# Infinite Sequence & Series

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CAT 性能优化中心

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# 1 Sequences

# Limits of Sequences

A **Sequence** is a function whose domain is a set of form $\{n \in \mathbb{Z} : n \geq m\}$; $m$ is usually 1 or 0. It is denoted by symbols $\{a_n\}$ or $(s_n)$;$(s_n) = (s_1, s_2 \ldots, s_n)$.

**Example**

**(a)** Consider a sequence $(s_n)_{n\in\mathbb{N}}$ where $(s_n) = \frac{1}{n^2}$. This is the sequence $\left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots\right)$ which is a function with domain $\mathbb{N}$ whose values at each $n$ is $\frac{1}{n^2}$.

**(b)** If $a_n = (n)^{\frac{1}{n}}, n \in \mathbb{N}$ then sequence is $\left(1, (2)^{\frac{1}{2}}, (3)^{\frac{1}{3}}, \ldots\right)$. It turns out $a_{100} \approx 1.0471 \wedge a_{1000} \approx 1.0069$.

**Definition**

A Sequence $(s_n)$ of real numbers is said to be *converge* to real number $s$ provided that for each $\varepsilon > 0$ $\exists$ a number $N$

$$n > N \Rightarrow |s_n - s| < \varepsilon$$

If $(s_n)$ convergers to $s$ ($\lim_{n\to\infty} s_n = s$).The number $s$ is called limit of sequence$(s_n)$.

A sequence that does not *converge* to some real number is said to **diverge**.

# Discussion about Proofs

**Example** Prove $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

*Discussion*. For each $\varepsilon > 0$

$$|\frac{3n+1}{7n-4} - \frac{3}{7}| < \varepsilon \Rightarrow \frac{19}{49\varepsilon} + \frac{4}{7} < n$$

Our steps are reversible, so we'll put $N = \frac{19}{49\varepsilon} + \frac{4}{7}$

**Formal Proof** Let $\varepsilon > 0$ and $N = \frac{19}{49\varepsilon} + \frac{4}{7}$.Then $n > N \Rightarrow n > \frac{19}{49\varepsilon} + \frac{4}{7}$

$$\therefore \frac{19}{7(7n-4)} < \varepsilon \Rightarrow |\frac{3n+1}{7n-4} - \frac{3}{7}| < \varepsilon$$

This proves $\lim \frac{3n+1}{7n-4} = \frac{3}{7}$.

**Sandwich Theorem**

Let $\{a_n\}, \{b_n\} \wedge \{c_n\}$ be sequences of real numbers. If $a_n \leq b_n \leq c_n \forall n > N$ if $\lim a_n = \lim c_n = L$ then $\lim b_n = L$.

**Example** $\lim_{n\to\infty} \frac{\cos(n)}{n}$

Sol. As $-1 \leq \cos(n) \leq 1 \Rightarrow -\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n} \wedge \lim_{n\to\infty} \frac{1}{|n|} = 0 = L$. Hence limit converges to 0.

# Johann Bernoulli Rule (L'Hôpital's Rule)

It states that functions $f \wedge g$ which are defined on open interval $I$ and differentiable on $\{I - c\}$ for a (possibly infinite) accumulation point $c$ of $I$,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

**Definition**

A sequence $\{a_n\}$ is **nondecreasing** if $a_n \leq a_{n+1} \forall n$. The sequence is **nonincreasing** if $a_n \geq a_{n+1} \forall n$. The sequence is said to be **monotonic** if it is either nondecreasing or nonincreasing.

**Monotonic sequence Theorem**

If a sequence $\{a_n\}$ is both bounded and monotonic, then sequence $converges$.

**Stirling's Approximation**

$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

# An Infinite Series

**Definition**

Given a sequence of numbers $\{a_n\}$, an expression of form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots$$

is an **infinite series**. $a_n$ is **nth term** of series. The sequence $\{s_n\}$ is defined by

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$\vdots$$

$$s_n = a_1 + a_2 + \dots + a_n = \sum_{k=0}^{n} a_k$$

is the sequence of partial sums of series, the number $s_n$ being nth partial sum. If sequence of partial sum converges to limit **L**

$$s_n = L$$

# Integral Test

**Definition**

Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where $f$ is a continuous, positive, decreasing function of $x \forall x \geq N$.Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)dx$ both diverge or both converge.

**Example** Show that **p-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \ldots + \frac{1}{n^p} + \ldots, p \in \mathbb{R}$$

converges if $p > 1$ and diverges if $p \leq 1$.

**Proof** If $p > 1$, then $f(x) = \frac{1}{x^p}$ is nonincreasing function $\forall x \in \mathbb{R}^+$

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^p} = \lim_{\varepsilon \to \infty} \left[ \frac{x^{-p+1}}{p+1} \right]_1^{\varepsilon} = \frac{1}{p-1}$$

If $0 < p < 1$ then $1 - p > 0 \wedge \int_1^{\infty} \frac{\mathrm{d}x}{x^p} = \frac{1}{1-p} \lim(b^{1-p} - 1) = \infty$

Series diverges by integral test.

# Comparison Test

### Definition

Let $\sum a_n, \sum c_n \wedge \sum d_n$ be series with non negative terms. Suppose that for some integer $N$

$$d_n \leq a_n \leq c_n \forall n > N$$

1. If $\sum c_n$ converges then $\sum a_n$ also converges.
2. If $\sum d_n$ diverges then $\sum a_n$ also diverges.

## Limit Comparison Test

### Definition

Suppose that $a_n > 0 \wedge b_n > 0 \ \forall n \geq N, N \in \mathbb{Z}$

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n \wedge \sum b_n$ both converge or diverge.
2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ coverges.
3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

## Absolute Convergence Test

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$.

# The Ratio Test

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho$$

Then

1. Series **converges** absolutely if $\rho < 1$
2. Series **diverges** if $\rho > 1$
3. Test is inconclusive if $\rho = 1$

**The Root Test**

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} (|a_n|)^{\frac{1}{n}} = \rho$$

Then

1. Series **converges** absolutely if $\rho < 1$
2. Series **diverges** if $\rho > 1$
3. Test is inconclusive if $\rho = 1$

# Raabe's Test

Let $\sum a_n$ be any series, then

$$\rho_n \equiv n\left(\frac{a_n}{a_{n+1}} - 1\right)$$

1. Converge if $\rho = \lim_{n\to\infty} \rho_n > 1$
2. Diverge if $\lim_{n\to\infty} \rho_n < 1$
3. Test is inconclusive if $\rho = 1$

**Betrand Test**

Let $\sum a_n (a_n > 0, \forall n \in \mathbb{N})$ be any series, then

$$\rho_n \equiv \left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) \ln(n)$$

1. Converges if $\lim_{n\to\infty} \rho_n > 1$
2. Diverges if $\lim_{n\to\infty} \rho_n < 1$

# The Alternating Series Test

The series

$$\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \ldots$$

converges if

1. The $u_n{}'s$ are all positive.
2. The positive $u_n{}'s$ are nonincreasing $u_n \geq u_{n+1} \forall n \geq N$ for some $N \in \mathbb{Z}$
3. $u_n \to 0$

# Exercises

1. Show that if $\sum_{n=1}^{\infty} s_n$ converges then

$$\sum_{n=1}^{\infty} \left( \frac{1 + \sin(s_n)}{2} \right)^n$$

converges.

2. Prove that **(The Basel problem)**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

(HINT: Use fourier series $f(x) = x^2 \wedge x \in [-\pi, \pi]$)

## References


1. Springer, "Elementary Analysis",$Theory\ of\ Calculus$,Kenneth A. Ross, edition 2., pp. 33-46,2013
2. Pearson, "Early Transcendentals",$Thomas's\ Calculus$,George B. Thomas, Jr.,edition 13.,pp. 572-652,2014