Solving Multi-Item Lot-Sizing Problems with an MIP Solver Using Classification and Reformulation

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Based on research on the polyhedral structure of lot-sizing models over the last 20 years, we claim that there is a nontrivial fraction of practical lot-sizing problems that can now be solved by nonspecialists just by taking an appropriate a priori reformulation of the problem, and then feeding the resulting formulation into a commercial mixed-integer programming solver.

This claim uses the fact that many multi-item problems decompose naturally into a set of single-item problems with linking constraints, and that there is now a large body of knowledge about single-item problems. To put this knowledge to use, we propose a classification of lot-sizing problems (in large part single-item) and then indicate in a set of tables, what is known about a particular problem class and how useful it might be. Specifically, we indicate for each class (i) whether a tight extended formulation is known, and its size; (ii) whether one or more families of valid inequalities are known defining the convex hull of solutions, and the complexity of the corresponding separation algorithms; and (iii) the complexity of the corresponding optimization algorithms (which would be useful if a column generation or Lagrangian relaxation approach was envisaged).

Three distinct multi-item lot-sizing instances are then presented to demonstrate the approach, and comparative computational results are presented. Finally, we also use the classification to point out what appear to be some of the important open questions and challenges.

(*Lot Sizing; Production Planning; Classification; Convex Hull; Extended Formulation; Mixed-Integer Programming*)

1. Introduction

Production planning problems involving lot sizing have been an area of active research since the seminal paper of Wagner and Whitin (1958). Work on the polyhedral structure of the uncapacitated problem started with Barany et al. (1984) and on extended formulations with Krarup and Bilde (1977) and Eppen and Martin (1987). Since then, there has been a considerable amount of research extending these results for the single-item problem to incorporate other important features such as backlogging, start-ups, constant and varying capacities, etc. See Pochet and Wolsey (1995) for a survey and Pochet (2001) and Wolsey (1999) for two recent tutorials. On the other hand, although almost all practical problems are multi-item, and also often multimachine and multilevel, the polyhedral results concerning such models are limited. See Constatino (1998), Karmarkar and Schrage (1985), and Miller et al. (2000a) for some notable exceptions. As a result, the approach of choice in solving such problems has been implicitly or explicitly some form of decomposition, namely the development of

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solution methods such as Lagrangian relaxation, column generation, or branch and cut that explicitly use algorithms for optimization or for separation of *single-item* problems.

In two recent papers, we have described ways to formulate certain constraints that arise in practical lot-sizing models and thereby improve solution times (Belvaux and Wolsey 2001), and presented a special purpose modelling and branch-and-cut system BC-PROD designed for lot-sizing problems (Belvaux and Wolsey 2000). Here we would like to suggest that, based on the research cited above and the progress of commercial MIP systems, certain multi-item lot-sizing problems can now be solved just using *standard reformulations* and an *off-the-shelf MIP solver*. To achieve this, we present a simple classification of single-item lot-sizing problems, and then indicate in the form of tables our present knowledge about such problems. This knowledge consists of extended formulations, families of valid inequalities that provide or approximate the convex hull of solutions, and separation algorithms allowing one to use the valid inequalities as cutting planes along with their complexity. This is the knowledge typically needed when solving the problems directly as MIPs using branch and cut, the approach favored here. For those interested in developing column generation or Lagrangian relaxation approaches, the tables also indicate the complexity of optimization and give references. We then indicate a few of the characteristics of multi-item problems for which useful modelling results are available, and finally, we show by three examples how the classification and the corresponding reformulations can be used to obtain guaranteed high-quality solutions using nothing but a basic MIP system. Earlier classification schemes can be found in Bitran and Yanasse (1982) and Kuik et al. (1994). The former is mostly concerned with the varying capacity single-item problem, classifying problems according to how the four parameters f_t, h'_t, p'_t, C_t, namely set-up cost/storage cost/unit production cost/capacity are general/constant/nonincreasing/zero over time, and whether the resulting problem is polynomially solvable or not. The latter considers very general batching and scheduling problems. Our classification, on the other hand, concentrates mainly on the uncapacitated and constant capacity variants, which are polynomially solvable, and for which tight formulations can potentially be found.

The outline of this paper is as follows. In §2, we present a brief description of three multi-item problems. Just from these descriptions, we obtain a first verbal classification as an indication of what needs to be classified formally. In §3, we present the single-item classification that we have found useful. In §4, we present tables indicating the status of the most important problems concerning (i) families of valid inequalities, whether they describe the convex hull, and the complexity of the separation problem for these families of inequalities, (ii) the existence of tight or "good" extended formulations giving the convex hull exactly or approximately, and (iii) the complexity of optimization. In §5, we extend the classification to some aspects of multi-item problems and discuss briefly the important results available. In §6, we show how the classification and tables of §§3 and 4 can be used to obtain effective formulations in practice, giving computational results for the three multi-item problems presented earlier. Finally, in §7, we indicate several important open problems.

2. Three Multi-Item Problems

Here, we take the description of three multi-item lot-sizing problems and use the description to derive a verbal classification of each problem, suggesting what will be the important points in the formal classification presented later. In §6, we will translate these verbal classifications into our formal scheme, and use this to reformulate and solve one or more instances of each problem.

Problem 1. This is a bottling line problem with a 30-day planning horizon. There are four products. The line is available 16 hours per day, and only one product can be produced per day. There are storage, set-up and start-up costs that are all constant over time. The minimum production per day is 7 hours.

Classification. (i) Multi-item constraints and costs. At most one item can be produced per period. (ii) Individual item constraints and costs. When produced, each item is produced for between 7 and 16 hours, so both the upper bound and the lower bounds on production per period are time-invariant. Also, the unit production and storage costs are time-invariant, and there are start-up costs.

Problem 2. This is a lot-sizing problem with 10 items with sequence-dependent changeover costs and storage costs studied by Fleischmann (1994). Production is at full capacity and, at most, one item is produced per period.

Classification. (i) Multi-item constraints and costs. At most one item can be produced per period, and there are sequence dependent set-up costs.

(ii) Individual item constraints and costs. Production is all or nothing with constant capacities. There are no unit production costs, and storage costs are nonnegative and constant over time.

Problem 3. This is a general model for multilevel problems with assembly product structure proposed in Simpson and Erenguc (1998), involving product families consisting of one or more items, where each family can in turn have a fixed cost, a set-up time, or a resource constraint associated with it. Instances of this problem come from the construction of bottle racks and the production of animal feed. Instances of this problem have been tackled earlier with the special purpose systems BC-PROD (Belvaux and Wolsey 2000) and BC-OPT (Cordier et al. 1999).

Classification. (i) Multilevel structure. Assembly-type product structure.

(ii) Multi-item constraints and costs. Many items can be produced in each period, and the capacity constraints limiting production in each period involve both production levels and set-up times for families.

(iii) Individual item constraints and costs. There are no individual capacity constraints, but there are storage costs and implicit fixed costs through the families.

3. Single-Item Classification

We start by defining the basic lot-sizing problem (LS). There is a time horizon of n periods, and in each period there is a demand to be satisfied d_t, and a production limit C_t. There is a per unit production cost p_t, a fixed set-up cost f_t if production takes place in t for $t = 1, \ldots, n$, and a cost h'_t per unit of stock at the end of period t for $t = 0, \ldots, n$. Note that in principle, a variable amount of initial stock is allowed.

3.1. The Basic Classification

There are three fields *PROB-CAP-VAR*. We use $[x, y, z]$ to denote exactly one element from the set $\{x, y, z\}$, and $[x, y, z]^*$ to denote any subset of $\{x, y, z\}$. Fields that are empty are dropped. In the first field *PROB*, there is a choice of four problem versions $[LS, WW, DLSI, DLS]$.

LS (Lot Sizing). This is the general problem defined above.

WW (Wagner-Whitin). This is problem *LS*, except that the variable production and storage costs satisfy $h_t = h'_t + p_t - p_{t+1} \geq 0$ for $t = 0, \ldots, n-1$.

DLSI (Discrete Lot Sizing with Variable Initial Stock). This is problem *LS* with the restriction that there is either no production or production at full capacity C_t in each period t.

DLS (Discrete Lot Sizing). This is problem *DLSI* without an initial stock variable.

The second field *CAP* concerns the production limits or capacities $[C, CC, U]$.

PROB-C (Capacitated). Here, the capacities C_t vary over time.

PROB-CC (Constant Capacity). This is the case where $C_t = C$, a constant, for all t.

PROB-U (Uncapacitated). This is the case when there is no limit on the amount produced in each period, i.e., C_t exceeds the sum of all present and future demands.

Before presenting the third parameter involving the many possible extensions, we now present mixed-integer programming formulations of the four basic variants with varying capacities *PROB-C*.

3.2. Formulations

The standard formulation of *LS* as a mixed-integer program involves the variables x_t—the amount produced in period t for $t = 1, \ldots, n$, s_t—the stock at the end of period t for $t = 0, \ldots, n$, and $y_t = 1$ if the machine is set up to produce in period t, and $y_t = 0$ otherwise. We also use the notation $d_{kt} \equiv \sum_{u=k}^{t} d_u$ throughout.

LS-C now has the formulation

$$\min \sum_{t=1}^{n} p_t x_t + \sum_{t=0}^{n} h'_t s_t + \sum_{t=1}^{n} f_t y_t \tag{1}$$

$$s_{t-1} + x_t = d_t + s_t \quad \text{for } t = 1, \ldots, n, \tag{2}$$

$$x_t \le C_t y_t \quad \text{for } t = 1, \ldots, n, \tag{3}$$

$$x \in R_+^n, \quad s \in R_+^{n+1}, \quad y \in \{0,1\}^n. \tag{4}$$

WW-C can be formulated just in the space of the s, y variables:

$$\min \sum_{t=0}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t \tag{5}$$

$$s_{k-1} + \sum_{u=k}^{t} C_u y_u \ge d_{kt} \quad \text{for } 1 \le k \le t \le n, \tag{6}$$

$$s \in R_+^{n+1}, \quad y \in \{0,1\}^n. \tag{7}$$

To derive this formulation, one first uses (2) to eliminate x_t from the objective function (1). To within a constant, the resulting objective function is

$$\sum_{t=0}^{n} (h'_t + p_t - p_{t+1}) s_t + \sum_{t=1}^{n} f_t y_t = \sum_{t=0}^{n} h_t s_t + \sum_{t=1}^{n} f_t y_t.$$

Then, as $h_t \ge 0$ for all t, it follows that once the set-up periods are fixed, the stocks will be as low as possible compatible with satisfying the demand. Thus,

$$s_{k-1} = \max \left(0, \max_{t=k,\ldots,n} \left[d_{kt} - \sum_{u=k}^{t} C_u y_u \right] \right),$$

(see Pochet and Wolsey 1994). It follows that the proposed formulation is valid, though its (s, y) feasible region is not the same as that of $LS-C$. Specifically, (s, y) is feasible in (6)–(7) if and only if there exists (x, s', y) feasible in (2)–(4) with $s' \le s$.

DLSI-C can be formulated by adding $x_t = C_t y_t$ in the formulation of *LS-C*. However, after elimination of the variables $s_t = s_0 + \sum_{u=1}^{t} x_u - d_{1t} \ge 0$ and $x_t = C_t y_t$, we obtain an equivalent formulation just in the space of the s_0 and the y variables:

$$\min \tilde{h}_0 s_0 + \sum_{t=1}^{n} \tilde{f}_t y_t \tag{8}$$

$$s_0 + \sum_{u=1}^{t} C_u y_u \ge d_{1t} \quad \text{for } 1 \le t \le n, \tag{9}$$

$$s_0 \in R_+^1, \quad y \in \{0,1\}^n, \tag{10}$$

where $\tilde{f}_t = f_t + C_t(\sum_{u=t}^{n} h_u)$ for $t = 1, \ldots, n$, and $\tilde{h}_0 = \sum_{t=0}^{n} h_t$.

DLS-C can be formulated just in the space of the y variables:

$$\min \sum_{t=1}^{n} \tilde{f}_t y_t \tag{11}$$

$$\sum_{u=1}^{t} C_u y_u \ge d_{1t} \quad \text{for all } 1 \le t \le n, \tag{12}$$

$$y \in \{0,1\}^n. \tag{13}$$

Without introducing a new problem class, we say that *DLS* has *Wagner-Whitin costs* if $\tilde{f}_t \ge \tilde{f}_{t+1}$ for all t.

3.3. Complexity

Observation 1. All eight constant or uncapacitated instances *PROB*-[*CC*, *U*] are polynomially solvable. The dynamic programming algorithm of Florian and Klein (1971) solves *LS-CC*, and the other seven problems can all be seen as special cases.

Observation 2. All four varying capacity instances *PROB-C* are *NP*-hard. All four problems are polynomially reducible to the 0-1 knapsack problem (see Bitran and Yanasse 1982).

The above imply that we can only reasonably hope to have complete convex hull descriptions, and/or tight reformulations when *CAP* is selected from $[U, CC]$.

We now consider the relationships between the different problems.

Notation 1. We let $X^{PROB\text{-}CAP}$ denote the feasible region of *PROB-CAP* as formulated in §2.2 in the corresponding space of variables. $\text{proj}_w(Y)$ denotes the projection of the solution set Y onto the space of variables w.

$$X_k^{DLSI-C} = \left\{ (s, y) \in R_+^{n+1} \times [0,1]^n : \right.$$
$$\left. s_{k-1} + \sum_{u=k}^{t} C_u y_u \ge d_{kt} \quad \text{for } t = k, \ldots, n \right\}.$$

The following proposition states more formally the links between the different formulations introduced in the previous subsection.

Proposition 1. (i) $\text{proj}_{s,y} X^{LS\text{-}C} \subseteq X^{WW\text{-}C}$.
(ii) $\text{proj}_{s_0,y} X^{WW\text{-}C} = X^{DLSI\text{-}C}$.
(iii) $X^{WW\text{-}C} = \bigcap_{k=1}^{n} X_k^{DLSI\text{-}C}$ *with* $X_1^{DLSI\text{-}C} = X^{DLSI\text{-}C}$.
(iv) $X^{LS\text{-}C} \subseteq X^{LS\text{-}CC} \subseteq X^{LS\text{-}U}$ *if we take* $\max_t C_t$ *as the constant capacity.*

On the other hand, it is clear that in the (x, s, y) space, *DLSI* is a restriction of *LS*.

Corollary. *Every valid inequality for WW-CAP in* (s, y) *variables is valid for LS-CAP, and every valid inequality for DLSI-CAP in* (s_0, y) *variables is valid for WW-CAP. Also, every valid inequality for PROB-U is valid for PROB-*$[C, CC]$.

3.4. Extensions

The third field *VAR* concerns extensions/variants $[B, SC, ST, LB, SL, SS]^*$ to 1 of the 12 problems *PROB-CAP* considered so far.

B (Backlogging). Demand must be satisfied, but the items can be produced later than requested. The cumulated shortfall $\max\{0, d_{1t} - s_0 - \sum_{j=1}^{t} x_j\}$ in satisfaction of the demand in period t is charged at a cost of b_t per unit.

SC (Start-Up Costs). If a sequence of set-ups starts in period t, a start-up cost g_t is incurred.

ST (Start-Up Times). If a sequence of set-ups starts in period t, the capacity C_t is reduced by an amount ST_t ($ST(C)$) denotes constant start-up times.

LB (Minimum Production Levels). If production takes place in period t, a minimum amount LB_t must be produced. $LB(C)$ denotes constant lower bounds.

SL (Sales). In addition to the demand d_t that must be satisfied in each period, an additional amount up to u_t can be sold at a unit price of c_t.

SS (Safety Stocks). There is a lower bound $\underline{S}_t$ on the stock level at the end of period t.

Now, we have the three fields that describe a single-item lot-sizing problem:

$$[LS, WW, DLSI, DLS]\text{-}[C, CC, U]$$
$$\text{-}[B, SC, ST, ST(C), SL, LB, LB(C), SS]^*,$$

where one entry is required from each of the first two fields, and any number of entries from the third.

Example 1. (i) *WW-U-Ø* (or just *WW-U*) denotes the uncapacitated Wagner-Whitin problem.

(ii) *DLSI-CC-B*, *ST* denotes the constant capacity discrete lot-sizing problem with initial stock variable, backlogging, and start-up times.

It is not difficult to show, by looking at the structure of the regeneration intervals, that the variants are still polynomially solvable in versions *PROB*-$[CC, U]$-*VAR* provided that the start-up times or lower bounds, if any, are constant (versions $ST(C)$, $LB(C)$). See, for instance, Vanderbeck (1998) for the case with constant capacity and constant start-up times *LS-CC-ST*(C).

4. Knowledge About *PROB-CAP-VAR*

In this section, we catalogue our state of knowledge about the most important polynomially solvable variants. Specifically, we present three tables for *PROB*-$[U, CC]$, *PROB*-$[U, CC]$-*B*, and *PROB*-$[U, CC]$-*SC*, respectively. We also indicate the relatively few results known for more complicated variants.

For each problem *PROB-CAP-VAR*, we present a table with three parts. The first part, FORMULATION, deals with extended formulations whose projection is the convex hull of $X^{PROB\text{-}CAP\text{-}VAR}$. First, some indication of the name of the reformulation (if any) is given, along with the number of constraints and variables in the formulation, and then references. The second part, VALID INEQUALITIES and SEPARATION, gives the family of valid inequalites describing the convex hull, the complexity of their separation, and references. The third, OPTIMIZATION, gives the complexity of the best known algorithm and references.

In the tables an asterisk (*) indicates that the family of inequalities only gives a partial description of the convex hull of solutions. A triple asterisk (***) indicates that we do not know of any result specific to the particular problem class, and a dash (−) indicates that the problem is considered trivial.

4.1. *PROB*-$[U, CC]$

Table 1 contains results for *PROB*-$[U, CC]$. The cases $[DLSI, DLS]$-U have been left blank as the results and algorithms are trivial.

Remarks Concerning Table 1. *FL* denotes the facility location reformulation from Krarup and Bilde (1977). *SP* denotes the shortest path reformulation from Eppen and Martin (1987). (l, S) denotes the (l, S)-inequalities $\sum_{j\in S} x_j + \sum_{j\in L\setminus S} d_{jl} y_j \geq d_{1l}$ for $L = \{1, \ldots, l\}$ and $S \subseteq L$, derived in Barany et al. (1984). (l, S, WW) denotes the subclass of (l, S)-inequalities needed for Wagner-Whitin costs (see Pochet and

Table 1 Models ***PROB*-[*U*, *CC*]**

	LS	*WW*	*DLSI*	*DLS*
FORMULATION				
U	*SP* $O(n) \times O(n^2)$ *FL* $O(n^2) \times O(n^2)$ Krarup and Bilde (1977) Eppen and Martin (1987)	*WW* $O(n^2) \times O(n)$ Pochet and Wolsey (1994)	—	—
CC	$O(n^3) \times O(n^3)$ Van Vyve (2001)	$O(n^2) \times O(n^2)$ Pochet and Wolsey (1994)	$O(n) \times O(n)$ Miller and Wolsey (2001a), Pochet and Wolsey (1994)	$O(n) \times O(n)$ Folklore
VALID INEQUALITIES and SEPARATION				
U	(I, S) $O(n \log n)$ Barany et al. (1984)	(I, S, WW) $O(n)$ Pochet and Wolsey (1994)	—	—
CC	$klSI^*$ — Pochet and Wolsey (1993)	$(klSI, WW)$ $O(n^2 \log n)$ Pochet and Wolsey (1994)	Mixing $O(n \log n)$ Günlük and Pochet (2001), Miller and Wolsey (2001a), Pochet and Wolsey (1994)	Gomory Folklore
OPTIMIZATION				
U	$O(n \log n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	$O(n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	—	—
CC	$O(n^3)$ Florian and Klein (1971), van Hoesel and Wagelmans (1996)	$O(n^2 \log n)$ Van Vyve (2002)	$O(n^2 \log n)$ Van Vyve (2002)	$O(n \log n)$ Van Vyve (2002)

Wolsey 1994). Specifically, $S = \{1, \ldots, k-1\}$, so the resulting inequalities can be rewritten in the form $s_{k-1} + \sum_{j=k}^{l} d_{jl} y_j \geq d_{kl}$. *klSI* denotes the *klSI*-inequalities derived in Pochet and Wolsey (1993) and $(klSI, WW)$ denotes a restricted subclass of *klSI*-inequalities (see Pochet and Wolsey 1994) that suffice for the Wagner-Whitin case. There is an exact separation algorithm for the subclass $(klSI, WW)$, which can be used as a basis for a heuristic separation algorithm for the class of *klSI* inequalities.

Mixing denotes essentially the $(klSI, WW)$-inequalities (see Günlük and Pochet 2001). Gomory indicates that Gomory fractional cuts give a tight $O(n) \times O(n)$ formulation for *DLS-CC*. The basic algorithm for *LS-CC*, due to Florian and Klein (1971), was an $O(n^4)$ algorithm based on a shortest path over regeneration intervals. This algorithm extends easily to *LS-CC-B* and *LS-CC-SC*. For *LS-CC*, van Hoesel and Wagelmans (1996) show how the costs of the regeneration intervals can be calculated more efficiently, leading to an $O(n^3)$ implementation. Recently, Van Vyve (2002) has generalized this approach even further, obtaining improved algorithms for [*WW-DLSI-DLS*]-*CC* and also for cases with backlogging.

Varying Capacities: Valid Inequalities and Separation. In Pochet (1988), it is shown how flow cover inequalities (Padberg et al. 1985) can be used to derive a class of valid inequalities for *LS-C*. Recently, a dynamic knapsack model was studied (Loparic 2001, Loparic et al. 2002, Marchand 1998), leading to new families of valid inequalities for *DLSI-C*, *WW-C*, and *LS-C*, as well as a separation heuristic. A fully polynomial approximation scheme is given in van Hoesel and Wagelmans (2001).

We now consider what results are known for the most important variants, in particular, those with backlogging and start-up costs, respectively.

4.2. Backlogging *PROB*-[*U*, *CC*]-*B*

The basic formulation for *LS-C-B* has as additional data b'_t the per unit cost of backlogging demand in period t. Its formulation requires the introduction of new variables: r_t is the amount backlogged at the end of period t for $t = 1, \ldots, n$. It is assumed throughout that r_0 is undefined, or equivalently that $r_0 = 0$.

LS-C-B now has the formulation

$$\min \sum_{t=0}^{n} h'_t s_t + \sum_{t=1}^{n} b'_t r_t + \sum_{t=1}^{n} p_t x_t + \sum_{t=1}^{n} f_t y_t \tag{14}$$

$$s_{t-1} - r_{t-1} + x_t = d_t + s_t - r_t \quad \text{for } t = 1, \ldots, n, \tag{15}$$

$$x_t \le C_t y_t \quad \text{for } t = 1, \ldots, n, \tag{16}$$

$$x, r \in R^n_+, \quad s \in R^{n+1}_+, \quad y \in \{0, 1\}^n. \tag{17}$$

WW-C-B. With backlogging, the costs are said to be *Wagner-Whitin* if both $h_{t-1} = p_{t-1} + h'_{t-1} - p_t \ge 0$ and $b_t = p_{t+1} + b'_t - p_t \ge 0$ for all t. However, it is not known if there is a simple formulation similar to that of *WW-C*, involving just the s, r, y variables.

DLSI-C-B has the formulation in the (s, r, y) space

$$s_0 + \sum_{u=1}^{t} C_u y_u = d_{1t} + s_t - r_t \quad \text{for } t = 1, \ldots, n,$$
$$s \in R^{n+1}_+, \quad r \in R^n_+, \quad y \in [0, 1]^n.$$

Now the variables $s_1, \ldots, s_n$ (or alternatively $r_1, \ldots, r_n$) can be eliminated, giving the feasible region

$$s_0 + r_t + \sum_{u=1}^{t} C_u y_u \ge d_{1t} \quad \text{for } t = 1, \ldots, n,$$
$$s_0 \in R^1_+, \quad r \in R^n_+, \quad y \in [0, 1]^n.$$

DLS-C-B is obtained from *DLSI-C-B* by setting $s_0 = 0$. The results for *PROB*-[*U*, *CC*]-*B* are given in Table 2.

Remarks Concerning Table 2. *SP* and *FL* are again shortest path and facility location like formulations. *RI* indicates a formulation based on regeneration intervals. It is simple to add backlog variables to the (l, S) inequalities to make them valid for *LS-U-B*. A larger family of inequalities, called cycle inequalities (Pochet and Wolsey 1994) suffice to generate $\text{conv}(X^{WW\text{-}U\text{-}B})$, and can be separated in polynomial time using network flow algorithms to find a negative cost cycle in an appropriate graph. Ext(*lS*) denotes an even larger family of inequalities. In similar fashion, Ext(*klSI*) denotes the family of *klSI* inequalities extended to be valid for *LS-CC-B*. *FC* denotes flow-cover inequalities, *RC* reduced capacity inequalities, *GMIX* denotes mixing inequalities made feasible by

Table 2 Model ***PROB*-[*U*, *CC*]-*B* with Backlogging**

	LS	*WW*	*DLSI*	*DLS*
FORMULATION				
U	*SP*(*B*) $O(n) \times O(n^2)$ *FL*(*B*) $O(n^2) \times O(n^2)$ Barany et al. (1986), Pochet and Wolsey (1988)	$O(n^2) \times O(n)$ Pochet and Wolsey (1994)	–	–
CC	*RI* $O(n^3) \times O(n^3)$ Van Vyve (2001)	*** Miller and Wolsey (2001a), Van Vyve (2001)	$O(n^2) \times O(n^2)^*$ Miller and Wolsey (2001a)	$O(n) \times O(n)$ Miller and Wolsey (2001a)
VALID INEQUALITIES and SEPARATION				
U	Ext$(l, S)^*$ Pochet and Wolsey (1988)	Cycles $O(n^3)$ Pochet and Wolsey (1994)	–	–
CC	Ext$(klSI)^*$	*FC*, *RC*, *GMix** Pochet (1988), Leung et al. (1989), Miller and Wolsey (2001a)	*GMix** Miller and Wolsey (2001a)	*MIR* Miller and Wolsey (2001a)
OPTIMIZATION				
U	$O(n \log n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	$O(n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	–	–
CC	$O(n^4)$	***	***	$O(n^2)$ Van Vyve (2002)

the addition of appropriate backlog variables, and *MIR* denotes mixed-integer rounding inequalities.

4.3. Start-up Costs (*SC*)

The basic formulation for *LS-C-SC* has, as additional data, the start-up costs g_t for $t = 1, \ldots, n$. It requires the introduction of new variables: $z_t = 1$ if there is a start-up in period t, i.e., there is a set-up in period t, but there was not in period $t-1$, and $z_t = 0$ otherwise. The resulting formulation is

$$\min \sum_{t=1}^{n} p_t x_t + \sum_{t=0}^{n} h'_t s_t + \sum_{t=1}^{n} f_t y_t + \sum_{t=1}^{n} g_t z_t \qquad (18)$$

$$s_{t-1} + x_t = d_t + s_t \quad \text{for } t = 1, \ldots, n, \qquad (19)$$

$$x_t \le C_t y_t \quad \text{for } t = 1, \ldots, n, \qquad (20)$$

$$z_t \ge y_t - y_{t-1} \quad \text{for } t = 1, \ldots, n, \qquad (21)$$

$$z_t \le y_t \quad \text{for } t = 1, \ldots, n, \qquad (22)$$

$$z_t \le 1 - y_{t-1} \quad \text{for } t = 1, \ldots, n, \qquad (23)$$

$$x \in R^n_+, \quad s \in R^{n+1}_+, \quad y, z \in \{0, 1\}^n, \qquad (24)$$

where we assume that y_0, the state of the machine at time 0, is given as data.

The formulations of $[WW, DLSI, DLS]$-*C-SC* are obtained by just adding the constraints (21)–(23) and $z \in \{0, 1\}^n$ to the earlier formulations given in §2.

The results for *PROB*-$[U, CC]$-*SC* are given in Table 3.

Remarks concerning Table 3. Eppen and Martin (1987) provided a first shortest path formulation for *LS-U-SC* with $O(n^3)$ variables. Again for *LS-U-SC*, Rardin and Wolsey (1993) showed that the separation problem for (l, R, S) inequalities can be solved by a single max flow calculation in a graph with $O(n^3)$ nodes. For *WW-U-SC*, the (l, S, SC) inequalities are a simple modification of the (l, S, WW) inequalities to include start-up variables.

In Constantino (1996), $O(n^2)$ separation algorithms are given for the classes of left and right submodular inequalities that are valid for *LS-C-SC* with varying capacities. Also, an $O(n^3)$ separation algorithm is given for the family of left *klSI* inequalities valid

Table 3 Model ***PROB-[U, CC]-SC*** with Start-ups

	LS	*WW*	*DLSI*	*DLS*
FORMULATION				
U	*SP*(*SC*) $O(n^2) \times O(n^2)$ *FL*(*SC*) $O(n^3) \times O(n^2)$ van Hoesel et al. (1994), Wolsey (1989)	$O(n^2) \times O(n)$ Pochet and Wolsey (1994)	–	–
CC			$O(n^3) \times O(n^3)$ *WW* $O(n^3) \times O(n^2)$	$O(n^2) \times O(n^2)$ van Hoesel and Kolen (1994) *WW* $O(n^2) \times O(n)$ van Eijl and van Hoesel (1997)
VALID INEQUALITIES and SEPARATION				
U	(l, R, S) $O(n^3)$ van Hoesel et al. (1994), Wolsey (1989)	(l, S, SC) ***	–	–
CC	left/right,submod* Constantino (1996)	***	***	hole/bucket* van Hoesel and Kolen (1993)
OPTIMIZATION				
U	$O(n \log n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	$O(n)$ Aggarwal and Park (1993), Federgruen and Tsur (1991), Wagelmans et al. (1992)	–	–
CC	$O(n^4)$ Florian and Klein (1971)	***	***	$O(n^2)$ Fleischmann (1990) *WW* $O(n \log n)$ van Hoesel (1991)

for *LS-CC-SC*. In van Eijl (1996), polynomial separation algorithms are given for several classes of hole/bucket inequalities for *DLS-CC-SC*. Formulations for *DLSI-CC-ST* can be obtained by viewing the set $X^{DLSI\text{-}CC\text{-}SC}$ as the union of $n+1$ sets of the form $X^{DLS\text{-}CC\text{-}SC}$, depending on the possible values taken by the initial stock variable s_0.

4.4. Other Variants

We indicate a series of results concerning either formulations or familes of valid inequalities that can be useful.

- *WW-U-B, SC*. In Agra and Constantino (1999), an $O(n^2) \times O(n)$ reformulation is presented generalizing those for *WW-U-B* and *WW-U-SC*.
- *LS-U-SS, SL*. In Loparic et al. (2001), a family of valid inequalites describing the convex hull is presented as well as tight extended formulations in certain special cases.
- *LS-CC-SC*. In Constantino (1996), several families of valid inequalities are presented as well as efficient separation algorithms.
- *LS-U-LB*. In Constantino (1998), models are studied that provide relaxations of both *LS-U-LB*, and also of single-period relaxations of multi-item models.
- *LS-CC-ST(C)*. In Vanderbeck (1998), a dynamic programming algorithm for the optimization problem is presented.

5. Classification of Multi-Item/Machine/Level Problems

Here, we present a minimal extension of the classification scheme to deal with a limited class of multi-item and/or multimachine problems. We assume that there are several items and one or more machines.

Machines {*NK* = #, [*LT*]*, [*SB1*, *SB2*, *BB*], [*SET*, *ST*, *SQT*, *SQC*]*}. The first subfields are simple. *NK* is the number of machines. *LT* indicates that there are lead times.

If a machine produces more than one item, there are typically joint capacity constraints across items. When periods are short, so that only one or two items are produced by the machine in a period, the production order is completely specified by the set-up and start-up variables, and one talks of *small time buckets*. When the time periods are longer and more than two set-ups are permitted per period, the order of the items within each time period may or may not be important. For such problems, one talks of *big time buckets*.

The following subfield gives information about the time buckets. *SB1*, *SB2* indicate a small bucket model in which either at most one or at most two set-ups are permitted per period, respectively. *SB1* is often referred to as a model with *mode* constraints. *BB* denotes a big bucket model with at least one joint capacity constraint k imposing a limit L_t^k on the amount of capacity available in each period. a^{ik} denotes the capacity consumption rate per unit of item i.

The last subfield gives information about the capacity utilization. *SET* indicates that there are also set-up times b^{ik} that reduce the capacity available. *ST* indicates that there are start up times e^{ik} . *SQT* indicates that there are sequence-dependent changeover times qt^{ijk}. *SQC* indicates that there are sequence dependent changeover costs qc^{ijk} whether it is a big or small bucket model. Our second example, Fleischmann (1994), is an *SB1* example of this type, whereas CHES problems (1989) are big bucket problems with sequence dependent changeover costs.

Multilevel Production {*NL* = #, [*G*, *A*, *S*]}. The production structure classification is simple. *NL* denotes the number of levels, with ρ_t^{ijk} the number of units of item i needed to produce one item of j on machine k in period t for each item $j \in S(i)$, the set of successors of i. *G* denotes a general product structure. *A* denotes assembly structure. *S* denotes in series product structure, i.e., linear. Finally, to complete this partial classification, we may wish to add $NT = n$ the number of time periods and *NI* the number of items.

5.1. MIP Formulation

Introducing additional suffices i or j for items, and k for machines, we also require new variables u_t^{ijk} to model sequence-dependent changeovers. Most of the problems covered by the above classification can now be represented by the MIP

$$\min \sum_{i,k,t} \text{Cost}(x_t^{ik}, y_t^{ik}, s_t^i, r_t^i, z_t^{ik}) + \sum_{i,j,k,t} qc_t^{ijk} u_t^{ijk}$$

$$s_{t-1}^i - r_{t-1}^i + \sum_k x_t^{ik} = d_t^i + \sum_{j \in S(i)} \rho^{ijk} x_t^{jk} + s_t^i - r_t^i \quad \text{for all } i, t, \tag{25}$$

$$\sum_i \left(a^{ik} x_t^{ik} + b^{ik} y_t^{ik} + e^{ik} z_t^{ik} + \sum_{j \neq i} qt^{ijk} u_t^{ijk} \right) \leq L_t^k \quad \text{for all } k, t, \tag{26}$$

constraints modelling start-ups, (27)

constraints modelling sequence dependence, etc. (28)

...

We note that in *SB1* models, a^{ik}, e^{ik}, and qt^{ijk} are zero, and inequality (26) reduces to

$$\sum_i y_t^{ik} \leq 1 \quad \text{for all } k, t. \tag{29}$$

One possible model for *SB2* has the constraints

$$\sum_i y_t^{ik} \leq 2 \quad \text{for all } k, t,$$

$$\sum_i (y_t^{ik} - z_t^{ik}) \leq 1 \quad \text{for all } k, t.$$

The latter constraint imposes that there is only one set-up per period that is not a start-up.

5.2. Known Results for Multi-Item Single-Machine Problems

We present a few basic results on polynomial solvability, reformulation, and valid inequalities. In all the special cases below, there is a single machine ($NK = 1$).

- Multilevel Uncapacitated Lot Sizing in Series. $NL > 1$, *S/LS-U* is polynomially solvable by dynamic programming (Zangwill 1969).
- Multilevel Lot Sizing. $NL > 1$, *G/LS-CC-VAR*. Using an echelon stock reformulation (Clark and Scarf 1960) leads to a formulation with a single-item lot-sizing problem for each item.
- Multi-Item Single-Mode Constant Capacity, Discrete Lot Sizing. *SB1/DLS-CC* reduces to a network flow problem. This is part of the folklore; see for example, Miller and Wolsey (2001a).
- Multi-Item Single-Mode Constant Capacity, Discrete Lot Sizing with Backlogging. *SB1/DLS-CC-B*. The convex hull of solutions is obtained using the convex hull formulation for $NI = 1$ plus the mode constraints (29) (see Miller and Wolsey 2001a).
- Big Bucket Problems with Set-Up Times. *BB*, *SET/LS-C*. Valid inequalities have been proposed by Miller et al. (2000a, 2000b)
- $[BB, SB1, SB2], [SQT, SQC]^*$. Formulations for sequence-dependent changeovers for small buckets and big buckets can be found in Belvaux and Wolsey (2001), Constantino (1995), Karmarkar and Schrage (1985), and Wolsey (1989).

6. Three Problems: Reformulation by Classification

Here we show how to profit from the classification of §§3 and 4 to obtain a good formulation. We then demonstrate the approach on three problem instances. In each case, we first classify the instance. Then, we use the tables to derive a strong reformulation of the instance that is then fed into a standard MIP solver. Results obtained are compared either with those provided by alternative formulations, or with those obtained earlier using one or more special purpose systems.

6.1. Use of the Classification

As an illustration of how to use the classification, we consider a multi-item, single-level, single-machine problem. Suppose that the problem is single mode with backlogging and constant capacities, namely $NK = 1$, *SB1/LS-CC-B*.

Step 1. Check to see if the costs are Wagner-Whitin, as this property is unaffected by mode constraints. We assume that the answer is positive.

Step 2. Check *WW-CC-B* in Table 2. An approximate reformulation is proposed, but $O(n^3) \times O(n^3)$ appears too large.

Step 3. We can move upward or toward the right in Table 2 to find a relaxation. Moving upward from *CC* to U, the relaxation *WW-U-B* is obtained for which a tight $O(n^2) \times O(n)$ reformulation is indicated in Table 2.

Step 4. Moving right from *WW* to *DLSI*, we obtain the relaxations $DLSI_k$-*CC-B* for which a good $O(n^2) \times O(n^2)$ reformulation is again known for each k. However, this leads to an $O(n^3) \times O(n^3)$ formulation, which is again rejected as being too big.

Step 5. Decide to use the reformulation of Step 3, which has $NI \times O(n^2)$ constraints and $NI \times O(n)$ variables, and is of reasonable size.

A similar approach has been taken in tackling the three instances treated below, starting from the verbal classification derived in §2.

6.2. Problem 1: Bottling

(i) Multi-item constraints and costs. At most, one item can be produced per period.

(ii) Individual item constraints and costs. When produced, each item is produced for between 7 and 16 hours, so both the upper bound and the lower bounds on production per period are time-invariant. Also, the unit production and storage costs are time-invariant and there are start-up costs.

From this, the problem can be classified as $NK = 1$, *SB1*/*WW-CC-SC*, *LB* with formulation

$$\min \sum_{i,t} (p_t^i x_t^i + h_t^i s_t^i + f_t^i y_t^i + g_t^i z_t^i) \tag{30}$$

$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \text{for all } i, t, \tag{31}$$

$$x_t^i \le C^i y_t^i \quad \text{for all } i, t, \tag{32}$$

$$x_t^i \ge LB^i y_t^i \quad \text{for all } i, t, \tag{33}$$

$$\sum_i y_t^i \le 1 \quad \text{for all } t, \tag{34}$$

$$z_t^i \ge y_t^i - y_{t-1}^i \quad \text{for all } i, t, \tag{35}$$

$$z_t^i \le y_t^i \quad \text{for all } i, t, \tag{36}$$

$$x, s \ge 0, \quad y, z \in \{0, 1\}. \tag{37}$$

In Table 3, we see that the reformulation of *WW-CC-SC*, *LB* is blank. However, there is an $O(n^2) \times O(n)$ reformulation of *WW-U-SC*. Also, in Table 1, we see that there is an $O(n^2) \times O(n^2)$ reformulation of *WW-CC*.

The reformulation for *WW-U-SC* is obtained by just adding the $O(n^2)$ inequalities

$$s_{t-1} \ge \sum_{j=t}^{l} d_j (1 - y_t - z_{t+1} - \cdots - z_j) \quad \text{for all } t, l \text{ with } t \le l. \tag{38}$$

The reformulation for *WW-CC* for each item is

$$s_{k-1} \ge C \sum_{t=k}^{n} f_t^k \delta_t^k + C\mu^k \quad \text{for all } k, \tag{39}$$

$$\sum_{u=k}^{t} y_u \ge \sum_{\tau \in \{0\} \cup [k,n]} \left\lceil \frac{d_{kt}}{C} - f_\tau^k \right\rceil \delta_\tau^k - \mu^k \quad \text{for all } k, t, k \le t, \tag{40}$$

$$\sum_{\tau \in \{0\} \cup [k,n]} \delta_t^k = 1 \quad \text{for all } k, \tag{41}$$

$$\mu^k \ge 0, \delta_t^k \ge 0 \quad \text{for } t \in \{0\} \cup [k, n] \quad \text{for all } k, \tag{42}$$

$$0 \le y_t \le 1 \quad \text{for } t = 1, \ldots, n, \tag{43}$$

where

$$f_0^k = 0, \quad f_\tau^k = \frac{d_{k\tau}}{C} - \left\lfloor \frac{d_{k\tau}}{C} \right\rfloor$$

and $[k, t]$ denotes the interval $\{k, k+1, \ldots, t\}$. The additional variables δ_t^k indicate that $s_{k-1} = Cf_t^k$ (modulo C).

In Table 4, we present computational results showing the effects of the reformulations. Instance cl-1a is the original formulation (30)–(37). Instance cl-1b is with the addition of the inequalities (38) for *WW-U-SC*. Instance cl-1c has, in addition, the reformulation (39)–(43) of *WW-CC* for each item. The nine columns represent the instance, the number of rows, columns and 0-1 variables, followed by the initial LP value, the value XLP after the system has automatically added cuts, IP the optimal value, the total number of seconds required to prove optimality, and finally, the number of nodes in the branch-and-

Table 4 Results for Problem 1

Instance	m	n	Int	LP	XLP	IP	Secs	Nodes
cl-1a	511	720	120	1509.1	3549.6	4414.2	5000*	3.8×10^5
cl-1b	2354	720	120	3800.6	4305.1	4404.5	383	3826
cl-1c	4454	2824	120	4309.9	4310.5	4404.5	82	175

cut tree. All runs were carried out with the default version of the XPRESS-MP Optimiser (2001) Release 12.50 running on a 500-MHz Pentium III under Windows NT.

An asterisk (*) indicates that the run was terminated before optimality was proved. For formulation cl-1a, the best lower bound on termination was 4,251.2 leaving a gap of 3.7%.

6.3. Problem Instance 2: Discrete Lot-Sizing and Sequence-Dependent Changover Costs

(i) Multi-item constraints and costs. At most, one item can be produced per period, and there are sequence-dependent set-up costs.

(ii) Individual item constraints and costs. Production is all or nothing with constant capacities. There are no unit production costs, and storage costs are nonnegative and constant over time.

The problem can be classified as $NK = 1, SB1$, *SQC/DLS-CC*.

As observed in Fleischmann (1994), there is no backlogging, so demands can be normalized with $d_t^i \in \{0, 1\}$. A basic formulation is then

$$\min \sum_{i,t} h^i s_t^i + \sum_{i,j,t} q^{ij} u_t^{i,j}$$
$$s_{t-1}^i + x_t^i = d_t^i + s_t^i \quad \text{for all} \quad i, t,$$
$$x_t^i \le y_t^i \quad \text{for all} \quad i, t,$$
$$\sum_i y_t^i = 1 \quad \text{for all} \quad t,$$
$$u_t^{ij} \ge y_{t-1}^i + y_t^j - 1 \quad \text{for all} \quad i, j, t,$$
$$x, y \in \{0, 1\}, \quad s, u \ge 0.$$

Observation 3. The reformulation of changeover variables (Karmarkar and Schrage 1985, Wolsey 1989) indicated in §5.2 leads to the constraints

$$\sum_i u_t^{ij} = y_t^j \quad \text{for all } j, t,$$
$$\sum_j u_t^{ij} = y_{t-1}^i \quad \text{for all } i, t,$$
$$\sum_i y_0^i = 1,$$
$$u_t^{ij} \ge 0 \quad \text{for all } i, j, t,$$

representing the flow of a single unit passing from item set-up to item set-up over time. Here, the set-up variable y_t^i is the flow through node (i, t) and u_t^{ij} is the flow from node $(i, t-1)$ to node (j, t), indicating a switch from a set-up of item i in period $t-1$ to a set-up of item j in t.

Observation 4. Inclusion of Start up Variables. When there are changeover variables, there are implicitly start-up variables for which we know tighter formulations. Thus, we introduce the equation

$$z_t^j = \sum_{i:i\neq j} u_t^{ij}$$

to define the start-up variables. Switch-off variables w_t^i can be defined similarly. This means that it is possible to use results for the single-item model *DLS-CC-SC*.

Observation 5. Reformulation of *DLS-CC-SC*. From Table 3, we see that there is a tight $O(n^2) \times O(n)$ reformulation under the assumption of Wagner-Whitin costs. This consists of the inequalities

$$s_{t-1}^i + \sum_{u=t}^{t+p-1} y_u^i + \sum_{u=t+1}^{t+p-1} (d_{ul} - (t+p-u)) z_u + \sum_{u=t+p}^{l} d_{ul} z_u \ge p$$

for all t, l such that $d_l = 1, l \ge t$, where we suppose that $d_{t_1} = \ldots = d_{t_p} = 1$ with $t < t_1 < \ldots < t_p = l$ and $d_\tau = 0$ in intervening periods in $\{t, \ldots, l\}$.

In Table 5, we present computational results showing the effects of the reformulations. Instance cl2-NTa is the initial formulation, instance cl2-NTb is the formulation with reformulation from Observation 3, and instance cl2-NTc also includes the reformulation of *DLS-CC-SC*(*WW*) from Observations 4 and 5. Instances with $NT = 35$ and $NT = 60$ periods were solved. Table 5 has the same structure as Table 4. Note that cl2-35a and cl2-35b are unsolved after

Table 5 Results for Problem 2

Instance	m	n	Int	LP	XLP	IP	Secs	Nodes
cl2-35a	3797	4110	350	27.2	34.7	2056	1800*	51500*
cl2-35b	2062	5130	690	180.9	531.6	1599	1800*	8000*
cl2-35c	2599	5130	690	1361.5	1361.5	1387	9	17
cl2-60c	4817	8880	1190	1453.6	1454.0	1560	17579	8117

1,800 seconds. The best lower bounds obtained are 240.9 and 804.3, respectively.

6.4. Problem 3: Multilevel Assembly

(i) This is a multilevel problem with assembly-type product structure.

(ii) Multi-item constraints and costs. Many items can be produced in each period, and the capacity constraints limiting production in each period involve both production levels and set-up times for families.

(iii) Individual item constraints and costs. There are no individual capacity constraints, but there are storage costs and implicit fixed costs through the families.

This gives the classification $NL > 1$, $A/NK > 1$, BB, $ST(Family)/LS\text{-}U$.

We now present the initial formulation from Simpson and Erenguc (1998), except for the replacement of the stock variables s_t^i by echelon stock variables e_t^i, where $s_t^i = e_t^i - e_t^{\sigma(i)}$ and $\sigma(i)$ is the unique successor, if any, of item i. This gives

$$\min \sum_{i,t} \bar{h}^i e_t^i + \sum_{f,t} c^f \eta_t^f \tag{44}$$

$$e_{t-1}^i + x_t^i = d_t^{q(i)} + e_t^i \quad \text{for all } i, t, \tag{45}$$

$$e_t^i \geq e_t^{\sigma(i)} \quad \text{for all } i, t, \tag{46}$$

$$x_t^i \leq M y_t^i \quad \text{for all } i, t, \tag{47}$$

$$y_t^i \leq \eta_t^f \quad \text{for all } i, f, t, \text{ with } i \in F(f), \tag{48}$$

$$\sum_{i \in F(f)} a^{if} x_t^i + \sum_{g \in V(f)} \beta^{gf} \eta_t^g \leq C_t^f \eta_t^f \quad \text{for all } f, t, \tag{49}$$

$$y_t^i, \eta_t^f \in \{0,1\}, x_t^i, s_t^i \geq 0 \quad \text{for all } i, f, t, \tag{50}$$

where $q(i)$ is the final product containing item i, $\bar{h}_t^i = h_t^i - \sum_{j \in P(i)} h_t^j$, where $P(i)$ is the set of immediate predecessors of item i, η_t^f is the set-up variable for family f in period t, $F(f)$ is the set of items in family f and $V(f)$ is a set of families appearing in the budget constraint of family f.

This model can also be reformulated by eliminating the y_t^i variables giving

$$x_t^i \leq M \eta_t^f \quad \text{for all } i, f, t \text{ with } i \in F(f), \tag{51}$$

in place of the constraints (47)–(48).

As observed in §5.2, the echelon stock formulation is such that the constraints (45)–(47) give a model of the form $LS\text{-}U$. Rather than use an $O(n) \times O(n^2)$ reformulation of $LS\text{-}U$ involving many new variables, we have used the reformulation $WW\text{-}U$ (see Table 1). In addition, to avoid adding too many constraints, we have added only a subset of the (l, S, WW) inequalities

$$e_{t-1}^i + \sum_{u=t}^{l} d_{ul}^{q(i)} y_u^i \geq d_{tl}^{q(i)} \quad \text{for all } t, l, l-t \leq PAR,$$

where PAR is an integer. We denote the resulting formulation by cl3-NT-#c, where $\# \in \{1, 2\}$ is the number of the instance.

In the model with the y_t^i variables eliminated, we can do something similar by adding the constraints

$$e_{t-1}^i + \sum_{u=t}^{l} d_{ul}^{q(i)} \eta_u^{f(i)} \geq d_{tl}^{q(i)} \quad \text{for all } t, l, l-t \leq PAR,$$

where $f(i)$ is any family containing item i. Clearly, these inequalities are only unique when each item belongs to just one family. We denote the resulting formulations by cl3-NT-#b.

In Table 6, we present results for the four instances tackled in Belvaux and Wolsey (2001). In all cases, $NT = 16$. The two 78-item instances have each item belonging to a single family, so for these, we have used the more compact formulation cl3-78-#b. These two instances were run with $PAR = 4$.

The 80-item instances were run with the larger formulation cl3-80-#c and with $PAR = 8$.

The columns of Table 6 contain the same information as in Tables 4 and 5, except that the last column has been replaced by the Gap % on termination, where $GAP = (BIP - BLB)/BIP \times 100$ with BLB, the value of the best lower bound.

The best results obtained in Belvaux and Wolsey (2001) were gaps of 8.1, 4.9 % running bc-opt on the two 78-item instances with the echelon stock formulation (44)–(50), but with (47) replaced by (51), and gaps of 13.5, 13.8 % running bc-prod on the two 80-item instances using the original formulation without echelon stock variables. There, all four instances were run for 1,800 seconds on a 350-MHz Pentium running under Windows NT.

Table 6 Results for Problem 3

Instance	*r*	*c*	Int	LP	XLP	BIP	Secs	BLB	Gap %
cl3-78-1b	7607	2688	192	10777.0	10839.9	11592.0	450	10934.0	5.7
cl3-78-2b	7618	2688	192	10464.8	10511.1	10926.0	450	10550.9	3.4
cl3-80-1c	13725	4128	288	21376.9	21551.7	25160.3	900	21869.3	13.1
cl3-80-2c	13700	4128	288	21951.6	22152.5	26377.4	900	22417.3	15.0

7. Conclusions

The three examples treated in the last section suggest that certain practical lot-sizing problems can now be effectively tackled with nothing but appropriate tight a priori reformulations and a commercial mixed-integer programming system. Another such example can be found in Miller and Wolsey (2001a).

The classification scheme for single-item problems introduced and detailed in §§2 and 3 shows that there are still a number of open questions whose solutions would allow us to tackle an even larger range of lot-sizing problems. Here, we list a few that we believe are the most important or challenging.

(i) *DLSI-CC-B*. Find a compact tight reformulation and establish whether the $O(n^2) \times O(n^2)$ formulation from Miller and Wolsey (2001a) is tight. This question is also of importance for *WW-CC-B*.

(ii) *DLSI-CC-SC* and *DLS-CC-B*, *SC*. Find compact formulations and/or strong valid inequalities.

(iii) *LS-CC-SS*. Find formulations and valid inequalities.

(iv) *PROB-C*. Find fast and effective separation heuristics for the dynamic knapsack inequalities proposed in Loparic et al. (2002).

(v) $NK > 1$, $NI = 1$. Study the multimachine single-item problem. Do the dynamic knapsack inequalities suffice computationally? For problems with two machines, do the recent two variable knapsack results of Agra and Constantino (2001) provide useful inequalities?

(vi) A further question involves the effect of explicit upper bounds on the stocks. The optimization problem has been examined in Love (1973), but the effect on formulations has apparently not been examined.

There are also obviously a wealth of questions when one turns to multi-item problems. Some important ones are

(vii) *SB1/WW-U*. For the simplest possible single-mode problem, find valid inequalities involving multiple items.

(viii) *BB-ST/LS-CAP*. Find valid inequalities to deal with start-up times in big bucket models, extending the results of Miler et al. (2000a, 2000b)

(ix) *BB*-$[SQC, SQT]^*$/*PROB-CC*. Find valid inequalities for big bucket models with sequence-dependent costs and/or times.

It is also, perhaps, worth pointing out that there is to our knowledge still no complete convex hull description or compact convex hull reformulation for the basic uncapacitated lot sizing in series problem $NL > 1$, *S/LS-U*.

The approach advocated here also raises algorithmic questions, such as finding ways to combine valid inequalities and tight reformulations, finding approximate but more compact, reformulations that are tight for many instances, or using the reformulations with LP to solve the separation problems. Given that some reformulations provide good bounds but are too large to be effective during enumeration, one could also, perhaps, imagine working simultaneously with more than one formulation. Finally, there is the largely untouched question of whether the classification and reformulations can be used to develop effective primal heuristics.

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