Anderson's Mixing for Nonlinear Magnetostatic Poisson Problem

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Problem Description

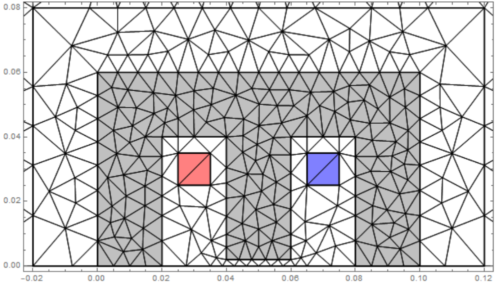
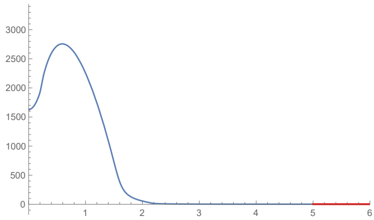


Figure: z cross-section: iron magnet (gray), copper wires (red & blue)

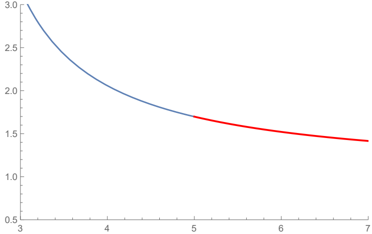
Problem: given current density $\mathbf{J} = (0, 0, J_z)$, $J_z := \pm j \left[A\, m^{-2}\right]$ in wires, and magnetic permeability $\mu \left[N\, A^{-2}\right]$ of the magnet, find resulting magnetic field $\mathbf{B}\,[T]$

Magnetic Permeability

One may write $\mu = \mu_0 \, \hat{\mu}$ with $\mu_0 := 4\,\pi\,10^{-7}\,\left[N\,A^{-2}\right]$ permeability of vacuum. If we assume $\hat{\mu}_{\text{iron}} = 1000$, resulting problem will be **linear**. However, in reality $\hat{\mu}_{\text{iron}}$ depends on $\mathbf{B}$:



(a) $\hat{\mu}_{\text{iron}} = \hat{\mu}_{\text{iron}}(||\mathbf{B}||)$



(b) Extrapolated part

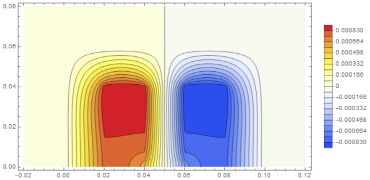
In this case we end up with **nonlinear** problem.

Mathematical Model

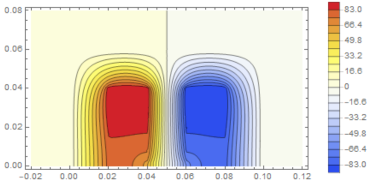
1. Start with Maxwell's equations $\nabla \times \frac{1}{\mu}\mathbf{B} = \mathbf{J}$, $\nabla \cdot \mathbf{B} = 0$
2. Introduce vector potential $\mathbf{A}$ via $\mathbf{B} = \nabla \times \mathbf{A}$ and rewrite (1) as $\nabla \times \frac{1}{\mu}\nabla \times \mathbf{A} = \mathbf{J}$
3. Assume that magnet is very long in z-direction and note that only z-component J_z of $\mathbf{J}$ is nonzero and $J_z = J_z(x, y)$; then (2) simplifies to Poisson problem $-\nabla \cdot (\frac{1}{\mu}\nabla A_z) = J_z$
4. Equip (3) with appropriate boundary conditions and discretize using e.g finite elements; then we have to solve
   - Linear system $\mathbf{S}\,\mathbf{x} = \mathbf{b}$ $[\mu = \mu(x, y)]$ or
   - Nonlinear system $\mathbf{S}(\mathbf{x})\,\mathbf{x} = \mathbf{b}$ $[\mu = \mu(||\mathbf{B}|| = ||\nabla A_z||)]$

Linear Post–processing

Figure: A_z



(a) $|J_z| = 10^6\ A\, m^{-2}$



(b) $|J_z| = 10^{11}\ A\, m^{-2}$

For linear problem ($\hat{\mu}_{\text{iron}} = 1000$) only scaling changes as we change current density

Nonlinear Solvers (1/2)

We need to solve $f(x) = 0$ ($x = g(x)$):

- **Fixed Point Method**:

$$x^{(k+1)} = g(x^{(k)})$$

- **Anderson's Mixing**:

$$x^{(k+1)} = \sum_{i=0}^{m} \alpha_i \, g(x^{(k-i)})$$

with $\boldsymbol{\alpha} = \arg\min_{\sum_{i=0}^{m} \alpha_i = 1} || \sum_{i=0}^{m} \alpha_i \, f(x^{(k-i)})||_2$.

This unconstrained minimization problem may be written as a small least-squares problem. We solve it (e.g using Gaussian elimination or (better) Householder reflections) at each iteration

Nonlinear Solvers (2/2)

In our case $\mathbf{f}(\mathbf{x}) := \mathbf{x} - \mathbf{S}(\mathbf{x})\,\mathbf{x}$. For the iteration function we may choose

- $\mathbf{g}(\mathbf{x}) := \mathbf{S}^{-1}(\mathbf{x})\,\mathbf{b}$ (Simplified Newton's method / Picard's method) or
- $\mathbf{g}_\omega(\mathbf{x}) := \omega\,\mathbf{S}^{-1}(\mathbf{x})\,\mathbf{b} + (1-\omega)\,\mathbf{x}$ with $0 < \omega \leq 1$ (relaxed version).

In practice, especially for strong nonlinearities, $\mathbf{g}$ rarely satisfies contraction property. Thus we need to introduce a relaxation parameter ω

Fixed Point vs. Anderson's Mixing

Table: Numb of iterations / execution time for different methods

$\lvert J_z \rvert$	10^6	10^7	10^8	10^9	10^{10}	10^{11}
Picard	1 0.4 s	6 1.3 s	— —	— —	203 25.2 s	7 1 s
Relaxed Picard	1 0.5 s	6 2 s	84 71 s	70 52.4 s	164 54.3 s	7 2.6 s
Anderson's Mixing	1+0 0.7 s	0+4 1 s	22+20 20 s	21+21 14.5 s	35+6 11.4 s	4+2 1.9 s

We used linear Lagrange elements with a mesh from the first slide (362 DOFs). We stopped when $\mathbf{f}(\mathbf{x}^{(k)}) < 10^{-8}$. For Anderson's Mixing $k = i + j$ means i *relaxed* Picard's iterations ($\mathbf{f}(\mathbf{x}^{(i)}) < 10^{-4}$) and j *accelerated* Picard's iterations. We used numb of mixings $m = 10$.

Nonlinear Post–processing

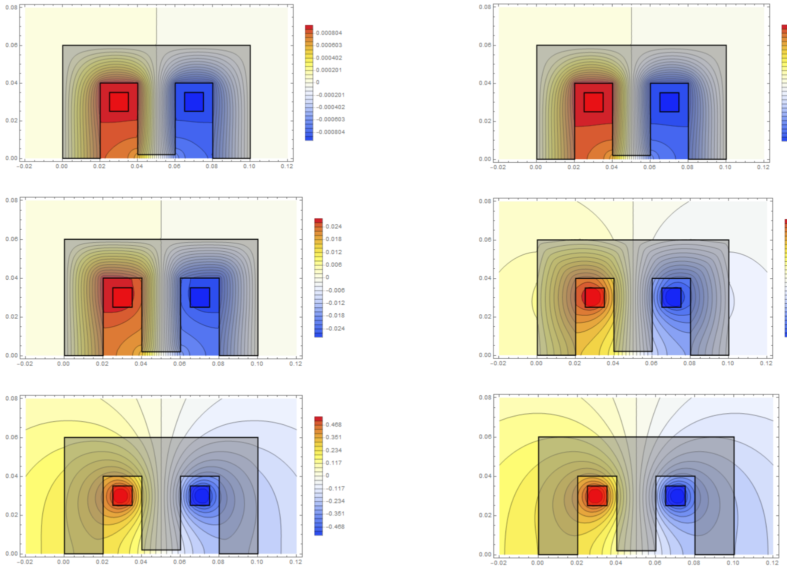


Figure: $|J_z| = 10^6, 10^7, \ldots, 10^{11}\, A\, m^{-2}$