# Kalman Filter

## 1 First exercise

### Exercise

Consider the following dynamical system:

$$\begin{aligned} x(t+1) &= 2 \cdot x(t) \\ y(t) &= x(t) + v(t) \end{aligned}$$

where $v(t) \sim WN(0, 2)$.

1. Compute the difference Riccati equation (D-DRE)
2. Compute the arithmetic Riccati equation (D-ARE)
3. Explain the result of the arithmetic Riccati equation (D-ARE) considering the given system.

### Solution

**1) Compute the difference Riccati equation (D-DRE):**

Observe that the system is given in the form:

$$\begin{aligned} x(t+1) &= A \cdot x(t) + B \cdot u(t) + D \cdot w(t) \\ y(t) &= C \cdot x(t) + v(t) \end{aligned}$$

Thus:

$$A = 2$$

$$C = 1$$

$$B = 0$$

$$D = 0$$

$$Q = 0$$

$$R = 2$$

Notice that the matrices turn into scalar values, meaning we are dealing with a system with only one hidden variable (dimensione of state variables is 1), that is the univariate Kalman Filter.

From theory we know that the DRE is:

$$P(k+1) = A_k \cdot P(k) \cdot A_k^T - A_k \cdot P(k) \cdot C_k^T \cdot \delta(k)^{-1} \cdot C_k \cdot P(k) \cdot A_k^T + D_k \cdot Q_k \cdot D_k^T$$

Where

$$\delta(k) = C_k \cdot P(k) \cdot C_k^T + R_k$$

In this case we also have that:

$$A_k = A \quad \forall k$$

$$C_k = C \quad \forall k$$
$$D_k = D \quad \forall k$$

$$Q_k = Q \quad \forall k$$

$$R_k = R \quad \forall k$$

Thus:

$$\begin{aligned}
P(k+1) &= A \cdot P(k) \cdot A^T - A \cdot P(k) \cdot C^T \cdot \delta(k)^{-1} \cdot C \cdot P(k) \cdot A^T + D \cdot Q \cdot D^T \\
&= A \cdot P(k) \cdot A^T - A \cdot P(k) \cdot C^T \cdot \delta(k)^{-1} \cdot C \cdot P(k) \cdot A^T \\
&= A^2 \cdot P(k) - \frac{(A \cdot P(k) \cdot C)^2}{C^2 \cdot P(k) + R} \\
&= 4 \cdot P(k) - \frac{(2 \cdot P(k))^2}{P(k) + 2} = 4 \cdot P(k) - \frac{4 \cdot P(k)^2}{P(k) + 2} = \frac{\cancel{4 \cdot P(k)^2} + 8 \cdot P(k) + \cancel{4 \cdot P(k)^2}}{P(k) + 2} \\
&= \frac{8 \cdot P(k)}{P(k) + 2}
\end{aligned}$$

**2) Compute the arithmetic Riccati equation (D-ARE):**

From theory we know that the ARE is when asymptotically:

$$P(k+1) \overset{k\to\infty}{=} P(k) = P$$

Thus the DRE turns into the ARE:

$$P = \frac{8 \cdot P}{P + 2}$$

$$P^2 + 2 \cdot P - 8 \cdot P = 0$$

$$P \cdot (P - 6) = 0$$

The two feasible ($P \geq 0$ since it's a covariance matrix) solutions are:

$$\begin{cases} P = 0 \\ P = 6 \end{cases}$$

**3) Explain the result of the arithmetic Riccati equation (D-ARE):**

Observe that there is no process noise acting on the system. Thus the system dynamics is deterministic. We conclude that the covariance of the error estimate (which embeds information on the distance between the actual state and the estimate) falls into the following two cases:

1. if $P_0 = 0 \quad \overset{k\to\infty}{\Longrightarrow} P = 0$ (the Kalman filter converges). Note that if $P_0 = 0$ the initial state condition is known, thus the estimates will be accurate and the covariance of the error estimate will be always equal to 0.

2. if $P_0 > 0 \quad \overset{k\to\infty}{\Longrightarrow} P = 6$ (the Kalman filter converges). Note that if $P_0 > 0$ the initial state condition is not known, thus the estimates will be innacurate (to a certain degree) and the covariance of the error estimate will be greater than 0.

# 2 Second exercise

## Exercise

Consider the system:

$$\begin{aligned} x_1(t+1) &= 2 \cdot x_1(t) + w_{11}(t) \\ x_2(t+1) &= \alpha \cdot x_1(t) + x_2(t) + w_{12}(t) \\ y(t) &= x_2(t) + v(t) \end{aligned}$$

where $w_{11}(t) \sim WN(0,1)$, $w_{12}(t) \sim WN(0,1)$, $v(t) \sim WN(0,1)$ and $w_{11}(t) \perp w_{12}(t) \perp v(t)$ and $\alpha \in \mathbb{R}$

1. Suppose the Kalman Filter is applied to the system. Establish whether the covariance matrix of the estimation error keeps bounded or not in the case $\alpha = 1$ and in the case $\alpha = 0$.

## Solution

- if $\alpha = 1$:

Observe that the system is given in the form:

$$\begin{aligned} x(t+1) &= A \cdot x(t) + B \cdot u(t) + D \cdot w(t) \\ y(t) &= C \cdot x(t) + v(t) \end{aligned}$$

Let's compute the matrices of the system:

$$\begin{aligned} x_1(t+1) &= 2 \cdot x_1(t) + 0 \cdot x_2(t) + 1 \cdot w_{11}(t) \\ x_2(t+1) &= \alpha \cdot x_1(t) + 1 \cdot x_2(t) + 0 \cdot w_{11}(t) + 1 \cdot w_{12}(t) \\ y(t) &= 0 \cdot x_1(t) + 1 \cdot x_2(t) + v(t) \end{aligned}$$

Thus:

$$A = \begin{bmatrix} 2 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$B = 0$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

with $w(t) = \begin{bmatrix} w_{11}(t) \\ w_{12}(t) \end{bmatrix}$ and $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$.

We also have that the covariance matrix $E$ of the process noise is:

$$\begin{aligned} E = var[D \cdot w(t)] &= \mathbb{E}\left[(D \cdot w(t)) \cdot (D \cdot w(t))^T\right] \\ &= \mathbb{E}\left[D \cdot w(t) \cdot w(t)^T \cdot D^T\right] \\ &= D \cdot \mathbb{E}\left[w(t) \cdot w(t)^T\right] \cdot D^T \\ &= D \cdot var[w(t)] \cdot D^T \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

because:

$$
\begin{aligned}
var\left[w(t)\right] &= \mathbb{E}\left[w(t)\cdot w(t)^T\right] \\
&= \mathbb{E}\left[\begin{bmatrix} w_{11}(t) \\ w_{12}(t)\end{bmatrix}\cdot\begin{bmatrix} w_{11}(t) & w_{12}(t)\end{bmatrix}\right] \\
&= \mathbb{E}\left[\begin{bmatrix} w_{11}(t)^2 & w_{11}(t)\cdot w_{12}(t) \\ w_{11}(t)\cdot w_{12}(t) & w_{12}(t)^2\end{bmatrix}\right] \\
&= \begin{bmatrix} \underbrace{\mathbb{E}\left[w_{11}(t)^2\right]}_{\lambda^2_{w_{11}}} & \underbrace{\cancel{\mathbb{E}\left[w_{11}(t)\cdot w_{12}(t)\right]}}_{w_{11}\perp w_{12}} \\ \underbrace{\cancel{\mathbb{E}\left[w_{11}(t)\cdot w_{12}(t)\right]}}_{w_{11}\perp w_{12}} & \underbrace{\mathbb{E}\left[w_{12}(t)^2\right]}_{\lambda^2_{w_{12}}}\end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}
\end{aligned}
$$

We need to compute the observability matrix $O$ and the reachability matrix $R$. In this case we have that the system has order equal to $n = 2$, thus:

$$
O = \begin{bmatrix} C \\ C\cdot A \\ \vdots \\ C\cdot A^{n-1}\end{bmatrix} = \begin{bmatrix} C \\ C\cdot A\end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1\end{bmatrix} \\ \begin{bmatrix} 0 & 1\end{bmatrix}\cdot\begin{bmatrix} 2 & 0 \\ 1 & 1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1\end{bmatrix}
$$

$$
\begin{aligned}
R &= \begin{bmatrix} E & A\cdot E & \cdots & A^{n-1}\cdot E\end{bmatrix} = \begin{bmatrix} A & A\cdot E\end{bmatrix} \\
&= \begin{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1\end{bmatrix}
\end{aligned}
$$

The observability matrix $O$ and the reachability matrix $R$ have full rank equal to $n = 2$ thus the system is *completely observable* and *completely reachable*.

We can use then the theorem:

Given the system:

$$
\begin{aligned}
x(k+1) &= A\cdot x(k) + B\cdot u(k) + D\cdot w(k) \\
y(k) &= C\cdot x(k) + v(k)
\end{aligned}
$$

with $w(k) \sim WN(0, Q)$ and $v(k) \sim WN(0, R)$. Let $E = D\cdot Q^{\frac{1}{2}} \quad \left(E\cdot E^T = D\cdot Q\cdot D^T\right)$
If:

- $(A, C)$ is *completely observable.*
- $(A, E)$ is *completely reachable.*

Then:

- $\exists \lim_{k\to\infty} P(k) = P = P^T \geq 0, \quad \forall P_0$, with $P$ being the unique solution of the *D-ARE.*
- ...
- ...

We conclude that the covariance matrix $P(k)$ of the estimation error keeps bounded to the value $P$.

- if $\alpha = 0$:

In this case the only change is the system matrix $A$:

$$A = \begin{bmatrix} 2 & 0 \\ \alpha & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

We need to compute again the observability matrix $O$ and the reachability matrix $R$:

$$O = \begin{bmatrix} C \\ C \cdot A \\ \vdots \\ C \cdot A^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ C \cdot A \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} R &= \begin{bmatrix} E & A \cdot E & \cdots & A^{n-1} \cdot E \end{bmatrix} = \begin{bmatrix} A & A \cdot E \end{bmatrix} \\ &= \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \end{aligned}$$

The reachability matrix $R$ have full rank equal to $n = 2$ thus the system is *completely reachable* but the observability matrix $O$ doesn't have full rank, thus the system *is not completely observable.* We can not use then second asymptotic theorem.
Yet by looking out at the system:

$$\begin{aligned} x_1(t+1) &= 2 \cdot x_1(t) + w_{11}(t) \\ x_2(t+1) &= x_2(t) + w_{12}(t) \\ y(t) &= x_2(t) + v(t) \end{aligned}$$

We see that there is no interaction between $x_1(t)$ and $x_2(t)$ (notice that $w_{11}(t) \perp w_{12}(t)$ and there are no direct dependences between $x_1(t)$ and $x_2(t)$).
Moreover $y(t)$ depends on $x_2(t)$ only and $v(t) \perp x_1(t)$. Hence $y(t)$ carries no information on $x_1(t)$ which is totally unpredictable. Since the variance of $x_1(t)$ diverges ($x_1(t+1) = 2 \cdot x_1(t) + w_{11}(t)$ is unstable) we have that the filter is not able to track $x_1(t)$ and the estimation error covariance matrix will diverge.

# 3 Third exercise

## Exercise

Consider the system:

$$
\begin{aligned}
x(t+1) &= \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot w(t) \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(t) + v(t)
\end{aligned}
$$

with $w(t) \sim WN(0,1)$ and $v(t) \sim WN(0,r)$ with $r > 0$.

1. Study the observability of the system.
2. Compute, if possible, the stationary Kalman filter gain and the covariance matrix of the estimation error.
3. Discuss the role of $r$ in the covariance of the estimation error.

## Solution

Observe that the system is given in the form:

$$
\begin{aligned}
x(t+1) &= A \cdot x(t) + B \cdot u(t) + D \cdot w(t) \\
y(t) &= C \cdot x(t) + v(t)
\end{aligned}
$$

with $w(t) \sim WN(0,1)$ and $v(t) \sim WN(0,r)$ with $r > 0$
In this case, we have that:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$B = 0$$

$$D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We also have that the covariance matrix $E$ of the process noise is:

$$
\begin{aligned}
E = var[D \cdot w(t)] &= \mathbb{E}\left[(D \cdot w(t)) \cdot (D \cdot w(t))^T\right] \\
&= \mathbb{E}\left[D \cdot w(t) \cdot w(t)^T \cdot D^T\right] \\
&= D \cdot \mathbb{E}\left[w(t) \cdot w(t)^T\right] \cdot D^T \\
&= D \cdot var[w(t)] \cdot D^T = D \cdot \lambda_w^2 \cdot D^T = D \cdot Q \cdot D^T \\
&= 1 \cdot D \cdot D^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\end{aligned}
$$

while the covariance matrix $R$ of the measurement noise is:

$$R = var[v(t)] = \lambda_v^2 = r$$

**1) Study the observability of the system.**

We know that a linear system is observable if and only if $rank(O) = n$.

In this case we have that the system has order equal to $n = 2$, thus:

$$O = \begin{bmatrix} C \\ C \cdot A \\ \vdots \\ C \cdot A^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ C \cdot A \end{bmatrix}$$

Thus:

$$O = \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

and the system is observable since $rank\,(O) = n = 2$ because $\det\,(O) \neq 0$

**2) Compute, if possible, the stationary Kalman filter gain and the covariance matrix of the estimation error.**

We need to compute the reachability matrix $R$:

$$R = \begin{bmatrix} E & A \cdot E & \cdots & A^{n-1} \cdot E \end{bmatrix} = \begin{bmatrix} A & A \cdot E \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The reachability matrix $R$ have full rank equal to $n = 2$ thus the system is *completely reachable.*

We know that the stationary Kalman filter gain is:

$$L = A \cdot P \cdot C^T \cdot \left(C \cdot P \cdot C^T + R\right)^{-1}$$

while the covariance matrix of the estimation error is given by the *D-ARE*:

$$\begin{aligned} P &= A \cdot P \cdot A^T + D \cdot Q \cdot D^T - A \cdot P \cdot C^T \cdot \left(C \cdot P \cdot C^T + R\right)^{-1} \cdot C \cdot P \cdot A^T \\ &= A \cdot P \cdot A^T + E - A \cdot P \cdot C^T \cdot \left(C \cdot P \cdot C^T + R\right)^{-1} \cdot C \cdot P \cdot A^T \end{aligned}$$

$$P = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\right)^{-1} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot P \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = P^T$(thus $b = c$), we have that:

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\right)^{-1} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 2 \cdot b & 2 \cdot d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 \cdot b & 2 \cdot d \\ a & b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r\right)^{-1} \cdot \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 4 \cdot d & 2 \cdot b \\ 2 \cdot b & a \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 \cdot b \\ a \end{bmatrix} \cdot (a + r)^{-1} \cdot \begin{bmatrix} 2 \cdot b & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 4 \cdot d & 2 \cdot b \\ 2 \cdot b & a + 1 \end{bmatrix} - \frac{1}{a + r} \cdot \begin{bmatrix} 2 \cdot b \\ a \end{bmatrix} \cdot \begin{bmatrix} 2 \cdot b & a \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} 4 \cdot d & 2 \cdot b \\ 2 \cdot b & a + 1 \end{bmatrix} - \frac{1}{a + r} \cdot \begin{bmatrix} 4 \cdot b^2 & 2 \cdot a \cdot b \\ 2 \cdot a \cdot b & a^2 \end{bmatrix}$$

Thus we have to solve a parametric $(r)$ system with 3 equations and 3 unknowns:

$$\begin{cases} a + \frac{4 \cdot b^2}{a+r} = 4 \cdot d \\ b + \frac{2 \cdot a \cdot b}{a+r} = 2 \cdot b \\ d + \frac{a^2}{a+r} = a + 1 \end{cases}$$

$$\begin{cases} a + \frac{4 \cdot b^2}{a+r} = 4 \cdot d \\ \cancel{b} + \frac{2 \cdot a \cdot b}{a+r} = \cancel{2} \cdot b \\ d = \frac{\cancel{a^2} + a + a \cdot r + r - \cancel{a^2}}{a+r} \end{cases}$$

Notice that the covariace of the measurement noise is $r > 0$ (given but reasonable because there is always measurement noise):

$$\begin{cases} a + \frac{4 \cdot b^2}{a+r} = 4 \cdot d \\ \frac{2 \cdot a \cdot b}{a+r} = b \\ d = \frac{a + a \cdot r + r}{a+r} \end{cases}$$

$$\begin{cases} a + \frac{4 \cdot b^2}{a+r} = 4 \cdot d \\ \cancel{2} \cdot a \cdot b = \cancel{a \cdot b} + b \cdot r \\ d = \frac{a + a \cdot r + r}{a+r} \end{cases}$$

$$\begin{cases} a + \frac{4 \cdot b^2}{a+r} = 4 \cdot d \\ b \cdot (a - r) = 0 \\ d = \frac{a + a \cdot r + r}{a+r} \end{cases}$$

$$\begin{cases} a + \cancel{\frac{4 \cdot b^2}{a+r}} = 4 \cdot d \\ b = 0 \\ d = \frac{a + a \cdot r + r}{a+r} \end{cases}$$

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ d = \frac{a + a \cdot r + r}{a+r} \end{cases}$$

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ d \cdot (a + r) = a + a \cdot r + r \end{cases}$$

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ d \cdot (4 \cdot d + r) = 4 \cdot d + 4 \cdot d \cdot r + r \end{cases}$$

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ 4 \cdot d^2 - (3 \cdot r + 4) \cdot d - r = 0 \end{cases}$$

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ d = \frac{3 \cdot r + 4 \pm \sqrt{(9 \cdot r + 4) \cdot (r + 4)}}{8} \end{cases}$$

Notice that we must impose also that $a \geq 0$ and $d \geq 0$ because they are related to the variances on the diagonal of the covariance matrix $P$. Thus the solution is:

$$\begin{cases} a = 4 \cdot d \\ b = 0 \\ d = \frac{3 \cdot r + 4}{8} + \frac{\sqrt{(9 \cdot r + 4) \cdot (r + 4)}}{8} \end{cases}$$

$$P = \begin{bmatrix} 7.53 & 0 \\ 0 & 1.88 \end{bmatrix}, r = 1; \qquad P = \begin{bmatrix} 10.74 & 0 \\ 0 & 2.69 \end{bmatrix}, r = 2; \qquad P = \begin{bmatrix} 13.87 & 0 \\ 0 & 3.47 \end{bmatrix}, r = 3$$

**3) Discuss the role of $r$ in the covariance of the estimation error.**

If we look at covariance matrix of the error estimate $P$ we notice that:

- The off-diagonal is always 0 because of the fixed $b = 0$. Thus $r$ doesn't affect the cross-covariaces between $\tilde{x}_1$and $\tilde{x}_2$. We could expect it by looking at the system:

$$x(t+1) = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \cdot x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot w(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(t) + v(t)$$

Indeed we notice that the dynamic relationship between $x_1$ and $x_2$ is deterministic (the two states are decoupled).

- The process and measurement noise affect directly only on $x_2$. Thus $r$ affects the covariance component of $\tilde{x}_2$, that its the variance of the estimation error $\tilde{x}_2$, directly and the covariance component of $\tilde{x}_1$ only through the deterministic relationship between $x_1$ and $x_2$.