Dynamics
Linearisation
Control Structure and Root Locus
Optimal Control
Disturbance Estimator
Network Effects
Results
Best-Case Results
Pathological Synchronisation
References 56

[1] S. S. Rathore, "Optimal Control in Water Distribution Network," 2019.

[2] G. Pannocchia, M. Gabiccini, and A. Artoni, "Offset-free MPC explained: Novelties, subtleties, and applications," *IFAC-PapersOnLine*, vol. 48, no. 23, pp. 342–351, jan 2015.

[3] A. K. Singh and B. C. Pal, "An extended linear quadratic regulator for LTI systems with exogenous inputs," *Automatica*, vol. 76, pp. 10–16, feb 2017.

[4] A. Sheth, S. Nedevschi, R. Patra, S. Surana, E. Brewer, and L. Subramanian, "Packet loss characterization in wifi-based long distance networks," *Proceedings - IEEE INFOCOM*, pp. 312–320, 2007.

[5] S. Hu and W. Y. Yan, "Stability robustness of networked control systems with respect to packet loss," *Automatica*, vol. 43, no. 7, pp. 1243–1248, jul 2007.

Open for questions!

AALBORG UNIVERSITY
DENMARK