Ex. 28.12 [FTRL and Changing Potentials]

**28.12** (FOLLOW-THE-REGULARISED-LEADER AND CHANGING POTENTIALS) Like in the previous exercise, let $\mathcal{A}$ be non-empty and convex and $y_1, \dots, y_n \in \mathcal{L} \subseteq \mathbb{R}^d$. Let $F_1, \dots, F_n, F_{n+1}$ be a sequence of convex functions and $\Phi_t(a) = F_t(a) + \sum_{s=1}^{t-1} \langle a, y_s \rangle$ and $a_t = \operatorname{argmin}_{a \in \mathcal{A}} \Phi_t(a)$, which you may assume are well defined.

(a) Show that

$$R_n(a) \le \sum_{t=1}^{n} (\langle a_t - a_{t+1}, y_t \rangle - D_{F_t}(a_{t+1}, a_t)) + F_{n+1}(a) - F_1(a_1) + \sum_{t=1}^{n} (F_t(a_{t+1}) - F_{t+1}(a_{t+1})) .$$

(b) Show that if $F_t = F/\eta_t$ and $(\eta_t)_{t=1}^{n+1}$ is decreasing with $\eta_n = \eta_{n+1}$, then

$$R_n(a) \le \frac{F(a) - \min_{b \in \mathcal{A}} F(b)}{\eta_n} + \sum_{t=1}^{n} \left( \langle a_t - a_{t+1}, y_t \rangle - \frac{D_F(a_{t+1}, a_t)}{\eta_t} \right) .$$

Again, the statement applies to any sequence of Legendre functions, including those that are constructed based on the past.

Proof. (a).
$$R_n = \sum_{t=1}^{n} \langle a_t - a_{t+1}, y_t \rangle + \sum_{t=1}^{n} \langle a_{t+1} - a, y_t \rangle$$

To bound $\sum_{t=1}^{n} \langle a_{t+1} - a, y_t \rangle$, let $\Phi_t(b) = \sum_{s=1}^{t-1} \langle b, y_s \rangle + F_t(b)$.

Then

$\Phi_{t+1}(a_{t+1}) = \sum_{s=1}^{t} \langle b, y_s \rangle + F_{t+1}(a_{t+1})$

$$\sum_{t=1}^{n} \langle a_{t+1} - a, y_t \rangle$$

$$= \sum_{t=1}^{n} \langle a_{t+1}, y_t \rangle - \sum_{t=1}^{n} \langle a, y_t \rangle$$

$(2,2) \to (n+1, n+1)$ $\quad (1,2) \to (n, n+1)$

$$= \sum_{t=1}^{n} ( \Phi_{t+1}(a_{t+1}) - \Phi_t(a_{t+1}) - (F_{t+1}(a_{t+1}) - F_t(a_{t+1}))) - \Phi_{n+1}(a) + F_{n+1}(a)$$

$$= -\Phi_1(a_1) + \sum_{t=1}^{n} ( \Phi_t(a_t) - \Phi_t(a_{t+1})) + \Phi_{n+1}(a_{n+1}) - \Phi_{n+1}(a)$$

$F_t(a_{t+2})$ $\quad F_{t+1}(a_{t+2})$. $\quad (0,2) \to (n-1, n+1)$. $\quad (1,2) \to (n, n+1)$

$$+ \sum_{t=1}^{n} \left( F_t(a_{t+1}) - F_{t+1}(a_{t+1}) \right) + F_{n+1}(a)$$

$$\leq -F_1(a_1) - \sum_{t=1}^{n} D_{F_t}(a_{t+1}, a_t) + \sum_{t=1}^{n} \left( F_t(a_{t+1}) - F_{t+1}(a_{t+1}) \right)$$

$$+ F_{n+1}(a)$$

(b) Choosing $F_t(b) := \left( F(b) - \min_{c \in \mathcal{A}} F(c) \right) / \eta_t$, together with the fact that adding a constant does not change the Bregman divergence or the policy, concludes the proof. ∎

$D_{F_t}(\cdot,\cdot) = \frac{1}{\eta_t} D_F(\cdot,\cdot)$ $\quad (a_t)_{t=1}^{n+1}$ is not changed