Analysis of stochastic processes
Part I

White Noise (WN) brief review

A WSS process $e(t)$ is a white noise $e(t) \sim WN\left(\mu, \lambda^2\right)$ if:

1. MEAN
$$m_e(t) = \mathbb{E}\left[e(t)\right] = \mu, \qquad \forall t$$
2. VARIANCE
$$\gamma_e(0) = \mathbb{E}\left[\left(e(t) - m_e\right)^2\right] = \lambda^2, \qquad \forall t$$
3. COVARIANCE FUNCTION
$$\gamma_e(\tau) = \mathbb{E}\left[\left(e(t) - m_e\right)\left(e(t-\tau) - m_e\right)\right] = \mathbb{E}\left[\left(e(t) - \mu\right)\left(e(t-\tau) - \mu\right)\right] = 0, \qquad \forall t, \tau \neq 0$$

Notation

- The autocovariance function of the process output is called simply covariance function and is denoted with the following notation: $\gamma_y(\tau)$.
- The autospectrum of the process output is called simply spectrum and is denoted with the following notation: $\mathit{\Gamma}_y(\omega)$.

Exercise 1

Consider the following process:

$$y(t) = e(t) + \frac{1}{5} \cdot e(t-1), \qquad e(t) \sim WN(0, 2)$$

1. What kind of process is this?
2. Is the process WSS?
3. Compute the process mean m_y and the covariance function $\gamma_y(\tau)$.
4. Compute the spectrum $\mathit{\Gamma}_y(\omega)$ of the process.
5. What happens to the process mean, covariance function and spectrum when $e(t) \sim WN(1, 2)$

1) What kind of process is this?

The process is a MA(1):

$$y(t) = c_0 \cdot e(t) + c_1 \cdot e(t-1)$$

Where, in this case, $c_0 = 1, c_1 = \frac{1}{5}$.

2) Is the process WSS?

Since the process is MA(1) we can conclude that it is a WSS process too (recall the basic proprieties of MA(n) processes). However, let's verify the stationarity.

$$y(t) = c_0 \cdot e(t) + c_1 \cdot e(t-1)$$

$$y(t) = \left(1 + \frac{1}{5} \cdot z^{-1}\right) \cdot e(t)$$

$$y(t) = \left(\frac{z + \frac{1}{5}}{z}\right) \cdot e(t) = W(z) \cdot e(t)$$

Notice that the pole of $W(z)$ is located in $z = 0$. Thus $W(z)$ is asymptotically stable. Since $e(t)$ is a WSS process (it is a WN indeed), then we can conclude that $y(t)$ is a WSS process too.
We can conclude that ***MA processes are always weak sense stationary***.

We can now compute its mean and covariance function.

3) Compute the process mean m_y and the covariance function $\gamma_y(\tau)$.

MEAN m_y

$$y(t) = e(t) + \frac{1}{5} \cdot e(t-1), \qquad e(t) \sim WN(0, 2)$$

Since $W(z)$ is asymptotically stable and $m_e = \mathbb{E}[e(t)] = 0$, we can conclude that $m_y = \mathbb{E}[y(t)] = 0$. Let's verify it.

$$\begin{aligned} m_y = \mathbb{E}[y(t)] &= \mathbb{E}\left[e(t) + \frac{1}{5} \cdot e(t-1)\right] \\ &= \mathbb{E}[e(t)] + \frac{1}{5} \cdot \mathbb{E}[e(t-1)] \\ &= m_e + \frac{1}{5} \cdot m_e \\ &= 0 + \frac{1}{5} \cdot 0 = 0 \end{aligned}$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

$$y(t) = e(t) + \frac{1}{5} \cdot e(t-1), \qquad e(t) \sim WN(0, 2)$$

- $\tau = 0$ (variance of the process)

$$\begin{aligned} \gamma_y(0) &= \mathbb{E}\left[(y(t) - m_y)^2\right] = \mathbb{E}\left[y(t)^2\right] = \mathbb{E}\left[\left(e(t) + \frac{1}{5} \cdot e(t-1)\right)^2\right] \\ &= \mathbb{E}\left[e(t)^2 + \frac{2}{5} \cdot e(t) \cdot e(t-1) + \frac{1}{25} \cdot e(t-1)^2\right] \\ &= \mathbb{E}\left[e(t)^2\right] + \frac{2}{5} \cdot \underbrace{\cancel{\mathbb{E}[e(t) \cdot e(t-1)]}}_{e \sim WN} + \frac{1}{25} \cdot \mathbb{E}\left[e(t-1)^2\right] = 2 + \frac{1}{25} \cdot 2 \\ &= \frac{52}{25} \end{aligned}$$

We could have computed the result directly from theory:

$$\gamma_y(0) = \lambda^2 \cdot \sum_{i=0}^{n} c_i^2 = \lambda^2 \cdot \left(c_0^2 + c_1^2\right)$$

Thus:

$$\gamma_y(0) = \lambda^2 \cdot \sum_{i=0}^{n} c_i^2 = \lambda^2 \cdot \left(c_0^2 + c_1^2\right) = 2 \cdot \left(1 + \frac{1}{25}\right) = \frac{52}{25}$$

- $\tau = 1$

$$\begin{aligned}
\gamma_y(1) &= \mathbb{E}\left[(y(t) - m_y) \cdot (y(t-1) - m_y)\right] = \mathbb{E}\left[y(t) \cdot y(t-1)\right] \\
&= \mathbb{E}\left[\left(e(t) + \frac{1}{5} \cdot e(t-1)\right) \cdot \left(e(t-1) + \frac{1}{5} \cdot e(t-2)\right)\right] \\
&= \mathbb{E}\left[e(t) \cdot e(t-1) + \frac{1}{5} \cdot e(t) \cdot e(t-2) + \frac{1}{5} \cdot e(t-1)^2 + \frac{1}{25} \cdot e(t-1) \cdot e(t-2)\right] \\
&= \underbrace{\cancel{\mathbb{E}\left[e(t) \cdot e(t-1)\right]}}_{e \sim WN} + \frac{1}{5} \cdot \underbrace{\cancel{\mathbb{E}\left[e(t) \cdot e(t-2)\right]}}_{e \sim WN} + \frac{1}{5} \cdot \mathbb{E}\left[e(t-1)^2\right] + \frac{1}{25} \cdot \underbrace{\cancel{\mathbb{E}\left[e(t-1) \cdot e(t-2)\right]}}_{e \sim WN} \\
&= \frac{1}{5} \cdot \mathbb{E}\left[e(t-1)^2\right] = \frac{1}{5} \cdot \gamma_e(0) = \frac{1}{5} \cdot \lambda^2 = \frac{1}{5} \cdot 2 = \frac{2}{5}
\end{aligned}$$

We could have computed the result directly from theory:

$$\gamma_y(1) = \lambda^2 \cdot \sum_{i=0}^{n-1} c_i \cdot c_{i-1} = \lambda^2 \cdot (c_0 \cdot c_1)$$

Thus:

$$\gamma_y(1) = \lambda^2 \cdot \sum_{i=0}^{n-1} c_i \cdot c_{i-1} = \lambda^2 \cdot (c_0 \cdot c_1) = 2 \cdot \left(1 \cdot \frac{1}{5}\right) = \frac{2}{5}$$

- $\tau = 2$

$$\begin{aligned}
\gamma_y(2) &= \mathbb{E}\left[(y(t) - m_y) \cdot (y(t-2) - m_y)\right] = \mathbb{E}\left[y(t) \cdot y(t-2)\right] \\
&= \mathbb{E}\left[\left(e(t) + \frac{1}{5} \cdot e(t-1)\right) \cdot \left(e(t-2) + \frac{1}{5} \cdot e(t-3)\right)\right] \\
&= \mathbb{E}\left[e(t) \cdot e(t-2) + \frac{1}{5} \cdot e(t) \cdot e(t-3) + \frac{1}{5} \cdot e(t-1) \cdot e(t-2) + \frac{1}{25} \cdot e(t-1) \cdot e(t-3)\right] \\
&= \underbrace{\cancel{\mathbb{E}\left[e(t) \cdot e(t-2)\right]}}_{e \sim WN} + \frac{1}{5} \cdot \underbrace{\cancel{\mathbb{E}\left[e(t) \cdot e(t-3)\right]}}_{e \sim WN} + \frac{1}{5} \cdot \underbrace{\cancel{\mathbb{E}\left[e(t-1) \cdot e(t-2)\right]}}_{e \sim WN} + \frac{1}{25} \cdot \underbrace{\cancel{\mathbb{E}\left[e(t-1) \cdot e(t-3)\right]}}_{e \sim WN} \\
&= 0
\end{aligned}$$

- $\tau = 3$

$$\gamma_y(3) = 0$$

Infact, from the theory holds:

$$\gamma_y(\tau) = 0, \qquad \forall \tau > n$$

where $n = 1$ in this case since we have a MA(1) SSP.

4) Compute the spectrum $\Gamma_y(\omega)$ of the process.

SPECTRAL DENSITY $\Gamma_y(\omega)$

- From the definition:

$$\begin{aligned}\Gamma_y(\omega) &= \sum_{\tau=-\infty}^{+\infty} \gamma_y(\tau) \cdot e^{-j\cdot\omega\cdot\tau} \\ &= \gamma_y(0) \cdot e^{-j\cdot\omega\cdot 0} + \gamma_y(-1) \cdot e^{-j\cdot\omega\cdot(-1)} + \gamma_y(1) \cdot e^{-j\cdot\omega\cdot 1} \\ &= \frac{52}{25} + \gamma_y(1) \cdot \left(e^{-j\cdot\omega} + e^{j\cdot\omega}\right) = \frac{52}{25} + \frac{2}{5} \cdot \left(e^{-j\cdot\omega} + e^{j\cdot\omega}\right)\end{aligned}$$

Euler representation of the complex exponential

Recall that:

$$A \cdot e^{j\cdot\theta} = A \cdot \cos(j \cdot \theta) + j \cdot A \cdot \sin(j \cdot \theta)$$

Thus:

$$e^{-j\cdot\omega} + e^{j\cdot\omega} = \cos(\omega) + j \cdot \sin(\omega) + \cos(\omega) - j \cdot \sin(\omega) = 2 \cdot \cos(\omega)$$

So:

$$\Gamma_y(\omega) = \frac{52}{25} + \frac{4}{5} \cdot \cos(\omega)$$

- From the transfer function:

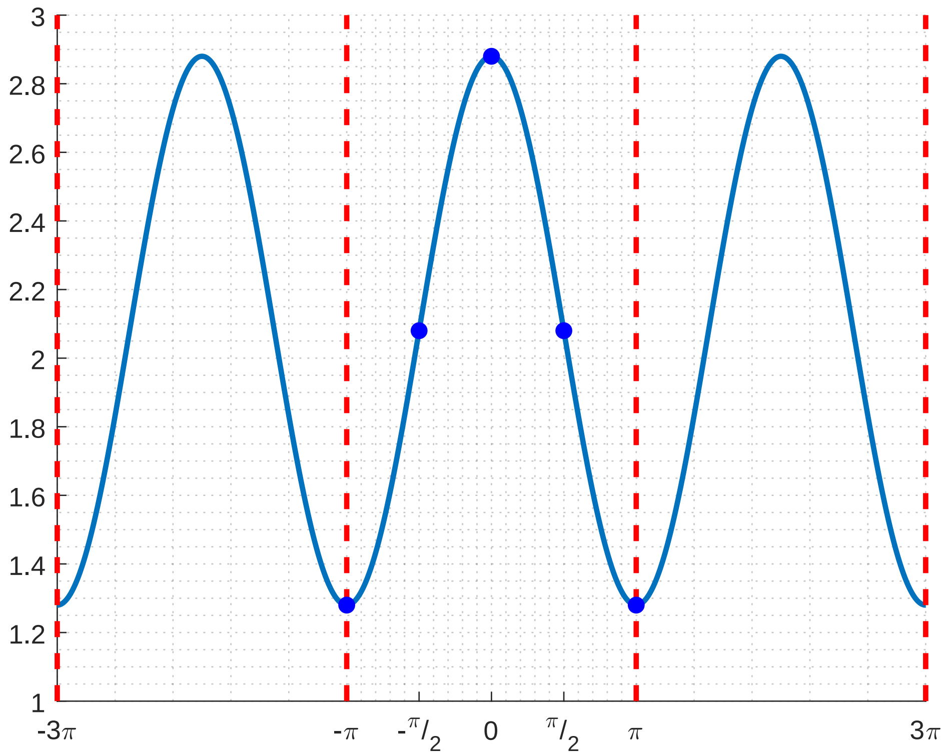
$$\begin{aligned}\Gamma_y(\omega) &= W\left(e^{j\cdot\omega}\right) \cdot W\left(e^{-j\cdot\omega}\right) \cdot \lambda^2 \\ &= \frac{e^{j\cdot\omega} + \frac{1}{5}}{e^{j\cdot\omega}} \cdot \frac{e^{-j\cdot\omega} + \frac{1}{5}}{e^{-j\cdot\omega}} \cdot 2 \\ &= 2 \cdot \left(1 + \frac{1}{25} + \frac{1}{5} \cdot \left(e^{-j\cdot\omega} + e^{j\cdot\omega}\right)\right) = \frac{52}{25} + \frac{4}{5} \cdot \cos(\omega)\end{aligned}$$

Notice that the two methods lead to the same result. The obtained spectrum is:

- a real function;
- a positive function;
- an even function;
- a periodic function (with period $2 \cdot \pi$, it depends on the cosine).

We draw the spectrum plot based on some simply computable points:

- $\Gamma_y(0) = \frac{52}{25} + \frac{4}{5} = \frac{72}{25}$
- $\Gamma_y\left(\pm\frac{\pi}{2}\right) = \frac{52}{25}$
- $\Gamma_y(\pm\pi) = \frac{52}{25} - \frac{4}{5} = \frac{32}{25}$



Low frequencies are more relevant. We expect a very slow-varying signal.

5) What happens to the process mean, covariance function and spectrum when $e(t) \sim WN(1,2)$

When $e(t) \sim WN(1,2)$ the stationarity propriety does not change ($e(t)$ is again a white noise and the filter $W(z)$ is asymptotically stable): $y(t)$ is a WSS process. But...

MEAN m_y

$$\mathbb{E}[y(t)] = \mathbb{E}\left[e(t) + \frac{1}{5}\cdot e(t-1)\right] = \mathbb{E}[e(t)] + \frac{1}{5}\cdot\mathbb{E}[e(t-1)] = 1 + \frac{1}{5}\cdot 1 = \frac{6}{5}$$

The process is MA(1) with non null-mean!

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$ (variance of the process)

$$\begin{aligned}\gamma_y(0) &= \mathbb{E}\left[(y(t)-m_y)^2\right] = \mathbb{E}\left[\left(y(t)-\frac{6}{5}\right)^2\right] = \mathbb{E}\left[\left(e(t)+\frac{1}{5}\cdot e(t-1)-\frac{6}{5}\right)^2\right] \\ &= \mathbb{E}\left[e(t)^2 + \frac{1}{25}\cdot e(t-1)^2 + \frac{36}{25} - \frac{12}{5}\cdot e(t) + \frac{2}{5}\cdot e(t)\cdot e(t-1) - \frac{12}{25}\cdot e(t-1)\right] \\ &= \mathbb{E}\left[e(t)^2\right] + \frac{1}{25}\cdot\mathbb{E}\left[e(t-1)^2\right] + \frac{36}{25} - \frac{12}{5}\cdot\mathbb{E}[e(t)] + \frac{2}{5}\cdot\mathbb{E}[e(t)\cdot e(t-1)] - \frac{12}{25}\cdot\mathbb{E}[e(t-1)]\end{aligned}$$

Observe that:

$$\mathbb{E}\left[e(t)^2\right] = \mathbb{E}\left[e(t-1)^2\right] = \mathbb{E}\left[(e(t)-m_e)^2\right] + m_e^2 = \gamma_e(0) + m_e^2 = \lambda^2 + m_e^2 = 2 + 1 = 3$$

$$\mathbb{E}[e(t)\cdot e(t-1)] = \gamma_e(1) + m_e^2 = 0 + 1 = 1$$

So:

$$\begin{aligned}\gamma_y(0) &= \mathbb{E}\left[e(t)^2\right] + \frac{1}{25}\cdot\mathbb{E}\left[e(t-1)^2\right] + \frac{36}{25} - \frac{12}{5}\cdot\mathbb{E}\left[e(t)\right] + \frac{2}{5}\cdot\mathbb{E}\left[e(t)\cdot e(t-1)\right] - \frac{12}{25}\cdot\mathbb{E}\left[e(t-1)\right]\\ &= 3 + \frac{1}{25}\cdot 3 + \frac{36}{25} - \frac{12}{5}\cdot 1 + \frac{2}{5}\cdot 1 - \frac{12}{25}\cdot 1\\ &= \frac{26}{25}\cdot 3 + \frac{36}{25} - \frac{12}{5} + \frac{2}{5} - \frac{12}{25} = \frac{52}{25}\end{aligned}$$

The variance of the non-null mean MA(1) process is equal to the variance of the same MA(1) process with null mean. But using the classical formula to compute the covariance function of non-null mean processes involves too many computations.

So we solve the problem again considering the **UNBIASED PROCESSES**.

Recall that:

$$m_y = \frac{6}{5}$$

$$m_e = 1$$

Define the new processes:

$$\tilde{y}(t) = y(t) - m_y = y(t) - \frac{6}{5}$$

$$\tilde{e}(t) = e(t) - m_e = e(t) - 1$$

Observe that:

$$m_{\tilde{y}} = \mathbb{E}\left[\tilde{y}(t)\right] = \mathbb{E}\left[y(t)\right] - m_y = 0$$

$$m_{\tilde{e}} = \mathbb{E}\left[\tilde{e}(t)\right] = \mathbb{E}\left[e(t)\right] - m_e = 0$$

The processes $\tilde{y}$ and $\tilde{e}$ are the unbiased processes. We have to derive the dynamical relation between these new signals:

$$\begin{cases} y(t) = \tilde{y}(t) + \frac{6}{5} \\ e(t) = \tilde{e}(t) + 1 \end{cases} \overset{y(t)=e(t)+\frac{1}{5}\cdot e(t-1)}{\Longrightarrow} \tilde{y}(t) + \frac{6}{5} = \tilde{e}(t) + 1 + \frac{1}{5}\cdot\left(\tilde{e}(t-1) + 1\right)$$

$$\tilde{y}(t) = \tilde{e}(t) + \frac{1}{5}\cdot\tilde{e}(t-1)$$

We can now evaluate the covariance function $\gamma_{\tilde{y}}(\tau)$. Observe that the structure of the process is equivalent to the zero-mean MA(1) process, so that:

$$\gamma_{\tilde{y}}(0) = \frac{52}{25}$$

$$\gamma_{\tilde{y}}(1) = \frac{2}{5}$$

$$\gamma_{\tilde{y}}(\tau) = 0, \quad \forall \tau > 1$$

The following relation holds:

$$\gamma_{\tilde{y}}(\tau) = \gamma_y(\tau) \quad \forall \tau$$

The covariance function of the process $y(t)$ is equal to the covariance function of the unbiased process $\tilde{y}(t)$. Thus, also the spectrum does not change!

Exercise 2

Consider the following covariance functions:

1. $\gamma_1(0) = -1, \quad \gamma_1(1) = 0.5, \quad \gamma_1(-1) = 0.5, \quad \gamma_1(\tau) = 0, \ |\tau| \geq 2$
2. $\gamma_2(0) = 1, \quad \gamma_2(1) = 0.5, \quad \gamma_2(-1) = -0.5, \quad \gamma_2(\tau) = 0, \ |\tau| \geq 2$
3. $\gamma_3(0) = 2, \quad \gamma_3(1) = -0.2, \quad \gamma_3(-1) = -0.2, \quad \gamma_3(\tau) = 0, \ |\tau| \geq 2$
4. $\gamma_4(0) = 0.5, \quad \gamma_4(1) = 1, \quad \gamma_4(-1) = 1, \quad \gamma_4(\tau) = 0, \ |\tau| \geq 2$

Find the underlying zero-mean WSS process.

Obviously just the third one satisfies the proprieties for the covariance function of a WSS process. In fact, recall that the **covariance function properties**:

1. $\gamma(0) \geq 0$ **variance is positive** (not satisfied by number 1);
2. $\gamma(\tau) = \gamma(-\tau)$ **simmetry** (i.e. $\gamma(\tau)$ is a **even** function) (not satisfied by number 2);
3. $|\gamma(\tau)| \leq \gamma(0)$ **greatest correlation for** $\tau = 0$ (not satisfied by number 4);

Every time you work with covariance functions, **check** whether your covariance function satisfies these proprieties.

Since $\gamma_3(\tau) = 0, \ |\tau| \geq 2$ we can conclude that the underlying process is a MA(1). In the previous exercise we have proven that, for a MA(1) process, we have:

$$\gamma_y(0) = \lambda^2 \cdot \left(c_0^2 + c_1^2\right)$$

$$\gamma_y(1) = c_0 \cdot c_1 \cdot \lambda^2$$

We have to find the coefficients c_0, c_1 of the MA(1) and the white noise variance λ^2. So we solve the following system of equations:

$$\begin{cases} \lambda^2 \cdot \left(c_0^2 + c_1^2\right) & = 2 \\ c_0 \cdot c_1 \cdot \lambda^2 & = -0.2 \end{cases}$$

Notice that this system has three unknowns and two equations. There are infinite solutions, i.e. there are infinite MA(1) processes that have the same covariance function!
So we fix one parameter, e.g. $c_0 = 1$. Thus (assuming $c_1 \neq 0$):

$$\begin{cases} \lambda^2 \cdot \left(1 + c_1^2\right) & = 2 \\ c_1 \cdot \lambda^2 & = -0.2 \end{cases} \Rightarrow \begin{cases} c_1^2 + 10 \cdot c_1 + 1 & = 0 \\ \lambda^2 & = -\frac{0.2}{c_1} \end{cases} \Rightarrow \begin{cases} c_1 & = -5 \pm 2 \cdot \sqrt{6} \\ \lambda^2 & = -\frac{0.2}{c_1} \end{cases}$$

We now choose for example $c_1 = -5 \pm 2 \cdot \sqrt{6}$, so we get:

$$\begin{cases} c_1 & = -5 \pm 2 \cdot \sqrt{6} \backsimeq -0.101 \\ \lambda^2 & = -\frac{0.2}{c_1} \backsimeq 1.980 \end{cases}$$

Exercise 3

Consider the process:

$$y(t) = \theta \cdot y(t-1) + e(t) \qquad e(t) \sim WN(0,1), \quad \theta \in \mathbb{R}$$

1. What kind of process is this?

2. What are the values of θ which make the process WSS?
3. Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for ($|\tau| \leq 2$)
4. Compute the spectrum $\Gamma_y(\omega)$ of the process.
5. What happens if $e(t) \sim WN(2,1)$?

1) What kind of process is this?

The process is an AR(1):

$$y(t) = a_1 \cdot y(t-1) + e(t)$$

where $a_1 = \theta$.

2) What are the values of θ which make the process WSS?

Immediately, we cannot conclude anything about the stationarity of $y(t)$, since the process has 1 non trivial pole. We must verify the stationarity by writing its operatorial representation.

OPERATORIAL REPRESENTATION

$$y(t) = \theta \cdot y(t-1) + e(t)$$

$$y(t) = \theta \cdot z^{-1} \cdot y(t) + e(t)$$

$$\left(1 - \theta \cdot z^{-1}\right) \cdot y(t) = e(t)$$

$$y(t) = \frac{1}{(1 - \theta \cdot z^{-1})} \cdot e(t)$$

$$y(t) = \frac{z}{(z-\theta)} \cdot e(t) = W(z) \cdot e(t)$$

The system has 1 trivial zero located at the origin. What about the pole?
Obviously (root of the denominator) it is located in:

$$z - \theta = 0 \implies z = \theta$$

The digital filter is asymptotically stable when the pole is inside the unit circle, so:

$$|\theta| < 1 \quad (i.e. -1 < \theta < 1) \iff W(z) \text{ is asymptotically stable}$$

Since $e(t)$ is a WSS process, then $y(t)$ is a WSS process if $W(z)$ is asymptotically stable. Hence:

$$|\theta| < 1 \quad (i.e. -1 < \theta < 1) \iff y(t) \text{ is a WSS process}$$

In particular, when $\theta = -\frac{1}{4}$, $y(t)$ is a WSS process.

3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $(|\tau| \leq 2)$

Analysis for $\theta = -\frac{1}{4}$

MEAN m_y
Since $e(t)$ is a zero-mean WN and $W(z)$ is asymptotically stable, then:

$$m_y = \mathbb{E}[y(t)] = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

- $\tau = 0$

$$\begin{aligned}
\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] &= \mathbb{E}\left[\left(-\frac{1}{4} \cdot y(t-1) + e(t)\right)^2\right] \\
&= \frac{1}{16} \cdot \mathbb{E}\left[y(t-1)^2\right] + \mathbb{E}\left[e(t)^2\right] - \frac{1}{2} \cdot \cancel{\mathbb{E}[y(t-1) \cdot e(t)]} \\
&= \frac{1}{16} \cdot \gamma_y(0) + 1
\end{aligned}$$

$$\left(1 - \frac{1}{16}\right) \cdot \gamma_y(0) = 1 \implies \gamma_y(0) = \frac{16}{15}$$

- $\tau = 1$

$$\begin{aligned}
\gamma_y(1) = \mathbb{E}[y(t) \cdot y(t-1)] &= \mathbb{E}\left[\left(-\frac{1}{4} \cdot y(t-1) + e(t)\right) \cdot y(t-1)\right] \\
&= -\frac{1}{4} \cdot \mathbb{E}\left[y(t-1)^2\right] + \cancel{\mathbb{E}[y(t-1) \cdot e(t)]} \\
&= -\frac{1}{4} \cdot \gamma_y(0) = -\frac{4}{15}
\end{aligned}$$

- $\tau = 2$

$$\begin{aligned}
\gamma_y(2) = \mathbb{E}[y(t) \cdot y(t-2)] &= \mathbb{E}\left[\left(-\frac{1}{4} \cdot y(t-1) + e(t)\right) \cdot y(t-2)\right] \\
&= -\frac{1}{4} \cdot \mathbb{E}[y(t-1) \cdot y(t-2)] + \cancel{\mathbb{E}[y(t-2) \cdot e(t)]} \\
&= -\frac{1}{4}\gamma_y(1) = \frac{1}{15}
\end{aligned}$$

[...]

Recursive equations...?

Yule-Walker equations for AR(1) processes

$$\gamma_y(0) = \frac{\lambda^2}{1 - a_1^2}$$

$$\gamma_y(\tau) = a_1 \cdot \gamma_y(\tau - 1), \qquad \tau \geq 1$$

- $\tau = 0$

$$\gamma_y(0) = \frac{\lambda^2}{1-a_1^2} = \frac{1}{1-\frac{1}{16}} = \frac{16}{15}$$

- $\tau = 1$

$$\gamma_y(1) = -\frac{1}{4}\cdot\gamma_y(0) = -\frac{4}{15}$$

- $\tau = 2$

$$\gamma_y(2) = -\frac{1}{4}\cdot\gamma_y(1) = \frac{1}{15}$$

Yule-Walker equations (for AR(1) processes) provide the same results of the computation using the definition of covariance function.

4) Compute the spectrum $\mathit{\Gamma}_y(\omega)$ of the process.

SPECTRAL DENSITY $\mathit{\Gamma}_y(\omega)$

The better way of evaluating the spectral density of an AR(1) process is via the transfer function:

- From the transfer function:

$$\begin{aligned}
\mathit{\Gamma}_y(\omega) &= W\left(e^{j\cdot\omega}\right)\cdot W\left(e^{-j\cdot\omega}\right)\cdot\lambda^2 \\
&= \frac{e^{j\cdot\omega}}{e^{j\cdot\omega}+\frac{1}{4}}\cdot\frac{e^{-j\cdot\omega}}{e^{-j\cdot\omega}+\frac{1}{4}}\cdot 1 \\
&= \frac{1}{1+\frac{1}{4}\cdot\left(e^{j\cdot\omega}+e^{-j\cdot\omega}\right)+\frac{1}{16}} = \frac{1}{1+\frac{1}{2}\cdot\cos(\omega)+\frac{1}{16}} \\
&= \frac{16}{16+8\cdot\cos(\omega)+1} = \frac{16}{17+8\cdot\cos(\omega)}
\end{aligned}$$

The obtained spectrum is:

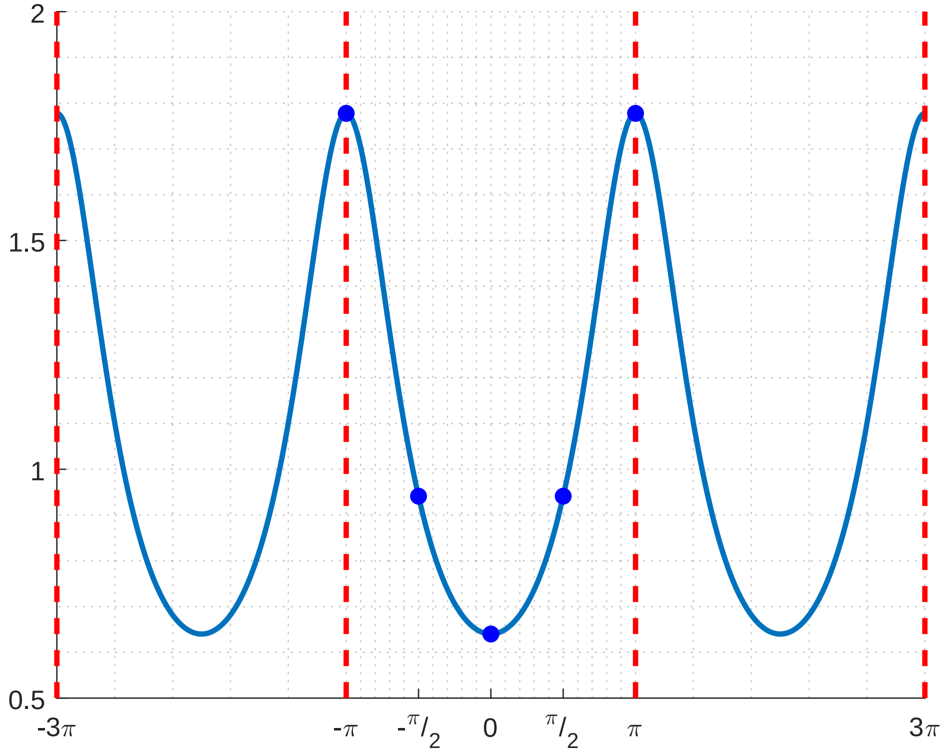
- a real function;
- a positive function;
- an even function;
- a periodic function (with period $2\cdot\pi$, it depends on the cosine).

We draw the spectrum plot based on some simply computable points:

$$\mathit{\Gamma}_y(0) = \frac{16}{17+8} = \frac{16}{25}$$

$$\mathit{\Gamma}_y\left(\pm\frac{\pi}{2}\right) = \frac{16}{17+0} = \frac{16}{17}$$

$$\mathit{\Gamma}_y(\pm\pi) = \frac{16}{17-8} = \frac{16}{9}$$



High frequencies are more relevant. We expect a rapidly-varying signal.

5) What happens if $e(t) \sim WN(2,1)$**?**

Analysis of the process:

$$y(t) = -\frac{1}{4} \cdot y(t-1) + e(t), \qquad e(t) \sim WN(2,1)$$

MEAN m_y

$$m_y = \mathbb{E}[y(t)] = -\frac{1}{4} \cdot \mathbb{E}[y(t-1)] + \mathbb{E}[e(t)] = -\frac{1}{4} \cdot m_y + 2$$

$$\left(1 + \frac{1}{4}\right) \cdot m_y = 2 \implies m_y = 2 \cdot \frac{4}{5} = \frac{8}{5}$$

UNBIASED PROCESSES
Define the new processes:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y = y(t) - \frac{8}{5} \\ \tilde{e}(t) = e(t) - m_e = e(t) - 2 \end{cases} \implies \begin{cases} y(t) = \tilde{y}(t) + \frac{8}{5} \\ e(t) = \tilde{e}(t) + 2 \end{cases}$$

Thus:

$$y(t) = -\frac{1}{4} \cdot y(t-1) + e(t), \qquad e(t) \sim WN(2,1)$$

$$\tilde{y}(t) + \frac{8}{5} = -\frac{1}{4} \cdot \left(\tilde{y}(t-1) + \frac{8}{5}\right) + \tilde{e}(t) + 2$$

$$\tilde{y}(t) = -\frac{1}{4} \cdot \tilde{y}(t-1) + \tilde{e}(t) \cancel{- \frac{8}{5} - \frac{1}{4} \cdot \frac{8}{5} + 2}$$

$$\tilde{y}(t) = -\frac{1}{4} \cdot \tilde{y}(t-1) + \tilde{e}(t), \qquad \tilde{e}(t) \sim WN(0,1)$$

which is exactly the previous process.

COVARIANCE FUNCTION $\gamma_y(\tau)$ AND SPECTRAL DENSITY $\Gamma_y(\omega)$
Notice that the structure of the process is the same of the initial case. So the covariance function of $\tilde{y}(t)$ is equal to the covariance function of the initial case. As a consequence the spectrum does not change.

Exercise 4

Consider the following process:

$$y(t) = \frac{1}{2} \cdot y(t-1) - \frac{1}{4} \cdot y(t-2) + e(t), \qquad e(t) \sim WN(0,1)$$

1. What kind of process is this?
2. Is the process WSS?
3. Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.

1) What kind of process is this?

The process is an AR(2):

$$y(t) = a_1 \cdot y(t-1) + a_2 \cdot y(t-2) + e(t)$$

where $a_1 = \frac{1}{2}$, $a_2 = -\frac{1}{4}$.

2) Is the process WSS?

Immediately, we cannot conclude anything about the stationarity of $y(t)$, since the process has 2 non trivial poles. We must verify the stationarity by writing its operatorial representation.

OPERATIONAL REPRESENTATION

$$y(t) = \frac{1}{2} \cdot y(t-1) - \frac{1}{4} \cdot y(t-2) + e(t)$$

$$y(t) = \frac{1}{2} \cdot z^{-1} \cdot y(t) - \frac{1}{4} \cdot z^{-2} \cdot y(t) + e(t)$$

$$\left(1 - \frac{1}{2} \cdot z^{-1} + \frac{1}{4} \cdot z^{-2}\right) \cdot y(t) = e(t)$$

$$y(t) = \frac{1}{\left(1 - \frac{1}{2} \cdot z^{-1} + \frac{1}{4} \cdot z^{-2}\right)} \cdot e(t)$$

$$y(t) = \frac{z^2}{\left(z^2 - \frac{1}{2} \cdot z + \frac{1}{4}\right)} \cdot e(t) = W(z) \cdot e(t)$$

We have 2 trivial zeroes in the origin ($z^2 = 0$). What about the poles?

$$z^2 - \frac{1}{2} \cdot z + \frac{1}{4} = 0$$

$$z_{1,2} = \frac{1}{4} \pm \frac{\sqrt{\frac{1}{4} - 1}}{2} = \frac{1}{4} \pm j \cdot \frac{\sqrt{3}}{4}$$

Two complex conjugate poles! In order to prove the asymptotical stability of $W(z)$ we have to verify that:

$$|z_1| = |z_2| < 1$$

Hence:

$$|z_1| = |z_2| = \left|\frac{1}{4} \pm j \cdot \frac{\sqrt{3}}{4}\right| = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2} = \sqrt{\frac{1}{16} + \frac{3}{16}} = \sqrt{\frac{4}{16}} = \frac{1}{2} < 1$$

So the digital filter $W(z)$ is asymptotically stable.
Since $e(t)$ is a WSS process (it's a WN) and $W(z)$ is asymptotically stable, then $y(t)$ is a WSS process too.

3) Compute the process mean m_y and covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.

MEAN m_y

Since $e(t)$ is a zero-mean White Noise and $W(z)$ is asymptotically stable, we can conclude that:

$$m_y = \mathbb{E}[y(t)] = 0$$

COVARIANCE FUNCTION $\gamma_y(\tau)$

$$y(t) = \frac{1}{2} \cdot y(t-1) - \frac{1}{4} \cdot y(t-2) + e(t)$$

- $\tau = 0$

$$\begin{aligned}
\gamma_y(0) = \mathbb{E}\left[y(t)^2\right] &= \mathbb{E}\left[\left(\frac{1}{2} \cdot y(t-1) - \frac{1}{4} \cdot y(t-2) + e(t)\right)^2\right] \\
&= \frac{1}{4} \cdot \mathbb{E}\left[y(t-1)^2\right] + \frac{1}{16} \cdot \mathbb{E}\left[y(t-2)^2\right] + \mathbb{E}\left[e(t)^2\right] \\
&- \frac{1}{4} \cdot \mathbb{E}[y(t-1) \cdot y(t-2)] + \cancel{\mathbb{E}[y(t-1) \cdot e(t)]} - \frac{1}{2} \cdot \cancel{\mathbb{E}[y(t-2) \cdot e(t)]} \\
&= \frac{1}{4} \cdot \gamma_y(0) + \frac{1}{16} \cdot \gamma_y(0) + 1 - \frac{1}{4} \cdot \gamma_y(1)
\end{aligned}$$

Notice that $\gamma_y(0)$ depends on $\gamma_y(1)$. We cannot solve this equation alone. But if we write the expression for $\gamma_y(1)$:

- $\tau = 1$

$$\begin{aligned}
\gamma_y(1) = \mathbb{E}[y(t) \cdot y(t-1)] &= \mathbb{E}\left[\left(\frac{1}{2} \cdot y(t-1) - \frac{1}{4} \cdot y(t-2) + e(t)\right) \cdot y(t-1)\right] \\
&= \frac{1}{2} \cdot \mathbb{E}\left[y(t-1)^2\right] - \frac{1}{4} \cdot \mathbb{E}[y(t-2) \cdot y(t-1)] + \cancel{\mathbb{E}[e(t) \cdot y(t-1)]} \\
&= \frac{1}{2} \cdot \gamma_y(0) - \frac{1}{4} \cdot \gamma_y(1)
\end{aligned}$$

We observe that $\gamma_y(1)$ depends on$\gamma_y(0)$. So this system of equations:

$$\begin{cases} \gamma_y(0) = \frac{1}{4} \cdot \gamma_y(0) + \frac{1}{16} \cdot \gamma_y(0) + 1 - \frac{1}{4} \cdot \gamma_y(1) \\ \gamma_y(1) = \frac{1}{2} \cdot \gamma_y(0) - \frac{1}{4} \cdot \gamma_y(1) \end{cases}$$

has just one possible solution, since we have two equations and two unknowns, i.e. $\gamma_y(1)$ and $\gamma_y(0)$. Hence:

$$\begin{cases}\gamma_y(0) = \frac{1}{4}\cdot\gamma_y(0) + \frac{1}{16}\cdot\gamma_y(0) + 1 - \frac{1}{4}\cdot\gamma_y(1)\\ \gamma_y(1) = \frac{1}{2}\cdot\gamma_y(0) - \frac{1}{4}\cdot\gamma_y(1)\end{cases} = \begin{cases}\left(1-\frac{1}{4}-\frac{1}{16}\right)\cdot\gamma_y(0) = 1 - \frac{1}{4}\cdot\gamma_y(1)\\ \frac{5}{4}\cdot\gamma_y(1) = \frac{1}{2}\cdot\gamma_y(0)\end{cases}$$

$$= \begin{cases}\gamma_y(1) = \frac{2}{5}\cdot\gamma_y(0)\\ \left(1-\frac{1}{4}-\frac{1}{16}\right)\cdot\gamma_y(0) = 1 - \frac{1}{4}\cdot\gamma_y(1)\end{cases}$$

$$= \begin{cases}\gamma_y(1) = \frac{2}{5}\cdot\gamma_y(0)\\ \left(1-\frac{1}{4}-\frac{1}{16}\right)\cdot\gamma_y(0) = 1 - \frac{1}{10}\cdot\gamma_y(0)\end{cases}$$

$$= \begin{cases}\gamma_y(1) = \frac{2}{5}\cdot\gamma_y(0)\\ \left(1-\frac{1}{4}-\frac{1}{16}+\frac{1}{10}\right)\cdot\gamma_y(0) = 1\end{cases}$$

$$= \begin{cases}\gamma_y(1) = \frac{2}{5}\cdot\frac{80}{63} = \frac{32}{63}\\ \gamma_y(0) = \frac{80}{63}\end{cases}$$

The computation of the next terms of the covariance function is simpler:

- $\tau = 2$

$$\begin{aligned}\gamma_y(2) &= \mathbb{E}\left[y(t)\cdot y(t-2)\right] = \mathbb{E}\left[\left(\frac{1}{2}\cdot y(t-1) - \frac{1}{4}\cdot y(t-2) + e(t)\right)\cdot y(t-2)\right]\\ &= \frac{1}{2}\cdot\mathbb{E}\left[y(t-1)\cdot y(t-2)\right] - \frac{1}{4}\cdot\mathbb{E}\left[y(t-2)^2\right] + \cancel{\mathbb{E}\left[e(t)\cdot y(t-2)\right]}\\ &= \frac{1}{2}\cdot\gamma_y(1) - \frac{1}{4}\cdot\gamma_y(0) = \frac{16}{63} - \frac{20}{63} = -\frac{4}{63}\end{aligned}$$

- $\tau = 3$

$$\begin{aligned}\gamma_y(3) &= \mathbb{E}\left[y(t)\cdot y(t-3)\right] = \mathbb{E}\left[\left(\frac{1}{2}\cdot y(t-1) - \frac{1}{4}\cdot y(t-2) + e(t)\right)\cdot y(t-3)\right]\\ &= \frac{1}{2}\cdot\mathbb{E}\left[y(t-1)\cdot y(t-3)\right] - \frac{1}{4}\cdot\mathbb{E}\left[y(t-2)\cdot y(t-3)\right] + \cancel{\mathbb{E}\left[e(t)\cdot y(t-3)\right]}\\ &= \frac{1}{2}\cdot\gamma_y(2) - \frac{1}{4}\cdot\gamma_y(1) = -\frac{2}{63} - \frac{8}{63} = -\frac{10}{63}\end{aligned}$$

Recursive equations. . . ?

Yule-Walker equations for AR(2) processes

Generic AR(2) process:

$$y(t) = a_1\cdot y(t-1) + a_2\cdot y(t-2) + e(t)$$

$\gamma_y(0)$ is a function of $\gamma_y(1)$, starting from the process time domain representation

$$\gamma_y(\tau) = a_1\cdot\gamma_y(\tau-1) + a_2\cdot\gamma_y(\tau-2), \qquad \tau\geq 1$$