Public Key Encryption from trapdoor permutations

The RSA trapdoor permutation

Review: trapdoor permutations

Three algorithms: (G, F, F^{-1})

- G: outputs pk, sk. pk defines a function $F(pk, \cdot): X \to X$

- F(pk, x): evaluates the function at x

- $F^{-1}(sk, y)$: inverts the function at y using sk

Secure trapdoor permutation:

The function $F(pk, \cdot)$ is one-way without the trapdoor sk

Review: arithmetic mod composites

Let $N = p \cdot q$ where p,q are prime

$Z_N = \{0,1,2,\ldots,N-1\}$; $(Z_N)^*$ = {invertible elements in Z_N}

<u>Facts</u>: $x \in Z_N$ is invertible $\Leftrightarrow$ $\gcd(x,N) = 1$

– Number of elements in $(Z_N)^*$ is $\varphi(N) = (p-1)(q-1) = N-p-q+1$

<u>Euler's thm</u>: $\boxed{\forall\, x \in (Z_N)^* \;:\; x^{\varphi(N)} = 1}$

The RSA trapdoor permutation

First published: Scientific American, Aug. 1977.

Very widely used:

- SSL/TLS: certificates and key-exchange
- Secure e-mail and file systems

… many others

The RSA trapdoor permutation

G(): choose random primes $p,q \approx 1024$ bits. Set **N=pq**.

choose integers **e , d** s.t. $\mathbf{e \cdot d = 1 \pmod{\varphi(N)}}$

output $pk = (N, e)$, $sk = (N, d)$

F(pk, x): $\mathbb{Z}_N^* \to \mathbb{Z}_N^*$; $\mathbf{RSA(x) = x^e}$ (in Z_N)

$\mathbf{F^{-1}(sk, y)} = y^d$; $y^d = \mathbf{RSA(x)^d} = x^{ed} = x^{k\varphi(N)+1} = \left(x^{\varphi(N)}\right)^k \cdot \mathbf{x} =$

x

The RSA assumption

RSA assumption: RSA is one-way permutation

For all efficient algs. A:

$$\Pr\Big[\ A(N,e,y) = y^{1/e}\ \Big] < \text{negligible}$$

where $p,q \xleftarrow{R}$ n-bit primes, $N \leftarrow pq$, $y \xleftarrow{R} Z_N^*$

Review: RSA pub-key encryption (ISO std)

(E_s, D_s): symmetric enc. scheme providing auth. encryption.

$H: Z_N \to K$ where K is key space of (E_s, D_s)

- **G**(): generate RSA params: $pk = (N,e)$, $sk = (N,d)$
- **E**(pk, m): (1) choose random x in Z_N

  (2) $y \leftarrow RSA(x) = x^e$, $k \leftarrow H(x)$

  (3) output $(y, E_s(k,m))$
- **D**(sk, (y, c)): output $D_s\big(H\big(RSA^{-1}(y)\big), c\big)$

Textbook RSA is insecure

Textbook RSA encryption:

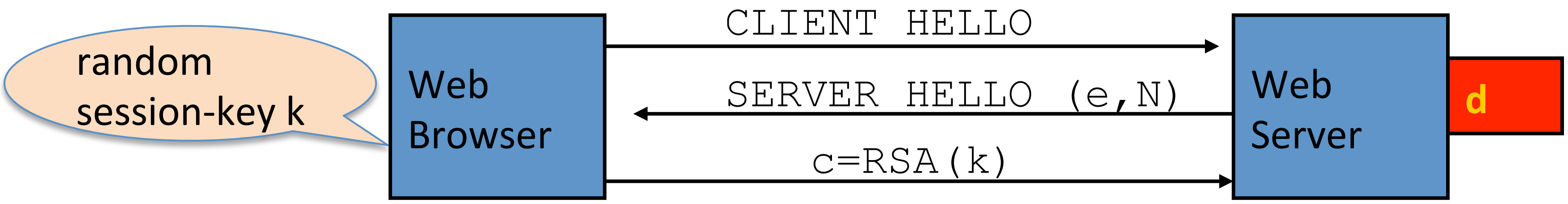
- public key: **(N,e)** Encrypt: $\mathbf{c} \longleftarrow \mathbf{m}^{\mathbf{e}}$ (in Z_N)
- secret key: **(N,d)** Decrypt: $\mathbf{c}^{\mathbf{d}} \longrightarrow \mathbf{m}$

Insecure cryptosystem !!

- Is not semantically secure and many attacks exist

$\Rightarrow$ The RSA trapdoor permutation is not an encryption scheme !

A simple attack on textbook RSA



Suppose k is 64 bits: $k \in \{0,\ldots,2^{64}\}$. Eve sees: $c = k^e$ in Z_N

If $\mathbf{k = k_1 \cdot k_2}$ where $k_1, k_2 < 2^{34}$ (prob. ≈20%) then $\mathbf{c/k_1^e = k_2^e}$ in Z_N

Step 1: build table: $c/1^e, c/2^e, c/3^e, \ldots, c/2^{34e}$. time: 2^{34}

Step 2: for $k_2 = 0,\ldots, 2^{34}$ test if k_2^e is in table. time: 2^{34}

Output matching (k_1, k_2). Total attack time: $\approx 2^{40} \ll 2^{64}$

End of Segment