

# Computing Partial Walsh Transform from the Algebraic Normal Form of a Boolean Function

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## Abstract

We study the relationship between the Walsh transform and the algebraic normal form of a Boolean function. In the first part of the paper, we carry out a combinatorial analysis to obtain a formula for the Walsh transform at a certain point in terms of parameters derived from the algebraic normal form. The second part of the paper is devoted to simplify this formula and develop an algorithm to evaluate it. Our algorithm can be applied in situations where it is practically impossible to use the fast Walsh transform algorithm. Experimental results show that under certain conditions it is possible to execute our algorithm to evaluate the Walsh transform (at a small set of points) of functions on a few scores of variables having a few hundred terms in the algebraic normal form.

**Keywords :** Boolean function, Algebraic Normal Form, Walsh Transform.

## 1 Introduction

Boolean functions are used for a wide variety of applications in engineering and computer science. An  $m$ -variable Boolean function  $g(x)$  is a map  $g : \{0, 1\}^m \rightarrow \{0, 1\}$ . One of the most useful tools for the study of Boolean functions is the Walsh transform, which is essentially the Fourier transform applied to the function  $(-1)^{g(x)}$ . The Walsh transform measures the correlations between an  $m$ -variable Boolean function and all the  $m$ -variable linear functions. These correlations uniquely determine the function and hence it is possible to work entirely with the Walsh transform. In fact, many properties of Boolean functions are most easily stated in terms of Walsh transform.

For practical applications it is often useful to be able to compute the Walsh transform of a Boolean function. It turns out that there is an excellent algorithm to do so, namely, the fast Walsh transform [1]. For an  $m$ -variable function the fast Walsh transform takes time  $O(m2^m)$  and hence can be used for functions of around 30 variables. The fast Walsh transform is computed from the description of the function itself. More precisely, the fast Walsh transform takes as input the bit string  $f(\sigma_0) \dots f(\sigma_{2^m-1})$  of length  $2^m$ , where for  $0 \leq i \leq 2^m - 1$ ,  $\sigma_i$  is the  $m$ -bit binary representation of  $i$ .

There is another way to uniquely represent a Boolean function, namely by its algebraic normal form, which expresses a Boolean function as a multivariate polynomial over  $F_2$ , the finite field of two elements. The number of nonzero terms in the polynomial can be  $2^m$  in the worst case. However, many interesting

classes of Boolean functions do have compact algebraic normal form representation. (For example, an  $m = 2k$ -variable bent function can have as few as  $k$  many terms in their ANF.)

In this paper, we study the relationship between the algebraic normal form and the Walsh transform of a Boolean function. We obtain a formula for the Walsh transform of a Boolean function at a certain point  $v \in F_2^m$  in terms of certain parameters derived from the algebraic normal form. We present an efficient algorithm to evaluate the formula and hence compute the Walsh transform at  $v$ .

Our algorithm can be used to compute the Walsh transform in cases where it is not possible to use the fast Walsh transform. For example, it is possible in certain cases to run our algorithm for 50 to 100 variable functions having a few hundred terms in their algebraic normal form. For such functions it is possible to compute the Walsh transform for a small set of points  $v$ . Note that it is practically impossible to compute the Walsh transform of an  $m$ -variable function at all points in  $F_2^m$  if  $m$  is around 50 or more. Our algorithm provides a method to probe the spectral domain of large variable functions. This will provide some useful information about the function such as the size of its support and an estimate of its nonlinearity. Note that for small number of variables, the fast Walsh transform is faster than our algorithm. Hence we do not provide a substitute for the fast Walsh transform; rather we provide a tool to analyse a Boolean function in situations where the fast Walsh transform cannot be used.

Carlet and Guillot [3] study an alternative representation of Boolean functions – the numerical normal form (NNF). In Theorem 5 of [3], they obtain a formula for computing the NNF from the ANF representation and in Equation (6) of [3], they obtain a formula for computing the Walsh transform from the NNF representation. Using these two results, it is possible to obtain a formula for computing the Walsh transform from the ANF representation. In fact, in principle this formula can be used to derive the explicit relationship between the Walsh transform and ANF that we obtain in this paper. However, carrying out this task appears to be a non-trivial exercise. More importantly, we do more than just obtain the relationship between the Walsh transform and the ANF. We analyse this relationship and ultimately obtain an algorithm to compute the Walsh transform from the ANF. Obtaining this algorithm is the major motivation of our paper. We note that our analysis and algorithm is not present in [3]. (Actually, the purpose of [3] is to study the NNF and its relationship with the other representations.)

Boolean functions are studied extensively from different perspectives – coding theory [7, 6], circuit complexity [2, 8] and cryptography [4, 9] are some examples. In all these areas, the Walsh transform is the main tool in the analysis of Boolean functions. However, to the best of our knowledge, the only previously known algorithm for computing the Walsh transform is the fast Walsh transform [1]. Hopefully the present work will motivate researchers to study the algorithmic issues of the Walsh transform more deeply.

## 2 Preliminaries

Let  $F_2 = GF(2)$ . An  $m$ -variable Boolean function  $g(x)$  is a map  $g : F_2^m \rightarrow F_2$ . We consider the domain of an  $m$ -variable Boolean function to be the vector space  $(F_2^m, \oplus)$  over  $F_2$ , where  $\oplus$  is used to denote the addition operator over both  $F_2$  and the vector space  $F_2^m$ . The symbol  $+$  is used to denote addition over integers. The inner product of two vectors  $u, v \in F_2^m$  will be denoted by  $\langle u, v \rangle$ . The weight of an  $m$ -bit vector  $u$  is the number of ones in  $u$  and will be denoted by  $wt(u)$ . The *support* of a Boolean function  $g$  is denoted by  $Sup(g)$  and is defined to be  $Sup(g) = \{x : g(x) = 1\}$ . The weight of  $g$  is denoted by  $wt(g)$  and is defined to be  $wt(g) = |Sup(g)|$ . The *Walsh Transform* of a Boolean function  $g(x)$  is an integer valued function  $W_g : \{0, 1\}^m \rightarrow [-2^m, 2^m]$  defined by (see [7, page 414])

$$W_g(u) = \sum_{w \in F_2^m} (-1)^{g(w) \oplus \langle u, w \rangle}. \quad (1)$$

Let  $H_m$  be the  $2^m \times 2^m$  Hadamard matrix (see [7, page 44]) whose rows and columns are indexed by the elements of  $F_2^m$  such that  $H_m(u, v) = (-1)^{\langle u, v \rangle}$ . Then we can write

$$[(-1)^{g(0)}, \dots, (-1)^{g(2^m-1)}]H_m = [W_g(0), \dots, W_g(2^m-1)]. \quad (2)$$

An  $m$ -variable Boolean function  $g(x_1, x_2, \dots, x_m)$  can be uniquely written as

$$g(x_1, x_2, \dots, x_m) = \bigoplus_{(\alpha_1, \alpha_2, \dots, \alpha_m) \in F_2^m} A_g(\alpha_1, \alpha_2, \dots, \alpha_m) x_1^{\alpha_1} x_2^{\alpha_2} \dots x_m^{\alpha_m} \quad (3)$$

where  $A_g(x_1, x_2, \dots, x_m)$  is a Boolean function. This representation is called the *algebraic normal form (ANF)* of  $g$ . We will call the Boolean function  $A_g$  the ANF of  $g$ .

**To illustrate different definitions, terms and notations we take two small examples.**

**Example 1 :**  $g_1(x_1, x_2, x_3) = x_1x_2 \oplus x_2x_3 \oplus x_3x_1$ , and  $Sup(A_{g_1}) = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$ .

**Example 2 :**  $g_2(x_1, x_2, x_3) = x_1x_2 \oplus x_2x_3 \oplus x_1 \oplus x_2$ , and  $Sup(A_{g_2}) = \{(1, 1, 0), (0, 1, 1), (1, 0, 0), (0, 1, 0)\}$ .

### 3 Walsh Transform

In this section, we express the Walsh transform of a Boolean function in terms of certain parameters derived from its ANF. The approach that we take is the following. Equation (2) expresses the relation between  $(-1)^{g(x)}$  and  $W_g(u)$ . Our first task is to obtain a formula for  $(-1)^{g(x)}$  in terms of the ANF of  $g(x)$ . Then using Equation (2) we obtain the desired relationship between the Walsh transform and the ANF.

In obtaining a formula for  $(-1)^{g(x)}$  in terms of the ANF, we first tackle the special case when  $g(x) = g(x_1, \dots, x_m) = x_1 \dots x_m$ . The result of this case is used to analyse the general case.

#### 3.1 Case $g(x_1, x_2, \dots, x_m) = x_1 x_2 \dots x_m$

For  $x = (x_1, \dots, x_m) \in \{0, 1\}^m$ , define

$$N_r^{(m)}(x) \triangleq N_r^{(m)}(x_1, \dots, x_m) \triangleq \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} (-1)^{x_{i_1} \oplus \dots \oplus x_{i_r}}.$$

**Lemma 3.1** *Let  $x = (x_1, \dots, x_m) \in F_2^m$  be such that  $wt(x) = k$ . Then*

$$N_r^{(m)}(x) = \sum_{j=0}^r (-1)^j \binom{k}{j} \binom{m-k}{r-j}.$$

Consequently,  $N_r^{(m)}(x)$  is the Krawtchouk polynomial  $p_r(k, m)$  [7, page 130].

**Proof :** In the vector  $(x_1, \dots, x_m)$ ,  $k$  of the  $x_i$ 's are 1 and  $(m - k)$  of the  $x_i$ 's are 0. Hence the number of terms of the type  $(x_{i_1} \oplus \dots \oplus x_{i_r})$  with  $j$  number of 1's ( $0 \leq j \leq r$ ) and  $(r - j)$  number of 0's is equal to  $\binom{k}{j} \binom{m-k}{r-j}$ . Since  $j$  of the  $x_i$ 's are ones, we have,  $(-1)^{x_{i_1} \oplus \dots \oplus x_{i_r}} = (-1)^j$ . Hence we get  $N_r^{(m)}(x_1, \dots, x_m) = \sum_{j=0}^r (-1)^j \binom{k}{j} \binom{m-k}{r-j}$  which is the Krawtchouk polynomial  $p_r(k, m)$  [7, page 130]. ■

**Corollary 3.1** 1.  $\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(x_1, \dots, x_m) = 0$  for any vector  $x = (x_1, \dots, x_m) \neq (1, \dots, 1)$ .

2.  $\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(1, \dots, 1) = -2^m$ .

**Proof :** From [7, Equation (16), page 130] we have  $\sum_{r=0}^m (-1)^{r-1} p_r(k, m) = 0$  for  $0 \leq k < m$  and  $\sum_{r=0}^m (-1)^{r-1} p_r(k, m) = -2^m$  for  $k = m$ . Hence using Lemma 3.1 the result follows. ■

**Theorem 3.1** *Let  $g(x) = x_1 x_2 \cdots x_m$ . Then*

$$(-1)^{g(x)} = \frac{1}{2^{m-1}} \left( 2^{m-1} + \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(x) \right).$$

**Proof :** *Case 1:*  $x \in \{0, 1\}^m$  such that  $x = (1, \dots, 1)$ . Then L.H.S =  $(-1)^1 = -1$ . By Corollary 3.1(2) we have  $\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(x) = \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(1, \dots, 1) = -2^m$ . Hence R.H.S =  $\frac{1}{2^{m-1}}(2^{m-1} - 2^m) = -1$ . *Case 2:*  $x \in \{0, 1\}^m$  such that  $x \neq (1, \dots, 1)$ . Then L.H.S =  $(-1)^0 = 1$ . By Corollary 3.1(1) we have  $\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(x) = 0$ . Hence R.H.S =  $\frac{1}{2^{m-1}}(2^{m-1} - 0) = 1$ . ■

### 3.2 Arbitrary $g$

Let  $g$  be a Boolean function and  $A_g$  be its ANF. For  $\alpha, x \in F_2^m$  and  $r \in [m] = \{1, \dots, m\}$  define

$$N_r^{(m)}(\alpha, x) \triangleq \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} (-1)^{x_{i_1} \alpha_{i_1} \oplus \dots \oplus x_{i_r} \alpha_{i_r}}.$$

For  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m) \in F_2^m$  and  $x = (x_1, \dots, x_m)$ , define  $x^\alpha \triangleq x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$ . Note that if  $\alpha_i = 1$ , then  $x_i^{\alpha_i} = x_i$  else  $x_i^{\alpha_i} = 1$ .

**Proposition 3.1**

$$(-1)^{x^\alpha} = \frac{1}{2^{m-1}} \left( 2^{m-1} + \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) \right)$$

**Proof :** Let  $\mathbf{1} = (1, \dots, 1)$  and note that  $N_r^{(m)}(\mathbf{1}, x) = N_r^{(m)}(x)$ . We define  $y = (y_1, \dots, y_m)$  as follows. For  $1 \leq i \leq m$ , if  $\alpha_i = 1$  then  $y_i = x_i$  else  $y_i = 1$ . Now it is easy to check the following

1.  $x_i^{\alpha_i} = y_i$  and hence  $(-1)^{x^\alpha} = (-1)^y$ .
2.  $N_r^{(m)}(\alpha, x) = N_r^{(m)}(y)$ .

From this the result follows. ■

The next result expresses  $(-1)^{g(x)}$  in terms of  $A_g$ .

**Proposition 3.2**

$$(-1)^{g(x)} = \left( \frac{1}{2^{m-1}} \right)^{wt(A_g)} \prod_{\alpha: A_g(\alpha)=1} \left( 2^{m-1} + \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) \right)$$

**Proof :**

$$(-1)^{g(x)} = (-1)^{\bigoplus_{\alpha \in F_2^m} A_g(\alpha) x^\alpha} = \prod_{\alpha \in F_2^m} (-1)^{A_g(\alpha) x^\alpha} = \prod_{\alpha \in F_2^m} \left( (-1)^{x^\alpha} \right)^{A_g(\alpha)} = \prod_{\alpha: A_g(\alpha)=1} (-1)^{x^\alpha}$$

$$\begin{aligned}
&= \prod_{\alpha: A_g(\alpha)=1} \frac{1}{2^{m-1}} \left( 2^{m-1} + \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) \right) \\
&= \left( \frac{1}{2^{m-1}} \right)^{wt(A_g)} \prod_{\alpha: A_g(\alpha)=1} \left( 2^{m-1} + \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) \right)
\end{aligned}$$

For  $\alpha = (\alpha_1, \dots, \alpha_m)$  and  $u = (u_1, \dots, u_m)$  we define  $u \leq \alpha$  if  $u_i \leq \alpha_i$  for  $1 \leq i \leq m$ . In the rest of the paper we will denote  $(wt(u) \bmod 2)$  by  $wt_2(u)$ . In the next two results we present a two step simplification of the sum  $\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x)$  which occurs in Proposition 3.2. ■

**Proposition 3.3**

$$N_r^{(m)}(\alpha, x) = (-1)^r \sum_{\{u \in F_2^m : u \leq \alpha, wt(u) \leq r\}} \binom{m-wt(\alpha)}{r-wt(u)} (-1)^{\langle u, x \rangle \oplus wt_2(u)}$$

**Proof :** Note  $x_{i_1}^{\alpha_{i_1}} = x_{i_1}$  if  $\alpha_{i_1} = 1$  and  $x_{i_1}^{\alpha_{i_1}} = 1$  if  $\alpha_{i_1} = 0$ . So  $x_{i_1}^{\alpha_{i_1}} = 1 \oplus (1 \oplus x_{i_1})\alpha_{i_1} = 1 \oplus (\bar{x}_{i_1})\alpha_{i_1}$  and hence  $x_{i_1}^{\alpha_{i_1}} \oplus \dots \oplus x_{i_r}^{\alpha_{i_r}} = (r \bmod 2) \oplus \langle (\bar{x}_{i_1}, \dots, \bar{x}_{i_r}), (\alpha_{i_1}, \dots, \alpha_{i_r}) \rangle$ . We have

$$\begin{aligned}
N_r^{(m)}(\alpha, x) &= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} (-1)^{x_{i_1}^{\alpha_{i_1}} \oplus \dots \oplus x_{i_r}^{\alpha_{i_r}}} \\
&= \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} (-1)^{(r \bmod 2) \oplus \langle (\bar{x}_{i_1}, \dots, \bar{x}_{i_r}), (\alpha_{i_1}, \dots, \alpha_{i_r}) \rangle} \\
&= (-1)^r \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq m} (-1)^{\langle (\bar{x}_{i_1}, \dots, \bar{x}_{i_r}), (\alpha_{i_1}, \dots, \alpha_{i_r}) \rangle}.
\end{aligned}$$

Given  $\alpha = (\alpha_1, \dots, \alpha_m)$  and  $1 \leq i_1 < i_2 < \dots < i_r \leq m$ , define an  $m$ -bit vector  $u$  in the following manner. For  $1 \leq j \leq m$ , if  $j \in \{i_1, \dots, i_r\}$ , then  $u_j = \alpha_j$  else  $u_j = 0$ . We say that  $\{i_1, \dots, i_r\}$  produces  $u$  from  $\alpha$ . If  $\{i_1, \dots, i_r\}$  produces  $u$  from  $\alpha$ , then it is easy to verify the following relations.

1.  $\langle (\bar{x}_{i_1}, \dots, \bar{x}_{i_r}), (\alpha_{i_1}, \dots, \alpha_{i_r}) \rangle = \langle (\bar{x}_1, \dots, \bar{x}_m), (u_1, \dots, u_m) \rangle = \langle (x_1, \dots, x_m), (u_1, \dots, u_m) \rangle \oplus wt_2(u)$ ;
2.  $u \leq \alpha$  and  $wt(u) \leq r$ .

It is possible that two distinct sets  $\{i_1, i_2, \dots, i_r\}$  and  $\{i'_1, i'_2, \dots, i'_r\}$  produce the same  $u \leq \alpha$ . This will happen if and only if  $wt(\alpha_{i_1}, \dots, \alpha_{i_r}) = wt(\alpha_{i'_1}, \dots, \alpha_{i'_r}) = wt(\alpha_{j_1}, \dots, \alpha_{j_s})$ , where  $\{j_1, \dots, j_s\} = \{i_1, \dots, i_r\} \cap \{i'_1, \dots, i'_r\}$ . We now claim that the number of distinct sets  $\{i_1, i_2, \dots, i_r\}$  which produce the same  $u$  is  $\binom{m-k}{r-l}$ , where  $k = wt(\alpha)$  and  $l = wt(u)$ . To see this fix  $u \leq \alpha$  with  $wt(u) = l \leq r$ . Let  $j_1, \dots, j_l$  be such that  $u_{j_1} = \dots = u_{j_l} = 1$  and  $u_j = 0$  for  $j \notin \{j_1, \dots, j_l\}$ . If  $\{i_1, \dots, i_r\}$  produces  $u$ , we must have  $\{j_1, \dots, j_l\} \subseteq \{i_1, \dots, i_r\}$  and  $\alpha_j = 0$  for  $j \in S = \{i_1, \dots, i_r\} \setminus \{j_1, \dots, j_l\}$ .

Thus the number of sets  $\{i_1, \dots, i_r\}$  which produce  $u$  is the number of ways we can choose the set  $S$  of  $r-l$  elements such that  $\alpha_j = 0$  for  $j \in S$ . Since  $wt(\alpha) = k$ , the number of  $j \in \{1, \dots, m\}$  such that  $\alpha_j = 0$  is  $m-k$ . Since  $S$  has  $r-l$  elements, the number of possible sets  $S$  is  $\binom{m-k}{r-l}$ , which proves our claim. Hence we can write

$$N_r^{(m)}(\alpha, x) = (-1)^r \sum_{\{u \in F_2^m : u \leq \alpha, wt(u) \leq r\}} \binom{m-k}{r-l} (-1)^{\langle u, x \rangle \oplus wt_2(u)}$$

This completes the proof. ■

**Proposition 3.4**

$$\sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) = (-1) \times \sum_{\{u \in F_2^m : u \leq \alpha\}} 2^{m-wt(\alpha)} (-1)^{\langle u, x \rangle \oplus wt_2(u)}$$

**Proof :** By Proposition 3.3

$$\begin{aligned} \sum_{r=0}^m (-1)^{r-1} N_r^{(m)}(\alpha, x) &= (-1)^{2r-1} \sum_{r=0}^m \sum_{\{u \in F_2^m : u \leq \alpha, wt(u) \leq r\}} \binom{m-wt(\alpha)}{r-wt(u)} (-1)^{\langle u, x \rangle \oplus wt_2(u)} \\ &= (-1) \sum_{\{u \in F_2^m : u \leq \alpha\}} \sum_{r=wt(u)}^m \binom{m-wt(\alpha)}{r-wt(u)} (-1)^{\langle u, x \rangle \oplus wt_2(u)} \\ &= - \sum_{\{u \in F_2^m : u \leq \alpha\}} 2^{m-wt(\alpha)} (-1)^{\langle u, x \rangle \oplus wt_2(u)}. \end{aligned}$$

This completes the proof. ■

Let  $Sup(A_g) = \{\alpha^{(1)}, \dots, \alpha^{(p)}\}$  and  $k_i = wt(\alpha^{(i)})$ . Define

$$V(Sup(A_g)) \triangleq V(\{\alpha^{(1)}, \dots, \alpha^{(p)}\}) \triangleq \{u^{(i_1)} \oplus \dots \oplus u^{(i_r)} : u^{(i_j)} \leq \alpha^{(i_j)}, \{i_1, \dots, i_r\} \subseteq [p], 1 \leq j \leq r\}.$$

**Example 1 (continued)** For Boolean function  $g_1(x_1, x_2, x_3) = x_1x_2 \oplus x_2x_3 \oplus x_3x_1$ ,

$Sup(A_{g_1}) = \{(1, 1, 0), (0, 1, 1), (1, 0, 1)\}$  and  $k_1 = k_2 = k_3 = 2$ .

Note that  $V(Sup(A_g))$  is a subspace of  $(F_2^m, \oplus)$ . A Boolean function  $g(x_1, \dots, x_n)$  is said to be degenerate on variable  $x_i$  if  $g(x_1, \dots, x_{i-1}, 1, x_{i+1}, \dots, x_n) = g(x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_n)$ . The function  $g$  is said to be degenerate if it is degenerate on some variable, else it is said to be non-degenerate. It is easy to verify that  $V(Sup(A_g)) = F_2^m$  if and only if  $g$  is non-degenerate. Suppose  $v = u^{(i_1)} \oplus \dots \oplus u^{(i_r)}$  where  $u^{(i_j)} \leq \alpha^{(i_j)}$  then we say that  $R = \{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\}$  is a *representation* of  $v$ . For  $v \in F_2^m$ , define

$$\mathcal{S}(v) \triangleq \text{set of all representations of } v \text{ in the vector space } V(Sup(A_g)).$$

**Example 1 (continued)** For  $v = 000$ , we have  $\mathcal{S}(v) = \mathcal{S}(000) = \{R_1, R_2, R_3, \dots, R_{15}, R_{16}, R_{17}\}$  where

$$\begin{aligned} R_1 &= \{((0, 0, 0), (1, 1, 0))\}, R_2 = \{((0, 0, 0), (0, 1, 1))\}, R_3 = \{((0, 0, 0), (1, 0, 1))\}, \\ R_4 &= \{((0, 0, 0), (1, 1, 0)), ((0, 0, 0), (0, 1, 1))\}, R_5 = \{((0, 0, 0), (0, 1, 1)), ((0, 0, 0), (1, 0, 1))\}, \\ R_6 &= \{((0, 0, 0), (1, 1, 0)), ((0, 0, 0), (1, 0, 1))\}, R_7 = \{((0, 1, 0), (1, 1, 0)), ((0, 1, 0), (0, 1, 1))\}, \\ R_8 &= \{((1, 0, 0), (1, 1, 0)), ((1, 0, 0), (1, 0, 1))\}, R_9 = \{((0, 0, 1), (0, 1, 1)), ((0, 0, 1, 1, 0, 1))\}, \\ R_{10} &= \{((0, 0, 0), (1, 1, 0)), ((0, 0, 0), (0, 1, 1)), ((0, 0, 0), (1, 0, 1))\}, \\ R_{11} &= \{((0, 1, 0), (1, 1, 0)), ((0, 1, 0), (0, 1, 1)), ((0, 0, 0), (1, 0, 1))\}, \\ R_{12} &= \{((1, 0, 0), (1, 1, 0)), ((0, 0, 0), (0, 1, 1)), ((1, 0, 0), (1, 0, 1))\}, \\ R_{13} &= \{((0, 0, 0), (1, 1, 0)), ((0, 0, 1), (0, 1, 1)), ((0, 0, 1), (1, 0, 1))\}, \\ R_{14} &= \{((1, 1, 0), (1, 1, 0)), ((0, 1, 1), (0, 1, 1)), ((1, 0, 1), (1, 0, 1))\}, \\ R_{15} &= \{((1, 1, 0), (1, 1, 0)), ((0, 1, 0), (0, 1, 1)), ((1, 0, 0), (1, 0, 1))\}, \\ R_{16} &= \{((0, 1, 0), (1, 1, 0)), ((0, 1, 1), (0, 1, 1)), ((0, 0, 1), (1, 0, 1))\}, \\ R_{17} &= \{((1, 0, 0), (1, 1, 0)), ((0, 0, 1), (0, 1, 1)), ((1, 0, 1), (1, 0, 1))\}. \end{aligned}$$

**Note:**

1.  $\mathcal{S}(v) = \phi$  if and only if  $v \notin V(Sup(A_g))$ .
2. If  $v = (0, \dots, 0)$ , then  $\mathcal{S}(v) \neq \phi$ .

Let  $v \in F_2^m$  and  $R = \{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\} \in \mathcal{S}(v)$ . Define

$$\left. \begin{aligned} C(R) &\triangleq \frac{(-1)^r}{2^{(k_{i_1}-1)+\dots+(k_{i_r}-1)}} \\ C_v &\triangleq (-1)^{wt_2(v)} \sum_{R \in \mathcal{S}(v)} C(R). \end{aligned} \right\} \quad (4)$$

**Example 1 (continued)**  $C_{000} = C(R_1) + C(R_2) + C(R_3) + \dots + C(R_{15}) + C(R_{16}) + C(R_{17}) = \frac{(-1)^1}{2^{2-1}} + \frac{(-1)^1}{2^{2-1}} + \frac{(-1)^1}{2^{2-1}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^2}{2^{4-2}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^3}{2^{6-3}} + \frac{(-1)^3}{2^{6-3}} = -1$

Note that if  $\mathcal{S}(v) = \emptyset$ , then  $\sum_{R \in \mathcal{S}(v)} C(R) = 0$  and hence, if  $v \notin V(\text{Sup}(A_g))$  then  $C_v = 0$ .

The parameters  $C_v$  for  $v \in F_2^m$  are derived entirely from the ANF of  $g$ . Our next result expresses  $(-1)^{g(x)}$  in terms of  $C_v$ .

**Theorem 3.2**

$$(-1)^{g(x)} = 1 + \sum_{v \in F_2^m} C_v (-1)^{\langle v, x \rangle}$$

where  $C_v$  is as defined by Equation 4.

**Proof :** As before let  $\text{Sup}(A_g) = \{\alpha^{(1)}, \dots, \alpha^{(p)}\}$  where  $wt(\alpha^{(i)}) = k_i$ . Using Proposition 3.2 and Proposition 3.4 we can write.

$$(-1)^{g(x)} = \prod_{\alpha: A_g(\alpha)=1} \left( 1 - \frac{1}{2^{m-1}} \sum_{\{u \in F_2^m: u \leq \alpha\}} 2^{m-wt(\alpha)} (-1)^{\langle u, x \rangle \oplus wt_2(u)} \right)$$

The first term in the expansion of the above expression is clearly 1. For  $1 \leq r \leq p = wt(A_g)$ , the general term is of the form

$$\left( \frac{-1}{2^{m-1}} \right)^r \sum_{u^{(i_1)} \leq \alpha^{(i_1)}} \dots \sum_{u^{(i_r)} \leq \alpha^{(i_r)}} 2^{(m-k_{i_1})+\dots+(m-k_{i_r})} (-1)^{\langle u^{(i_1)} \oplus \dots \oplus u^{(i_r)}, x \rangle \oplus wt_2(u^{(i_1)} \oplus \dots \oplus u^{(i_r)})}$$

Let  $v = u^{(i_1)} \oplus \dots \oplus u^{(i_r)}$ . Then  $R = \{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\}$  is a representation of  $v$ . Therefore the general term is of the form

$$\sum_{v \in F_2^m} \sum_{R \in \mathcal{S}(v)} C(R) (-1)^{\langle v, x \rangle \oplus wt_2(v)} = \sum_{v \in F_2^m} C_v (-1)^{\langle v, x \rangle}.$$

This gives us the desired result. ■

Now we are in a position to state the main result of this section which relates  $W_g(v)$  to  $C_v$ .

**Theorem 3.3** *If  $g$  is an  $m$ -variable Boolean function then*

$$W_g(v) = 2^m (C_v + \delta_v) \quad (5)$$

where  $\delta_v = 1$  if  $v = 0$  else  $\delta_v = 0$ .

**Proof :** From Theorem 3.2 we have

$$(-1)^{g(x)} - 1 = \sum_{v \in F_2^m} C_v (-1)^{\langle v, x \rangle}.$$

So  $[(-1)^{g(0)} - 1, \dots, (-1)^{g(2^m-1)} - 1] = [C_0, \dots, C_{2^m-1}] H_m$ , where  $H_m$  is the  $(2^m \times 2^m)$  Hadamard matrix (see [7, page 44]). Post multiplying both sides by  $H_m$  and noting that  $H_m H_m = 2^m I_{2^m}$  we get

$$[(-1)^{g(0)}, \dots, (-1)^{g(2^m-1)}] H_m - [1, \dots, 1] H_m = 2^m [C_0, \dots, C_{2^m-1}].$$

i.e.  $[W_g(0), \dots, W_g(2^m-1)] - [2^m, 0, \dots, 0] = 2^m [C_0, \dots, C_{2^m-1}]$ . Hence  $2^m C_v = W_g(v) - 2^m \delta_v$  ■

## 4 Simplifying $C_v$

Theorem 3.3 relates  $W_g(v)$  to  $C_v$ . Thus to compute  $W_g(v)$  it is sufficient to compute  $C_v$ . However, the definition of  $C_v$  given by 4 is in purely algebraic terms. We need to obtain a formula for  $C_v$  which can be computed by an algorithm. In this section we perform this task of simplifying  $C_v$ .

We start by defining certain terms. As in Section 3, we assume  $Sup(A_g) = \{\alpha^{(1)}, \dots, \alpha^{(p)}\}$ . Given  $\emptyset \neq S \subseteq [p] = \{1, 2, \dots, p\}$  and  $v \in F_2^m$ , we define  $\Delta(S, v)$  as follows. Let  $v = (v_1, \dots, v_m)$  and  $S = \{i_1, \dots, i_r\}$ . Write  $\alpha^{(i_j)} = (\alpha_{j,1}, \dots, \alpha_{j,m})$  where  $\alpha_{j,k} \in \{0, 1\}$  for  $1 \leq j \leq r$ ,  $1 \leq k \leq m$ . Let  $\phi(S, v)$  be a Boolean formula which is true if and only if there is a  $k \in [m]$ , such that  $\alpha_{j,k} = 0$  for all  $j \in [r]$  and  $v_k = 1$ . Let

$$\begin{aligned} \sigma(S) &= \bigvee_{i \in S} \alpha^{(i)} = \alpha^{(i_1)} \vee \dots \vee \alpha^{(i_r)}; \\ \mu(S) &= wt(\sigma(S)). \end{aligned} \quad (6)$$

Here  $\vee$  represents the bitwise logical OR of two binary strings of the same length. Define

$$\begin{aligned} \Delta(S, v) &= \begin{cases} 0 & \text{if } \phi(S, v) = 1 \\ \frac{1}{2^{\mu(S) - |S|}} & \text{otherwise.} \end{cases} \end{aligned} \quad (7)$$

**Remark:** If  $\Delta(S, v) > 0$ , then the value of  $\Delta(S, v)$  is independent of  $v$  and depends only on  $S$ .

### Theorem 4.1

$$C_v = (-1)^{wt_2(v)} \sum_{\emptyset \neq S \subseteq [p]} (-1)^{|S|} \Delta(S, v).$$

**Proof :** Fix a set  $S$ , such that,  $\emptyset \neq S = \{i_1, \dots, i_r\} \subseteq [p]$  and a vector  $v = (v_1, \dots, v_m) \in F_2^m$ . Define  $\Gamma(S) = \frac{2^r}{2^{k_{i_1} + \dots + k_{i_r}}}$  where  $k_{i_j} = wt(\alpha^{(i_j)})$ . As before, let  $\alpha^{(i_j)} = (\alpha_{j,1}, \dots, \alpha_{j,m})$ ,  $1 \leq j \leq r$  and define  $\lambda_k = \sum_{j=1}^r \alpha_{j,k}$ . For  $1 \leq k \leq m$ , define

$$\begin{aligned} b_k &= \begin{cases} 2^{\lambda_k - 1} & \text{if } \lambda_k \neq 0 \\ 1 & \text{if } \lambda_k = 0, v_k = 0 \\ 0 & \text{if } \lambda_k = 0, v_k = 1 \end{cases} \end{aligned} \quad (8)$$

Define  $n(S, v) = b_1 \dots b_m$ .

**Claim 1:**  $\Delta(S, v) = n(S, v) \Gamma(S)$

*Proof of Claim 1:* There are two cases to consider.

*Case  $\phi(S, v) = 1$  :* In this case,  $\Delta(S, v) = 0$  by Equation 7. Also  $\phi(S, v) = 1$  implies that there is a  $k \in [m]$  such that  $\alpha_{j,k} = 0$  for all  $j \in [r]$  and  $v_k = 1$ . This implies that  $\lambda_k = 0$  and  $v_k = 1$ . Hence  $b_k = 0$  and so  $n(S, v) = 0$ . Thus in this case we have  $\Delta(S, v) = n(S, v) \Gamma(S)$ .

*Case  $\phi(S, v) = 0$  :* In this case  $\Delta(S, v) = \frac{1}{2^{m_1 - r}}$  by Equation 7, where  $m_1 = \mu(S)$  and  $r = |S|$ . Also  $b_k = 2^{\lambda_k - z_k}$  where  $z_k = 0$  if  $\lambda_k = 0$ ; and  $z_k = 1$  if  $\lambda_k > 0$ . So

$$n(S, v) = (2^{\lambda_1 - z_1})(2^{\lambda_2 - z_2}) \dots (2^{\lambda_m - z_m}) = 2^{(\lambda_1 + \dots + \lambda_m) - (z_1 + \dots + z_m)}.$$

Since  $m_1 = \mu(S) = wt(\alpha^{(i_1)} \vee \dots \vee \alpha^{(i_r)})$  we have  $(z_1 + z_2 + \dots + z_m) = m_1$ . Also we have

$$\sum_{k=1}^m \lambda_k = \sum_{k=1}^m \sum_{j=1}^r \alpha_{j,k} = \sum_{j=1}^r \sum_{k=1}^m \alpha_{j,k} = \sum_{j=1}^r k_{i_j}.$$

So we get

$$n(S, v) = 2^{(\lambda_1 + \dots + \lambda_m) - m_1} = \frac{2^{(k_{i_1} + \dots + k_{i_r})}}{2^{m_1}}.$$



By definition  $\Gamma(S) = \frac{2^r}{2^{k_{i_1} + \dots + k_{i_r}}}$  and so  $\Delta(S, v) = \frac{1}{2^{m_1 - r}} = n(S, v)\Gamma(S)$ . This completes the proof of Claim 1.

**Claim 2:**

$$C_v = (-1)^{wt_2(v)} \sum_{\emptyset \neq S \subseteq [p]} (-1)^r n(S, v) \Gamma(S).$$

*Proof of Claim 2:* From Equation 4, we have  $C_v = (-1)^{wt_2(v)} \sum_{R \in \mathcal{S}(v)} C(R)$ .

Given  $S = \{i_1, \dots, i_r\}$ , the vectors  $\alpha^{(i_1)}, \dots, \alpha^{(i_r)}$  are fixed. Let  $D(S, v)$  be the set of all representations of  $v$  of the form  $\{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\}$ . The value of  $C(R)$  for any such representation  $R$  is equal to  $\frac{2^r}{2^{k_{i_1} + \dots + k_{i_r}}}$  and depends only on  $S$ . From definition, this value is equal to  $\Gamma(S)$ . In evaluating  $C_v$  we have to sum over all representations of  $v$ . The contribution of the set  $S$  to this sum is clearly  $|D(S, v)|\Gamma(S)$ . Thus we obtain

$$C_v = (-1)^{wt_2(v)} \sum_{\emptyset \neq S \subseteq [p]} |D(S, v)|\Gamma(S).$$

From this it is sufficient to show that  $n(S, v) = |D(S, v)|$ . There are two cases to consider.

*Case  $b_k = 0$  for some  $k \in [m]$ :* This implies that  $v$  cannot be represented as  $\{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\}$  for any choice of  $u^{(i_j)} \leq \alpha^{(i_j)}$ . Hence  $D(S, v) = \emptyset$  and so we have  $n(S, v) = 0 = |D(S, v)|$ .

*Case  $b_k > 0$  for all  $k \in [m]$ :* In this case  $D(S, v) \neq \emptyset$ . Suppose  $R = \{(u^{(i_1)}, \alpha^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)})\} \in D(S, v)$ , where for  $1 \leq j \leq r$ ,  $u^{(i_j)} = (u_{j,1}, \dots, u_{j,m}) \in F_2^m$  and  $\alpha^{(i_j)} = (\alpha_{j,1}, \dots, \alpha_{j,m}) \in F_2^m$ .

Define two  $r \times m$  matrices  $M_1$  and  $M_2$  in the following manner. The  $(j, k)$ th entry of  $M_1$  (resp.  $M_2$ ) is  $\alpha_{j,k}$  (resp.  $u_{j,k}$ ). The conditions  $u^{(i_j)} \leq \alpha^{(i_j)}$  is equivalent to  $M_2 \leq M_1$ , where the matrices are compared entrywise. Since  $R$  is a representation of  $v$ , we must have,

$$u_{1,k} \oplus \dots \oplus u_{r,k} = v_k \text{ for each } k \in [m]. \quad (9)$$

Thus the problem of enumerating the set  $D(S, v)$  is equivalent to enumerating matrices  $M_2$  such that  $M_2 \leq M_1$  and Equation (9) holds. Let  $c_k$  be the number of possible choices for the  $k$ th column of  $M_2$ . Then  $|D(S, v)| = c_1 \dots c_m$  and it is sufficient to show that  $c_k = b_k$  for each  $k \in [m]$ . There are two cases to consider.

*Subcase  $\lambda_k = 0$ :* In this case  $v_k = 0$  and  $b_k = 1$ . Since  $\lambda_k = 0$ , the  $k$ th column of  $M_1$  is the all zero column. Hence the only possible choice for the  $k$ th column of  $M_2$  is also the all zero column and so  $c_k = 1 = b_k$ .

*Subcase  $\lambda_k > 0$ :* In this case  $b_k = 2^{\lambda_k - 1}$ . We first observe that the XOR of the ones in the  $k$ th column of  $M_2$  must be equal to  $v_k$ . Thus we have to consider two cases according as  $v_k = 0$  or  $v_k = 1$ . First suppose  $v_k = 0$ . The  $k$ th column of  $M_1$  contains  $\lambda_k$  ones. Since  $v_k = 0$ , we can choose an even number of these ones to form a column for the matrix  $M_2$ . Thus the number of choices for the  $k$ th column of  $M_2$  is  $c_k = \binom{\lambda_k}{0} + \binom{\lambda_k}{2} + \dots + \binom{\lambda_k}{2 \lfloor \frac{\lambda_k}{2} \rfloor} = 2^{\lambda_k - 1} = b_k$ . A similar argument holds when  $v_k = 1$ .

This completes the proof of Claim 2. Theorem 4.1 is a direct consequence of Claim 1 and Claim 2. ■

Theorem 4.1 provides a method for computing  $C_v$ . However, this requires an algorithm to consider all possible subsets of  $[p] = \{1, \dots, p\}$ . If  $p$  is even moderately large, this yields an impractical algorithm. Our next task is to show that it is actually not required to consider all the subsets of  $[p]$ . In this section we present one such special case and in the next section we present a general algorithm to compute  $C_v$  without generating all the subsets of  $[p]$ .

**Theorem 4.2** *Let  $\text{Sup}(A_g) = \{\alpha_1, \dots, \alpha_p\}$  and  $t > 0$  be such that for any  $\{i_1, \dots, i_{t+1}\} \subseteq \{1, \dots, p\}$ , we have  $wt(\alpha_{i_1} \vee \dots \vee \alpha_{i_{t+1}}) = m$ . Then*

$$C_v = (-1)^{wt_2(v)} \left( \sum_{j=t+1}^p \frac{(-1)^j}{2^{m-j}} \binom{p}{j} + \sum_{\emptyset \neq S \subseteq [p], |S| \leq t} (-1)^{|S|} \Delta(S, v) \right)$$

**Proof :** We have from Theorem 4.1

$$\begin{aligned} C_v &= (-1)^{wt_2(v)} \sum_{\emptyset \neq S \subseteq [p]} (-1)^{|S|} \Delta(S, v) \\ &= (-1)^{wt_2(v)} \left( \sum_{\emptyset \neq S \subseteq [p], |S| \geq t+1} (-1)^{|S|} \Delta(S, v) + \sum_{\emptyset \neq S \subseteq [p], |S| \leq t} (-1)^{|S|} \Delta(S, v) \right). \end{aligned}$$

Under the given condition if  $|S| \geq t+1$  then using Equation 7 we have  $\mu(S) = m$ . Hence

$$\begin{aligned} C_v &= (-1)^{wt_2(v)} \left( \sum_{\emptyset \neq S \subseteq [p], |S| \geq t+1} (-1)^{|S|} \frac{1}{2^{m-r}} + \sum_{\emptyset \neq S \subseteq [p], |S| \leq t} (-1)^{|S|} \Delta(S, v) \right) \\ &= (-1)^{wt_2(v)} \left( \sum_{j=t+1}^p (-1)^j \frac{1}{2^{m-r}} \binom{p}{j} + \sum_{\emptyset \neq S \subseteq [p], |S| \leq t} (-1)^{|S|} \Delta(S, v) \right) \end{aligned}$$

This completes the proof. ■

Under the condition of Theorem 4.2, to evaluate  $C_v$  we only have to consider all non empty subsets of  $[p]$  of cardinality at most  $t$ . If  $t$  is reasonably small, this is much better than considering all non empty subsets of  $[p]$ . In the next section, we develop this idea to obtain an algorithm to compute  $C_v$ .

## 5 Algorithm

In general we are interested in computing the Walsh transform of  $g$  at all the points  $u \in F_2^m$ . However, if  $m$  is relatively large (say around 50), then it will be practically impossible to compute the Walsh transform at all the  $2^m$  points. In such a situation, it will be of interest to compute the Walsh transform at a particular point or for a small set of points. The fast Walsh transform takes time  $O(m2^m)$  and computes the Walsh transform at all the  $2^m$  points. In fact, to the best of our knowledge, there is no known algorithm which can compute the Walsh transform at a particular point in time less than  $2^m$ .

Our approach is to design an algorithm that consists of two parts. In the first part, the algorithm does a certain amount of preprocessing and prepares a list. In the second part, the algorithm takes as input a particular  $v \in F_2^m$  and computes  $C_v$ . (Using Theorem 3.3 this also gives us  $W_g(v)$ ). Once the preprocessing is complete, the second part can be run for different  $v$  without running the first part. This makes it efficient to compute  $W_g(v)$  for a set of  $v$ .

We start by defining certain parameters. For  $\emptyset \neq S \subseteq [p]$  recall from equation (6) that  $\sigma(S) = \bigvee_{i \in S} \alpha^{(i)}$  and  $\mu(S) = wt(\sigma(S))$ . For  $j = 0, \dots, m$  define

$$B_j \triangleq \sum_{\mu(S)=j} (-2)^{|S|} \text{ and } B \triangleq \sum_{j=0}^{m-1} \sum_{\mu(S)=j} (-1)^{|S|} \Delta(S, v).$$

Note that if  $S = \{i\}$  and  $\alpha^{(i)} = (0, \dots, 0)$ , then  $\mu(S) = 0$ . The vector  $(0, \dots, 0) \in Sup(A_g)$  implies that the constant term in the ANF of  $g$  is equal to 1. The values of  $B_0, \dots, B_m$  are independent of  $v$  and only the value of  $B$  depends on  $v$ .

**Theorem 5.1** *For any  $v \in F_2^m$ , we have*

$$C_v = (-1)^{wt_2(v)} \left[ \frac{(-1)^p - 1 - B_0 - \dots - B_{m-1}}{2^m} + B \right]. \quad (10)$$

**Proof :** Define

$$A \triangleq \sum_{\emptyset \neq S \subseteq [p]} (-2)^{|S|} = \sum_{j=1}^p (-2)^j \binom{p}{j} = -1 + (-1)^p.$$

Then  $A = \sum_{j=0}^m B_j$ . Consequently,  $B_m = A - B_0 - \dots - B_{m-1}$ . To see this note that  $\sum_{j=0}^m B_j = \sum_{j=0}^m \sum_{\mu(S)=j} (-2)^{|S|} = \sum_{\emptyset \neq S \subseteq [p]} (-2)^{|S|} = A$ .

From Theorem 4.1 we have

$$\begin{aligned} C_v &= (-1)^{wt_2(v)} \sum_{\emptyset \neq S \subseteq [p]} (-1)^{|S|} \Delta(S, v) \\ &= (-1)^{wt_2(v)} \left[ \sum_{j=0}^m \sum_{\mu(S)=j} (-1)^{|S|} \Delta(S, v) \right] \\ &= (-1)^{wt_2(v)} \left[ \sum_{\mu(S)=m} (-1)^{|S|} \Delta(S, v) + \sum_{j=0}^{m-1} \sum_{\mu(S)=j} (-1)^{|S|} \Delta(S, v) \right] \\ &= (-1)^{wt_2(v)} \left[ \frac{B_m}{2^m} + B \right] \\ &= (-1)^{wt_2(v)} \left[ \frac{A - B_0 - \dots - B_{m-1}}{2^m} + B \right] \end{aligned}$$

This completes the proof. ■

Our algorithm is based on Equation (10). The intuition behind the algorithm is the following. For most sets  $S$ , the value of  $\mu(S)$  will be equal to  $m$  and will be accounted for by  $B_m$ . Thus if we can avoid computing  $B_m$  directly, then we will be saving a lot of computation. We will compute  $B_0, \dots, B_{m-1}$  and then use Equation (10) to compute the value of  $C_v$ . However, the value of  $B$  has to be computed. We next describe how this is done.

Define

$$\left. \begin{aligned} \mathcal{S}_1 &= \{S \subseteq [p] : \mu(S) < m\}; \\ \mathcal{S}_2 &= \sigma(\mathcal{S}_1) = \{\sigma(S) : S \in \mathcal{S}_1\}. \end{aligned} \right\} \quad (11)$$

**Example 2 (continued)** From now onward, we consider the Boolean function  $g_2(x_1, x_2, x_3) = x_1 x_2 \oplus x_2 x_3 \oplus x_1 \oplus x_2$ , so  $Sup(A_{g_2}) = \{(1, 1, 0), (0, 1, 1), (1, 0, 0), (0, 1, 0)\}$ . This Boolean function is taken different from the previous one so that  $|\mathcal{S}_1|$  and  $|\mathcal{S}_2|$  are different. Here  $p = 4$ ,  $m = 3$ . It is easy to see

$$\mathcal{S}_1 = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}\},$$

$\mathcal{S}_2 = \{\sigma(\{1\}) = (1, 1, 0), \sigma(\{2\}) = (0, 1, 1), \sigma(\{3\}) = (1, 0, 0), \sigma(\{4\}) = (0, 1, 0)\}$  and hence  $|\mathcal{S}_1| = 9$  and  $|\mathcal{S}_2| = 4$ . Note by definition of  $\sigma$ ,  $\sigma(\{1, 3\}) = (1, 1, 0) \vee (1, 0, 0) = (1, 1, 0) = \sigma(\{1\})$ . And this is the reason for  $|\mathcal{S}_2| \leq |\mathcal{S}_1|$ .

Write  $\mathcal{S}_2 = \{str_1, \dots, str_n\}$  for some  $n > 0$ . For  $1 \leq i \leq n$  define  $val_i = \sum_{\sigma(S)=str_i} (-2)^{|S|}$ . In the first part of our algorithm we prepare the list  $\mathcal{L} = ((str_1, val_1), \dots, (str_n, val_n))$ . Note that this part does not depend on  $v$ .

**Example 2 (continued)**  $\mathcal{S}_2 = \{str_1 = (1, 1, 0), str_2 = (0, 1, 1), str_3 = (1, 0, 0), str_4 = (0, 1, 0)\}$ . Now  $val_1 = \sum_{\sigma(S)=str_1} (-2)^{|S|} = (-2)^1 + (-2)^2 + (-2)^2 + (-2)^2 + (-2)^3 = -2$ . Similarly  $val_2 = 2$ ,  $val_3 = -2$  and  $val_4 = -2$ .

For  $str, v \in F_2^m$ , define  $\psi(str, v) \triangleq \overline{str} \wedge v$  where  $\overline{str}$  is bitwise complement of  $str$  and  $\wedge$  is the bitwise logical AND. The operation  $\wedge$  is performed bitwise on  $str$  and  $v$ . Suppose  $\emptyset \neq S \subseteq [p]$  such that  $\sigma(S) = str$

and  $v \in F_2^m$ . Then  $\Delta(S, v) > 0$  if and only if  $\psi(str, v) = (0, \dots, 0)$ . Thus

$$B = \sum_{\psi(str_i, v) = (0, \dots, 0)} \frac{val_i}{2^{wt(str_i)}}.$$

Also for  $j = 0, \dots, m-1$ ,

$$B_j = \sum_{wt(str_i) = j} val_i.$$

Hence once the list  $\mathcal{L}$  is prepared, it is easy to compute  $B_0, \dots, B_{m-1}$  and  $B$ . Now we describe a method for preparing the list  $\mathcal{L}$ .

**Example 2 (continued)** We take  $v = (1, 0, 0)$ , then  $\psi((1, 1, 0), (1, 0, 0)) = (0, 0, 1) \wedge (1, 0, 0) = (0, 0, 0)$ . Similarly  $\psi((0, 1, 1), (1, 0, 0)) = (1, 0, 0)$ ,  $\psi((1, 0, 0), (1, 0, 0)) = (0, 0, 0)$  and  $\psi((0, 1, 0), (1, 0, 0)) = (1, 0, 0)$ . So  $B = \frac{val_1}{2^{wt(str_1)}} + \frac{val_3}{2^{wt(str_3)}} = \frac{2}{4} + \frac{-2}{2} = -\frac{1}{2}$ . Again from definition  $B_0 = 0$ ,  $B_1 = val_3 + val_4 = -2 + (-2) = -4$ ,  $B_2 = val_1 + val_2 = 2 + 2 = 4$ . Now  $A = -1 + (-1)^p = 0$ . So  $C_v = (-1)^{wt_2(v)} \left[ \frac{A - B_0 - B_1 - B_2}{2^m} + B \right] = -\left[ \frac{0 - 0 + 4 - 4}{2^3} + \frac{1}{2} \right] = -\frac{1}{2}$ .

First we describe a rooted directed tree  $\mathcal{T}$  whose nodes are the subsets of  $[p] = \{1, \dots, p\}$ . The root node of  $\mathcal{T}$  is the empty set. The children of a set  $S$  are the sets  $S_1, \dots, S_k$ , where for  $1 \leq i \leq k$ ,  $S_i = S \cup \{\max(S) + i\}$  and  $k = p - \max(S)$ . This ensures that if  $S'$  is a node in the subtree rooted at  $S$ , then  $S \subset S'$ .

Our algorithm will traverse all the nodes  $S$  of  $\mathcal{T}$  for which  $\mu(S) < m$ . (Note that  $\mathcal{T}$  has  $2^p$  nodes and if an algorithm is required to traverse all the nodes of  $\mathcal{T}$ , then the algorithm will be exponential in  $p$ .) From the structure of  $\mathcal{T}$  we know that if  $\mu(S) = m$  for some node  $S$ , then  $\mu(S') = m$  for all nodes  $S'$  in the subtree rooted at  $S$ . This crucial fact makes the traversal particularly efficient. If during the traversal we reach a node  $S$  with  $\mu(S) = m$ , then we need not visit any of the nodes in the subtree rooted at  $S$ . This means that we are effectively *pruning* the subtree rooted at  $S$  from  $\mathcal{T}$ . The more we encounter this pruning effect, the more efficient is our algorithm.

While traversing  $\mathcal{T}$  we prepare the list  $\mathcal{L}$  in the following manner. Initially  $\mathcal{L}$  is the empty list. Let  $\text{First}(\mathcal{L}) = \{str : (str, val) \in \mathcal{L}\}$ . Suppose we have reached a node  $S$  with  $\mu(S) < m$ . If  $\sigma(S) \notin \text{First}(\mathcal{L})$ , then we add  $(\sigma(S), (-2)^{|S|})$  to  $\mathcal{L}$ . On the other hand, if  $\sigma(S) \in \text{First}(\mathcal{L})$ , then we update  $val$  to  $val + (-2)^{|S|}$ .

Thus the operations on  $\mathcal{L}$  are search and insert. We implement  $\mathcal{L}$  using a height balanced binary tree (see [5]). Hence each search/insert operation requires time  $O(\log \mathcal{L})$ . One such search or insert operation is required for each  $S$  such that  $\mu(S) < m$ . Also the total time spent at any node which is visited is  $O(m)$ . Hence the total time required by the algorithm is  $O(m|S_1| \log(|S_2|))$ . We next present the algorithm for computing  $C_v$ .

**Algorithm** ComputeCv

Inputs :

1.  $\text{sup}(A_g)$ , where  $g$  is an  $m$ -variable Boolean function.
2.  $v \in F_2^m$ .

Output :  $C_v$ .

*Part 1* : Computation of list  $\mathcal{L}$ .

Set  $\mathcal{L}$  equal to the empty list.

Set  $str$  equal to the empty string.

Traverse( $str$ ).

Assume  $\mathcal{L} = ((str_1, val_1), \dots, (str_n, val_n))$  at the end of Traverse.

*Part 2* : Computation of  $C_v$ .

Set  $B_0 = \dots = B_{m-1} = 0$ .

For  $i = 1$  to  $n$  do  
   if  $(wt(str_i) = j)$  then  $B_j = B_j + val_i$ .  
   if  $(\psi(str_i, v) = (0, \dots, 0))$  then  $B = B + \frac{val_i}{2^{wt(str_i)}}$ .  
 End For.  
 $C_v = (-1)^{wt_2(v)} (\frac{1}{2^m} ((-1)^p - 1 - \sum_{i=0}^{m-1} B_i) + B)$ .  
 Return  $C_v$ .

**End Algorithm** ComputeCv.

The subroutine `Traverse()` performs a depth first search on the tree  $\mathcal{T}$  described before. The details of the algorithm `Traverse()` are given below. In the algorithm we use the notation  $\sigma(tstr)$  for a  $p$ -bit string  $tstr$  to mean  $\sigma(S)$  where  $S = \{i : tstr_i = 1\}$ .

**Algorithm** `Traverse(str)`

Input : a binary string  $str$  of length at most  $p$ .

For  $i = |str|$  to  $p - 1$  do  
   Set  $j = i - |str|$ .  
   Set  $tstr = str || 0^j || 1$ .  
   If  $(\mu(tstr) < m)$  then  
     If  $(\sigma(tstr) \notin \text{First}(\mathcal{L}))$ , then  
       Add  $(\sigma(tstr), (-2)^{wt(tstr)})$  to  $\mathcal{L}$ .  
     Else suppose  $(\sigma(tstr), val)$  is present in  $\mathcal{L}$ .  
       Set  $val = val + (-2)^{wt(tstr)}$ .  
   End If.  
   If  $(|tstr| < p)$ , then `Traverse(tstr)`.  
 End For.

**End Algorithm** `Traverse`

From the above discussion we obtain the following result.

**Theorem 5.2** *Let  $v_1, \dots, v_t \in F_2^m$  and  $g$  be an  $m$ -variable Boolean function. Then  $\{W_g(v_i) : 1 \leq i \leq t\}$  can be computed in time  $O(m(|\mathcal{S}_1| \log(|\mathcal{S}_2|) + t|\mathcal{S}_2|))$ .*

**Proof :** Part 1 of Algorithm `ComputeCv` has to be executed only once for all the vectors  $v_1, \dots, v_t$ . This takes time  $O(|\mathcal{S}_1| \log(|\mathcal{S}_2|))$ . Part 2 of Algorithm `ComputeCv` has to run once for each of the vectors  $v_1, \dots, v_t$ . Each execution of Part 2 takes time  $O(|\mathcal{S}_2|)$ . This gives the time complexity of the algorithm. Correctness of the algorithm follows from the previous discussion. ■

**Remark :** It is important to note that for small  $t$  it is possible to have  $|\mathcal{S}_1| \log(|\mathcal{S}_2|) + t|\mathcal{S}_2| \ll \min(2^p, 2^m)$ . In such situations it is possible to efficiently compute the Walsh transform of  $g$  for a small set of points. Cardinality of  $|\mathcal{S}_1|$  and  $|\mathcal{S}_2|$  depends on ANF and in the worst case can be  $O(2^p)$ . To keep  $|\mathcal{S}_1|$  and  $|\mathcal{S}_2|$  within a controllable limit, the ANF should be a sparse multinomial and support of ANF should contain terms with very high weights (nearly  $m$ ). In Section 6, we provide some experimental data to support this intuition.

## 6 Experimental Results

Similar to Section 3, let  $Sup(A_g) = \{\alpha^{(1)}, \dots, \alpha^{(p)}\}$ . We write  $\alpha^{(i)} = (\alpha_{i,1}, \dots, \alpha_{i,m})$  where  $\alpha_{i,j} \in \{0, 1\}$  for  $1 \leq i \leq p$ ,  $1 \leq j \leq m$ . For  $1 \leq j \leq m$ , define  $\rho_j = p - \sum_{i=1}^p \alpha_{i,j}$ . Let  $\rho_{max} = \max_{j \in [m]}(\rho_j)$ ,  $\rho_{avg} = \sum_{i=1}^m \frac{\rho_i}{m}$  and  $w_{avg} = \sum_{i=1}^p \frac{wt(\alpha^{(i)})}{p}$ . We use the notation  $\mathcal{S}_1$  and  $\mathcal{S}_2$  as defined in Equation 11.

Tables 1 and 2 consider the cases  $p > m$  and  $p < m$  respectively. (Here we note that the class of Boolean functions for which  $p$  is less than  $m$  is also very rich. For example this class includes an important subset of the class of bent functions.) The results in these two tables show that  $\rho_i$ 's (specially  $\rho_{max}$ ) are crucial to the complexity of the algorithm. If  $\rho_{max}$  is comparatively larger than the other  $\rho_i$ 's then the size of  $|\mathcal{S}_1|$  is exponential in  $\rho_{max}$ . Also note that in general  $|\mathcal{S}_1| \geq 2^{\rho_{max}}$ .

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
100, 250	546	2135470, 1048576	20, 15, 94
100, 250	547	3184044, 2097152	21, 15, 94
100, 250	549	5288601, 4194304	22, 15, 94
100, 250	549	9482897, 8388608	23, 15, 94
100, 250	550	17871500, 16777216	24, 15, 94
100, 250	552	34648698, 33554432	25, 15, 94
100, 250	553	68203122, 67108864	26, 15, 94

Table 1

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
100, 50	2258	365465, 131072	17, 15, 70
100, 50	2392	1340716, 1048576	20, 15, 70
100, 50	2429	8678491, 8388608	23, 15, 70
100, 50	2465	67386792, 67108864	26, 15, 70

Table 2

From the definition, the parameters  $m, p, \rho_{avg}$  and  $w_{avg}$  satisfy the following relationship:  $m \times \rho_{avg} = p \times (m - w_{avg})$ . For fixed  $m$  and  $p$ ,  $\rho_{avg}$  decreases as  $w_{avg}$  increases. Similarly, for fixed  $m$  and  $\rho_{avg}$ , as  $p$  increases so does  $w_{avg}$ . In Table 3, we show the change of  $w_{avg}$  with  $p$  for  $m = 100$  and  $\rho_{avg} = 16$ .

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
100, 50	9976	5313186, 524288	19, 16, 68
100, 100	2934	6482400, 524288	19, 16, 84
100, 200	1414	4609688, 524288	19, 16, 92
100, 400	827	6644436, 524288	19, 16, 96

Table 3

It is possible to run the algorithm in cases where  $m$  is large but  $\rho_{max}$  is small. Table 4 provides some data which illustrates this fact.

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
200, 200	725	96213, 32768	15, 12, 188
400, 200	2175	207969, 16384	14, 12, 376
800, 200	1938	528605, 65536	16, 12, 752
3200, 100	2290	243188, 16384	14, 12, 2816
3200, 100	2502	4436706, 4194304	22, 12, 2816

Table 4

In the above examples we see that  $|\mathcal{S}_2|$  is small. In Table 5 we provide some examples for which  $|\mathcal{S}_2|$  is large but  $w_{avg}$  is small.

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
50, 25	134260	21432167, 1048576	20, 16, 18
60, 30	150554	48852312, 8388608	23, 16, 28
70, 35	92849	61196173, 16777216	24, 16, 38
80, 40	13403	17968626, 33554432	26, 16, 48
90, 45	22728	585495610, 134217728	27, 16, 58

Table 5

As long as  $\rho_{max}$  is small (say less than 25), it is possible to run the algorithm for quite large values of both  $m$  and  $p$ . Table 6 provides some evidence of this fact.

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 , 2^{\rho_{max}}$	$\rho_{max}, \rho_{avg}, w_{avg}$
400, 800	1578	1442226, 32768	15, 14, 393
400, 800	1588	2489671, 1048576	20, 14, 393

Table 6

Table 7 provides the actual running times taken by a simple C language implementation (Linux operating system on Pentium IV, 2.4 GHz CPU, 1 GB RAM).

$m, p$	$ \mathcal{S}_2 $	$ \mathcal{S}_1 $	time in seconds
100, 50	9976	5313186	50
100, 100	625	103455	2
100, 200	1297	71257808	830
400, 800	1588	2489671	1059

Table 7

Note that the following are true for the experiments performed above.

1.  $|\mathcal{S}_1|$  is exponential in  $\rho_{max}$  and hence the run time of the algorithm is exponential in  $\rho_{max}$ .
2. The bounds on  $p$  are  $0 \leq p \leq 2^m$ . But for our algorithm to be useful the value of  $p$  can not be very large because in that case  $\rho_{max}$  can not be restricted to low values (say 25) and our algorithm will not work.
3. The value of  $w_{avg}$  should be high i.e. ANF should be very sparse multinomial and support of ANF should contain terms with very high weight.
4. In most cases,  $|\mathcal{S}_2|$  is small compared to  $|\mathcal{S}_1|$ . Also  $|\mathcal{S}_2|$  tends to grow as  $w_{avg}$  decreases.
5. In all the examples given above, the chosen functions were non-degenerate on all the variables.

Finally, in summary we note that for certain types of Boolean functions on large number of variables algorithm ComputeCv can be used to compute the Walsh transform for a small set of points. This can provide some useful information about the Boolean function. For example, one can obtain the weight (or size of support) of  $g$  by computing  $W_g(0)$ . Also the ability to probe the spectral domain of the function at random points can provide an estimate of the nonlinearity of the function.

## 7 Possible Improvements

In this section we discuss possible ways of improving the algorithm. From Theorem 5.2, we see that the complexity of the algorithm depends on  $|\mathcal{S}_1|$  and  $|\mathcal{S}_2|$ . These two parameters are determined by the ANF of the function. We discuss two ways of improving the efficiency.

Let  $g$  be the function under consideration whose ANF is  $A_g$ . Suppose we apply an affine transformation to the variables of  $g$  to obtain  $f$  such that  $|sup(A_f)| < |sup(A_g)|$ . In such a situation it might be possible to improve the run time of the algorithm. The problem is in obtaining an affine transformation which will perform this task.

Another approach to improve the efficiency is to look for a more general and compact algebraic representation of a function. We briefly discuss this approach below and indicate the difficulty of this method.

Let  $g$  be of the form

$$g(x_1, x_2, \dots, x_m) = \bigoplus_{(\alpha, \beta) \in list} (x_1 \oplus \beta_1)^{\alpha_1} \cdots (x_m \oplus \beta_m)^{\alpha_m}$$

where  $list = ((\alpha^{(1)}, \beta^{(1)}), \dots, (\alpha^{(p)}, \beta^{(p)}))$  and  $\alpha^{(i)}, \beta^{(i)} \in F_2^m$ . The ANF of  $g$  is a special case of the above form where  $\beta_1 = \dots = \beta_m = 0$ .

Let  $r$  be such that  $1 \leq r \leq |list|$  and suppose  $v = u^{(i_1)} \oplus \dots \oplus u^{(i_r)}$  where  $u^{(i_j)} \leq \alpha^{(i_j)}$  then we say  $R = \{(u^{(i_1)}, \alpha^{(i_1)}, \beta^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)}, \beta^{(i_r)})\}$  is a representation of  $v$ .

Let  $R \in \mathcal{S}(v)$ , such that  $R = \{(u^{(i_1)}, \alpha^{(i_1)}, \beta^{(i_1)}), \dots, (u^{(i_r)}, \alpha^{(i_r)}, \beta^{(i_r)})\}$  where  $u^{(i_j)} \leq \alpha^{(i_j)}$  and for  $1 \leq j \leq r$ , let  $k_{i_j} = wt(\alpha^{(i_j)})$ . Define

$$C(R) = \frac{(-1)^r}{2^{(k_{i_1}-1)+\dots+(k_{i_r}-1)}} (-1)^{\langle u^{(i_1)}, \beta^{(i_1)} \rangle \oplus \dots \oplus \langle u^{(i_r)}, \beta^{(i_r)} \rangle}.$$

Now it is simple to check that Theorem 3.3 holds. But in this situation it seems difficult to simplify the formula and obtain an algorithm to compute  $C_v$ .

## 8 Concluding Remarks

In this paper we have developed an algorithm to compute the Walsh transform of Boolean function at a point from its algebraic normal form. This can also be extended to evaluate the Walsh transform for a set of points. The advantage of our method is that in certain situations it is possible to run our algorithm to evaluate the Walsh transform (at a small set of points) of Boolean functions with large number of variables and hence can be used in cases where the fast Walsh transform is not applicable.

An important parameter in the study of a Boolean function is its nonlinearity. To compute nonlinearity, it is necessary to compute the Walsh transform at all the points. For Boolean functions on a large number of variables ( $\geq 40$ ), it is practically impossible to perform this task. In certain situations, our algorithm can be used to evaluate the Walsh transform at any particular point. In such situations, it is possible to use our algorithm in conjunction with some randomized heuristic (simulated annealing/ hill climbing/ genetic algorithms) to estimate the nonlinearity of the function. Thus we feel that the algorithm developed in this paper is a valuable technique for the study and analysis of Boolean functions.

**Acknowledgements:** We wish to thank the reviewers for reading the paper and providing several suggestions.



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