The 1$^{\text{ST}}$ MWIT-KVIS Integration Bee
Qualifying Exam Solutions

Acknowledgements

The organizers would like to thank PolarBear, Chanatip Sujsuntinukul, Pannathut Chitpakdee, Tanupat Trakulthongchai, Thitiwat Kosolpattanadurong, and Pommekung for submitting problems to the shortlist.

Solutions

Problem 1 [*] $\displaystyle\int (23x+1)(2x+3)^6\,dx$

Solution. Using integration by parts, choose $f(x) = 23x+1$ and $g'(x) = (2x+3)^6$. Then $f'(x) = 23$ and $g(x) = \frac{1}{14}(2x+3)^7$. Hence,

$$\begin{aligned}\int (23x+1)(2x+3)^6\,dx &= \frac{1}{14}(23x+1)(2x+3)^7 - \int \frac{23}{14}(2x+3)^7\,dx \\ &= \frac{1}{14}(23x+1)(2x+3)^7 - \frac{23}{14}\cdot\frac{1}{16}(2x+3)^8 + C \\ &= \boxed{\frac{1}{14}(2x+3)^7\left(\frac{161}{8}x - \frac{53}{16}\right) + C.}\end{aligned}$$

□

Proposer: Tanupat Trakulthongchai

Problem 2 [*] $\displaystyle\int \tan(x)\ln(\cos^2 x)\,dx$

Solution. Substitute $u = \ln(\cos^2 x)$, so that $du = -2\tan(x)dx$. Hence,

$$\begin{aligned}\int \tan(x)\ln(\cos^2 x)dx &= \int -\frac{1}{2}udu \\ &= -\frac{u^2}{4} \\ &= \boxed{-\frac{1}{4}\ln^2(\cos^2 x) + C.}\end{aligned}$$

□

Proposer: Thitiwat Kosolpattanadurong

Problem 3 [**] $\int x + e^{x+e^{x+e}}\,dx$

Solution.

$$\begin{aligned}\int x + e^{x+e^{x+e}}\,dx &= \int x\,dx + \int e^{x+e^{x+e}}\,dx \\ &= \frac{x^2}{2} + \int e^{x+e^{x+e}}\,dx\end{aligned}$$

Substitute $u = e^x$, we get

$$\begin{aligned}\int e^{x+e^{x+e}}\,dx &= \int ue^{u\cdot e^e} \cdot \frac{1}{u}\,du \\ &= \frac{e^{u\cdot e^e}}{e^e} + C.\end{aligned}$$

So,

$$\int x + e^{x+e^{x+e}}\,dx = \boxed{\frac{x^2}{2} + e^{e^{x+e}-e} + C.}$$

□

Proposer: Tanupat Trakulthongchai

Problem 4 [**] $\int_0^1 \frac{x}{x^4+x^2+1}\,dx$

Solution.

$$\begin{aligned}I := \int_0^1 \frac{x}{x^4+x^2+1}\,dx &= \frac{1}{2}\left(\int_0^1 \frac{1}{x^2-x+1} - \frac{1}{x^2+x+1}\,dx\right) \\ &= \frac{1}{2}\left(\int_0^1 \frac{1}{(x-\frac{1}{2})^2+\frac{3}{4}} - \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx\right)\end{aligned}$$

For the first term, let $x = \sqrt{\frac{3}{4}}\tan\theta + \frac{1}{2}$, so that $dx = \sqrt{\frac{3}{4}}\sec^2\theta d\theta$ Hence,

$$\begin{aligned}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} &= \int \frac{\sqrt{\frac{3}{4}}\sec^2\theta d\theta}{\frac{3}{4}\sec^2\theta} \\ &= \frac{2}{\sqrt{3}}\int d\theta \\ &= \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right]\end{aligned}$$

Similarly,

$$\int \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}} \arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]$$

Therefore,

$$\begin{aligned} I &= \frac{1}{\sqrt{3}} \arctan\left[\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right]\Bigg|_0^1 - \frac{1}{\sqrt{3}} \arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]\Bigg|_0^1 \\ &= \frac{\pi}{3\sqrt{3}} - \frac{\pi}{6\sqrt{3}} \\ &= \boxed{\frac{\pi}{6\sqrt{3}}.} \end{aligned}$$

□

Proposer: Thitiwat Kosolpattanadurong

Problem 5 [**] $\int \frac{1}{x^{2\pi}-x} dx.$

Solution. For convenience, let

$$I := \int \frac{1}{x^{2\pi}-x} dx.$$

It is easy to see that

$$\begin{aligned} I &= -\int \frac{1}{x} dx + \frac{1}{2\pi} \int \frac{2\pi x^{2\pi-1}-1}{x^{2\pi}-x} dx + \frac{1}{2\pi} \int \frac{1}{x^{2\pi}-x} dx \\ &= -\int \frac{1}{x} dx + \frac{1}{2\pi} \int \frac{1}{x^{2\pi}-x} d(x^{2\pi}-x) + \frac{1}{2\pi} I \\ &= -\ln|x| + \frac{1}{2\pi} \ln|x^{2\pi}-x| + \frac{1}{2\pi} I + C. \end{aligned}$$

This implies that

$$I = \boxed{-\frac{2\pi}{2\pi-1} \ln|x| + \frac{1}{2\pi-1} \ln|x^{2\pi}-x| + C.}$$

□

Proposer: PolarBear

Problem 6 [**] $\int_0^1 (2566)^{\arcsin x} dx.$

Solution. Let $u = \arcsin x$, then $du = \frac{1}{\sqrt{1-x^2}}dx$. So, $\cos u du = dx$ and

$$I := \int_0^1 (2566)^{\arcsin x} dx = \int_0^{\frac{\pi}{2}} (2566)^u \cos u du.$$

By using integrate by part twice, we have

$$\begin{aligned}
I &= (2566)^u \sin u\Big]_{u=0}^{u=\frac{\pi}{2}} - \ln(2566)\int_0^{\frac{\pi}{2}} (2566)^u \sin u du \\
&= 2566^{\frac{\pi}{2}} - \ln(2566)\Bigg(- (2566)^u \cos u\Big]_{u=0}^{u=\frac{\pi}{2}} + \ln(2566)\int_0^{\frac{\pi}{2}} (2566)^u \cos u du\Bigg) \\
&= 2566^{\frac{\pi}{2}} - \ln(2566) - (\ln(2566))^2 I.
\end{aligned}$$

Hence,

$$I = \boxed{\frac{2566^{\frac{\pi}{2}} - \ln(2566)}{(\ln(2566))^2 + 1}.}$$

□

Proposer: PolarBear

Problem 7 [*] $\displaystyle\int_0^{\frac{\pi}{2}} \lceil \sin^2 x + \sin 2x \rceil dx$

Solution. Let $f : \left[0, \frac{\pi}{2}\right]$ by $f(x) = \sin^2 x + \sin 2x$. Note that

$$0 < f(x) \le 1 \iff x \in \left(0, \arcsin\left(\frac{1}{\sqrt{5}}\right)\right],$$

$$1 < f(x) \le 2 \iff x \in \left(\arcsin\left(\frac{1}{\sqrt{5}}\right), \frac{\pi}{2}\right].$$

Hence,

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \lfloor \sin^2 x + \sin 2x \rfloor dx &= \int_0^{\arcsin\left(\frac{1}{\sqrt{5}}\right)} 1 dx + \int_{\arcsin\left(\frac{1}{\sqrt{5}}\right)}^{\frac{\pi}{2}} 2dx \\
&= \arcsin\left(\frac{1}{\sqrt{5}}\right) + 2\left(\frac{\pi}{2} - \arcsin\left(\frac{1}{\sqrt{5}}\right)\right) \\
&= \boxed{\pi - \arcsin\left(\frac{1}{\sqrt{5}}\right).}
\end{aligned}$$

□

Proposer: PolarBear

Problem 8 [***] $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}dx$

Solution. Let $x = \dfrac{1-t}{1+t}$, we have $\dfrac{dx}{dt} = -\dfrac{2}{(1+t)^2}$. Note that $t \to 1$ where $x \to 0$ and $t \to 0$ where $x \to 1$. So

$$\begin{aligned}\int_0^1 \frac{\ln(x+1)}{x^2+1}dx &= \int_1^0 \frac{\ln\frac{2}{(1+t)}}{(\frac{1-t}{1+t})^2+1}\left(-\frac{2}{(1+t)^2}\right)dt \\ &= 2\int_0^1 \frac{\ln\frac{2}{(1+t)}}{(1+t)^2+(1-t)^2}dt \\ &= \int_0^1 \frac{\ln 2 - \ln(1+t)}{t^2+1}dt \\ &= \int_0^1 \frac{\ln 2}{x^2+1}dx - \int_0^1 \frac{\ln(x+1)}{x^2+1}dx.\end{aligned}$$

Therefore

$$\int_0^1 \frac{\ln(x+1)}{x^2+1}dx = \frac{1}{2}\int_0^1 \frac{\ln 2}{x^2+1}dx = \frac{\ln 2}{2}\big[\arctan x\big]_0^1 = \boxed{\frac{\pi\ln 2}{8}.}$$

□

Proposer: Pommekung (Putnam 2005 A5)

Problem 9 [**] $\displaystyle\int \sin(\sin x)\sin 2x\,dx$

Solution. Note that

$$I = \int \sin(\sin x)\sin 2x\,dx = \int \sin(\sin x) 2\sin x\cos x\,dx.$$

Substitute $u = \sin x$, so that $du = \cos x\,dx$. We then have that

$$I = \int 2u\sin u\,du = 2\int u\sin u\,du.$$

Use integration by parts by choosing $f(u) = u$ and $g'(u) = \sin u\,du$. Then $f'(u) = du$ and $g(u) = -\cos u$. So

$$\begin{aligned}I &= 2\left[-u\cos u - \int(-\cos u)\,du\right] \\ &= 2(-u\cos u + \sin u) + C \\ &= \boxed{2(\sin(\sin x) - \sin x\cos(\sin x)) + C.}\end{aligned}$$

□

Proposer: Chanatip Sujsuntinukul

Problem 10 [**] $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\cos x}{1+2023^x}dx$

Solution.

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^x}dx &= \int_{-\frac{\pi}{2}}^{0} \frac{x^2\cos x}{1+2023^x}dx + \int_{0}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^x}dx \\
&= \int_{\frac{\pi}{2}}^{0} \frac{(-x)^2\cos(-x)}{1+2023^{-x}}d(-x) + \int_{0}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^x}dx \\
&= \int_{0}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^{-x}}dx + \int_{0}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^x}dx \\
&= \int_{0}^{\frac{\pi}{2}} \frac{(2023^x)x^2\cos x}{1+2023^x}dx + \int_{0}^{\frac{\pi}{2}} \frac{x^2\cos x}{1+2023^x}dx \\
&= \int_{0}^{\frac{\pi}{2}} x^2\cos x dx \\
&= \left[x^2\sin x\right]_0^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 2x\sin x dx \\
&= \left[x^2\sin x\right]_0^{\frac{\pi}{2}} + \left[2x\cos x\right]_0^{\frac{\pi}{2}} - \int_{0}^{\frac{\pi}{2}} 2\cos x dx \\
&= \left[x^2\sin x\right]_0^{\frac{\pi}{2}} + \left[2x\cos x\right]_0^{\frac{\pi}{2}} - \left[2\sin x\right]_0^{\frac{\pi}{2}} \\
&= \boxed{\frac{\pi^2}{4} - 2.}
\end{aligned}$$

□

Proposer: Pommekung

Problem 11 [**] $\displaystyle\int \frac{e^{x/12}}{e^{x/3}+e^{x/2}}dx$

Solution. Substitutes $u = e^{x/12}$, then $u^{12} = e^x$ and $du = \dfrac{1}{12}e^{x/12}dx$. So,

$$\int \frac{e^{x/12}}{e^{x/3}+e^{x/2}}dx = 12\int \frac{1}{u^4+u^6}du.$$

Next, we consider

$$\frac{1}{u^4+u^6}$$

For convenience, let $u^2 = w$. Using partial fraction decomposition, we obtain

$$\frac{1}{u^4+u^6} = \frac{1}{w^3+w^2} = -\frac{1}{w} + \frac{1}{w^2} + \frac{1}{w+1} = -\frac{1}{u^2} + \frac{1}{u^4} + \frac{1}{u^2+1}.$$

Hence,

$$\begin{aligned}
\int \frac{e^{x/12}}{e^{x/3}+e^{x/2}}dx &= 12\left(-\int \frac{1}{u^2}du + \int \frac{1}{u^4}du + \int \frac{1}{u^2+1}du\right) \\
&= 12\left(\frac{1}{u} - \frac{1}{3u^3} + \arctan u\right) + C \\
&= \boxed{12e^{-x/12} - 4e^{-x/4} + 12\arctan(e^{x/12}) + C.}
\end{aligned}$$

□

Proposer: PolarBear

Problem 12 [***] $\displaystyle\int_0^{\pi/2} \frac{\tan x}{\ln^2(\tan x)+1}dx$

Solution. Using the fact that

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\begin{aligned}
\int_0^{\pi/2} \frac{\tan x}{\ln^2(\tan x)+1}dx &= \int_0^{\pi/2} \frac{\tan x}{(\ln(\sin x) - \ln(\cos x))^2+1}dx \\
&= \int_0^{\pi/2} \frac{\cot x}{(\ln(\sin x) - \ln(\cos x))^2+1}dx
\end{aligned}$$

$$2I = \int_0^{\pi/2} \frac{\tan x + \cot x}{(\ln(\sin x) - \ln(\cos x))^2+1}dx$$

Put $u = (\ln(\sin x) - \ln(\cos x))^2$. So, $du = (\tan x + \cot x)dx$. Moreover, $u \to \infty$ if $x \to \frac{\pi}{2}^-$ and $u \to -\infty$ if $x \to 0^+$. Hence,

$$2I = \int_{-\infty}^{\infty} \frac{du}{u^2+1} = \int_0^{\infty} \frac{du}{u^2+1} + \int_{-\infty}^{0} \frac{du}{u^2+1} = 2\int_0^{\infty} \frac{du}{u^2+1}.$$

This implies that

$$I = \int_0^{\infty} \frac{du}{u^2+1} = \lim_{s\to\infty} \int_0^{s} \frac{du}{u^2+1} = \lim_{s\to\infty} \arctan x\Big]_{x=0}^{x=s} = \lim_{s\to\infty} \arctan s = \boxed{\frac{\pi}{2}}.$$

□

Proposer: Pannathut Chitpakdee

Problem 13 [*] $\displaystyle\int \frac{x^{1282}}{1-x^{2566}}\,dx$

Solution.

$$I = \int \frac{x^{1282}}{1-x^{2566}}\,dx = \int \frac{x^{1282}}{(1-x^{1283})(1+x^{1283})}\,dx$$

Substitute $u = x^{1283}$, so that $du = 1283x^{1282}\,dx$. So we have

$$\begin{aligned}
I &= \int \frac{du}{1283(1-u)(1+u)} \\
&= -\frac{1}{1283}\int \frac{du}{(u+1)(u-1)} \\
&= -\frac{1}{1283}\int \left[\frac{1}{2}\left(\frac{1}{u-1} - \frac{1}{u+1}\right)\right] du \\
&= -\frac{1}{2566}(\ln|u-1| - \ln|u+1|) + C \\
&= -\frac{1}{2566}\ln\left|\frac{u-1}{u+1}\right| + C \\
&= \boxed{-\frac{1}{2566}\ln\left|\frac{x^{1283}-1}{x^{1283}+1}\right| + C.}
\end{aligned}$$

□

Proposer: Chanatip Sujsuntinukul

Problem 14 [*] $\displaystyle\int_0^1 \left(1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - 4\cdot 5x^3 + \cdots\right)\,dx$

Solution. Clearly, $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$ for $|x| < 1$. This gives

$$-1 + 2x - 3x^2 + 4x^3 - \cdots = \frac{d}{dx}\left(\frac{1}{1+x}\right) = -\frac{1}{(1+x)^2},$$

and

$$1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - 4\cdot 5x^3 + \cdots = \frac{d}{dx}\left(-\frac{1}{(1+x)^2}\right).$$

Therefore,

$$\int_0^1 \left(1\cdot 2 - 2\cdot 3x + 3\cdot 4x^2 - 4\cdot 5x^3 + \cdots\right)\,dx = -\frac{1}{(1+x)^2}\bigg|_0^1 = \boxed{\frac{3}{4}.}$$

□

Proposer: Tanupat Trakulthongchai

Problem 15 [**] $\displaystyle\int \frac{\sqrt{\cot x}}{1+\sin 2x}dx$

Solution. For the given integral, we can write the following

$$I = \int \frac{\sqrt{\cot x}}{1+\sin 2x}dx = \int \frac{\sqrt{\cot x}}{(\sin x+\cos x)^2}dx = \int \frac{\sqrt{\cot x}}{(1+\cot x)^2\sin^2 x}dx = \int \frac{\sqrt{\cot x}\csc^2 x}{(1+\cot x)^2}dx$$

Substitute $\cot x = u^2$, so that $\csc^2 x dx = -2udu$. So, we will write

$$I = \int \frac{u\cdot(-2udu)}{(1+u^2)^2}dx = -2\int \frac{u^2}{(1+u^2)^2}du.$$

Substitute $u = \tan\theta$, so that $du = \sec^2\theta d\theta$. Hence,

$$\begin{aligned}
I &= -2\int \frac{\tan^2\theta}{(1+\tan^2\theta)^2}\sec^2\theta d\theta \\
&= -2\int \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta d\theta \\
&= -2\int \sin^2\theta d\theta \\
&= \frac{1}{2}\sin 2\theta - \theta \\
&= \frac{\tan\theta}{1+\tan^2\theta} - \theta \\
&= \frac{u}{1+u^2} - \arctan u \\
&= \boxed{\frac{\sqrt{\cot x}}{1+\cot x} - \arctan\sqrt{\cot x} + C.}
\end{aligned}$$

□

Proposer: Thitiwat Kosolpattanadurong