CADMAS-STR Program Description

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1．Program Outline

The developed structural analysis code for performing coupled analyses with the gas-liquid two-phase flow analysis code CADMAS-SURF/3D-2F is described below.

［１］Overview of Functions

（１）Analysis type：Direct transient response analysis（static analysis is used for initial conditions）

（２）Nonlinear analysis：Geometric nonlinear（Total Lagrange method）
Material nonlinear（elastoplasticity）
Contact（static friction , dynamic friction ）

（３）Element library：Tetrahedron, Pentahedron, Hexahedron
Each primary and secondary element
Beam element（rectangular section）

（４）Special materials：Geotechnical（based on Biot's formula）
Porous Material

（５）Matrix solver：Multi-Frontal method, Domain Decomposition CG method（both can be computed in parallel）

［２］Coupled with CADMAS-SURF/3D-2F

The following two types of coupled analysis are possible. A flow diagram is shown on next page.

（１）One-way coupling

① At first, the CADMAS calculation is performed. At that time, the structural analysis mesh data is read by CADMAS and considered as obstruction.

② When CADMAS calculations executed, the pressure on the surface of the structure is obtained by interpolation of cell pressure in CADMAS, and the history is output to a file.

③ Next, the structural analysis is performed. The file② is read in and the pressure on the surface of the structure is take into account as a load.

（２）Bidirectional coupling

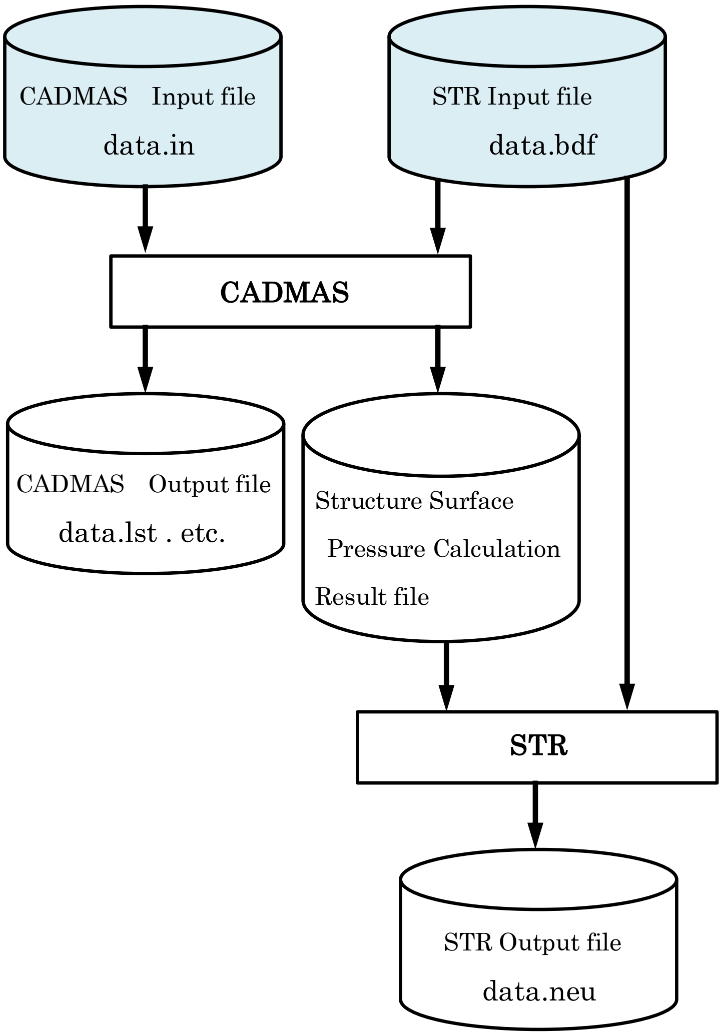
① CADMAS and structural analysis are executed in parallel. The parallel computation method is MPMD Multiple Program Multiple Data).

② The calculation proceeds by exchanging the following data with each other.

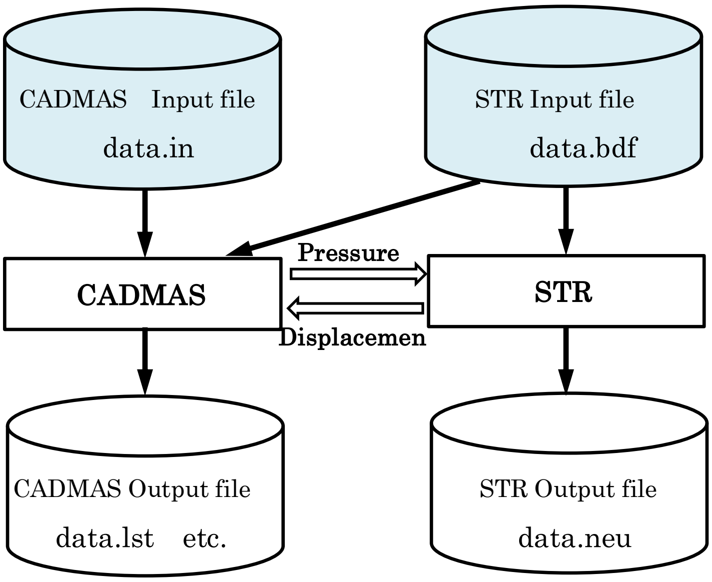
Structural analysis → CADMAS：Structure location

CADMAS → Structural Analysis：Structure Surface Pressure

One-way coupling



Bidirectional coupling



2．Formulation

2．1　Governing equations, discretization

2．1．1　Structural Analysis (Geotechnical)

［1］Governing equations

（1）Equation of motion

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} = \rho \ddot{\boldsymbol{u}} \quad -①$$

$\boldsymbol{\sigma}$: Stress tensor

ρ : Density of the geotechnical

$\rho = (1-n)\rho_s + n\rho_f$

n : Porosity

ρ_s : Density of soil particles

ρ_f : Density of pore water

$\boldsymbol{g}$: Gravitational acceleration

$\boldsymbol{u}$: Displacement

（2）Relation between stress and effective stress

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p\boldsymbol{I} \quad -②$$

$\boldsymbol{\sigma}'$: Effective stress tensor

p : Pressure of pore water

$\boldsymbol{I}$: Unit tensor

［2］Application of the weighted residual method

Applying the weighted residual method to the equation if motion in ① with the weight function as $\delta\boldsymbol{u}$

$$\int_V \delta\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\sigma})dV + \int_V \delta\boldsymbol{u}\cdot\rho\boldsymbol{g}dV = \int_V \delta\boldsymbol{u}\cdot\rho\ddot{\boldsymbol{u}}dV$$

$$-\int_V \boldsymbol{\sigma}:\delta\boldsymbol{\epsilon}dV + \int_S \delta\boldsymbol{u}\cdot(\boldsymbol{\sigma}^T\cdot\boldsymbol{n})dS + \int_V \delta\boldsymbol{u}\cdot\rho\boldsymbol{g}dV = \int_V \delta\boldsymbol{u}\cdot\rho\ddot{\boldsymbol{u}}dV$$

$$-\int_V \boldsymbol{\sigma}:\delta\boldsymbol{\epsilon}dV + \int_{S_2} \delta\boldsymbol{u}\cdot\boldsymbol{t}dS + \int_V \delta\boldsymbol{u}\cdot\rho\boldsymbol{g}dV = \int_V \delta\boldsymbol{u}\cdot\rho\ddot{\boldsymbol{u}}dV$$

$$\int_V \delta\boldsymbol{u}\cdot\rho\ddot{\boldsymbol{u}}dV + \int_V \boldsymbol{\sigma}:\delta\boldsymbol{\epsilon}dV = \int_{S_2} \delta\boldsymbol{u}\cdot\boldsymbol{t}dS + \int_V \delta\boldsymbol{u}\cdot\rho\boldsymbol{g}dV$$

$\boldsymbol{\epsilon}$: Strain tensor

$\boldsymbol{n}$: Outward normal vector to the surface

$\boldsymbol{t}$: Surface forces

S : Boundary surface

$S = S_1 + S_2$

S_1: Type 1 boundary（on S_1 $\delta\boldsymbol{u} = \boldsymbol{0}$）

S_2: Type 2 boundary（on S_2 $\boldsymbol{\sigma}^T \cdot \boldsymbol{n} = \boldsymbol{t}$）

Substituting equation (2) into this equation

$$\int_V \delta\boldsymbol{u} \cdot \rho\ddot{\boldsymbol{u}} dV + \int_V (\boldsymbol{\sigma}' - p\boldsymbol{I}) : \delta\boldsymbol{\epsilon} dV = \int_{S_2} \delta\boldsymbol{u} \cdot \boldsymbol{t} dS + \int_V \delta\boldsymbol{u} \cdot \rho\boldsymbol{g} dV$$

$$\int_V \delta\boldsymbol{u} \cdot \rho\ddot{\boldsymbol{u}} dV + \int_V \boldsymbol{\sigma}' : \delta\boldsymbol{\epsilon} dV - \int_V p\boldsymbol{I} : \delta\boldsymbol{\epsilon} dV = \int_{S_2} \delta\boldsymbol{u} \cdot \boldsymbol{t} dS + \int_V \delta\boldsymbol{u} \cdot \rho\boldsymbol{g} dV$$

$$\int_V \delta\boldsymbol{u} \cdot \rho\ddot{\boldsymbol{u}} dV + \int_V \boldsymbol{\sigma}' : \delta\boldsymbol{\epsilon} dV - \int_V \delta\epsilon_v p dV = \int_{S_2} \delta\boldsymbol{u} \cdot \boldsymbol{t} dS + \int_V \delta\boldsymbol{u} \cdot \rho\boldsymbol{g} dV$$

Change the tensor to vector notation

$$\int_V \delta\boldsymbol{u} \cdot \rho\ddot{\boldsymbol{u}} dV + \int_V \delta\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\sigma}}' dV - \int_V \delta\epsilon_v p dV = \int_{S_2} \delta\boldsymbol{u} \cdot \boldsymbol{t} dS + \int_V \delta\boldsymbol{u} \cdot \rho\boldsymbol{g} dV \quad -③$$

$$\epsilon_v = \epsilon_x + \epsilon_y + \epsilon_z$$

$$\hat{\boldsymbol{\epsilon}} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \qquad \hat{\boldsymbol{\sigma}}' = \begin{Bmatrix} \sigma'_x \\ \sigma'_y \\ \sigma'_z \\ \tau'_{xy} \\ \tau'_{yz} \\ \tau'_{zx} \end{Bmatrix}$$

［3］Spatial discretization

For interpolation function and their derivatives, as well as calculation methods for volume and areas integrals, see "Motoki Yagawa and Shinobu Yoshimura, Finite Element Method, Computational Mechanics and CAE Series 1, Baifukan, 1995, Chapter 5". Also, for interpolation function for pentahedral elements and the calculation of shear strain see MacNeal, R.H., The PENTA Solid Element, MSC Internal Memo No. RHM-43, Oct. 22, 1976”.

（1）To discretize equation (3) spatially, define the following matrix

$$p = [N_1 \quad N_2 \quad \cdots] \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \end{Bmatrix} = [N]\{P\}$$

$$\boldsymbol{u} = \begin{bmatrix} N_1 & & & N_2 & & & \cdots \\ & N_1 & & & N_2 & & \\ & & N_1 & & & N_2 & \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \\ U_2 \\ V_2 \\ W_2 \\ \\ \vdots \end{Bmatrix} = [N^3]\,\{U\}$$

$$\hat{\boldsymbol{\epsilon}} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x} & & \\ & \dfrac{\partial}{\partial y} & \\ & & \dfrac{\partial}{\partial z} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & \\ & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial z} & & \dfrac{\partial}{\partial x} \end{bmatrix} [N^3]\,\{U\} = [B]\,\{U\}$$

$$\epsilon_v = \nabla \cdot \boldsymbol{u} = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{bmatrix} [N^3]\,\{U\} = \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial z} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial z} & \cdots \end{bmatrix} \{U\}$$
$$= [E]\,\{U\}$$

P_i： Pressure of pore water at node i

U_i： Displacement of node i (x component)

V_i： Displacement of node i (y component)

W_i： Displacement of node i (z component)

N_i： Interpolation function corresponding to node i

（2）Using the above matrix, discretize each term in equation ③.

$$\int_V \delta\boldsymbol{u} \cdot \rho \ddot{\boldsymbol{u}} dV = \{\delta U\}^T \int_V \rho [N^3]^T [N^3] dV \{\ddot{U}\} = \{\delta U\}^T [M] \{\ddot{U}\}$$

$$\int_V \delta\hat{\boldsymbol{\epsilon}} \cdot \hat{\boldsymbol{\sigma}}' dV = \{\delta U\}^T \int_V [B]^T \hat{\boldsymbol{\sigma}}'\, dV = \{\delta U\}^T \{F\}$$

※ See Appendix 3, 3 [3] for {F} when geometric nonlinearity is taken into account.

$$\int_V \delta\epsilon_v p dV = \{\delta U\}^T \int_V [E]^T p\, dV = \{\delta U\}^T \{F^p\}$$

$$\int_{S_2} \delta\boldsymbol{u} \cdot \boldsymbol{t} dS = \{\delta U\}^T \int_{S_2} [N^3]^T \boldsymbol{t} dS = \{\delta U\}^T \{R_1\}$$

※ Refer t Appendix 1 for the calculation of $\{R_1\}$

$$\int_V \delta\boldsymbol{u}\cdot\rho\boldsymbol{g}dV = \{\delta U\}^T \int_V \rho[N^3]^T\, dV\boldsymbol{g} = \{\delta U\}^T\{R_2\}$$

Substituting these into each term of equation ③

$$\{\delta U\}^T\left([M]\{\ddot{U}\} + \{F\} - \{F^p\}\right) = \{\delta U\}^T(\{R_1\} + \{R_2\})$$

$$[M]\{\ddot{U}\} + \{F\} - \{F^p\} = \{R_1\} + \{R_2\} = \{R\} \quad -④$$

［4］Temporal discretization

Discretize equation ④ in time. The incremental form is used for nonlinear analysis. In Eq.

$\{\ \}_n$: Value at time n

$\{\ \}'$: Nonlinear iterative computation of the value at pre *iteration*

Δt_1 : Time $n-1$, time increment between time n

Δt_2 : Time n, time increment between time $n+1$

β : Time integration parameter $(0.25 \leq \beta < 0.5)$

Also add on the left side $C_M[M]\{\dot{U}\}_n$, $C_K\{\dot{F}\}_n$ as the attenuation term. See Appendix 2 for how to specify C_K.

$$[M]\{\ddot{U}\}_n + C_M[M]\{\dot{U}\}_n + C_K\{\dot{F}\}_n + \{\bar{F}\}_n - \{\bar{F}^p\}_n = \{\bar{R}\}_n \quad -⑤$$

$$\{\dot{U}\}_n = \frac{\{U\}_{n+1} - \{U\}_{n-1}}{\Delta t_1 + \Delta t_2} = \frac{\{\Delta U\} + \{U\}'_{n+1} - \{U\}_{n-1}}{2\Delta t_{12}} = \frac{\{\Delta U\}}{2\Delta t_{12}} + \{\dot{U}\}'_n$$

$$\{\ddot{U}\}_n = \frac{\dfrac{\{U\}_{n+1} - \{U\}_n}{\Delta t_2} - \dfrac{\{U\}_n - \{U\}_{n-1}}{\Delta t_1}}{\dfrac{\Delta t_1 + \Delta t_2}{2}} = \frac{\dfrac{\{\Delta U\} + \{U\}'_{n+1} - \{U\}_n}{\Delta t_2} - \dfrac{\{U\}_n - \{U\}_{n-1}}{\Delta t_1}}{\Delta t_{12}}$$

$$= \frac{\{\Delta U\}}{\Delta t_2 \cdot \Delta t_{12}} + \{\ddot{U}\}'_n$$

$$\{\bar{F}\}_n = \beta\{F\}_{n+1} + (1-2\beta)\{F\}_n + \beta\{F\}_{n-1} = \beta(\{\Delta F\} + \{F\}'_{n+1}) + (1-2\beta)\{F\}_n + \beta\{F\}_{n-1}$$

In this time

$$\{\Delta F\} = \int_V [B]^T\Delta\hat{\boldsymbol{\sigma}}'\, dV \fallingdotseq \int_V [B]^T\,[D]\Delta\hat{\epsilon}dV = \int_V [B]^T\,[D][B]\{\Delta U\}dV = \int_V [B]^T\,[D][B]dV\{\Delta U\}$$

$$= [K]\{\Delta U\}$$

$[D]$: Constitutive law matrix

※ For $[D]$ in the case of plasticity see "Y. Yamada, Plasticity and Viscoelasticity, Fundamentals and Applications of Finite Element Method Series 6, Baifukan, 1995, pp. 93".

※ For [K] when geometric nonlinearity is considered, see Appendix 3, 3 [1], [2].

Base on

$$\{\bar{F}\}_n \simeq \beta([K]\{\Delta U\} + \{F\}'_{n+1}) + (1-2\beta)\{F\}_n + \beta\{F\}_{n-1} = \beta[K]\{\Delta U\} + \{\bar{F}\}'_n$$

$$\{\bar{F}^p\}_n = \beta\{F^p\}_{n+1} + (1-2\beta)\{F^p\}_n + \beta\{F^p\}_{n-1}$$

$$= \beta(\{\Delta F^p\} + \{F^p\}'_{n+1}) + (1-2\beta)\{F^p\}_n + \beta\{F^p\}_{n-1}$$

In this time

$$\{\Delta F^p\} = \int_V [E]^T \Delta p\, dV = \int_V [E]^T [N]\{\Delta P\} dV = \int_V [E]^T [N] dV \{\Delta P\} = [K^p]\{\Delta P\}$$

Base on

$$\{\bar{F}^p\}_n = \beta([K^p]\{\Delta P\} + \{F^p\}'_{n+1}) + (1-2\beta)\{F^p\}_n + \beta\{F^p\}_{n-1}$$

$$= \beta[K^p]\{\Delta P\} + \{\bar{F}^p\}'_n$$

$$\{\dot{F}\}_n = \frac{1}{2}\left(\frac{\{F\}_{n+1} - \{F\}_{n,new}}{\Delta t_2} + \frac{\{F\}_{n,old} - \{F\}_{n-1}}{\Delta t_1}\right)$$

$$\simeq \frac{1}{2}\left(\frac{[K]\{\Delta U\} + \{F\}'_{n+1} - \{F\}_{n,new}}{\Delta t_2} + \frac{\{F\}_{n,old} - \{F\}_{n-1}}{\Delta t_1}\right)$$

$$= \frac{1}{2\Delta t_2}[K]\{\Delta U\} + \{\dot{F}\}'_n$$

$$\{\bar{R}\}_n = \beta\{R\}_{n+1} + (1-2\beta)\{R\}_n + \beta\{R\}_{n-1}$$

Substituting the above discretized vectors into equation ⑤ and organizing

$$[M]\left(\frac{\{\Delta U\}}{\Delta t_2 \cdot \Delta t_{12}} + \{\ddot{U}\}'_n\right) + C_M[M]\left(\frac{\{\Delta U\}}{2\Delta t_{12}} + \{\dot{U}\}'_n\right) + C_K\left(\frac{1}{2\Delta t_2}[K]\{\Delta U\} + \{\dot{F}\}'_n\right) + (\beta[K]\{\Delta U\} + \{\bar{F}\}'_n)$$

$$-(\beta[K^p]\{\Delta P\} + \{\bar{F}^p\}'_n) = \{\bar{R}\}_n$$

$$\left(\frac{1}{\Delta t_{12}}\left(\frac{1}{\Delta t_2}+\frac{C_M}{2}\right)[M] + \left(\frac{C_K}{2\Delta t_2}+\beta\right)[K]\right)\{\Delta U\} - \beta[K^p]\{\Delta P\}$$

$$= \{\bar{R}\}_n - \left([M]\{\ddot{U}\}'_n + C_M[M]\{\dot{U}\}'_n + C_K\{\dot{F}\}'_n + \{\bar{F}\}'_n - \{\bar{F}^p\}'_n\right) \quad - ⑥$$

$$[A^{uu}]\{\Delta U\} + [A^{up}]\{\Delta P\} = \{Q^u\} \quad - ⑦$$

［5］ Correspondence with variables in the program

The correspondence between each term on the right side of equation ⑥ (the ' indicating the value of the previous iteration is omitted) and the variables in the program is shown below (blue letters: variables in the program).

$$\overbrace{\{\bar{R}\}_n}^{\text{FTO}} - \overbrace{\left([M]\{\ddot{U}\}_n + C_M[M]\{\dot{U}\}_n + C_K\{\dot{F}\}_n + \{\bar{F}\}_n - \{\bar{F}^p\}_n\right)}^{\text{FTI}}$$

$$\{\bar{R}\}_n = \beta\overbrace{\{R\}_{n+1}}^{\text{FCO}(:,:,3)} + (1-2\beta)\overbrace{\{R\}_n}^{\text{FCO}(:,:,2)} + \beta\overbrace{\{R\}_{n-1}}^{\text{FCO}(:,:,1)}$$

$$[M]\{\ddot{U}\}_n = \frac{1}{\Delta t_{12}}\left(\frac{1}{\Delta t_2}\overbrace{[M]\{U\}_{n+1}}^{\text{FCM}(:,:,3)} - \left(\frac{1}{\Delta t_2}+\frac{1}{\Delta t_1}\right)\overbrace{[M]\{U\}_n}^{\text{FCM}(:,:,2)} + \frac{1}{\Delta t_1}\overbrace{[M]\{U\}_{n-1}}^{\text{FCM}(:,:,1)}\right)$$

$$C_M[M]\{\dot{U}\}_n = \frac{\overbrace{C_M[M]\{U\}_{n+1}}^{\text{FCMD}(:,:,3)} - \overbrace{C_M[M]\{U\}_{n-1}}^{\text{FCMD}(:,:,1)}}{2\Delta t_{12}}$$

$$C_K\{\dot{F}\}_n = \frac{1}{2}\left(\frac{\overbrace{C_K\{F\}_{n+1}}^{\text{FCD}(:,:,4)} - \overbrace{C_K\{F\}_{n,new}}^{\text{FCD}(:,:,3)}}{\Delta t_2} + \frac{\overbrace{C_K\{F\}_{n,old}}^{\text{FCD}(:,:,2)} - \overbrace{C_K\{F\}_{n-1}}^{\text{FCD}(:,:,1)}}{\Delta t_1}\right)$$

$$\{\bar{F}\}_n = \beta\overbrace{\{F\}_{n+1}}^{\text{FCK}(:,:,3)} + (1-2\beta)\overbrace{\{F\}_n}^{\text{FCK}(:,:,2)} + \beta\overbrace{\{F\}_{n-1}}^{\text{FCK}(:,:,1)}$$

FCP(:, :, 1)

$$\{\bar{F}^p\}_n = \beta\{F^p\}_{n+1} + (1 - 2\beta)\{F^p\}_n + \beta\{F^p\}_{n-1}$$

2．1．2 **Seepage Flow（Pore water）**

［1］Governing equations

（1）Equation of motion

$$\dot{\boldsymbol{w}} = k(-\nabla p + \rho_f \boldsymbol{g} - \rho_f \ddot{\boldsymbol{u}}) \quad -⑧$$

$\boldsymbol{w}$： Displacement of pore water relative to geotechnical

$\boldsymbol{w} = n(\boldsymbol{U} - \boldsymbol{u})$

$\boldsymbol{U}$： Displacement of pore water

k： Permeability coefficient

（2）Mass conservation equation

$$\nabla \cdot \dot{\boldsymbol{w}} = -\nabla \cdot \dot{\boldsymbol{u}} - C_{Kf}\dot{p} \quad -⑨$$

$$C_{Kf} = \frac{n}{K_f}$$

K_f： Volumetric modulus of pore water

［2］Application of the weighted residual method

Applying the weighted residual method to the mass conservation equation in ⑨ with the weight function as δp.

$$\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{w}} dV = -\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV - \int_V \delta p \, C_{Kf}\dot{p} dV$$

$$-\int_V \nabla \delta p \cdot \dot{\boldsymbol{w}} dV + \int_S \delta p \, \dot{\boldsymbol{w}} \cdot \boldsymbol{n} dS = -\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV - \int_V \delta p \, C_{Kf}\dot{p} dV$$

$$-\int_V \nabla \delta p \cdot \dot{\boldsymbol{w}} dV + \int_{S_2} \delta p \, q dS = -\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV - \int_V \delta p \, C_{Kf}\dot{p} dV$$

$\boldsymbol{n}$： Outward normal vector to the surface

q： Runoff flux from surface

S： Boundary surface

$S = S_1 + S_2$

S_1: Type 1 boundary（on S_1 $\delta \boldsymbol{u} = \boldsymbol{0}$）

S_2: Type 2 boundary（on S_2 $\dot{\boldsymbol{w}} \cdot \boldsymbol{n} = q$）

Substituting equation ⑧ into this equation

$$-\int_V \nabla \delta p \cdot k(-\nabla p + \rho_f \boldsymbol{g} - \rho_f \ddot{\boldsymbol{u}}) dV + \int_{S_2} \delta p \, q dS = -\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV - \int_V \delta p \, C_{Kf}\dot{p} dV$$

$$k\rho_f \int_V \nabla \delta p \cdot \ddot{\boldsymbol{u}} dV + \int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV + C_{Kf} \int_V \delta p \, \dot{p} dV + k \int_V \nabla \delta p \cdot \nabla p dV$$

$$= -\int_{S_2} \delta p\, q dS + k\rho_f \int_V \nabla \delta p \cdot \boldsymbol{g} dV \quad - ⑩$$

［３］ Spatial Discretization

For interpolation function and their derivatives, as well as calculation methods for volume and areas integrals, see "Motoki Yagawa and Shinobu Yoshimura, Finite Element Method, Computational Mechanics and CAE Series 1, Baifukan, 1995, Chapter 5". Also, for interpolation function for pentahedral elements and the calculation of shear strain see MacNeal, R.H., The PENTA Solid Element, MSC Internal Memo No. RHM-43, Oct. 22, 1976".

（１） To discretize equation ⑩ spatially, define the following matrix

$$p = [N_1 \quad N_2 \quad \cdots] \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \end{Bmatrix} = [N]\{P\}$$

$$\boldsymbol{u} = \begin{bmatrix} N_1 & & & N_2 & & & \cdots \\ & N_1 & & & N_2 & & \\ & & N_1 & & & N_2 & \end{bmatrix} \begin{Bmatrix} U_1 \\ V_1 \\ W_1 \\ U_2 \\ V_2 \\ W_2 \\ \vdots \end{Bmatrix} = [N^3]\{U\}$$

$$\nabla p = \begin{Bmatrix} \frac{\partial p}{\partial x} \\ \frac{\partial p}{\partial y} \\ \frac{\partial p}{\partial z} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}[N]\{P\} \\ \frac{\partial}{\partial y}[N]\{P\} \\ \frac{\partial}{\partial z}[N]\{P\} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x}[N] \\ \frac{\partial}{\partial y}[N] \\ \frac{\partial}{\partial z}[N] \end{bmatrix}\{P\} = [A]\{P\}$$

$$\nabla \cdot \boldsymbol{u} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix} [N^3]\{U\} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial z} & \frac{\partial N_2}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial z} & \cdots \end{bmatrix} \{U\}$$
$$= [E]\{U\}$$

P_i : Pressure of pore water at node i

U_i : Displacement of node i (x component)

V_i : Displacement of node i (y component)

W_i : Displacement of node i (z component)

N_i : Interpolation function corresponding to node i

（２） Using the above matrix, discretize each term in equation ⑩.

$$k\rho_f \int_V \nabla\delta p \cdot \ddot{\boldsymbol{u}} dV = \{\delta P\}^T k\rho_f \int_V [A]^T [N^3] dV \{\ddot{U}\} = \{\delta P\}^T [M^{pu}] \{\ddot{U}\}$$

$$\int_V \delta p \, \nabla \cdot \dot{\boldsymbol{u}} dV = \{\delta P\}^T \int_V [N]^T [E] dV \{\dot{U}\} = \{\delta P\}^T [C^{pu}] \{\dot{U}\}$$

$$C_{Kf} \int_V \delta p \, \dot{p} dV = \{\delta P\}^T C_{Kf} \int_V [N]^T [N] dV \{\dot{P}\} = \{\delta P\}^T [C^{pp}] \{\dot{P}\}$$

$$k \int_V \nabla\delta p \cdot \nabla p dV = \{\delta P\}^T k \int_V [A]^T [A] dV \{P\} = \{\delta P\}^T [K^{pp}] \{P\}$$

$$\int_{S_2} \delta p \, q dS = \{\delta P\}^T \int_{S_2} [N]^T q dS = \{\delta P\}^T \{R_1^p\}$$

$$k\rho_f \int_V \nabla\delta p \cdot \boldsymbol{g} dV = \{\delta P\}^T k\rho_f \int_V [A]^T dV \boldsymbol{g} = \{\delta P\}^T \{R_2^p\}$$

Substituting these into each term of equation ⑩

$$\{\delta P\}^T \left([M^{pu}]\{\ddot{U}\} + [C^{pu}]\{\dot{U}\} + [C^{pp}]\{\dot{P}\} + [K^{pp}]\{P\}\right) = \{\delta P\}^T \left(-\{R_1^p\} + \{R_2^p\}\right)$$

$$[M^{pu}]\{\ddot{U}\} + [C^{pu}]\{\dot{U}\} + [C^{pp}]\{\dot{P}\} + [K^{pp}]\{P\} = -\{R_1^p\} + \{R_2^p\} = \{R^p\} \quad - ⑪$$

［4］Temporal discretization

Discretize equation ⑪ in time. The incremental form is used for nonlinear analysis. In Eq.

$\{\ \}_n$: Value at time n

$\{\ \}'$: Nonlinear iterative computation of the value at pre *iteration*

Δt_1 : Time $n-1$, time increment between time n

Δt_2 : Time n, time increment between time $n+1$

α: Time integration parameter $(0 \leq \alpha \leq 1)$

$$[M^{pu}]\{\ddot{U}\}_n + [C^{pu}]\{\dot{U}\}_n + [C^{pp}]\{\dot{P}\}_n + [K^{pp}]\{\bar{P}\}_n = \{R^p\}_n \quad - ⑫$$

$$\{\ddot{U}\}_n = \frac{\frac{\{U\}_{n+1} - \{U\}_n}{\Delta t_2} - \frac{\{U\}_n - \{U\}_{n-1}}{\Delta t_1}}{\frac{\Delta t_1 + \Delta t_2}{2}} = \frac{\frac{\{\Delta U\} + \{U\}'_{n+1} - \{U\}_n}{\Delta t_2} - \frac{\{U\}_n - \{U\}_{n-1}}{\Delta t_1}}{\Delta t_{12}}$$

$$= \frac{\{\Delta U\}}{\Delta t_2 \cdot \Delta t_{12}} + \{\ddot{U}\}'_n$$

$$\{\dot{U}\}_n = \frac{\{U\}_{n+1} - \{U\}_{n-1}}{\Delta t_1 + \Delta t_2} = \frac{\{\Delta U\} + \{U\}'_{n+1} - \{U\}_{n-1}}{2\Delta t_{12}} = \frac{\{\Delta U\}}{2\Delta t_{12}} + \{\dot{U}\}'_n$$

$$\{\dot{P}\}_n = \frac{\{P\}_{n+1} - \{P\}_{n-1}}{\Delta t_1 + \Delta t_2} = \frac{\{\Delta P\} + \{P\}'_{n+1} - \{P\}_{n-1}}{2\Delta t_{12}} = \frac{\{\Delta P\}}{2\Delta t_{12}} + \{\dot{P}\}'_n$$

$$\{\bar{P}\}_n = (1-\alpha)\{P\}_{n-1} + \alpha\{P\}_{n+1} = (1-\alpha)\{P\}_{n-1} + \alpha(\{P\}'_{n+1} + \{\Delta P\})$$

$$= \{\bar{P}\}'_n + \alpha\{\Delta P\}$$

Substituting the above discretized vectors into ⑫ and rearranging

$$[M^{pu}]\left(\frac{\{\Delta U\}}{\Delta t_2 \cdot \Delta t_{12}} + \{\ddot{U}\}'_n\right) + [C^{pu}]\left(\frac{\{\Delta U\}}{2\Delta t_{12}} + \{\dot{U}\}'_n\right) + [C^{pp}]\left(\frac{\{\Delta P\}}{2\Delta t_{12}} + \{\dot{P}\}'_n\right)$$

$$+[K^{pp}](\{\bar{P}\}'_n + \alpha\{\Delta P\}) = \{R^p\}_n$$

$$\left(\frac{1}{\Delta t_2 \cdot \Delta t_{12}}[M^{pu}] + \frac{1}{2\Delta t_{12}}[C^{pu}]\right)\{\Delta U\} + \left(\frac{1}{2\Delta t_{12}}[C^{pp}] + \alpha[K^{pp}]\right)\{\Delta P\}$$

$$= \{R^p\}_n - \left([M^{pu}]\{\ddot{U}\}'_n + [C^{pu}]\{\dot{U}\}'_n + [C^{pp}]\{\dot{P}\}'_n + [K^{pp}]\{\bar{P}\}'_n\right) \quad - ⑬$$

$$[A^{pu}]\{\Delta U\} + [A^{pp}]\{\Delta P\} = \{Q^p\} \quad - ⑭$$

［5］ Correspondence with variables in the program

The correspondence between each term on the right side of equation ⑥ (the ' indicating the value of the previous iteration is omitted) and the variables in the program is shown below (blue letters: variables in the program).

$$\{R^p\}_n - \left([M^{pu}]\{\ddot{U}\}_n + [C^{pu}]\{\dot{U}\}_n + [C^{pp}]\{\dot{P}\}_n + [K^{pp}]\{\bar{P}\}_n\right)$$

$$= -\{R_1^p\}_n + \{R_2^p\} - \left([M^{pu}]\{\ddot{U}\}_n + [C^{pu}]\{\dot{U}\}_n + [C^{pp}]\{\dot{P}\}_n + [K^{pp}]\{\bar{P}\}_n\right)$$

$$= -[C^{pu}]\{\dot{U}\}_n - [C^{pp}]\{\dot{P}\}_n - \{R_1^p\}_n - \left([K^{pp}]\{\bar{P}\}_n - \{R_2^p\} + [M^{pu}]\{\ddot{U}\}_n\right)$$

$$= -\int_V [N]^T [E] dV \{\dot{U}\}_n - C_{Kf}\int_V [N]^T [N] dV \{\dot{P}\}_n - \int_{S_2} [N]^T q_n dS$$
$$- \left(k\int_V [A]^T [A] dV \{\bar{P}\}_n - k\rho_f \int_V [A]^T dV \boldsymbol{g} + k\rho_f \int_V [A]^T [N^3] dV \{\ddot{U}\}_n\right)$$

$$= -\int_V [N]^T [E] \{\dot{U}\}_n dV - \int_V [N]^T C_{Kf} [N] \{\dot{P}\}_n dV - \int_{S_2} [N]^T q_n dS$$
$$- \left(-\int_V [A]^T k \left(-[A]\{\bar{P}\}_n + \rho_f \boldsymbol{g} - \rho_f [N^3]\{\ddot{U}\}_n\right) dV\right)$$

$$= -\int_V [N]^T \nabla \cdot \dot{\boldsymbol{u}}_n dV - \int_V [N]^T C_{Kf} \dot{p}_n dV - \int_{S_2} [N]^T q_n dS$$
$$- \left(-\int_V [A]^T k\left(-\nabla p_n + \rho_f \boldsymbol{g} - \rho_f \ddot{\boldsymbol{u}}_n\right) dV\right)$$

$$= -\int_V [N]^T \nabla \cdot \dot{\boldsymbol{u}}_n dV - \underbrace{\int_V [N]^T C_{Kf} \dot{p}_n dV}_{\uparrow\ \text{FLO}} - \int_{S_2} [N]^T q_n dS - \left(-\underbrace{\int_V [A]^T \dot{\boldsymbol{w}}_n dV}_{\uparrow\ \text{FLI}}\right)$$

2．1．3 Coupled discretization equations for structural and seepage flow analysis

The discretization equations ⑦ and ⑭ for the structural analysis and seepage flow analysis, respectively, are summarized to obtain the final equation below.

$$\begin{bmatrix} A^{uu} & A^{up} \\ A^{pu} & A^{pp} \end{bmatrix} \begin{Bmatrix} \Delta U \\ \Delta P \end{Bmatrix} = \begin{Bmatrix} Q^{u} \\ Q^{p} \end{Bmatrix} \quad - ⑮$$

2．1．4 Separation of discretization equations for structural and seepage flow analysis

Equation ⑮ solves the discretization equation coupling the structural analysis and the seepage flow analysis to obtain $\{\Delta U\}$ and $\{\Delta P\}$ simultaneously. However, the order difference of the coefficients in the coefficient matrix is large, which is a bad condition for the convergence of the parallel CG method described in section 3. Therefore, as an approximate solution method that can be used when the CG method does not converge, we have also prepared a solution method in which the structural analysis and seepage flow analysis are separated and $\{\Delta U\}$ and $\{\Delta P\}$ are obtained by separate equations. The method is shown below.

（1） Assume that the solutions (i.e., $\{U\}_n, \{P\}_n$) of time n are known.

（2） First, perform a structural analysis to obtain $\{\Delta U\}$ (i.e., $\{U\}_{n+1}$) . In doing so, $\{\bar{F}^p\}_n \rightarrow \{F^p\}_n$ in equation ⑤, and $\{\Delta P\}$ is eliminated from equation ⑤. Equation ⑥ is modified as follows.

$$\left(\frac{1}{\Delta t_2 \cdot \Delta t_{12}}[M] + \left(\frac{G}{2\Delta t_{12}} + \beta\right)[K]\right)\{\Delta U\}$$

$$= \{R\}_n - \left([M]\{\ddot{U}\}_n' + G\{\dot{F}\}_n' + \{\bar{F}\}_n' - \{F^p\}_n\right) \quad - ⑯$$

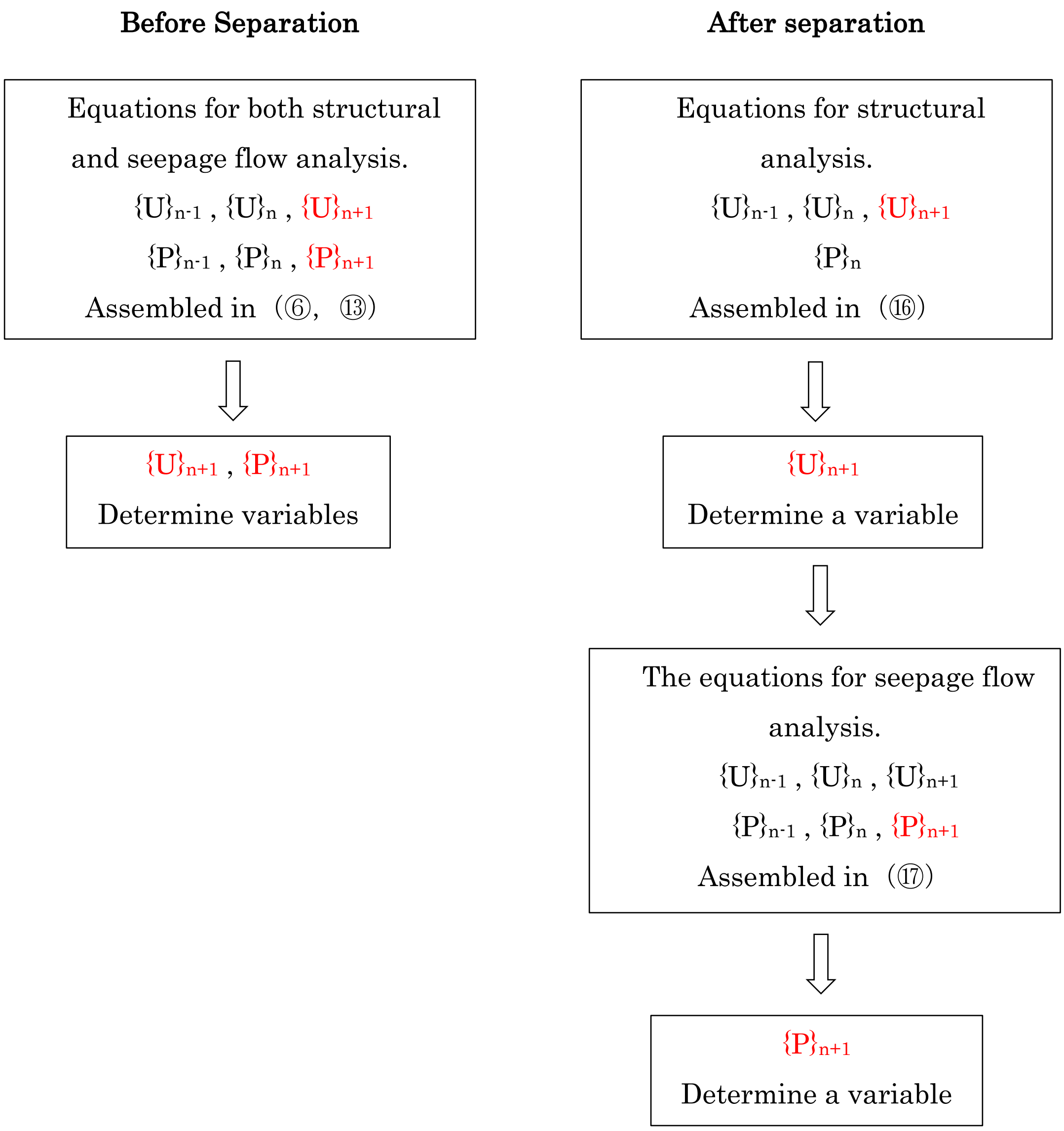
（3） Next, seepage flow analysis is performed to obtain $\{\Delta P\}$ (i.e., $\{P\}_{n+1}$). Since $\{U\}_{n+1}$ is known in equation ⑫, $\{\Delta U\}$ is not included in equation ⑫ and $\{\ddot{U}\}_n$ and $\{\dot{U}\}_n$ are also known. Equation ⑬ is modified as follows.

$$\left(\frac{1}{2\Delta t_{12}}[C^{pp}] + \alpha[K^{pp}]\right)\{\Delta P\}$$

$$= \{R^p\}_n - \left([M^{pu}]\{\ddot{U}\}_n + [C^{pu}]\{\dot{U}\}_n + [C^{pp}]\{\dot{P}\}_n' + [K^{pp}]\{\bar{P}\}_n'\right) \quad - ⑰$$

The equations $\{\Delta U\}$ and $\{\Delta P\}$ can be obtained by solving each equation independently in the order of ⑯ and ⑰. The difference in the solution flow before and after the separation is shown on the next page.

Find the state at time n from the state at time n+1. Unknown values are shown in red.



２．１．５ **Porous Material**

By specifying the porosity as a physical property value, the material can be made porous with the following characteristics. It can be used as a piled stone material, etc.

（１） The porosity of the CADMAS calculation cell interfering with the porous material structure shall be the porosity specified by the porous material properties.

（２） Deformed by effective stress as in the geotechnical (consider $\{F^p\}_n$ in equation ⑯)

（３） No seepage flow calculations are performed.

（４） Since pore water pressure is not calculated from (3), the pressure for effective stress calculation is the interpolated value of the pressure in the calculation cell of CADMAS.

Also If the void ratio (a separate input from the void ratio for the seepage flow calculation) is specified for the soil material, the characteristics described in (1) above are added to the soil. This allows water to permeate through the ground in CADMAS

2．2 Multipoint Constraint (MPC) Processing

［1］ Conversion Matrix

Given a multipoint constraint (MPC) condition, the coefficient matrix and right-hand side vector in a simultaneous linear equation are transformed as follows

As an example, assume the following MPC conditional equation with the vector of unknowns $\{U\}$ and U_5 as the dependent variable.

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \end{Bmatrix}$$

$$U_5 = A_1 U_1 + A_2 U_2$$

$\{U\}$ can be expressed using the transformation matrix $[\lambda]$ and a vector $\{\bar{U}\}$ of independent variables only as follows

$$\{U\} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \\ U_7 \end{Bmatrix} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ A_1 & A_2 & & & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_6 \\ U_7 \end{Bmatrix} = [\lambda]\{\bar{U}\}$$

Substituting this relationship into the FEM discretization equation yields a simultaneous linear equation with MPC conditions.

$$\{\delta U\}^T [K]\{U\} = \{\delta U\}^T \{R\}$$

$$([\lambda]\{\delta\bar{U}\})^T [K]([\lambda]\{\bar{U}\}) = ([\lambda]\{\delta\bar{U}\})^T \{R\}$$

$$\{\delta\bar{U}\}^T [\lambda]^T [K][\lambda]\{\bar{U}\} = \{\delta\bar{U}\}^T [\lambda]^T \{R\}$$

$$\{\delta\bar{U}\}^T [\bar{K}]\{\bar{U}\} = \{\delta\bar{U}\}^T \{\bar{R}\}$$

$$[\bar{K}]\{\bar{U}\} = \{\bar{R}\}$$

［2］Element-by-element processing in the overall stiffness matrix assembly

Let the component node numbers of an element be 1, 2, and 3. Two of the 9 degrees of freedom, 3 at each node, are dependent degrees of freedom expressed by the following MPC conditional equation (U_j^i denotes the displacement of node i, component j).

$$U_1^2 = A_1U_3^1 + A_2U_1^{10} + A_3U_2^{11}$$

$$U_2^3 = B_1U_2^2 + B_2U_3^8 + B_3U_1^{10}$$

8, 10, 11 は別要素の節点番号

The MPC transformation matrix ［λ］

NCR = NC-Number of dependent degrees of freedom +

number of independent degrees of freedom outside the element

$$\{U\} = \begin{Bmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ U_1^2 \\ U_2^2 \\ U_3^2 \\ U_1^3 \\ U_2^3 \\ U_3^3 \end{Bmatrix} = \mathrm{NC}\left\{ \begin{bmatrix} 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & & 1 & & & & & & & \\ & & A_1 & & & & & A_2 & A_3 & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & 1 & & & & \\ & & & B_1 & & & & B_3 & & B_2 \\ & & & & & & 1 & & & \end{bmatrix} \right. \begin{Bmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ U_2^2 \\ U_3^2 \\ U_1^3 \\ U_3^3 \\ U_1^{10} \\ U_2^{11} \\ U_3^8 \end{Bmatrix} = [\,\lambda\,]\{\bar{U}\}$$

IDCR

node	1	1	1	2	2	3	3	10	11	8
component	1	2	3	2	3	1	3	1	2	3

The kth degree of freedom of the element stiffness matrix $[\bar{K}] = [\,\lambda\,]^T[K][\,\lambda\,]$ is embedded in the overall stiffness matrix at

node i = IDCR(1, k)

component j = IDCR(2, k)

. i,e., INDOF(j, i).

2．3 Calculation of constraining force and residual force

［1］ Forces balance formula

The force balance relationship in the FEM discretization equation is shown below.

Internal force = external force + external constraint force + internal constraint force - residual force

FTI FTO RFCO RFCI RES

Internal force：All the force required to cause the present deformed state

External force：Externally applied forces

External constraint force：binding force

Internal constraint force：The force exerted by the MPC on the dependent and independent Degrees of freedom on each other

Residual force：Disproportionate forces in the convergence process of nonlinear calculations

［2］ Calculation of unknown power

When the displacement is calculated, the displacement is

Known power： FTI, FTO

Unknown power： RFCO, RFCI, RES

The force balance relationship for each type of degree of freedom is as follows

INDOF =degrees of freedom for -2 ：FTI = FTO + RFCI − ①

INDOF =degrees of freedom for 0,-1 ：FTI = FTO + RFCO + RFCI − ②

INDOF > degrees of freedom ：FTI = FTO + RFCI − RES − ③

Therefore, the following procedure is used to calculate the unknown power.

（1） Calculate the RFCI of the MPC dependent degrees of freedom from equation ①. Based on this, the RFCI of the MPC independent degrees of freedom in equations ② and ③ are calculated. The calculation method is, for example, as follows for the following MPC conditional equation

$$U_3 = A_1 U_1 + A_2 U_2$$

Let RFCI_3 be the internal constraint on the dependent degree of freedom 3

Internal constraint on 1 $\mathrm{RFCI}_1 = -A_1 \cdot \mathrm{RFCI}_3$

Internal constraint on 2 $\mathrm{RFCI}_2 = -A_2 \cdot \mathrm{RFCI}_3$

（2） determine RFCO from equation ②.

（3） determine RES from equation ③.

2．4　Overview of Processing in Contact Analysis

［1］The procedure for contact analysis is as follows.

（1）Separate the two contacting bodies into a master and a slave.

（2）Detects that a node of the slave body (hereinafter referred to as slave point) has penetrated a surface element of the master body (hereinafter referred to as master surface).

（3）Return the slave point to the through point on the master surface.

（4）Under the constraint that the slave point moves on the master plane (MPC condition of the displacement of the slave point and the displacement of the constituent nodes of the master plane), the equilibrium state of the forces is recalculated.

（5）Since the constraint condition equation in (4) is a linear approximation, the slave point is not perfectly placed on the master plane after determination. Therefore, the slave point is placed back on the master surface, and the solution is performed again under the new MPC conditions. This kind of iterative calculation is used to obtain the equilibrium state of the force.

［2］The above MPC conditional equation is set as follows. The coordinate vector of each point i is denoted by $\boldsymbol{X}_i$ and the displacement vector by $\boldsymbol{U}_i$.

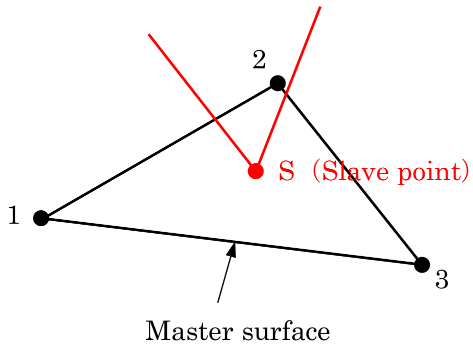
（1）Case when the master is a triangular surface

The MPC conditionals between the slave point S and the constituent nodes 1,2,3 of the master surface are

$$\boldsymbol{U}_s \cdot \boldsymbol{n} = L_1 \boldsymbol{U}_1 \cdot \boldsymbol{n} + L_2 \boldsymbol{U}_2 \cdot \boldsymbol{n} + L_3 \boldsymbol{U}_3 \cdot \boldsymbol{n}$$

L_i： Area coordinates of slave point S in triangle 123

$\boldsymbol{n}$： Normal vector of triangle 123



（2）Case when the master is a quadrilateral surface

The coordinate vector of the physical center of gravity point 5 of nodes 1,2,3,4 is

$$\boldsymbol{X}_5 = \frac{1}{4}(\boldsymbol{X}_1 + \boldsymbol{X}_2 + \boldsymbol{X}_3 + \boldsymbol{X}_4)$$

The displacement vector shall be as follows

$$\boldsymbol{U}_5 = \frac{1}{4}(\boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 + \boldsymbol{U}_4)$$

The MPC conditionals between the slave point S and the constituent nodes 1,2,3,4 of the master surface are

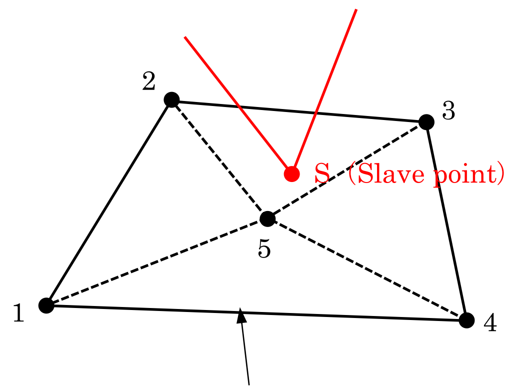
$$\boldsymbol{U}_s \cdot \boldsymbol{n} = L_2\boldsymbol{U}_2 \cdot \boldsymbol{n} + L_3\boldsymbol{U}_3 \cdot \boldsymbol{n} + L_5\boldsymbol{U}_5 \cdot \boldsymbol{n}$$

$$= L_2\boldsymbol{U}_2 \cdot \boldsymbol{n} + L_3\boldsymbol{U}_3 \cdot \boldsymbol{n} + L_5\frac{1}{4}(\boldsymbol{U}_1 + \boldsymbol{U}_2 + \boldsymbol{U}_3 + \boldsymbol{U}_4) \cdot \boldsymbol{n}$$

$$= \frac{1}{4}L_5\boldsymbol{U}_1 \cdot \boldsymbol{n} + \left(L_2 + \frac{1}{4}L_5\right)\boldsymbol{U}_2 \cdot \boldsymbol{n} + \left(L_3 + \frac{1}{4}L_5\right)\boldsymbol{U}_3 \cdot \boldsymbol{n} + \frac{1}{4}L_5\boldsymbol{U}_4 \cdot \boldsymbol{n}$$

L_i : Area coordinates of slave point S in triangle 235

$\boldsymbol{n}$: Normal vector c of triangle 235



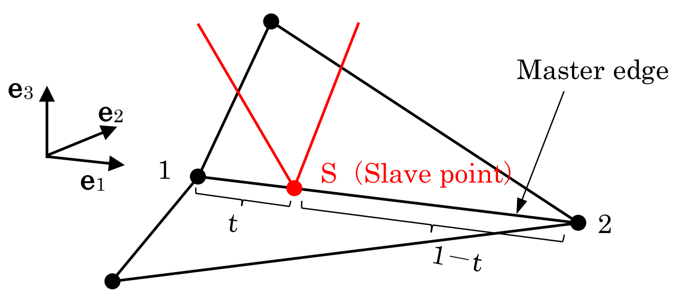
（3） Case when the master is on edge

The MPC conditionals between the slave point S and the constituent nodes 1 and 2 of the master edge are

$$\boldsymbol{U}_s \cdot \boldsymbol{e}_2 = (1-t)\,\boldsymbol{U}_1 \cdot \boldsymbol{e}_2 + t\,\boldsymbol{U}_2 \cdot \boldsymbol{e}_2$$

$$\boldsymbol{U}_s \cdot \boldsymbol{e}_3 = (1-t)\,\boldsymbol{U}_1 \cdot \boldsymbol{e}_3 + t\,\boldsymbol{U}_2 \cdot \boldsymbol{e}_3$$

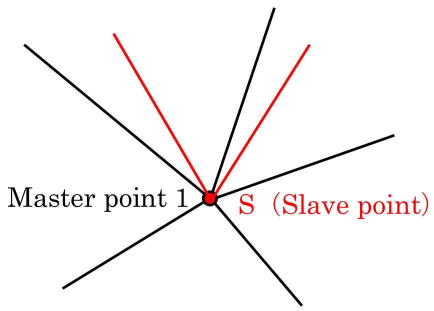
$\boldsymbol{e}_i$: Unit vectors orthogonal to each other (direction of $\boldsymbol{e}_1$ coincides with 12)



（4） Cases when the master is on the dot

The MPC conditionals between slave point S and master point 1 are

$$\boldsymbol{U}_s = \boldsymbol{U}_1$$



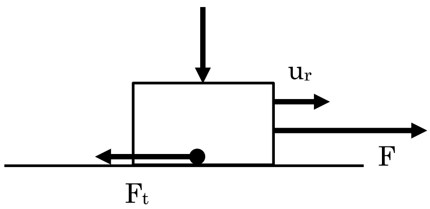
2． 5　**Friction model**

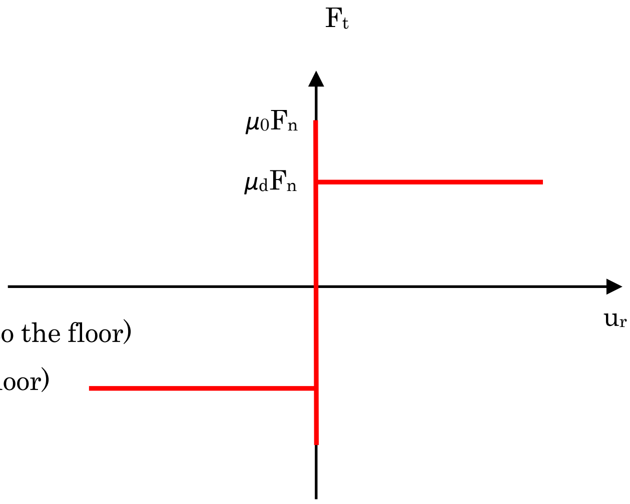
（1）The stick-slip model is used for the friction model.

（2）The lower right graph shows the relationship between u_r and F_t when an external force F is applied as shown in the lower left figure.

（3）While FF ≦ $\mu_0 F_n$, the object does not move (i.e., $u_r = 0$) and a frictional force Ft of the same magnitude as F.

（4）When $F > \mu_0 F_n$, the object starts to move, but the frictional force acting on the object is then constant and $F_t = \mu_d F_n$, irrespective of u_r .





F_n : External force acting on an object (perpendicular to the floor)

F : External force acting on an object (parallel to the floor)

u_r : Relative velocity of an object relative to the floor

F_t : Frictional force acting on an object

μ_0 : coefficient of static friction

μ_d : kinetic coefficient of friction

3．Input data

3． 1 Composition of input data

The input data format follows NASTRAN and consists of the following

- ・Case control section
- ・BEGIN BULK（a keyword indicating the beginning of the bulk data）
- ・Bulk data section

3． 2 Case control section

This section is used to select constraint conditions, loading conditions, etc., for the data set in the bulk data section. The correspondence between the case control commands and the bulk data to be selected is shown below.

Case control commnd	Conditions to be set	Bulk data to be selected
SPC	constraint	SPCADD, SPC1, SPC
DLOAD	dynamic load	DLOAD, TLOAD1
TSTEP	dynamic analysis step"	TSTEP
BCSET	contact	BCTADD, BCTSET

3．3 Bulk data section

The first field in the first row is the keywords. The first field of the first row is the keywords, and if the keywords are marked with an asterisk (*), the data format is a 16-column field.

［1］Node

GRID

Definition of node coordinates.

Format：

1	2	3	4	5	6	7	8	9	10
GRID	ID		X	Y	Z				

Format（Support for 16 columns）：

<table>
<tr><th>1</th><th>2</th><th>3</th><th>4</th><th>5</th><th>6</th><th>7</th><th>8</th><th>9</th><th>10</th></tr>
<tr><td>GRID*</td><td colspan="2">ID</td><td colspan="2"></td><td colspan="2">X</td><td colspan="2">Y</td><td></td></tr>
<tr><td></td><td colspan="2">Z</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td><td></td></tr>
</table>

Field	Type	Contents
ID	I	Node numbers
X, Y, Z	R	Node coordinates

［2］ Element, element characterization

PSOLID

Solid element characterization definition.

Format：

1	2	3	4	5	6	7	8	9	10
PSOLID	PID	MID							

Field	Type	Contents
PID	I	Node numbers
MID	I	Node coordinates

CTETRA

Definition of a tetrahedral element.

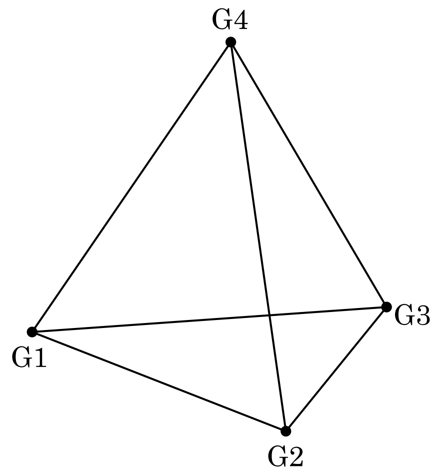
Format：

1	2	3	4	5	6	7	8	9	10
CTETRA	EID	PID	G1	G2	G3	G4			

Field	Type	Contents
EID	I	Element number
PID	I	Property number
G1−G4	I	Configuration node number (see Remark 1.)

Remarks：

1. The sequence of the configuration node numbers is shown in the figure below.



CPENTA

Definition of a pentahedral element.

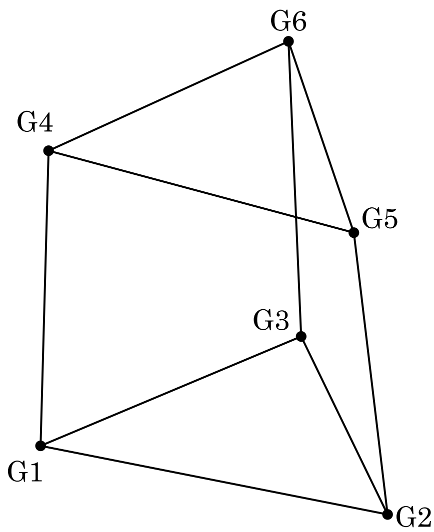
Format :

1	2	3	4	5	6	7	8	9	10
CPENTA	EID	PID	G1	G2	G3	G4	G5	G6	

Field	Type	Contents
EID	I	Element number
PID	I	Property number
G1－G6	I	Configuration node number (see Remark 1.)

Remarks :

1. The sequence of the configuration node numbers is shown in the figure below.



CHEXA

６面体要素の定義.

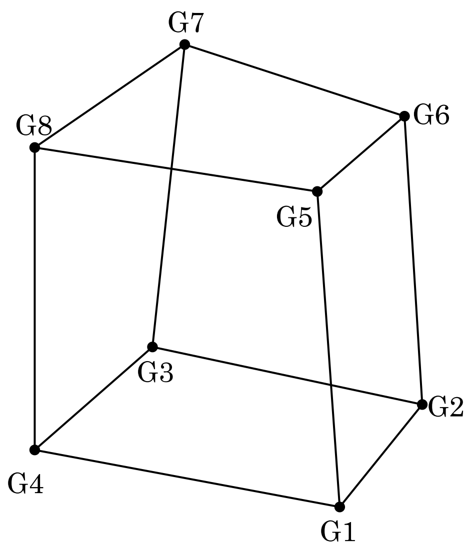
Format：

1	2	3	4	5	6	7	8	9	10
CHEXA	EID	PID	G1	G2	G3	G4	G5	G6	
	G7	G8							

Field	Type	Contents
EID	I	Element number
PID	I	Property number
G1－G8	I	Configuration node number (see Remark 1.)

Remarks：

1. The sequence of the configuration node numbers is shown in the figure below.



PROD

トラス要素の特性定義.

Format：

1	2	3	4	5	6	7	8	9	10
PROD	PID	MID	A						

Field	Type	Contents
PID	I	Property number
MID	I	Referenced material number
A	R	Cross-sectional area

CROD

Definition of truss elements.

Format：

1	2	3	4	5	6	7	8	9	10
CROD	EID	PID	G1	G2					

Field	Type	Contents
EID	I	Element number
PID	I	Property number
G1, G2	I	Configuration node number

PBARL

Characteristic definition of beam elements.

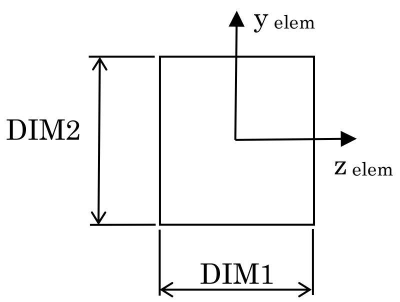
Format：

1	2	3	4	5	6	7	8	9	10
PBARL	PID	MID		TYPE					
	DIM1	DIM2							

Field	Type	Contents
PID	I	Property beam number
MID	I	Referenced material number
TYPE	A	Cross-sectional shape (for "BAR" only, see Remark 1.)
DIMi	R	Dimensions of the cross-sectional profile (see Remark 1.)

Remarks：

1. Beam cross-sectional profile types and dimensions are shown in the figure below.



TYPE = “BAR”

※ y_{elem}， z_{elem} are coordinate axes in the element coordinate system

CBAR

Sefinition of beam elements.

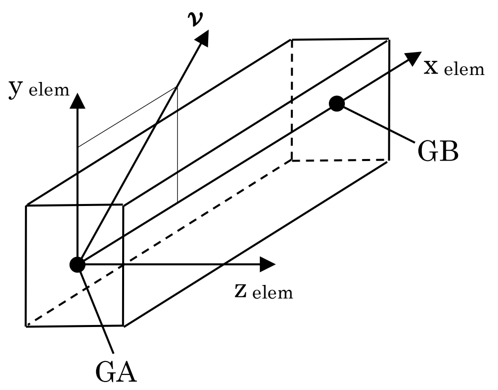
Format：

1	2	3	4	5	6	7	8	9	10
CBAR	EID	PID	GA	GB	X1	X2	X3		

Field	Type	Contents
EID	I	Element beam number
PID	I	Property number
GA, GB	I	Configuration node number (see Remark 1.)
X1, X2, X3	R	Vector **v** determining the element coordinate axes (see Remark 1.)

Remarks：

1. Configuration node number, v, The relationship between the element coordinate axes is shown in the figure below.



［３］Material

MAT1

Definition of material properties.

Format：

1	2	3	4	5	6	7	8	9	10
MAT1	MID	E		NU	RHO	CM	NC	GE	
	N	KF	K						

Field	Type	Contents
MID	I	Material Identification Number（MID＜100：general material, MID≧100：site material）
E	R	Young's modulus
NU	R	Poisson's ratio
RHO	R	Mass density (see Remark 1.)
CM	R	mass attenuation coefficient
NC	R	Porosity used in CADMAS (see Remark 2.)
GE	R	Damping coefficient (twice the damping ratio)
N	R	Porosity of geomaterials or Tensile strength of general materials
KF	R	Volumetric Modulus of Elasticity of Water in Geomaterials
K	R	Permeability of geomaterials

Remarks：

1. Mass density is the density of the material itself without voids (piled stones: density of stones themselves, soil: density of soil particles).

2. Porosity NC is used as the porosity of structural elements interfering with cells in CADMAS during coupled analysis with CADMAS for both general materials and geomaterials.

MATS1

Definition of plastic material properties.

Format：

1	2	3	4	5	6	7	8	9	10
MATS1	MID			H	YF				

Field	Type	Contents
MID	I	Material Identification Number (Material Identification Number of MAT1) (see Remark 1.)
H	R	Strain hardening rate (incremental stress/incremental plastic strain)
YF	I	Descending conditions (=1: von Mieses, =4: Drucker-Prager)

Remarks：

1. The plastic material properties defined here apply to materials defined in MAT1 with the same material property number.

［4］Single-point restraint

SPCADD

Define constraints as a combination of the constraints defined in SPC1 and SPC.

Format：

1	2	3	4	5	6	7	8	9	10
SPCADD	SID	S1	S2	S3	S4	S5	S6	S7	
	S8	S9	…	…					

Field	Type	Contents
SID	I	Constraint set number
Si	I	Constraint set number defined in SPC1, SPC

SPC1

Define node constraint..

Format：

1	2	3	4	5	6	7	8	9	10
SPC1	SID	C	G1	G2	G3	G4	G5	G6	
	G7	G8	…	…					

Alternate Format：

1	2	3	4	5	6	7	8	9	10
SPC1	SID	C	G1	“THRU”	G2				

Field	Type	Contents
SID	I	Constraint set number
C	I	Component number (see Remark 1.)
Gi	I	Node number

Remarks：

1. The correspondence between component numbers and displacement components is shown below. For example, the designation of U_x, θ_x and θ_y is 145.

U_x:1, U_y:2, U_z:3, θ_x:4, θ_y:5, θ_z:6

2. If "THRU" is used in the fifth field, the specified condition is applied to all nodes from G1 to G2; non-existent nodes between G1 and G2 are ignored.

SPC

Definition of node constraints and forced displacements.

Format：

1	2	3	4	5	6	7	8	9	10
SPC	SID	G1	C1	D1	G2	C2	D2		

Field	Type	Contents
SID	I	Constraint set number
Gi	I	Node number
Ci	I	Component number (see Remark 1.)
Di	R	Forced displacement values for Gi and Ci (0. in case of constraint)

Remarks：

1. The correspondence between component numbers and displacement components is shown below. For example, the designation of U_x, θ_x and θ_y is 145.

U_x:1, U_y:2, U_z:3, θ_x:4, θ_y:5, θ_z:6

SPCD

Definition of forced displacement of a node.

Format：

1	2	3	4	5	6	7	8	9	10
SPCD	SID	G1	C1	D1	G2	C2	D2		

Field	Type	Contents
SID	I	Load set number
Gi	I	Node number
Ci	I	Component number (see Remark 1.)
Di	R	Forced displacement values for Gi and Ci

Remarks：

1. The correspondence between component numbers and displacement components is shown below. For example, the designation of U_x, θ_x and θ_y is 145.

U_x:1, U_y:2, U_z:3, θ_x:4, θ_y:5, θ_z:6

［5］ Static load

LOAD

Define loads as a linear combination of load sets defined using FORCE, MOMENT, PLOAD4, and GRAV.

Format：

1	2	3	4	5	6	7	8	9	10
LOAD	SID	S	S1	L1	S2	L2	S3	L3	
	S4	L4	…	…					

Field	Type	Contents
SID	I	Load set number
S	R	Overall scale Factor
Si	R	Li scale Factor
Li	I	Load set number of FORCE, MOMENT, PLOAD4, GRAV

Remarks：

1. The load vector f is shown below.

$$\mathbf{f} = S \cdot \sum_i S_i \cdot \mathbf{f}_{Li}$$

FORCE

Definition of concentrated loads acting on nodes.

Format：

1	2	3	4	5	6	7	8	9	10
FORCE	SID	G		F	N1	N2	N3		

Field	Type	Contents
SID	I	Load set number
G	I	Node number of Load acting on
F	R	Scale factor (see Remark 1.)
N1, N2, N3	R	Load vector component (see Remark 1.)

Remarks：

1. The load vector f acting on node G is shown below.

$$\mathbf{f} = F \cdot \begin{Bmatrix} N1 \\ N2 \\ N3 \end{Bmatrix}$$

MOMENT

Definition of moments acting on nodes.

Format :

1	2	3	4	5	6	7	8	9	10
MOMENT	SID	G		M	N1	N2	N3		

Field	Type	Contents
SID	I	Load set number
G	I	Node number of moment acts on
M	R	Scale factor (see Remark 1.)
N1, N2, N3	R	Moment component (see Remark 1.)

Remarks :

1. The moment m acting on node G is shown below.

$$\mathbf{m} = \begin{Bmatrix} Mx \\ My \\ Mz \end{Bmatrix} = M \cdot \begin{Bmatrix} N1 \\ N2 \\ N3 \end{Bmatrix}$$

PLOAD4

Definition of loads acting on solid element surfaces.

Format：

1	2	3	4	5	6	7	8	9	10
PLOAD4	SID	EID	P				G1	G3 or G4	
		N1	N2	N3					

Field	Type	Contents
SID	I	Load set number
EID	I	Solid element number of the face on which the surface load is applied.
P	R	Magnitude of load per unit area（see Remark 1.)
G1	I	Representative node number of the surface on which the surface load is applied (see Remark 3.)
G3	I	Node number of Surface load acting on surface diagonal to G1 (only for CHEXA, CPENTA quadrilateral surfaces) (see Remark 3.)
G4	I	Node number of Not on plane on which surface load acts (CTETRA only) (see Remark 3.)
N1, N2, N3	R	Vector indicating the direction of the load (see Remark 1,2.)

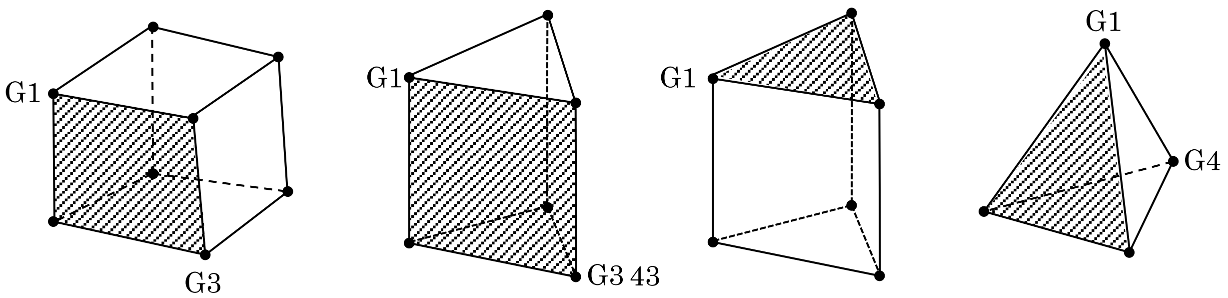
Remarks：

1. The load vector f per unit area is shown below.

$$\mathbf{f} = P \cdot \frac{1}{|\mathbf{N}|}\begin{Bmatrix} N1 \\ N2 \\ N3 \end{Bmatrix}$$

2. For the case N1,N2,N3 = 0, the direction of the load is perpendicular to the plane. The direction is positive from the front to the back of the plane.

3. The positions of G1, G3, and G4 for each element type are as follows.



GRAV

Definition of Gravity.

Format :

1	2	3	4	5	6	7	8	9	10
GRAV	SID		A	N1	N2	N3			

Field	Type	Contents
SID	I	load set number
A	R	Scale factor (see Remark 1.)
N1, N2, N3	R	Load vector component (see Remark 1.)

Remarks :

1. The gravity vector g is shown below.

$$\mathbf{g} = A \cdot \begin{Bmatrix} N1 \\ N2 \\ N3 \end{Bmatrix}$$

［６］動荷重

DLOAD

Define the loads as a linear combination of the load sets defined using TLOAD1 .

Format：

1	2	3	4	5	6	7	8	9	10
DLOAD	SID	S	S1	L1	S2	L2	S3	L3	
	S4	L4	…	…					

Field	Type	Contents
SID	I	load set number
S	R	Overall scale factor
Si	R	Li scale factor
Li	I	Load set number of TLOAD1

Remarks：

1. The load vector f is shown below.

$$\mathbf{f} = S \cdot \sum_i S_i \cdot \mathbf{f}_{Li}$$

TLOAD1

Definition of dynamic load.

Format：

1	2	3	4	5	6	7	8	9	10
TLOAD1	SID	EXCITEI D		TYPE	TID				

Field	Type	Contents
SID	I	Load set number
EXCITEID	I	Static load set number of Use　(see Remark 1.)
TYPE	I or A	Load type（see Remark 2.)
TID	I	Table number defining the time variation of the load factor（see Remark 1.)

Remarks：

1. The dynamic load vector **f** (t) is shown below.

$\mathbf{f}(t) = \mathbf{f} \cdot T(t)$

f：Static load vector specified by EXCITED

T(t): Time variation of load factor according to the table specified by TID

2. The load types defined by TYPE are as follows.

TYPE	Load type
0, LOAD	Load
1, DISP	Forced displacement (specified by SPCD)
2, VELO	Velocity (specified by SPCD)
3, ACCE	Acceleration (specified by SPCD)

TABLED2

Defines the time variation of the dynamic load.

Format :

1	2	3	4	5	6	7	8	9	10
TABLED2	TID								
	X1	Y1	X2	Y2	X3	Y3	X4	Y4	
	X5	Y5	…	…	ENDT				

Field	Type	Contents
TID	I	Table number
Xi, Yi	R	Table data

［7］ Contact

BCTADD

Definition of Contact pairs as a combination of contact pairs defined in BCTSET.

Format：

1	2	3	4	5	6	7	8	9	10
BCTADD	CSID	S1	S2	S3	S4	S5	S6	S7	
	S8	S9	…	…					

Field	Type	Contents
CSID	I	Contact set number
Si	I	Contact set number as defined in BCTSET

BCTSET

The definition of contact pair

Format：

1	2	3	4	5	6	7	8	9	10
BCTSET	CSID	SID1	TID1						
		SID2	TID2						
		…	…						

Field	Type	Contents
CSID	I	Contact set number
SIDi	I	Slave (source) region number
TIDi	I	Master (target) region number

BCTPARA

Setting of Friction Characteristics.

Format：

1	2	3	4	5	6	7	8	9	10
BCTPARA	CSID		MU0		MUD				

Field	Type	Contents
CSID	I	Contact set number (BCTSET number) (see Remark 1.)
MU0	R	Coefficient of static friction
MUD	R	Coefficient of kinetic friction

Remarks：

1. The friction properties defined here apply to contact pairs defined in BCTSET with the same contact set number.

BSURFS

Definition of Contact Regions.

Format：

1	2	3	4	5	6	7	8	9	10
BSURFS	ID				EID1	G1	G2	G3	
	EID2	G1	G2	G3	EID3	G1	G2	G3	
	…	…	…	…					

Field	Type	Contents
ID	I	Contact region number
EIDi	I	Element number to which the contact surface is attached
G1,G2,G3	I	Number of 3 of the component nodes of the contact surface (tri- or tetragonal surface)

［８］Analysis Control

TSTEP

Definition of time steps in dynamic analysis.

Format：

1	2	3	4	5	6	7	8	9	10
TSTEP	SID	N1	DT1	NO1					
		N2	DT2	NO2					
		…	…	…					

Field	Type	Contents
SID	I	Set number
Ni	I	Number of DTi
DTi	A	Time step
NOi	I	Result output interval

［９］ Other

Specify the following key word followed by the value shown in parentheses ( ).

PARAM,LGDISP, (= 1 : Consider large deformations)
PARAM,W4, (Eigen angular frequencies of dominant vibration modes)
PARAM,EPSC, (Tolerance for initial contact determination. Initially, a slave-master that is less than or equal to this distance is considered to be in contact.. default= 10^{-6})

CADMAS,PLOWER2, (Used in CADMAS, the cell porosity is set to 0 if the cells between obstacles are less than or equal to this value.)

4. Data table

4. 1 Control data table

[1] KK Table

The * (region) indicates the value in the relevant region at the time of region segmentation.

No.	Variable name	contents
1		Structural and seepage flow coupling analysis method (0: Simultaneous, 1: Separate)
2		Geometry of non-linear shapes (0 : micro-deformation, 1 : deformation, 2 : Large deformation & driven load)
3		Memory used (MB)
4		SUBCASE Number
5		
6		Maximum number of iterations in nonlinear analysis
7	NSTEP	Number of Steps
8	NNOD	Number of nodes
9	NSHL	Number of SHELL elements
10	NSOL	Number of SOLID elements
11	NMAT	Number of material types
12	NELM	Total number of elements
13	NTHK	Number of SHELL board thickness
14	NROD	Number of ROD elements
15	NRODA	Number of ROD cross-sectional area species
16	NBAR	Number of BEAM elements
17	NBARD	Number of BEAM cross-sectional shapes
18	NELAS	Number of SPRING elements
19	NEQ	Full degrees of freedom (math.)
20	NCGSPC	Number of non-zero elements in the overall stiffness matrix
21		Matrix Solver Type (1 : CG Method, 2 : ASE, 3 : PARDISO)
22		Maximum number of repetitions of the CG method
23		CG method pre-processing (0 : Diagonal component only, 1 : Incomplete Cholesky disassembly)
24		Maximum number of non-zero elements per degree of freedom in the overall stiffness matrix (structural analysis)
25		Maximum number of non-zero elements per degree of freedom in the overall stiffness matrix (seepage flow analysis)
26		(region) Number of inside
27		(region) Number of degrees of freedom corresponding to inside number
28	NNODC	(region) Number of nodes added for contact determination
29	NELMC	(region) Number of elements added for contact determination

30		Mass matrix form（0 : lumped, 1 : consistent）
31	NNODX	（region）Number of nodes added for equation assembly
32	NELMX	（region）Number of elements added for equation assembly
33		Maximum size of element stiffness matrix
34		Maximum number of degrees of freedom of the element stiffness matrix
35		Maximum size of the transformation matrix for MPC processing
36	MGP	Maximum number of integral points of an element
37	NM	Maximum size per element of array IELM
38	NISPD	4.2[6] see SPCD
39	NNSPD	
40	NLOAD	4.2[6] see LOAD
41	NNLOAD	
42	NIFC	4.2[6] see FORCE,MOMENT
43	NNFC	
44	NIPL4	4.2[6] see PLOAD4
45	NNPL4	
46	NISPA	4.2[5] see SPCADD
47	NNSPA	
48	NISP1	4.2[5] see SPC1
49	NNSP1	
50		
51		
52		
53		
54		
55		
56	NISPC	4.2[5] see SPC
57	NNSPC	
58		
59		
60	NIGRV	4.2[6] see GRAV
61		
62		
63		
64		
65		
66	NIDLD	4.2[7] see DLOAD
67	NSIDL	

68		
69		
70	NITD1	4.2[7] see TABLED2
71	NTBD1	
72		
73		
74		
75		
76		
77	NITL1	4.2[7] see TLOAD1
78	NISTP	4.2[9] see TSTEP
79	NNSTP	
80		Material non-linearity (0: none, 1: yes)
81	NPFC	Number of surface elements
82	NPTIM	Number of times the data is output to the CADMAS result file (data.prs) in one-directional coupled
83	NRST	Number of restart file output times (for bi-directional coupled, restart file output flag for CADMAS)
84	IRST	Restart number calculation start step (in bidirectional coupling, this is the the restart calculation start step number in CADMAS)
85		
86	NRANK	Number of priorities for position correction in contact analysis
87	NIDEP	Number of Dependent Degrees of Freedom of the MPC Relational Equation in Contact Analysis
88	NICPA	4.2[8] see BCTADD
89	NNCPA	
90	NICPR	4.2[8] see BCTSET,BCTPARA
91	NNCPR	
92	NICRG	4.2[8] see BSURFS
93	NNCRG	
94	NIGSF	Number of face center-of-gravity points of the quadrilateral faces among the component faces of the element connected to the component nodes of the contact surface.
95	NINDC	Number of contact points
96		Friction model (0: no friction, 1: arctangent, 2: stick-slip)
97	NIELC	Number of contact surfaces
98	NIELG	Number of contact surfaces attached to the ground surface
99	NIEDG	Number of contact surface edges
100	NIBTE	Number of split tetras (elements connected to the constituent nodes of the contact surface divided into tetrahedra)
101	NIELQ	Number of contact 4-angled surfaces

102	NINDC0	(region)Number of contact points for the entire area
103	NINDCX	(region) Number of contact points added for equation assembly
104	NIELCX	(region) Number of contact surfaces added for equation assembly
105	NIEDGX	(region) Number of contact surface edges added for equation assembly
106	NIELQX	(region) Number of contact quadrilateral faces added for equation assembly
107	NIGSFX	(region) Number of quadrilateral surface center-of-gravity points added for equation assembly
108	NIGSFC	(region) Number of quadrilateral surface center-of-gravity points added for contact determination
109		
110	MITER0	Number of calculation iterations to switch convergence method when considering static friction
111	MITER1	Number of calculation iterations to switch friction characteristics when considering static friction
112	MITERD0	Number of calculation iterations to switch friction characteristics and convergence method when considering dynamic friction
113	MITERD1	Number of calculation iterations to switch friction characteristics when considering dynamic friction

［2］RR table

No.	contents
1	Convergence tolerance of CG method
2	
3	
4	Time-integral parameter β for transient response analysis
5	Eigen angular frequencies of deformation modes for which damping ratios are valid in transient response analysis
6	Convergence tolerance of unbalanced forces in nonlinear analysis
7	Time-integral parameter for seepage flow analysis α
8	Water level in CADMAS model
9	Z-coordinate of the top boundary of the CADMAS model
10	Parameters used to calculate friction spring constant (static friction)
11	Parameters used to calculate friction spring constant (dynamic friction)
12	Dynamic friction state → Limit value of relative displacement used to determine static friction state
13	Parameters used to determine penetration in contact analysis
14	Limits of master-slave distance for initial contact determination

［３］IFL table

No.	拡張子	contents
10	.dat	Input data
11	.log	Log
12	.wk1	Work for ASE
13	.wk2	Work for ASE
14	.prs	CADMAS result files for one-way coupling
15	.neu	Result file（FEMAP neutral file format）
16		
17	.rst	Restart calculation file
18	.rtm	Data output time file for restart calculation

4．2　Input Data Table

The following data tables are used to read input data (in NASTRAN BULK data format) and set its contents. Bold type indicates the entry name of the NASTRAN BULK data corresponding to each data table.

［1］Case control

		1	2	・・・	NSUB = KK(4)
ISUB	Static load condition 1				
	Single-point constraint condition 2				
	Multipoint constraint condition 3				
	Eigenvalue analysis condition 4				
	Temperature load condition 5				
	Initial temperature conditions 6				
	Dynamic load condition 7				
	Load set 8				
	Frequency response analysis conditions 9				
	Material temperature conditions 10				
	Nonlinear analysis conditions11				
	Transient response analysis time condition 12				
	Contact set 13				

［2］Node

GRID

		1	2	・・・	NNOD = KK(8)
INDG	node number				
IGCD	Coordinate system number				
GRID	node coordinate X				
	Y				
	Z				

［３］ Element, element characterization

PSOLID, CTETRA, CPENTA, CHEXA
PROD, CROD
PBARL, CBAR

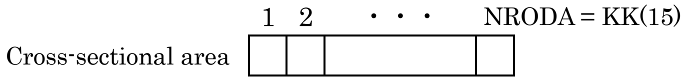
The contents per element of the array IELM are shown.

element type	SOLID	ROD	BEAM
1	Element number		
2	(= 2 : structure = 6 : ground)	(= 3)	(= 4)
3	Constituent node number		
4	material number		
5	Stone material flag (= 0 : non-stone = 1 : stone)	Cross section number (see RODA)	Cross-sectional shape number (see BARD)
6			Cross Section Type (= 2 : "BAR" only)
7			Beam element number (BVEC 参照)
8	Configuration node number 1		
9	Configuration node number 2		
·	·		
·	·		
·	·		
NM=KK(37)	·		

Total number of each element type	NSOL = KK(10)	NROD = KK(14)	NBAR = KK(16)
Total number of elements	NELM = KK(12)		

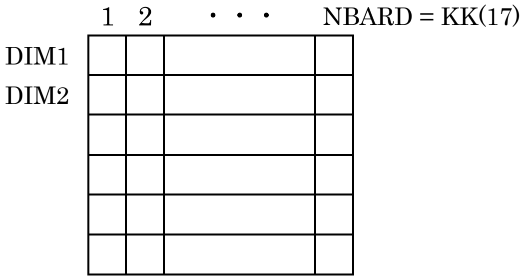
PROD

RODA



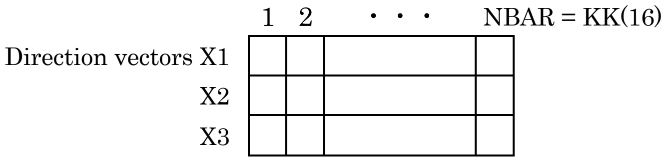
PBARL

BARD



CBAR

BVEC



［４］Material

MAT1

MAT

	1	2	・・・	NMAT = KK(11)
Empty				
Elastic-plasticity				

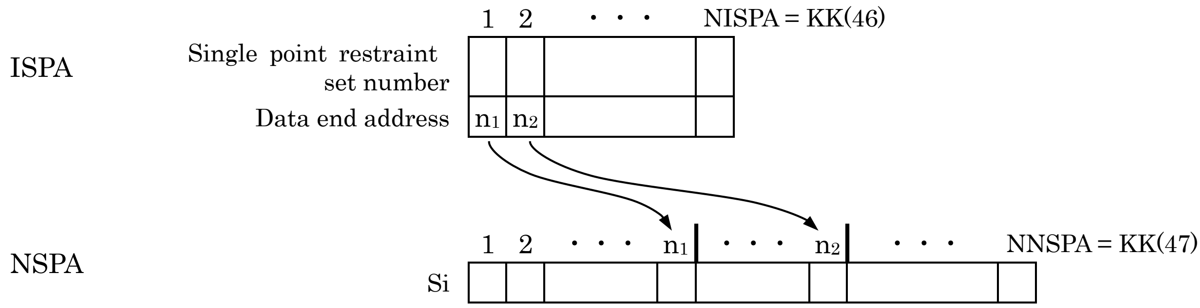
AMAT

E	1				
ν	2				
ρ	3				
GE	4				
CM	5				
n(CADMAS 用)	6				
n	7				
K_f	8				
k	9				
ρ_f	10				
σ_y	11				
H’	12				
α	13				
σ_t	14				
	・				
	・				
	33				

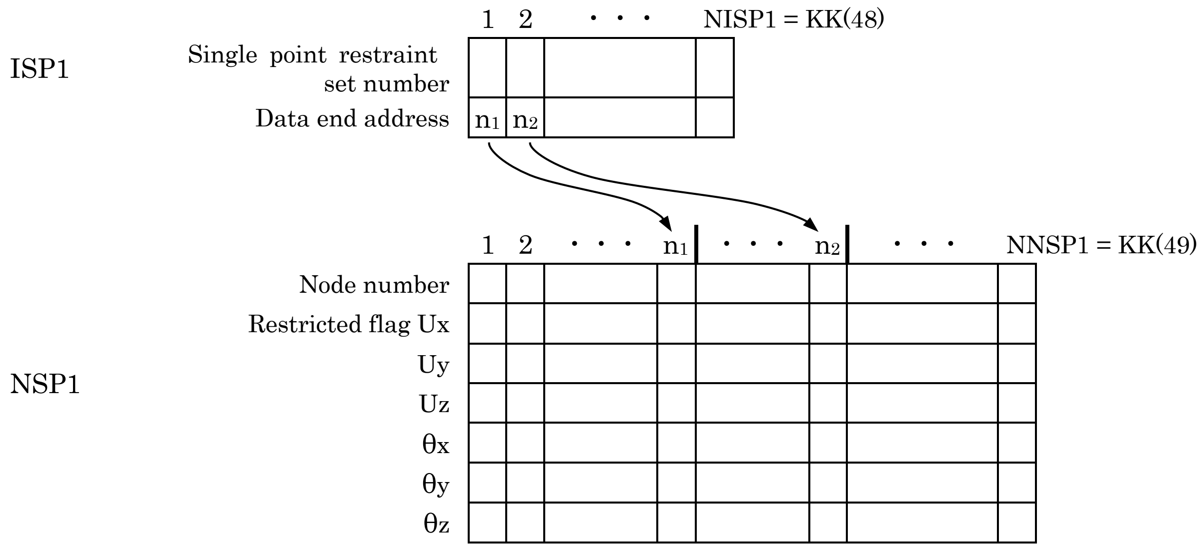
※ Elastoplasticity =0：Elasticity，=1：Mieses，=2：Drucker-Prager

［5］ Single-point restraint

SPCADD

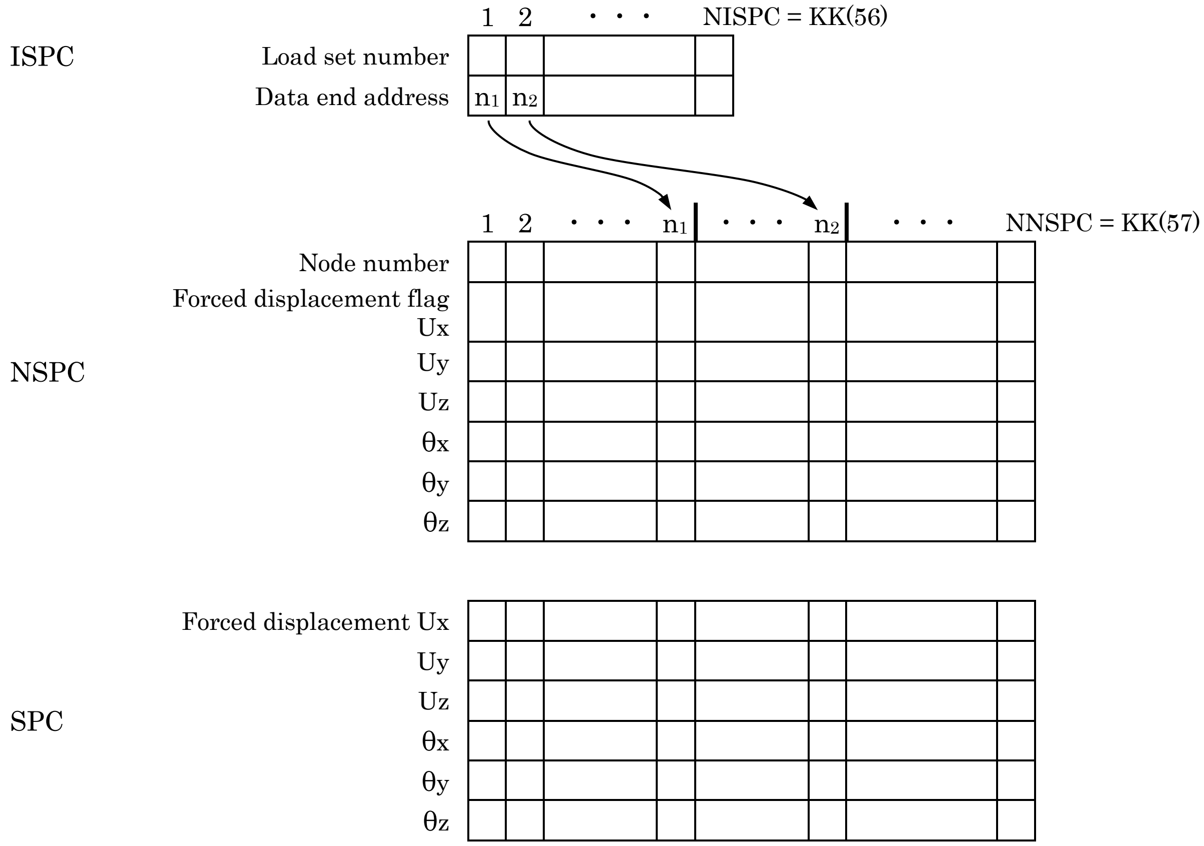


SPC1



※ Restricted flag =1 : restraint, =0 : free

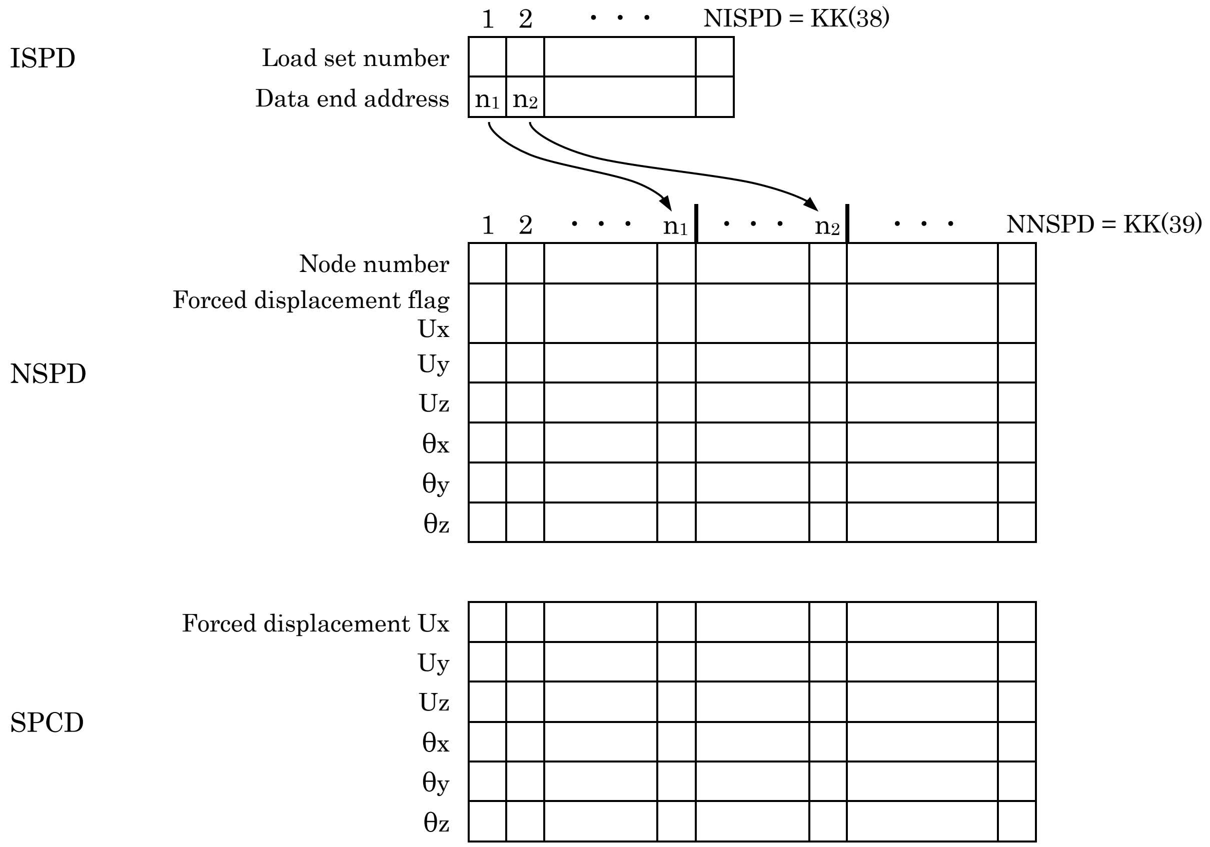
SPC



※ Forced displacement flag = 1 : Forced displacement, = 0 : No forced displacement

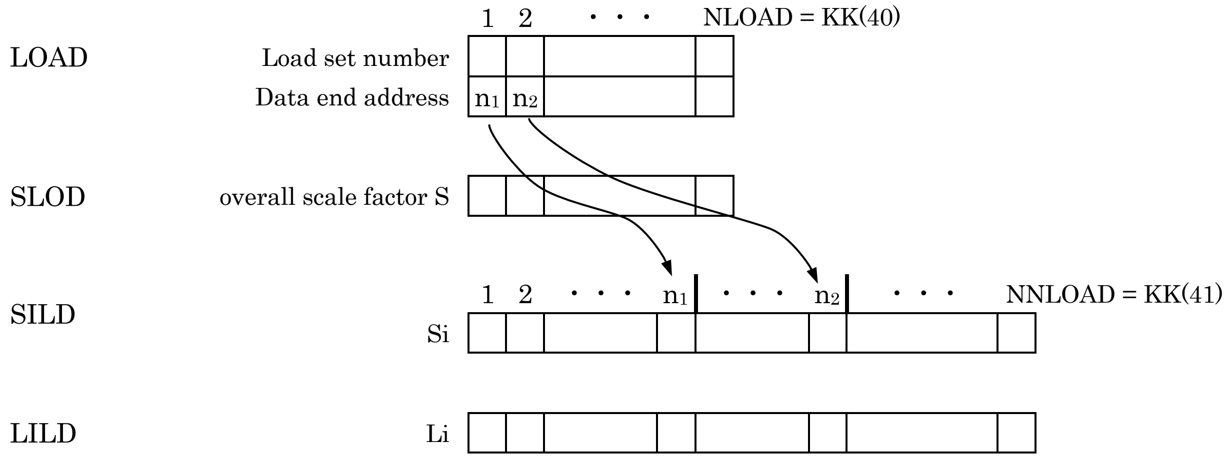
[6] Static load

SPCD

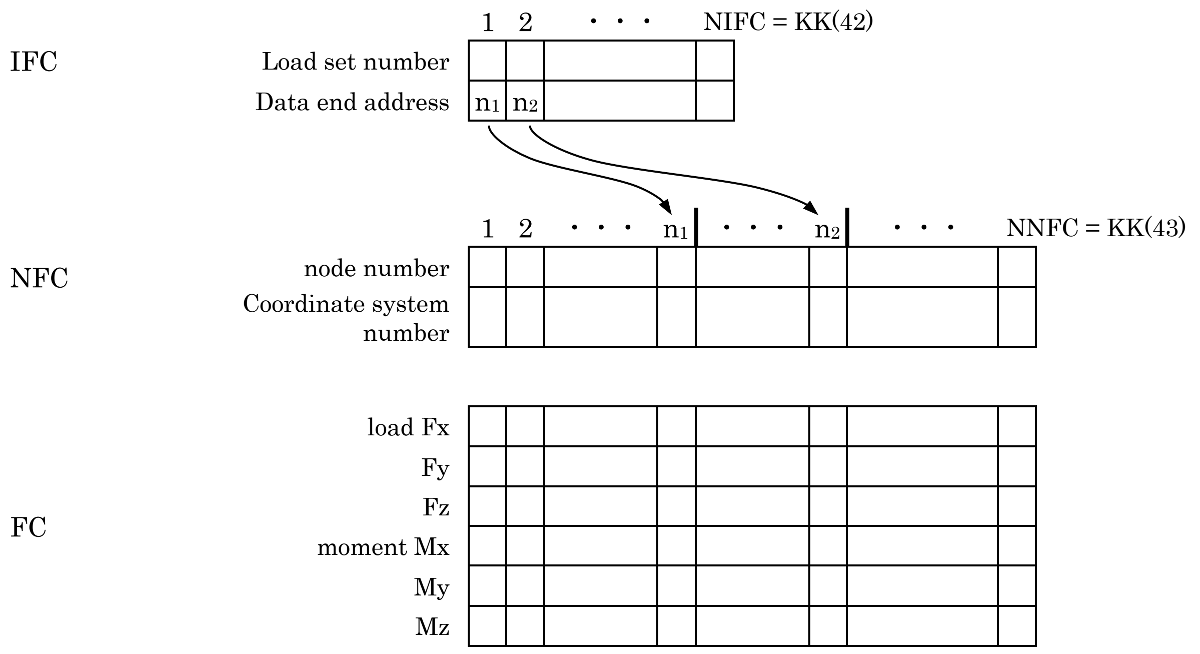


※ Forced displacement flag =1 : Forced displacement, = 0 : No forced displacement

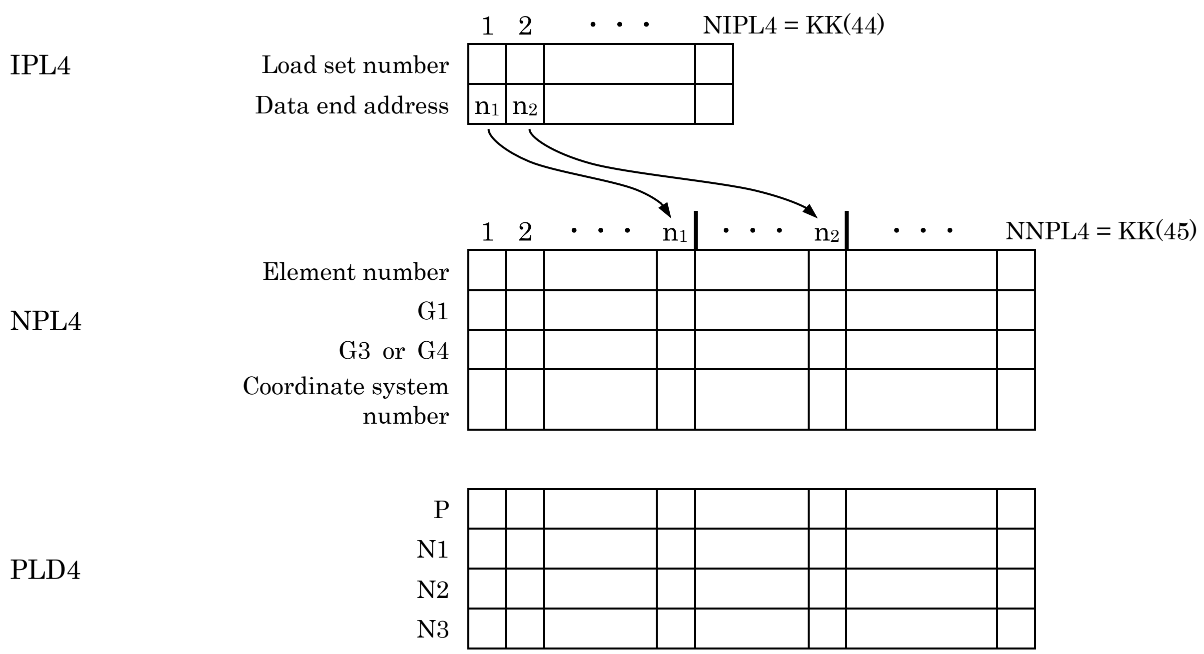
LOAD



FORCE, MOMENT

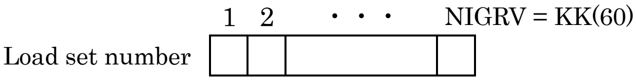


PLOAD4

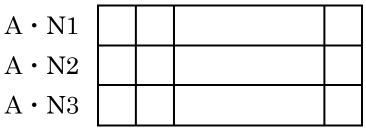


GRAV

IGRV

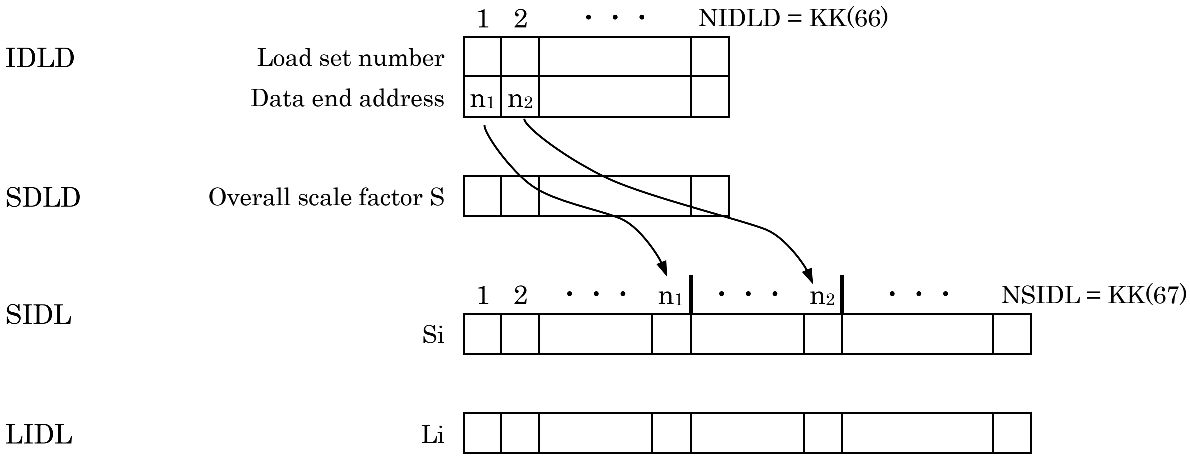


GRAV

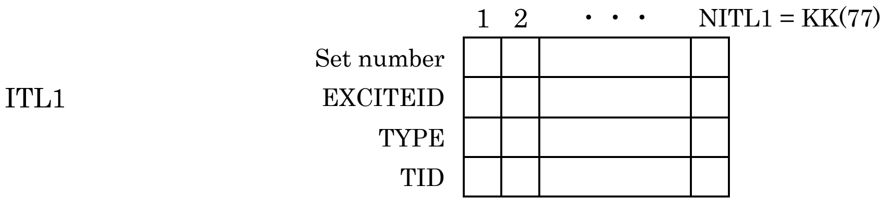


［7］ Dynamic load

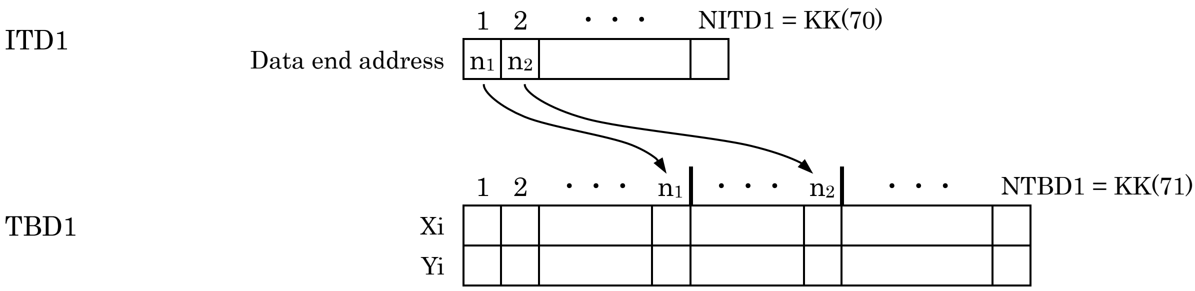
DLOAD



TLOAD1

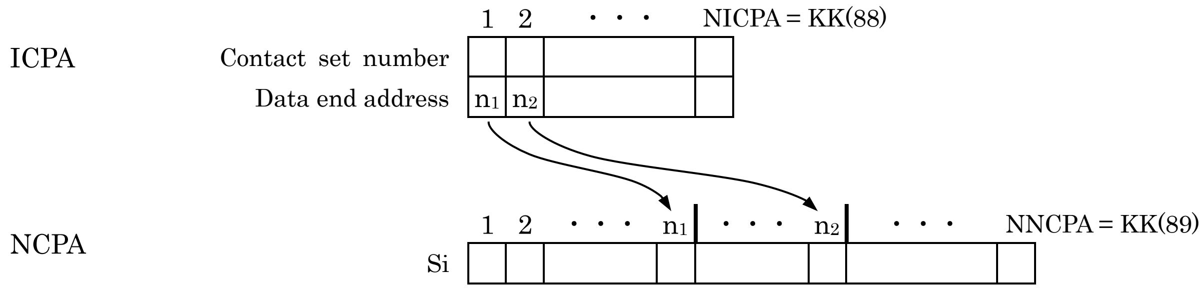


TABLED2

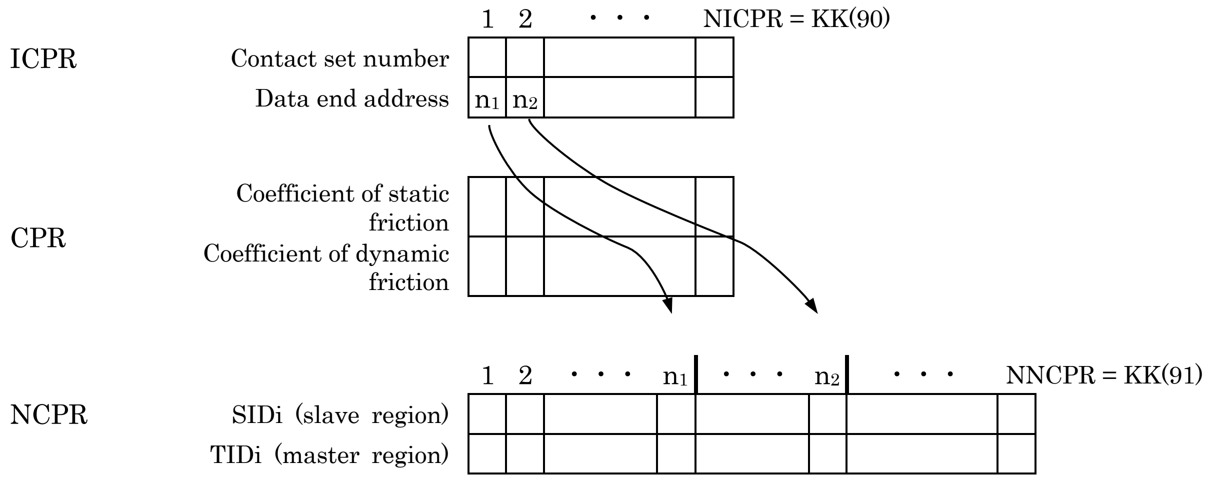


［8］Contact

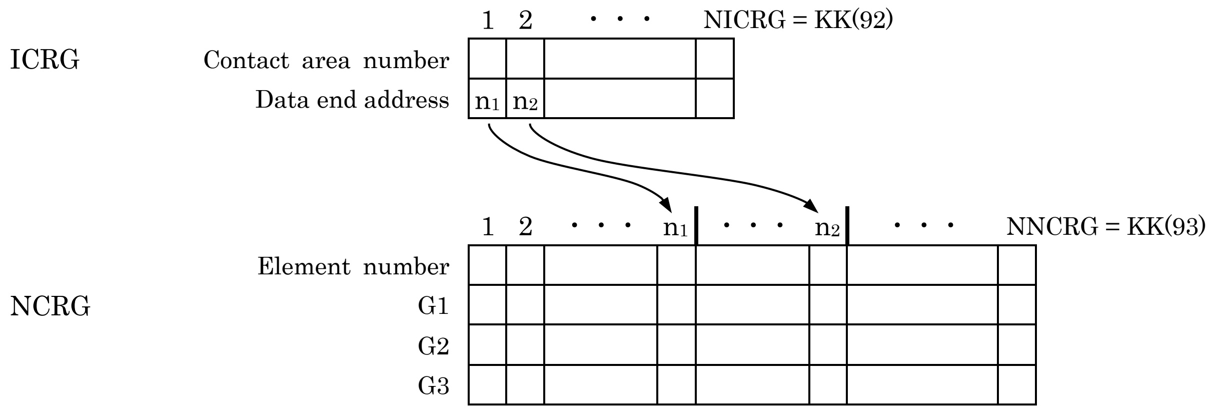
BCTADD



BCTSET, BCTPARA

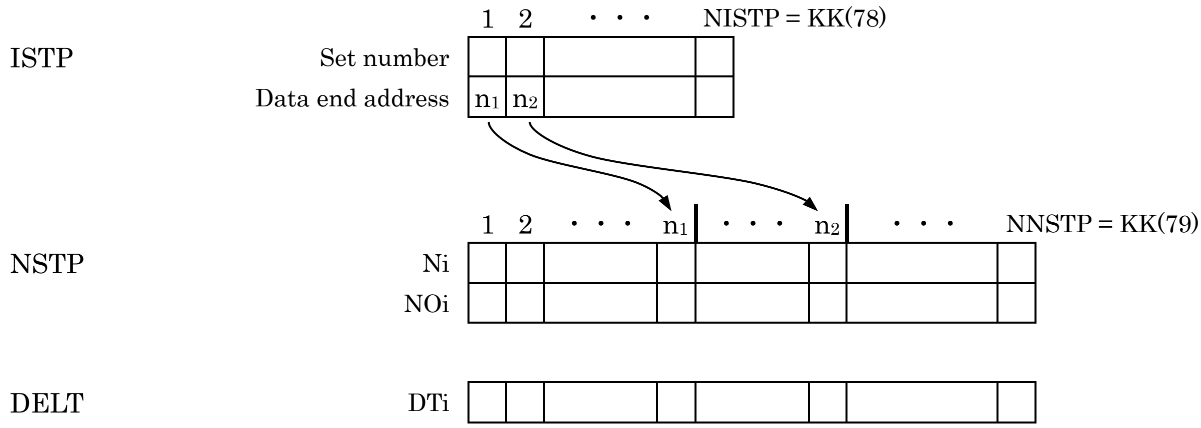


BSURFS



［９］Analysis control

TSTEP



［１０］ Other

REST, TIME
, DT
, RUN

	Restart calculation start time	1	
	Data output time increment for restart calculation	2	
	Data output time for restart calculation 1	3	
RTIM	Data output time for restart calculation 2	4	
		.	
		.	
	Data output time for restart calculation NRST	2+NRST	

NRST = KK(83)

4．3　Global variable table

The global variables defined in the module are listed below.

4．3．1 **Module name**：M_VAL（file name：m_val.f90）

see KK table for array size variables.

Classification.	variable name	type	contents
input data	see 4.2		
displacement node force	UG1(6,NNOD+NIGSF)	R8	Displacement（at n-1）
	UG2(6,NNOD+NIGSF)	R8	Displacement（at n）
	UG3(6,NNOD+NIGSF)	R8	Displacement（at n+1）
	UGP(6,NNOD+NIGSF)	R8	Displacement（at n+1）（before iter.）
	DUG(6,NNOD)	R8	Displacement step
	POS(3,NNOD+NIGSF)	R8	Position
	POSO(3,NNOD+NIGSF)	R8	Position　（before contact iter.）
	VG0(3,2,2,NBAR)	R8	Beam element director（initial value）
	VG(3,2,2,NBAR)	R8	Beam element director
	VGP(3,2,2,NBAR)	R8	Beam element director（before iter.）
	FTO(6,NNOD)	R8	see 2.1.1[5]
	FTI(6,NNOD)	R8	
	FTID(6,NNOD)	R8	
	FCO(6,NNOD,3)	R8	
	FCK(6,NNOD,3)	R8	
	FCD(6,NNOD,4)	R8	
	FCM(6,NNOD,3)	R8	
	FCP(3,NNOD,2)	R8	
	RFCO(6,NNOD)	R8	Restraint reaction force
	RFCI(6,NNOD)	R8	Contact reaction forcr
	FRCI(6,NNOD)	R8	Friction force
pressure velocity	PPND(NNOD)	R8	Pressure obtained by interpolating CADMAS cell pressure
	PG1(NNOD+NIGSF)	R8	Pore pressure（at n-1）
	PG2(NNOD+NIGSF)	R8	Pore pressure（at n）
	PG3(NNOD+NIGSF)	R8	Pore pressure（at n+1）
	DPG(NNOD)	R8	Incremental pore water pressure
	FLO(NNOD)	R8	see 2.1.2[5]
	FLI(NNOD)	R8	

condition of the integration point	DMT(21*MGP,NELM)	R8	Constitutive matrix
	EPSG(6*MGP,NELM)	R8	Distorted
	SIGG(6*MGP,NELM)	R8	Stress
	IST(MGP,NELM)	I4	State（=0: elasticity, =1: crack, =2 : plasticity）
	SIGY(MGP,NELM)	R8	Yield stress
seepage flow	VELG(3,MGP,NELM)	R8	Darcy velocity（element integral point）
	VELE(3,NELM)	R8	Darcy velocity（element center）
data for coupled analysis from CADMAS	IPFC(10,NPFC)	I4	Component node data of surface elements
	AFC(NPFC)	R8	Rate of area under pressure
	IPND(NNOD)	I4	State of nodes in CADMAS（≤0: non-pressure, =1: pressure loading, =2 : 1 other points on the ground surface）
	PTIM(NPTIM)	R8	CADMAS result files for one-way coupling Time of data output to (data.prs)
	PND(NNOD,NPITM)	R8	Node pressure obtained by interpolating cell pressure in CADMAS
time step	D_T(NSTEP)	R8	Calculation time step
	IOUT(NSTEP)	I4	Output flag（=0:non-output, =1 : output）
restart	IROUT(NSTEP)	I4	Restart file output flag （=0:non-output, =1 : output）
index	INDOF0(6,NNOD)	I4	Initial value of INDOF
	INDOF(6,NNOD)	I4	The degrees of freedom in a simultaneous linear equation（=-2: follow the MPC , =-1: forced displacement, =0 : fixation, >0 : degree of freedom）
	INDMPC(2,6,NNOD)	I4	The start and end addresses of the MPCF corresponding to the nodes and components that are subject to the MPC.（see 4.3.1[1]）
	MPCF(2,NIRH)	I4	Nodes of independent degrees of freedom in the MPC relation, component number
	RMPC(NIRH)	R8	Coefficient for the above
	INDOP0(NNOD)	I4	Initial value of INDOP
	INDOP(NNOD)	I4	Degrees of freedom in a simultaneous linear equation for pressure（=-1: fixation, =0: outside scope of calculation, >0 : degree of freedom）
matrix solver	RHV(NEQ)	R8	The right-hand side of a simultaneous linear equation
	X(NEQ)	R8	Solutions to simultaneous linear equations
	IDSK(NEQ+1)	I4	see 4.3.1[2]
	IDCG(NCGSPC)	I4	
	CGWK(NEQ,6)	R8	CG Method work
	STF(NGCPSC)	R8	Coefficient matrix of simultaneous linear equations
	LOW(NCGSPC)	R8	Work used for incomplete LU decomposition of CG method

work	WRK1-3(6,NNOD)	R8	Work
contact	see 4.3.1[3]		

The array sizes shown above are for the case without domain segmentation. The nodal and element data sizes with domain division are as follows.

Node data size

KK(8)	KK(28)	KK(94)	KK(108)	KK(31)	KK(107)
NNOD	NNODC	NIGSF	NIGSFC	NNODX	NIGSFX
Nodes in the area	Nodes in other domains (Added for contact determination)	Surface center of gravity point in the area	Surface center of gravity point of other area (for contact determination)	Nodes in other domains (Added for equation assembly)	Surface center of gravity point of another domain (Added for equation assembly)

Element data size

KK(12)	KK(29)	KK(32)
NELM	NELMC	NELMX
Nodes in the area	Nodes in other elements (Added for contact determination)	Nodes in other domains (Added for equation assembly)

［1］Examples of INDMPC, MPCF, and RMPC

The MPC conditional equation is

U(j, i) = R1・U(j1, i1) + R2・U(j2, i2) + R3・U(j3, i3)

the data of this conditional expression is stored as follows.

INDOF(j, i) = -2

INDMPC(1, j, i) = IS

INDMPC(2, j, i) = IE

	・・・	IS		IE	・・・
MPCF		i1	i2	i3	
		j1	j2	j3	
RMPC		R1	R2	R3	

［２］Examples of IDSK and IDCG

（１）The Case for the CG Method

Non-zero elements of the coefficient matrix（The number indicates the address in the STF where the value of the coefficient at that position is stored.）

$$\begin{bmatrix} 1 & & & & & \\ & 2 & & & & \\ 4 & 5 & 3 & & & \\ 7 & & 8 & 6 & & \\ & 10 & 11 & & 9 & \\ & & 13 & 14 & & 12 \end{bmatrix}$$

NEQ = 6

IDSK：Non-zero element top address of each line（Diagonal component first in each row）

1, 2, 3, 6, 9, 12, 15

NEQ+1 puts non-zero element final address +1 in the first piece

IDCG：Column number of non-zero element

1, | 2, | 3, 1, 2, | 4, 1, 3, | 5, 2, 3, | 6, 3, 4 |

（２）The Case for the Direct Method

Non-zero elements of the coefficient matrix（The number indicates the address in the STF where the value of the coefficient at that position is stored.）

$$\begin{bmatrix} 1 & & 2 & 3 & & \\ & 4 & & 5 & & 6 \\ & & 7 & & 8 & 9 \\ & & & 10 & & 11 \\ & & & & 12 & \\ & & & & & 13 \end{bmatrix}$$

NEQ = 6

IDSK：Non-zero element top address of each line（Diagonal component first in each row）

1, 4, 7, 10, 12, 13, 14

NEQ+1 puts non-zero element final address +1 in the first piece

IDCG：Column number of non-zero element

1, 3, 4, | 2, 4, 6, | 3, 5, 6, | 4, 6, | 5, | 6 |

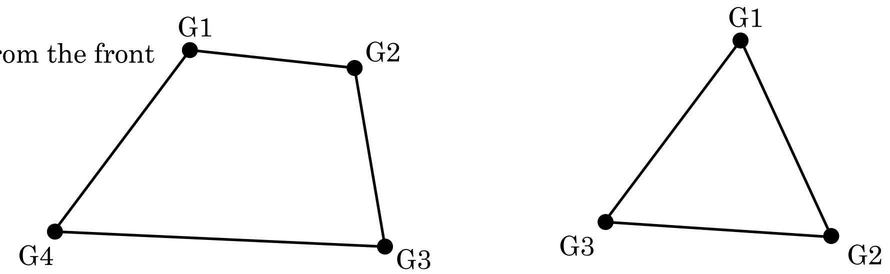
［３］ Contact Data Table

（１） Contact surface (corresponding to input data table NCRG)

ICSF

	1	2	・・・	NNCRG = KK(93)
Surface number				
Number of Constituent Nodes				
Configuration nodeG1				
G2				
G3				
G4				

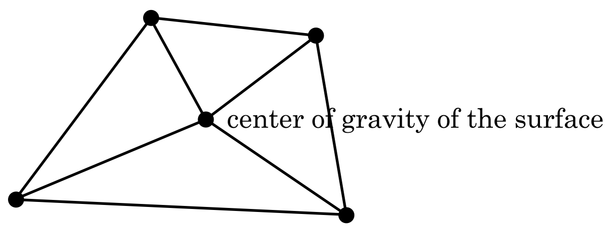
Figure from the front



Quadrilateral surface

Triangular surface

The quadrilateral surface is divided into four triangles and stored in the IELC table as follows



（2） Element surface center-of-gravity number

IGSFR

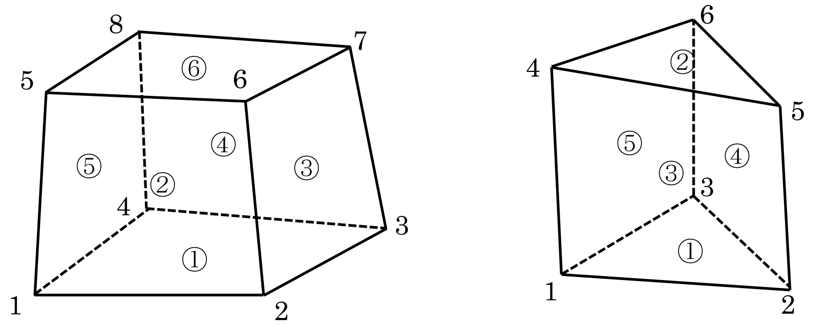
Element number	1	2	3	4	5	6	・・・	NELM = KK(12)
Surface number 1		1						
2		2						
3		3		7	10			
4		4		8	11			
5		5		9	12			
6		6						
		HEXA		PENTA		TETRA		

IBEL

connection flags to the constituent nodes of the contact surface	0	1	0	1	1	1		

※ The surface center-of-gravity point number is the above number plus NNOD = KK(8).

Surface number : ①, ②, ⋯



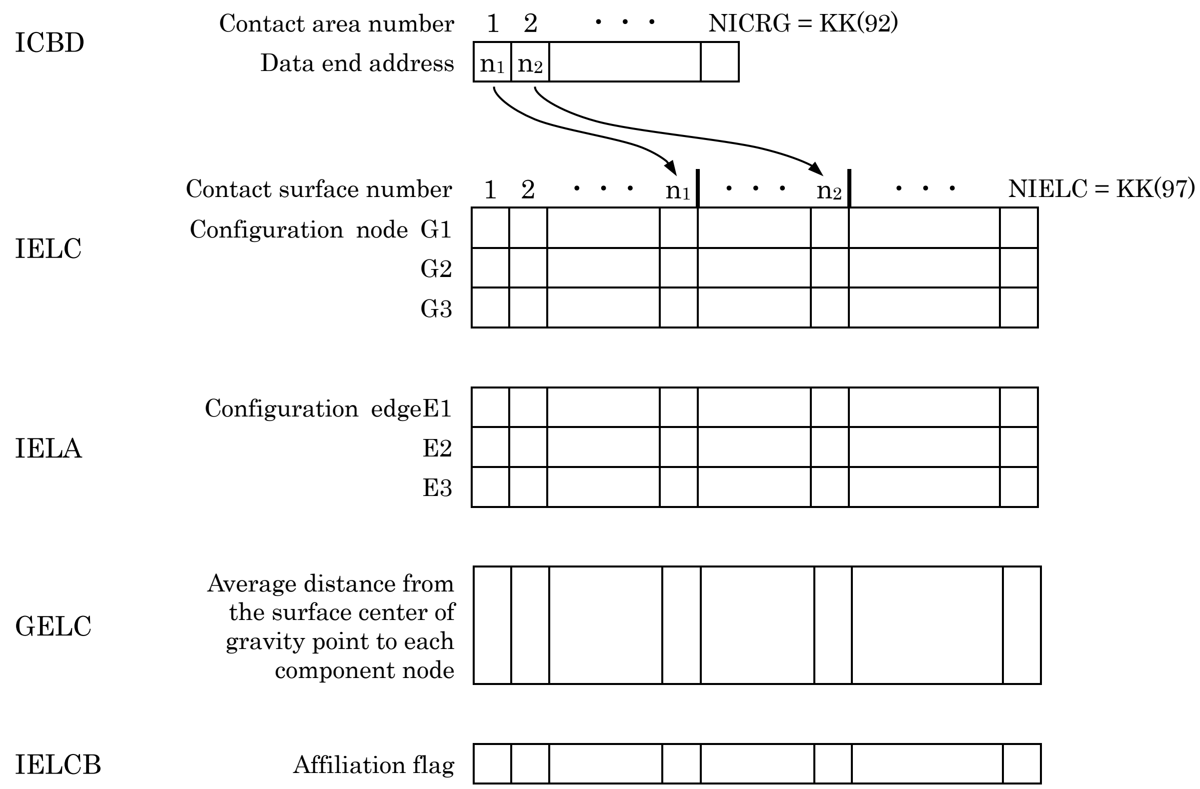
（3） Split tetrahedra (elements connected to the constituent nodes of the contact surface are divided into tetrahedra)

IBTE

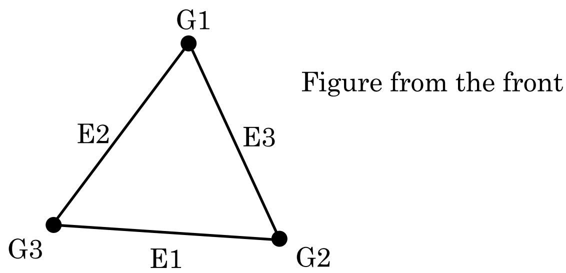
Split tetra number	1	2	・・・	NIBTE = KK(100)
Configuration node 1				
2				
3				
4				

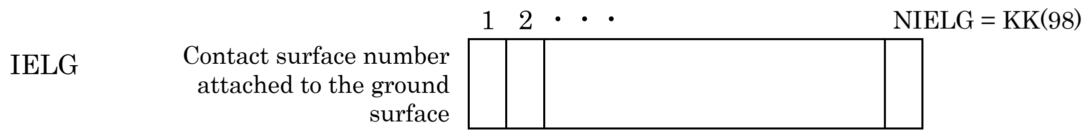
※Hexahedral element is divided into 24 tetrahedral elements and the pentahedral element into 11 tetrahedral elements, using the element's component nodes and face center of gravity points as component nodes.

（4）Contact surface (after dividing into triangular surfaces)

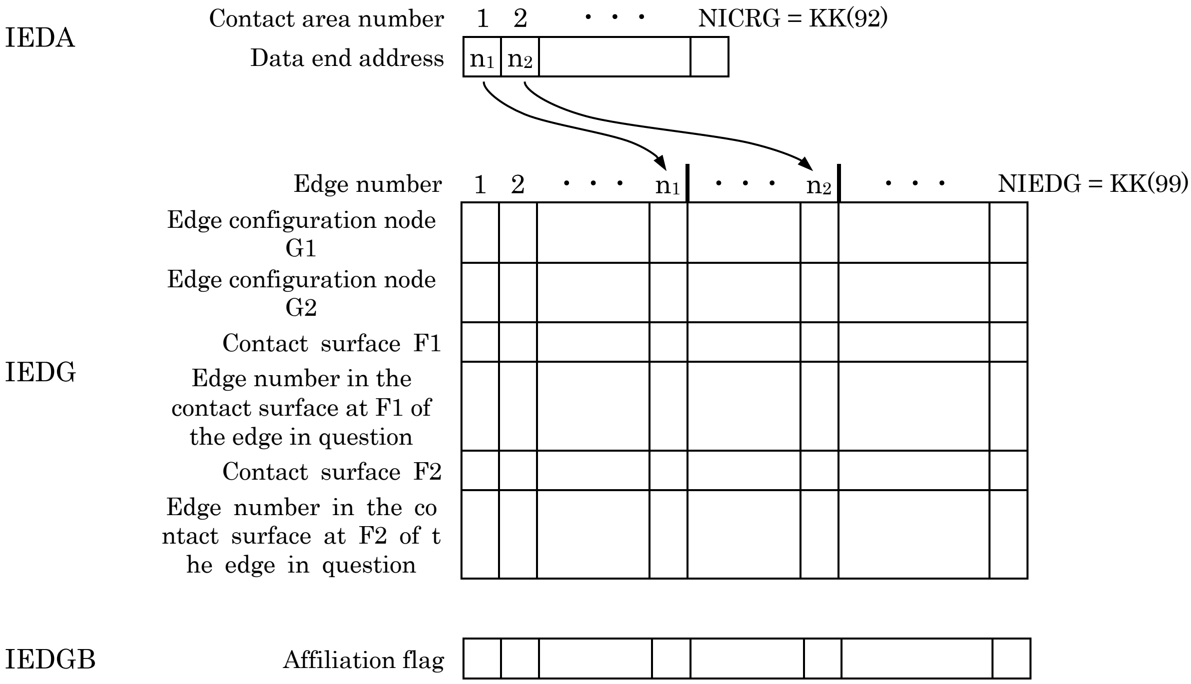


※ Affiliation flag =1 : at the area is split,the affiliation of the contact surface with the process in question, =0 : the other





（5）Contact surface edge



※ Affiliation flag =1 : at the area is split,the affiliation of the contact surface with the process in question, =0 : the other

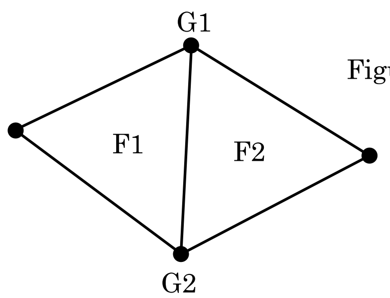
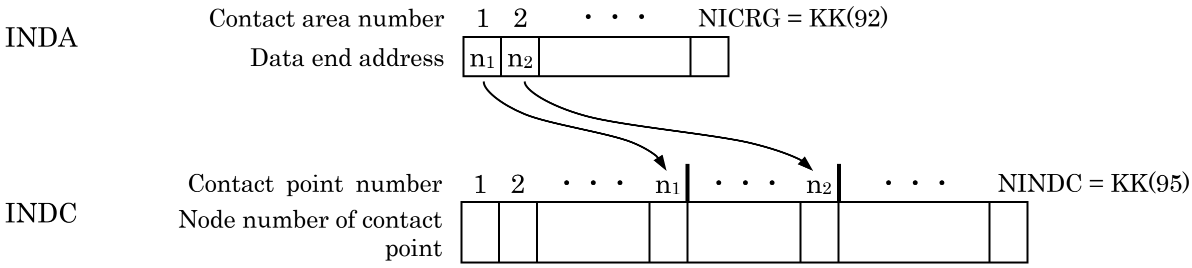
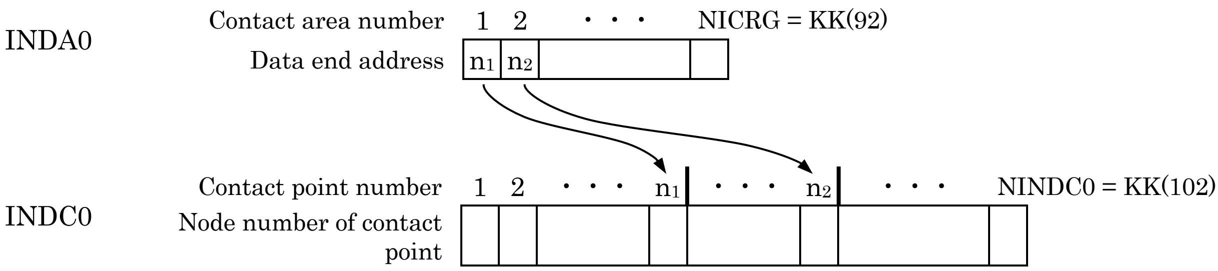


Figure from the front

（6） Contact point



（7） Contact point (data table for the entire domain during region partitioning)



（8） 4 角形面

IELQ

Tetragon surface number　1　2　・・・　NIELQ = KK(101)

Configuration nodeG1

G2

G3

G4

IFCQ

Contact surface number　1　2　・・・　NIELC = KK(97)

Tetragon surface number to which the contact surface belongs.

IEDQ

Edge number　1　2　・・・　NIEDG = KK(99)

Tetragon surface number to which the edge belongs

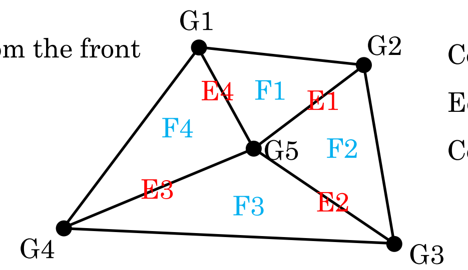
IVRQ

Element surface center-of-gravity number　1　2　・・・　NIGSF = KK(94)

Tetragon surface number to which the element face center-of-gravity point belongs

※ put the above data in IVRQ (NNOD+1:NNOD+NIGSF).

Figure from the front

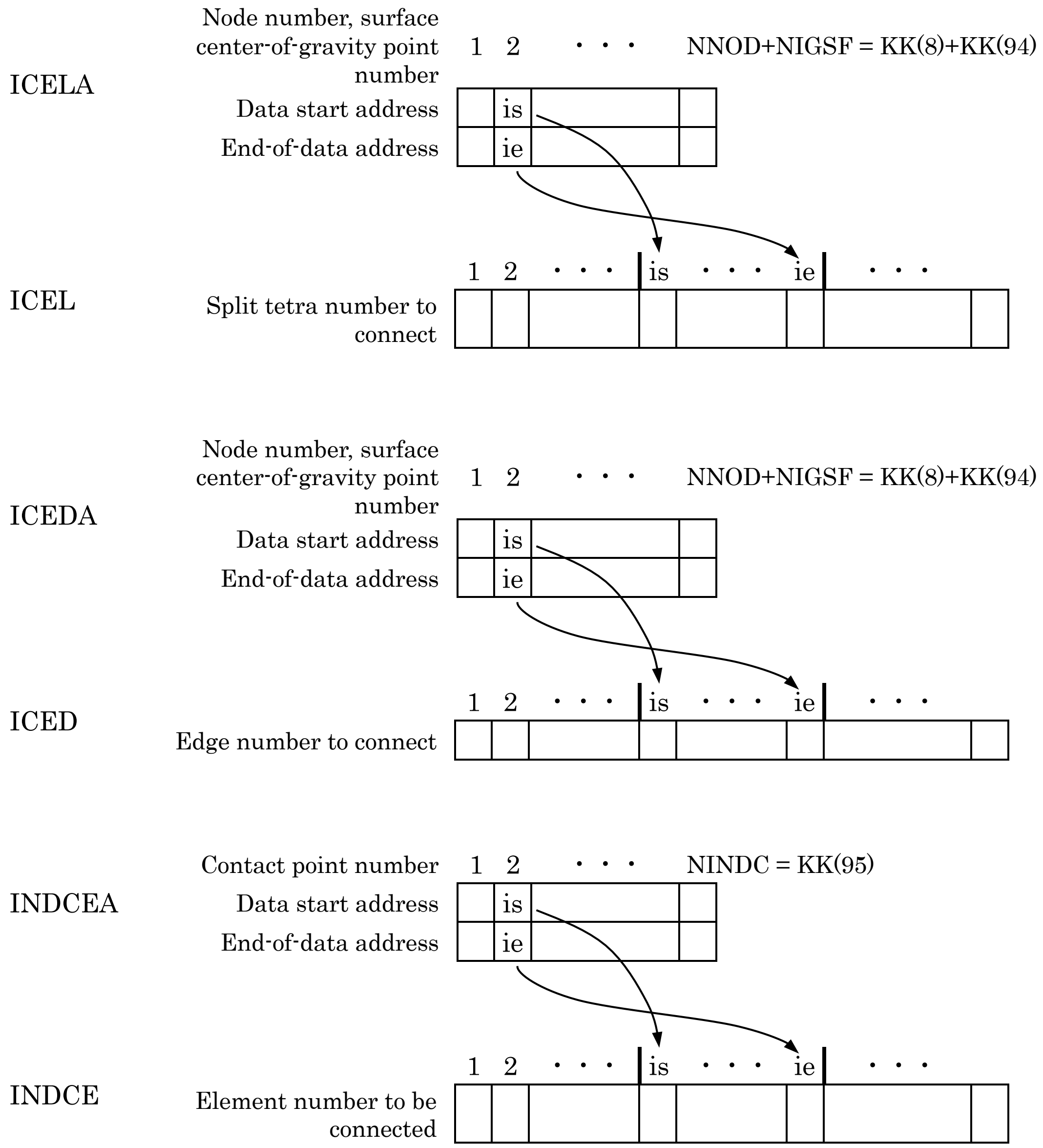


Contact surface：F1, F2, F3, F4

Edge：E1, E2, E3, E4

Center of gravity point：G5 belongs to this square surface

（9）Node-connected element data



※When dividing a region, the table is set only for contact points that are interior points of the process in question.

（１０）Contact point state

Contact point number 1 2 ・・・ NINDC = KK(95)

ISLV
- State of contact
- Master number

ISLVO
- State of contact (before iter.)
- Master number (before iter.)

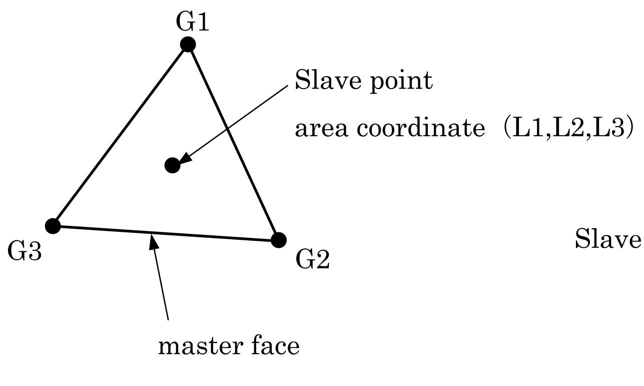
RSLV
- L1 or t
- L2
- L3

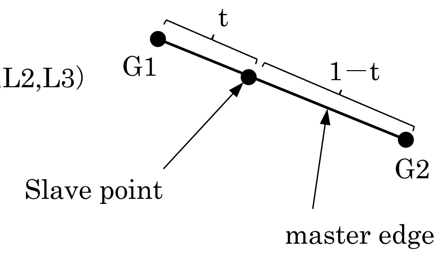
IRANK
- Priority of position correction

※ state of contact =0 : free, =1 : point restraint, =2 : line restraint, =3 : face restraint, =4 : straight-line adherence, =5 : surface adherence

The above number + 10 represents "new" (e.g., = 13: new face constraint).

※ Master number, master Node number, edge number or contact surface number





（１１）Contact point state（data table for the entire region at the time of region partitioning）

	Contact point number	1	2	・・・	NINDC0 = KK(102)
ISLV0	State of contact				
	Master number				
ISLVP	Process number to which the master belongs				
RSLV0	L1 or t				
	L2				
	L3				
PSLV	Position coordinates in previous iter.				
	Current position coordinates				
	Contact reaction force vector				
	Average normal vector of connected surfaces				

※ For PSLV, put 3 components for each item.

(１２) Frictional state of contact point

IFRIC

Contact point number	1	2	・・・	NINDC = KK(95)
Statue 1				
Mode 2				
Type 3				
Continuation of surface contact flag 4				
Slave belonging Area 5				
Master's affiliation area 6				
Unconverged flag 7				
Reaction flag 8				
Friction model 9				
10				

※ state =0 : non-friction, =1 : new, =2 : cotinuous

※ type =0 : dynamic friction, =1 : static friction

※ reaction force =1 : non-reaction force of previous iter. in terms of considering friction, =0 : the other

FRIC

Contact point number	1	2	・・・	NINDC = KK(95)
Coefficient of static friction 1				
2				
Contact reaction force 3				
Coefficient of kinetic friction 4				
Relative displacement with contact surface 5				
Friction spring constant 6				
Displacement at mode 3 transition 7				
8				
9				
10				

U0

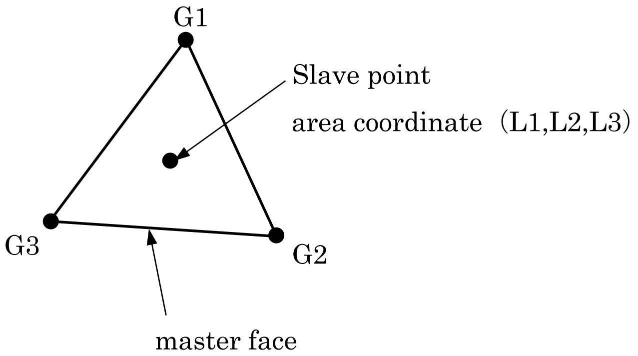
Contact point number	1	2	・・・	NINDC = KK(95)
Master configuration node G1 displacement				
G2 displacement				
G3 displacement				
Slave point displacement				

RL0

Initial L1				

L2				
L3				

※ For U0, put 3 components for each item.



（１３）Friction state of contact point (data table in the entire region at the time of region division)

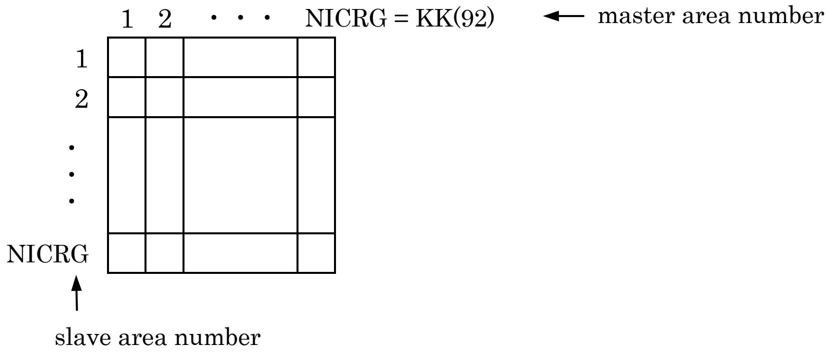
ISTICK

Contact point number	1	2	・・・	NINDC0 = KK(102)
Friction type				

※ friction type =0：kinetic friction, =1：static friction

（１４）Contact pair flag

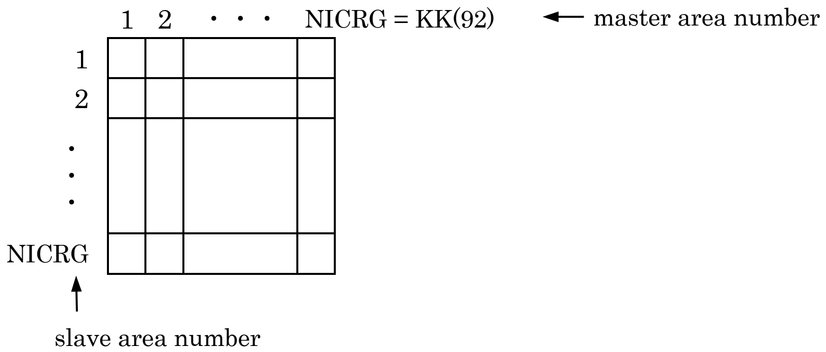
ICTB



※ contact pair flag =1：consider contact, =0：not consider

（１５）Friction type

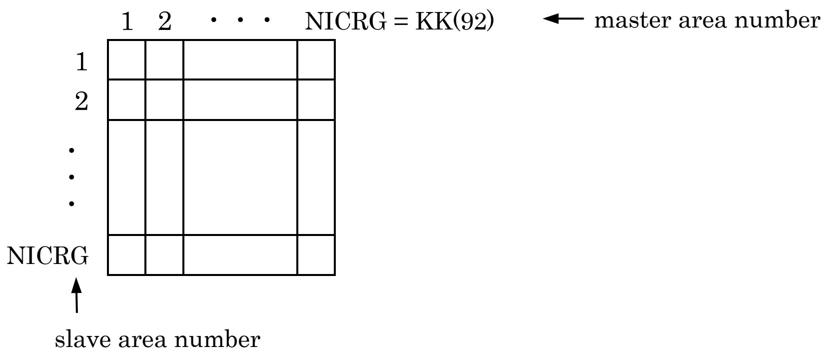
ISTK



※ friction type =0：kinetic friction, =1：static friction

（１６）coefficient of friction

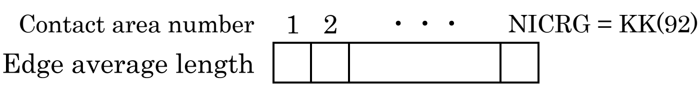
FRTB



※ put 3 components for each item
(Component 1: coefficient of static friction, component 3: coefficient of kinetic friction).

（１７）Edge average length

EDML



4．3．2 **Module name**：M_PART（file name：m_part.f90）

classification.	variable name	type	contents
data for each calculation process maintained by the root process	NN_INT(NPECG)	I4	Number of inside
	NN_EXT(NPECG)	I4	Total number of nodes (inner points + outer points)
	NE(NPECG)	I4	Number of elements
	NPF(NPECG)	I4	Number of face elements
	NOD(2*MINT,NPECG)	I4	Node number
	IEL(2*ME,NPECG)	I4	Element number
	IPF(MPF,NPECG)	I4	Surface element number
	NPEW-NODEXPW	I4	Work to create NPE-NODEXP
Node data communication table for each calculation process	NPE	I4	Number of processes communicating
	NIMP	I4	Number of nodes to receive data
	NEXP	I4	Number of nodes to send data
	IPE(NPE)	I4	Process number for communication
	IDXIMP(2,NPE)	I4	Start and end address of NODIMP where the list of node numbers to receive data from each process is stored.
	NODIMP(NIMP)	I4	Node number to receive data
	IDXEXP(2,NPE)	I4	Start and end address of NODEXP where the list of node numbers to send data to each process is stored.
	NODEXP(NEXP)	I4	Node number to send data

Data for contact analysis

classification.	variable name	type	contents
Data for each calculation process maintained by the root process	NG(NPECG)	I4	Surface gravity points
	NIEC(NPECG)	I4	Number of contact surfaces
	NINC(NPECG)	I4	Number of contact points
	NIEG(NPECG)	I4	Number of contact surface edges
	NIEQ(NPECG)	I4	Number of contact quadrilateral faces
	NIBT(NPECG)	I4	Number of split tetras
	NODG(MAX(NG),NPECG)	I4	Face gravity point number
	IEC(MAX(NIEC),NPECG)	I4	Contact surface number
	INC(MAX(NINC),NPECG)	I4	Contact point number
	IEG(MAX(NIEG),NPECG)	I4	Contact surface edge number
	IEQ(MAX(NIEQ),NPECG)	I4	Contact quadrilateral faces number
	IBT(MAX(NIBT),NPECG)	I4	Split tetra number

Process to which each data held in the root process belongs, local number	NODP(2,NNOD+NIGSF)	I4	Node, center of gravity point
	IELCP(2,NIELC)	I4	Contact surface
	INDCP(2,NINDC0)	I4	Contact point
	IEDGP(2,NIEDG)	I4	Contact surface edge
Data for each calculation process held by the root process (additional for contact determination)	NN_EXTC(NPECG)	I4	Number of nodes
	NGC(NPECG)	I4	Number of centers of gravity
	NEC(NPECG)	I4	Number of elements
	NODC(MAX(NN_EXTC),NPECG)	I4	Node number
	NODGC(MAX(NGC),NPECG)	I4	Face gravity point number
	IELMC(MAX(NEC),NPECG)	I4	Element number
Data for each calculation process retained by the root process (additional for equation assembly)	NN_EXTX(NPECG)	I4	Number of nodes
	NODX(MAX(NN_INT),NPECG)	I4	Node number
Nodal data communication table for each calculation process (additional for equation assembly)	NPIMPX	I4	Number of processes receiving data
	NIMPX	I4	Number of nodes to receive data
	IPIMPX(NPIMPX)	I4	Process number to receive data
	IDXIMPX(2,NPIMPX)	I4	Start and end address of NODIMPX where the list of node numbers to receive data from each process is stored.
	NODIMPX(NIMPX)	I4	Node number to receive data
	NPEXPX	I4	Number of processes sending data
	NEXPX	I4	Number of nodes to send data
	IPEXPX(NPEXPX)	I4	Process number to send data
	IDXEXPX(2,NPEXPX)	I4	Start and end address of NODEXPX where the list of node numbers to send data to each process is stored.
	NODEXPX(NEXPX)	I4	Node number to send data

Overview of distributed data structures and communication

In parallel computation, all mesh data is first read by the root process (rank=0), where the data is divided into regions and distributed to each computation process (rank>0). Then, the parallel computation is executed while communicating data among the processes.

The mesh data for each process is based on nodal-based domain segmentation, and nodes are classified into the following three types in terms of communication

・Inner points: Nodes assigned to each domain

・Outer points: Nodes that belong to other regions but are included in the elements of each region.

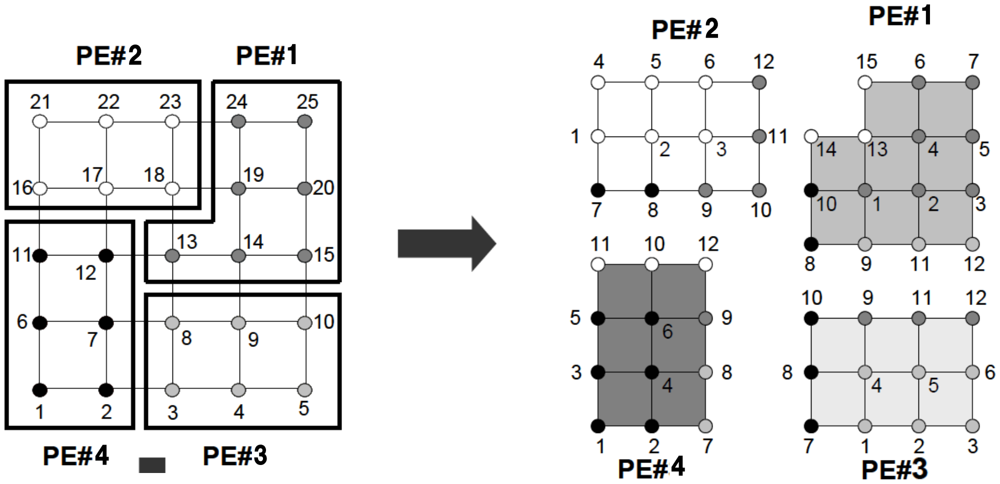
・Boundary point: An interior point that is an exterior point of another domain

The figure below shows an example of region segmentation data. In PE#3, nodes are classified as follows

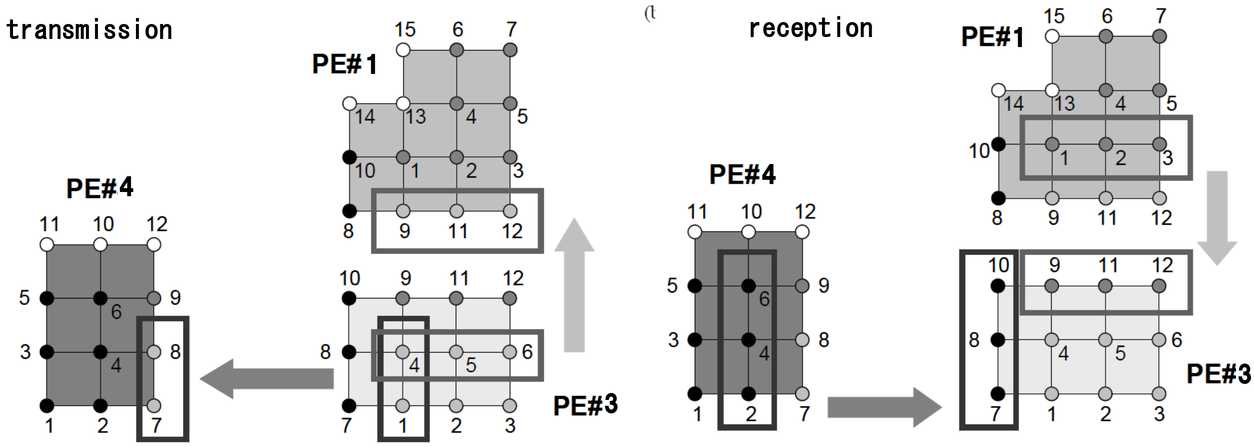
・Inner points　　{ 1, 2, 3, 4, 5, 6 }

・Outer points　　{ 7, 8, 9, 10, 11, 12 }

・Boundary point　{ 1, 4, 5, 6 }



The value at the boundary point is sent to the adjacent region, where it is received as an outer point. By communicating with each other, each process obtains the value of the boundary point.

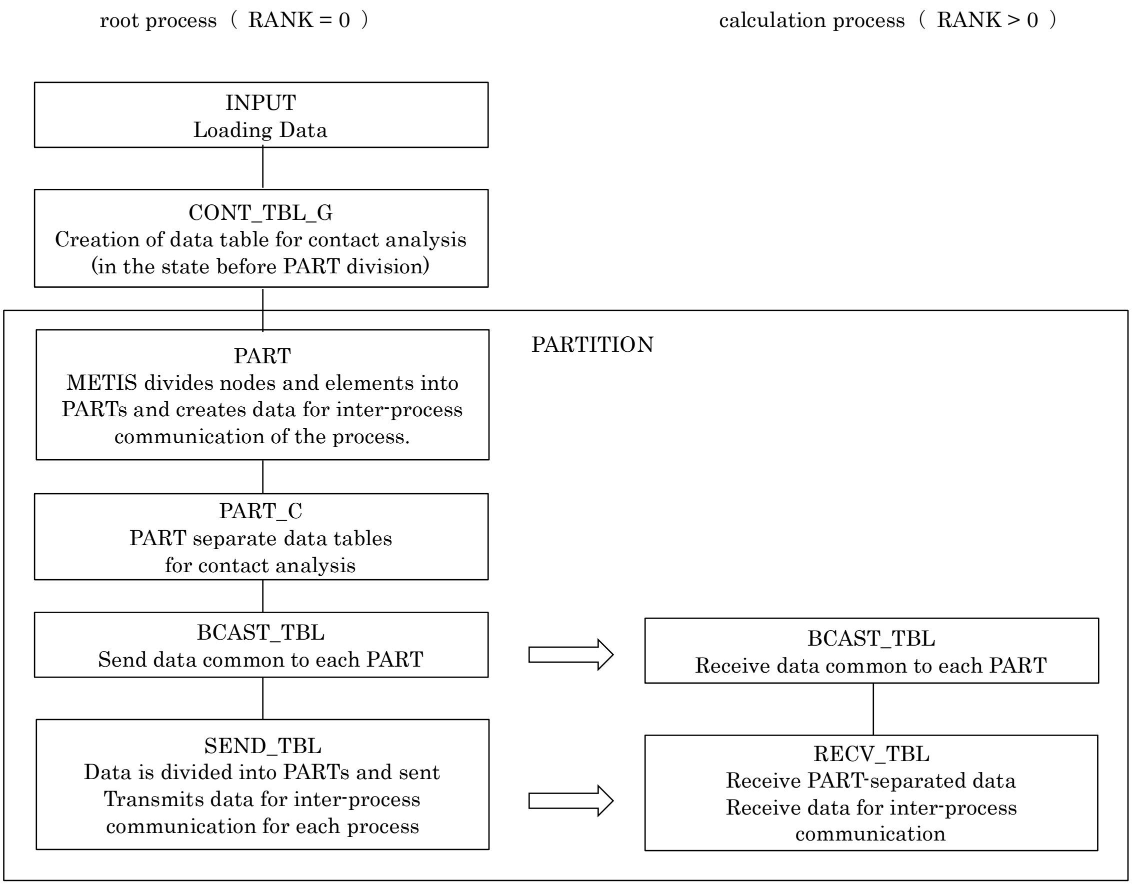


4．3．3 **Module name**：MPI_PARAM（file name：mpi_param.f90）

classification.	variable name	type	contents
Overall (CADMAS & Structural Analysis) Communicator	CPLWORLD	I4	Overall (CADMAS & Structural Analysis) communicator
	MYRANK0	I4	Rank your process in the overall (CADMAS & Structural Analysis) communicator
	IROOTC	I4	Rank of CADMAS root processes in the overall (CADMAS & Structural Analysis) communicator
Structural Analysis Communicator	MYWORLD	I4	Structural Analysis Communicator
	NPROCS	I4	Number of processes for structural analysis
	MYRANK	I4	Rank your process in the Structural Analysis Communicator
CG Law Communicator	CGWORLD	I4	Communicator that calculates the CG method (structural analysis communicator minus the root process)
coupled analysis	ICPL	I4	Types of coupled analysis (=0:non-coupled, =1: one-way coupling, =2：bidirectional coupling)

5．Program processing flow

［1］Data input, PART division (in case of parallel CG selection)



［２］ Communication between SUB.GLB_COMM and SUB.ASTEA_MECHANICAL (in case of parallel CG selection)

root process （ RANK = 0 ）		calculation process （ RANK > 0 ）
GLB_COMM		ASTEA_MECHANICAL
RESFRC0 Received residual norm for each PART Convergence determination by adding, Send convergence decision flag to each calculation process	⇔	RESFRC Transmit residual norm, convergence decision flag received
NPFLOW0 Residual norm of each PART, flow rate from pressure fixed point is received, added and message output	⇐	NPFLOW Transmits residual norm, flow rate from pressure fixed point
FEMAP_OUT0 Receive and output calculation results for each PART	⇐	FEMAP_OUT Send calculation results
RDPRESS0 Loads a CADMAS calculation result (pressure) file during one-way coupled analysis and sends it to each calculation process.	⇒	RDPRESS Receive CADMAS calculation results (pressure)
RECV_SURF0 Receives surface data from CADMAS during bidirectional coupled analysis and sends it to each calculation process	⇒	RECV_SURF Receive surface data
RECV_PRES0 Receive calculation results (pressure) from CADMAS during bidirectional coupled analysis and send them to each calculation process	⇒	RECV_PRES Receive CADMAS calculation results (pressure)
SEND_POS0 Receives the structural nodal point positions of each PART during bidirectional coupled analysis and sends them to CADMAS.	⇐	SEND_POS Send structural node point position
BCTSET0 Contact pairs, friction coefficient tables received	⇐	BCTSET Send contact pair, friction coefficient
SEND_CONT0 Send contact surface data to CADMAS	⇐	SEND_CONT Send contact analysis flag
CONT_INIT_0 Receive nodal information of each PART and send node information outside PART to each calculation process, Create and send contact point information Receive contact status, update and send	⇔	CONT_INIT_P Send and receive node point information Receive contact point information Determines contact state, transmits, and receives

PART_SEND_TBLX
Receive element information
for each PART
Prepare additional data necessary for
equation assembly for each PART and
send to each calculation process.
Prepares and sends data for inter-process
communication (additional data)
for each process

CONTACT0
Receive nodal information of each PART
and send nodal information outside PART
to each calculation process
Create and send contact point
information
Receive contact status, update and send
Convergence judgment, send
convergence flag
Update friction state

MDPRESS0
Receives provisional water pressure
values for surface nodes with
undetermined water pressure from each
PART, determines the water pressure,
and sends them to
each calculation process.

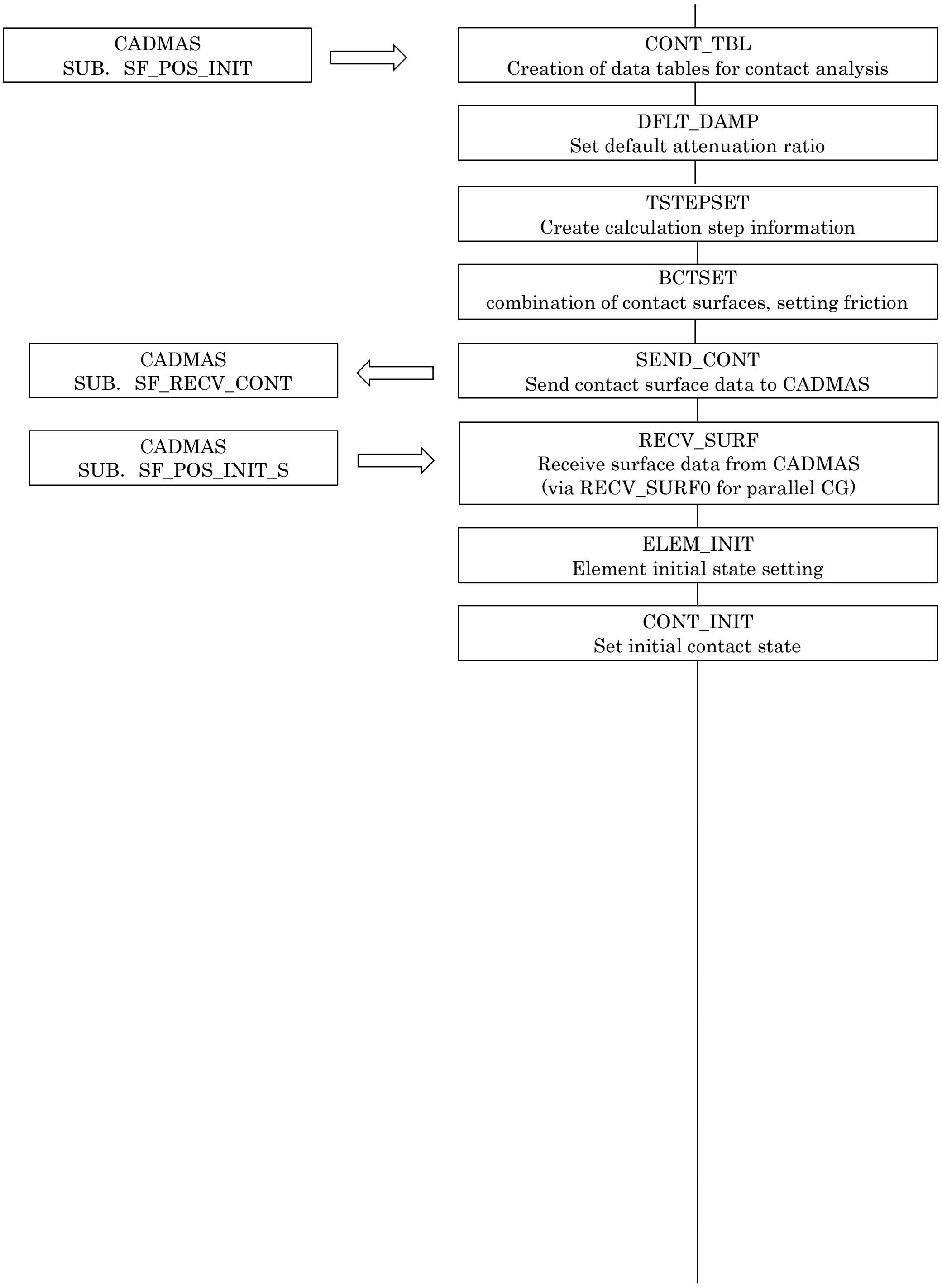
RECV_TBLX
Send element information
Receive additional data needed for
equation assembly
Receive additional data for inter-process
communication

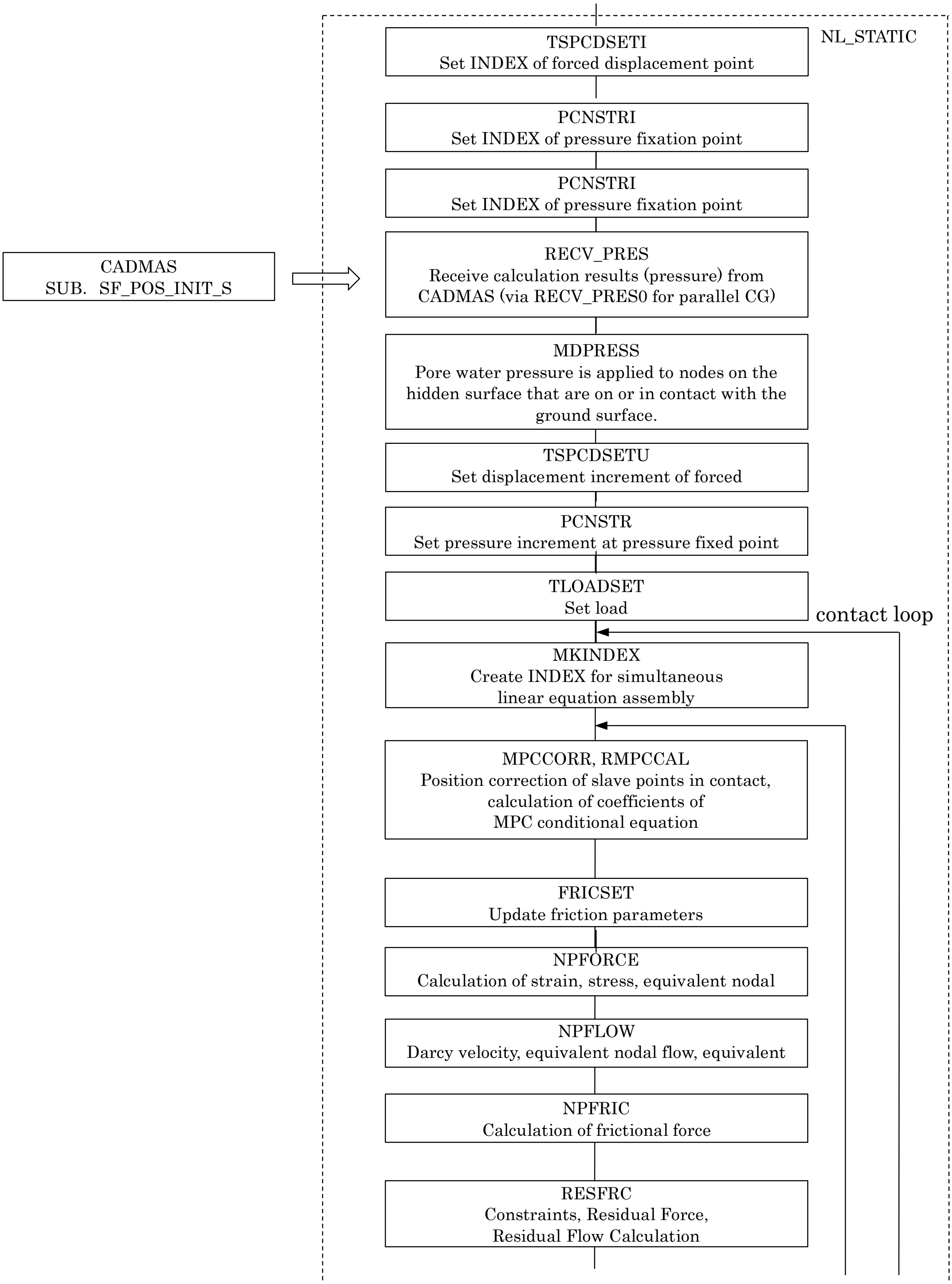
COMM_TBLX
Bidirectional communication between
processes for additional nodal data
needed for equation assembly

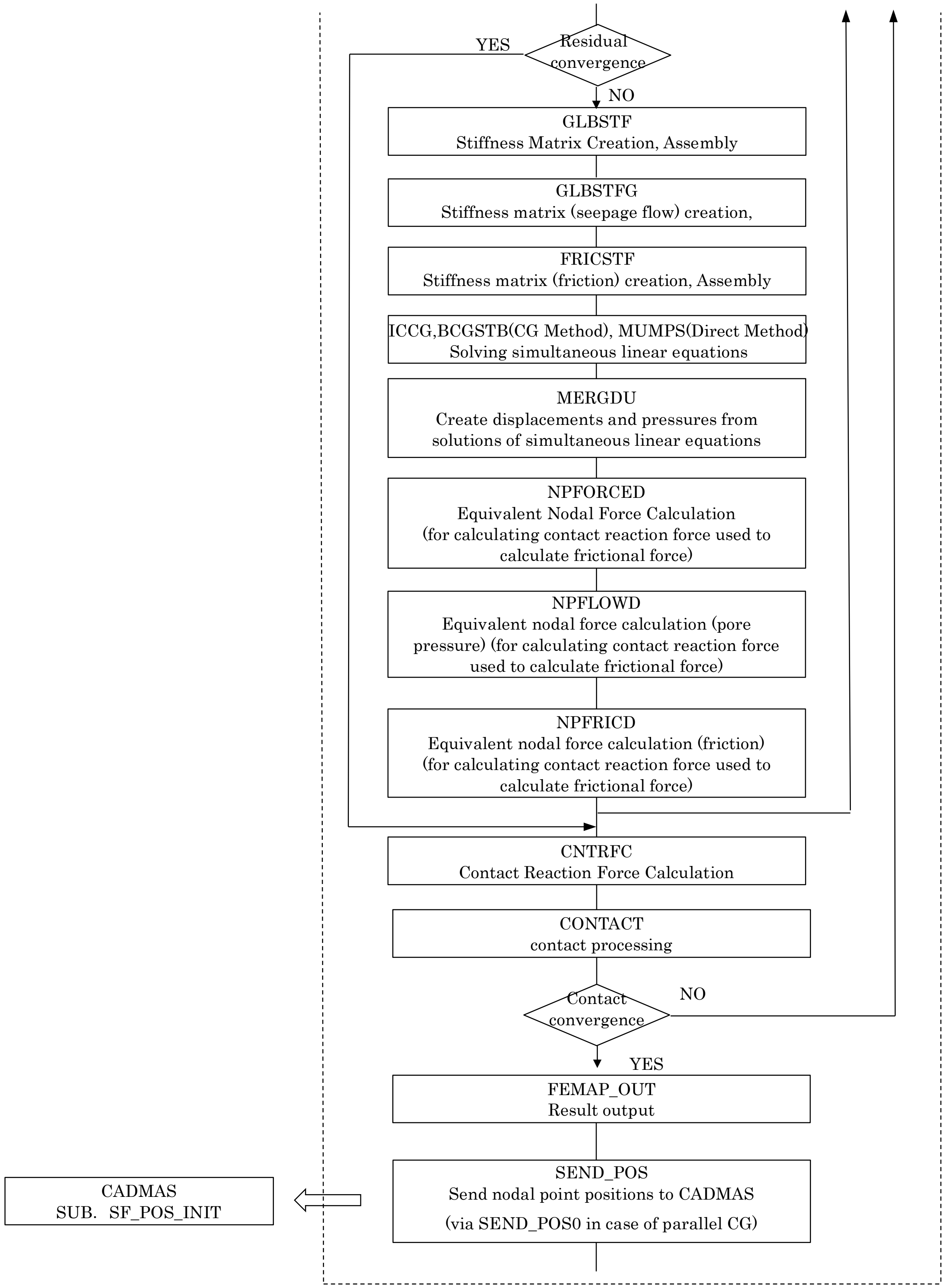
CONTACTP
Send and receive nodal point information
Receive contact point information
Determines contact state, transmits,
and receives
Convergence flag is received
Transmit friction state

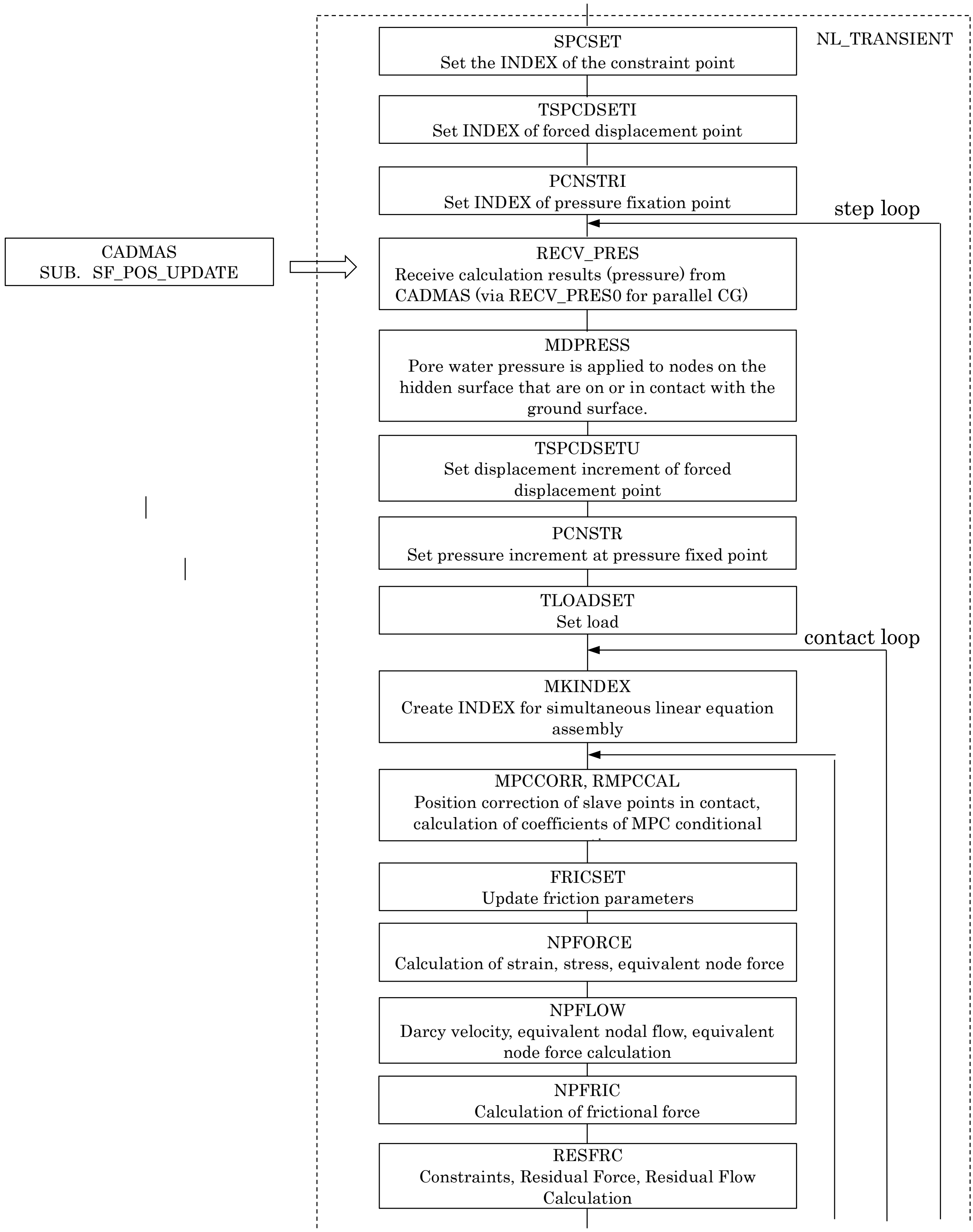
MDPRESSP
The water pressure at the surface node
with undetermined water pressure is
calculated by pore water pressure
interpolation on the ground surface
where the node is in contact with,
transmitted, and the determined value is
received.

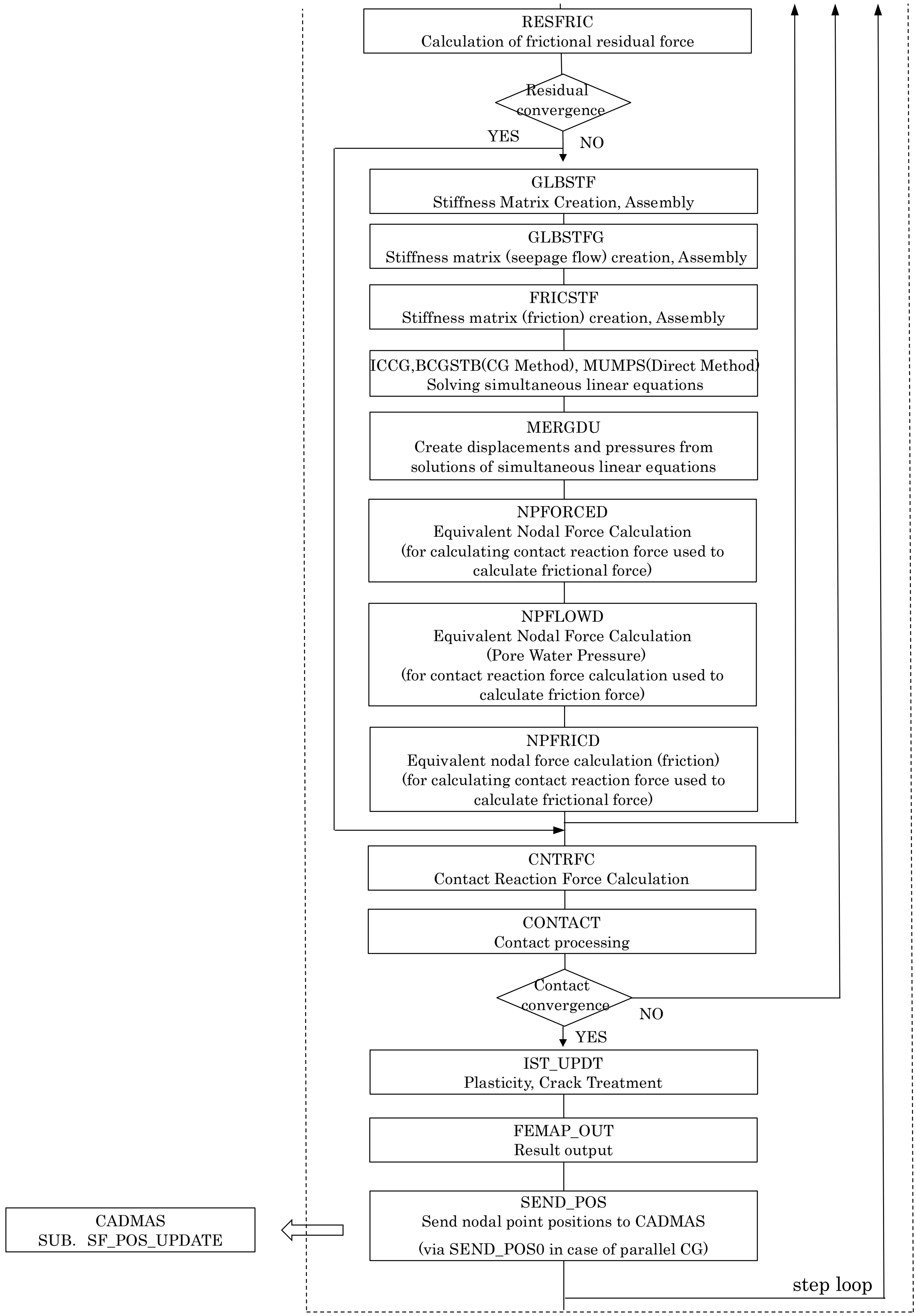
［３］SUB. ASTEA_MECHANICAL Flow (bidirectional coupled, with contact)




NL_STATIC
TSPCDSETI
Set INDEX of forced displacement point
PCNSTRI
Set INDEX of pressure fixation point
PCNSTRI
Set INDEX of pressure fixation point
CADMAS
SUB. SF_POS_INIT_S
RECV_PRES
Receive calculation results (pressure) from CADMAS (via RECV_PRES0 for parallel CG)
MDPRESS
Pore water pressure is applied to nodes on the hidden surface that are on or in contact with the ground surface.
TSPCDSETU
Set displacement increment of forced
PCNSTR
Set pressure increment at pressure fixed point
TLOADSET
Set load
contact loop
MKINDEX
Create INDEX for simultaneous linear equation assembly
MPCCORR, RMPCCAL
Position correction of slave points in contact, calculation of coefficients of MPC conditional equation
FRICSET
Update friction parameters
NPFORCE
Calculation of strain, stress, equivalent nodal
NPFLOW
Darcy velocity, equivalent nodal flow, equivalent
NPFRIC
Calculation of frictional force
RESFRC
Constraints, Residual Force, Residual Flow Calculation


Residual convergence
YES
NO
GLBSTF
Stiffness Matrix Creation, Assembly
GLBSTFG
Stiffness matrix (seepage flow) creation,
FRICSTF
Stiffness matrix (friction) creation, Assembly
ICCG,BCGSTB(CG Method), MUMPS(Direct Method)
Solving simultaneous linear equations
MERGDU
Create displacements and pressures from solutions of simultaneous linear equations
NPFORCED
Equivalent Nodal Force Calculation
(for calculating contact reaction force used to calculate frictional force)
NPFLOWD
Equivalent nodal force calculation (pore pressure) (for calculating contact reaction force used to calculate frictional force)
NPFRICD
Equivalent nodal force calculation (friction)
(for calculating contact reaction force used to calculate frictional force)
CNTRFC
Contact Reaction Force Calculation
CONTACT
contact processing
Contact convergence
NO
YES
FEMAP_OUT
Result output
SEND_POS
Send nodal point positions to CADMAS
(via SEND_POS0 in case of parallel CG)
CADMAS
SUB. SF_POS_INIT


NL_TRANSIENT
SPCSET
Set the INDEX of the constraint point
TSPCDSETI
Set INDEX of forced displacement point
PCNSTRI
Set INDEX of pressure fixation point
step loop
CADMAS
SUB. SF_POS_UPDATE
RECV_PRES
Receive calculation results (pressure) from CADMAS (via RECV_PRES0 for parallel CG)
MDPRESS
Pore water pressure is applied to nodes on the hidden surface that are on or in contact with the ground surface.
TSPCDSETU
Set displacement increment of forced displacement point
PCNSTR
Set pressure increment at pressure fixed point
TLOADSET
Set load
contact loop
MKINDEX
Create INDEX for simultaneous linear equation assembly
MPCCORR, RMPCCAL
Position correction of slave points in contact, calculation of coefficients of MPC conditional
FRICSET
Update friction parameters
NPFORCE
Calculation of strain, stress, equivalent node force
NPFLOW
Darcy velocity, equivalent nodal flow, equivalent node force calculation
NPFRIC
Calculation of frictional force
RESFRC
Constraints, Residual Force, Residual Flow Calculation


RESFRIC
Calculation of frictional residual force
Residual convergence
YES
NO
GLBSTF
Stiffness Matrix Creation, Assembly
GLBSTFG
Stiffness matrix (seepage flow) creation, Assembly
FRICSTF
Stiffness matrix (friction) creation, Assembly
ICCG,BCGSTB(CG Method), MUMPS(Direct Method)
Solving simultaneous linear equations
MERGDU
Create displacements and pressures from solutions of simultaneous linear equations
NPFORCED
Equivalent Nodal Force Calculation
(for calculating contact reaction force used to calculate frictional force)
NPFLOWD
Equivalent Nodal Force Calculation
(Pore Water Pressure)
(for contact reaction force calculation used to calculate friction force)
NPFRICD
Equivalent nodal force calculation (friction)
(for calculating contact reaction force used to calculate frictional force)
CNTRFC
Contact Reaction Force Calculation
CONTACT
Contact processing
Contact convergence
NO
YES
IST_UPDT
Plasticity, Crack Treatment
FEMAP_OUT
Result output
SEND_POS
Send nodal point positions to CADMAS
(via SEND_POS0 in case of parallel CG)
CADMAS
SUB. SF_POS_UPDATE
step loop

Appendix 1 . Method of calculating nodal forces due to surface loads

When the pressure p acts perpendicular to the surface, the nodal force on node i, for a given surface element, is

$$\int_{S_2^e} N_i \boldsymbol{t} dS = \int_{S_2^e} N_i p \boldsymbol{n} dS = \int_{-1}^{1}\int_{-1}^{1} N_i p \boldsymbol{n} | J \mid d\xi d\eta$$

$$\boldsymbol{r} = \begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \\ z(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} N_1(\xi,\eta)\, X_1 + N_2(\xi,\eta)\, X_2 + \cdots \\ N_1(\xi,\eta)\, Y_1 + N_2(\xi,\eta)\, Y_2 + \cdots \\ N_1(\xi,\eta)\, Z_1 + N_2(\xi,\eta)\, Z_2 + \cdots \end{Bmatrix}$$

$$\boldsymbol{r}_1 = \frac{\partial \boldsymbol{r}}{\partial \xi} = \frac{\partial}{\partial \xi} \begin{Bmatrix} N_1(\xi,\eta)\, X_1 + N_2(\xi,\eta)\, X_2 + \cdots \\ N_1(\xi,\eta)\, Y_1 + N_2(\xi,\eta)\, Y_2 + \cdots \\ N_1(\xi,\eta)\, Z_1 + N_2(\xi,\eta)\, Z_2 + \cdots \end{Bmatrix} = \frac{\partial N_1}{\partial \xi}(\xi,\eta) \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \frac{\partial N_2}{\partial \xi}(\xi,\eta) \begin{Bmatrix} X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} + \cdots$$

$$\boldsymbol{r}_2 = \frac{\partial \boldsymbol{r}}{\partial \eta} = \frac{\partial}{\partial \eta} \begin{Bmatrix} N_1(\xi,\eta)\, X_1 + N_2(\xi,\eta)\, X_2 + \cdots \\ N_1(\xi,\eta)\, Y_1 + N_2(\xi,\eta)\, Y_2 + \cdots \\ N_1(\xi,\eta)\, Z_1 + N_2(\xi,\eta)\, Z_2 + \cdots \end{Bmatrix} = \frac{\partial N_1}{\partial \eta}(\xi,\eta) \begin{Bmatrix} X_1 \\ Y_1 \\ Z_1 \end{Bmatrix} + \frac{\partial N_2}{\partial \eta}(\xi,\eta) \begin{Bmatrix} X_2 \\ Y_2 \\ Z_2 \end{Bmatrix} + \cdots$$

$$\boldsymbol{n} = \frac{\boldsymbol{r}_1 \times \boldsymbol{r}_2}{\|\boldsymbol{r}_1 \times \boldsymbol{r}_2\|} \quad \text{: Normal vector of a surface}$$

$$| J | = \|\boldsymbol{r}_1 \times \boldsymbol{r}_2\| = \sqrt{\|\boldsymbol{r}_1\|^2 \|\boldsymbol{r}_2\|^2 - (\boldsymbol{r}_1 \cdot \boldsymbol{r}_2)^2}$$

※ For the interpolation function N_i for a quadrilateral, see "Bathe, K.-J., Finite Element Procedures in Engineering Analysis," Prentice-Hall, 1982, pp.

Case of triangular primary element

$$\int_{S_2^e} N_i \boldsymbol{t} dS = \int_{S_2^e} N_i p \boldsymbol{n} dS = \int_{S_2^e} L_i\, (L_1 P_1 + L_2 P_2 + L_3 P_3) \boldsymbol{n} dS = \int_{S_2^e} (L_i L_1 P_1 + L_i L_2 P_2 + L_i L_3 P_3)\, \boldsymbol{n} dS$$

This is then integrated using the formula (Motoki Yagawa, Shinobu Yoshimura, Finite Element Method, Computational Mechanics and CAE Series 1, Baifukan, 1995, pp. 262 (11)).

Appendix 2 . How to specify damping in transient response analysis

(1) Consider a one-degree-of-freedom system consisting of a mass m, a spring constant k, and a viscous damping coefficient b. The equations of motion for this system are shown below.

$$\mathrm{m}\ddot{u} + b\dot{u} + ku = f$$

The damping characteristics of this system are determined by the following damping ratios(Ratio of attenuation coefficient to critical attenuation coefficient).

$$\zeta = \frac{b}{b_{cr}} = \frac{b}{2\sqrt{mk}}$$

ζ : attenuation ratio, b_{cr} : critical decay coefficient

(2) The equations of motion for a multi-degree-of-freedom system are as follows.

$$[M]\{\ddot{u}\} + [B]\{\dot{u}\} + [K]\{u\} = \{F\} \quad -①$$

Free oscillation when {F} is nothing. For simplicity, we assume {F}={0} and consider free vibration. The solution is determined by the initial conditions and is a superposition of multiple vibration modes.

(3) Mode decompose the equation ①. Substitute expression for ① expression

$$\{u\} = \sum_{i=1}^{N} \xi_i\{\phi_i\} = [\phi]\{\xi\} \quad -②$$

$\{\phi_i\}$: モードベクトル

$[\phi]$: $[\{\phi_1\} \quad \{\phi_2\} \quad \cdots \quad \{\phi_N\}]$

$\{\xi\}$: $\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_N \end{Bmatrix}$

Multiply $[\phi]^T$ from left to right

$$[\phi]^T[M][\phi]\{\ddot{\xi}\} + [\phi]^T[B][\phi]\{\dot{\xi}\} + [\phi]^T[K][\phi]\{\xi\} = \{0\}$$

In this time, diagonalized as follows ($[\phi]^T[B][\phi]$, $[\phi]^T[K][\phi]$ as well)

$$[\phi]^T[M][\phi] = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_N \end{bmatrix}$$

So the equations of motion are mode decomposed, leading to

$$m_i\ddot{\xi}_i + b_i\dot{\xi}_i + k_i\xi_i = 0$$

Solving for this and substituting ξ_i into ② yields the free vibration solution of ① in the form of superposition of vibration modes.

（4）In the NASTRAN and CADMAS-STR damping models $[B] = C_K[K]$, so the damping ratio for each mode is as follows.

$$\zeta_i = \frac{b_i}{2\sqrt{m_i k_i}} = \frac{C_K k_i}{2\sqrt{m_i k_i}} = \frac{C_K}{2}\sqrt{\frac{k_i}{m_i}} = \frac{C_K}{2}\omega_i$$

The damping ratio is proportional to the natural angular frequency of each vibration mode. Therefore, in$\{u\}$, which is obtained by superposition of the mode vectors, the modes with larger eigenangular frequencies will decay faster.

（5）How to specify attenuation (input method from FEMAP)

Determine the CK to achieve the intended damping ratio for the dominant vibration mode (the mode with large ξ_i , usually i = 1).

Therefore

$$\zeta_i = \frac{C_K}{2}\omega_i \quad , \quad C_K = \frac{\boxed{2\zeta_i}}{\boxed{\omega_i}} \begin{matrix} \leftarrow \text{Substituting } 2C/C_0 \\ \leftarrow \text{Substituting W4} \end{matrix}$$

If you want to suppress vibration modes above ω_i, set

$$C_K = \frac{2 \cdot 1}{\omega_i}$$

（6）When no damping is used, minimal damping is applied by default to suppress harmonic oscillations caused by contact processing, etc. therefore

$$\omega_{i,max} = \left(\frac{2}{l}\sqrt{\frac{E}{\rho}}\right)_{max}$$

$\omega_{i,max}$ Maximum value of the natural angular frequency of the element

l ：Element size

E Young's modulus of elements

ρ ：Density of elements

In this mode, the default C_K is set so that$\zeta_i = 1$ as follows.

$$C_K = \frac{2 \cdot \zeta_i}{\omega_i} = \frac{2 \cdot 1}{\omega_{i,max}}$$

Appendix 3 . FEM formulation when geometric nonlinearity is taken into account

1 . Hypothetical work principle equation

The principle expression for virtual work in the state at time t+Δt in total Lagrange form (Bathe, K.-J., Finite Element Procedures in Engineering Analysis, Prentice-Hall, 1982, equation (6-72)) is

$$\int_{{}^{0}V} {}^{t+\Delta t}_{\ \ \ 0}S_{ij}\, \delta\, {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij} d\, {}^{0}V = {}^{t+\Delta t}\mathcal{R} \quad -(1)$$

${}^{t+\Delta t}_{\ \ \ 0}S_{ij}$: $second\ Piola-Kirchhoff$ stress tensor at time$t+\Delta t$ with respect to the state at time 0

${}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}$: $Green-Lagrange$ strain tensor at time $t+\Delta t$ with respect to the state at time 0

$\delta\, {}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}$ ${}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}$ $corresponding\ to\ the\ possible\ virtual\ displacements\ at\ time\ t+\Delta t$

${}^{0}V$: Volume at time 0

${}^{t+\Delta t}\mathcal{R}$: $principle\ of\ virtual\ work$ due to external force at time $t+\Delta t$

2 . Incremental decomposition and linear approximation

Decompose ${}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij}$, ${}^{t+\Delta t}_{\ \ \ 0}S_{ij}$ incrementally as follows

$${}^{t+\Delta t}_{\ \ \ 0}\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial\, {}^{t+\Delta t}u_i}{\partial\, {}^{0}x_j} + \frac{\partial\, {}^{t+\Delta t}u_j}{\partial\, {}^{0}x_i} + \frac{\partial\, {}^{t+\Delta t}u_k}{\partial\, {}^{0}x_i}\frac{\partial\, {}^{t+\Delta t}u_k}{\partial\, {}^{0}x_j}\right)$$

$$= \frac{1}{2}\left(\frac{\partial(\,{}^{t}u_i + u_i)}{\partial\, {}^{0}x_j} + \frac{\partial(\,{}^{t}u_j + u_j)}{\partial\, {}^{0}x_i} + \frac{\partial(\,{}^{t}u_k + u_k)}{\partial\, {}^{0}x_i}\frac{\partial(\,{}^{t}u_k + u_k)}{\partial\, {}^{0}x_j}\right)$$

$$= \frac{1}{2}\left(\frac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j} + \frac{\partial u_i}{\partial\, {}^{0}x_j} + \frac{\partial\, {}^{t}u_j}{\partial\, {}^{0}x_i} + \frac{\partial u_j}{\partial\, {}^{0}x_i} + \frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_i}\frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_j} + \frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_i}\frac{\partial u_k}{\partial\, {}^{0}x_j} + \frac{\partial u_k}{\partial\, {}^{0}x_i}\frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_j} + \frac{\partial u_k}{\partial\, {}^{0}x_i}\frac{\partial u_k}{\partial\, {}^{0}x_j}\right)$$

$$= \frac{1}{2}\left(\frac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j} + \frac{\partial\, {}^{t}u_j}{\partial\, {}^{0}x_i} + \frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_i}\frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_j}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial\, {}^{0}x_j} + \frac{\partial u_j}{\partial\, {}^{0}x_i} + \frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_i}\frac{\partial u_k}{\partial\, {}^{0}x_j} + \frac{\partial u_k}{\partial\, {}^{0}x_i}\frac{\partial\, {}^{t}u_k}{\partial\, {}^{0}x_j}\right) + \frac{1}{2}\frac{\partial u_k}{\partial\, {}^{0}x_i}\frac{\partial u_k}{\partial\, {}^{0}x_j}$$

$$= {}^{t}_{0}\epsilon_{ij} + {}_{0}e_{ij} + {}_{0}\eta_{ij}$$

First second ← Regarding displacement increments

$$= {}^{t}_{0}\epsilon_{ij} + {}_{0}\epsilon_{ij} \quad -(2)$$

${}^{0}x_i$: $Coordinates\ of\ the\ material\ point\ at\ time$ 0

${}^{t}u_i$: Displacement of material point at time t

u_i : ${}^{t+\Delta t}u_i - {}^{t}u_i$ (displacement of material point at time t)

$${}^{t+\Delta t}_{\ \ \ 0}S_{ij} = {}^{t}_{0}S_{ij} + {}_{0}S_{ij} \quad -(3)$$

Furthermore, a linear approximation of ${}_{0}S_{ij}$ (leaving only a first-order term with respect to the displacement increment) is shown below.

$$ {}_0S_{ij} \doteq \left.\frac{\partial\, {}^t_0S_{ij}}{\partial\, {}^t_0\epsilon_{rs}}\right|_t {}_0e_{rs} = {}_0C_{ijrs}\,{}_0e_{rs} \quad -(4) $$

Using (2) and (3), the integrand function in (1) is incrementally decomposed into the following.

$$ {}^{t+\Delta t}_{\ \ 0}S_{ij}\,\delta\,{}^{t+\Delta t}_{\ \ 0}\epsilon_{ij} = \left({}^t_0S_{ij} + {}_0S_{ij}\right)\delta\left({}^t_0\epsilon_{ij} + {}_0\epsilon_{ij}\right) = \left({}^t_0S_{ij} + {}_0S_{ij}\right)\delta\,{}_0\epsilon_{ij} = \left({}^t_0S_{ij} + {}_0S_{ij}\right)\left(\delta\,{}_0e_{ij} + \delta\,{}_0\eta_{ij}\right) $$

Furthermore, a linear approximation using (4) (leaving only the terms below the first order with respect to the displacement increment) is shown below.

$$ \doteq \left({}^t_0S_{ij} + {}_0C_{ijrs}\,{}_0e_{rs}\right)\left(\delta\,{}_0e_{ij} + \delta\,{}_0\eta_{ij}\right) \doteq {}^t_0S_{ij}\,\delta\,{}_0e_{ij} + {}^t_0S_{ij}\,\delta\,{}_0\eta_{ij} + {}_0C_{ijrs}\,{}_0e_{rs}\,\delta\,{}_0e_{ij} \quad -(5) $$

zero first zero first ← Regarding displacement increments

Substituting (5) into (1), we obtain the following expression, which is an incremental decomposition and linear approximation of the virtual work principle expression (total Lagrange form).

$$ \int_{{}^0V} \delta\,{}_0e_{ij}\;{}_0C_{ijrs}\;{}_0e_{rs}\,d\,{}^0V + \int_{{}^0V} {}^t_0S_{ij}\,\delta\,{}_0\eta_{ij}\,d\,{}^0V = {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^t_0S_{ij}\,\delta\,{}_0e_{ij}\,d\,{}^0V \quad -(6) $$

3. Discretization

Discretizing (6) by FEM, we obtain

$$ \{\delta U\}^T[{}^t_0K_L]\{U\} + \{\delta U\}^T[{}^t_0K_{NL}]\{U\} = \{\delta U\}^T\{{}^{t+\Delta t}R\} - \{\delta U\}^T\{{}^t_0F\} $$

$$ ([{}^t_0K_L] + [{}^t_0K_{NL}])\{U\} = \{{}^{t+\Delta t}R\} - \{{}^t_0F\} $$

The calculation methods for each term are shown in [1]-[3] below.

［1］ First term of the left-hand side

$$\int_{{}^0V} \delta\ {}_0e_{ij}\ {}_0C_{ijrs}\ {}_0e_{rs} d\ {}^0V = \int_{{}^0V} \{\delta\ {}_0e\}^T [\ {}_0C]\{\ {}_0e\}\ d\ {}^0V = \int_{{}^0V} \left([{}_0^tB_L\ ]\{\delta U\}\right)^T [\ {}_0C] \left([{}_0^tB_L\ ]\{U\}\right) d\ {}^0V$$

$$= \{\delta U\}^T \int_{{}^0V} \left[{}_0^tB_L\ \right]^T [\ {}_0C]\left[{}_0^tB_L\ \right] d\ {}^0V\{U\} = \{\delta U\}^T \left[{}_0^tK_L\ \right]\{U\}$$

In this time

$[\ {}_0C]$ ： Tangential stiffness matrix

$$\{\ {}_0e\} = \begin{Bmatrix} {}_0e_{11} \\ {}_0e_{22} \\ {}_0e_{33} \\ 2\ {}_0e_{12} \\ 2\ {}_0e_{23} \\ 2\ {}_0e_{31} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2}\left(\frac{\partial u_1}{\partial\ {}^0x_1} + \frac{\partial u_1}{\partial\ {}^0x_1} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_1} + \frac{\partial u_k}{\partial\ {}^0x_1}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\right) \\ \frac{1}{2}\left(\frac{\partial u_2}{\partial\ {}^0x_2} + \frac{\partial u_2}{\partial\ {}^0x_2} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_2} + \frac{\partial u_k}{\partial\ {}^0x_2}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\right) \\ \frac{1}{2}\left(\frac{\partial u_3}{\partial\ {}^0x_3} + \frac{\partial u_3}{\partial\ {}^0x_3} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_3} + \frac{\partial u_k}{\partial\ {}^0x_3}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\right) \\ 2\cdot\frac{1}{2}\left(\frac{\partial u_1}{\partial\ {}^0x_2} + \frac{\partial u_2}{\partial\ {}^0x_1} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_2} + \frac{\partial u_k}{\partial\ {}^0x_1}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\right) \\ 2\cdot\frac{1}{2}\left(\frac{\partial u_2}{\partial\ {}^0x_3} + \frac{\partial u_3}{\partial\ {}^0x_2} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_3} + \frac{\partial u_k}{\partial\ {}^0x_2}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\right) \\ 2\cdot\frac{1}{2}\left(\frac{\partial u_3}{\partial\ {}^0x_1} + \frac{\partial u_1}{\partial\ {}^0x_3} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_1} + \frac{\partial u_k}{\partial\ {}^0x_3}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\right) \end{Bmatrix}$$

$$= \begin{Bmatrix} \frac{\partial u_1}{\partial\ {}^0x_1} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_1} \\ \frac{\partial u_2}{\partial\ {}^0x_2} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_2} \\ \frac{\partial u_3}{\partial\ {}^0x_3} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_3} \\ \frac{\partial u_1}{\partial\ {}^0x_2} + \frac{\partial u_2}{\partial\ {}^0x_1} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_2} + \frac{\partial u_k}{\partial\ {}^0x_1}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_2} \\ \frac{\partial u_2}{\partial\ {}^0x_3} + \frac{\partial u_3}{\partial\ {}^0x_2} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_3} + \frac{\partial u_k}{\partial\ {}^0x_2}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_3} \\ \frac{\partial u_3}{\partial\ {}^0x_1} + \frac{\partial u_1}{\partial\ {}^0x_3} + \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_1} + \frac{\partial u_k}{\partial\ {}^0x_3}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_1} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_1}{\partial\ {}^0x_1} \\ \frac{\partial u_2}{\partial\ {}^0x_2} \\ \frac{\partial u_3}{\partial\ {}^0x_3} \\ \frac{\partial u_1}{\partial\ {}^0x_2} + \frac{\partial u_2}{\partial\ {}^0x_1} \\ \frac{\partial u_2}{\partial\ {}^0x_3} + \frac{\partial u_3}{\partial\ {}^0x_2} \\ \frac{\partial u_3}{\partial\ {}^0x_1} + \frac{\partial u_1}{\partial\ {}^0x_3} \end{Bmatrix} + \begin{Bmatrix} \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_1} \\ \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_2} \\ \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_3} \\ \frac{\partial\ {}^tu_k}{\partial\ {}^0x_1}\frac{\partial u_k}{\partial\ {}^0x_2} + \frac{\partial u_k}{\partial\ {}^0x_1}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_2} \\ \frac{\partial\ {}^tu_k}{\partial\ {}^0x_2}\frac{\partial u_k}{\partial\ {}^0x_3} + \frac{\partial u_k}{\partial\ {}^0x_2}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_3} \\ \frac{\partial\ {}^tu_k}{\partial\ {}^0x_3}\frac{\partial u_k}{\partial\ {}^0x_1} + \frac{\partial u_k}{\partial\ {}^0x_3}\frac{\partial\ {}^tu_k}{\partial\ {}^0x_1} \end{Bmatrix}$$

$$= \{\ {}_0e\}_0 + \{\ {}_0e\}_1$$

$$= [{}_0^tB_{L0}]\{U\} + [{}_0^tB_{L1}]\{U\}$$

$$= ([{}_0^tB_{L0}] + [{}_0^tB_{L1}])\{U\}$$

$$= [{}_0^tB_L]\{U\}$$

$$
\left\{ {}_0e \right\}_0 = \left\{ \begin{matrix} \dfrac{\partial u_1}{\partial\, {}^0x_1} \\ \dfrac{\partial u_2}{\partial\, {}^0x_2} \\ \dfrac{\partial u_3}{\partial\, {}^0x_3} \\ \dfrac{\partial u_1}{\partial\, {}^0x_2} + \dfrac{\partial u_2}{\partial\, {}^0x_1} \\ \dfrac{\partial u_2}{\partial\, {}^0x_3} + \dfrac{\partial u_3}{\partial\, {}^0x_2} \\ \dfrac{\partial u_3}{\partial\, {}^0x_1} + \dfrac{\partial u_1}{\partial\, {}^0x_3} \end{matrix} \right\} = \left\{ \begin{matrix} \dfrac{\partial}{\partial\, {}^0x_1}(N_1U_1^1 + N_2U_1^2 + \cdots + N_nU_1^n) \\ \dfrac{\partial}{\partial\, {}^0x_2}(N_1U_2^1 + N_2U_2^2 + \cdots + N_nU_2^n) \\ \dfrac{\partial}{\partial\, {}^0x_3}(N_1U_3^1 + N_2U_3^2 + \cdots + N_nU_3^n) \\ \dfrac{\partial}{\partial\, {}^0x_2}(N_1U_1^1 + N_2U_1^2 + \cdots + N_nU_1^n) + \dfrac{\partial}{\partial\, {}^0x_1}(N_1U_2^1 + N_2U_2^2 + \cdots + N_nU_2^n) \\ \dfrac{\partial}{\partial\, {}^0x_3}(N_1U_2^1 + N_2U_2^2 + \cdots + N_nU_2^n) + \dfrac{\partial}{\partial\, {}^0x_2}(N_1U_3^1 + N_2U_3^2 + \cdots + N_nU_3^n) \\ \dfrac{\partial}{\partial\, {}^0x_1}(N_1U_3^1 + N_2U_3^2 + \cdots + N_nU_3^n) + \dfrac{\partial}{\partial\, {}^0x_3}(N_1U_1^1 + N_2U_1^2 + \cdots + N_nU_1^n) \end{matrix} \right\}
$$

$$
= \begin{bmatrix}
\frac{\partial N_1}{\partial\, {}^0x_1} & & & \frac{\partial N_2}{\partial\, {}^0x_1} & & & & \frac{\partial N_n}{\partial\, {}^0x_1} & & \\
& \frac{\partial N_1}{\partial\, {}^0x_2} & & & \frac{\partial N_2}{\partial\, {}^0x_2} & & & & \frac{\partial N_n}{\partial\, {}^0x_2} & \\
& & \frac{\partial N_1}{\partial\, {}^0x_3} & & & \frac{\partial N_2}{\partial\, {}^0x_3} & \cdots & & & \frac{\partial N_n}{\partial\, {}^0x_3} \\
\frac{\partial N_1}{\partial\, {}^0x_2} & \frac{\partial N_1}{\partial\, {}^0x_1} & & \frac{\partial N_2}{\partial\, {}^0x_2} & \frac{\partial N_2}{\partial\, {}^0x_1} & & & \frac{\partial N_n}{\partial\, {}^0x_2} & \frac{\partial N_n}{\partial\, {}^0x_1} & \\
& \frac{\partial N_1}{\partial\, {}^0x_3} & \frac{\partial N_1}{\partial\, {}^0x_2} & & \frac{\partial N_2}{\partial\, {}^0x_3} & \frac{\partial N_2}{\partial\, {}^0x_2} & & & \frac{\partial N_n}{\partial\, {}^0x_3} & \frac{\partial N_n}{\partial\, {}^0x_2} \\
\frac{\partial N_1}{\partial\, {}^0x_3} & & \frac{\partial N_1}{\partial\, {}^0x_1} & \frac{\partial N_2}{\partial\, {}^0x_3} & & \frac{\partial N_2}{\partial\, {}^0x_1} & & \frac{\partial N_n}{\partial\, {}^0x_3} & & \frac{\partial N_n}{\partial\, {}^0x_1}
\end{bmatrix}
\left\{ \begin{matrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ U_1^2 \\ U_2^2 \\ U_3^2 \\ \vdots \\ U_1^n \\ U_2^n \\ U_3^n \end{matrix} \right\}
$$

$$
= [{}_0^tB_{L0}]\{U\}
$$

$$
\{{}_{0}e\}_1 = \left\{\begin{matrix}
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_1}\dfrac{\partial u_k}{\partial\ {}^{0}x_1} \\
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_2}\dfrac{\partial u_k}{\partial\ {}^{0}x_2} \\
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_3}\dfrac{\partial u_k}{\partial\ {}^{0}x_3} \\
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_1}\dfrac{\partial u_k}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_2}\dfrac{\partial u_k}{\partial\ {}^{0}x_1} \\
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_2}\dfrac{\partial u_k}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_3}\dfrac{\partial u_k}{\partial\ {}^{0}x_2} \\
\dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_3}\dfrac{\partial u_k}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_k}{\partial\ {}^{0}x_1}\dfrac{\partial u_k}{\partial\ {}^{0}x_3}
\end{matrix}\right\}
$$

$$
= \left\{\begin{matrix}
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_1}\dfrac{\partial u_1}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_1}\dfrac{\partial u_2}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_1}\dfrac{\partial u_3}{\partial\ {}^{0}x_1} \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_2}\dfrac{\partial u_1}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_2}\dfrac{\partial u_2}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_2}\dfrac{\partial u_3}{\partial\ {}^{0}x_2} \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_3}\dfrac{\partial u_1}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_3}\dfrac{\partial u_2}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_3}\dfrac{\partial u_3}{\partial\ {}^{0}x_3} \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_1}\dfrac{\partial u_1}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_1}\dfrac{\partial u_2}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_1}\dfrac{\partial u_3}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_2}\dfrac{\partial u_1}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_2}\dfrac{\partial u_2}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_2}\dfrac{\partial u_3}{\partial\ {}^{0}x_1} \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_2}\dfrac{\partial u_1}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_2}\dfrac{\partial u_2}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_2}\dfrac{\partial u_3}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_3}\dfrac{\partial u_1}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_3}\dfrac{\partial u_2}{\partial\ {}^{0}x_2} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_3}\dfrac{\partial u_3}{\partial\ {}^{0}x_2} \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_3}\dfrac{\partial u_1}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_3}\dfrac{\partial u_2}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_3}\dfrac{\partial u_3}{\partial\ {}^{0}x_1} + \dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_1}\dfrac{\partial u_1}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_1}\dfrac{\partial u_2}{\partial\ {}^{0}x_3} + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_1}\dfrac{\partial u_3}{\partial\ {}^{0}x_3}
\end{matrix}\right\}
$$

$$
= \left\{\begin{matrix}
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_3^1 + \cdots + N_nU_3^n) \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_3^1 + \cdots + N_nU_3^n) \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_3^1 + \cdots + N_nU_3^n) \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_1}\dfrac{\partial}{\partial\ {}^{0}x_2}(N_1U_3^1 + \cdots + N_nU_3^n) \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_2}\dfrac{\partial}{\partial\ {}^{0}x_3}(N_1U_3^1 + \cdots + N_nU_3^n) \\
\dfrac{\partial\ {}^{t}u_1}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_1^1 + \cdots + N_nU_1^n) + \dfrac{\partial\ {}^{t}u_2}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_2^1 + \cdots + N_nU_2^n) + \dfrac{\partial\ {}^{t}u_3}{\partial\ {}^{0}x_3}\dfrac{\partial}{\partial\ {}^{0}x_1}(N_1U_3^1 + \cdots + N_nU_3^n)
\end{matrix}\right.
$$

$$
\left.
\begin{aligned}
&+\ \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial}{\partial\,{}^{0}x_1}(N_1U_1^1+\cdots+N_nU_1^n)+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial}{\partial\,{}^{0}x_1}(N_1U_2^1+\cdots+N_nU_2^n)+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial}{\partial\,{}^{0}x_1}(N_1U_3^1+\cdots+N_nU_3^n)\\
&+\ \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial}{\partial\,{}^{0}x_2}(N_1U_1^1+\cdots+N_nU_1^n)+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial}{\partial\,{}^{0}x_2}(N_1U_2^1+\cdots+N_nU_2^n)+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial}{\partial\,{}^{0}x_2}(N_1U_3^1+\cdots+N_nU_3^n)\\
&+\ \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial}{\partial\,{}^{0}x_3}(N_1U_1^1+\cdots+N_nU_1^n)+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial}{\partial\,{}^{0}x_3}(N_1U_2^1+\cdots+N_nU_2^n)+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial}{\partial\,{}^{0}x_3}(N_1U_3^1+\cdots+N_nU_3^n)
\end{aligned}
\right\}
$$

$$
=\begin{bmatrix}
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_1} & & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_1}\\
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_2} & & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_2}\\
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_3} & \cdots & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_3}\\
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_1} & & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_1} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_2}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_1}\\
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_1}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_2} & & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_2} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_2}\frac{\partial N_n}{\partial\,{}^{0}x_3}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_2}\\
\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_1}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_1}{\partial\,{}^{0}x_3} & & \frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_1}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_2}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_3} & \frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_3}\frac{\partial N_n}{\partial\,{}^{0}x_1}+\frac{\partial\,{}^{t}u_3}{\partial\,{}^{0}x_1}\frac{\partial N_n}{\partial\,{}^{0}x_3}
\end{bmatrix}
\begin{Bmatrix}
U_1^1\\ U_2^1\\ U_3^1\\ U_1^2\\ U_2^2\\ U_3^2\\ \vdots\\ U_1^n\\ U_2^n\\ U_3^n
\end{Bmatrix}
$$

$$
=[{}_{0}^{t}B_{L1}]\{U\}
$$

［2］ Second term of the left-hand side

$$\int_{{}^0V} {}_0^tS_{ij}\, \delta\ {}_0\eta_{ij}\ d\ {}^0V = \int_{{}^0V} {}_0^tS_{ij} \left(\{\delta U\}^T\left[\ {}_0E_{ij}\right]\{U\}\right) d\ {}^0V = \{\delta U\}^T \int_{{}^0V} {}_0^tS_{ij}\left[\ {}_0E_{ij}\right] d\ {}^0V\{U\} = \{\delta U\}^T[{}_0^tK_{NL}]\{U\}$$

In this case

$${}_0\eta_{ij} = \frac{1}{2}\frac{\partial u_k}{\partial\ {}^0x_i}\frac{\partial u_k}{\partial\ {}^0x_j} = \frac{1}{2}\frac{\partial \boldsymbol{u}}{\partial\ {}^0x_i}\cdot\frac{\partial \boldsymbol{u}}{\partial\ {}^0x_j}$$

$$\frac{\partial \boldsymbol{u}}{\partial\ {}^0x_i} = \frac{\partial}{\partial\ {}^0x_i}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\partial}{\partial\ {}^0x_i}\begin{Bmatrix} N_1U_1^1 + N_2U_1^2 + \ \cdots + N_nU_1^n \\ N_1U_2^1 + N_2U_2^2 + \ \cdots + N_nU_2^n \\ N_1U_3^1 + N_2U_3^2 + \ \cdots + N_nU_3^n \end{Bmatrix}$$

$$= \frac{\partial}{\partial\ {}^0x_i}\begin{bmatrix} N_1 & & & N_2 & & & & & N_n & & \\ & N_1 & & & N_2 & & \cdots & & & N_n & \\ & & N_1 & & & N_2 & & & & & N_n \end{bmatrix}\begin{Bmatrix} U_1^1 \\ U_2^1 \\ U_3^1 \\ \\ U_1^2 \\ U_2^2 \\ U_3^2 \\ \vdots \\ U_1^n \\ U_2^n \\ U_3^n \end{Bmatrix}$$

$$= [A_i]\{U\}$$

If it is assumed that

$${}_0\eta_{ij} = \frac{1}{2}([A_i]\{U\})^T\left([A_j]\{U\}\right) = \frac{1}{2}\{U\}^T[A_i]^T[A_j]\{U\}$$

$$\delta\ {}_0\eta_{ij} = \frac{1}{2}\left(\{\delta U\}^T[A_i]^T[A_j]\{U\} + \{U\}^T[A_i]^T[A_j]\{\delta U\}\right) = \frac{1}{2}\left(\{\delta U\}^T[A_i]^T[A_j]\{U\} + \{\delta U\}^T[A_j]^T[A_i]\{U\}\right)$$

$$= \{\delta U\}^T\frac{1}{2}\left([A_i]^T[A_j] + [A_j]^T[A_i]\right)\{U\}$$

$$= \{\delta U\}^T\left[\ {}_0E_{ij}\right]\{U\}$$

［3］The second term of the right-hand side

$$\int_{{}^0V} {}_0^tS_{ij}\,\delta\ {}_0e_{ij}\,d\ {}^0V = \int_{{}^0V} \{\delta\ {}_0e\}^T \{{}_0^tS\}\,d\ {}^0V = \int_{{}^0V} \left([{}_0^tB_L\ ]\{\delta U\}\right)^T \{{}_0^tS\}d\ {}^0V = \{\delta U\}^T \int_{{}^0V} \left[{}_0^tB_L\ \right]^T \{{}_0^tS\}d\ {}^0V$$

$$= \{\delta U\}^T\{{}_0^tF\}$$

In this case

$$\{{}_0^tS\} = \begin{Bmatrix} {}_0^tS_{11} \\ {}_0^tS_{22} \\ {}_0^tS_{33} \\ {}_0^tS_{12} \\ {}_0^tS_{23} \\ {}_0^tS_{31} \end{Bmatrix}$$