Basic-Concepts-of-Asymptotic-Theory

September 8, 2021

1 DATA 5600: Introduction to Regression and Machine Learning for Analytics

1.1 Review of Basic Concepts in Asymptotic Theory

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
```

[2]:
```
%matplotlib inline
```

1.2 The Law of Large Numbers

Definition

The law which states that the larger a sample, the nearer its mean is to that of the parent population from which the sample is drawn. More formally: for every $\varepsilon > 0$, the probability

$$\{|\bar{Y} - Y| > \varepsilon\} \to 0 \quad \text{as} \quad n \to \infty$$

where n is the sample size, $\bar{Y}$ is the sample mean, and μ is the population mean.

More rigorous definitions are the following:

For i.i.d sequences of one-dimensional random variables $Y_1, Y_2, \ldots$, let $\bar{Y}_n = \frac{1}{n}\sum\limits_{i=1}^{n} Y_i$.

The *weak law of large numbers* states that $\bar{Y}_n$ converges in probability to $\mu = E\{Y_i\}$ if $E\{|Y_i|\} < \infty$.

The *strong law of large numbers* states that $\bar{Y}_n$ converges almost surely to μ if $E\{|Y_i|\} < \infty$.

Both results hold under the more stringent but easily checked condition that $var\{Y_i\} = \sigma^2 < \infty$.

1.2.1 Using Simulation to Check the Law of Large Numbers

We can use simulation to check the Law of Large Numbers. Consider a fair die with six sides and outcomes $Y = \{1, 2, 3, 4, 5, 6\}$, each with $P[Y_i = y] = \frac{1}{6}$. The true mean is

$$\mu = E\{Y\} = \frac{1}{6}[1 + 2 + 3 + 4 + 5 + 6] = 3.5$$

We can verify this in `Python`:

[6]:
```
x = np.arange(1,7)
mu = (1/6) * x.sum()
x
```

[6]:
```
array([1, 2, 3, 4, 5, 6])
```

[7]:
```
mu
```

[7]:
```
3.5
```

Now let's simulate some rolls of the die and collect some data. We will let our sample size increase and plot the estimated mean.

We can simulate a single roll of the die as follows:

[10]:
```
## roll the dice a single time and observe the outcome
np.random.randint(1, 7)
```

[10]:
```
5
```

We can also simulate many draws at once as follows:

[11]:
```
## roll the dice 100 times and observe the outcomes
np.random.randint(1, 7, size=100)
```

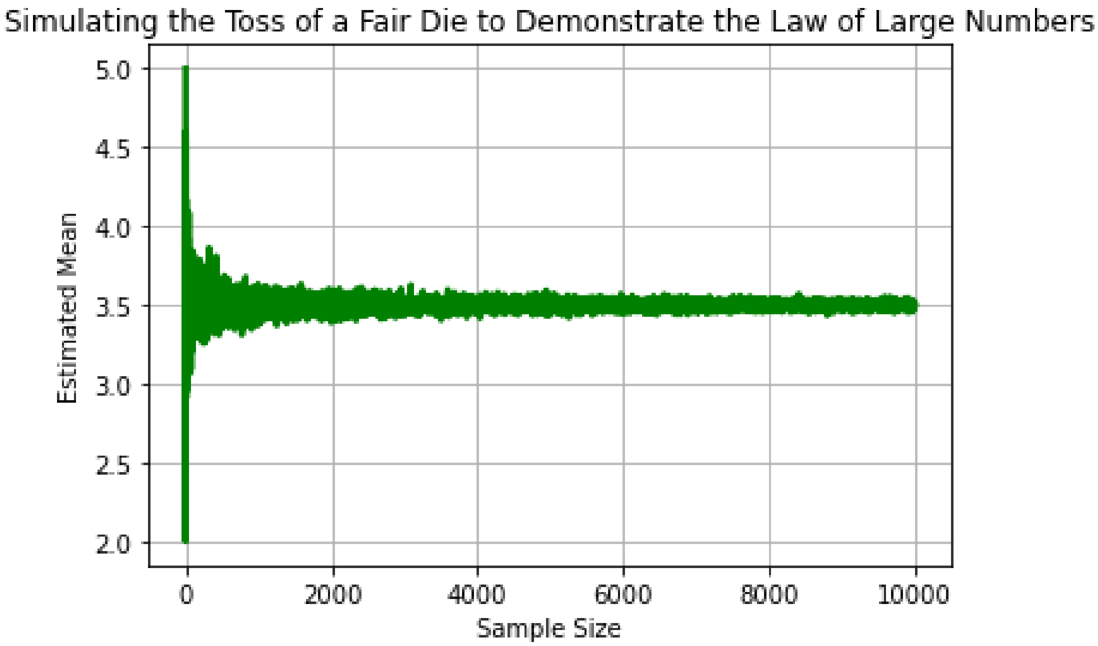
[11]:
```
array([2, 6, 5, 3, 3, 1, 1, 6, 4, 5, 5, 1, 3, 6, 4, 1, 1, 3, 2, 4, 4, 2,
       6, 1, 5, 6, 4, 3, 2, 3, 6, 3, 2, 4, 6, 5, 6, 5, 5, 4, 5, 2, 6, 5,
       4, 1, 3, 6, 4, 1, 2, 2, 4, 3, 6, 3, 5, 3, 1, 5, 4, 3, 4, 6, 6, 5,
       2, 6, 1, 2, 4, 5, 2, 1, 5, 6, 1, 2, 5, 1, 1, 1, 5, 3, 2, 5, 2, 3,
       2, 3, 6, 2, 2, 4, 2, 3, 3, 5, 1, 1])
```

[12]:
```
m = 10000
sizes = np.arange(1,m + 1)
means = np.zeros((m,))

for i in range(len(sizes)):
    y = np.random.randint(1,7, size=sizes[i])
    means[i] = y.mean()

plt.plot(means, 'g', lw = 2.5)
plt.grid(True)
plt.title("Simulating the Toss of a Fair Die to Demonstrate the Law of Large␣
 ↪Numbers")
plt.xlabel("Sample Size")
plt.ylabel("Estimated Mean")
```

[12]:
```
Text(0, 0.5, 'Estimated Mean')
```



[13]:
```
## peak at the first 10 estimated means
means[:10]
```

[13]:
```
array([5.        , 2.        , 2.66666667, 3.        , 4.6       ,
       3.16666667, 3.14285714, 4.5       , 3.11111111, 3.5       ])
```

[15]:
```
## peak at the last 10 estimated means
means[-10:]
```

[15]:
```
array([3.48473626, 3.48208567, 3.51816271, 3.50130078, 3.49874937,
       3.50440176, 3.483345  , 3.49089818, 3.50085009, 3.4922    ])
```

We can do a similar simulation for the flipping of a fair coin. We can simulate the flip of a coin with the Binomial distribution as follows:

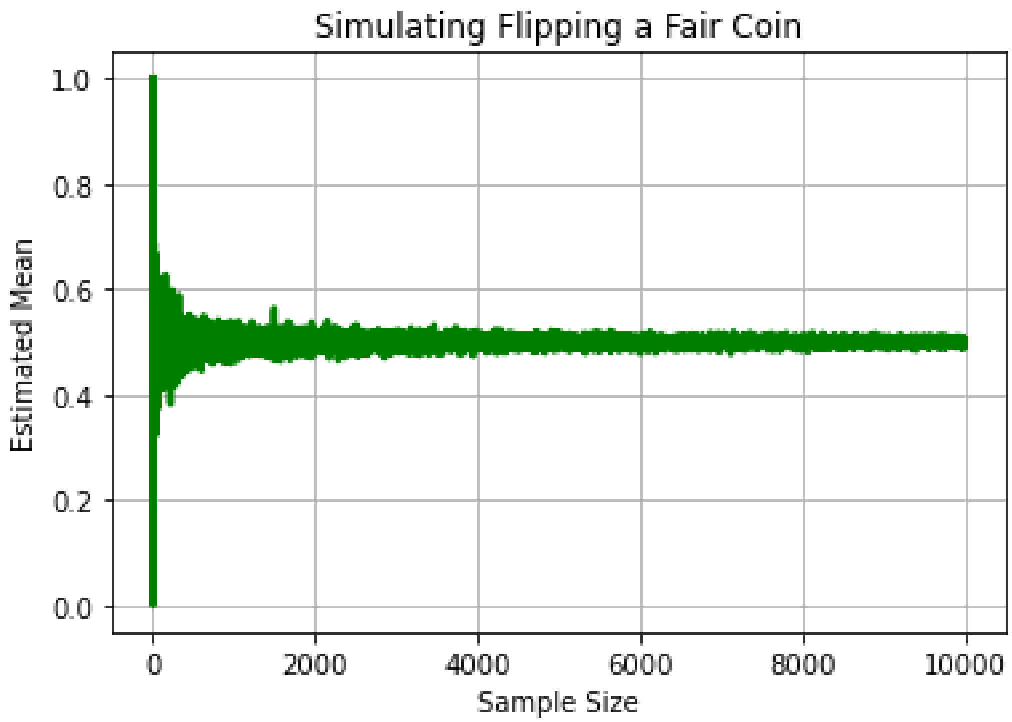
[16]:
```
m = 10000
sizes = np.arange(1,m + 1)
means = np.zeros((m,))

for i in range(len(sizes)):
    y = np.random.binomial(1, 0.5, sizes[i])
    means[i] = y.mean()

plt.plot(means, 'g', lw = 2.5)
plt.grid(True)
plt.title("Simulating Flipping a Fair Coin")
```

```
plt.xlabel("Sample Size")
plt.ylabel("Estimated Mean")
```

[16]: Text(0, 0.5, 'Estimated Mean')



[17]:
```
## peak at the first 10 estimated means
means[:10]
```

[17]:
```
array([1.        , 0.        , 0.33333333, 0.        , 1.        ,
       0.83333333, 0.42857143, 0.375     , 0.77777778, 0.7       ])
```

[18]:
```
## peak at the last 10 estimated means
means[-10:]
```

[18]:
```
array([0.50395356, 0.50110088, 0.50175123, 0.50190114, 0.50325163,
       0.48929572, 0.49024707, 0.50280056, 0.49664966, 0.5029    ])
```

Example: The Exponential Distribution Let's do a simulation for the ***exponential distribution*** together.

See the wikipedia article for details: https://en.wikipedia.org/wiki/Exponential_distribution

[19]:
```
## let's take a single draw from the exponential distribution
np.random.exponential?
```

Docstring:

```
exponential(scale=1.0, size=None)

Draw samples from an exponential distribution.

Its probability density function is

.. math:: f(x; \frac{1}{\beta}) = \frac{1}{\beta} \exp(-\frac{x}{\beta}),

for ``x > 0`` and 0 elsewhere. :math:`\beta` is the scale parameter,
which is the inverse of the rate parameter :math:`\lambda = 1/\beta`.
The rate parameter is an alternative, widely used parameterization
of the exponential distribution [3]_.

The exponential distribution is a continuous analogue of the
geometric distribution.  It describes many common situations, such as
the size of raindrops measured over many rainstorms [1]_, or the time
between page requests to Wikipedia [2]_.

.. note::
    New code should use the ``exponential`` method of a ``default_rng()``
    instance instead; please see the :ref:`random-quick-start`.

Parameters
----------
scale : float or array_like of floats
    The scale parameter, :math:`\beta = 1/\lambda`. Must be
    non-negative.
size : int or tuple of ints, optional
    Output shape.  If the given shape is, e.g., ``(m, n, k)``, then
    ``m * n * k`` samples are drawn.  If size is ``None`` (default),
    a single value is returned if ``scale`` is a scalar.  Otherwise,
    ``np.array(scale).size`` samples are drawn.

Returns
-------
out : ndarray or scalar
    Drawn samples from the parameterized exponential distribution.

See Also
--------
Generator.exponential: which should be used for new code.

References
----------
.. [1] Peyton Z. Peebles Jr., "Probability, Random Variables and
       Random Signal Principles", 4th ed, 2001, p. 57.
.. [2] Wikipedia, "Poisson process",
```

```
       https://en.wikipedia.org/wiki/Poisson_process
.. [3] Wikipedia, "Exponential distribution",
       https://en.wikipedia.org/wiki/Exponential_distribution
Type:      builtin_function_or_method
```

Example: The Poisson Distribution Let's do a simulation for the ***Poisson distribution*** together.

See the wikipedia article for details: https://en.wikipedia.org/wiki/Poisson_distribution

More Examples:

- Normal distribition
- Binomial experiment (number of heads after n tosses)

[ ]:

```
```

[ ]:

```
```

[ ]:

```
```