21-127 Homework 11

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Section J

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Complete the following problems. Fully justify each response.

1. Let $(X, 0, 1, +, \cdot)$ be a field, where X is a finite set. Prove that there is no ordering $\leq$ on X under which X is an ordered field.

2. Let $(X, 0, 1, +, \cdot, \leq)$ be an ordered field. Prove each of the following basic ordered field properties, from axioms.

   (a) For all $x \in X$, $x^2 > 0$.

   (b) For all $w, x, y, z \in X$, if $w \leq x$ and $y \leq z$, then $w + y \leq x + z$.

   (c) For all $x, y, z \in X$, if $x \geq 0$ and $y \leq z$, then $xy \leq xz$.

   (d) For all $x, y, z \in X$, if $x \leq 0$ and $y \leq z$, then $xy \geq xz$.

3. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$. Prove that $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$.

   Proof. Recall the triangle inequality: $\|a+b\| = \|a\|+\|b\|$. Let $a = \vec{x} - \vec{y}$ and let $b = \vec{y} - \vec{z}$. Then we have

   $$\|\vec{x} - \vec{y} + \vec{y} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$

   $$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$

   and we are done.

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4. Let $x_n = \frac{n+2}{n+1}$. Prove that x_n converges to 1.

Proof. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon} - 1$. Let $n \geq N$. Then $|\frac{n+2}{n+1} - 1| = \frac{1}{n+1} < \frac{1}{N+1} < \frac{1}{\frac{1}{\epsilon}-1+1} = \epsilon$. Therefore, by definition, $(\frac{n+2}{n+1}) \to 1$. ∎

5. Let (x_n) and (y_n) be sequences of real numbers, with $(x_n) \to a$ and $(y_n) \to b$. Let $z_n = x_n y_n$ for all $n \in \mathbb{N}$. Prove that $(z_n) \to ab$.

Proof. Let $\epsilon > 0$. Since every convergent sequence of real numbers must be bounded, there exists some $M > 0$, $N_1 \in \mathbb{N}$, where

$$\forall n \geq N_1, |x_n| < M$$

In addition, since (x_n) and (y_n) both converge, then there exists $N_2, N_3 \in \mathbb{N}$ such that

$$\forall n \geq N_2, |x_n - a| < \frac{\epsilon}{2|b|}$$

$$\forall n \geq N_3, |y_n - b| < \frac{\epsilon}{2M}$$

So $\forall n \geq N$, $N = \max\{N_1, N_2, N_3\}$.

Now,

$$\begin{aligned} |(x_n - y_n) - ab| &= |x_n y_n - x_n b + x_n b - ab| \\ &\leq |x_n y_n - x_n b| + |x_n b - ab| \\ &\leq |x_n(y_n - b)| + |b(x_n - a)| \end{aligned}$$

Substituting, we have

$$M \cdot \frac{\epsilon}{|2M|} + |b| \cdot \frac{\epsilon}{|2b|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, $|x_n y_n - ab| < \epsilon$. Thus, we conclude $x_n y_n \to ab$. ∎

6. Prove that if (x_n) is a monotonically decreasing sequence, having a lower bound, then (x_n) converges.