Modelling and Networked Control of Water Distribution Networks

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CA733

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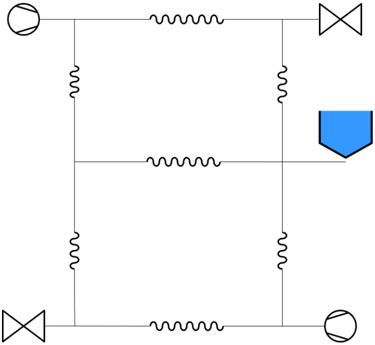
Agenda

- Critical societal infrastructure, responsible for the provision of water to both domestic and industrial consumers.
- Network pressure must be controlled.
  - Underpressure $\rightarrow$ insufficient service pressure $\rightarrow$ dissatisfied end users.
  - Overpressure $\rightarrow$ component failure $\rightarrow$ repair costs, supply intermittency, etc.
- Network pressure tied to level in Elevated Water Reservoir(s) (EWR) $\rightarrow$ level control is key!
- Other key components are pumps, valves, and pipes.

We analyse a small-scale test WDN with the following layout:



Larger WDNs are typically not amenable to first-principles modelling.

- General principles for modelling fast regime very similar to circuit analysis.
    - Water flows = currents.
    - Pressures = voltages.
    - Network components behave similarly to circuit components, but resistors generally non-linear.
- Graph theory is combined with physics-induced boundary conditions to yield model of fast regime.

Graph theory describes a networked system as a directed graph composed of edges and vertices.

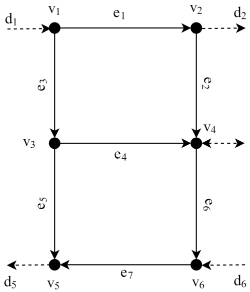


Figure: Graph of simplified WDN network. From [1].

Auxiliary matrices can be defined to mathematically describe the network:

$$H_{i,j} = \begin{cases} 1 & \text{If } j\text{th edge leaves } i\text{th node} \\ -1 & \text{If } j\text{th edge enters } i\text{th node} \\ 0 & \text{If } j\text{th edge and } i\text{th node unconnected} \end{cases}$$

$$B_{i,j} = \begin{cases} 1 & \text{If direction of } i\text{th loop and } j\text{th edge agree} \\ -1 & \text{If direction of } i\text{th loop and } j\text{th edge disagree} \\ 0 & \text{If } i\text{th loop excludes } j\text{th edge} \end{cases}$$

Example:

$$H = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{1}$$

$$\bar{H} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 \end{bmatrix} \tag{2}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 1 \end{bmatrix} \tag{3}$$

Whether a node is open to atmosphere or not, or connected to a tank, can be described with matrices F and G respectively.

$$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

The edges of the WDN can be described with a pressure and flow relationship, analogous to the current and voltage of an electrical component.

This project uses pipes, valves and pumps as edge components.

The general relationship between pressure and flow can be defined as:

$$\Delta p = \mathcal{J}\dot{q} + \lambda(q) + \mu(q,\Theta) + \alpha(q,\omega) - \Delta h \tag{5}$$

The pressure drop across a pipe is defined as:

$$\Delta p_k = \mathcal{J}\dot{q} + \lambda(q) + \Delta z \tag{6}$$

where:

$$\lambda_k(q_k) = \left(f \cdot \frac{8 \cdot L \cdot q^2}{\pi^2 \cdot g \cdot D^5} + k_f \cdot \frac{8 \cdot q^2}{\pi^2 \cdot g \cdot D^4}\right) \cdot g \cdot \rho \tag{7}$$

$$\mathcal{J} = \frac{L \cdot \rho}{A} \tag{8}$$

$$\Delta z_k = \rho \cdot g \cdot \Delta h_k \tag{9}$$

The pressure drop across a valve is defined as:

$$\Delta p_k = \mu(q, OD) = \frac{1}{K_{valve}(\Theta)^2} \cdot |q| \cdot q \tag{10}$$

where OD is the opening degree of the valve.
Deriviation:

$$\frac{\Delta p_1}{q_1^2} = \frac{\Delta p_2}{q_2^2} \Leftrightarrow q_1 = q_2 \cdot \sqrt{\frac{\Delta p_1}{\Delta p_2}} \tag{11}$$

$$q = q_n(\Theta) \cdot \sqrt{\frac{\Delta p_1}{1}} = K_{valve}(\Theta) \cdot \sqrt{\Delta p_1} \tag{12}$$

12

The pressure drop across a pump is defined as:

$$\Delta p_k = a_0 \cdot \omega^2 + a_1 \cdot \omega \cdot q - a_2 \cdot |q| \cdot q \qquad (13)$$

where $[a_0, a_1, a_2]$ is a tuple of coefficients that describe the pump's characteristic curve, q is the flow rate through the pump, and ω is the rotational velocity of the pump.

Kirchhoff's node and mesh law:

$$Hq = d \tag{14}$$

$$B\Delta p = BH^T p = 0 \wedge B\Delta h = BH^T h = 0 \tag{15}$$

Mass conservation:

$$d_n = -\sum_{i=1}^{n-1} d_i \tag{16}$$

Lemma 4.1:

$$H_T \bar{H}_T^{-1} = \begin{bmatrix} I_{n-1} \\ -\mathbf{1}^T \end{bmatrix} \tag{17}$$

where $\mathbf{1}$ is a vector of ones and $I_{n-1} \in \mathbb{R}^{n-1 \times n-1}$ is an identity matrix.

Lemma 4.2:

$$q = B^T q_C + \begin{bmatrix} 0_{C \times n-1} \\ \bar{H}_T^{-1} \end{bmatrix} \bar{d} \tag{18}$$

Non linear differential equation:

$$\Phi\mathcal{J}\Phi^T\dot{q} = -\Phi\Big(\lambda(q_n)+\mu(q_n)+\alpha(q_n)\Big)+\Psi(\bar{h}-\mathbf{1}h_0)+\mathcal{I}(p_\tau-\mathbf{1}p_0) \tag{19}$$

Where the matrices $\Phi, \Psi, \mathcal{I}$ are defined as:

$$\Phi \triangleq \begin{bmatrix} I & -\bar{H}_C^T\bar{H}_T^{-T} \\ 0 & \bar{F}^T\bar{H}_T^{-T} \\ 0 & \bar{G}^T\bar{H}_T^{-T} \end{bmatrix}, \qquad \Psi \triangleq \begin{bmatrix} 0 \\ \bar{F}^T \\ \bar{G}^T \end{bmatrix}, \qquad \mathcal{I} \triangleq \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \tag{20}$$

$$\mathcal{P} : (\Phi\mathcal{J}\Phi^T)^{-1} \tag{21}$$

$$\dot{q}_n = -\mathcal{P}\Phi\Big(\lambda(q_n)+\mu(q_n)+\alpha(q_n)\Big)+\mathcal{P}\Big(\Psi(\bar{h}-\mathbf{1}h_0)+\mathcal{I}(p_\tau-\mathbf{1}p_0) \tag{22}$$

- Fundamentals

$$p \propto h$$

$$\dot{V} = q$$

- Assume constant cross sectional area A

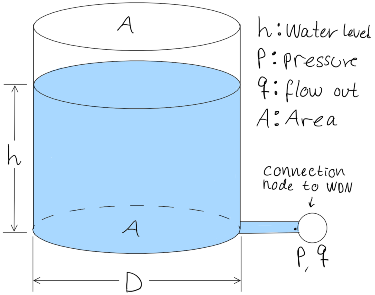
$$V \propto h \implies V \propto p$$

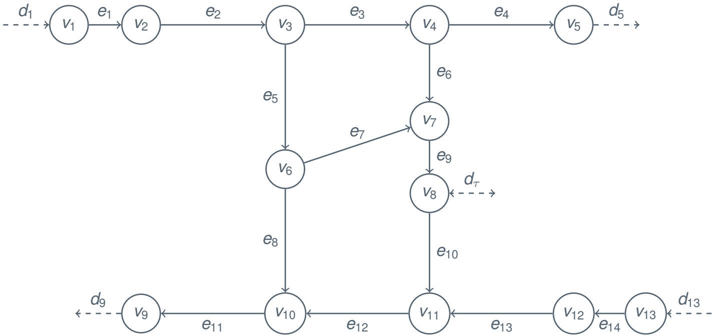
$$\dot{p} \propto \dot{V} \wedge \dot{V} = q \implies \dot{p} \propto q$$

- We arrive at

$$\dot{p} = -\tau q, \text{ where}$$

$$\tau = \rho g \frac{1}{A}$$





In context of WDN we now consider flows to and from network as external demands d_i

Mass conservation holds, and such

$$d_n = -\sum_{i=1}^{n-1} d_i \implies d_\tau = -(d_p + d_c)$$

$$\dot{p} = -\tau d_\tau = \tau(d_p + d_c)$$

When discretised by forward Euler:

$$p_\tau(k+1) = p_\tau(k) - \tau d_\tau(k) t_s = p_\tau(k) + \tau(d_p(k) + d_c(k)) t_s$$

Which corresponds to a discrete, linear state-space model:

$$p_\tau(k+1) = A p_\tau(k) + B_p d_p(k) + B_c d_c(k) \tag{23}$$

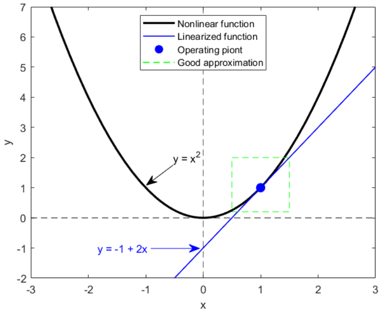
$$d_p = \begin{bmatrix} d_1 \\ d_{13} \end{bmatrix}, d_c = \begin{bmatrix} d_5 \\ d_9 \end{bmatrix}, B_p = B_c = t_s \begin{bmatrix} \tau & \tau \end{bmatrix}, A = 1$$

- Fast dynamics are non-linear
- Linearisation required

In near vicinity of linearisation point x_0,

$$\dot{x} \approx f(x_0) + \nabla f\Big|_{x_0} (x - x_0)$$

Linearising around equilibrium point preferred



Recalling the fast dynamics differential equation is given as

$$\dot{q}_n = -\mathcal{P}\Phi\Big(\lambda(q_n) + \mu(q_n,\Theta) + \alpha(q_n,\omega)\Big) + \mathcal{P}\Big(\Psi(\bar{h} - \mathbf{1}h_0) + \mathcal{I}(p_\tau - \mathbf{1}p_0)\Big) \tag{24}$$

The linear model such becomes

$$\dot{q}_n \approx f(x_0) + \frac{\partial f}{\partial q_n}\bigg|_{x_0}\tilde{q}_n + \frac{\partial f}{\partial \Theta}\bigg|_{x_0}\tilde{\Theta} + \frac{\partial f}{\partial \omega}\bigg|_{x_0}\tilde{\omega} + \frac{\partial f}{\partial p_\tau}\bigg|_{x_0}\tilde{p}_\tau \tag{25}$$

where $x_0 = \{q_0, \Theta_0, \omega_0, p_{\tau 0}\}$, $\tilde{q} = q - q_0$, likewise for $\tilde{\Theta}$, $\tilde{\omega}$, $\tilde{p}_\tau$

The full linearised model is then obtained as

$$\begin{aligned}\dot{q}_n \approx f(x_0) - \mathcal{P}\Phi\Bigg(& a_1\omega_0 + \Big(|q_0| + \mathrm{sign}(q_0)q_0\Big) \\ & \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\Bigg) \\ & -\mathcal{P}\Phi\left(\left(-|q_0|q_0\frac{2}{K_v^2\Theta_0^3}\right)\tilde{\Theta}\right) \\ & -\mathcal{P}\Phi\Big(\big(a_1 q_0 + 2a_0\omega_0\big)\tilde{\omega}\Big) \\ & +\mathcal{P}\mathcal{I}\tilde{p}_\tau\end{aligned} \tag{26}$$

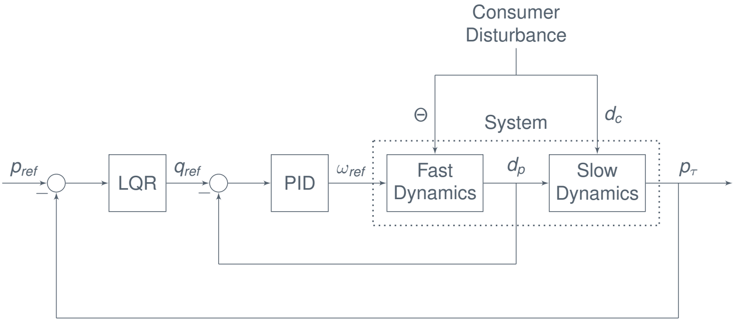
The equation can be simplified further making some assumptions

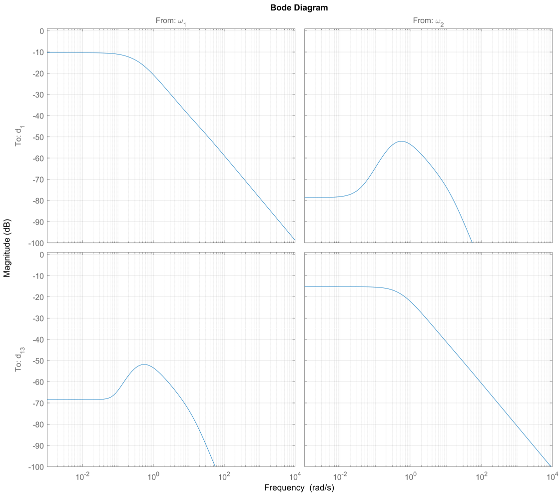
Equilibrium disappears, valve and tank dynamics assumed to be constant disturbances

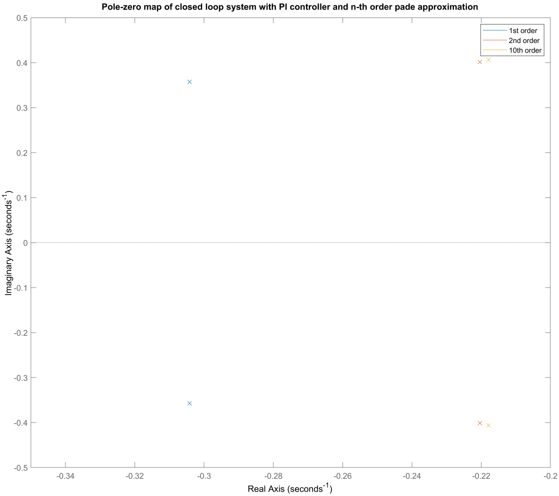
$$\begin{aligned}\dot{q}_n \approx -\mathcal{P}\Phi\Bigg(& a_1\omega_0 + \Big(|q_0| + \operatorname{sign}(q_0)q_0\Big) \\ & \left(K_\lambda + a_2 + \frac{1}{(K_v\Theta_0)^2}\right)\tilde{q}_n\Bigg) \\ -\mathcal{P}\Phi\Bigg(& \Big(a_1 q_0 + 2a_0\omega_0\Big)\tilde{\omega}\Bigg)\end{aligned} \tag{27}$$

Control Structure

We elect for a cascaded control structure with optimal central control and local PI control.

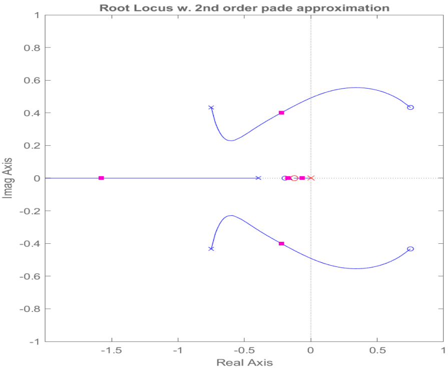



Bode Diagram
From: ω_1
From: ω_2
To: d_1
To: d_{13}
Magnitude (dB)
Frequency (rad/s)
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100
10^{-2}
10^{0}
10^{2}
10^{4}



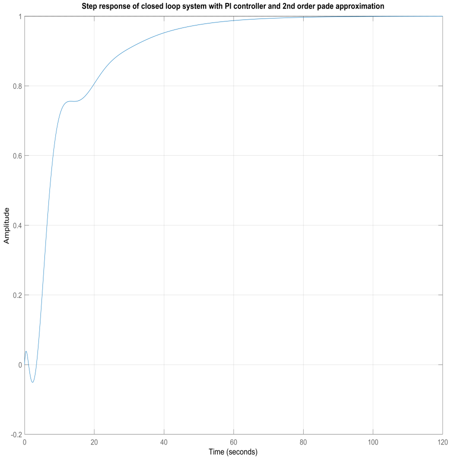
Pole-zero map of closed loop system with PI controller and n-th order pade approximation
1st order
2nd order
10th order
Imaginary Axis (seconds^{-1})
Real Axis (seconds^{-1})
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.34
-0.32
-0.3
-0.28
-0.26
-0.24
-0.22
-0.2

- Faster than outer loop.
- No steady state error.
- No overshoot, and low amount of oscillation.
- Stability.



Results in controller transfer function:

$$C(s) = \frac{K_p s + K_i}{s} = \frac{1.8s + 0.225}{s} \quad (28)$$



Step response of closed loop system with PI controller and 2nd order pade approximation
Amplitude
Time (seconds)
1
0.8
0.6
0.4
0.2
0
-0.2
0
20
40
60
80
100
120

The basic structure of an optimal control problem is sketched in the language of calculus of variations.

$$\dot{x} = f(x, u, t) \tag{29}$$

$$x(t_0) = x_0,\ x \in \mathbb{R}^n,\ u \in U \subseteq \mathbb{R}^m \tag{30}$$

where $t \in \mathbb{R}$ is the time and x, u are functions of t, with U the set of admissible controls.

Cost functional:

$$J = \mathcal{M}(x(T)) + \int_0^T \mathcal{L}(x(t), u(t))dt \tag{31}$$

Infinite-Horizon Linear-Quadratic Regulator

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The LQR time-invariant case with no terminal cost.
System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{32}$$

Cost functional:

$$J = \int_{t_0}^{\infty} \left(x^T(t)Qx(t) + u^T(t)Ru(t) \right) dt \tag{33}$$

State feedback control law:

$$u^*(t) = -R^{-1}B^TPx^*(t) \tag{34}$$

P is time-invariant and fullfills the *algebraic Riccati equation*:

$$PA + A^TP + Q - PBR^{-1}B^TP = 0 \tag{35}$$

Let $\hat{x} = x - x_r$ and $\hat{u} = u - u_r$. Shifted coordinate system cost functional framed as an output tracking problem:

$$J = \int_{t_0}^{\infty} \left(\left(C^T \hat{x}^T \right) Q_y \left(C\hat{x} \right) + \hat{u}^T R \hat{u} \right) dt = \int_{t_0}^{\infty} \left(\hat{y}^T Q_y \hat{y} + \hat{u}^T R \hat{u} \right) dt \tag{36}$$

- If linearised around some equilibrium point, these can be used as reference.

States are extended with integral state x_i

$$u = -\bar{K}\bar{x} \tag{37}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u + B_r r \tag{38}$$

$$y = \bar{C}\bar{x} \tag{39}$$

$$\bar{A} = \begin{bmatrix} A & 0 \\ -C & 0 \end{bmatrix}, \ \bar{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}, \ B_r = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \tag{40}$$

$$\bar{K} = -\begin{bmatrix} K & -K_i \end{bmatrix} \tag{41}$$

K_i can be an awkward weight to choose.

Velocity-Form LQR

Deviation variables:

$$\Delta x_k = x_k - x_{k-1}, \ \Delta y_k = y_k - r_k, \ \Delta u_k = u_k - u_{k-1} \tag{42}$$

Extended vectors and matrices:

$$\begin{aligned} \tilde{\zeta}_k &= \begin{bmatrix} \Delta x_k \\ \Delta y_k \end{bmatrix}, \ \tilde{u}_k = \Delta u_k, \\ \tilde{A} &= \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} B \\ CB \end{bmatrix}, \ \tilde{C} = \begin{bmatrix} 0 & I \end{bmatrix} \end{aligned} \tag{43}$$

Cost functional:

$$J = \sum_{k_0}^{\infty} \left(\tilde{\zeta}^T Q \tilde{\zeta} + \tilde{u}^T R \tilde{u} \right) \tag{44}$$

Origin regulation! If $\tilde{\zeta} \to 0 \Rightarrow \Delta y \to 0 \Rightarrow y \to r$.

Control input applied at time k is:

$$u^*(k) = \sum_{i=1}^{k} \Delta u^*(i) \tag{45}$$

with

$$\tilde{K} = (\tilde{B}^T P \tilde{B} - R)^{-1} (\tilde{B}^T P \tilde{A}) \tag{46}$$

Standard LQR does not accommodate exogenous inputs (such as the the model of the consumer demand flows) but can be modified to do so:

$$u(k) = \sum_{i=1}^{k} \Delta u^*(i) - B^\dagger \mathcal{B}\delta(i) \tag{47}$$

where $B^\dagger$ is the Moore-Penrose pseudoinverse of B and $\mathcal{B}$ is the disturbance input matrix.

Objectives with the use of disturbance estimation

- VF-LQR controller needs an estimate of the consumer demand.
- The optimal estimator is the **Kalman Filter**.

- Need a linear model of consumer behaviour for the Kalman Filter
- Under normal circumstances the Kalman gain is found recursively.
- In the case of LTI system the Kalman filter itself becomes time invariant $\rightarrow$ constant Kalman gain.
  ARE:

$$\begin{aligned} \Pi &= A(\Pi^{-1} + C^T R^{-1} C)^{-1} A^T + Q \\ K &= \Pi C^T (C \Pi C^T + R)^{-1} \end{aligned} \quad (48)$$

- Kalman filter is also very interesting from a leakage detection POV.

Disturbance Estimation

Water Consumption Data

- Data of consumption pattern over a 35 day period courtesy of CSK and Grundfos.

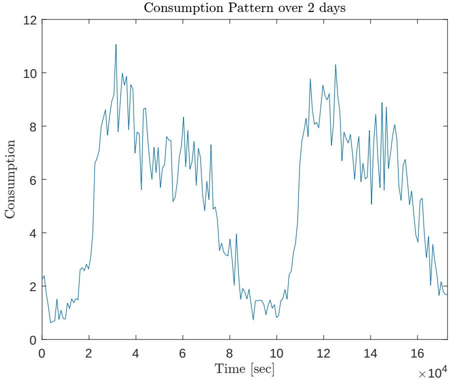


Figure: Consumption pattern over two days

- Frequency analysis of the data.

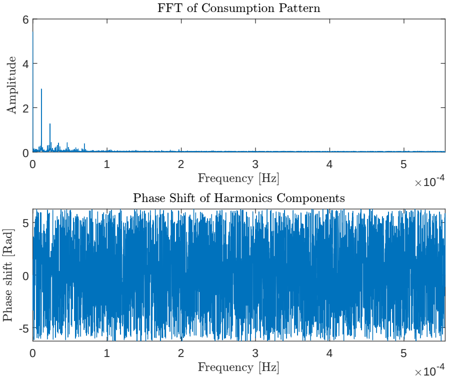


Figure: Amplitude and phase content of full consumption pattern

We want a sparse representation. Largest frequency components are: DC, 24.05hr, 12.03hr, 8.02hr and 5.99hr.

- Fourth-order Fourier approximation of consumer demand:

$$\begin{aligned} d_c(t) \approx\; & k_0 + k_1 cos(\omega_1 t + \phi_1) + k_2 cos(\omega_2 t + \phi_2) \\ & + k_3 cos(3\omega_3 t + \phi_3) + k_4 cos(4\omega_4 t + \phi_4) \end{aligned} \tag{49}$$

- We wish to model equation 49 as a state-space model:

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

- Need to represent the evolution of the "states" in the approximation as a linear combination of states.
- This can not be achieved using the current states.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\omega_1 & 0 & 0 \\ 0 & \omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega_2 \\ 0 & 0 & 0 & \omega_2 & 0 \end{bmatrix} \begin{bmatrix} k_0 \\ k_1 cos(\omega_1 t) \\ k_1 sin(\omega_1 t) \\ k_2 cos(\omega_2 t) \\ k_2 sin(\omega_2 t) \end{bmatrix} \tag{50}$$

$$C = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \end{bmatrix}, \quad \dot{x} = \begin{bmatrix} 0 \\ -k_1\omega_1 sin(\omega_1 t) \\ k_1\omega_1 cos(\omega_1 t) \\ -k_2\omega_2 sin(\omega_2 t) \\ k_2\omega_2 cos(\omega_2 t) \end{bmatrix} \tag{51}$$

- Fourth order doesn't fit the page!

Disturbance Estimation

Model vs. Data

- The model compared to the real data - shown over 2 days.

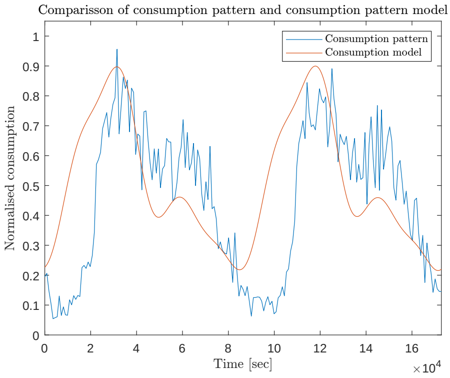


Figure: Comparison of raw historical data, and model

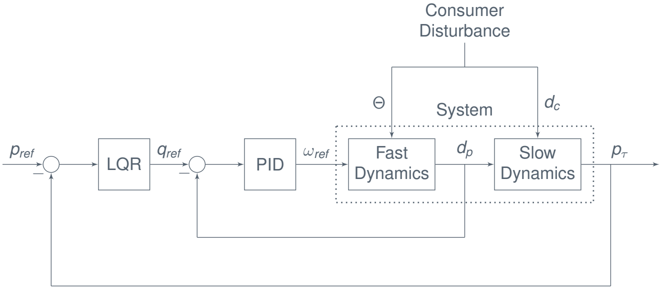
- The model follows the visual trend in real data.

Considerations when designing the KF.

- In practice the Kalman gain is found using lqr in matlab.
- Stiffness of the filter is decided by Q-R ratio.
- Big R means uncertainty concerning observation and that we trust model much $\rightarrow$ stiff filter.

Network Effects

- Outer loop
  - Central control unit
  - Transmit data
- Urban environment
  - Packet loss scaling with distance
  - 60% or more expected at 20 km
- Try-Once-Discard protocol used
  - Packet loss $\rightarrow$ assume states = 0



Network Effects

For Specific Packet Loss

- Packet loss is stochastic, must think in terms of mean-square stability:

$$\lim_{k\to\infty} \mathrm{E}[x(k)^2] = 0 \tag{52}$$

- Assuming 60% as upper bound for loss, α, the condition for mean-square stability is [2]:

$$\mathcal{S}\Big(\alpha A \otimes A + (1-\alpha)(A-BK)\otimes(A-BK)\Big) < 1 \tag{53}$$

- This is because propagation matrix of the state covariance is:

$$\alpha A \otimes A + (1-\alpha)(A-BK)\otimes(A-BK) \tag{54}$$

- We get $\mathcal{S} = 0.9453$

- Upper bound on packet loss Ξ. All values $\alpha < \Xi$ will also be stable.

$$\Xi = \frac{1}{\|\sigma_+(V)\|_\infty}$$
$$V = \begin{bmatrix} (S \otimes \hat{S} + \hat{S} \otimes S)(I - S \otimes S)^{-1} & \hat{S} \otimes \hat{S} \\ (I - S \otimes S)^{-1} & 0 \end{bmatrix} \tag{55}$$
$$S = (A - BK) \otimes (A - BK)\,, \quad \hat{S} = A \otimes A - S$$

- In the limit of zero control:

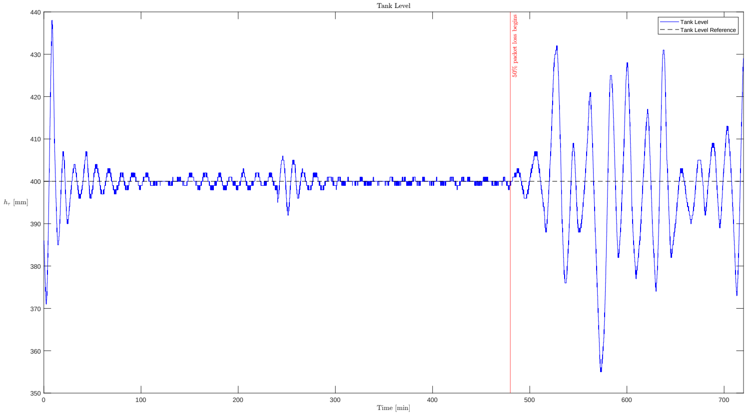
$$\begin{aligned} \lim_{\tilde{K} \to 0} & \left(\alpha \tilde{A} \otimes \tilde{A} + (1 - \alpha)(\tilde{A} - \tilde{B}\tilde{K}) \otimes (\tilde{A} - \tilde{B}\tilde{K})\right) \\ & = \left(\alpha \tilde{A} \otimes \tilde{A} + (1 - \alpha)\tilde{A} \otimes \tilde{A}\right) \end{aligned} \tag{56}$$

$$\forall \alpha \in \{0 \dots 1\}: \quad \left(\alpha \tilde{A} \otimes \tilde{A} + (1-\alpha)\tilde{A} \otimes \tilde{A}\right) = \tilde{A} \otimes \tilde{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad (57)$$

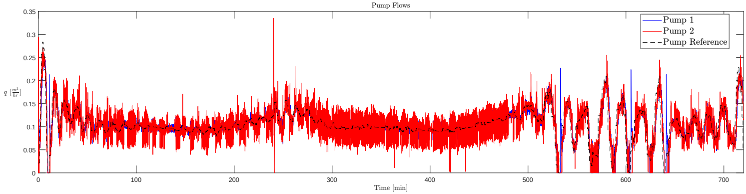
Thus *any* EWR is marginally stable in the zero limit of control for any packet loss as $\mathcal{S}(1,1,1,1) = 1$.

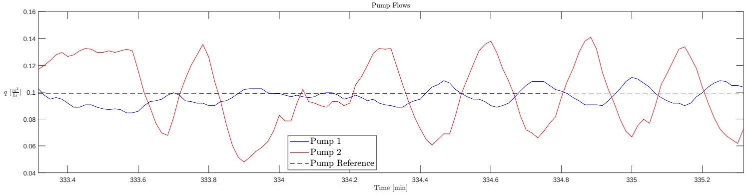
Results
SWIL Laboratory Setup

Results
Level Control



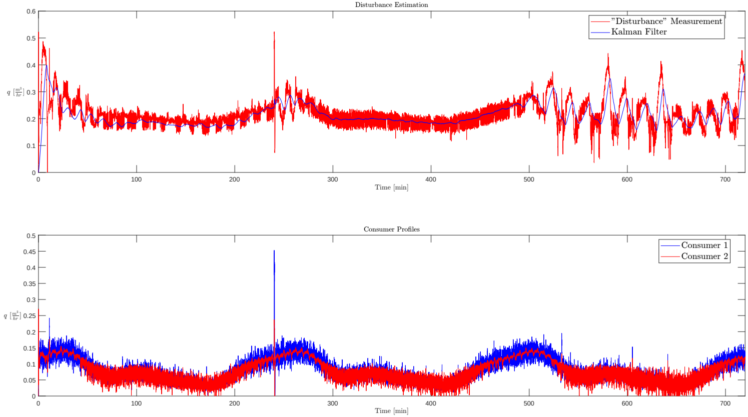
Results
Flow Control





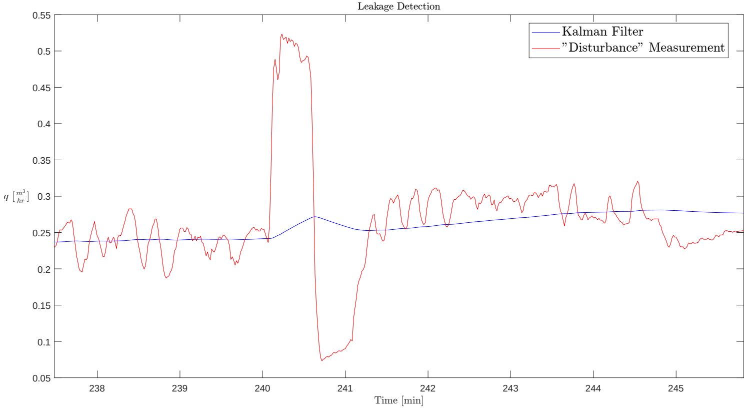
Results

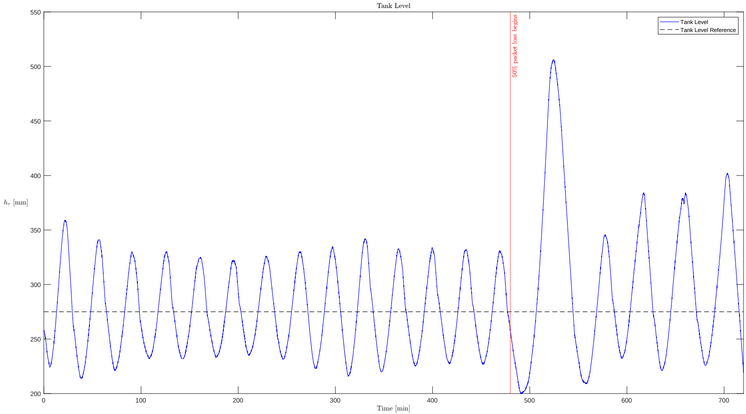
Disturbance Estimation



Results

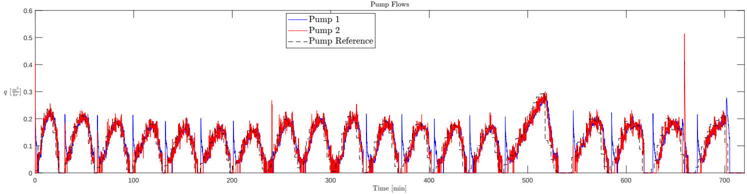
Leakage Detection

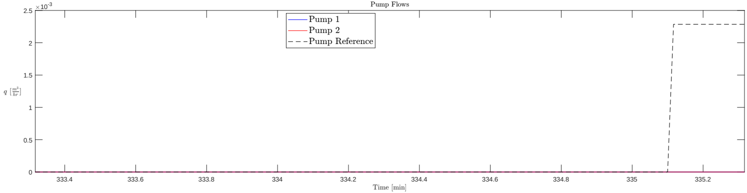



Tank Level
Tank Level
Tank Level Reference
50% packet loss begins
h_τ [mm]
Time [min]
550
500
450
400
350
300
250
200
0
100
200
300
400
500
600
700

Results

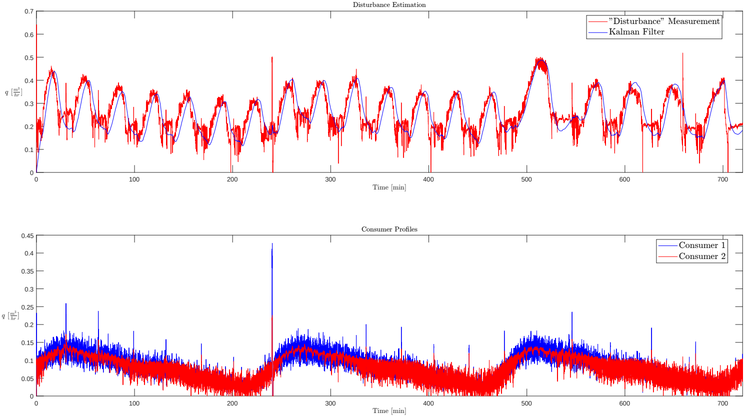
Flow Control - Pathological Behaviour





Results

Disturbance Estimation - Pathological Behaviour



References

[1] S. S. Rathore, “Optimal Control in Water Distribution Network,” 2019.

[2] S. Hu and W. Y. Yan, “Stability robustness of networked control systems with respect to packet loss,” *Automatica*, vol. 43, no. 7, pp. 1243–1248, jul 2007.

Open for questions!

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