Modelling and Networked Control of Water Distribution Networks

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CA733

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Agenda

Introduction

Modelling of WDN

Fast Dynamics and Graph theory

Slow Dynamics and Linearisation
- Slow dynamics
- Linearisation

Optimal Control

Disturbance Estimator

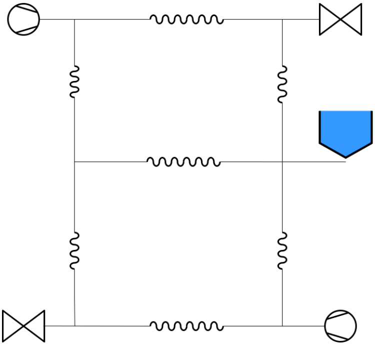
References

Introduction

Introduction to Water Distribution Networks

- Critical societal infrastructure, responsible for the provision of water to both domestic and industrial consumers.
- Network pressure must be controlled.
  - Underpressure $\rightarrow$ insufficient service pressure $\rightarrow$ dissatisfied end users.
  - Overpressure $\rightarrow$ component failure $\rightarrow$ repair costs, supply intermittency, etc.
- Network pressure tied to level in Elevated Water Reservoir(s) (EWR) $\rightarrow$ level control is key!
- Other key components are pumps, valves, and pipes.

We analyse a small-scale test WDN with the following layout:



Larger WDNs are typically not amenable to first-principles modelling.

divided in two sections..?

- Fundamentals

$$p \propto h$$

$$\dot{V} = q$$

- Assume constant cross sectional area A

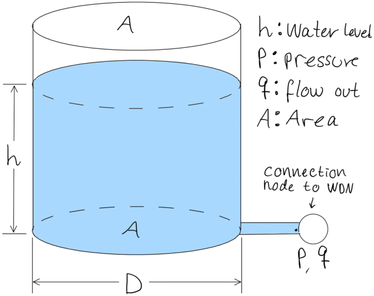
$$V \propto h \implies V \propto p$$

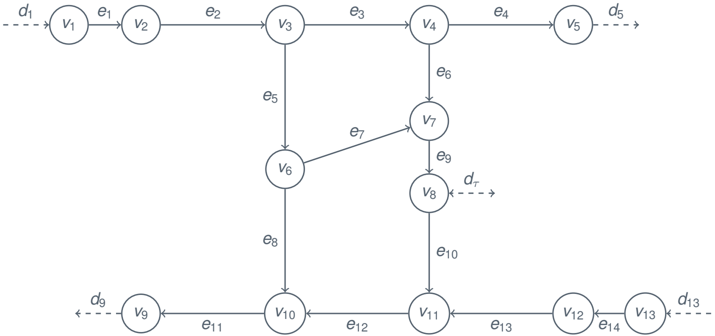
$$\dot{p} \propto \dot{V} \wedge \dot{V} = q \implies \dot{p} \propto q$$

- We arrive at

$$\dot{p} = -\tau q, \text{ where}$$

$$\tau = \rho g \frac{1}{A}$$





In context of WDN we now consider flows to and from network as external demands d_i

Mass conservation holds, and such

$$d_n = -\sum_{i=1}^{n-1} d_i \implies d_\tau = -(d_p + d_c)$$

$$\dot{p} = -\tau d_\tau = \tau(d_p + d_c)$$

When discretised by forward Euler:

$$p_\tau(k+1) = p_\tau(k) - \tau d_\tau(k) t_s = p_\tau(k) + \tau(d_p(k) + d_c(k)) t_s$$

Which corresponds to a discrete, linear state-space model:

$$p_\tau(k+1) = A p_\tau(k) + B_p d_p(k) + B_c d_c(k) \tag{1}$$

$$d_p = \begin{bmatrix} d_1 \\ d_{13} \end{bmatrix}, d_c = \begin{bmatrix} d_5 \\ d_9 \end{bmatrix}, B_p = B_c = t_s \begin{bmatrix} \tau & \tau \end{bmatrix}, A = 1$$

Fast dynamics are non linear - linearisation is needed.
In near vicinity of linearisation point x_0,

$$\dot{x} \approx f(x_0) + \nabla f\Big|_{x_0} (x - x_0)$$

Recalling the fast dynamics differential equation is given as

$$\begin{aligned}\dot{q}_n = &-\mathcal{P}\Phi\Big(\lambda(q_n) + \mu(q_n, \Theta) + \alpha(q_n, \omega)\Big) + \\ &\mathcal{P}\Big(\Psi(\bar{h} - \mathbf{1}h_0) + \mathcal{I}(p_\tau - \mathbf{1}p_0)\Big)\end{aligned} \tag{2}$$

The linear model such becomes

$$\dot{q}_n \approx f(x_0) + \frac{\partial f}{\partial q_n}\Big|_{x_0} \tilde{q}_n + \frac{\partial f}{\partial \Theta}\Big|_{x_0} \tilde{\Theta} + \frac{\partial f}{\partial \omega}\Big|_{x_0} \tilde{\omega} + \frac{\partial f}{\partial p_\tau}\Big|_{x_0} \tilde{p}_\tau \tag{3}$$

where $x_0 = \{q_0, \Theta_0, \omega_0, p_{\tau_0}\}$, $\tilde{q} = q - q_0$, likewise for $\tilde{\Theta}$, $\tilde{\omega}$, $\tilde{p_\tau}$

The full linearised model is then obtained as

$$
\begin{aligned}
\dot{q}_n \approx f(x_0) - \mathcal{P}\Phi \Bigg(& a_1\omega_0 + \Big(|q_0| + \mathrm{sign}(q_0)q_0\Big) \\
& \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\Bigg) \\
& -\mathcal{P}\Phi\left(\left(-|q_0|q_0\frac{2}{K_v^2\Theta_0^3}\right)\tilde{\Theta}\right) \\
& -\mathcal{P}\Phi\left(\Big(a_1 q_0 + 2a_0\omega_0\Big)\tilde{\omega}\right) \\
& +\mathcal{P}\mathcal{I}\tilde{p}_\tau
\end{aligned}
\tag{4}
$$

The equation can be simplified further making some assumptions

10

Equilibrium disappears, valve and tank dynamics assumed to be constant disturbances

$$\dot{q}_n \approx -\mathcal{P}\Phi\left(a_1\omega_0 + \left(|q_0| + \mathrm{sign}(q_0)q_0\right)\right.$$
$$\left.\left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\right) \tag{5}$$
$$-\mathcal{P}\Phi\left(\left(a_1 q_0 + 2a_0\omega_0\right)\tilde{\omega}\right)$$

General optimal control problem

The basic structure of an optimal control problem is sketched in the language of calculus of variations.

$$\dot{x} = f(x, u, t) \tag{6}$$

$$x(t_0) = x_0,\ x \in \mathbb{R}^n,\ \ u \in U \in \mathbb{R}^m \tag{7}$$

where $t \in \mathbb{R}$ is the time and x, u are functions of t, with U the set of admissible controls.

Cost functional:

$$J = \mathcal{M}(x(T)) + \int_0^T \mathcal{L}(x(t), u(t))dt \tag{8}$$

Infinite-Horizon Linear-Quadratic Regulator

The LQR timeinvariant case with no terminal cost.
System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{9}$$

Cost functional:

$$J = \int_{t_0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t)\right) dt \tag{10}$$

State feedback control law:

$$u^*(t) = -R^{-1}B^T P x^*(t) \tag{11}$$

P is time-invariant and fullfills the *algebraic Riccati equation*:

$$PA + A^T P + Q - PBR^{-1}B^T P = 0 \tag{12}$$

Let $\hat{x} = x - x_r$ and $\hat{u} = u - u_r$. Shifted coordinate system cost functional framed as an output tracking problem:

$$J = \int_{t_0}^{\infty} \left((C\hat{x})^T Q_y (C\hat{x}) + \hat{u}^T R \hat{u}\right) dt = \int_{t_0}^{\infty} \left(\hat{y}^T Q_y \hat{y} + \hat{u}^T R \hat{u}\right) dt \tag{13}$$

Tracking LQR and Integral Action

Integral Action

States are extended with integral state x_i

$$u = -\bar{K}\bar{x} \tag{14}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + B_r r \tag{15}$$

$$y = \bar{C}\bar{x} \tag{16}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{17}$$

$$\bar{K} = -\begin{bmatrix} K & -K_i \end{bmatrix} \tag{18}$$

K_i can be an awkward weight to choose.

Velocity-Form LQR

Deviation variables:

$$\Delta x_k = x_k - x_{k-1},\ \Delta y_k = y_k - r_k,\ \Delta u_k = u_k - u_{k-1} \tag{19}$$

Extended vectors and matrices:

$$\begin{aligned}\tilde{\zeta}_k &= \begin{bmatrix}\Delta x_k \\ \Delta y_k\end{bmatrix},\ \tilde{u}_k = \Delta u_k, \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ CA & I\end{bmatrix},\ \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix},\ \tilde{C} = \begin{bmatrix} 0 & I \end{bmatrix}\end{aligned} \tag{20}$$

Cost functional:

$$J = \sum_{k_0}^{\infty} \left(\tilde{\zeta}^T Q \tilde{\zeta} + \tilde{u}^T R \tilde{u}\right) \tag{21}$$

Origin regulation! If $\tilde{\zeta} \to 0 \Rightarrow \Delta y \to 0 \Rightarrow y \to r$.

If linearising around equilibrium point x_e and corresponding operating point u_{op}

$$\Delta\tilde{x} = x - x_e, \; \Delta\tilde{u} = u - u_{op} \tag{22}$$

VF-LQR will penalize deviations from the point $\{x_e, u_{op}, r\}$!

Incremental feedback law:

$$\Delta u^*(k) = -\Delta K(k)\Delta x(k) \tag{23}$$

Control input applied at time k is:

$$u^*(k) = \sum_{i=1}^{k} \Delta u^*(i) \tag{24}$$

with

$$\tilde{K} = (\tilde{B}^T P \tilde{B} - R)^{-1}(\tilde{B}^T P \tilde{A}) \tag{25}$$

Exogenous input disturbance-accommodating LQR

Standard LQR does not accommodate exogenous inputs (such as the the model of the consumer demand flows) but can be modified to do so:

$$u(k) = \sum_{i=1}^{k} \Delta u^*(i) - B^{\dagger}\mathcal{B}\delta(i) \tag{26}$$

where $B^{\dagger}$ is the Moore-Penrose pseudoinverse of B and $\mathcal{B}$ is the disturbance input matrix.

Objectives with the use of Kalman filte

- VF-LQR controller needs an estimate of the consumer demand.
- The optimal estimator is the **Kalman Filter**.
- Recursively finds optimal Kalman gain.
- Kalman filter is also very interesting when from a leakage detection POV.

Water consumption data

- Data of consumption pattern over a 35 day period obtained by CSK.

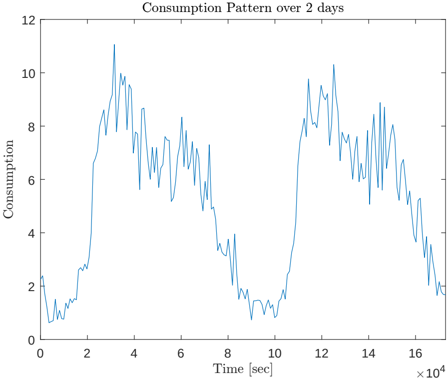


Figure: Consumption pattern over two days

FFT of consumption pattern

- Frequency analysis of the data.

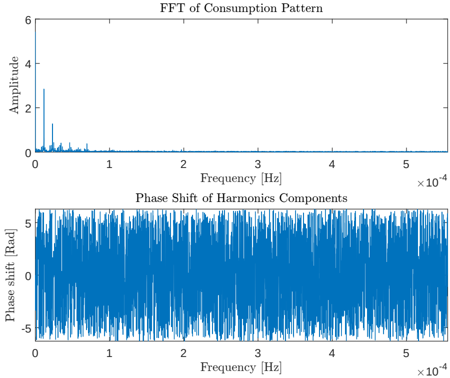


Figure: Consumption pattern over two days

- Highest frequency contents at periods: DC, 24.05h, 12.03h, 8.02h and 5.99h.

Approximation of consumption

- Fourth order Fourier approximation of consumer demand:

$$\begin{aligned} d_c(t) \approx & k_0 + k_1 cos(\omega_1 t + \phi_1) + k_2 cos(\omega_2 t + \phi_2) \\ & + k_3 cos(3\omega_3 t + \phi_3) + k_4 cos(4\omega_4 t + \phi_4) \end{aligned} \tag{27}$$

- We wish to model 27 as a state space model:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

State space representation

- Need to represent the evolution of the "states" in the approximation as a linear combination of states.
- This can not be achieved using the current states.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & \omega_2 & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 cos(\omega_1 t) \\ k_1 sin(\omega_1 t) \\ k_2 cos(\omega_2 t) \\ k_2 sin(\omega_2 t) \end{bmatrix} \tag{28}$$

$$y = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 cos(\omega_1 t) \\ k_1 sin(\omega_1 t) \\ k_2 cos(\omega_2 t) \\ k_2 sin(\omega_2 t) \end{bmatrix} \tag{29}$$

- Fourth order doesn't fit the page..

Model vs. data

- The model compared to the real data - shown over 2 days.

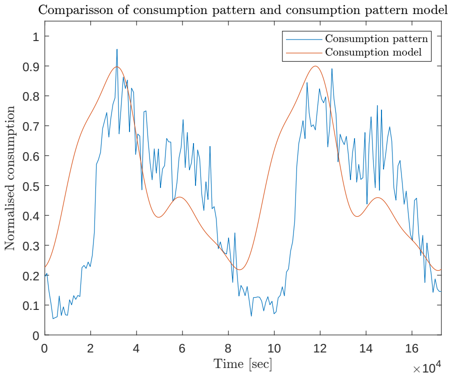


Figure: Comparison of raw historical data, and model

- The model follows the visual trend in real data.

Considerations when designing the KF.

- Stiffness of the filter

References

Open for questions!

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