Analysis of stochastic processes
Part II

Exercise 1

Consider the following process:

$$y(t) = \frac{1}{3} \cdot y(t-1) + \eta(t) - 2 \cdot \eta(t-1), \qquad \eta(t) \sim WN(1, 9)$$

1. What kind of process is this?
2. Is the process WSS?
3. Compute the process mean m_y and the covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.
4. Compute the spectrum $\Gamma_y(\omega)$ of the process.

1) What kind of process is this?

The process is a ARMA$(1, 1)$:

$$y(t) = a_1 \cdot y(t-1) + \eta(t) + c_1 \cdot \eta(t-1), \qquad \eta(t) \sim WN(1, 9)$$

Where, in this case, $a_1 = \frac{1}{3}$, $c_1 = -2$.

Notice that the noise is a white noise with non-null mean.

2) Is the process WSS?

OPERATORIAL REPRESENTATION

$$\begin{aligned}
y(t) &= \frac{1}{3} \cdot y(t-1) + \eta(t) - 2 \cdot \eta(t-1), \qquad \eta(t) \sim WN(1, 9) \\
y(t) &= \frac{1}{3} \cdot z^{-1} \cdot y(t) + \eta(t) - 2 \cdot z^{-1} \cdot \eta(t) \\
\left(1 - \frac{1}{3} \cdot z^{-1}\right) \cdot y(t) &= \left(1 - 2 \cdot z^{-1}\right) \cdot \eta(t) \\
y(t) &= \frac{\left(1 - 2 \cdot z^{-1}\right)}{\left(1 - \frac{1}{3} \cdot z^{-1}\right)} \cdot \eta(t) \\
y(t) &= \frac{(z-2)}{\left(z - \frac{1}{3}\right)} \cdot \eta(t) \\
y(t) &= W(z) \cdot \eta(t)
\end{aligned}$$

The zero $(z = 2)$ is located outside the unit circle: we say it's a non-minimum phase system.

The pole $(z = \frac{1}{3})$ is inside the unit circle: $W(z)$ is asymptotically stable.

Since $\eta(t)$ is a WSS process and $W(z)$ is asymptotically stable, then $y(t)$ is a WSS process too.

3) Compute the process mean m_y and the covariance function $\gamma_y(\tau)$ for $|\tau| \leq 3$.

MEAN m_y

$$y(t) = \frac{1}{3} \cdot y(t-1) + \eta(t) - 2 \cdot \eta(t-1), \qquad \eta(t) \sim WN(1, 9)$$

$$\begin{aligned} m_y = \mathbb{E}[y(t)] &= \mathbb{E}\left[\frac{1}{3} \cdot y(t-1) + \eta(t) - 2 \cdot \eta(t-1)\right] \\ \mathbb{E}[y(t)] &= \frac{1}{3} \cdot \mathbb{E}[y(t-1)] + \mathbb{E}[\eta(t)] - 2 \cdot \mathbb{E}[\eta(t-1)] \\ m_y &= \frac{1}{3} \cdot m_y + m_\eta - 2 \cdot m_\eta \\ \left(1 - \frac{1}{3}\right) \cdot m_y &= -m_\eta = -1 \\ m_y &= -\frac{3}{2} \end{aligned}$$

UNBIASED PROCESS

We will work with the unbiased processes $y(t)$ and $\eta(t)$ defined as:

$$\begin{cases} \tilde{y}(t) = y(t) - m_y = y(t) + \frac{3}{2} \\ \tilde{\eta}(t) = \eta(t) - m_\eta = \eta(t) - 1 \end{cases} \implies \begin{cases} y(t) = \tilde{y}(t) - \frac{3}{2} \\ \eta(t) = \tilde{\eta}(t) + 1 \end{cases}$$

Remembering that the process equation is:

$$y(t) = \frac{1}{3} \cdot y(t-1) + \eta(t) - 2 \cdot \eta(t-1), \qquad \eta(t) \sim WN(1, 9)$$

The new unbiased process equation is:

$$\begin{aligned} \tilde{y}(t) - \frac{3}{2} &= \frac{1}{3} \cdot \left(\tilde{y}(t-1) - \frac{3}{2}\right) + \tilde{\eta}(t) + 1 - 2 \cdot (\tilde{\eta}(t-1) + 1) \\ \tilde{y}(t) &= \frac{1}{3} \cdot \tilde{y}(t-1) + \tilde{\eta}(t) - 2 \cdot \tilde{\eta}(t-1) + \cancel{\frac{3}{2} - \frac{1}{3} \cdot \frac{3}{2} + 1 - 2} \\ \tilde{y}(t) &= \frac{1}{3}\tilde{y}(t-1) + \tilde{\eta}(t) - 2\tilde{\eta}(t-1), \qquad \tilde{\eta}(t) \sim WN(0, 9) \end{aligned}$$

COVARIANCE FUNCTION $\gamma_{\tilde{y}}(\tau)$

- $\tau = 0$ (variance of the process)

$$\begin{aligned} \gamma_{\tilde{y}}(0) &= \mathbb{E}\left[(\tilde{y}(t) - m_{\tilde{y}})^2\right] = \mathbb{E}\left[\tilde{y}(t)^2\right] = \mathbb{E}\left[\left(\frac{1}{3} \cdot \tilde{y}(t-1) + \tilde{\eta}(t) - 2 \cdot \tilde{\eta}(t-1)\right)^2\right] \\ &= \mathbb{E}\left[\frac{1}{9} \cdot \tilde{y}(t-1)^2 + \tilde{\eta}(t)^2 + 4 \cdot \tilde{\eta}(t-1)^2 + \frac{2}{3} \cdot \tilde{y}(t-1) \cdot \tilde{\eta}(t) - \frac{4}{3} \cdot \tilde{y}(t-1) \cdot \tilde{\eta}(t-1) - 4 \cdot \tilde{\eta}(t) \cdot \tilde{\eta}(t-1)\right] \\ &= \frac{1}{9} \cdot \mathbb{E}\left[\tilde{y}(t-1)^2\right] + \mathbb{E}\left[\tilde{\eta}(t)^2\right] + 4 \cdot \mathbb{E}\left[\tilde{\eta}(t-1)^2\right] + \\ &+ \frac{2}{3} \cdot \cancel{\mathbb{E}[\tilde{y}(t-1) \cdot \tilde{\eta}(t)]} - \frac{4}{3} \cdot \mathbb{E}[\tilde{y}(t-1) \cdot \tilde{\eta}(t-1)] - 4 \cdot \cancel{\mathbb{E}[\tilde{\eta}(t) \cdot \tilde{\eta}(t-1)]} \\ &= \frac{1}{9} \cdot \gamma_{\tilde{y}}(0) + \lambda^2 + 4 \cdot \lambda^2 - \frac{4}{3} \cdot \mathbb{E}[\tilde{y}(t-1) \cdot \tilde{\eta}(t-1)] \end{aligned}$$

We compute the term $\mathbb{E}[\tilde{y}(t-1) \cdot \tilde{\eta}(t-1)]$ separately:

$$\begin{aligned}
\mathbb{E}\left[\tilde{y}\left(t-1\right)\cdot\tilde{\eta}\left(t-1\right)\right] &= \mathbb{E}\left[\left(\frac{1}{3}\cdot\tilde{y}\left(t-2\right)+\tilde{\eta}\left(t-1\right)-2\cdot\tilde{\eta}\left(t-2\right)\right)\cdot\tilde{\eta}\left(t-1\right)\right] \\
&= \frac{1}{3}\cdot\cancel{\mathbb{E}\left[\tilde{y}\left(t-2\right)\cdot\tilde{\eta}\left(t-1\right)\right]}+\mathbb{E}\left[\tilde{\eta}\left(t-1\right)^2\right]-2\cdot\cancel{\mathbb{E}\left[\tilde{\eta}\left(t-2\right)\cdot\tilde{\eta}\left(t-1\right)\right]} \\
&= \lambda^2 = 9
\end{aligned}$$

So:

$$\begin{aligned}
\gamma_{\tilde{y}}\left(0\right) &= \frac{1}{9}\cdot\gamma_y\left(0\right)+\lambda^2+4\cdot\lambda^2-\frac{4}{3}\cdot\mathbb{E}\left[\tilde{y}\left(t-1\right)\cdot\tilde{\eta}\left(t-1\right)\right] \\
&= \frac{1}{9}\cdot\gamma_y\left(0\right)+9+4\cdot 9-\frac{4}{3}\cdot 9 \\
\left(1-\frac{1}{9}\right)\gamma_{\tilde{y}}\left(0\right) &= 33 \\
\gamma_{\tilde{y}}\left(0\right) &= \frac{9}{8}\cdot 33 = \frac{297}{8}
\end{aligned}$$

- $\tau = 1$

$$\begin{aligned}
\gamma_{\tilde{y}}\left(1\right) &= \mathbb{E}\left[\left(\tilde{y}\left(t\right)-m_{\tilde{y}}\right)\cdot\left(\tilde{y}\left(t-1\right)-m_{\tilde{y}}\right)\right] = \mathbb{E}\left[\tilde{y}\left(t\right)\cdot\tilde{y}\left(t-1\right)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{3}\cdot\tilde{y}\left(t-1\right)+\tilde{\eta}\left(t\right)-2\cdot\tilde{\eta}\left(t-1\right)\right)\cdot\tilde{y}\left(t-1\right)\right] \\
&= \frac{1}{3}\cdot\mathbb{E}\left[\tilde{y}\left(t-1\right)^2\right]+\cancel{\mathbb{E}\left[\tilde{\eta}\left(t\right)\cdot\tilde{y}\left(t-1\right)\right]}-2\cdot\mathbb{E}\left[\tilde{\eta}\left(t-1\right)\cdot\tilde{y}\left(t-1\right)\right] \\
&= \frac{1}{3}\cdot\gamma_{\tilde{y}}\left(0\right)-2\cdot\lambda^2 = \frac{1}{3}\cdot\frac{297}{8}-2\cdot 9 = -\frac{45}{8}
\end{aligned}$$

- $\tau = 2$

$$\begin{aligned}
\gamma_{\tilde{y}}\left(2\right) &= \mathbb{E}\left[\left(\tilde{y}\left(t\right)-m_{\tilde{y}}\right)\cdot\left(\tilde{y}\left(t-2\right)-m_{\tilde{y}}\right)\right] = \mathbb{E}\left[\tilde{y}\left(t\right)\cdot\tilde{y}\left(t-2\right)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{3}\cdot\tilde{y}\left(t-1\right)+\tilde{\eta}\left(t\right)-2\cdot\tilde{\eta}\left(t-1\right)\right)\cdot\tilde{y}\left(t-2\right)\right] \\
&= \frac{1}{3}\cdot\mathbb{E}\left[\tilde{y}\left(t-1\right)\cdot\tilde{y}\left(t-2\right)\right]+\cancel{\mathbb{E}\left[\tilde{\eta}\left(t\right)\cdot\tilde{y}\left(t-2\right)\right]}-2\cdot\cancel{\mathbb{E}\left[\tilde{\eta}\left(t-1\right)\cdot\tilde{y}\left(t-2\right)\right]} \\
&= \frac{1}{3}\cdot\gamma_{\tilde{y}}\left(1\right) = -\frac{15}{8}
\end{aligned}$$

- $\tau = 3$

$$\begin{aligned}
\gamma_{\tilde{y}}\left(3\right) &= \mathbb{E}\left[\left(\tilde{y}\left(t\right)-m_{\tilde{y}}\right)\cdot\left(\tilde{y}\left(t-3\right)-m_{\tilde{y}}\right)\right] = \mathbb{E}\left[\tilde{y}\left(t\right)\cdot\tilde{y}\left(t-3\right)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{3}\cdot\tilde{y}\left(t-1\right)+\tilde{\eta}\left(t\right)-2\cdot\tilde{\eta}\left(t-1\right)\right)\cdot\tilde{y}\left(t-3\right)\right] \\
&= \frac{1}{3}\cdot\mathbb{E}\left[\tilde{y}\left(t-1\right)\cdot\tilde{y}\left(t-3\right)\right]+\cancel{\mathbb{E}\left[\tilde{\eta}\left(t\right)\cdot\tilde{y}\left(t-3\right)\right]}-2\cdot\cancel{\mathbb{E}\left[\tilde{\eta}\left(t-1\right)\cdot\tilde{y}\left(t-3\right)\right]} \\
&= \frac{1}{3}\cdot\gamma_{\tilde{y}}\left(2\right) = -\frac{5}{8}
\end{aligned}$$

[...]

4) Compute the spectrum $\Gamma_y\left(\omega\right)$ of the process.

SPECTRAL DENSITY $\Gamma_y\left(\omega\right)$

Since $\gamma_{\tilde{y}}(\tau) = \gamma_y(\tau)$ then:

$$\mathit{\Gamma}_{\tilde{y}}(\omega) = \mathit{\Gamma}_y(\omega)$$

Thus we can work with the undiased processes $\tilde{y}(t)\,,\ \tilde{\eta}(t)$.

- From the transfer function definition:

$$\begin{aligned}
\mathit{\Gamma}_y(\omega) = \mathit{\Gamma}_{\tilde{y}}(\omega) &= \left|W\left(e^{j\cdot\omega}\right)\right|^2 \cdot \mathit{\Gamma}_{\tilde{\eta}}(\omega) \\
&= W\left(e^{j\cdot\omega}\right) \cdot W\left(e^{-j\cdot\omega}\right) \cdot \lambda^2 \\
&= \frac{e^{j\cdot\omega} - 2}{e^{j\cdot\omega} - \frac{1}{3}} \cdot \frac{e^{-j\cdot\omega} - 2}{e^{-j\cdot\omega} - \frac{1}{3}} \cdot 9 \\
&= \frac{1 + 4 - 2 \cdot \left(e^{j\omega} + e^{-j\omega}\right)}{1 + \frac{1}{9} - \frac{1}{3} \cdot \left(e^{j\omega} + e^{-j\omega}\right)} \cdot 9 \\
&= \frac{5 - 4 \cdot \cos(\omega)}{\frac{10}{9} - \frac{2}{3} \cdot \cos(\omega)} \cdot 9 \\
&= 81 \cdot \frac{5 - 4 \cdot \cos(\omega)}{10 - 6 \cdot \cos(\omega)}
\end{aligned}$$

We draw the spectrum plot based on some simply computable points:

$$\mathit{\Gamma}_y(0) = 81 \cdot \frac{5 - 4 \cdot \cos(0)}{10 - 6 \cdot \cos(0)} = 81 \cdot \frac{5 - 4}{10 - 6} = \frac{81}{4} = 20.25$$

$$\mathit{\Gamma}_y\left(\frac{\pi}{2}\right) = 81 \cdot \frac{5 - 4 \cdot \cos\left(\frac{\pi}{2}\right)}{10 - 6 \cdot \cos\left(\frac{\pi}{2}\right)} = 81 \cdot \frac{5}{10} = \frac{81}{2} = 40.5 = \mathit{\Gamma}_y\left(-\frac{\pi}{2}\right)$$

$$\mathit{\Gamma}_y(\pi) = 81 \cdot \frac{5 - 4 \cdot \cos(\pi)}{10 - 6 \cdot \cos(\pi)} = 81 \cdot \frac{5 + 4}{10 + 6} = \frac{729}{16} = 45.56 = \mathit{\Gamma}_y(-\pi)$$

High frequencies are more relevant. We expect a very rapidly-varying signal.

Exercise 2 (December 2018 DSI exam)

Consider the following process:

$$y(t) = \frac{z+5}{\left(z+\frac{1}{5}\right)\cdot\left(z-\frac{1}{3}\right)}\cdot e_1(t) + \frac{z}{z-\frac{1}{3}}\cdot e_2(t-1)$$

with $e_1 \perp e_2$, $e_1 \sim WN(0,1)$, $e_2 \sim WN(0,2)$

1. Assess if the process is WSS.
2. Compute the mean of the process m_y.
3. Compute the autocorrelation function $\gamma_y(\tau)$.
4. Compute the spectral density function $\Gamma_y(\omega)$.
5. Plot an approximation of $\Gamma_y(\omega)$.

Preliminary computation.
Before answering the actual points, it's possible to do some simplification on the stochastic process in order to put it in its *canonical form*. This will simplify the computation of the next points as well as obtain a more familiar stochastic process class according to the ones already seen in the theory.

Spectral factorization theorem
Given the generic WSS process equation

$$y(t) = \frac{C(z)}{A(z)}\cdot e(t), \qquad e(t) \sim WN\left(\mu, \lambda^2\right)$$

in order to see whether the process is in its *canonical form* the following points must be checked:

1. $C(z)$ and $A(z)$ are polynomials with the same degree (or equivalently, the relative degree of the transfer function is zero, same number of poles and zeros);
2. $C(z)$ and $A(z)$ are coprime (which means that they share no root, i.e. we do not have any pole-zero cancellations);
3. $C(z)$ and $A(z)$ are monic (a polynomial $Q(z)$ is said to be monic when the nonzero coefficient of the highest power of z is equal to 1);
4. $C(z)$ and $A(z)$ have roots inside the unit circle (i.e., all poles and zeros are such that $|z| < 1$)

First of all, note that there is an all-pass filter:

$$\begin{aligned} y(t) &= \frac{z+5}{\left(z+\frac{1}{5}\right)\cdot\left(z-\frac{1}{3}\right)}\cdot e_1(t) + \frac{z}{z-\frac{1}{3}}\cdot e_2(t-1) \\ &= 5\cdot \underbrace{\left(\frac{1}{5}\cdot\frac{z+5}{z+\frac{1}{5}}\right)}_{\text{stable all-pass filter}}\cdot\frac{1}{z-\frac{1}{3}}\cdot e_1(t) + \frac{z}{z-\frac{1}{3}}\cdot e_2(t-1) \\ &= 5\cdot\frac{1}{z-\frac{1}{3}}\cdot e_1(t) + \frac{z}{z-\frac{1}{3}}\cdot e_2(t-1) \end{aligned}$$

Furthermore, we can note that the z at the numerator of the second part can be put inside the noise e_2 to deal with its time delay. Obtaining:

$$\begin{aligned}
y(t) &= 5 \cdot \frac{1}{z-\frac{1}{3}} \cdot e_1(t) + \frac{z}{z-\frac{1}{3}} \cdot e_2(t-1) \\
&= 5 \cdot \frac{1}{z-\frac{1}{3}} \cdot e_1(t) + \frac{1}{z-\frac{1}{3}} \cdot e_2(t)
\end{aligned}$$

Now it's possible to recollect the common part in the two terms:

$$\begin{aligned}
y(t) &= 5 \cdot \frac{1}{z-\frac{1}{3}} \cdot e_1(t) + \frac{1}{z-\frac{1}{3}} \cdot e_2(t) \\
&= \frac{1}{z-\frac{1}{3}} \cdot \underbrace{(5 \cdot e_1(t) + e_2(t))}_{\eta(t)} \\
&= \frac{1}{z-\frac{1}{3}} \cdot \eta(t)
\end{aligned}$$

where $\eta(t)$ is a new defined stochastic process, with the following properties:
MEAN m_η

$$\begin{aligned}
m_\eta = \mathbb{E}[\eta(t)] &= \mathbb{E}[5 \cdot e_1(t) + e_2(t)] \\
&= 5 \cdot \mathbb{E}[e_1(t)] + \mathbb{E}[e_2(t)] \\
&= 5 \cdot m_{e_1} + m_{e_2} \\
&= 5 \cdot 0 + 0 = 0
\end{aligned}$$

COVARIANCE FUNCTION $\gamma_\eta(\tau)$

$$\begin{aligned}
\gamma_\eta(0) &= \mathbb{E}\left[(\eta(t) - m_\eta)^2\right] = \mathbb{E}\left[\eta(t)^2\right] = \mathbb{E}\left[(5 \cdot e_1(t) + e_2(t))^2\right] \\
&= \mathbb{E}\left[25 \cdot e_1(t)^2 + e_2(t)^2 + 10 \cdot e_1(t) \cdot e_2(t)\right] \\
&= 25 \cdot \mathbb{E}\left[e_1(t)^2\right] + \mathbb{E}\left[e_2(t)^2\right] + 10 \cdot \mathbb{E}[e_1(t) \cdot e_2(t)] \\
&= 25 \cdot \lambda_{e_1}^2 + \lambda_{e_2}^2 + 10 \cdot \underbrace{\cancel{\mathbb{E}[e_1(t) \cdot e_2(t)]}}_{e_1 \perp e_2} \\
&= 25 \cdot 1 + 2 = 27
\end{aligned}$$

and for the other time instant $\tau \neq 0$:

$$\begin{aligned}
\gamma_\eta(\tau) &= \mathbb{E}[(\eta(t) - m_\eta) \cdot ((\eta(t-\tau) - m_\eta))] = \mathbb{E}[\eta(t) \cdot \eta(t-\tau)] = \mathbb{E}[(5 \cdot e_1(t) + e_2(t)) \cdot (5 \cdot e_1(t-\tau) + e_2(t-\tau))] \\
&= \mathbb{E}[25 \cdot e_1(t) \cdot e_1(t-\tau) + 5 \cdot e_2(t) \cdot e_1(t-\tau) + 5 \cdot e_1(t) \cdot e_2(t-\tau) + e_2(t) \cdot e_2(t-\tau)] \\
&= 25 \cdot \underbrace{\cancel{\mathbb{E}[e_1(t) \cdot e_1(t-\tau)]}}_{e_1 \sim WN} + 5 \cdot \underbrace{\cancel{\mathbb{E}[e_2(t) \cdot e_1(t-\tau)]}}_{e_1 \perp e_2} + 5 \cdot \underbrace{\cancel{\mathbb{E}[e_1(t) \cdot e_2(t-\tau)]}}_{e_1 \perp e_2} + 25 \cdot \underbrace{\cancel{\mathbb{E}[e_2(t) \cdot e_2(t-\tau)]}}_{e_2 \sim WN} \\
&= 0
\end{aligned}$$

therefore $\eta(t)$ is a white noise. In particular $\eta(t) \sim WN\left(m_\eta, \lambda_\eta^2\right) = WN(0, 27)$. Thus the new stochastic process equation is:

$$\begin{aligned}
y(t) &= \frac{1}{z-\frac{1}{3}} \cdot \eta(t), \qquad \eta(t) \sim WN(0,27) \\
&= \frac{z}{z-\frac{1}{3}} \cdot z^{-1} \cdot \eta(t) \\
&= \frac{z}{z-\frac{1}{3}} \cdot e(t), \qquad e(t) \sim WN(0,27) \\
&= \frac{1}{1-\frac{1}{3} \cdot z^{-1}} \cdot e(t) \\
&= \frac{C(z)}{A(z)} \cdot e(t) = W(z) \cdot e(t)
\end{aligned}$$

with $A(z) = 1 - \frac{1}{3} \cdot z^{-1}$, $C(z) = 1$ and $z^{-1} \cdot \eta(t) = \eta(t-1) = e(t)$since they have the same mean value and are spectral equivalent (they are "statistically equivalent").

Finally, we obtain the $AR(1)$ process:

$$y(t) = \frac{1}{1-\frac{1}{3} \cdot z^{-1}} \cdot e(t), \qquad e(t) \sim WN(0,27)$$

with recursive form:

$$\begin{aligned}
\left(1 - \frac{1}{3} \cdot z^{-1}\right) \cdot y(t) &= 1 \cdot e(t) \\
y(t) - \frac{1}{3} \cdot z^{-1} \cdot y(t) &= e(t) \\
y(t) - \frac{1}{3} \cdot y(t-1) &= e(t) \\
y(t) &= \frac{1}{3} \cdot y(t-1) + e(t), \qquad e(t) \sim WN(0,27)
\end{aligned}$$

These simplications aren't mandatory, but they simplify the other points a lot. Notice that the process is in its canonical representation.

1) Assess if the process is WSS.

The process is stationary if the poles of the transfer function of the dynamical representation are inside the unitary circle. In this case there is only one pole:

$$-1 \leq z = \frac{1}{3} \leq 1$$

that is inside the unitary circle. Thus the stochastic process is WSS.

2) Compute the mean of the process m_y.

$$y(t) = \frac{1}{3} \cdot y(t-1) + e(t), \qquad e(t) \sim WN(0,27)$$

$$\begin{aligned}
m_y = \mathbb{E}[y(t)] &= \mathbb{E}\left[\frac{1}{3} \cdot y(t-1) + e(t)\right] \\
&= \frac{1}{3} \cdot \mathbb{E}[y(t-1)] + \mathbb{E}[e(t)] \\
&= \frac{1}{3} \cdot m_y + m_e \\
\left(1 - \frac{1}{3}\right) \cdot m_y &= m_e \\
m_y &= \frac{3}{2} m_e = 0
\end{aligned}$$

3) Compute the autocorrelation function $\gamma_y(\tau)$

- for $\tau = 0$:

$$\begin{aligned}
\gamma_y(0) &= \mathbb{E}\left[(y(t)-m_y)^2\right] = \mathbb{E}\left[y(t)^2\right] \\
&= \mathbb{E}\left[\left(\frac{1}{3}\cdot y(t-1)+e(t)\right)^2\right] \\
&= \mathbb{E}\left[\frac{1}{9}\cdot y(t-1)^2+e(t)^2+\frac{2}{3}\cdot y(t-1)\cdot e(t)\right] \\
&= \frac{1}{9}\cdot\mathbb{E}\left[y(t-1)^2\right]+\mathbb{E}\left[e(t)^2\right]+\frac{2}{3}\cdot\cancel{\mathbb{E}\left[y(t-1)\cdot e(t)\right]} \\
&= \frac{1}{9}\cdot\gamma_y(0)+\lambda_e^2
\end{aligned}$$

$$\left(1-\frac{1}{9}\right)\cdot\gamma_y(0) = \lambda_e^2$$

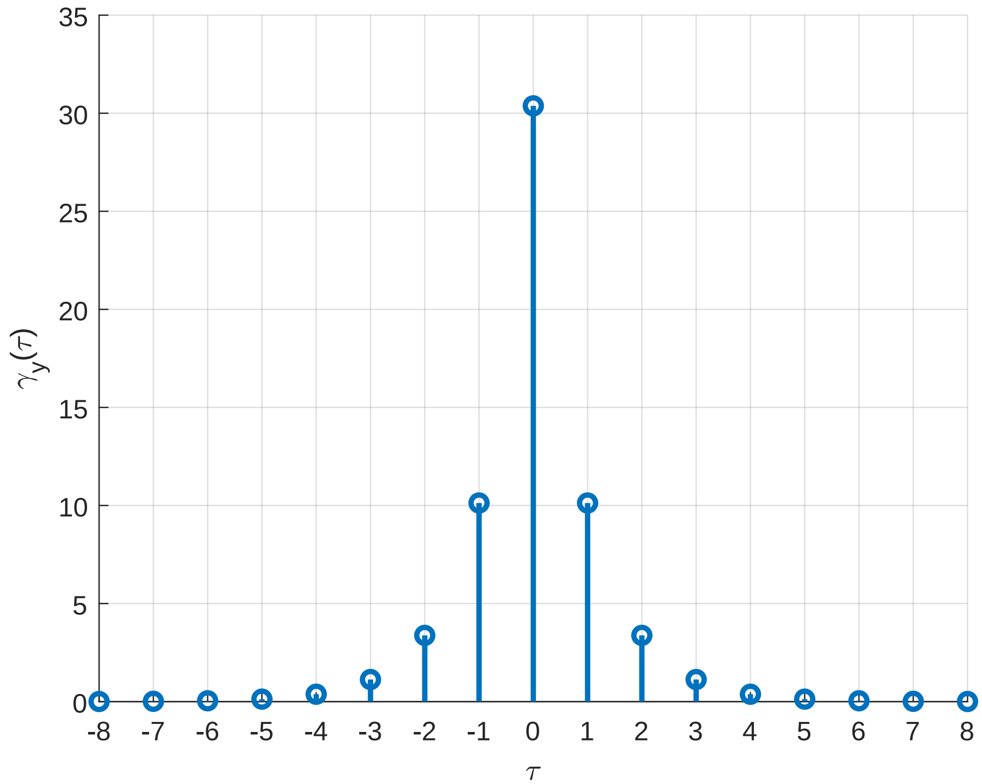
$$\gamma_y(0) = \frac{9}{8}\cdot\lambda_e^2 = \frac{9}{8}\cdot 27 = \frac{243}{8}$$

- for $\tau \neq 0$:

$$\begin{aligned}
\gamma_y(\tau) &= \mathbb{E}\left[(y(t)-m_y)\cdot(y(t-\tau)-m_y)\right] = \mathbb{E}\left[y(t)\cdot y(t-\tau)\right] \\
&= \mathbb{E}\left[\left(\frac{1}{3}\cdot y(t-1)+e(t)\right)\cdot y(t-\tau)\right] \\
&= \frac{1}{3}\cdot\mathbb{E}\left[y(t-1)\cdot y(t-\tau)\right]+\cancel{\mathbb{E}\left[e(t)\cdot y(t-\tau)\right]} \\
\gamma_y(\tau) &= \frac{1}{3}\cdot\gamma_y(\tau-1)
\end{aligned}$$

therefore:

$$\gamma_y(\tau) = \frac{243}{8}\cdot\left(\frac{1}{3}\right)^{|\tau|}$$



4) Compute the spectral density function $\Gamma_y(\omega)$.
The power spectral density function $\Gamma_y(\omega)$ can be computed using the *spectral theorem:*

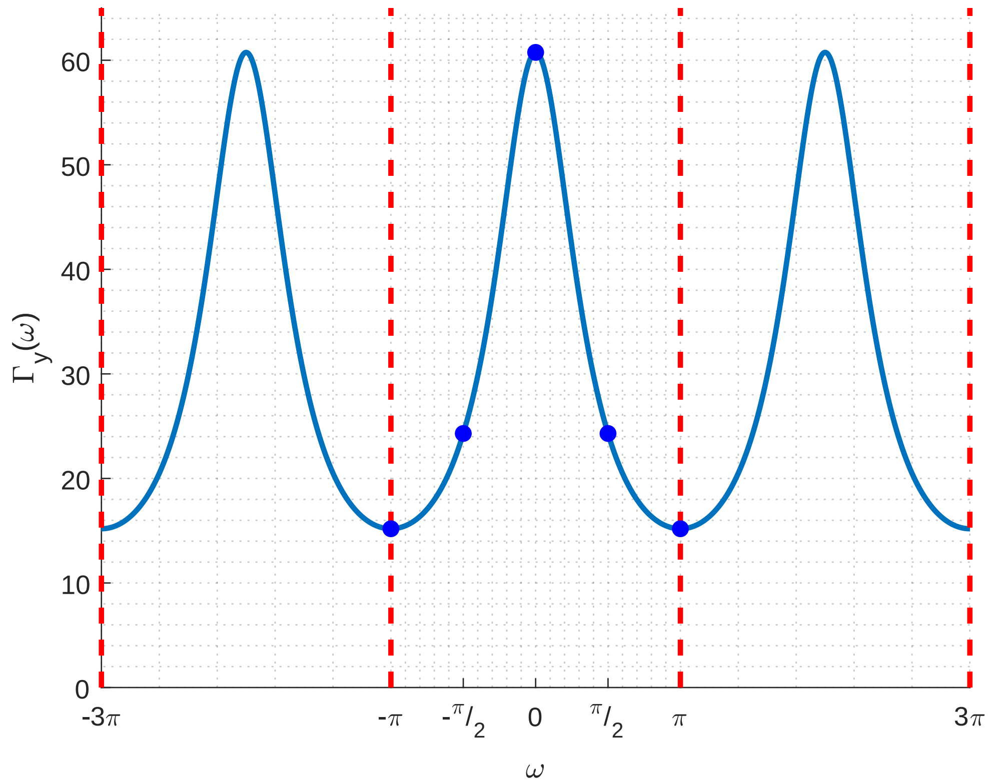
$$\begin{aligned}
\Gamma_y(\omega) &= \left|W\left(e^{j\cdot\omega}\right)\right|^2 \cdot \Gamma_e(\omega) \\
&= \left|\frac{1}{e^{j\cdot\omega}-\frac{1}{3}}\right|^2 \cdot \lambda_e^2 = \frac{1}{\left|e^{j\cdot\omega}-\frac{1}{3}\right|^2} \cdot 27 \\
&= \frac{1}{\left(e^{j\cdot\omega}-\frac{1}{3}\right)\cdot\left(e^{j\cdot\omega}-\frac{1}{3}\right)^*} \cdot 27 = \frac{1}{\left(e^{j\cdot\omega}-\frac{1}{3}\right)\cdot\left(e^{-j\cdot\omega}-\frac{1}{3}\right)} \cdot 27 \\
&= \frac{1}{1+\frac{1}{9}-\frac{2}{3}\cdot\left(e^{j\cdot\omega}+e^{-j\cdot\omega}\right)} \cdot 27 \\
&= \frac{1}{\frac{10}{9}-\frac{2}{3}\cdot\cos(\omega)} \cdot 27 = \frac{9}{10-6\cdot\cos(\omega)} \cdot 27 \\
&= \frac{243}{10-6\cdot\cos(\omega)}
\end{aligned}$$

5) Plot an approximation of $\Gamma_y(\omega)$.
Since the spectral density function is periodic with period $2\cdot\pi$, we can plot the function only between $-\pi$ and $+\pi$. Therefore, we can compute some of the points in this range:

$$\Gamma_y(0) = \frac{243}{10-6\cdot\cos(0)} = \frac{243}{10-6} = \frac{243}{4} = 60.75$$

$$\Gamma_y\left(\frac{\pi}{2}\right) = \frac{243}{10-6\cdot\cos\left(\frac{\pi}{2}\right)} = \frac{243}{10-6\cdot 0} = \frac{243}{10} = 24.3 = \Gamma_y\left(-\frac{\pi}{2}\right)$$

$$\Gamma_y(\pi) = \frac{243}{10-6\cdot\cos(\pi)} = \frac{243}{10-6\cdot(-1)} = \frac{243}{16} = 15.1875 = \Gamma_y(-\pi)$$



Low frequencies are more relevant. We expect a low rapidly-varying signal.

Exercise 3

The following discrete-time state-space model is given:

$$\begin{cases} x_1(t+1) = 0.8 \cdot x_1(t) + u(t) + e(t) \\ x_2(t+1) = x_1(t) + 0.5 \cdot x_2(t) + 4 \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

where $e(t) \sim WN(0,1)$ and $u(t)$ is an exogenous input (e.g. control variable).

1. Find the corresponding ARMAX model:

$$A(z) \cdot y(t) = B(z) \cdot u(t) + C(z) \cdot e(t)$$

2. Find the canonical representation of the process.

3. Find the steady-state effects on the output if the exogenous input is the sinusoid $u(t) = A \cdot \sin(2 \cdot \pi \cdot f \cdot T_s \cdot t + \varphi)$ with $A = 2$, $f = 0.5$, $T_s = 0.1$, $\varphi = 0$.

1) Find the corresponding ARMAX model
The ARMAX model can be simply computed:

$$\begin{cases} x_1(t+1) = 0.8 \cdot x_1(t) + u(t) + e(t) \\ x_2(t+1) = x_1(t) + 0.5 \cdot x_2(t) + 4 \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} z \cdot x_1(t) = 0.8 \cdot x_1(t) + u(t) + e(t) \\ z \cdot x_2(t) = x_1(t) + 0.5 \cdot x_2(t) + 4 \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} (z-0.8) \cdot x_1(t) = u(t) + e(t) \\ (z-0.5) \cdot x_2(t) = x_1(t) + 4 \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{(z-0.8)} \cdot u(t) + \frac{1}{(z-0.8)} \cdot e(t) \\ x_2(t) = \frac{1}{(z-0.5)} \cdot x_1(t) + \frac{4}{(z-0.5)} \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{(z-0.8)} \cdot u(t) + \frac{1}{(z-0.8)} \cdot e(t) \\ x_2(t) = \frac{1}{(z-0.5)} \cdot \frac{1}{(z-0.8)} \cdot u(t) + \frac{1}{(z-0.5)} \cdot \frac{1}{(z-0.8)} \cdot e(t) + \frac{4}{(z-0.5)} \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{(z-0.8)} \cdot u(t) + \frac{1}{(z-0.8)} \cdot e(t) \\ x_2(t) = \frac{1}{(z-0.5)\cdot(z-0.8)} \cdot u(t) + \frac{1+4\cdot(z-0.8)}{(z-0.5)\cdot(z-0.8)} \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} x_1(t) = \frac{1}{(z-0.8)} \cdot u(t) + \frac{1}{(z-0.8)} \cdot e(t) \\ x_2(t) = \frac{1}{(z-0.5)\cdot(z-0.8)} \cdot u(t) + \frac{4\cdot z-2.2}{(z-0.5)\cdot(z-0.8)} \cdot e(t) \\ y(t) = x_2(t) \end{cases}$$

So:

$$\begin{aligned}
y(t) &= x_2(t) \\
&= \frac{1}{(z-0.5)\cdot(z-0.8)}\cdot u(t) + \frac{4\cdot z - 2.2}{(z-0.5)\cdot(z-0.8)}\cdot e(t) \\
&= \frac{z^{-2}}{1-1.3\cdot z^{-1}+0.4\cdot z^{-2}}\cdot u(t) + \frac{4\cdot z^{-1}-2.2\cdot z^{-2}}{1-1.3\cdot z^{-1}+0.4\cdot z^{-2}}\cdot e(t)
\end{aligned}$$

We recognize the three different terms:

$$\begin{aligned}
\left(1-1.3\cdot z^{-1}+0.4\cdot z^{-2}\right)\cdot y(t) &= \left(z^{-2}\right)\cdot u(t) + \left(4\cdot z^{-1}-2.2\cdot z^{-2}\right)\cdot e(t) \\
\left(1-1.3\cdot z^{-1}+0.4\cdot z^{-2}\right)\cdot y(t) &= u(t-2) + \left(4\cdot z^{-1}-2.2\cdot z^{-2}\right)\cdot e(t)
\end{aligned}$$

Thus:

$$\begin{cases}
A(z) = 1-1.3\cdot z^{-1}+0.4\cdot z^{-2} \\
B(z) = 1 \\
C(z) = 4\cdot z^{-1}-2.2\cdot z^{-2} \\
k = 2
\end{cases}$$

$$y(t) = \frac{B(z)}{A(z)}\cdot u(t-k) + \frac{C(z)}{A(z)}\cdot e(t)$$

2) Find the canonical representation of the process.
In order to find the canonical representation of the process we consider it's stochastic part, that's the underlying ARMA model in it's operatorial representation:

$$\begin{aligned}
\frac{y_e(t)}{e(t)} &= W(z) \\
&= \frac{C(z)}{A(z)} \\
&= \frac{4\cdot z^{-1}-2.2\cdot z^{-2}}{1-1.3\cdot z^{-1}+0.4\cdot z^{-2}} \\
&= \frac{4\cdot z-2.2}{z^2-1.3\cdot z+0.4}
\end{aligned}$$

Let's consider the zeroes of the dynamic filter (the roots of the numerator):

$$4\cdot z - 2.2 = 0$$

$$z = \frac{2.2}{4} = 0.55$$

$$|z| < 1 \text{ inside the unit circle}$$

Let's consider the poles of the dynamic filter (the roots of the denominator):

$$z^2 - 1.3\cdot z + 0.4 = 0$$

$$\begin{aligned}
z_{1,2} &= \frac{1.3 \pm \sqrt{1.3^2 - 1.6}}{2} \\
&= \frac{1.3 \pm 0.3}{2}
\end{aligned}$$

$$z_1 = 0.5, \qquad z_2 = 0.8$$

$$|z_{1,2}| < 1 \text{ inside the unit circle}$$

But this representation is not canonical since:

- $C(z)$ and $A(z)$ have not the same degree;
- $C(z)$ is not monic;

So we have to multiply/divide by z and divide/multiply by 4:

$$\begin{aligned} y_e(t) &= W(z) \cdot e(t) \\ &= \frac{4 \cdot z - 2.2}{z^2 - 1.3 \cdot z + 0.4} \cdot \frac{z}{z} \cdot \frac{4}{4} \cdot e(t) \\ &= \frac{z^2 - 0.55 \cdot z}{z^2 - 1.3 \cdot z + 0.4} \cdot z^{-1} \cdot 4 \cdot e(t) \\ &= \frac{z^2 - 0.55 \cdot z}{z^2 - 1.3 \cdot z + 0.4} \cdot 4 \cdot e(t-1) \\ &= W_1(z) \cdot 4 \cdot e(t-1) \end{aligned}$$

Where we are considering only the stochastic part of $y(t)$ for the moment. Let's introduce this white noise:

$$\eta(t) = 4 \cdot e(t-1)$$

Notice that:

$$\begin{aligned} m_\eta &= \mathbb{E}[\eta(t)] \\ &= 4 \cdot \mathbb{E}[e(t-1)] \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lambda_\eta^2 = \gamma_\eta(0) &= \mathbb{E}\left[\eta(t)^2\right] \\ &= 16 \cdot \mathbb{E}\left[e(t-1)^2\right] \\ &= 16 \end{aligned}$$

So:

$$\eta(t) \sim WN(0, 16)$$

The canonical representation of the process is then:

$$\begin{aligned} y_e(t) &= W_1(z) \cdot 4 \cdot e(t-1) \\ &= \frac{z^2 - 0.55 \cdot z}{z^2 - 1.3 \cdot z + 0.4} \cdot \eta(t) \end{aligned}$$

3) Find the steady-state effects on the output if the exogenous input is the sinusoid $u(t) = A \cdot \sin(2 \cdot \pi \cdot f \cdot T_s \cdot t + \varphi)$ **with** $A = 2$, $f = 0.5$, $T_s = 0.1$, $\varphi = 0$.
We found that the system is described by the following equation:

$$A(z) \cdot y(t) = B(z) \cdot u(t-k) + C(z) \cdot e(t)$$

In particular we can separate the effects of the exogenous input and the white noise input on the output:

$$\begin{cases} \frac{y_e(t)}{e(t)} = \frac{C(z)}{A(z)} \\ \frac{y_u(t)}{u(t-k)} = \frac{B(z)}{A(z)} \end{cases}$$

In this example the exogenous input is the sinusoid:

$$\begin{aligned} u(t) &= 2 \cdot \sin(2 \cdot \pi \cdot 0.5 \cdot 0.1 \cdot t) \\ &= 2 \cdot \sin(0.1 \cdot \pi \cdot t) \end{aligned}$$

Frequency response theorem:
If an asymptotically stable dynamical system is fed by a sinusoid with sampling period T_s:

$$u(t) = A \cdot \sin(2 \cdot \pi \cdot f \cdot T_s \cdot t + \varphi)$$

then the steady-state output is a sinusoid, with the same frequency, whose amplitude and phase are "modified" according to the magnitude and phase of the frequency response of the system evaluated at the frequency f:

$$\begin{aligned} y(t) &= A \cdot \left|F\left(e^{j \cdot 2 \cdot \pi \cdot f \cdot T_s}\right)\right| \cdot \sin\left(2 \cdot \pi \cdot f \cdot T_s \cdot t + \varphi + \angle F\left(e^{j \cdot 2 \cdot \pi \cdot f \cdot T_s}\right)\right) \\ &= A_y \cdot \sin(2 \cdot \pi \cdot f \cdot T_s \cdot t + \varphi_y) \\ &= A_y \cdot \sin(\omega_s \cdot t + \varphi_y) \end{aligned}$$

with $A_y = A \cdot \left|F\left(e^{j \cdot 2 \cdot \pi \cdot f \cdot T_s}\right)\right|$, $\varphi_y = \varphi + \angle F\left(e^{j \cdot 2 \cdot \pi \cdot f \cdot T_s}\right)$ and $\omega_s = 2 \cdot \pi \cdot f \cdot T_s$.

We can then use the *Frequency response theorem* to find the steady-state effects on the output:

$$\begin{aligned} F\left(e^{j \cdot \omega_s}\right) &= \left.\frac{B(z)}{A(z)}\right|_{z=e^{j \cdot \omega_s}} \\ &= \left.\frac{1}{z^2 - 1.3 \cdot z + 0.4}\right|_{z=e^{j \cdot \omega_s}} \\ &= \frac{B\left(e^{j \cdot \omega_s}\right)}{A\left(e^{j \cdot \omega_s}\right)} \\ &= \frac{1}{e^{2 \cdot j \cdot \omega_s} - 1.3 \cdot e^{j \cdot \omega_s} + 0.4} \end{aligned}$$

In this case we have $\omega_s = 2 \cdot \pi \cdot f \cdot T_s = 2 \cdot \pi \cdot 0.5 \cdot 0.1 = 0.1 \cdot \pi$ so:

$$\begin{aligned} F\left(e^{j \cdot \omega_s}\right) &= \frac{1}{e^{j \cdot 0.2 \cdot \pi} - 1.3 \cdot e^{j \cdot 0.1 \cdot \pi} + 0.4} \\ &= 5.317 \cdot e^{j \cdot (-1.717)} \end{aligned}$$

$$\begin{aligned} A_y &= A \cdot \left|F\left(e^{j \cdot \omega_s}\right)\right| \\ &= 2 \cdot 5.317 \\ &= 10.63 \end{aligned}$$

$$\begin{aligned} \varphi_y &= \varphi + \angle F\left(e^{j \cdot \omega_s}\right) \\ &= -1.717 \, \mathrm{rad/s} \end{aligned}$$

The output of the system is then:

$$\begin{aligned} y_u(t) &= A_y \cdot \sin(\omega_s \cdot t + \varphi_y) \\ &= 10.63 \cdot \sin(0.1 \cdot \pi \cdot t - 1.717) \end{aligned}$$