Convex bilinear inversion of sparse vectors

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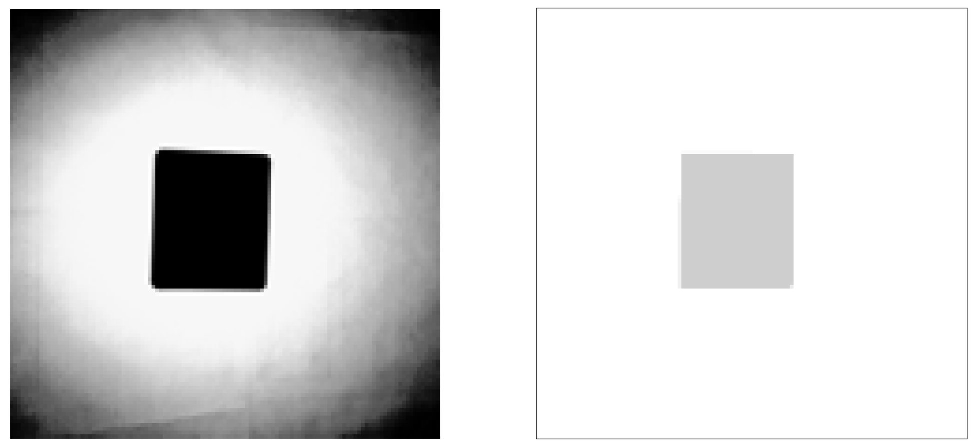
Mousepad data



(a) Distorted image (b) Recovered image

Panel (a) shows an image of a mousepad. The goal is to recover the piecewise constant block that correspond to the mousepad. Robust ℓ_1-BranchHull (3) is used to recover the signal. Panel (b) shows the output of (3).

Problem Statement

The bilinear inverse problem we consider is:
Let $\boldsymbol{y}$, $\boldsymbol{w}^\natural$, $\boldsymbol{x}^\natural \in \mathbb{R}^L$ such that

$$\boldsymbol{y} = \boldsymbol{w}^\natural \circ \boldsymbol{x}^\natural. \tag{1}$$

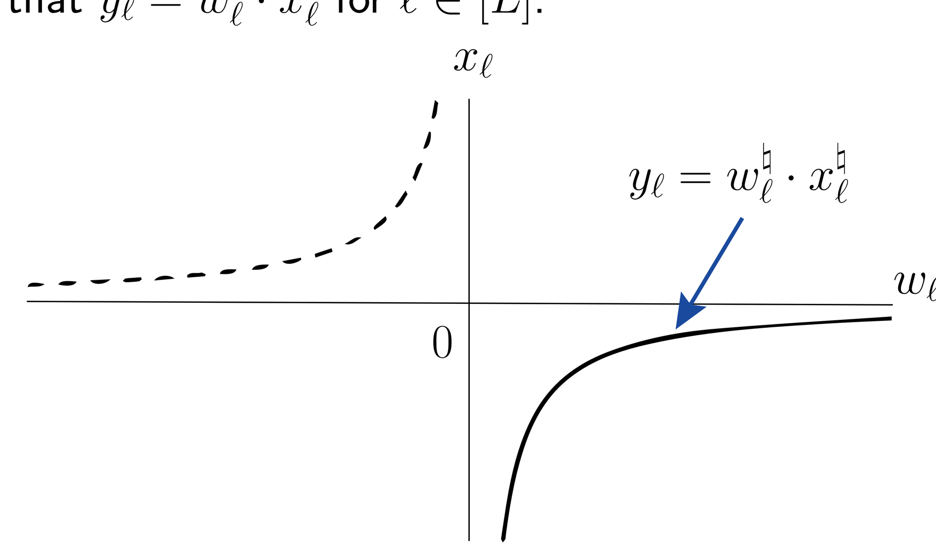
Let $\boldsymbol{B} \in \mathbb{R}^{L\times K}$, $\boldsymbol{C} \in \mathbb{R}^{L\times N}$ such that

$$\boldsymbol{w}^\natural = \boldsymbol{B}\boldsymbol{h}^\natural,$$
$$\boldsymbol{x}^\natural = \boldsymbol{C}\boldsymbol{m}^\natural.$$

Let $\|\boldsymbol{h}^\natural\|_0 = S_1$, $\|\boldsymbol{m}^\natural\|_0 = S_2$.
Given $\boldsymbol{y}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and $\boldsymbol{s} = \operatorname{sign}(\boldsymbol{w}^\natural)$,
Find $(\boldsymbol{h}^\natural, \boldsymbol{m}^\natural)$ up to the scaling ambiguity.

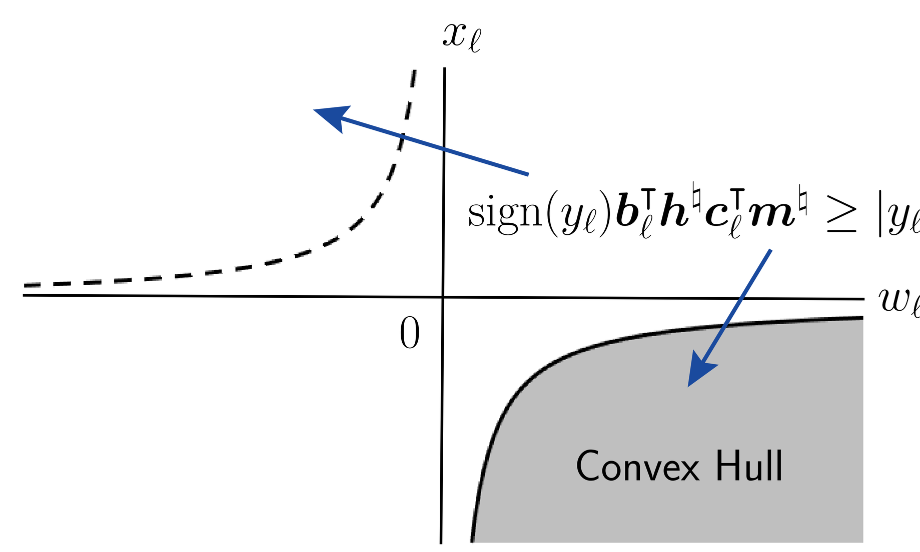
Identifiability from bilinear measurements

Let $\boldsymbol{y} \in \mathbb{R}^L$ be a bilinear measurement of $\boldsymbol{w}^\natural$ and $\boldsymbol{x}^\natural$ in $\mathbb{R}^L$ such that $y_\ell = w_\ell^\natural \cdot x_\ell^\natural$ for $\ell \in [L]$.



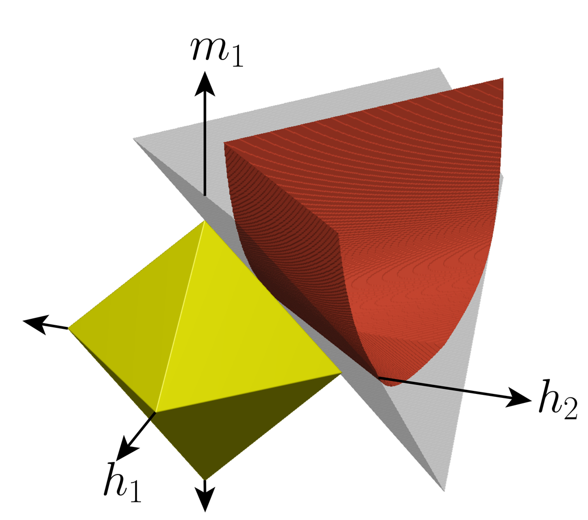
- Without additional structural assumption on $\boldsymbol{w}^\natural$ and $\boldsymbol{x}^\natural$, both $(\boldsymbol{w}^\natural, \boldsymbol{x}^\natural)$ and $(\boldsymbol{1}, \boldsymbol{w}^\natural \circ \boldsymbol{x}^\natural)$ solves the problem. We assume $\boldsymbol{w}^\natural$ and $\boldsymbol{x}^\natural$ live in known subspaces.
- For any $c \neq 0$, $(c\boldsymbol{w}^\natural, c^{-1}\boldsymbol{x}^\natural)$ solves the problem

Convex program



We introduce a convex program written in the natural parameter space for the bilinear inverse problem. The convex program ℓ_1-BranchHull (ℓ_1-BH) program is used to recover $(\boldsymbol{h}^\natural, \boldsymbol{m}^\natural)$.

$$(\boldsymbol{h}^*, \boldsymbol{m}^*) := \underset{(\boldsymbol{h},\boldsymbol{m})\in\mathbb{R}^{K+N}}{\operatorname{argmin}} \|\boldsymbol{h}\|_1 + \|\boldsymbol{m}\|_1 \quad (\ell_1\text{-BH})$$
$$\text{subject to} \quad \operatorname{sign}(y_\ell)\boldsymbol{b}_\ell^\intercal \boldsymbol{h} \boldsymbol{c}_\ell^\intercal \boldsymbol{m} \geq |y_\ell|$$
$$s_\ell \boldsymbol{b}_\ell^\intercal \boldsymbol{h} \geq 0, \quad \ell = 1, 2, \ldots, L.$$



In the above figure,
- the feasible set (red) is the intersection of L convex sets,
- the objective function (yellow) intersects the feasible set at a point $(\boldsymbol{h}, \boldsymbol{m})$ with $\|\boldsymbol{h}\|_1 = \|\boldsymbol{m}\|_1$, and
- the gray hyperplane segments correspond to linearization of the hyperbolic measurements.

Effective sparsity condition

We provide a recovery guarantee theorem for the class of sparse vectors $\boldsymbol{h}^\natural$ and $\boldsymbol{m}^\natural$ with comparable sparsity levels. Precisely, the vectors $\boldsymbol{h}^\natural$ and $\boldsymbol{m}^\natural$ have comparable effective sparsity if there exists an absolute constant C such that

$$\frac{\|\boldsymbol{h}^\natural\|_1}{\|\boldsymbol{h}^\natural\|_2} = \alpha \frac{\|\boldsymbol{m}^\natural\|_1}{\|\boldsymbol{m}^\natural\|_2} \tag{2}$$

with $\frac{1}{C} \leq \alpha \leq C$.

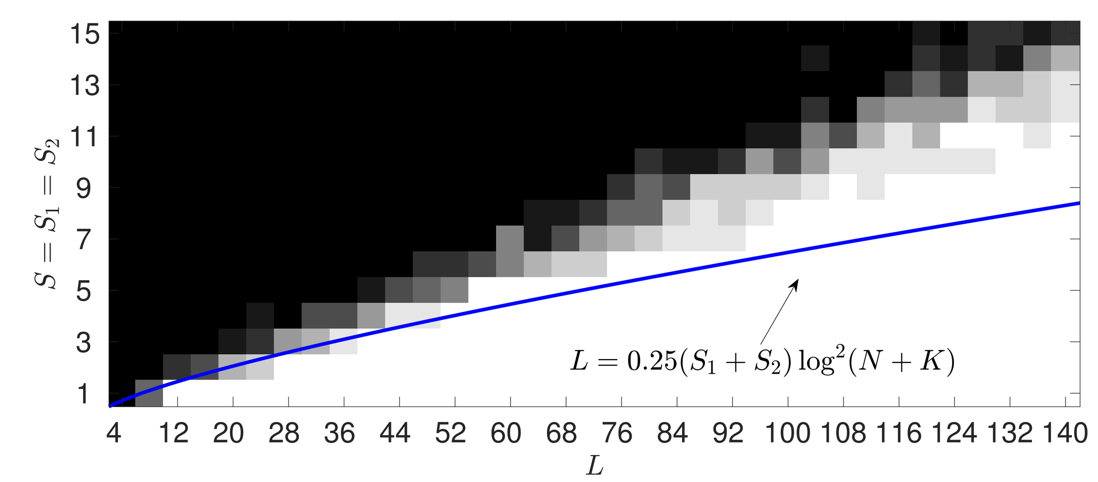
Recovery Theorem

Let $\boldsymbol{B}$ and $\boldsymbol{C}$ have i.i.d. $\mathcal{N}(0,1)$ entries.
If $(\boldsymbol{h}^\natural, \boldsymbol{m}^\natural$ satisfy (2) and $L \geq C_t(S_1 + S_2)\log^2(K+N)$, then the unique minimizer $(\boldsymbol{h}^*, \boldsymbol{m}^*)$ of (ℓ_1-BH) satisfies

$$(\boldsymbol{h}^*, \boldsymbol{m}^*) = \left(\boldsymbol{h}^\natural\sqrt{\frac{\|\boldsymbol{m}^\natural\|_1}{\|\boldsymbol{h}^\natural\|_1}}, \boldsymbol{m}^\natural\sqrt{\frac{\|\boldsymbol{h}^\natural\|_1}{\|\boldsymbol{m}^\natural\|_1}}\right)$$

with probability at least $1 - e^{-cLt^2}$. Here, c are absolute constants and C_t depends on $t > 0$.

Phase Plot



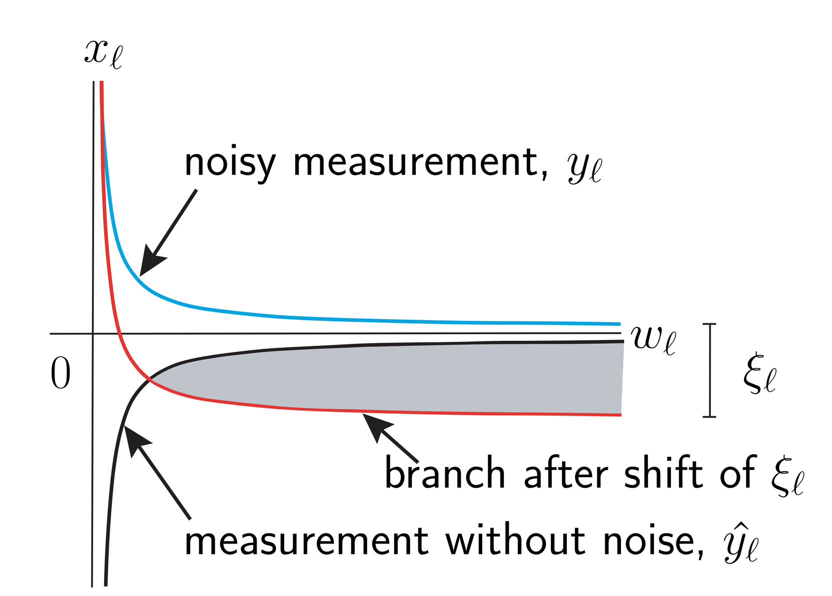
The empirical recovery probability for imbalanced synthetic data. The shades of black and white represents the fraction of successful simulation.

Robust formulation

Let $\boldsymbol{\nu} \in \mathbb{R}^L$ be multiplicative noise so that

$$\boldsymbol{y} = \left(\boldsymbol{w}^\natural \circ \boldsymbol{x}^\natural\right) \circ (\boldsymbol{1} + \boldsymbol{\nu}) = \hat{\boldsymbol{y}} \circ (\boldsymbol{1} + \boldsymbol{\nu})$$

If $\nu_\ell < -1$, the shape of the feasible set changes. $\boldsymbol{\xi}$ shifts noisy measurement to ensure a consistent feasible set.



$$\underset{(\boldsymbol{h},\boldsymbol{m},\boldsymbol{\xi})\in\mathbb{R}^{K+N+L}}{\text{minimize}} \|\boldsymbol{P}\boldsymbol{h}\|_1 + \|\boldsymbol{m}\|_1 + \lambda\|\boldsymbol{\xi}\|_1 \tag{3}$$
$$\text{subject to} \quad \operatorname{sign}(y_\ell)\boldsymbol{b}_\ell^\intercal \boldsymbol{h}(\boldsymbol{c}_\ell^\intercal \boldsymbol{m} + \xi_\ell) \geq |y_\ell|$$
$$t_\ell \boldsymbol{b}_\ell^\intercal \boldsymbol{h} \geq 0, \quad \ell = 1, 2, \ldots, L.$$

ADMM implementation

The ADMM scheme that solves (3) can be presented in closed form.

Let $\boldsymbol{u} = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{w}\\ \boldsymbol{\xi}\end{pmatrix}, \quad \boldsymbol{z} = \begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{h}\\ \lambda\boldsymbol{\xi}\end{pmatrix}.$

Let $\boldsymbol{E} = \begin{pmatrix}\boldsymbol{C} & 0 & 0\\ 0 & \boldsymbol{B} & 0\\ 0 & 0 & \lambda^{-1}\boldsymbol{I}_L\end{pmatrix}, \quad \boldsymbol{Q} = \begin{pmatrix}\boldsymbol{I}_N & 0 & 0\\ 0 & \boldsymbol{P} & 0\\ 0 & 0 & \boldsymbol{I}_L\end{pmatrix}.$

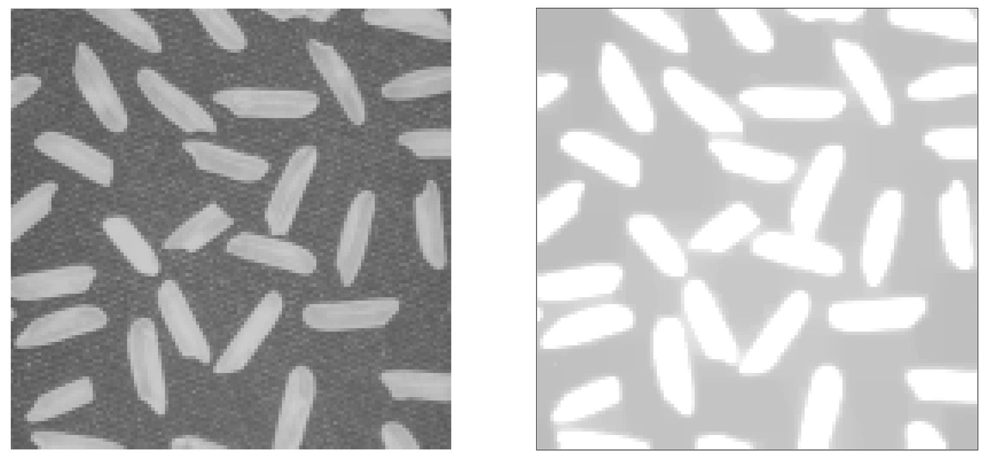
Let $\mathcal{C} = \Big\{\boldsymbol{u} \in \mathbb{R}^{3L} \,|\, \operatorname{sign}(y_\ell)(\xi_\ell + x_\ell)w_\ell \geq |y_\ell|, \; s_\ell w_\ell \geq 0, \; \ell \in [L]\Big\}.$

The ADMM steps are:

$$\boldsymbol{u}_{k+1} = \text{proj}_{\mathcal{C}}\left(\boldsymbol{E}\boldsymbol{z}_k - \boldsymbol{\alpha}_k\right), \tag{4}$$
$$\boldsymbol{v}_{k+1} = S_{1/\rho}\left(\boldsymbol{Q}\boldsymbol{z}_k - \boldsymbol{\beta}_k\right), \tag{5}$$
$$\boldsymbol{z}_{k+1} = \left(\boldsymbol{E}^\intercal\boldsymbol{E} + \boldsymbol{Q}^\intercal\boldsymbol{Q}\right)^{-1}\Big(\boldsymbol{E}^\intercal(\boldsymbol{\alpha}_k + \boldsymbol{u}_{k+1}) + \boldsymbol{Q}^\intercal(\boldsymbol{\beta}_k + \boldsymbol{v}_{k+1})\Big), \tag{6}$$
$$\boldsymbol{\alpha}_{k+1} = \boldsymbol{\alpha}_k + \boldsymbol{u}_{k+1} - \boldsymbol{E}\boldsymbol{z}_{k+1},$$
$$\boldsymbol{\beta}_{k+1} = \boldsymbol{\beta}_k + \boldsymbol{v}_{k+1} - \boldsymbol{Q}\boldsymbol{z}_{k+1}.$$

where $\text{proj}_{\mathcal{C}}(\boldsymbol{z})$ is the projection of $\boldsymbol{z}$ onto $\mathcal{C}$ and $S_c(\cdot)$ the soft-thresholding operator.

Rice grain data



(a) Distorted image (b) Recovered image

Panel (a) shows an image of rice grains. The goal is to remove distortions and recover a piecewise constant image that outlines the rice grains. Robust ℓ_1-BranchHull (3) is used to recover the signal. Panel (b) shows the output of (3).

Key Points

ℓ_1-BranchHull is a convex algorithm that is/has:
- posed in the natural parameter space,
- initialization free and does not require approximate solutions, and
- optimal sample complexity up to a log factor.