The 1st MWIT-KVIS Integration Bee
Playoff Solutions

Acknowledgements

We would like to thank Thitiwat Kosolpattanadurong, Atiratch Laoharenoo, Patthadon Phengpinij, Sirawit Pipittanaban, Wasanont Pongsawat, Chanatip Sujsantinukul, Tanupat Trakulthongchai for problem proposals.

1 Quarterfinal 1

Problem 1 [*] $\displaystyle\int \sin x \sin 2x \sin 3x \sin 4x \cos x \cos 2x \cos 3x \cos 4x \, dx$

Solution.

$$\begin{aligned}
&\int \sin x \sin 2x \sin 3x \sin 4x \cos x \cos 2x \cos 3x \cos 4x \, dx \\
&= \boxed{\frac{1}{16}\left(\frac{x}{8} - \frac{\sin(12x)}{96} - \frac{\sin(16x)}{128} + \frac{\sin(20x)}{160}\right)}.
\end{aligned}$$

□

Proposer: Tanupat Trakulthongchai

Problem 2 [***] $\displaystyle\int_0^{\pi/2} \left(\frac{x}{\sin x}\right)^2 dx$

Solution. Using integration by parts to remove the x^2 term, choose $f(x) = x^2$ and $g'(x) = \csc^2 x$. Then, $f'(x) = 2x$ and $g(x) = -\cot x$. Hence,

$$\begin{aligned}
I &= \int_0^{\pi/2} \left(\frac{x}{\sin x}\right)^2 dx \\
&= -x^2 \cot x \Big|_0^{\pi/2} + 2\int_0^{\pi/2} x \cot x \, dx \\
&= 2\int_0^{\pi/2} x \cot x \, dx
\end{aligned}$$

Using integration by parts to remove the x term, choose $f(x) = x$ and $g'(x) = \cot x$. Then, $f'(x) = 1$ and $g(x) = \ln \sin x$. Therefore,

$$\begin{aligned} I &= 2\int_0^{\pi/2} x \cot x \, dx \\ &= 2x \ln \sin x \Big|_0^{\pi/2} - 2\int_0^{\pi/2} \ln \sin x \, dx \\ &= -2\int_0^{\pi/2} \ln \sin x \, dx \end{aligned}$$

For convenience, define $I_1 = \displaystyle\int_0^{\pi/2} \ln \sin x \, dx$. Using the fact that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

The integral I_1 can be rewritten as

$$\begin{aligned} I_1 &= \int_0^{\pi/2} \ln \sin x dx \\ &= \int_0^{\pi/2} \ln \sin\left(\frac{\pi}{2} - x\right) dx \\ &= \int_0^{\pi/2} \ln \cos x dx \end{aligned}$$

$$\begin{aligned} 2I_1 &= \int_0^{\pi/2} \ln \sin x \, dx + \int_0^{\pi/2} \ln \cos x \, dx \\ &= \int_0^{\pi/2} (\ln \sin 2x - \ln 2) \, dx \\ &= \frac{1}{2}\int_0^{\pi} \ln \sin x \, dx - \frac{\pi \ln 2}{2} \\ &= \frac{1}{2}\int_0^{\pi/2} \ln \sin x \, dx + \frac{1}{2}\int_{\pi/2}^{\pi} \ln \sin x \, dx - \frac{\pi \ln 2}{2} \\ &= \frac{1}{2}I_1 + \frac{1}{2}I_1 - \frac{\pi \ln 2}{2} \\ &= I_1 - \frac{\pi \ln 2}{2} \end{aligned}$$

$$I_1 = -\frac{\pi \ln 2}{2}$$

$$\begin{aligned} I &= -2I_1 \\ &= \boxed{\pi \ln 2} \end{aligned}$$

□

Proposer: Cat Sodium

Problem 3 [*] $\displaystyle\int_{-1}^{1}\left(\sin^{-1}(x)\right)^2\,dx$

Solution. Given $\theta=\sin^{-1}(x)$, such that $x=\sin(\theta)$ and $\frac{dx}{d\theta}=\cos(\theta)$. Note that $\theta\to\frac{\pi}{2}$ where $x\to 1$ and $\theta\to-\frac{\pi}{2}$ where $x\to-1$.

$$I:=\int_{-1}^{1}\left(\sin^{-1}(x)\right)^2\,dx=\int_{-\pi/2}^{\pi/2}\theta^2\cdot\cos(\theta)\,d\theta$$

Using integration by parts,

$$\begin{aligned}
I&=\int_{-\pi/2}^{\pi/2}\theta^2\cdot\cos(\theta)\,d\theta\\
&=\int_{-\pi/2}^{\pi/2}\theta^2\,d\left(\sin(\theta)\right)\\
&=\left(\theta^2\right)\left(\sin(\theta)\right)\Bigg|_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}\sin(\theta)\,d\left(\theta^2\right)\\
&=\left\{\left(\left(\frac{\pi}{2}\right)^2\sin\left(\frac{\pi}{2}\right)\right)-\left(\left(-\frac{\pi}{2}\right)^2\sin\left(-\frac{\pi}{2}\right)\right)\right\}-\int_{-\pi/2}^{\pi/2}2\theta\sin(\theta)\,d\left(\theta\right)\\
&=\left\{\frac{\pi^2}{4}-\left(-\frac{\pi^2}{4}\right)\right\}-2\left[\int_{-\pi/2}^{\pi/2}\theta\sin(\theta)\,d\theta\right]\\
&=\frac{\pi^2}{2}-2\left[\int_{-\pi/2}^{\pi/2}\theta\,d\left(-\cos(\theta)\right)\right]\\
&=\frac{\pi^2}{2}-2\left[(\theta)\left(-\cos(\theta)\right)\Bigg|_{-\pi/2}^{\pi/2}-\int_{-\pi/2}^{\pi/2}-\cos(\theta)\,d\theta\right]\\
&=\frac{\pi^2}{2}-2\left[\left\{\left(-\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right)\right)-\left(-\left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}\right)\right)\right\}+\int_{-\pi/2}^{\pi/2}\cos(\theta)\,d\theta\right]\\
&=\frac{\pi^2}{2}-2\left[\sin(\theta)\Bigg|_{-\pi/2}^{\pi/2}\right]\\
&=\frac{\pi^2}{2}-2\left[\left(\sin\left(\frac{\pi}{2}\right)\right)-\left(\sin\left(-\frac{\pi}{2}\right)\right)\right]\\
&=\frac{\pi^2}{2}-2\left[1-(-1)\right]\\
&=\frac{\pi^2}{2}-4\\
&=\boxed{\frac{1}{2}\left(\pi^2-8\right).}
\end{aligned}$$

$\square$

Proposer: Patthadon Phengpinij

Problem 4 [*] $\displaystyle\int \frac{3e^{2x} - 3e^{x}}{e^{3x}+1} dx$

Solution. Let $u = e^x$. So $du = e^x dx$. Thus

$$\begin{aligned}
\int \frac{3e^{2x} - 3e^{x}}{e^{3x}+1} dx &= \int \frac{3u-3}{u^3+1} du \\
&= \int \frac{2u-1}{u^2-u+1} du - \int \frac{2}{u+1} du \\
&= \ln|u^2 - u + 1| - 2\ln|u+1| + C \\
&= \boxed{\ln|e^{2x} - e^{x} + 1| - 2\ln|e^{x}+1| + C.}
\end{aligned}$$

$\square$

Proposer: Pommekung

2 Quarterfinal 2

Problem 5 [**] $\displaystyle\int_0^\infty \frac{x\,(1-\ln(x))}{1+x^4}\, dx$

Solution. Notice that,

$$\begin{aligned}
\int_0^\infty \frac{x\,(1-\ln(x))}{1+x^4}\, dx &= \int_0^\infty \frac{x - x\ln(x)}{1+x^4}\, dx \\
&= \int_0^\infty \frac{x}{1+x^4}\, dx - \int_0^\infty \frac{x\ln(x)}{1+x^4}\, dx
\end{aligned}$$

Given $I := \displaystyle\int_0^\infty \frac{x\,(1-\ln(x))}{1+x^4}\, dx$, $I_1 := \displaystyle\int_0^\infty \frac{x}{1+x^4}\, dx$, and $I_2 := \displaystyle\int_0^\infty \frac{x\ln(x)}{1+x^4}\, dx$.
Consider I_1, substitute $u = x^2$, $du = 2x\, dx$;

$$
\begin{aligned}
I_1 &= \int_{x=0}^{x=\infty} \frac{x}{1+x^4}\,dx \\
&= \int_{u=0}^{u=\infty} \frac{x}{1+u^2}\,\frac{du}{2x} \\
&= \frac{1}{2}\int_0^\infty \frac{1}{1+u^2}\,du \\
&= \frac{1}{2}\tan^{-1}(u)\Big|_0^\infty \\
&= \frac{1}{2}\left(\frac{\pi}{2}-0\right) \\
&= \frac{\pi}{4}
\end{aligned}
$$

and, consider I_2, substitute $x = e^u$, $dx = e^u\,du$;

$$
\begin{aligned}
I_2 &= \int_{x=0}^{x=\infty} \frac{x\ln(x)}{1+x^4}\,dx \\
&= \int_{u=-\infty}^{u=\infty} \frac{e^u\ln\left(e^u\right)}{1+\left(e^u\right)^4}\cdot e^u\,du \\
&= \int_{-\infty}^{\infty} \frac{ue^{2u}}{1+e^{4u}}\,du \\
&= \int_{-\infty}^{0} \frac{ue^{2u}}{1+e^{4u}}\,du + \int_0^\infty \frac{ue^{2u}}{1+e^{4u}}\,du \\
&= \int_0^\infty \frac{ue^{2u}}{1+e^{4u}} + \frac{-ue^{-2u}}{1+e^{-4u}}\,du \\
&= \int_0^\infty \frac{ue^{2u}}{1+e^{4u}} - \frac{\left(ue^{-2u}\right)\times e^{4u}}{\left(1+e^{-4u}\right)\times e^{4u}}\,du \\
&= \int_0^\infty \frac{ue^{2u}}{1+e^{4u}} - \frac{ue^{2u}}{1+e^{4u}}\,du \\
&= 0
\end{aligned}
$$

Because $I = I_1 - I_2$, thus,

$$
\begin{aligned}
I &= I_1 - I_2 \\
&= \frac{\pi}{4} - 0 \\
&= \boxed{\frac{\pi}{4}.}
\end{aligned}
$$

□

Proposer: Patthadon Phengpinij

Problem 6 [**] $\int_0^1 x\cdot\arcsin\left(\sin\left(\frac{1}{x}\right)\right)\,dx$

Solution.

$$\int_0^1 x \cdot \arcsin\left(\sin\left(\frac{1}{x}\right)\right)\, dx = \boxed{\frac{1}{2}.}$$

□

Proposer: Patthadon Phengpinij

Problem 7 [*] $\displaystyle\int \frac{1}{ax^2+bx+c}\, dx \quad \text{when } b^2-4ac<0$

Solution.

$$\int \frac{1}{ax^2+bx+c}\, dx = \boxed{\frac{2}{\sqrt{4ac-b^2}}\arctan\left(\frac{2ax+b}{\sqrt{4ac-b^2}}\right)}$$

□

Proposer: Sirawit Pipittanaban

Problem 8 [*] $\displaystyle\int \ln(1+x^{\frac{1}{3}})\, dx$

Solution. Using integration by parts, choose $f(x)=\ln(1+x^{1/3})$ and $g(x)=x$,

$$\begin{aligned} I := \int \ln(1+x^{1/3})\, dx &= x\ln(1+x^{1/3}) - \int x \cdot \frac{\frac{1}{3}x^{-2/3}}{1+x^{1/3}}\, dx \\ &= x\ln(1+x^{1/3}) - \frac{1}{3}\int \frac{x^{1/3}}{1+x^{1/3}}\, dx \end{aligned}$$

For the latter term, substitute $x=(u-1)^3$ and $dx = 3(u-1)^2du$.

$$\begin{aligned} \frac{1}{3}\int \frac{x^{1/3}}{1+x^{1/3}}\, dx &= \int \frac{(u-1)^3}{u}\, du \\ &= \int u^2 - 3u + 3 - \frac{1}{u}\, du \\ &= \frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln u \\ &= \frac{1}{3}(x^{1/3}+1)^3 - \frac{3}{2}(x^{1/3}+1)^2 + 3(x^{1/3}+1) - \ln(x^{1/3}+1) \\ &= -\frac{x^{2/3}}{2} + \frac{x}{3} + x^{1/3} - \ln(x^{1/3}+1) \end{aligned}$$

$$\begin{aligned}\int \ln(1+x^{\frac{1}{3}})\,dx &= x\ln(1+x^{1/3}) - \left(-\frac{x^{2/3}}{2}+\frac{x}{3}+x^{1/3}-\ln(x^{1/3}+1)\right)\\ &= \boxed{\frac{1}{6}(3x^{2/3}-2x-6x^{1/3})+(x+1)\ln(x^{1/3}+1)+C}\end{aligned}$$

□

Proposer: Cat sodium

3 Quarterfinal 3

Problem 9 [*] $\displaystyle\int \frac{e^x\cos\left(\ln\left(\tan^{-1}(e^x)\right)\right)}{(1+e^{2x})\cdot\tan^{-1}(e^x)}\,dx$

Solution. Using u-substitution, given $u=\sin\left(\ln\left(\tan^{-1}(e^x)\right)\right)$.
Therefore,

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}\sin\left(\ln\left(\tan^{-1}(e^x)\right)\right)\\ &= \cos\left(\ln\left(\tan^{-1}(e^x)\right)\right)\cdot\frac{d}{dx}\ln\left(\tan^{-1}(e^x)\right)\\ &= \cos\left(\ln\left(\tan^{-1}(e^x)\right)\right)\cdot\frac{1}{\tan^{-1}(e^x)}\cdot\frac{d}{dx}\tan^{-1}(e^x)\\ &= \cos\left(\ln\left(\tan^{-1}(e^x)\right)\right)\cdot\frac{1}{\tan^{-1}(e^x)}\cdot\frac{1}{1+(e^x)^2}\cdot\frac{d}{dx}e^x\\ &= \cos\left(\ln\left(\tan^{-1}(e^x)\right)\right)\cdot\frac{1}{\tan^{-1}(e^x)}\cdot\frac{1}{1+e^{2x}}\cdot e^x.\end{aligned}$$

Hence,

$$\begin{aligned}\int \frac{e^x\cos\left(ln\left(\tan^{-1}(e^x)\right)\right)}{(1+e^{2x})\cdot\tan^{-1}(e^x)}\,dx &= \int \frac{du}{dx}\,dx\\ &= \int du\\ &= u+C\\ &= \boxed{\sin\left(\ln\left(\tan^{-1}(e^x)\right)\right)+C.}\end{aligned}$$

□

Proposer: Patthadon Phengpinij

Problem 10 [*] $\displaystyle\int \frac{e^x\,dx}{\sqrt{3e^x-2-e^{2x}}}$

Solution. First, we let $u = e^x$, then $du = e^x dx$. So,

$$\int \frac{e^x dx}{\sqrt{3e^x - 2 - e^{2x}}} = \int \frac{du}{\sqrt{3u - 2 - u^2}} = \int \frac{du}{\sqrt{\frac{1}{4} - \left(u - \frac{3}{2}\right)^2}}.$$

Next, let $u - \frac{3}{2} = \frac{1}{2}\sin\theta$, then $du = \frac{1}{2}\cos\theta d\theta$ and $\theta = \arcsin(2u - 3)$. Hence,

$$\int \frac{du}{\sqrt{\frac{1}{4} - \left(u - \frac{3}{2}\right)^2}} = \int d\theta = \theta + C = \arcsin(2u - 3) + C.$$

Substitutes $u = e^x$, we have

$$\int \frac{e^x dx}{3e^x - 2 - e^{2x}} = \boxed{\arcsin(2e^x - 3) + C.}$$

□

Proposer: PolarBear

Problem 11 [**] $\int (x^2 + 2)\frac{\sin x}{x^3} dx$

Solution. Firstly, note that $\int (x^2 + 2)\frac{\sin x}{x^3} dx = \int \frac{\sin x}{x} dx + 2\int \frac{\sin x}{x^3} dx$. We will consider the first integral. By using integration by-part twice, we have that

$$\int \frac{\sin x}{x} dx = -\frac{\cos x}{x} - \int \frac{\cos x}{x^2} dx = -\frac{\cos x}{x} - \frac{\sin x}{x^2} - 2\int \frac{\sin x}{x^3} dx.$$

Therefore, $\int (x^2 + 2)\frac{\sin x}{x^3} dx = \boxed{-\frac{\cos x}{x} - \frac{\sin x}{x^2} + C}$. □

Proposer: Chanatip S

Problem 12 [*] $\int \sin^{-1}\left(\sqrt{x}\right)\, dx$

Solution. Using substitution, let $u = \sqrt{x}$, then $u\, du = \frac{1}{2}dx$.
Given,

$$I := \int \sin^{-1}\left(\sqrt{x}\right)\, dx = \int \sin^{-1}(u) \cdot 2u\, du$$
$$I = \int 2u \sin^{-1}(u)\, du$$

Substitute $\theta = \sin^{-1}(u)$, $\sin(\theta) = u$ and $du = cos(\theta)d\theta$, thus,

$$I = \int 2\theta \sin(\theta) \cdot \cos(\theta)\, d\theta$$
$$= \int \theta \sin(2\theta)\, d\theta$$

Using integration by parts,

$$
\begin{aligned}
I &= \int \theta \sin(2\theta)\, d\theta \\
&= -\frac{1}{2}\theta\cos(2\theta) - \int -\frac{1}{2}\cos(2\theta)\, d\theta \\
&= -\frac{1}{2}\theta\cos(2\theta) + \frac{1}{2}\int \cos(2\theta)\, d\left(\frac{2\theta}{2}\right) \\
&= -\frac{1}{2}\theta\cos(2\theta) + \frac{1}{4}\sin(2\theta) + C \\
&= -\frac{1}{2}\theta\left(1 - 2\sin(\theta)^2\right) + \frac{1}{2}\sin(\theta)\cos(\theta) + C \\
&= -\frac{1}{2}\sin^{-1}(u)\left(1-2u^2\right) + \frac{1}{2}u\sqrt{1-u^2} + C \\
&= \boxed{\frac{1}{2}\left((2x-1)\sin^{-1}(\sqrt{x}) + \sqrt{x(1-x)}\right) + C.}
\end{aligned}
$$

□

Proposer: Patthadon Phengpinij

4 Quarterfinal 4

Problem 13 [*] $\int (11x^{11} + 10x^5 + 9x^3)\sqrt{x^9 + 2x^3 + 3x}\, dx$

Solution. The integral can be rewritten as

$$
\begin{aligned}
I &= \int (11x^{11} + 10x^5 + 9x^3)\sqrt{x^9 + 2x^3 + 3x}\, dx \\
&= \int (11x^{10} + 10x^4 + 9x^2)\sqrt{x^2(x^9 + 2x^3 + 3x)}\, dx \\
&= \int (11x^{10} + 10x^4 + 9x^2)\sqrt{x^{11} + 2x^5 + 3x^3}\, dx
\end{aligned}
$$

Substitute $t = x^{11} + 2x^5 + 3x^3$ such that $\frac{dt}{dx} = \frac{d}{dx}\left(x^{11} + 2x^5 + 3x^3\right) = 11x^{10} + 10x^4 + 9x^2$. Consequently,

$$
\begin{aligned}
I &= \int \sqrt{t}dt \\
&= \frac{2}{3}t^{3/2} + C \\
&= \boxed{\frac{2}{3}(x^{11} + 2x^5 + 3x^3)^{3/2} + C.}
\end{aligned}
$$

□

Proposer: Cat Sodium *(Inspired by CHMMC 2023 Finals)*

Problem 14 [**] $\displaystyle\int \frac{\ln(x+1)}{\sqrt{x}}dx$

Solution. Using integration by parts, choose $f(x) = \ln(x+1)$ and $g'(x) = \frac{1}{\sqrt{x}}$, such that $f'(x) = \frac{1}{x+1}$ and $g(x) = 2\sqrt{x}$. Therefore,

$$\begin{aligned} I &= \int \frac{\ln(x+1)}{\sqrt{x}}dx \\ &= 2\ln(x+1)\sqrt{x} - 2\int \frac{\sqrt{x}}{x+1}dx \end{aligned}$$

Substitute $u = \sqrt{x}$, so that $du = \frac{1}{2\sqrt{x}}dx$. Hence,

$$\begin{aligned} \int \frac{\sqrt{x}}{x+1}dx &= \int \frac{2u^2}{u^2+1}du \\ &= \int \left(2 - \frac{2}{u^2+1}\right)du \\ &= 2u - 2\arctan u \\ &= 2\sqrt{x} - 2\arctan\sqrt{x} \end{aligned}$$

Therefore,

$$I = \boxed{2\ln(x+1)\sqrt{x} - 4\sqrt{x} - 4\arctan\sqrt{x} + C}$$

□

Proposer: Cat Sodium

Problem 15 [*] $\displaystyle\int_0^\infty (x-1)(x-3)(x-5)x^4e^{-x}\,dx$

Solution.

$$\int_0^\infty (x-1)(x-3)(x-5)x^4e^{-x}\,dx = \boxed{960.}$$

□

Proposer: Patthadon Phengpinij

Problem 16 [*] $\displaystyle\int \frac{x^{2023}}{x^2+1}\,dx$

Solution. It is easy to see that

$$\begin{aligned}
\int \frac{x^{2024}}{x^2+1}\,dx &= \int \frac{x^{2024}-1}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= \int \frac{(x^4-1)(x^{2020}+x^{2016}+\cdots+1)}{x^2+1}\,dx + \arctan x + C \\
&= \int (x^2-1)(x^{2020}+x^{2016}+\cdots+1)\,dx + \arctan x + C \\
&= \int (x^{2022}-x^{2020}+x^{2018}-\cdots-1)\,dx + \arctan x + C \\
&= \boxed{\sum_{k=1}^{1011} \frac{(-1)^{k+1}x^{2k}}{2k} - \frac{1}{2}\ln(x^2+1) + C.}
\end{aligned}$$

□

Proposer: Very Funny

5 Semifinal 1

Problem 17 [**] $\int_0^{2024\pi} \lfloor 2024\sin(\sin(x)+\cos(x)) \rfloor \, dx$

Solution. Given $f(x) = 2024\sin(\sin(x)+\cos(x))$.
Consequently,

$$\begin{aligned}
f(x+\pi) &= 2024\sin(\sin(x+\pi)+\cos(x+\pi)) \\
&= 2024\sin(-\sin(x)-\cos(x)) \\
&= -2024\sin(\sin(x)+\cos(x)) \\
&= -f(x)
\end{aligned}$$

Because, $\sin(x+2\pi n) = \sin(x)$ and $\cos(x+2\pi n) = \cos(x)$ for all integer n,

$$\begin{aligned}
\int_0^{2024\pi} \lfloor 2024\sin(\sin(x)+\cos(x)) \rfloor \, dx &= 1012 \cdot \int_0^{2\pi} \lfloor 2024\sin(\sin(x)+\cos(x)) \rfloor \, dx \\
&= 1012 \cdot \int_0^{\pi} \lfloor f(x) \rfloor + \lfloor f(x+\pi) \rfloor \, dx \\
&= 1012 \cdot \int_0^{\pi} \lfloor f(x) \rfloor + \lfloor -f(x) \rfloor \, dx
\end{aligned}$$

By considering integer $k \in [-2024, 2024]$, that $k \le f(x) < k+1$,

$$\begin{aligned}
\lfloor f(x) \rfloor &= k \\
\lfloor f(x+\pi) \rfloor &= \lfloor -f(x) \rfloor \\
&= -(k+1) \\
&= -\lfloor f(x) \rfloor - 1
\end{aligned}$$

This implies that,

$$\begin{aligned}
\int_0^{2024\pi} \lfloor 2024\sin(\sin(x)+\cos(x))\rfloor\, dx &= 1012\cdot\int_0^{\pi} \lfloor f(x)\rfloor + \lfloor -f(x)\rfloor\, dx \\
&= 1012\cdot\int_0^{\pi} \lfloor f(x)\rfloor - \lfloor f(x)\rfloor - 1\, dx \\
&= 1012\cdot\int_0^{\pi} (-1)\, dx \\
&= \boxed{-1012\pi.}
\end{aligned}$$

□

Proposer: Patthadon Phengpinij

Problem 18 [**] $\displaystyle\int e^{e^{e^x}}\left(e^{e^x}+e^{e^{2x}}\right)\left(e^x+e^{2x}\right)\, dx$

Solution.

$$\int e^{e^{e^x}}\left(e^{e^x}+e^{e^{2x}}\right)\left(e^x+e^{2x}\right)\, dx = \boxed{e^{e^{e^x}}e^{e^x}e^x + e^{e^{e^x}}e^{e^x} - e^{e^{e^x}} + C}$$

□

Proposer: Sirawit Pipittanaban

Problem 19 [**] $\displaystyle\int \frac{e^x}{(e^x+1)^2}\ln\left(\frac{e^x}{e^x-1}\right)dx$

Solution. Using integration by parts, choose $f(x) = -\ln\left(\frac{e^x}{e^x-1}\right)$ and $g'(x) = -\frac{e^x}{(e^x+1)^2}$. Then $f'(x) = \frac{1}{e^x-1}$ and $g(x) = \frac{1}{e^x+1}$. Hence,

$$\begin{aligned}
I &= \int \frac{e^x}{(e^x+1)^2}\ln\left(\frac{e^x}{e^x-1}\right)dx \\
&= -\ln\left(\frac{e^x}{e^x-1}\right)\cdot\frac{1}{e^x+1} - \int \frac{1}{e^x+1}\cdot\frac{1}{e^x-1}dx \\
&= -\ln\left(\frac{e^x}{e^x-1}\right)\cdot\frac{1}{e^x+1} - \int \frac{1}{(e^{2x}-1)}dx
\end{aligned}$$

Substitute $u = e^{2x} - 1$, so that $dx = \frac{1}{2(u+1)} du$. Hence,

$$\begin{aligned}
\int \frac{1}{(e^{2x}-1)} dx &= \frac{1}{2} \int \frac{1}{u(u+1)} du \\
&= \frac{1}{2} \int \left(\frac{1}{u} - \frac{1}{u+1} \right) du \\
&= \frac{1}{2} \ln \left| \frac{u}{u+1} \right| \\
&= \frac{1}{2} \ln |e^{2x} - 1| - x
\end{aligned}$$

Therefore,

$$I = \boxed{-\ln\left(\frac{e^x}{e^x-1}\right) \cdot \frac{1}{e^x+1} - \frac{1}{2} \ln |e^{2x}-1| + x + C}$$

$\square$

Proposer: Cat Sodium

Problem 20 [**] $\displaystyle\int_{1/2}^{1} \frac{x^3 - x + 1}{x^2\sqrt{1-x^2}}\, e^x \, dx$

Solution. For convenience, let

$$I := \int \frac{x^3 - x + 1}{x^2\sqrt{1-x^2}}\, e^x \, dx$$

Substitute $x = \sin\theta$ such that $dx = \cos\theta \, d\theta$.

$$\begin{aligned}
I &= \int \frac{\sin^3\theta - \sin\theta + 1}{\sin^2\theta \cos\theta}\, e^{\sin\theta} \cos\theta \, d\theta \\
&= \int (\csc^2\theta - \cot\theta\cos\theta)\, e^{\sin\theta} \, d\theta
\end{aligned}$$

Using integration by parts,

$$\begin{aligned}
\int \csc^2\theta e^{\sin\theta} \, d\theta &= -\int e^{\sin\theta} \, d(\cot\theta) \\
&= -e^{\sin\theta}\cot\theta + \int \cot\theta \, d(e^{\sin\theta}) \\
&= -e^{\sin\theta}\cot\theta + \int \cot\theta\cos\theta e^{\sin\theta} \, d\theta
\end{aligned}$$

Therefore,

$$
\begin{aligned}
I &= \int (\csc^2\theta - \cot\theta\cos\theta)\, e^{\sin\theta}\, d\theta \\
&= -e^{\sin\theta}\cot\theta \\
&= -e^{x}\sqrt{\frac{1}{x^2}-1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\int_{1/2}^{1} \frac{x^3-x+1}{x^2\sqrt{1-x^2}}\, e^x\, dx &= -e^x\sqrt{\frac{1}{x^2}-1}\,\Bigg|_{1/2}^{1} \\
&= \boxed{\sqrt{3e}}
\end{aligned}
$$

□

Proposer: Cat Sodium

6 Semifinal 2

Problem 21 [***] $\displaystyle\int_{-2023}^{2023} e^{\overbrace{||||x|-1|-1|\ldots|-1|}^{\text{2023 (-1)'s.}}}\, dx$

Solution. Note that $f(x) := \overbrace{|||||x|-1|-1|\ldots|-1|}^{\text{2023 (-1)'s.}}$, and $I := \displaystyle\int_{-2023}^{2023} e^{f(x)}\, dx$.
Consider the value of $f(x)$ for $x \in [0, 2023]$, choose integer $k = \lfloor x \rfloor$ such that,

$$
f(x) = \overbrace{|||||x|-1|-1|\ldots|-1|}^{\text{2023 (-1)'s.}} = \overbrace{|||||(x-k)|-1|-1|\ldots|-1|}^{\text{2023-k (-1)'s.}} = \left|(x-k) + \sum_{n=1}^{2023-k}(-1)^n\right|.
$$

Because $f(x) = f(-x)$, therefore,

$$
f(x) = \begin{cases} \{x\}, & \text{if } \lfloor x \rfloor \text{ is odd} \\ 1-\{x\}, & \text{otherwise} \end{cases}
$$

for all $x \in [-2023,\, 2023]$

Hence,

$$
\begin{aligned}
I &= \int_{-2023}^{2022} e^{f(x)}\,dx \\
&= \sum_{k=-2023}^{2022} \int_{k}^{k+1} e^{f(x)}\,dx \\
&= \sum_{k=-2023,\ k \text{ is odd}}^{2022} \int_{k}^{k+1} e^{\{x\}}\,dx + \sum_{k=-2023,\ k \text{ is even}}^{2022} \int_{k}^{k+1} e^{1-\{x\}}\,dx \\
&= \sum_{k=-2023,\ k \text{ is odd}}^{2022} \int_{k}^{k+1} e^{x-k}\,dx + \sum_{k=-2023,\ k \text{ is even}}^{2022} \int_{k}^{k+1} e^{1-x+k}\,dx \\
&= \sum_{k=-2023,\ k \text{ is odd}}^{2022} e^{x-k}\Big|_{k}^{k+1} + \sum_{k=-2023,\ k \text{ is even}}^{2022} -e^{1-x+k}\Big|_{k}^{k+1} \\
&= \sum_{k=-2023,\ k \text{ is odd}}^{2022} (e-1) + \sum_{k=-2023,\ k \text{ is even}}^{2022} (e-1) \\
&= \sum_{k=-2023}^{2022} (e-1) \\
&= \boxed{4046(e-1).}
\end{aligned}
$$

□

Proposer: Patthadon Phengpinij

Problem 22 [**] $\displaystyle\int_0^2 \frac{(x-1)^2 e^{3x}}{e^2 e^x + e^4 e^{-x}}dx$

Solution. Consider

$$\int_0^2 \frac{(x-1)^2 e^{3x}}{e^2 e^x + e^4 e^{-x}}dx = \int_0^2 \frac{(x-1)^2 e^{3(x-1)}}{e^{x-1} + e^{-(x-1)}}dx$$

Let $u = x - 1$. So

$$\begin{aligned}
\int_0^2 \frac{(x-1)^2 e^{3(x-1)}}{e^{x-1}+e^{-(x-1)}}dx &= \int_{-1}^1 \frac{u^2e^{3u}}{e^u+e^{-u}}du \\
&= \int_0^1 \frac{u^2e^{3u}}{e^u+e^{-u}}du + \int_{-1}^0 \frac{u^2e^{3u}}{e^u+e^{-u}}du \\
&= \int_0^1 \frac{u^2e^{3u}}{e^u+e^{-u}}du + \int_0^1 \frac{u^2e^{-3u}}{e^u+e^{-u}}du \\
&= \int_0^1 \frac{u^2(e^{3u}+e^{-3u})}{e^u+e^{-u}}du \\
&= \int_0^1 u^2e^{2u} - u^2 + u^2e^{-2u}du \\
&= \left[\frac{u^2e^{2u}}{2} - \frac{ue^{2u}}{2} + \frac{e^{2u}}{4} - \frac{u^3}{3} - \frac{u^2e^{-2u}}{2} - \frac{ue^{-2u}}{2} - \frac{e^{-2u}}{4}\right]_0^1 \\
&= \boxed{\frac{e^2}{4} - \frac{1}{3} - \frac{5e^{-2}}{4}}
\end{aligned}$$

□

Proposer: Pommekung

Problem 23 [**] $\displaystyle\int_0^1 \left\{\ln\left(\frac{1}{x}\right)\right\} dx$

Solution. Let $u = \ln\left(\dfrac{1}{x}\right)$ such that $x = e^{-u}$ and $\dfrac{du}{dx} = -\dfrac{1}{x}$.

Thus,

$$
\begin{aligned}
\int_0^1 \left\{\ln\left(\frac{1}{x}\right)\right\} dx &= \int_{x=0}^{x=1} \{u\} \cdot (-x\,du) \\
&= \int_{u=0}^{u=\infty} \{u\} \cdot e^{-u}\,du \\
&= \int_0^\infty e^{-u} \cdot (u - \lfloor u \rfloor)\ du \\
&= \int_0^\infty u \cdot e^{-u}\,du - \int_0^\infty \lfloor u \rfloor \cdot e^{-u}\,du \\
&= \int_{u=0}^{u=\infty} -u\,d\left(e^{-u}\right) - \sum_{k=0}^{\infty} \int_k^{k+1} e^{-u} \cdot k\,du \\
&= \left[(-u)\left(e^{-u}\right)\Big|_{u=0}^{u=\infty} - \int_{u=0}^{u=\infty} e^{-u}\,d(-u)\right] - \sum_{k=0}^{\infty} \left(k \cdot \left\{-e^{-u}\Big|_k^{k+1}\right\}\right) \\
&= \left[(0-0) - \left(e^{-u}\right)\Big|_{u=0}^{u=\infty}\right] - \sum_{k=0}^{\infty} \left(k \cdot \left(\frac{1}{e^k} - \frac{1}{e^{k+1}}\right)\right) \\
&= [-(0-1)] - (e-1) \cdot \sum_{k=0}^{\infty} \left(\frac{k}{e^{k+1}}\right) \\
&= 1 - (e-1) \cdot \frac{1}{(e-1)^2} \\
&= \boxed{\frac{e-2}{e-1}.}
\end{aligned}
$$

□

Proposer: Patthadon Phengpinij

Problem 24 [*] $\displaystyle\int_0^{\pi/3} \frac{1}{1+\sin x}\,dx$

Solution. Substituting $u = \tan\frac{x}{2}$ such that $du = \sec^2\frac{x}{2}\,dx$.

$$\begin{aligned}
I &= \int_0^{\pi/3} \frac{1}{1+\sin x}\,dx \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{2\,du}{(u^2+1)\left(\frac{2u}{u^2+1}+1\right)} \\
&= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(u+1)^2}\,du \\
&= -\frac{2}{u+1}\bigg|_0^{\frac{1}{\sqrt{3}}} \\
&= \boxed{\sqrt{3}-1}
\end{aligned}$$

□

Proposer: Cat sodium

7 Final

Problem 25 [* or **] $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(2x^2+5+\sqrt{5})^2(2x^2+5-\sqrt{5})^2-16}$.

Solution. First, we factorize the denominator term to obtain

$$\begin{aligned}
(2x^2+5+\sqrt{5})^2(2x^2+5-\sqrt{5})^2-16 &= ((2x^2+5)^2-5)^2-16 \\
&= (4x^4+20x^2+20)^2-16 \\
&= (4x^4+20x^2+24)(4x^4+20x^2+16) \\
&= 16(x^2+1)(x^2+2)(x^2+3)(x^4+4).
\end{aligned}$$

Next, we use partial fraction decomposition to separate the denominator term. For convenience, let $x^2 = u$, then

$$\begin{aligned}
\frac{1}{16(x^2+1)(x^2+2)(x^2+3)(x^2+4)} &= \frac{1}{16(u+1)(u+2)(u+3)(u+4)} \\
&= \frac{1}{16}\left(\frac{1}{6(u+1)} - \frac{1}{2(u+2)} + \frac{1}{2(u+3)} - \frac{1}{6(u+4)}\right) \\
&= \frac{1}{16}\left(\frac{1}{6(x^2+1)} - \frac{1}{2(x^2+2)} + \frac{1}{2(x^2+3)} - \frac{1}{6(x^2+4)}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{dx}{(2x^2+5+\sqrt{5})^2(2x^2+5-\sqrt{5})^2-16} \\
&= \frac{1}{16}\left(\frac{1}{6}\int_{-\infty}^{\infty}\frac{dx}{x^2+1} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^2+2} + \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{x^2+3} - \frac{1}{6}\int_{-\infty}^{\infty}\frac{dx}{x^2+4}\right).
\end{aligned}$$

Note that

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+a^2} = 2\int_0^{\infty} \frac{dx}{x^2+a^2} = 2\lim_{t\to\infty}\int_0^t \frac{dx}{x^2+a^2} = \frac{2}{a}\lim_{t\to\infty} \arctan\left(\frac{t}{a}\right) = \frac{\pi}{a},$$

for $a > 0$. Therefore,

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{(2x^2+5+\sqrt{5})^2(2x^2+5-\sqrt{5})^2-16} &= \frac{\pi}{16}\left(\frac{1}{6}-\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{3}}-\frac{1}{12}\right)\\ &= \boxed{\frac{\pi}{192}(1-3\sqrt{2}+2\sqrt{3}).}\end{aligned}$$

□

Proposer: PolarBear

Problem 26 [F,**] $\displaystyle\int_0^{2\pi} \sqrt{1+\cos x+\sqrt{2+2\cos x}}\,dx$

Solution. The integral can be rewritten as

$$\begin{aligned}I &= \int_0^{2\pi} \sqrt{1+\cos x+\sqrt{2+2\cos x}}\,dx\\ &= \int_0^{2\pi} \sqrt{2\cos^2\frac{x}{2}+2\sqrt{\cos^2\frac{x}{2}}}\,dx\\ &= \sqrt{2}\int_0^{2\pi} \sqrt{\cos^2\frac{x}{2}+\left|\cos\frac{x}{2}\right|}\,dx\\ &= \sqrt{2}\int_0^{\pi} \sqrt{\cos^2\frac{x}{2}+\cos\frac{x}{2}}\,dx+\sqrt{2}\int_{\pi}^{2\pi} \sqrt{\cos^2\frac{x}{2}-\cos\frac{x}{2}}\,dx\end{aligned}$$

Substitute $u = \cos\frac{x}{2}$, then $du = -\frac{1}{2}\sin\frac{x}{2}dx$. Therefore,

$$\begin{aligned}I &= 2\sqrt{2}\int_0^1 \sqrt{\frac{u^2+u}{1-u^2}}\,du + 2\sqrt{2}\int_{-1}^0 \sqrt{\frac{u^2-u}{1-u^2}}\,du\\ &= 2\sqrt{2}\int_0^1 \sqrt{\frac{u}{1-u}}\,du + 2\sqrt{2}\int_{-1}^0 \sqrt{\frac{-u}{1+u}}\,du\end{aligned}$$

For the first term, substitute $u = \sin^2\theta$ such that $du = 2\sin\theta\cos\theta d\theta$.

$$\begin{aligned}I_1 &= \int_0^1 \sqrt{\frac{u}{1-u}}du\\ &= 2\int_0^{\pi/2} \sqrt{\frac{\sin^2\theta}{1-\sin^2\theta}}\sin\theta\cos\theta d\theta\\ &= 2\int_0^{\pi/2} \sin^2\theta d\theta\\ &= \frac{\pi}{2}\end{aligned}$$

Notice that replacing $u \mapsto -u$ for $\int_{-1}^{0} \sqrt{\frac{-u}{1+u}}\, du$ gives

$$\int_{-1}^{0} \sqrt{\frac{-u}{1+u}}\, du = \int_{0}^{1} \sqrt{\frac{u}{1-u}}\, du = I_1.$$

Thus,

$$I = (2\sqrt{2} + 2\sqrt{2})I_1 = \boxed{2\sqrt{2}\pi.}$$

□

Proposer: Cat Sodium

Problem 27 [***] $\int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n x \rfloor}{4^n} dx.$

Solution. This problem was inspired by MIT Integration Bee 2023. First, we let $x = -2y$, then

$$\begin{aligned}\int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n x \rfloor}{4^n} dx &= 2\int_{-\frac{1}{2}}^{0} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^{n+1} y \rfloor}{4^n} dy = 8\int_{-\frac{1}{2}}^{0} \sum_{n=2}^{\infty} \frac{\lfloor (-2)^n y \rfloor}{4^n} dy \\ &= 8\int_{-\frac{1}{2}}^{0} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n y \rfloor}{4^n} dy,\end{aligned}$$

since $-\frac{1}{2} < y < 0$ implies $0 < -2y < 1$. Next, we consider

$$\int_{-\frac{1}{2}}^{0} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n y \rfloor}{4^n} dy.$$

We let $y = z - \frac{k}{2}$, where $k = 1, 2$. Then

$$\begin{aligned}\int_{-\frac{1}{2}}^{0} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n y \rfloor}{4^n} dy &= \int_{\frac{k-1}{2}}^{\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n z + (-2)^{n-1} k \rfloor}{4^n} dz \\ &= \int_{\frac{k-1}{2}}^{\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n z \rfloor}{4^n} dz - \frac{k}{4} \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n \\ &= \int_{\frac{k-1}{2}}^{\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n z \rfloor}{4^n} dz + \frac{k}{12}.\end{aligned}$$

Hence,

$$2\int_{-\frac{1}{2}}^{0} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n y \rfloor}{4^n} dy = \sum_{k=1}^{2} \left(\int_{\frac{k-1}{2}}^{\frac{k}{2}} \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n z \rfloor}{4^n} dz + \frac{k}{12} \right) = \int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n z \rfloor}{4^n} dz + \frac{1}{4}.$$

Hence,

$$\int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n x \rfloor}{4^n} dx = 4 \int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n x \rfloor}{4^n} dx + 1.$$

This implies

$$\int_0^1 \sum_{n=1}^{\infty} \frac{\lfloor (-2)^n x \rfloor}{4^n} dx = \boxed{-\frac{1}{3}.}$$

□

Proposer: PolarBear

Problem 28 [**] $\displaystyle\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{2}+\sqrt{3}\tan 3x}$.

Solution. For convenience, let

$$I = \int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{2}+\sqrt{3}\tan 3x} = \int_0^{\frac{\pi}{4}} \frac{\cos 3x}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} dx.$$

Note that

$$\begin{aligned}
I &= \frac{1}{3\sqrt{3}} \int_0^{\frac{\pi}{4}} \frac{3\sqrt{3}\cos 3x - 3\sqrt{2}\sin 3x}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} dx + \sqrt{\frac{2}{3}} \int_0^{\frac{\pi}{4}} \frac{\sin 3x}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} dx \\
&= \frac{1}{3\sqrt{3}} \int_0^{\frac{\pi}{4}} \frac{d(\sqrt{2}\cos 3x+\sqrt{3}\sin 3x)}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} + \frac{\sqrt{2}}{3} \int_0^{\frac{\pi}{4}} \frac{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} dx \\
&\qquad - \frac{2}{3} \int_0^{\frac{\pi}{4}} \frac{\cos 3x}{\sqrt{2}\cos 3x+\sqrt{3}\sin 3x} dx \\
&= \frac{1}{3\sqrt{3}} \ln\left|\sqrt{2}\cos 3x+\sqrt{3}\sin 3x\right| + \frac{\sqrt{2}}{3} x \Bigg]_{x=0}^{x=\frac{\pi}{4}} - \frac{2}{3} I. \\
&= \frac{1}{3\sqrt{3}} \left(\ln\left|-1+\sqrt{\frac{3}{2}}\right| - \ln\sqrt{2} \right) + \frac{\pi}{6\sqrt{2}} - \frac{2}{3} I \\
&= \frac{\pi}{6\sqrt{2}} + \frac{1}{3\sqrt{3}} \ln\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right) - \frac{2}{3} I.
\end{aligned}$$

Hence,

$$\int_0^{\frac{\pi}{4}} \frac{dx}{\sqrt{2}+\sqrt{3}\tan 3x} = I = \boxed{\frac{\pi}{10\sqrt{2}} + \frac{1}{5\sqrt{3}} \ln\left(\frac{\sqrt{3}-\sqrt{2}}{2}\right).}$$

□

Proposer: PolarBear Bearrrrrrrrrrrrr~~~~~~~~~~!!

Problem 29 [**] $\displaystyle\int_0^{\pi} \frac{\cos 2\theta}{6+4\sin\theta+4\cos\theta+\sin 2\theta} d\theta$

Solution. Let $u = \sin\theta + \cos\theta + 2$. So $du = (\cos\theta - \sin\theta)d\theta$. Then

$$\begin{aligned}
\int_0^{\pi} \frac{\cos 2\theta}{6 + 4\sin\theta + 4\cos\theta + \sin 2\theta} d\theta &= \int_0^{\pi} \frac{\cos^2\theta - \sin^2\theta}{6 + 4\sin\theta + 4\cos\theta + 2\sin\theta\cos\theta} d\theta \\
&= \int_0^{\pi} \frac{(\sin\theta + \cos\theta)(\cos\theta - \sin\theta)}{1 + (2 + \sin\theta + \cos\theta)^2} d\theta \\
&= \int_3^1 \frac{u-2}{u^2+1} du \\
&= [\frac{1}{2}\ln|u^2+1| - 2\arctan u]_3^1 \\
&= \frac{\ln 2 - \ln 10}{2} - 2\arctan 1 + 2\arctan 3 \\
&= \boxed{-\frac{\ln 5}{2} + 2\arctan\frac{1}{2}}
\end{aligned}$$

□

Proposer: Pommekung

Problem 30 [***] $\displaystyle\int_0^{\frac{3}{4}} \frac{x^2}{\left(\sqrt{x^2+1} - x\right)^{\frac{1}{2}}} dx$

Solution. Using $x = i\ \sin\ \theta$, we get $dx = i\ \cos\ \theta$. So,

$$\int \frac{x^2}{\left(\sqrt{x^2+1} - x\right)^{\frac{1}{2}}} dx = \int x^2 \, (\sqrt{x^2+1} + x)^{\frac{1}{2}} \, dx$$

$$\int -1\, sin^2\,\theta e^{\frac{i\theta}{2}} i\, cos\,\theta\, d\theta = -\frac{i}{8} \int e^{\frac{i\theta}{2}} \, (e^{i\theta} + e^{-i\theta} - e^{3i\theta} + e^{-3i\theta}) \, d\theta$$

Next, let $u = e^{\frac{i\theta}{2}}$, $du = \frac{i}{2}\, e^{\frac{i\theta}{2}}\, d\theta$ So,

$$-\frac{i}{8} \int e^{\frac{i\theta}{2}} \, (e^{i\theta} + e^{-i\theta} - e^{3i\theta} - e^{-3i\theta}) \, d\theta = \frac{-1}{4} \int (u^2 + u^{-2} + u^6 + u^{-6}) \, du$$

Now, we have to consider the interval of the integral. It can be shown that $[0, \frac{3}{4}]$ will be changed to $[0, 2^{\frac{1}{2}}]$ in the integral w.r.t. the new substitution of u. Therefore,

$$\frac{-1}{4} \int_1^{2^{\frac{1}{2}}} (u^2 + u^{-2} + u^6 + u^{-6}) \, du = \boxed{\frac{799\sqrt{2} - 512}{3360}}$$

□

Proposer: Sirawit Pipittanaban

Problem 31 [***] $\displaystyle\int_0^3 \cos\left(\lfloor 2023x\rfloor + \frac{1}{2}\right) dx.$

Solution. Note that

$$\int_0^3 \cos\left(\lfloor nx\rfloor + \frac{1}{2}\right) dx = \sum_{k=0}^{3n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \cos\left(\lfloor nx\rfloor + \frac{1}{2}\right) dx = \frac{1}{n}\left(\sum_{k=0}^{3n-1} \cos\left(k + \frac{1}{2}\right)\right).$$

By using the sum of sine

$$\sum_{k=0}^{n} \cos(a + kd) = \frac{1}{2\sin\left(\frac{d}{2}\right)}\left(\sin\left(a + \frac{2n+1}{2}d\right) - \sin\left(a - \frac{d}{2}\right)\right),$$

where $a, d \in \mathbb{R}$ with $\sin\left(\frac{d}{2}\right) \neq 0$ and $n \in \mathbb{N}$, we have

$$\int_0^3 \cos\left(\lfloor nx\rfloor + \frac{1}{2}\right) dx = \frac{\sin(3n)}{2n\sin\left(\frac{1}{2}\right)}.$$

Hence,

$$\int_0^3 \cos\left(\lfloor 2023x\rfloor + \frac{1}{2}\right) dx = \boxed{\frac{\sin(6069)}{4046\sin\left(\frac{1}{2}\right)}.}$$

□

Proposer: PolarBear