21-127 Homework 8

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Section J

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Complete the following problems. Fully justify each response.

NOTE: due to the Spring Break, this homework set is a bit longer than is typical. You only need to turn in those problems marked with (*).

1. (*) Let X be a finite set, and suppose there is a surjection $f : X \to Y$. Prove that $|X| \geq |Y|$.

   Proof. Assume by contradiction, that $|X| < |Y|$. If f is surjective, then $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$. Because every $y \in Y$ must be mapped to an $x \in X$, and $|X| < |Y|$, there will not be enough elements in X to map to every y, and we have arrived at a contradiction. Since f is a function and surjective, each y must have an $x \in X$, so it follows that $|X| \geq |Y|$. ■

2. (*) Let

   $$X_2 = \{n \mid 1 \leq n \leq 200, n = k^2 \ \exists k \in \mathbb{Z}\},$$

   $$X_3 = \{n \mid 1 \leq n \leq 200, n = k^3 \ \exists k \in \mathbb{Z}\},$$

   and

   $$X_4 = \{n \mid 1 \leq n \leq 200, n = k^4 \ \exists k \in \mathbb{Z}\}.$$

   Determine $|X_2 \cup X_3 \cup X_4|$.

   The set of X_2 is defined as $\{1, 4, 9, 16, 25, 36 \dots 196\}$, so it has the squares of $k = 1$ through $k = 14$. Thus $|X_2| = 14$. The set X_3 is defined as $\{1, 8, 27, 64, 125\}$, with $k = 1$ through $k = 5$. So $|X_3| = 5$.

Finally, the set X_4 is defined as $\{1, 16, 81\}$. So it has values for $k = 1$ through $k = 3$, and $|X_4| = 3$.

To calculate $|X_2 \cup X_3 \cup X_4|$, we use the inclusion-exclusion rule to get $|X_2 \cup X_3 \cup X_4| = |X_2| + |X_3| + |X_4| - |X_2 \cap X_3| - |X_2 \cap X_4| - |X_3 \cap X_4| + |X_2 \cap X_3 \cap X_4|$. Now, we can simply insert the cardinalities. So, $14 + 5 + 3 - 2 - 3 - 1 + 1 = 17$. Thus, $|X_2 \cup X_3 \cup X_4| = 14$.

3. (*) Let $X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{0,1\} \forall i\} = \{0,1\}^n$. These are sometimes called bitstrings of length n.

   (a) Show that there is a bijection between X and $\{f : [n] \to \{0,1\}\}$, the set of functions from $[n]$ to $\{0,1\}$

   Proof. The function works by taking a bitstring of length n and mapping each index to 0 or 1. So the bitstring can be written as $(a_1, a_2 \ldots a_i)$, where each a_i is mapped to $\{0,1\}$ by $f(a_i)$. Let $Y = \{f : [n] \to \{0,1\}\}$ and $g : X \to Y$, where $g(x) = f([n])$, and $x = \{0,1\}^n$. We can show that this function is injective. Let $g(a) = g(b)$ for some $a, b \in \mathbb{Z}$. Then $\{0,1\}^a = \{0,1\}^b$, and $a = b$. Thus g is injective.

   Now, we will show g is surjective. For all $f([n])$ there exists some $\{0,1\}^n$. Thus, for all $f([n])$ there exists some $x \in X$.

   Since g is injective and surjective, it must be bijective. Thus, there is a bijection between X and $\{f : [n] \to \{0,1\}\}$. ■

   (b) Show that there is a bijection between X and $\mathcal{P}([n])$.

   Proof. Let $Y = \{f : [n] \to \{0,1\}\}$ and define $g : Y \to \mathcal{P}([n])$ and $g(f) = \{n \in [n] \mid f(n) = 1\}$.

   Thus, to show g is injective, we set $g(f_1) = g(f_2)$ for some f_1, f_2. Then $f_1(n) = 1$ when $f_2(n) = 1$ and $f_1(n) = 0$ when $f_2(n) = 0$. Thus, $f_1 = f_2$ and g is injective.

   Show g surjective. Let $A \in \mathcal{P}([n])$. For some f, $g(f)$ maps every element in A to 1 or 0. So g is surjective.

   Thus, there exists a bijection between X and $\mathcal{P}([n])$. ■

   (c) Determine $|X|$.

   There are 2 options for each of the n elements of X, so $|X| = 2^n$

4. (*) Let X and Y be finite sets. Define $X^Y = \{f : Y \to X\}$, the set of functions from Y to X. Prove that $|X^Y| = |X|^{|Y|}$.

   Proof. Since X, Y are finite sets, let $|X| = n$, $|Y| = m$ for some $n, m \in \mathbb{N}$. So clearly, $|X|^{|Y|} = n^m$. Then $X = \{x_1, x_2 \dots x_n\}$ and $Y = \{y_1, y_2 \dots y_m\}$. We need to map each $x \in X$ with some $y \in Y$. Each of the elements in X can be mapped to Y in m different ways. Because there are n elements in X, $|X^Y| = n^m$ ∎

5. (*) Let $n, k \in \mathbb{N}$ with $n \geq k$. Prove, by counting in 2 ways, that $k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$.

   Proof. Suppose we wish to select from a group of n people a committee of k people with a president (so there are $k-1$ members plus one president on the committee). There are two ways to do this. First, we could select the k people for the committee from the n total people, and then select the president from the committee of k people so we have $\binom{k}{1}\binom{n}{k} = k\binom{n}{k}$.

   On the other hand, we could first select the committee with $k-1$ members from the n total people. We then choose the president from the remaining group of people, which is now $n-k+1$ in size. Using this selection technique we get $\binom{n-k+1}{1}\binom{n}{k-1} = (n-k+1)\binom{n}{k-1}$.

   Thus, since both sides of the equality count the same set, they are equal. ∎

6. (*) How many subsets of [20] contain a multiple of 4? Prove that your answer is correct.

   Proof. We have previously proven that $|\mathcal{P}([n])| = 2^n$. We can use this to attain the fact that $|\mathcal{P}([20])| = 2^{20}$, so there are 2^{20} possible subsets of $\{1, 2, 3, \dots 20\}$. We write the set of possible multiples of 4 as $\{4, 8, 12, 16, 20\} = A$. There are 2^5 possible subsets of A. We can also subtract the number of non-multiples of 4 from the set of [20], so $2^{20} - 2^{15} = 2^5 = 32$. ∎

7. (*) Let $f : X \to Y$ be a bijection. Prove that X is countably infinite if and only if Y is countably infinite.

Proof. Assume X is countably infinite. So there exists a bijection $g : \mathbb{N} \to X$. We can therefore create a composition $g \circ f : \mathbb{N} \to Y$. Since there exists a bijection from the naturals to Y, we conclude Y is countably infinite.

Assume Y is countably infinite. So there exists $f^{-1} : Y \to X$. Since Y is countably infinite, there is a function $g : \mathbb{N} \to Y$. So if we take $f^-1 \circ g : \mathbb{N} \to X$, there is clearly a bijection between the natural numbers and X, so it is countably infinite. ■

8. (*) Let X be a finite set. Show that $\mathbb{N}^X = \{f : X \to \mathbb{N}\}$ is countably infinite.

Proof. We proceed by induction on X.

Base Case: $|X| = 1$. X can be mapped to any value in $\mathbb{N}$ by specifying some function f, where $f(x) = a$, $a \in \mathbb{N}$. Since there are a countably infinite number of possible values to set a to, there are countably infinite possible functions f mapping x to a. There are therefore $|X|^1$ possiblities, which is countably infinite.

Inductive Hypothesis: $\mathbb{N}^X = \{f : X \to \mathbb{N}\}$, $\mathbb{N}^X$ is countably infinite. We will show that $\mathbb{N}^{X+1}$ is countably infinite for $N^{X+1} = f : \{A \to \mathbb{N}\}$, where A is a set with the cardinality $|X|+1$. For $\mathbb{N}^{X+1}$ to be countably infinite, there must be some $g : \mathbb{N}^{X+1} \to \mathbb{N}$, and g is bijective. $|\mathbb{N}^{X+1}|$ can be found by $\mathbb{N}^X \times \mathbb{N}$, since there are $\mathbb{N}^X$ different functions that map X to $\mathbb{N}$. Now we must show that $\mathbb{N}^X \times \mathbb{N}$ forms a bijection with $\mathbb{N}$. We know $\mathbb{N}^X$ is countably infinite by IH and $\mathbb{N}$ is countably infinite as well. The cartesian product of two countably infinite sets is known to be countably infinite as well (proved in book), so we know $\mathbb{N}^X \times \mathbb{N}$ is countably infinite, so we conclude that $\mathbb{N}^{X+1}$ is countably infinite. ■