Let B be the backbone edge set for multilayer digraph G with weight function $w : E(G) \to \mathcal{W}$ computed using edge aggregation $g(a, b) = a + b$.

Let $\prec$ be a partial order on the edge weights of G such that there is a zero-element $0 \prec z$ for all $z \neq 0$ and for any $z \neq 0$, $a \precsim b \implies a \precsim b + z$ (here, $a \precsim b$ means $b \not\prec a$).

Proposition 1. *Let m be an edge connecting u to v such that a) any edge x_i ending at v satisfies $w(m) \precsim w(x_i)$ or b) any edge y_i starting at u satisfies $w(m) \precsim w(y_i)$. Then $m \in B$, i.e., $w(m) \precsim p$ for any path P of weight p connecting u to v.*

Proof. If P is direct, then $w(m) \precsim p$ by either assumption a or assumption b. Otherwise, P has more than one edge. If assumption a holds, then P contains some x_i ending at v and there exists some weight $z \neq 0$ such that $p = w(x_i) + z$. Because $w(m) \precsim w(x_i)$, it follows that $w(m) \precsim w(x_i) + z = p$. Similarly, if assumption b holds, $w(m) \precsim w(y_i) + z = p$. Thus, $w(m) \precsim p$, implying $m \in B$. □

Proposition 2. *Let $\{m_\alpha\}$ be any set edges in B beginning at u, and let e_i be the remaining edges beginning at u with $w(e_0) \precsim w(e_i)$ for all i and e_0 ending at v. If i) if any m_α connects u to v, then $w(e_0) \simeq w(m_\alpha)$ and ii) $w(e_0)$ is not longer than the weight of any path of two hops containing an edge in $\{m_\alpha\}$, then $e_0 \in B$.*

Proof. Consider a path P of weight p that connects u to v. If P is direct, then either $p = w(e_i)$ for some i, in which case e contains some e_i and $w(e_0) \precsim p$ by assumption, or $p = w(m_\alpha)$ for some alpha, and $w(e_0) \precsim p$ by condition i). Otherwise, P conains two or more edges. If P does not contain any m_α, then $p = w(e_i) + z$ for some $z \neq 0$, and thus $w(e_0) \precsim p$. Otherwise, P contains a path of two hops containing an edge in $\{m_\alpha\}$, implying by condition ii) that $w(e_0) \precsim p$. Thus, $w(e_0) \precsim p$ and so $e_0 \in B$. □