21-127 Homework 11

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Section J

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Complete the following problems. Fully justify each response.

1. Let $(\Omega, \mathbb{P})$ be a probability space. Prove that for all $A \subseteq \Omega$, $\mathbb{P}(\Omega \backslash A) = 1 - \mathbb{P}(A)$.

   Proof. We know that $\Omega \cap A = A$ and $\Omega \backslash A$ is disjoint. Thus, $(\Omega \backslash A) \cup A = \Omega$. Hence,
   $$1 = \mathbb{P}(\Omega) = \mathbb{P}(\Omega \backslash A) + \mathbb{P}(A)$$
   Therefore, this implies that $\mathbb{P}(\Omega \backslash A) = 1 - \mathbb{P}(A)$. ■

2. Let $(\Omega, \mathbb{P})$ be a probability space, and let $A, B \subseteq \Omega$. Prove the following:

   (a) For all $\omega \in \Omega$, $i_{A \cup B}(\omega) = i_A(\omega) + i_B(\omega) - i_A(\omega) i_B(\omega)$

   Proof. We know that $i_{A \cap B} = i_A i_B$. By the inclusion exclusion principle, $i_{A \cup B} = i_A + i_B - i_{A \cap B}$. Thus, we have that $i_{A \cup B} = i_A + i_B - i_A i_B$. ■

   (b) For all $\omega \in \Omega$, $i_{A^c}(\omega) = 1 - i_A(\omega)$.

   Proof. We know that $i_\Omega = 1$. Following from the first question, we know that $i_{\Omega \backslash A} = 1 - i_A$. Since $\Omega \backslash A = A^c$ by definition, we have that $i_{A^c}(\omega) = 1 - i_A(\omega)$. ■

   (c) For all $\omega \in \Omega$, $i_{A \backslash B}(\omega) = i_A(\omega)(1 - i_B(\omega))$

Proof. We knwo that $i_{A \setminus B} = i_{A \cap B^c} = i_A i_{B^c}$, by defintion of intersection. And using the previously proven fact (2B), we have that $i_A(1 - i_B)$. ∎

3. You play the following game with a friend: You flip a fair coin 5 times. If it comes up heads, your friend gives you a dollar, and if it comes up tails, you give your friend a dollar. What is the probability that you end the game with more money than you started with?

   In order to end up with more money than you started with, you would need to win at least 3 of the 5 coin flips. This means we are looking for the probability that there are 3 or more heads landed. So, we can add up the probabilities that 3, 4, and 5 coin flips are heads. So, by defintion of probability, we know that $\mathbb{P}(A) = \frac{|A|}{|\Omega|}$. In this situation, we add $\mathbb{P}(3 \text{ heads are landed}) + \mathbb{P}(4 \text{ heads are landed}) + \mathbb{P}(5 \text{ heads are landed})$, all the winning sinarios. We know that $|\Omega| = 2^5$, as there are 5 flips and 2 possible outcomes for each. Thus, $\frac{\binom{5}{3}}{2^5} + \frac{\binom{5}{4}}{2^5} + \frac{\binom{5}{5}}{2^5} = \frac{1}{2}$.

4. You have a drawer containing 6 socks, 3 black and 3 white. Every day for 3 days, you take 2 socks at random out of the drawer.

   (a) What is the probability that you choose k matching pairs of socks, for any choice of k between 0 and 3?

   (b) What is the expected number of matching pairs of socks you pick?

5. You get a job after college as a public pollster. Each week, you select 1000 people from the country at random and ask them if they approve or disapprove of the job the president is doing. You then report the average approval in a news article.

   (a) Explain how the average approval you report can be seen as a random variable.

   (b) What distribution does your random variable follow? Why? What do the parameters represent?

   (c) Without calculating, discuss (mathematically!) how you could consider the question of accuracy in your poll.

6. Use the fact that
$$\sum_{n\in\mathbb{N}} nx^{n-1} = \frac{1}{(1-x)^2}$$
to calculate the expected value a geometrically distributed random variable.

7. Suppose you have a box containing 3 coins. Two of these coins are normal, but one is a trick coin that has both sides as heads. You pick a coin and flip it twice.

   (a) What is the probability that you get heads twice?

   (b) Suppose the coin shows heads twice. What is the probability that it is the trick coin?

8. Let $(\Omega, \mathbb{P})$ be a probability space, and let $U_1, U_2, \ldots, U_m$ be a partition of Ω. Let $A, B \subseteq \Omega$. Suppose that for all k with $1 \leq k \leq m$, we have the property that
$$\mathbb{P}(A \cap B | U_k) = \mathbb{P}(A|U_k)\,\mathbb{P}(B|U_k)\,.$$
(This property is called conditional independence with respect to U_k.) Suppose, moreover, that B is independent from U_k for all k.

   Prove that A and B are independent.