

Modelling and Networked Control of Water Distribution Networks

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CA733

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- ▶ Critical societal infrastructure, responsible for the provision of water to both domestic and industrial consumers.
- ▶ Network pressure must be controlled.
 - ▶ Underpressure → insufficient service pressure → dissatisfied end users.
 - ▶ Overpressure → component failure → repair costs, supply intermittency, etc.
- ▶ Network pressure tied to level in Elevated Water Reservoir(s) (EWR) → level control is key!
- ▶ Other key components are pumps, valves, and pipes.

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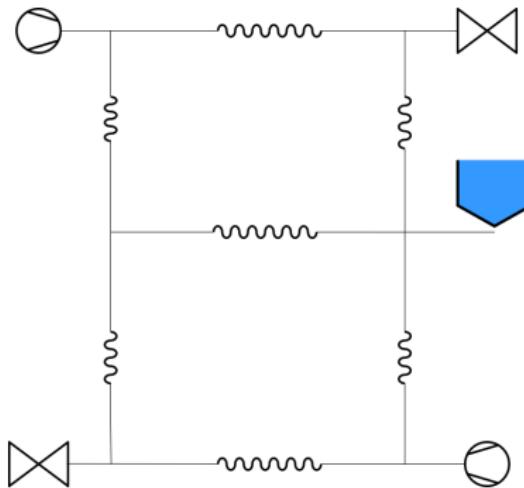
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We analyse a small-scale test WDN with the following layout:



Larger WDNs are typically not amenable to first-principles modelling.



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- ▶ General principles for modelling fast regime very similar to circuit analysis.
 - ▶ Water flows = currents.
 - ▶ Pressures = voltages.
 - ▶ Network components behave similarly to circuit components, but resistors generally non-linear.
- ▶ Graph theory is combined with physics-induced boundary conditions to yield model of fast regime.

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Graph Theory

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Graph theory describes a networked system as a directed graph composed of edges and vertices.

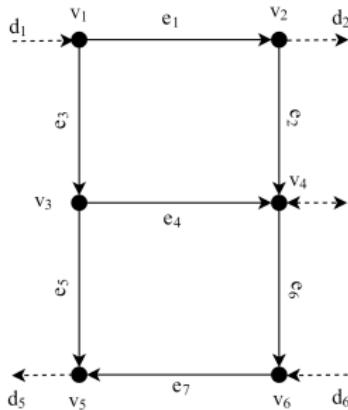


Figure: Graph of simplified WDN network. From [1].

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Incidence and Loop matrices

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Auxiliary matrices can be defined to mathematically describe the network:

$$H_{i,j} = \begin{cases} 1 & \text{If } j\text{th edge leaves } i\text{th node} \\ -1 & \text{If } j\text{th edge enters } i\text{th node} \\ 0 & \text{If } j\text{th edge and } i\text{th node unconnected} \end{cases}$$
$$B_{i,j} = \begin{cases} 1 & \text{If direction of } i\text{th loop and } j\text{th edge agree} \\ -1 & \text{If direction of } i\text{th loop and } j\text{th edge disagree} \\ 0 & \text{If } i\text{th loop excludes } j\text{th edge} \end{cases}$$

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Incidence and Loop matrices: Example

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Example:

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad (1)$$

$$\bar{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \quad (2)$$

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \end{bmatrix} \quad (3)$$



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Demand Matrices

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Whether a node is open to atmosphere or not, or connected to a tank, can be described with matrices F and G respectively.

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

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General Component Model

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The edges of the WDN can be described with a pressure and flow relationship, analogous to the current and voltage of an electrical component.

This project uses pipes, valves and pumps as edge components.

The general relationship between pressure and flow can be defined as:

$$\Delta p = \mathcal{J}\dot{q} + \lambda(q) + \mu(q, \Theta) + \alpha(q, \omega) - \Delta h \quad (5)$$

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Pipe Model

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The pressure drop across a pipe is defined as:

$$\Delta p_k = \mathcal{J} \dot{q} + \lambda(q) + \Delta z \quad (6)$$

where:

$$\lambda_k(q_k) = \left(f \cdot \frac{8 \cdot L \cdot q^2}{\pi^2 \cdot g \cdot D^5} + k_f \cdot \frac{8 \cdot q^2}{\pi^2 \cdot g \cdot D^4} \right) \cdot g \cdot \rho \quad (7)$$

$$\mathcal{J} = \frac{L \cdot \rho}{A} \quad (8)$$

$$\Delta z_k = \rho \cdot g \cdot \Delta h_k \quad (9)$$

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Valve Model

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The pressure drop across a valve is defined as:

$$\Delta p_k = \mu(q, \Theta) = \frac{1}{K_{valve}(\Theta)^2} \cdot |q| \cdot q \quad (10)$$

where Θ is the opening degree of the valve.

Derivation:

$$\frac{\Delta p_1}{q_1^2} = \frac{\Delta p_2}{q_2^2} \Leftrightarrow q_1 = q_2 \cdot \sqrt{\frac{\Delta p_1}{\Delta p_2}} \quad (11)$$

$$q = q_n(\Theta) \cdot \sqrt{\frac{\Delta p_1}{1}} = K_{valve}(\Theta) \cdot \sqrt{\Delta p_1} \quad (12)$$

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Pump Model

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The pressure drop across a pump is defined as:

$$\Delta p_k = a_0 \cdot \omega^2 + a_1 \cdot \omega \cdot q - a_2 \cdot |q| \cdot q \quad (13)$$

where $[a_0, a_1, a_2]$ is a tuple of coefficients that describe the pump's characteristic curve, q is the flow rate through the pump, and ω is the rotational velocity of the pump.



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Kirchhoff's node and mesh law:

$$Hq = d \quad (14)$$

$$B\Delta p = BH^T p = 0 \wedge B\Delta h = BH^T h = 0 \quad (15)$$

Mass conservation:

$$d_n = - \sum_{i=1}^{n-1} d_i \quad (16)$$



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Lemmas

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Lemma 4.1:

$$H_T \bar{H}_T^{-1} = \begin{bmatrix} I_{n-1} \\ -\mathbf{1}^T \end{bmatrix} \quad (17)$$

where $\mathbf{1}$ is a vector of ones and $I_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ is an identity matrix.

Lemma 4.2:

$$q = B^T q_C + \begin{bmatrix} 0_{C \times n-1} \\ \bar{H}_T^{-1} \end{bmatrix} \bar{d} \quad (18)$$

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Non linear differential equation:

$$\Phi \mathcal{J} \Phi^T \dot{q} = -\Phi \left(\lambda(q_n) + \mu(q_n) + \alpha(q_n) \right) + \Psi (\bar{h} - \mathbf{1} h_0) + \mathcal{I} (p_\tau - \mathbf{1} p_0) \quad (19)$$

Where the matrices Φ, Ψ, \mathcal{I} are defined as:

$$\Phi \triangleq \begin{bmatrix} I & -\bar{H}_C^T \bar{H}_T^{-T} \\ 0 & \bar{F}^T \bar{H}_T^{-T} \\ 0 & \bar{G}^T \bar{H}_T^{-T} \end{bmatrix}, \quad \Psi \triangleq \begin{bmatrix} 0 \\ \bar{F}^T \\ \bar{G}^T \end{bmatrix}, \quad \mathcal{I} \triangleq \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \quad (20)$$

$$\mathcal{P} : (\Phi \mathcal{J} \Phi^T)^{-1} \quad (21)$$

$$\dot{q}_n = -\mathcal{P} \Phi \left(\lambda(q_n) + \mu(q_n) + \alpha(q_n) \right) + \mathcal{P} \left(\Psi (\bar{h} - \mathbf{1} h_0) + \mathcal{I} (p_\tau - \mathbf{1} p_0) \right) \quad (22)$$

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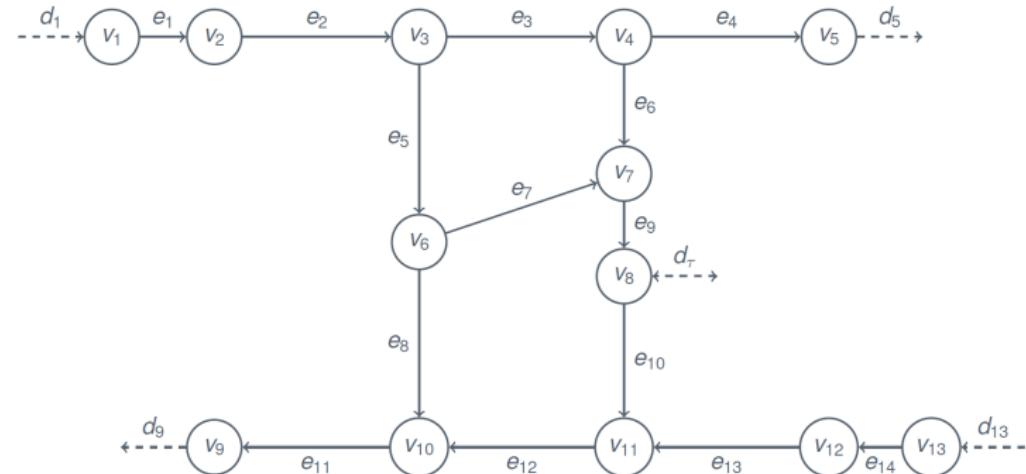
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► Fundamentals

$$p \propto h$$

$$\dot{V} = q$$

► Assume constant cross sectional area A

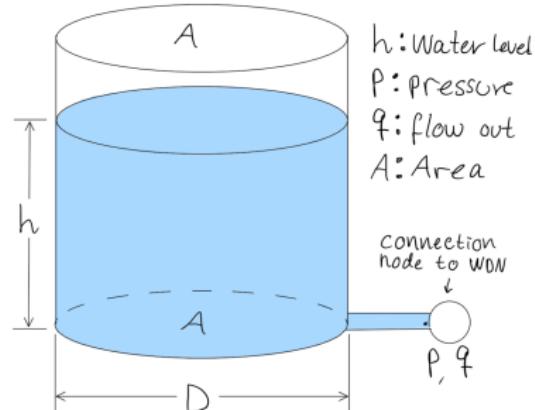
$$V \propto h \implies V \propto p$$

$$\dot{p} \propto \dot{V} \wedge \dot{V} = q \implies \dot{p} \propto q$$

► We arrive at

$$\dot{p} = -\tau q, \text{ where}$$

$$\tau = \rho g \frac{1}{A}$$



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State-space Formulation of Slow Dynamics

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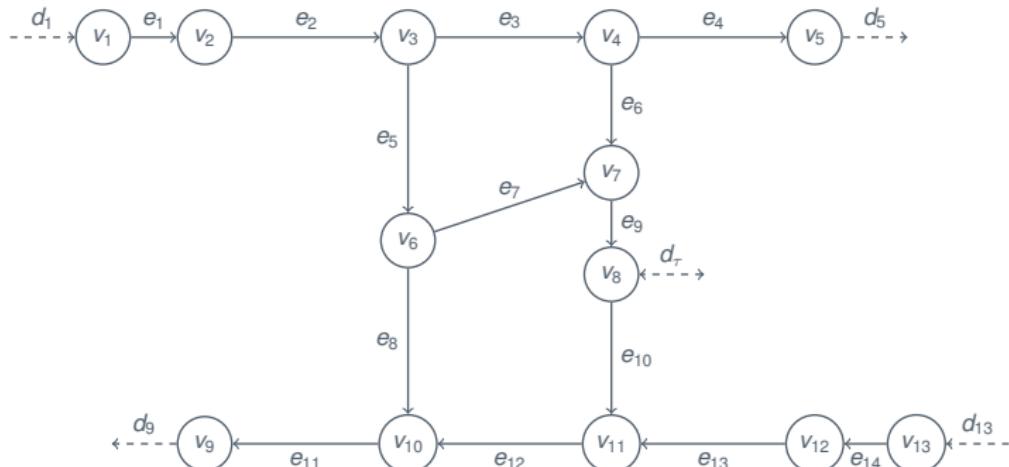
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In context of WDN we now consider flows to and from network
as external demands d_i



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Mass conservation holds, and such

$$d_n = - \sum_{i=1}^{n-1} d_i \implies d_\tau = -(d_p + d_c)$$

$$\dot{p} = -\tau d_\tau = \tau(d_p + d_c)$$

When discretised by forward Euler:

$$p_\tau(k+1) = p_\tau(k) - \tau d_\tau(k) t_s = p_\tau(k) + \tau(d_p(k) + d_c(k)) t_s$$

Which corresponds to a discrete, linear state-space model:

$$p_\tau(k+1) = Ap_\tau(k) + B_p d_p(k) + B_c d_c(k) \quad (23)$$

$$d_p = \begin{bmatrix} d_1 \\ d_{13} \end{bmatrix}, d_c = \begin{bmatrix} d_5 \\ d_9 \end{bmatrix}, B_p = B_c = t_s \begin{bmatrix} \tau & \tau \end{bmatrix}, A = 1$$

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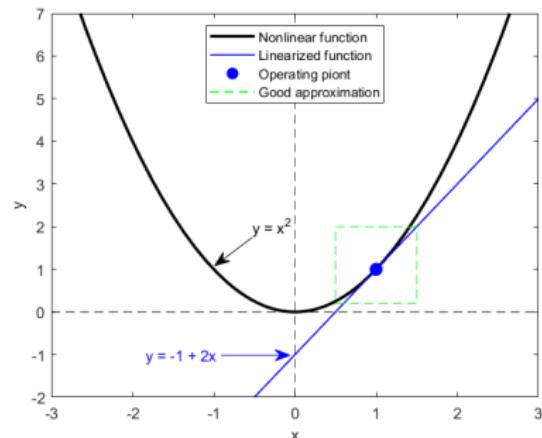
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In near vicinity of
linearisation point x_0 ,

$$\dot{x} \approx f(x_0) + \nabla f \Big|_{x_0} (x - x_0)$$

Linearising around
equilibrium point
preferred

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Recalling the fast dynamics differential equation is given as

$$\dot{q}_n = -\mathcal{P}\Phi\left(\lambda(q_n) + \mu(q_n, \Theta) + \alpha(q_n, \omega)\right) + \mathcal{P}\left(\Psi(\bar{h} - \mathbf{1}h_0) + \mathcal{I}(p_\tau - \mathbf{1}p_0)\right) \quad (24)$$

The linear model such becomes

$$\dot{q}_n \approx f(x_0) + \frac{\partial f}{\partial q_n} \Big|_{x_0} \tilde{q}_n + \frac{\partial f}{\partial \Theta} \Big|_{x_0} \tilde{\Theta} + \frac{\partial f}{\partial \omega} \Big|_{x_0} \tilde{\omega} + \frac{\partial f}{\partial p_\tau} \Big|_{x_0} \tilde{p}_\tau \quad (25)$$

where $x_0 = \{q_0, \Theta_0, \omega_0, p_{\tau_0}\}$, $\tilde{q} = q - q_0$, likewise for $\tilde{\Theta}$, $\tilde{\omega}$, \tilde{p}_τ



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The full linearised model is then obtained as

$$\begin{aligned}\dot{q}_n \approx f(x_0) - \mathcal{P}\Phi & \left(a_1\omega_0 + \left(|q_0| + \text{sign}(q_0)q_0 \right) \right. \\ & \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2} \right) \tilde{q}_n \Big) \\ & - \mathcal{P}\Phi \left(\left(-|q_0|q_0 \frac{2}{K_v^2\Theta_0^3} \right) \tilde{\Theta} \right) \\ & - \mathcal{P}\Phi \left(\left(a_1 q_0 + 2a_0\omega_0 \right) \tilde{\omega} \right) \\ & + \mathcal{P}\mathcal{I}\tilde{p}_\tau\end{aligned}\tag{26}$$

The equation can be simplified further making some assumptions

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Equilibrium disappears, valve and tank dynamics assumed to
be constant disturbances

$$\dot{q}_n \approx -\mathcal{P}\Phi \left(a_1\omega_0 + \left(|q_0| + \text{sign}(q_0)q_0 \right) \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2} \right) \tilde{q}_n \right) - \mathcal{P}\Phi \left(\left(a_1 q_0 + 2a_0\omega_0 \right) \tilde{\omega} \right) \quad (27)$$

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Control Structure

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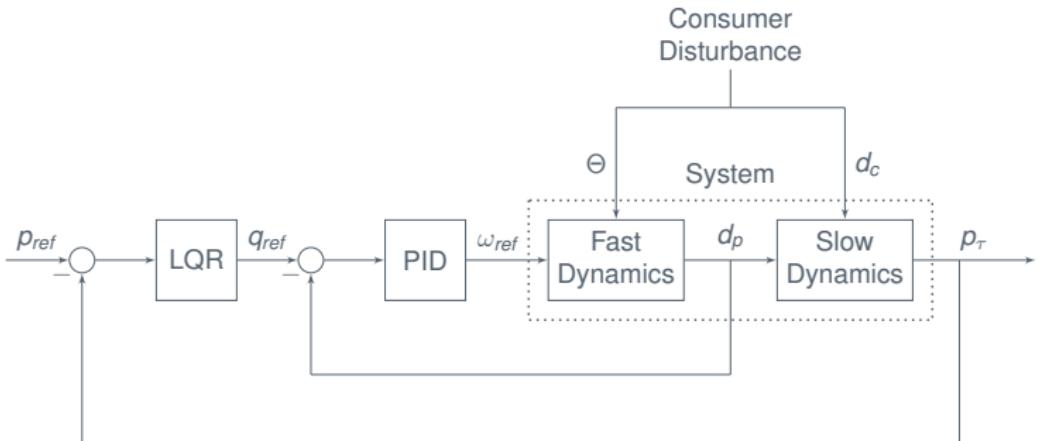
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We elect for a cascaded control structure with optimal central control and local PI control.





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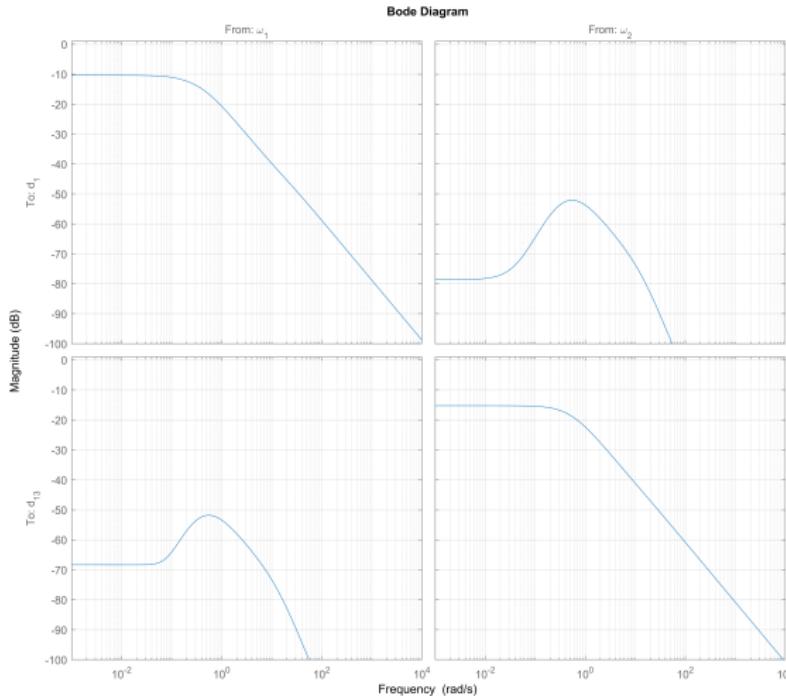
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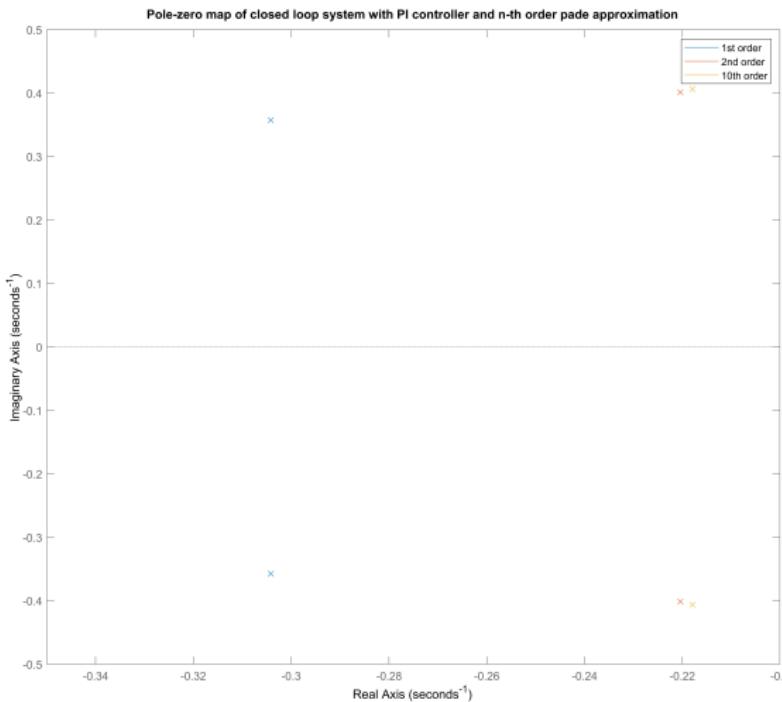
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Control Structure

Root Locus and Resulting Controller

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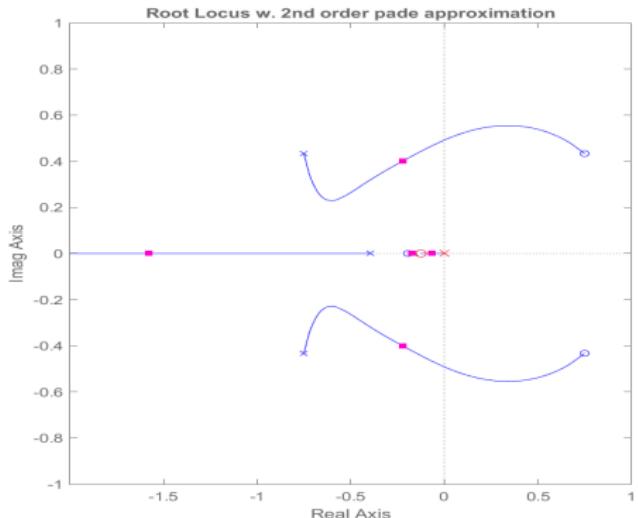
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- ▶ Faster than outer loop.
- ▶ No steady state error.
- ▶ No overshoot, and low amount of oscillation.
- ▶ Stability.



Results in controller transfer function:

$$C(s) = \frac{K_p s + K_i}{s} = \frac{1.8s + 0.225}{s} \quad (28)$$



Control Structure

Step Response

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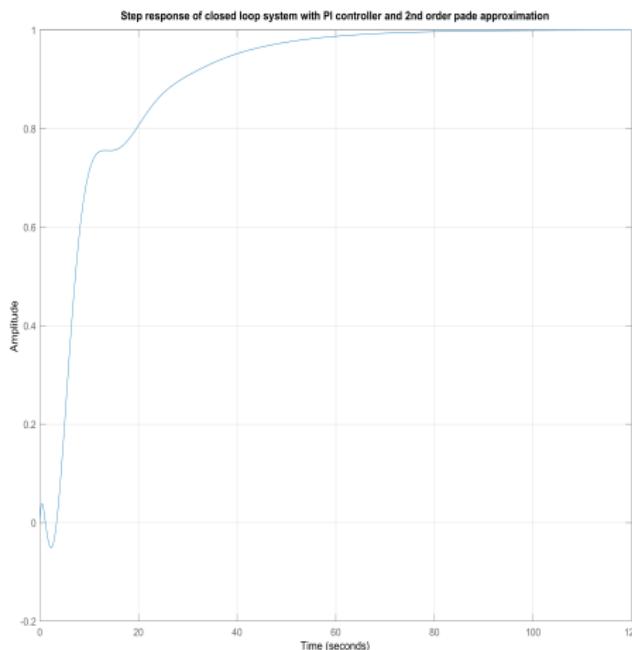
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Optimal Control

General Optimal Control Problem

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The basic structure of an optimal control problem is sketched in the language of calculus of variations.

$$\dot{x} = f(x, u, t) \quad (29)$$

$$x(t_0) = x_0, x \in \mathbb{R}^n, u \in U \subseteq \mathbb{R}^m \quad (30)$$

where $t \in \mathbb{R}$ is the time and x, u are functions of t , with U the set of admissible controls.

Cost functional:

$$J = \mathcal{M}(x(T)) + \int_0^T \mathcal{L}(x(t), u(t)) dt \quad (31)$$



Infinite-Horizon Linear-Quadratic Regulator

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The LQR time-invariant case with no terminal cost.
System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (32)$$

Cost functional:

$$J = \int_{t_0}^{\infty} (x^T(t)Qx(t) + u^T(t)Ru(t)) dt \quad (33)$$

State feedback control law:

$$u^*(t) = -R^{-1}B^T Px^*(t) \quad (34)$$

P is time-invariant and fulfills the *algebraic Riccati equation*:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \quad (35)$$



Tracking LQR and Integral Action

Tracking LQR

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Let $\hat{x} = x - x_r$ and $\hat{u} = u - u_r$. Shifted coordinate system cost functional framed as an output tracking problem:

$$J = \int_{t_0}^{\infty} ((C^T \hat{x}^T) Q_y (C \hat{x}) + \hat{u}^T R \hat{u}) dt = \int_{t_0}^{\infty} (\hat{y}^T Q_y \hat{y} + \hat{u}^T R \hat{u}) dt \quad (36)$$

- If linearised around some equilibrium point, these can be used as reference.



Tracking LQR and Integral Action

Integral Action

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States are extended with integral state x_i

$$u = -\bar{K}\bar{x} \quad (37)$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + B_r r \quad (38)$$

$$y = \bar{C}\bar{x} \quad (39)$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \quad B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \bar{C} = [C \quad 0] \quad (40)$$

$$\bar{K} = -[K \quad -K_i] \quad (41)$$

K_i can be an awkward weight to choose.



Velocity-Form LQR

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Deviation variables:

$$\Delta x_k = x_k - x_{k-1}, \quad \Delta y_k = y_k - r_k, \quad \Delta u_k = u_k - u_{k-1} \quad (42)$$

Extended vectors and matrices:

$$\begin{aligned}\tilde{\zeta}_k &= \begin{bmatrix} \Delta x_k \\ \Delta y_k \end{bmatrix}, \quad \tilde{u}_k = \Delta u_k, \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix}, \quad \tilde{C} = [0 \quad I]\end{aligned} \quad (43)$$

Cost functional:

$$J = \sum_{k_0}^{\infty} (\tilde{\zeta}^T Q \tilde{\zeta} + \tilde{u}^T R \tilde{u}) \quad (44)$$

Origin regulation! If $\tilde{\zeta} \rightarrow 0 \Rightarrow \Delta y \rightarrow 0 \Rightarrow y \rightarrow r$.



Velocity-Form LQR

Feedback Law

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with

$$u^*(k) = \sum_{i=1}^k \Delta u^*(i) \quad (45)$$

$$\tilde{K} = (\tilde{B}^T P \tilde{B} - R)^{-1} (\tilde{B}^T P \tilde{A}) \quad (46)$$

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Velocity-Form LQR

Disturbance-accommodating VF-LQR

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Standard LQR does not accommodate exogenous inputs (such as the model of the consumer demand flows) but can be modified to do so:

$$u(k) = \sum_{i=1}^k \Delta u^*(i) - B^\dagger \mathcal{B} \delta(i) \quad (47)$$

where B^\dagger is the Moore-Penrose pseudoinverse of B and \mathcal{B} is the disturbance input matrix.



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Objectives with the use of disturbance estimation

- ▶ VF-LQR controller needs an estimate of the consumer demand.
- ▶ The optimal estimator is the **Kalman Filter**.



Disturbance Estimation

The Kalman Filter

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- ▶ Need a linear model of consumer behaviour for the Kalman Filter
- ▶ Under normal circumstances the Kalman gain is found recursively.
- ▶ In the case of LTI system the Kalman filter itself becomes time invariant → constant Kalman gain.
ARE:

$$\begin{aligned}\Pi &= A(\Pi^{-1} + C^T R^{-1} C)^{-1} A^T + Q \\ K &= \Pi C^T (C \Pi C^T + R)^{-1}\end{aligned}\tag{48}$$

- ▶ Kalman filter is also very interesting from a leakage detection POV.

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Water Consumption Data

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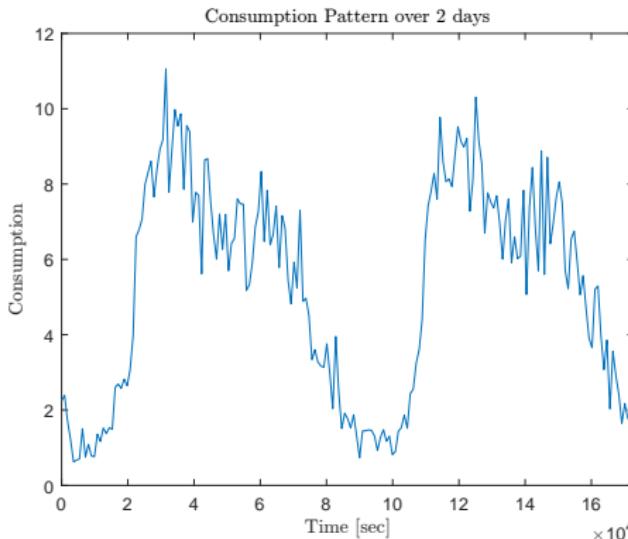


Figure: Consumption pattern over two days

Disturbance Estimation

FFT of Consumption Pattern

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- ▶ Frequency analysis of the data.

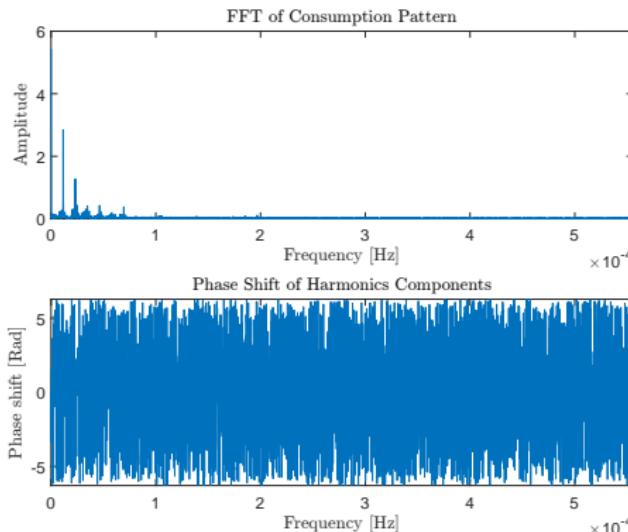


Figure: Amplitude and phase content of full consumption pattern

We want a sparse representation. Largest frequency components are: DC, 24.05hr, 12.03hr, 8.02hr and 5.99hr.



Disturbance Estimation

Approximation of Consumption

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- ▶ Fourth-order Fourier approximation of consumer demand:

$$d_c(t) \approx k_0 + k_1 \cos(\omega_1 t + \phi_1) + k_2 \cos(\omega_2 t + \phi_2) \\ + k_3 \cos(3\omega_3 t + \phi_3) + k_4 \cos(4\omega_4 t + \phi_4) \quad (49)$$

- ▶ We wish to model equation 49 as a state-space model:

$$\dot{x} = Ax$$

$$y = Cx$$



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State-space Representation

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- ▶ Need to represent the evolution of the "states" in the approximation as a linear combination of states.
- ▶ This can not be achieved using the current states.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & \omega_2 & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 \cos(\omega_1 t) \\ k_1 \sin(\omega_1 t) \\ k_2 \cos(\omega_2 t) \\ k_2 \sin(\omega_2 t) \end{bmatrix} \quad (50)$$

$$C = [1 \ 1 \ 0 \ 1 \ 0], \quad \dot{x} = \begin{bmatrix} 0 \\ -k_1 \omega_1 \sin(\omega_1 t) \\ k_1 \omega_1 \cos(\omega_1 t) \\ -k_2 \omega_2 \sin(\omega_2 t) \\ k_2 \omega_2 \cos(\omega_2 t) \end{bmatrix} \quad (51)$$

- ▶ Fourth order doesn't fit the page!

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Model vs. Data

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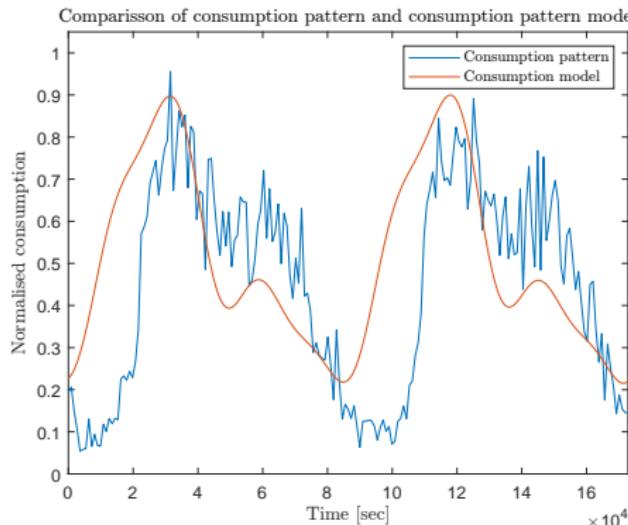


Figure: Comparison of raw historical data, and model

- The model follows the visual trend in real data.

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Disturbance Estimation

Kalman Filter Design

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Considerations when designing the KF.

- ▶ In practice the Kalman gain is found using lqr in matlab.
- ▶ Stiffness of the filter is decided by Q-R ratio.
- ▶ Big R means uncertainty concerning observation and that we trust model much → stiff filter.

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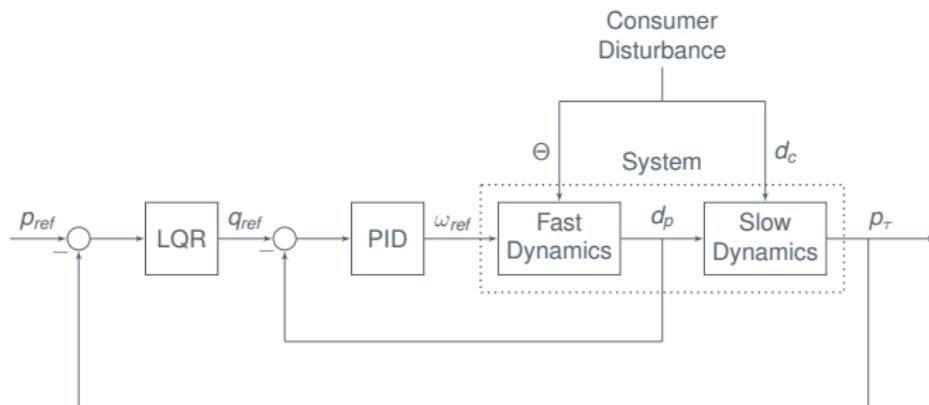
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- ▶ Outer loop
 - ▶ Central control unit
 - ▶ Transmit data
- ▶ Urban environment
 - ▶ Packet loss scaling with distance
 - ▶ 60% or more expected at 20 km
- ▶ Try-Once-Discard protocol used
 - ▶ Packet loss → assume states = 0





Network Effects

For Specific Packet Loss

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- ▶ Packet loss is stochastic, must think in terms of mean-square stability:

$$\lim_{k \rightarrow \infty} E[x(k)^2] = 0 \quad (52)$$

- ▶ Assuming 60% as upper bound for loss, α , the condition for mean-square stability is [2]:

$$\mathcal{S} \left(\alpha A \otimes A + (1 - \alpha)(A - BK) \otimes (A - BK) \right) < 1 \quad (53)$$

- ▶ This is because propagation matrix of the state covariance is:

$$\alpha A \otimes A + (1 - \alpha)(A - BK) \otimes (A - BK) \quad (54)$$

- ▶ We get $\mathcal{S} = 0.9453$



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Upper Loss Bound

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- ▶ Upper bound on packet loss Ξ . All values $\alpha < \Xi$ will also be stable.

$$\Xi = \frac{1}{\|\sigma_+(V)\|_\infty}$$

$$V = \begin{bmatrix} (S \otimes \hat{S} + \hat{S} \otimes S)(I - S \otimes S)^{-1} & \hat{S} \otimes \hat{S} \\ (I - S \otimes S)^{-1} & 0 \end{bmatrix} \quad (55)$$

$$S = (A - BK) \otimes (A - BK), \quad \hat{S} = A \otimes A - S$$

- ▶ In the limit of zero control:

$$\begin{aligned} \lim_{\tilde{K} \rightarrow 0} & \left(\alpha \tilde{A} \otimes \tilde{A} + (1 - \alpha)(\tilde{A} - \tilde{B}\tilde{K}) \otimes (\tilde{A} - \tilde{B}\tilde{K}) \right) \\ &= \left(\alpha \tilde{A} \otimes \tilde{A} + (1 - \alpha)\tilde{A} \otimes \tilde{A} \right) \end{aligned} \quad (56)$$

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Upper Loss Bound - Continued

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Thus *any* EWR is marginally stable in the zero limit of control
for any packet loss as $\mathcal{S}(1, 1, 1, 1) = 1$.

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$$\begin{aligned} \forall \alpha \in \{0 \dots 1\} : & \left(\alpha \tilde{\mathbf{A}} \otimes \tilde{\mathbf{A}} + (1 - \alpha) \tilde{\mathbf{A}} \otimes \tilde{\mathbf{A}} \right) \\ &= \tilde{\mathbf{A}} \otimes \tilde{\mathbf{A}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{aligned} \quad (57)$$

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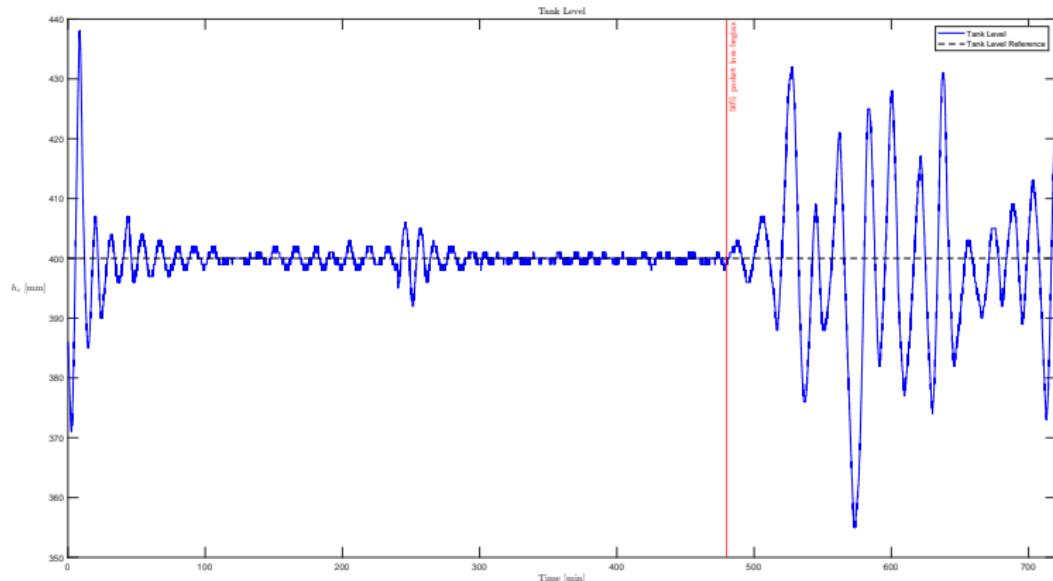
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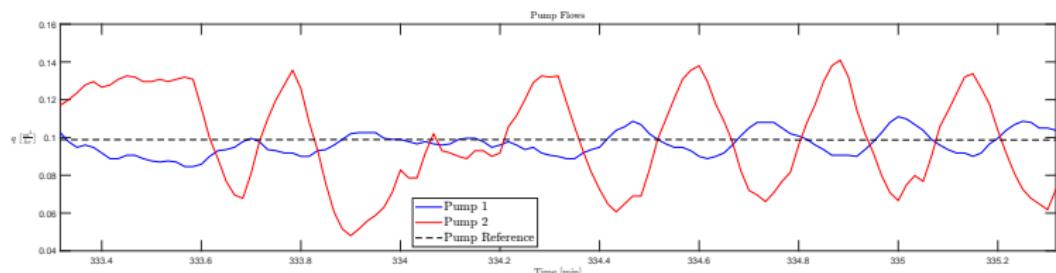
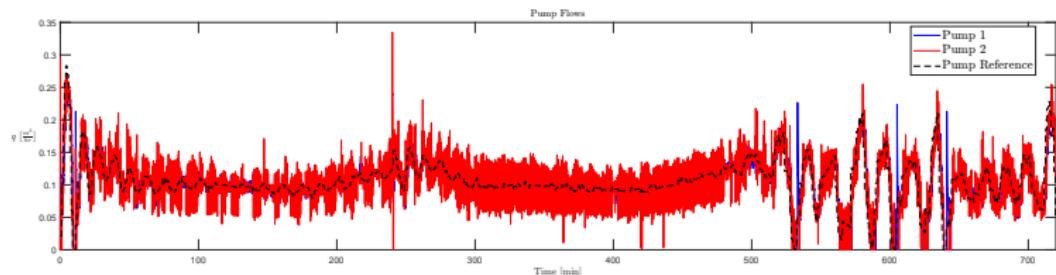
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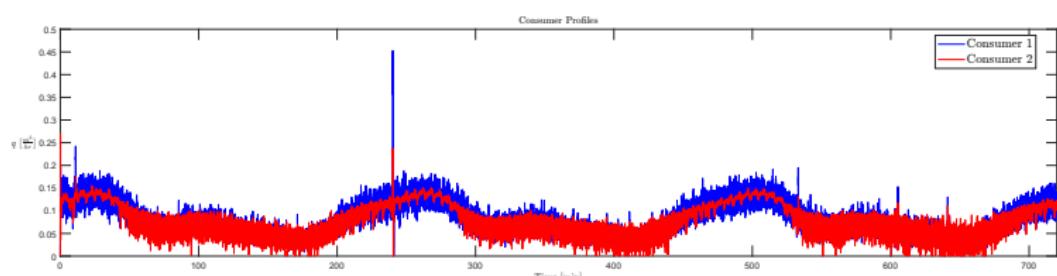
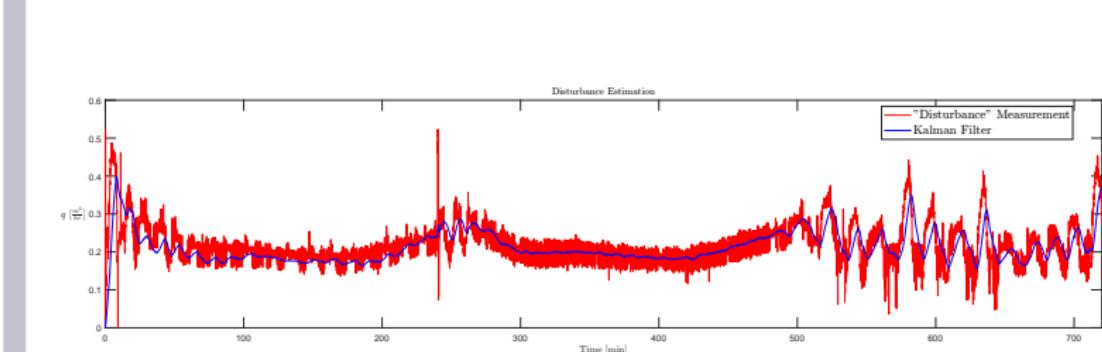
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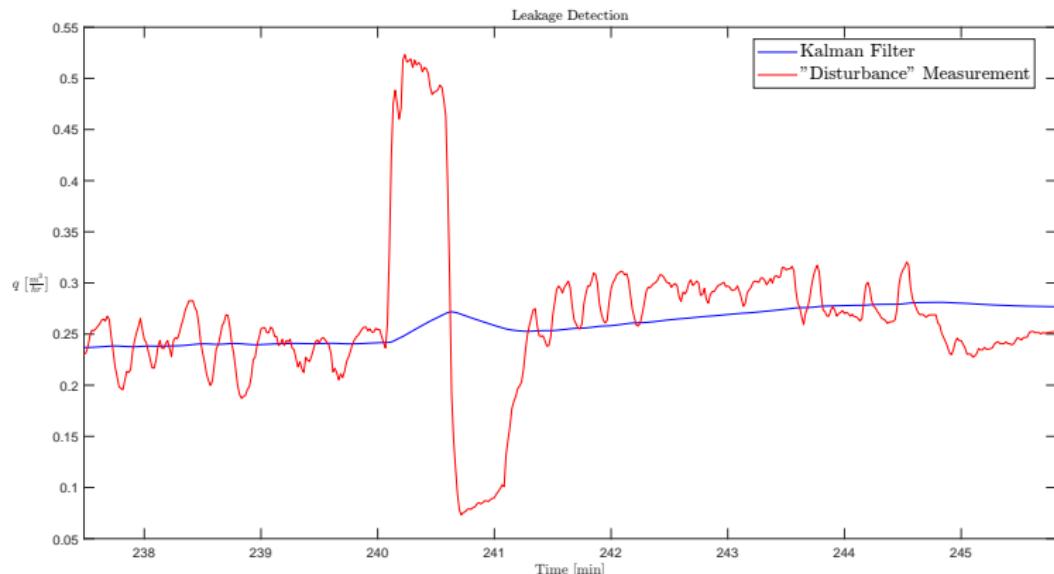
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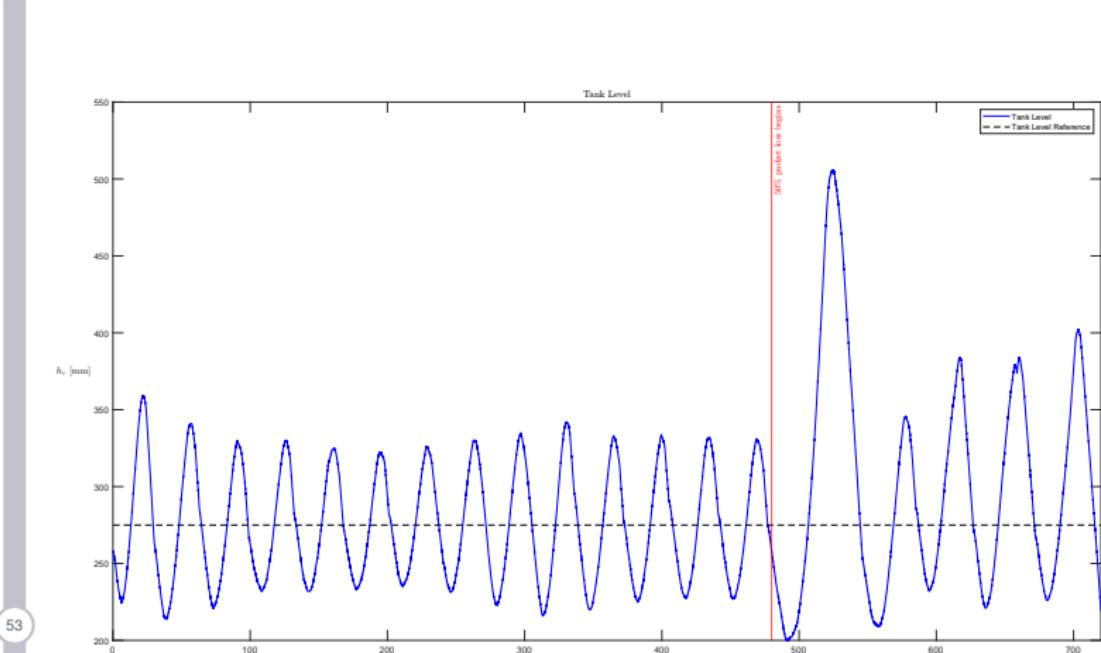
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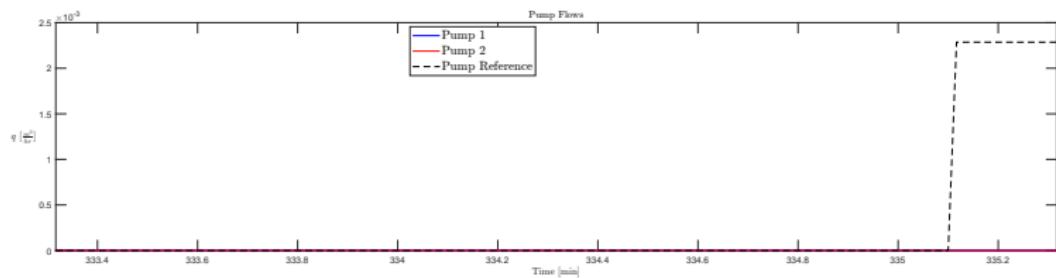
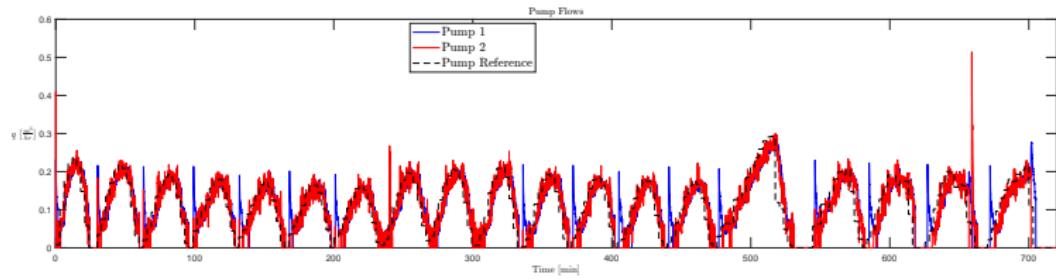
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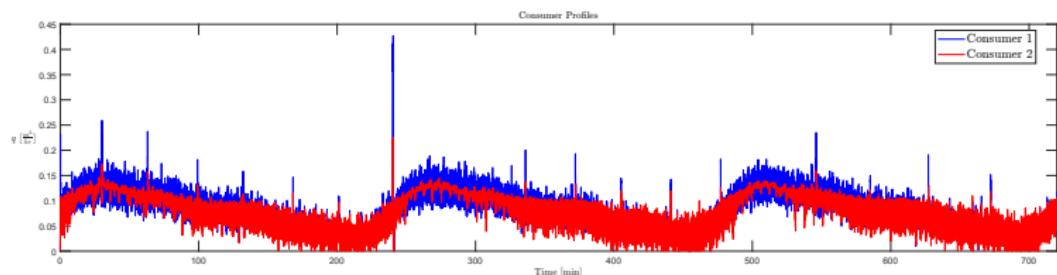
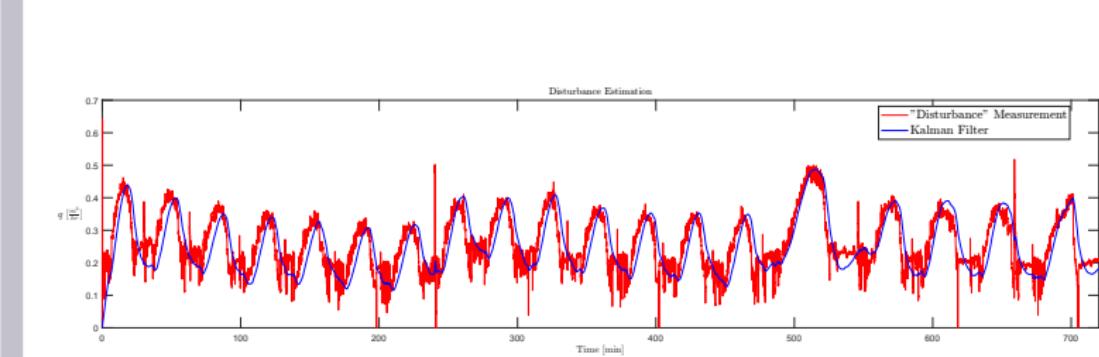
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Open for questions!



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