A General Correlation Theorem

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Abstract

In 2001, Nyberg proved three important correlation theorems and applied them to several cryptanalytic contexts. We continue the work of Nyberg in a more theoretical direction. We consider a general functional form and obtain its Walsh transform. Two of Nyberg's correlation theorems are seen to be special cases of our general functional form. S-box look-up, addition modulo 2^{2k} and X-OR are three frequently occuring operations in the design of symmetric ciphers. We consider two methods of combining these operations and in each apply our main result to obtain the Walsh transform.

Keywords : Boolean function, S-box, Walsh Transform, correlation, linear approximation, symmetric ciphers.

1 Introduction

Symmetric ciphers are a basic cryptographic primitive. In practice, symmetric ciphers are designed using nonlinear Boolean functions and S-boxes. One of the most effective methods of attacking symmetric ciphers is the technique of linear cryptanalysis [5]. The efficacy of this technique depends upon the ability to obtain good linear approximations of the constituent Boolean functions and S-boxes.

Linear approximations are studied using the technique of Walsh transform analysis. While it is usually easy to apply the Walsh transform to an individual constituent of a symmetric cipher, in general it is more difficult to apply the technique when a combination of primitives are used. This requires the development of a general methodology of Walsh transform applications.

One such important work has been done by Nyberg in [8]. This work unifies some of the previous approaches and obtains three key results on the Walsh transform of various functional forms. These are then applied to several typical cryptanalytic context.

The purpose of the current paper is to continue the direction of research initiated in [8]. We obtain the Walsh transform for the following general functional form:

$$h(x_1, x_2, \ldots, x_{t+1}) = f(g_1(x_1, x_2), g_2(x_2, x_3), \ldots, g_t(x_t, x_{t+1}))$$

where each g_i is a map from $\mathbb{F}_2^{m_i+m_{i+1}}$ to $\mathbb{F}_2^{n_i}$ and f is a Boolean function from $\mathbb{F}_2^{n_1+\cdots+n_t}$ to $\mathbb{F}_2$. We obtain a closed form expression for the Walsh transform of h in terms of the Walsh transform of f and $g_1, \ldots, g_t$. We show that two of Nyberg's results in [8] are special cases of our theorem. This underlines the importance of our result in the context of symmetric cipher cryptanalysis.

We also consider two applications of our result. The operations of S-box look-up, X-OR and addition modulo 2^{2k} typically occur in the design of block and stream ciphers. We consider two possible ways of combining these operations. The first method is the situation where an S-box is applied to the X-OR of two outputs of the application of the S-box to two input bit strings. The second method considers the situation where an S-box is combined with addition modulo 2^{2k}. In both cases, we obtain complete expressions for the Walsh transform.

2 Preliminaries

Let $\mathbb{F}_2 = GF(2)$ be the finite field of two elements. We consider the domain of an n-variable Boolean function to be the vector space $(\mathbb{F}_2^n, \oplus)$ over $\mathbb{F}_2$, where $\oplus$ is used to denote the addition operator over both $\mathbb{F}_2$ and the vector space $\mathbb{F}_2^n$. The inner product of two vectors $u, v \in \mathbb{F}_2^n$ will be denoted by $\langle u, v\rangle$. We will denote the weight of a binary string x by $\mathsf{wt}(x)$.

An n-variable Boolean function is a map $f : \mathbb{F}_2^n \to \mathbb{F}_2$. The Walsh Transform of an n-variable Boolean function f is an integer valued function $W_f : \mathbb{F}_2^n \to [-2^n, 2^n]$ defined by (see [4, page 414])

$$W_f(u) = \sum_{w\in\mathbb{F}_2^n} (-1)^{f(w)\oplus\langle u,w\rangle}. \tag{1}$$

The Walsh Transform is called the spectrum of f. The inverse Walsh Transform is given by

$$(-1)^{f(u)} = \frac{1}{2^n} \sum_{w\in\mathbb{F}_2^n} W_f(w)(-1)^{\langle u,w\rangle}. \tag{2}$$

An (n, m) S-box (or vectorial function) is a map $g : \{0,1\}^n \to \{0,1\}^m$. Let $g : \{0,1\}^n \to \{0,1\}^m$ be an S-box and $f : \{0,1\}^m \to \{0,1\}$ be an m-variable Boolean function. The composition of f and g, denoted by $f \circ g$ is an n-variable Boolean function defined by $(f \circ g)(x) = f(g(x))$.

Linear cryptanalysis [5] is a very powerful cryptanalytic method for block ciphers. The study of correlation between linear combinations of input and output of an S-box is therefore very important. If two functions are highly correlated, then they are “close” to each other and can be approximated one for the other. The correlation between two n-variable Boolean functions f and g is defined in the following manner (see for example [8]).

$$c(f, g) = 2^{-n} \sum_{x\in\mathbb{F}_2^n} (-1)^{f(x)\oplus g(x)}. \tag{3}$$

We have the following relationship $c(f, g) = 2^{-n} W_{f\oplus g}(0)$ and $c(f, l_v) = 2^{-n} W_f(v)$, where l_v is the linear function defined as $l_v(x) = \langle v, x\rangle$ for $x \in \mathbb{F}_2^n$. Thus correlation between a Boolean function and a linear function can be conveniently studied using Walsh transform analysis.

3 Convolution and Composition Theorems for S-Boxes

The Convolution for two n-variable Boolean functions is well known (see for example [1]).

Theorem 1 (Convolution Theorem) *Let f and g be n-variable Boolean functions and $h(x) = g(x) \oplus f(x)$. Then*

$$W_h(u) = \frac{1}{2^n} \sum_{v\in\mathbb{F}_2^n} W_g(v) W_f(v \oplus u). \tag{4}$$

We next prove a generalization of the Convolution Theorem.

Theorem 2 (Generalized Convolution Theorem) *Let $g_1, \ldots, g_k$ be n-variable Boolean functions and $h(x) = g_1(x) \oplus \cdots \oplus g_k(x)$. Then for $u \in \mathbb{F}_2^n$*

$$\begin{aligned} W_h(u) &= \frac{1}{2^s} \sum_{v \in \mathbb{F}_2^s} W_{g_k}(u \oplus u_{k-1}) W_{g_{k-1}}(u_{k-1} \oplus u_{k-2}) \cdots W_{g_2}(u_2 \oplus u_1) W_{g_1}(u_1) \quad (5) \\ &= \frac{1}{2^s} \sum_{v \in \mathbb{F}_2^s} \prod_{i=1}^{k} W_{g_i}(u_i \oplus u_{i-1}) \quad (6) \end{aligned}$$

where $s = n(k-1)$, $v = (u_1, \ldots, u_{k-1})$, $u_k = u$ and $u_0 = (0, \ldots, 0)$ with each $u_i \in \mathbb{F}_2^n$.

Proof : We prove the above Theorem by induction on k. For $k = 2$, the result follows from the Convolution Theorem. Assume that result holds for $(k-1) \geq 2$. We now apply the Convolution Theorem on the functions $g_k(x)$ and $f(x) = g_1(x) \oplus \cdots \oplus g_{k-1}(x)$. This gives

$$W_h(u) = \frac{1}{2^n} \sum_{u_{k-1} \in \mathbb{F}_2^n} W_f(u_{k-1}) W_{g_k}(u \oplus u_{k-1}) = \frac{1}{2^n} \sum_{u_{k-1} \in \mathbb{F}_2^n} W_f(u_{k-1}) W_{g_k}(u_k \oplus u_{k-1})$$

Now we invoke the induction hypothesis for $(k-1)$ on the function f to get

$$\begin{aligned} W_h(u) &= \frac{1}{2^n} \sum_{u_{k-1} \in \mathbb{F}_2^n} W_{g_k}(u_k \oplus u_{k-1}) \left(\frac{1}{2^{n(k-2)}} \sum_{(u_1, \ldots, u_{k-2})} \prod_{i=1}^{k-1} W_{g_i}(u_i \oplus u_{i-1}) \right) \\ &= \frac{1}{2^s} \sum_{v \in \mathbb{F}_2^s} \prod_{i=1}^{k} W_{g_i}(u_i \oplus u_{i-1}). \end{aligned}$$

This proves the result. ■

Now we provide the Walsh Transform of composition of an S-box and a Boolean function. A similar result is stated in [1] in terms of correlation matrices.

Theorem 3 (Composition Theorem) *Let $g : \{0,1\}^n \to \{0,1\}^m$ and $f : \{0,1\}^m \to \{0,1\}$. Then for any $w \in \mathbb{F}_2^n$,*

$$W_{(f \circ g)}(w) = \frac{1}{2^m} \sum_{v \in \mathbb{F}_2^m} W_f(v) W_{(l_v \circ g)}(w)$$

where $(l_v \circ g)(x) = \langle v, g(x) \rangle$.

Proof : From the inverse Walsh transform (2) we know $(-1)^{f(x)} = \frac{1}{2^m} \sum_{v \in \mathbb{F}_2^m} W_f(v) (-1)^{\langle v, x \rangle}$. Let $y = g(x)$. Then

$$\begin{aligned} (-1)^{(f \circ g)(x)} = (-1)^{f(g(x))} = (-1)^{f(y)} &= \frac{1}{2^m} \sum_{v \in \mathbb{F}_2^m} W_f(v) (-1)^{\langle v, y \rangle} \\ &= \frac{1}{2^m} \sum_{v \in \mathbb{F}_2^m} W_f(v) (-1)^{\langle v, g(x) \rangle} \\ &= \frac{1}{2^m} \sum_{v \in \mathbb{F}_2^m} W_f(v) (-1)^{(l_v \circ g)(x)}. \end{aligned}$$

So we have

$$\begin{aligned} W_{f\circ g}(w) &= \sum_{x\in \mathbb{F}_2^n} (-1)^{(f\circ g)(x)\oplus\langle w,x\rangle} \\ &= \frac{1}{2^m}\sum_{x\in \mathbb{F}_2^n}\sum_{v\in \mathbb{F}_2^m} W_f(v)(-1)^{(l_v\circ g)(x)\oplus\langle w,x\rangle} \\ &= \frac{1}{2^m}\sum_{v\in \mathbb{F}_2^m} W_f(v)\sum_{x\in \mathbb{F}_2^n}(-1)^{(l_v\circ g)(x)\oplus\langle w,x\rangle} \\ &= \frac{1}{2^m}\sum_{v\in \mathbb{F}_2^m} W_f(v)W_{(l_v\circ g)}(w). \end{aligned}$$

This proves the result. ■

4 Correlation Theorem

In this section we prove the main correlation theorem. Let $g_1,\ldots,g_t$ be S-boxes, where for $1\le i\le t$, $g_i:\mathbb{F}_2^{m_i+m_{i+1}}\to\mathbb{F}_2^{n_i}$. Let $f:\mathbb{F}_2^n\to\mathbb{F}_2$ be a Boolean function, where $n=n_1+\cdots+n_t$. Let $m=m_1+\cdots+m_{t+1}$ and define a Boolean function $h:\mathbb{F}_2^m\to\mathbb{F}_2$ in the following manner.

$$h(x_1,\ldots,x_{t+1}) \quad=\quad f(g_1(x_1,x_2),g_2(x_2,x_3),\ldots,g_t(x_t,x_{t+1})) \tag{7}$$

where $x_i\in\mathbb{F}_2^{m_i}$ for $1\le i\le t+1$. Our task in this section is to compute $W_h(u)$ for $u\in\mathbb{F}_2^m$. In Theorem 7(see later) we show that $W_h(w)$ is given by the following expression.

$$W_h(w) \quad=\quad \frac{2^{m_1+m_{t+1}}}{2^{n+m}}\sum_{v\in\mathbb{F}_2^n} W_f(v)\sum_{u\in\mathbb{F}_2^M}\prod_{i=1}^{t} W_{(l_{v_i}\circ g_i)}(w_{t,i}\oplus w_{i-1,i},w_{i,i+1})$$

where $v=(v_1,\ldots,v_t)$ with $v_i\in\mathbb{F}_2^{n_i}$, $M=m-m_1-m_{t+1}$, $w=w_t=(w_{t,1},\ldots,w_{t,t+1})$ with $w_{t,i}\in\mathbb{F}_2^{m_i}$, $u=(w_{1,2},\ldots,w_{t-1,t})$ and for $1\le i\le t$, $w_{i-1,i}\in\mathbb{F}_2^{m_i}$ with $w_{0,1}=(0,\ldots,0)$.
This result is obtained through a series of simplifications. Let $g^{'}:\mathbb{F}_2^m\to\mathbb{F}_2^n$ be an S-box defined by

$$g^{'}(x_1,\ldots,x_{t+1}) \quad=\quad (g_1(x_1,x_2),\ldots,g_t(x_t,x_{t+1})). \tag{8}$$

Using the Composition Theorem we have the following result.

Proposition 4 *Let h be defined by (7). Then*

$$W_h(u)=\frac{1}{2^n}\sum_{v\in\mathbb{F}_2^n} W_f(v)W_{(l_v\circ g^{'})}(u).$$

For $v\in\mathbb{F}_2^n$, write $v=(v_1,\ldots,v_t)$, where $v_i\in\mathbb{F}_2^{n_i}$. We can write

$$\begin{aligned} (l_v\circ g^{'})(x_1,\ldots,x_{t+1})=l_v(g^{'}(x_1,\ldots,x_{t+1}) &= l_v((g_1(x_1,x_2),\ldots,g_t(x_t,x_{t+1})) \\ &= l_{v_1}(g_1(x_1,x_2))\oplus\cdots\oplus l_{v_t}(g_t(x_t,x_{t+1})) \\ &= (l_{v_1}\circ g_1)(x_1,x_2))\oplus\cdots\oplus(l_{v_t}\circ g_t)(x_t,x_{t+1})). \end{aligned} \tag{9}$$

For $1\le i\le t$, we define

$$h_i^{'}(x_1,\ldots,x_{t+1}) \quad=\quad (l_{v_i}\circ g_i)(x_i,x_{i+1}). \tag{10}$$

Given bit strings $u_1,\ldots,u_k$ we define $\delta(u_1,\ldots,u_k)=1$ if each $u_1,\ldots,u_k$ are all-zero bit strings; otherwise $\delta(u_1,\ldots,u_k)=0$. The Walsh transform of $h_i^{'}$ is given by the following proposition.

Proposition 5 *Let $h_i^{'}$ be defined by (10). Then*

$$W_{h_i^{'}}(u) = 2^{M_i} W_{l_{v_i} \circ g_i}(u_i, u_{i+1}) \delta(\omega_i)$$

where $u = (u_1, \ldots, u_{t+1})$, $M_i = m - m_i - m_{i+1}$ and $\omega_i = (u_1, \ldots, u_{i-1}, u_{i+2}, \ldots, u_{t+1})$.

Proof : We compute as follows.

$$\begin{aligned}
W_{h_i^{'}}(u) &= \sum_{x \in \mathbb{F}_2^m} (-1)^{h_i^{'}(x) \oplus \langle u, x \rangle} \\
&= \sum_{x \in \mathbb{F}_2^m} (-1)^{l_{v_i}(g_i(x_i, x_{i+1})) \oplus \langle (u_1, \ldots, u_{t+1}), (x_1, \ldots, x_{t+1}) \rangle} \\
&= \sum_{y_i \in \mathbb{F}_2^{M_i}} (-1)^{\langle \omega_i, y_i \rangle} \times \sum_{(x_i, x_{i+1}) \in \mathbb{F}_2^{m_i + m_{i+1}}} (-1)^{l_{v_i}(g_i(x_i, x_{i+1})) \oplus \langle (u_i, u_{i+1}), (x_i, x_{i+1}) \rangle} \\
&= W_{(l_{v_i} \circ g_i)}(u_i, u_{i+1}) \sum_{y_i \in \mathbb{F}_2^{M_i}} (-1)^{\langle \omega_i, y_i \rangle} \\
&= 2^{M_i} W_{(l_{v_i} \circ g_i)}(u_i, u_{i+1}) \delta(\omega_i)
\end{aligned}$$

where $y_i = (x_1, \ldots, x_{i-1}, x_{i+2}, \ldots, x_{t+1})$. The last statement follows from the fact that $\sum_{z \in \mathbb{F}_2^r} (-1)^{\langle w, z \rangle} = 2^{|w|} \delta(w)$. This completes the proof. ■

Now we obtain the Walsh transform of $(l_v \circ g^{'})$.

Lemma 6 *Let $g^{'}$ be defined as in (8). Then*

$$W_{(l_v \circ g^{'})}(w) = \frac{2^{m_1 + m_{t+1}}}{2^m} \sum_{u \in \mathbb{F}_2^M} \prod_{i=1}^{t} W_{(l_{v_i} \circ g_i)}(w_{t,i} \oplus w_{i-1,i}, w_{i,i+1})$$

where $M = m - m_1 - m_{t+1}$, $w = w_t = (w_{t,1}, \ldots, w_{t,t+1})$ with $w_{t,i} \in \mathbb{F}_2^{m_i}$, $u = (w_{1,2}, \ldots, w_{t-1,t})$ and for $1 \leq i \leq t$, $w_{i-1,i} \in \mathbb{F}_2^{m_i}$ with $w_{0,1} = (0, \ldots, 0)$.

Proof : Using (9), (10) and the Generalized Convolution Theorem, we have

$$W_{(l_v \circ g^{'})}(w_t) = \frac{1}{2^{m(t-1)}} \sum_{(w_1, \ldots, w_{t-1})} \prod_{i=1}^{t} W_{h_i^{'}}(w_i \oplus w_{i-1})$$

where $w_i \in \mathbb{F}_2^m$ for $0 \leq i \leq t-1$ and $w_0 = (0, \ldots, 0)$. For $0 \leq i \leq t$, write $w_i = (w_{i,1}, \ldots, w_{i,t+1})$ with $w_{i,j} \in \mathbb{F}_2^{m_j}$. Set $\alpha_{i,j} = w_{i,j} \oplus w_{i-1,j}$ and $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,t+1}) = w_i \oplus w_{i-1}$.
Set $\beta_i^t = (\alpha_{i,1}, \ldots, \alpha_{i,i-1}, \alpha_{i,i+2}, \ldots, \alpha_{i,t+1})$. Using Proposition 5, we obtain

$$\begin{aligned}
W_{(l_v \circ g^{'})}(w_t) &= \frac{1}{2^{m(t-1)}} \sum_{(w_1, \ldots, w_{t-1})} \prod_{i=1}^{t} W_{h_i^{'}}(\alpha_i) \\
&= \frac{1}{2^{m(t-1)}} \sum_{(w_1, \ldots, w_{t-1})} \prod_{i=1}^{t} 2^{M_i} W_{l_{v_i} \circ g_i}(\alpha_{i,i}, \alpha_{i,i+1}) \delta(\beta_i^t) \\
&= \frac{2^{m_1 + m_{t+1}}}{2^m} \sum_{(w_1, \ldots, w_{t-1})} \prod_{i=1}^{t} W_{l_{v_i} \circ g_i}(\alpha_{i,i}, \alpha_{i,i+1}) \delta(\beta_i^t)
\end{aligned}$$

The proof now follows from the following claim.
Claim: For $t \geq 2$, we have

$$\sum_{(w_1,\ldots,w_{t-1})} \prod_{i=1}^{t} W_{l_{v_i}\circ g_i}(\alpha_{i,i}, \alpha_{i,i+1})\delta(\beta_i^t) \quad = \quad \sum_{(w_{1,2},\ldots,w_{t-1,t})} \prod_{i=1}^{t} W_{(l_{v_i}\circ g_i)}(w_{t,i} \oplus w_{i-1,i}, w_{i,i+1}) \qquad (11)$$

Proof : The claim is proved by induction on $t \geq 2$. Let $\mathcal{L}_t$ (resp. $\mathcal{R}_t$) be the left (resp. right) side of (11).
Base : Case $t = 2$. In this case, the left side becomes

$$\mathcal{L}_2 = \sum_{w_1} W_{l_{v_1}\circ g_1}(\alpha_{1,1}, \alpha_{1,2})\delta(\beta_1^2) W_{l_{v_2}\circ g_2}(\alpha_{2,2}, \alpha_{2,3})\delta(\beta_2^2).$$

Note that $\beta_1^2 = \alpha_{1,3} = w_{1,3} \oplus w_{0,3} = w_{1,3}$ and $\beta_2^2 = \alpha_{2,1} = w_{2,1} \oplus w_{1,1}$. Substituting the value of α's, we write the above expression as

$$\begin{aligned}
\mathcal{L}_2 &= \sum_{w_{1,1}} \sum_{w_{1,2}} \sum_{w_{1,3}} W_{l_{v_1}\circ g_1}(w_{1,1}, w_{1,2}) W_{l_{v_2}\circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3} \oplus w_{1,3})\delta(w_{1,3})\delta(w_{2,1} \oplus w_{1,1}) \\
&= \sum_{w_{1,1}} \sum_{w_{1,2}} W_{l_{v_1}\circ g_1}(w_{1,1}, w_{1,2}) W_{l_{v_2}\circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3})\delta(w_{2,1} \oplus w_{1,1}) \\
&= \sum_{w_{1,2}} \sum_{w_{1,1}} W_{l_{v_1}\circ g_1}(w_{1,1}, w_{1,2}) W_{l_{v_2}\circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3})\delta(w_{2,1} \oplus w_{1,1}) \\
&= \sum_{w_{1,2}} W_{l_{v_1}\circ g_1}(w_{2,1}, w_{1,2}) W_{l_{v_2}\circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3})
\end{aligned}$$

which is equal to $\mathcal{R}_2$, i.e., the right side of the claim for $t = 2$.
Induction hypothesis : Assume that the claim is true for $t-1$.
Induction step : We now prove the result for t. We will use the induction hypothesis in this case. The main difficulty arises from the fact that the length of the w's and the α's increases by one. We have to take care of this while applying the induction hypothesis. First note that for $1 \leq i \leq t-1$, we have $\delta(\beta_i^t) = \delta(\beta_i^{t-1})\delta(\alpha_{i,t+1})$. Now we compute

$$\begin{aligned}
\mathcal{L}_t &= \sum_{(w_1,\ldots,w_{t-2})} \prod_{i=1}^{t-1} W_{l_{v_i}\circ g_i}(\alpha_{i,i}, \alpha_{i,i+1})\delta(\beta_i^t) \sum_{w_{t-1}} W_{l_{v_t}\circ g_t}(\alpha_{t,t}, \alpha_{t,t+1})\delta(\beta_t^t) \\
&= \sum_{(w_1,\ldots,w_{t-2})} \prod_{i=1}^{t-1} W_{l_{v_i}\circ g_i}(\alpha_{i,i}, \alpha_{i,i+1})\delta(\beta_i^{t-1}) \\
&\quad \times \sum_{(w_{1,t+1},\ldots,w_{t-2,t+1})} \delta(\alpha_{1,t+1})\ldots\delta(\alpha_{t-2,t+1}) \\
&\quad \times \sum_{w_{t-1}} W_{l_{v_t}\circ g_t}(\alpha_{t,t}, \alpha_{t,t+1})\delta(\beta_t^t) \\
&= \sum_{(w_1,\ldots,w_{t-2})} \prod_{i=1}^{t-1} W_{l_{v_i}\circ g_i}(\alpha_{i,i}, \alpha_{i,i+1})\delta(\beta_i^{t-1}) \\
&\quad \times \sum_{(w_{t-1,1},\ldots,w_{t-1,t})} \delta(\alpha_{t,1})\delta(\alpha_{t,2})\ldots\delta(\alpha_{t,t-1}) \\
&\quad \times \sum_{(w_{1,t+1},\ldots,w_{t-2,t+1},w_{t-1,t+1})} W_{l_{v_t}\circ g_t}(\alpha_{t,t}, \alpha_{t,t+1})\delta(\alpha_{1,t+1})\ldots\delta(\alpha_{t-2,t+1}).
\end{aligned}$$

Note that $\alpha_{1,t+1} = w_{1,t+1} \oplus w_{0,t+1} = w_{1,t+1}$ and for $i > 1$, $\alpha_{i,t+1} = w_{i,t+1} \oplus w_{i-1,t+1}$. Thus the expression within the last sum evaluates to $W_{l_{v_t} \circ g_t}(\alpha_{t,t}, \alpha_{t,t+1})$ only under the condition $w_{1,t+1} = \cdots = w_{t-2,t+1} = (0, \ldots, 0)$. Also the expression within the second sum evaluates to 1 only under the condition $w_{t,i} = w_{t-1,i}$ for $1 \leq i \leq t-1$. We invoke the induction hypothesis on the first sum to obtain.

$$\begin{aligned}
\mathcal{L}_t &= \sum_{(w_{1,2},\ldots,w_{t-2,t-1})} \prod_{i=1}^{t-1} W_{(l_{v_i} \circ g_i)}(w_{t-1,i} \oplus w_{i-1,i}, w_{i,i+1}) \\
&\quad \times \sum_{w_{t-1,t}} W_{l_{v_t} \circ g_t}(\alpha_{t,t}, \alpha_{t,t+1}) \\
&= \mathcal{R}_t.
\end{aligned}$$

This completes the proof of the claim and the lemma. ■

Finally using Proposition 4 and Lemma 6 we obtain the Walsh transform of W_h.

Theorem 7 *Let h be defined as in (7). Then*

$$W_h(w) = \frac{2^{m_1+m_{t+1}}}{2^{n+m}} \sum_{v \in \mathbb{F}_2^n} W_f(v) \sum_{u \in \mathbb{F}_2^M} \prod_{i=1}^{t} W_{(l_{v_i} \circ g_i)}(w_{t,i} \oplus w_{i-1,i}, w_{i,i+1}) \tag{12}$$

where $v = (v_1, \ldots, v_t)$ with $v_i \in \mathbb{F}_2^{n_i}$, $M = m - m_1 - m_{t+1}$, $w = w_t = (w_{t,1}, \ldots, w_{t,t+1})$ with $w_{t,i} \in \mathbb{F}_2^{m_i}$, $u = (w_{1,2}, \ldots, w_{t-1,t})$ and for $1 \leq i \leq t$, $w_{i-1,i} \in \mathbb{F}_2^{m_i}$ with $w_{0,1} = (0, \ldots, 0)$.

5 Nyberg's Correlation Theorems

In [8], Nyberg stated three correlation theorems – Theorems 3, 4 and 5 – which have important applications to cryptanalysis. Of these, Theorem 4 has been proved in [7] and the other two theorems are proved in [8]. In this section, we show that Nyberg's correlation theorems – Theorem 3 and 5 of [8] – can be obtained as special cases of Theorem 7. First we rewrite Theorem 7 for $t = 2$.

Theorem 8 *Let h be defined as in (7) and $t = 2$. Then*

$$W_h(w_{2,1}, w_{2,2}, w_{2,3}) = \frac{2^{m_1+m_3}}{2^{m+n}} \sum_{v \in \mathbb{F}_2^n} W_f(v) \sum_{w_{1,2} \in \mathbb{F}_2^{m_2}} W_{l_{v_1} \circ g_1}(w_{2,1}, w_{1,2}) W_{l_{v_2} \circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3}) \tag{13}$$

where $v = (v_1, v_2)$ with $v_i \in \mathbb{F}_2^{n_i}$.

A special case is obtained when f is the linear function $f(a_1, \ldots, a_n) = a_1 \oplus \cdots \oplus a_n$. In this case $W_f(1, \ldots, 1) = 2^n$ and $W_f(v) = 0$ for $v \in \mathbb{F}_2^n \setminus \{(1, \ldots, 1)\}$. Also $v_1 = (1, \ldots, 1) \in \mathbb{F}_2^{n_1}$ and $v_2 = (1, \ldots, 1) \in \mathbb{F}_2^{n_2}$. We denote by $\mathbf{1}_k$ the all one vector of length k. When the value of k is clear from the context, we will simply write $\mathbf{1}$. We have the following corollary to Theorem 8.

Corollary 9 *Let $h(x_1, x_2, x_3) = \langle \mathbf{1}_n, (g_1(x_1, x_2), g_2(x_2, x_3)) \rangle$. Then*

$$W_h(w_{2,1}, w_{2,2}, w_{2,3}) = \frac{2^{m_1+m_3}}{2^m} \sum_{w_{1,2} \in \mathbb{F}_2^{m_2}} W_{l_{\mathbf{1}_{n_1}} \circ g_1}(w_{2,1}, w_{1,2}) W_{l_{\mathbf{1}_{n_2}} \circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3}).$$

Substituting $n_1 = n_2 = 1$ we obtain $h(x_1, x_2, x_3) = g_1(x_1, x_2) \oplus g_2(x_2, x_3)$. Further, substituting $w_{2,2} = 0$ in Corollary 9 we obtain the the following result of Nyberg [8] (stated in terms of Walsh transform).

Theorem 10 (Nyberg [8], Theorem 5) *Let* $h(x_1, x_2, x_3) = g_1(x_1, x_2) \oplus g_2(x_2, x_3)$. *Then*

$$W_h(w_{2,1}, 0, w_{2,3}) \;=\; \frac{2^{m_1+m_3}}{2^m} \sum_{w_{1,2} \in \mathbb{F}_2^{m_2}} W_{g_1}(w_{2,1}, w_{1,2}) W_{g_2}(w_{1,2}, w_{2,3}).$$

Now we turn to Theorem 3 of Nyberg [8]. For this we make the following substitution in the definition of h: $m_2 = 0$, $g = g_2$, $g_1(x) = x$ and hence $n_1 = m_1$. Thus h is now of the form $h(x_1, x_3) = f(x_1, g(x_3))$. In this situation, we have

$$\sum_{w_{1,2} \in \mathbb{F}_2^{m_2}} W_{l_{v_1} \circ g_1}(w_{2,1}, w_{1,2}) W_{l_{v_2} \circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3}) \;=\; W_{l_{v_1} \circ g_1}(w_{2,1}) W_{l_{v_2} \circ g_2}(w_{2,3}).$$

Since $g_1(x) = x$, we have $(l_{v_1} \circ g_1)(x) = l_{v_1}(g_1(x)) = l_{v_1}(x) = \langle v_1, x \rangle$ and hence $W_{l_{v_1} \circ g_1}(w_{2,1}) = 2^{m_1} \delta(v_1 \oplus w_{2,1}) = 2^{n_1} \delta(v_1 \oplus w_{2,1})$, since $m_1 = n_1$. Thus $W_{l_{v_1} \circ g_1}(w_{2,1}) = 2^{n_1}$ if $v_1 = w_{2,1}$ and is equal to 0 otherwise. Let the right side of the (13) be $\mathcal{A}$. In this case $\mathcal{A}$ becomes

$$\begin{aligned}
\mathcal{A} &= \frac{1}{2^{n_1+n_2}} \sum_{v \in \mathbb{F}_2^n} W_f(v) \sum_{w_{1,2} \in \mathbb{F}_2^{m_2}} W_{l_{v_1} \circ g_1}(w_{2,1}, w_{1,2}) W_{l_{v_2} \circ g_2}(w_{2,2} \oplus w_{1,2}, w_{2,3}) \\
&= \frac{1}{2^{n_1+n_2}} \sum_{(v_1, v_2) \in \mathbb{F}_2^n} W_f(v_1, v_2) W_{l_{v_1} \circ g_1}(w_{2,1}) W_{l_{v_2} \circ g_2}(w_{2,3}) \\
&= \frac{1}{2^{n_2}} \sum_{v_2 \in \mathbb{F}_2^{n_2}} W_f(w_{2,1}, v_2) W_{l_{v_2} \circ g}(w_{2,3}).
\end{aligned}$$

Now we have the following result of Nyberg [8] again stated in terms of Walsh transform.

Theorem 11 (Nyberg [8], Theorem 3) *Let* $h(x_1, x_3) = f(x_1, g(x_3))$. *Then*

$$W_h(w_{2,1}, w_{2,3}) \;=\; \frac{1}{2^{n_2}} \sum_{v_2 \in \mathbb{F}_2^{n_2}} W_f(w_{2,1}, v_2) W_{l_{v_2} \circ g}(w_{2,3}).$$

6 Applications

In this section, we consider two other applications of Theorem 7. These are based on operations which typically occur in design of symmetric ciphers, namely S-box look-up, addition modulo 2^{2k} and the X-OR operation. We consider two possible ways of combining these operations and obtain the Walsh transform in each case.

6.1 Brick Layering

In this section, we consider a map of the form

$$h(x, y, z) \;=\; g(g(x, y) \oplus g(y, z)). \tag{14}$$

Figure 1 gives a diagrammatic view of the map. The term brick layering was used in [1] to denote a map which consists of several parallel applications of different S-boxes on disjoint subsets of the inputs which also form a partition of the input. In our case the X-OR and the second application of g is used to “glue” the outputs of the first two applications of g. In block cipher applications g is usually a $2k$-bit to $2k$-bit S-box possibly the inverse function over $GF(2^{2k})$. Hence we assume that g is an $2k$-bit to $2k$-bit map and x, y and

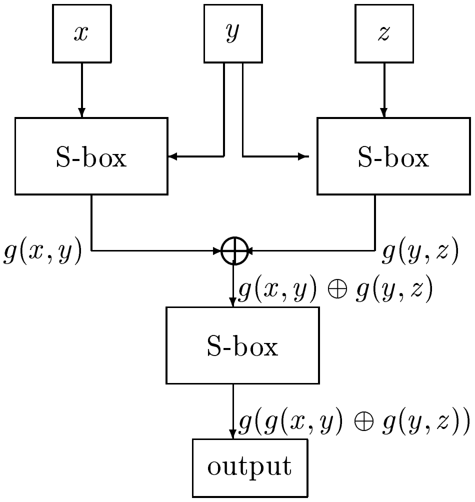

Figure 1: Brick layering transformation.

z are all k-bit strings. Let $g_1, \cdots, g_{2k}$ be the component functions of g. Let $\nu_i(x,y,z) = g_i(x,y) \oplus g_i(y,z)$ and $\nu = (\nu_1, \ldots, \nu_{2k})$. Applying Corollary 9 with $n_1 = n_2 = 1$ we obtain

$$W_{\nu_i}(\delta_1, \delta_2, \delta_3) \quad = \quad \frac{1}{2^k} \sum_{u \in \mathbb{F}_2^k} W_{g_i}(u \oplus \delta_2, \delta_3) W_{g_i}(u, \delta_1) \tag{15}$$

Let $h_1, \ldots, h_{2k}$ be the component functions of h. We have $h_i(x,y,z) = g_i(\nu_1(x,y,z), \ldots, \nu_{2k}(x,y,z))$. Now applying the Composition Theorem we get for $(\gamma_1, \gamma_2, \gamma_3) \in \{0,1\}^{3k}$,

$$W_{h_i}(\gamma_1, \gamma_2, \gamma_3) = \frac{1}{2^{2k}} \sum_{v \in \mathbb{F}_2^{2k}} W_{g_i}(v) W_{(l_v \circ \nu)}(\gamma_1, \gamma_2, \gamma_3) \tag{16}$$

The next step is to compute the Walsh transform of $(l_v \circ \nu)$. Let v be of weight r (i.e., $\mathsf{wt}(v) = r$) with the bits in the $j_1, \ldots, j_r$th positions to be 1 and all others to be 0. Then $(l_v \circ \nu)(x,y,z) = \mu_1(x,y,z) \oplus \cdots \oplus \mu_r(x,y,z)$, where $\mu_i = \nu_{j_i}$ for $1 \leq i \leq r$. Further, let $s = 3k(\mathsf{wt}(v) - 1)$, $w = (u_1, \ldots, u_{r-1})$, with each $u_i \in \mathbb{F}_2^{3k}$, $u_0 = (0, \ldots, 0) \in \mathbb{F}_2^{3k}$ and $u_r = (\gamma_1, \gamma_2, \gamma_3)$. Now applying the Generalized Convolution Theorem, we have

$$W_{(l_v \circ \nu)}(\gamma_1, \gamma_2, \gamma_3) \quad = \quad \frac{1}{2^s} \sum_{w \in \mathbb{F}_2^s} \prod_{i=1}^{r} W_{\mu_i}(u_i \oplus u_{i-1}) = \frac{1}{2^s} \sum_{w \in \mathbb{F}_2^s} \prod_{i=1}^{r} W_{\nu_{j_i}}(u_i \oplus u_{i-1}).$$

Substituting in (16) and using (15) we obtain the Walsh transform for any component function of the map defined in (14).

Theorem 12 *Let h be an S-box defined as in (14) and let $h_1, \ldots, h_{2k}$ be its component functions. Then*

$$\begin{aligned} W_{h_i}(\gamma_1, \gamma_2, \gamma_3) \quad &= \quad \frac{1}{2^{2k}} \sum_{v \in \mathbb{F}_2^{2k}} W_{g_i}(v) \frac{1}{2^{s+kr}} \\ &\times \sum_{(u_1, \ldots, u_{r-1})} \prod_{i=1}^{r} \sum_{w \in \mathbb{F}_2^k} W_{g_{j_i}}(w \oplus u_{i,2} \oplus u_{i-1,2}, u_{i,3} \oplus u_{i-1,3}) W_{g_{j_i}}(w, u_{i,1} \oplus u_{i-1,1}) \end{aligned}$$

where $u_i = (u_{i,1}, u_{i,2}, u_{i,3})$ *with* $u_{i,j} \in \mathbb{F}_2^k$ *and for each* $v \in \mathbb{F}_2^{2k}$*,* $r = \mathsf{wt}(v)$*,* $u_r = (\gamma_1, \gamma_2, \gamma_3)$*.*

Theorem 12 provides the complete expression for the Walsh transform of any h_i.

6.2 Substitute-and-Add

In this section, we consider the map obtained by alternate application of an S-box and sum modulo 2^{2k}. More precisely, we consider the following map.

$$h(x,y) \quad = \quad g(x)\boxed{+}g(y). \tag{17}$$

The map is shown diagrammatically in Figure 2. We obtain the complete Walsh transform of each component function of h. As before we consider g to be an $2k$-bit to $2k$-bit S-box whose component functions are $g_1, \dots, g_{2k}$.

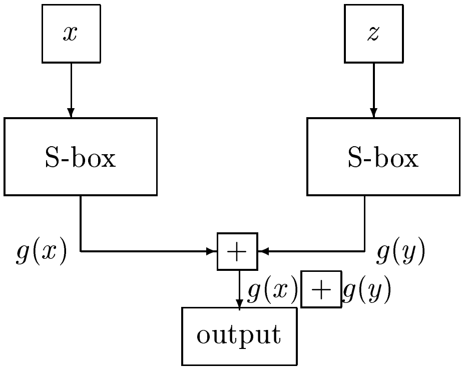

Figure 2: Substitute and add.

For the sake of convenience, we will denote $\nu(x,y) = x\boxed{+}y$. The map ν is conveniently described by separating the carry part. Suppose $(a_1, \dots, a_{2k})$ and $(b_1, \dots, b_{2k})$ are the inputs to $\boxed{+}$. Then the carry in the ith position is given by the function $c(a_1, \dots, a_i, b_1, \dots, b_i)$ and $\nu_i = a_i \oplus b_i \oplus c_i$. The Walsh transform of the carry function has been described in [11]. Let $h_1, \dots, h_{2k}$ be the component functions of h. We can now write

$$h_i(x,y) \quad = \quad g_i(x) \oplus g_i(y) \oplus c_i(g_1(x), \dots, g_i(x), g_1(y), \dots, g_i(y)).$$

Let $f_i(x,y) = c_i(g_1(x), \cdots, g_i(x), g_1(y), \cdots, g_i(y))) = c_i(\mu_i(x,y))$ where $\mu_i : \{0,1\}^{4k} \rightarrow \{0,1\}^{2i}$ is defined as $\mu_i(x,y) = (g_1(x), \cdots, g_i(x), g_1(y), \cdots, g_i(y))$. Applying the Composition Theorem for $(u,v) \in \{0,1\}^{2k}$, we have

$$W_{f_i}(u,v) \quad = \quad \frac{1}{2^{2i}} \sum_{w \in \mathbb{F}_2^{2i}} W_{c_i}(w) W_{(l_w \circ \mu_i)}(u,v)$$

Let $\lambda_i(x,y) = g_i(x) \oplus g_i(y)$ and hence $W_{\lambda_i}(u,v) = W_{g_i}(u)W_{g_i}(v)$. We have $h_i(x,y) = \lambda_i(x,y) \oplus f_i(x,y)$.

Using the convolution Theorem we have

$$\begin{aligned}
W_{h_i}(u,v) &= \frac{1}{2^{4k}} \sum_{(u_1,v_1)\in \mathbb{F}_2^{2k+2k}} W_{\lambda_i}(u_1,v_1) W_{f_i}(u \oplus u_1, v \oplus v_1) \\
&= \frac{1}{2^{4k}} \sum_{(u_1,v_1)\in \mathbb{F}_2^{2k+2k}} W_{g_i}(u_1) W_{g_i}(v_1) W_{f_i}(u \oplus u_1, v \oplus v_1) \\
&= \frac{1}{2^{4k+2i}} \sum_{(u_1,v_1)\in \mathbb{F}_2^{2k+2k}} W_{g_i}(u_1) W_{g_i}(v_1) \sum_{w \in \mathbb{F}_2^{2i}} W_{c_i}(w) W_{(l_w \circ \mu_i)}(u \oplus u_1, v \oplus v_1).
\end{aligned}$$

Thus we get the following result.

Theorem 13 *Let h be defined as in (17) and $h_1, \ldots, h_{2k}$ be its component functions. Then*

$$W_{h_i}(u,v) = \frac{1}{2^{4k+2i}} \sum_{(u_1,v_1)\in \mathbb{F}_2^{2k+2k}} W_{g_i}(u_1) W_{g_i}(v_1) \sum_{w \in \mathbb{F}_2^{2i}} W_{c_i}(w) W_{(l_w \circ \mu_i)}(u \oplus u_1, v \oplus v_1)$$

where $\mu_i(x,y) = (g_1(x), \cdots, g_i(x), g_1(y), \cdots, g_i(y))$.

The complete expression for $W_{h_i}(u,v)$ is obtained by computing the Walsh transform of $(l_w \circ \mu_i)$. This requires one more invocation of the Convolution Theorem and hence involves another summation.

7 Conclusion

We have proved a result which provides the Walsh transform of a general functional form. As special cases, we obtain two of Nyberg's correlation theorems proved in [8]. We consider two applications of our results. These applications combine S-box look-up with addition modulo 2^{2k} and the X-OR operation. In each case we obtain complete expressions for the Walsh transform. A possible future research problem is to apply our techniques to actual block ciphers.

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References

[1] J. Daemen and V. Rijmen. *The Design of Rijndael: AES* - The Advanced Encryption Standard (Information Security and Cryptography). Spriner-Verlag 2002.

[2] C. Harpes, G. G. Kramer and J. L. Massey. A Generalization of Linear Cryptanalysis and the Applicability of Matsui's Piling-up Lemma. In *Advances in Cryptology - EUROCRYPT 1995*, pages 24–38, Lecture Notes in Computer Science, Springer-Verlag, 1995.

[3] M. Hermelin and K. Nyberg. Correlation Properties of the Bluetooth Combiner Generator. *The Second International Conference on Information Security and Cryptology of ICISC '99*, pages 17–29, Lecture Notes in Computer Science, Springer-Verlag, 2000.

[4] F. J. MacWillams and N. J. A. Sloane. *The Theory of Error Correcting Codes.* North Holland, 1977.

[5] M. Matsui. Linear Cryptanalysis Method for DES Cipher. In *Advances in Cryptology - EUROCRYPT 1993*, pages 386–397, Lecture Notes in Computer Science, Springer-Verlag, 1994.

[6] W. Meier and O. Staffelbach. Correlation Properties of Combiners with Memory in Stream Cipher. J. Cryptology. 5 (1) (1992) 67-86

[7] K. Nyberg. Linear Approximation of Block Ciphers. In *Advances in Cryptology - EUROCRYPT 1994*, pages 439–444, Lecture Notes in Computer Science, Springer-Verlag, 1995.

[8] K. Nyberg. Correlation Theorems in Cryptanalysis. Discrete Applied Mathematics 111 (2001) 177-188.

[9] P. Sarkar and S. Maitra. Cross-Correlation Analysis of Cryptographically Useful Boolean Functions and S-Boxes. Theory of Computing Systems , 35(1): 39-57 (2002).

[10] O. Staffelbach and W. Meier. Cryptographic Significance of the Carry for Ciphers Based on Integer Addition. In *Advances in Cryptology - CRYPT0 1990*, pages 601–615, Lecture Notes in Computer Science, Springer-Verlag, 1991.

[11] J Wallén. Linear Approximation of Addition Modulo 2^n. *Tenth Annual Workshop on Fast Software Encryption* , February 24-26, 2003, AF-Borgen, Lund, Sweden, pages 277–290, Pre-proceedings.