An Introduction to Mathematical Notation

Mathematical Sciences Society
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Preface

The aim of this lecture was for attendees to familiarise themselves with mathematical notation. The aim is not a rigorous treatment of the subject of proofs and theorem proving, rather, the focus was entirely on developing familiarity and fluency with the language.

Section 1. Logical Quantifiers and Equivalence Relations

Mathematical statements often involve quantifiers and equivalence relations:

- $=$ — (equality symbol)
- $\equiv$ — (logical equivalence)
- $\therefore$ — (conclusion shorthand)
- $\forall$ — (universal quantifier)
- $\exists$ — (existential quantifier)
- $\exists!$ — (unique existential quantifier)
- $|$ or $:$ — (such that)
- $:=$ — (structural definition)
- $\implies$ — (implies)
- $\iff$ — (if and only if)
- ∞ — (infinity)
- $\emptyset$ — (empty set)
- $\approx$ — (approximately equal)

Section 2. Set Builder Notation

Sets are used to represent collections of elements. We say $x \in X$ to denote that x is an element of set X, and $x \notin X$ to denote that x is not an element of X. Set-builder notation expresses sets through properties. For example: $\{x \in X \mid P(x)\}$ denotes the set of all x in X satisfying property $P(x)$. Some examples are $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$. Common number sets include:

- Natural Numbers — $\mathbb{N} = \{1, 2, 3, \ldots n\}$
- Integer Numbers — $\mathbb{Z} = \{\ \ldots, -2, -1, 0, 1, 2, \ldots\}$
- Rational Numbers — $\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$
- Real Numbers — $\mathbb{R} = (-\infty, \infty)$ (Complete ordered Archimedean field)
- Embeddings: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

2.1 Set Operations

For sets X and Y:

- Union — $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$
- Intersection — $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$
- Difference — $X \setminus Y = \{x \mid x \in X \text{ and } x \notin Y\}$
- Cartesian Product — $X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1 \text{ and } x_2 \in X_2\}$
- Complement — $X^c = \{x \in U \mid x \notin X\}$ (relative to universal set U)

2.2 Set Theory Examples

Let $L := 2 + 3$ and $M := 3 + 2$. Are these statements equivalent?

$$L = 2 + 3 = 5 \text{ and } M = 3 + 2 = 5, \quad L = M$$

Let $X = \{1, 2, 3\}$ and $Y = \{4, 5, 6\}$.

$$2 \in X? \text{ YES} \quad 2 \in Y? \text{ NO} \quad 4 \in X \cup Y \text{ YES}, \quad 2 \in X \cap Y \text{ NO}$$

Let $X = \{1, 2, 3, 4, 5\}$

$$\exists x \in X, x > 3? \text{ YES } (x = 4/x = 5), \quad \exists x \in X, x < 0? \text{ NO}$$

Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

$$A \cup B = \{1, 2, 3, 4, 5\}, \quad A \cap B = \{3\}$$

$$1 \in A \cap B \text{ NO}, \quad 3 \in A \cap B \text{ YES}, \quad 5 \in A \cup B \text{ YES}$$

Let $U = \{1, 2, 3, 4, 5\}$ (universal set) and $A = \{1, 2\}$

$$A^c = \{3, 4, 5\}, \quad 1 \in A^c \text{ NO}, \quad 4 \in A^c \text{ YES}$$

Let x be a subset of the integers contained in the interval $(0, 4)$ such that $x \in (0, 4) \cap \mathbb{Z}$. What is the set of values belonging to this?

$$\forall x \in \mathbb{Z},\ \exists\ x \in (0 < x < 4), \quad x = \{1, 2, 3\}$$

Let x be a subset of the integers such that $x \in \mathbb{Z}$. There exists a unique set containing one element which is the value of x. What is that set and what is the value of the element?

$$\forall x \in \mathbb{Z},\ \exists!\ x \in [0, 2] \cap \mathbb{Z}, \quad 0 < x < 2, \implies x = \{1\}$$

Let $\mathbb{R}$ be the set of real numbers on the interval $(-\infty, \infty)$. We are looking for a unique value x in a bounded subset of the reals $[\ ,\ ] \subseteq \mathbb{R}$ which also belongs the bounded subset of integers $[4, 6] \subset \mathbb{Z}$.

$$\mathbb{R} := (-\infty, \infty) \implies \{x \in [\ ,\ ] \cap \mathbb{Z} \mid 4 < x < 6\}, \quad x = 5$$

Section 3. Functions and Maps

3.1 Functions

A function $f : X \to Y$ assigns each element $x \in X$ to exactly one element $f(x) \in Y$. We call X the *domain* and Y the *codomain*. The notation $f : X \to Y,\ x \mapsto f(x)$ indicates the function's domain/codomain, as well as its action on elements. The arrow $\to$ describes the function (the mapping), while $\mapsto$ shows where individual elements are sent. For example, the function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$ squares real numbers. A simple example of this is $f(3) = 9$.

3.2 Classes of Functions

A function $f : X \to Y$ is:

- **Injective** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- **Surjective** if $\forall y \in Y$, $\exists x \in X$ s.t. $f(x) = y$.
- **Bijective** if it is both injective and surjective.

3.3 Composition and Inverses

Function composition $(g \circ f)(x) = g(f(x))$ combines functions. If $f : X \to Y$ is bijective, then it has an inverse function $f^{-1} : Y \to X$ satisfying:

$$
\begin{aligned}
f^{-1}(f(x)) &= x, \quad \forall x \in X, \\
f(f^{-1}(y)) &= y, \quad \forall y \in Y.
\end{aligned}
$$

Section 4. Calculus & Analysis

For $x \in \mathbb{R}$:

4.1 Basic Derivatives

$$\frac{d}{dx}(x^2) = 2x, \quad \frac{d}{dx}(x^3) = 3x^2$$

4.2 Basic Integrals

$$\int x\,dx = \frac{x^2}{2} + c, \quad \int x^2\,dx = \frac{x^3}{3} + c$$

4.3 Partial Derivatives

Let $f(x, y) = x^2 + y^2$.

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

At the point $(1, 2) \in \mathbb{R}^2$:

$$\left.\frac{\partial f}{\partial x}\right|_{(1,2)} = 2, \quad \left.\frac{\partial f}{\partial y}\right|_{(1,2)} = 4$$

Section 5. Summation Notation and Products

5.1 Sigma Notation

The summation symbol $\sum$ compactly expresses sums: $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$. The index variable ($i$ or k above) is a "dummy variable"—its name doesn't matter. We could equally write $\sum_{j=1}^{n} a_j$.

$$\sum_{i=1}^{5} i^2 = 1+2+3+4+5 = 55, \quad \sum_{i=1}^{2}\sum_{j=1}^{3} ij = (1\cdot 1 + 1\cdot 2 + 1\cdot 3) + (2\cdot 1 + 2\cdot 2 + 2\cdot 3) = 18$$

5.2 Product Notation

Similarly, $\prod$ denotes products: $\prod_{i=1}^{n} a_i = a_1 \cdot a_2 \cdots a_n$.

$$\prod_{i=1}^{5} i = 1\times 2\times 3\times 4\times 5 = 120 = 5!, \quad \prod_{i=1}^{3}(2i) = 2\times 4\times 6 = 48$$

Section 6. Linear Algebra

6.1 Vectors

$$\mathbf{v} = (1,2,3), \quad \mathbf{u} = (4,5,6), \quad \mathbf{u},\mathbf{v} \in \mathbb{R}^3$$

6.2 Addition & Product

$$\mathbf{u}+\mathbf{v} = (5,7,9), \quad 2\mathbf{v} = (2,4,6)$$

$$\mathbf{u}\cdot\mathbf{v} = 4(1) + 5(2) + 6(3) = 32$$

6.3 Tensor Products

$$\mathbf{u}\otimes\mathbf{v}, \quad \mathbf{u},\mathbf{v}\in\mathbb{R}^n$$

$$\mathbf{u} = (1,2), \quad \mathbf{v} = (3,4), \quad \mathbf{u}\otimes\mathbf{v} = \begin{bmatrix} 1\cdot 3 & 1\cdot 4 \\ 2\cdot 3 & 2\cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

In this case, $\mathbb{R}^{2\times 2} \cong \mathcal{M}_{2\times 2}(\mathbb{R})$ is the set of all 2×2 real matrices.

$$V\otimes W, \quad \mathbb{R}^2\otimes\mathbb{R}^3 \cong \mathbb{R}^6$$

$$\bigotimes_{i=1}^{3} \mathbf{v}_i = \mathbf{v}_1\otimes\mathbf{v}_2\otimes\mathbf{v}_3 \textit{ rank-3 tensor}, \quad \bigotimes_{i=1}^{2}\mathbf{e}_i, \quad \mathbf{e}_1 = (1,0),\ \mathbf{e}_2 = (0,1)$$

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad R(90^\circ) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \ R\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Section 7. Gradient Differential Operator

For a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, let $f(x, y, z) = x^2 + y^2 + z^2$.

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = (2x, 2y, 2z)$$

At the point $(1, 0, 2) \in \mathbb{R}^3$:

$$\nabla f \big|_{(1,0,2)} = (2, 0, 4)$$

The gradient gives a vector field that points in the direction of steepest increase.

Section 8. Continuity and Differentiability Classes

8.1 C^0, C^1, and C^∞ Classes

$$f(x) = |x| \in C^0(\mathbb{R}), \quad C^0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$$

$$f(x) = x^2 \in C^1(\mathbb{R}), \quad f(x) = |x| \notin C^1(\mathbb{R}) \text{ as } x = 0$$

$$f(x) = e^x \in C^\infty(\mathbb{R}), \quad f(x) = \sin(x) \in C^\infty(\mathbb{R})$$

$$C^\infty(\mathbb{R}) \subset C^2(\mathbb{R}) \subset C^1(\mathbb{R}) \subset C^0(\mathbb{R})$$

Section 9. Greek Letters and Common Symbols

9.1 The Greek Alphabet

Greek letters appear throughout mathematics and physics. We will not explore specific areas of mathematics, physics, engineering, or computer science, as the same symbol can have different contexts. This is rather a reference for the reader to return to if needed.

α (alpha)	ν (nu)	A (Alpha)	N (Nu)
β (beta)	ξ (xi)	B (Beta)	Ξ (Xi)
γ (gamma)	o (omicron)	Γ (Gamma)	O (Omicron)
δ (delta)	π (pi)	Δ (Delta)	Π (Pi)
ϵ (epsilon)	ρ (rho)	E (Epsilon)	P (Rho)
ζ (zeta)	σ (sigma)	Z (Zeta)	Σ (Sigma)
η (eta)	τ (tau)	H (Eta)	T (Tau)
θ (theta)	υ (upsilon)	Θ (Theta)	Υ (Upsilon)
ι (iota)	ϕ (phi)	I (Iota)	Φ (Phi)
κ (kappa)	χ (chi)	K (Kappa)	X (Chi)
λ (lambda)	ψ (psi)	Λ (Lambda)	Ψ (Psi)
μ (mu)	ω (omega)	M (Mu)	Ω (Omega)