21-127 Homework 1

Christian Broms
Section J

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1. Let a, b, c be integers. Suppose that $c|a$ and $c|b$. Prove that $c|(a + b)$.

   Proof. Since $c|a$ and $c|b$ there must be $p, q \in \mathbb{Z}$ such that $a = c \cdot p$ and $b = c \cdot q$. Thus

   $$a + b = (c \cdot p) + (c \cdot q)$$
   $$a + b = c(p + q)$$

   Therefore, $c|(a + b)$. ■

2. Let a and b be irrational numbers, and let x be a rational number. Determine if the following are irrational, rational, or whether it cannot be determined. Prove that your answer is correct.

   (a) $a + b$
   Cannot be determined.

   Proof. Consider two distinct cases. In the first, let $a = -\pi + 1$ and $b = \pi$. Therefore $a + b = 1$, which is rational.

   On the other hand, let $a = 2\pi$ and $b = \pi$. Thus, $a + b = 3\pi$, which is irrational.

   Hence, there is no way to determine if the sum of two arbitrary irrationals is rational or irrational. ■

(b) $a + x$

Irrational.

Proof. Assume that $a + x = q$, where $q \in \mathbb{Q}$. We can express x as an rational number, $\frac{c}{d}$, and q as a rational number $\frac{m}{n}$ such that

$$\begin{aligned} a + x &= q \\ a + \frac{c}{d} &= \frac{m}{n} \\ a &= \frac{m}{n} - \frac{c}{d} \\ a &= \frac{m \cdot d - n \cdot c}{n \cdot d} \end{aligned}$$

Because a can be expressed as the ratio of two integers, a must be rational. Therefore, by contradiction, q must be irrational. ■

(c) ax

Cannot be determinded.

Proof. Consider two cases. In the first, let $a = \pi$ and $x = 0$. Then $a \cdot x = 0 \in \mathbb{Q}$. Thus the product can be rational.

In the other case, Assume that $ax = q$, where $q \in \mathbb{Q}$. We can express x as an rational number, $\frac{c}{d}$, and q as a rational number $\frac{m}{n}$ such that

$$\begin{aligned} a \cdot x &= q \\ a \cdot \frac{c}{d} &= \frac{m}{n} \\ a &= \frac{m}{n} \cdot \frac{d}{c} \\ a &= \frac{m \cdot d}{n \cdot c} \end{aligned}$$

Because a can be expressed as the ratio of two integers, a must be rational. Therefore, by contradiction, q must be irrational. Therefore, $a \cdot x$ can be both rational and irrational. ■

(d) ab

Cannot be determined.

Proof. Consider two distinct cases. First, let $a = \frac{1}{\pi}$ and $b = \pi$. Therefore $ab = 1$, which is rational.

On the other hand, let $a = \pi$ and $b = \pi$. Thus, $ab = \pi^2$, which is irrational.

There is no way to know if the product of two arbitrary irrationals is rational or irrational. ■

3. Let a, b, r be integers, and suppose that a leaves a remainder of r when divided by b. Prove that a also leaves a remainder of r when divided by $-b$.

   Proof. We know that when dividing a by b we get a remainder of r. By the Division Theorem, we can write $a = bq + r$ where $r, q \in \mathbb{Z}$ and $0 \leq r < |b|$. We can substitute in $-b$:

$$\begin{aligned} a &= -bq + r \\ a &= r - bq \\ a + bq &= r \end{aligned}$$

   Therefore, when dividing a by $-b$ we get a remainder of r. ■

4. Let $a, b \in R$ and let $p(x) = x^2 + ax + b$. The value $\Delta = a^2 - 4b$ is called the discriminant of p. Prove that p has no real roots if $\Delta < 0$, one real root if $\Delta = 0$, and two real roots if $\Delta > 0$.

   Proof. In the first case, $\Delta < 0$ so $a^2 - 4b < 0$. We can substitute in Δ in the quadratic formula to find roots:

$$\begin{aligned} r_i &= \frac{-a \pm \sqrt{\Delta}}{2} \\ r_i &= \frac{-a \pm \sqrt{|a^2 - 4b|} \cdot i}{2} \end{aligned}$$

Thus there are no real roots when $\Delta < 0$. In the second case, $\Delta = 0$ so $a^2 - 4b = 0$. Substituting again:

$$r_1 = \frac{-a \pm \sqrt{\Delta}}{2}$$
$$r_1 = \frac{-a \pm \sqrt{0}}{2}$$
$$r_1 = \frac{-a}{2}$$

This yields one real root when $\Delta = 0$. Finally, consider $\Delta > 0$ so $a^2 - 4b > 0$. Substituting again:

$$r_{1,2} = \frac{-a \pm \sqrt{\Delta}}{2}$$
$$r_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$
$$r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$
$$r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Therefore, there are two real roots when $\Delta > 0$. ■

5. Let $p(x)$ be a polynomial with coefficients in $\mathbb{R}$. Suppose that α is a root of $p(x)$. Prove that α is also a root of $(x - a)p(x)$ for all $a \in \mathbb{R}$.

6. Identify which of the following are propositions, and discuss what might be required to prove them if they are.

   (a) This is the tallest sheep in the world.
   A proposition. In order to prove, we would need to have the heights of all sheep in the world and establish that the sheep in question is the tallest of all of them. This is a very hard proposition to prove.

   (b) My dog Rufus.
   Not a proposition.

(c) The square of every even integer is also an even integer.
A proposition. To prove, make an arbitrary even number and square it. Note that when distributed and reduced, the resulting integer can be expressed in the origional same form. Example: $a = 2b$ is an even integer such that $a^2 = (2b)^2$. Therefore, $a^2 = 4b^2$ and $a^2 = 2(2b^2)$, which is in the form $2b$.

(d) A kitty is orange.
A proposition. To prove, we would have to establish that there is at least one kitty that exists in the world that is orange in color.

(e) This kitty is orange.
A propostion. Prove it in the same way as above, except for a single predefined kitty.

(f) Go to the store.
Not a proposition.