2022.04.22

## 1. [Matrix Norm]

Assume a vecturm norm $\|\cdot\|_\alpha$ on $\mathbb{R}^n$ is given. Then any matrix $A \in \mathbb{R}^{n\times n}$ induces a linear operator from $\mathbb{R}^n \to \mathbb{R}^n$, whose matrix norm (operator norm) w.r.t. $\|\cdot\|_\alpha$ is defined as

$$\|A\|_\alpha := \sup_{x:\, x\neq 0} \frac{\|Ax\|_\alpha}{\|x\|_\alpha}.$$

$$= \sup_{x:\, \|x\|_\alpha = 1} \|A\cdot x\|_\alpha .$$

## 2. [On the equivalence between the eigenvalues and singular values]

Assume $A \in \mathbb{C}^{m\times n}$. Then its singular values are the square roots of $X^*X \in \mathbb{R}^{n\times n}$, where $X^*$ denotes the conjugate transpose of $X$. Hence if $X$ is a real symmetric matrix, then $|\lambda_i(X)| = \sigma_i(X)$, where $\lambda_i(X)$ is the i-th eigenvalue of $X$, and $\sigma_i(X)$ is the i-th singular value of $X$.

## 3. [Spectral radius & Spectral norm]

- Spectral radius is the maximum of the absolute values of all the eigenvalues of matrix $A \in \mathbb{C}^{n\times n}$, defined as

  $\rho(A) = |\lambda_1(A)|$.

- $\rho(A) \leq \|A\|_2$ :

Let $v$ be the unit-eigenvector of $A$ associated with $\lambda_1(A)$, then

$$\|A\|_2 \geq \|A \cdot v\|_2 = \|\lambda_1(A) \cdot v\|_2 = |\lambda_1(A)| = \rho(A).$$

- Spectral norm is the largest singular value of $A \in \mathbb{C}^{n \times n}$.
- $\|A\|_2 \leq \sigma_1(A)$ :

Since $A^T A$ is diagonalizable due to that $A^T A$ is symmetric,

$$\|A \cdot x\|_2^2 = x^T \cdot A^T \cdot A \cdot x = y^T \cdot D \cdot y.$$

If $\|x\|_2 = 1$, then $\|y\|_2 = 1$. Hence

$$\|A\|_2 = \sup_{x:\|x\|_2=1} \|Ax\|_2 = \sqrt{y^T \cdot D \cdot y} = \sqrt{\sum_i y_i^2 \cdot \sigma_i^2(A)} \leq \sigma_1(A),$$

where $y = Q \cdot x$ and $Q^T \cdot D \cdot Q = A^T A$.

- Hence $\rho(A) \leq \sigma_1(A)$ with the equality holds when $A$ is symmetric.