21-127 Bonus

Christian Broms

Exercise 1. Prove that if A is a cut, and $r \notin A$, then $p < r$ for every $p \in A$.

Proof. By the first property of a cut, we have that for $x, y \in \mathbb{Q}$, if $y \leq x$, then $y \in A$, where $A \subseteq \mathbb{Q}$. If $r \notin A$, $p \in A$, then $r \nleq p$ because this would contradict the defintion of a cut. Thus, it must be that $r > p$, and therefore we have that $p < r$ for every $p \in A$. $\square$

Exercise 2. Prove that for any $q \in \mathbb{Q}$, there exists a cut A_q for which q is a least upper bound.

Proof. Assume for some $q \in \mathbb{Q}$, $A_q \subseteq \mathbb{Q}$. By the second property of a cut, A_q has an upper bound at q. Thus, we have that $q \geq a \ \forall a \in A_q$. Additionally, if $q' \geq a \ \forall a \in A_q$, then $q' \geq q$. Therefore the properties of a least upper bound are fufilled and we can conclude that for any $q \in \mathbb{Q}$, there exists a cut A_q for which q is a least upper bound. $\square$

Exercise 3. Prove that if $p, q \in \mathbb{Q}$, with $p < q$, then $A_p \subset A_q$.

Proof. Let $x \in A_p$. By Exercise 2 we know that $x < p$. Since $p < q$, then $x < q$ due to the transitive property of $\mathbb{Q}$, an ordered field. If $x < q$, and q is the least upper bound of A_q, then $x \in A_q$ by definition of least upper bound. Thus, we have that $A_p \subset A_q$. $\square$

Exercise 4. Prove that $\{x \in \mathbb{Q} \mid x^2 \leq 2 \text{ or } x < 0\}$ is a cut.

Proof. Set $A = \{x \in \mathbb{Q} \mid x^2 \leq 2 \text{ or } x < 0\}$. We wish to show that A fufills the properties of a cut. There are two cases, either $x < 0$ or $x^2 \leq 2$.

Case 1: $x < 0$.

1. Suppose $x, y \in \mathbb{Q}$, $x \in A$, and $y \leq x$. Since $x \in A$, $x < 0$, and since $y \leq x$, it must be that $y < 0$ as well, by transitivity. So, $y \in A$.
2. The upper bound of $x < 0$ is 0, since $\forall x \in A$, where $x < 0$, $x \ngeq 0$. Thus, 0 is an upper bound.
3. We know that $A \subseteq \mathbb{Q}$ and $\mathbb{Q}$ is infinite. So, given that $x > 0$, it follows that $\forall x \in A$, $\exists y \in A$ such that $x < y$.

Case 2: $x^2 \leq 2$

1. Suppose $x, y \in \mathbb{Q}$ and $y \geq 0, y < x$. We know that $x - y > 0$, as proven previously. Additionally, we have that $x + y > y$ since $x > 0$, and since $y \geq 0$ and $x + y > y$, then $x + y > 0$. So, we have $x + y > 0$ and $x - y > 0$, thus $(x - y)(x + y) \geq 0$, from the ordered field axioms. Simplifying, we get $x^2 - y^2 \geq 0$ and $x^2 \geq y^2$. Since $x^2 \leq 2$ and $y^2 \leq x^2$, we have that $y^2 \leq 2$, so we conclude that $y \in A$.

2. The upper bound of $x^2 \leq 2$ is $\sqrt{2}$, which we can derive from the fact that $0 < 1$, so $0 < \sqrt{2}$. To prove the existence of a rational upper bound, we get that $2 \geq \sqrt{2}$, so $\forall a \in A$, $a \leq \sqrt{2}$, so 2 is also an upper bound. Given that $2 \in \mathbb{Q}$, we can conclude that A has a rational upper bound.

3. Let $x \in A$. Since $A \subseteq \mathbb{Q}$, we know that x is rational. So, we know that $\forall x \in A$, $\exists y \in A$ such that $x < y$ since $\sqrt{2}$ is an upper bound that is not rational, so all elements must be less than $\sqrt{2}$ and are infinitely decreasing.

$\square$

Exercise 5. Prove that the ordering defined above is a total order; that is, it is a partial order, and any two elements of $\mathcal{C}$ are comparable.

Exercise 6. Prove the following are true about this definition of addition:

1. Closure: For all $A, B \in \mathcal{C}$, $A + B$ is a cut (so that when we add cuts, we stay within the universe of cuts).

2. Associativity: $A + (B + C) = (A + B) + C$ for all $A, B, C \in \mathcal{C}$

3. Identity: $A + A_0 = A$ for all $A \in \mathcal{C}$

4. Existence of inverses: For all $A \in \mathcal{C}$ there exists $B \in \mathcal{C}$ such that $A + B = A_0$. (Hint: set $B = \{p \in \mathbb{Q} \mid \exists r \notin A \text{ such that } p < -r\}$. Then prove that B is a cut, and $A + B = A_0$.)

5. Commutativity: For all $A, B \in \mathcal{C}$, $A + B = B + A$.

6. Order arithmetic: For all $A, B, C \in \mathcal{C}$, if $A \leq B$, then $A + C \leq B + C$.

Proof. 1. Closure: We will prove that $A + B$ fufills the properties of a cut.

(a) We will prove if $x, y \in \mathbb{Q}$, $x \in A + B$, and $y \leq x$, then $y \in A + B$. There are two cases, either $y < x$ or $y = x$.

Case 1: $y < x$. Let $y = a + b$, such that $a \in A$ with $a < p$, and $b \in B$, with $b < q$. We have the property that $a + b < p + q$ since $a < p$ and $b < q$, as proven in a previous homework. Since A and B are cuts, and A contains all elements less than p and B has all elements less than q, then $a + b$ represents all elements less than $p + q$. So, we have that $y \in A + B$.

Case 2: $y = x$. Let $y = p + q$. Clearly, it follows that $y \in A + B$.

Thus, in both cases, we get $y \in A + B$.

(b) Since A and B are cuts, they both have upper bounds. Let the upper bounds be denoted a and b, respectively. Let $p + q \in A + B$ such that $p \in A$ and $q \in B$. So, $p \leq a$ and $q \leq b$, so we have the property that $p + q \leq a + b$, as proven previously. Thus, by defintion, $a + b$ is an upper bound.

(c) Let $x = a + b$, with $a \in A$ and $b \in B$. Then $\forall a \in A$, $\exists p \in A$ such that $a < p$. Similarly, $\forall b \in B$, $\exists q \in A$ such that $b < q$. So, $a + b < p + q$, as previously proven. Thus, there exists some $y = p + q$ such that $\forall x \in A + B$, $\exists y \in A + B$ with $x < y$.

Thus, since $A + B$ fufills all three properties, it must be a cut.

2. Associativity: Let $A + (B + C) = \{a + (b + c) \mid a \in A, b \in B, c \in C\}$ and $(A + B) + C = \{(a + b) + c \mid a \in A, b \in B, c \in C\}$. We have that $a+(b+c) = (a+b)+c$ by the axiom of associativity of addition. Suppose $p \in A+(B+C)$ and $q \in (A+B)+C$, so $p = a+(b+c)$ and $q = (a+b)+c$. Thus, $p = q$. So, $p \in (A + B) + C$ and $q \in A + (B + C)$. Since p and q are elements of both sets, it follows that $A + (B + C) \subseteq (A + B) + C$ and $(A + B) + C \subseteq A + (B + C)$, so $(A + B) + C = A + (B + C)$.

3. Identity: Let $A + A_0 = \{p + q \mid p \in A, q \in A_0\}$. Suppose $q = 0$. Then $p+q = p+0 = p$ by the first field axiom. Since $p+q = p \; \forall p$ when $q = 0$, then $\forall p \in A$, $p \in A + A_0$. So, $A \subseteq A + A_0$. To prove that $A + A_0 \subseteq A$, there are two cases, $q = 0$ and $q < 0$.

   Case 1: Let $q < 0$. Then , $p + q < p$. So, $p + q \in A$.

   Case 2: Let $q = 0$.Then $p+q \in A+A_0$ and $p+q = p+0 = p$, so $p+q \in A$.

   Thus, $A + A_0 \subseteq A$. Since we have shown both sides of containment, we conclude that $A = A + A_0$.

4. Existence of Inverses: (prove this case)

5. Communitivity: Let $A + B = \{a + b \mid a \in A, b \in B\}$ and $B + A = \{b + a \mid a \in A, b \in B\}$. So, we know that $\forall a + b \in A + B, \exists a + b \in B + A$, since $a+b = b+a$ by communitivity of addition. So, $A+B \subseteq B+A$. Using the same logic, we get $B+A \subseteq A+B$, so we conclude that $A+B = B+A$.

6. Order Arithmatic: (prove this case).

□

Exercise 7. Let $p \in \mathbb{Q}$. Prove that $-A_p = A_{-p}$.

Exercise 8. Prove that for $A, B > A_0$, the definition of $A \cdot B$ above yields a cut; that is, $A \cdot B \in \mathcal{C}$.

Exercise 9. Prove that if $p, q \in \mathbb{Q}$, and $p, q > 0$, then $A_p \cdot A_q = A_{pq}$.

Exercise 10. For all $A, B \in \mathcal{C}$, prove that $-(A \cdot B) = (-A) \cdot B = A \cdot (-B)$.

(Note: You will need to consider cases, depending on the sign of A and B.)

Exercise 11. Prove the following are true about this definition of multiplication:

1. Identity: $A \cdot A_1 = A$ for all $A \in \mathcal{C}$.

2. Inverse: For all $A \in \mathcal{C}$ with $A \neq A_0$, there exists $B \in \mathcal{C}$ such that $A \cdot B = A_1$. (Hint: set $B = \{p \in \mathbb{Q} \mid \exists r \notin A, p < \frac{1}{r}\}$ when $A > 0$.)

3. Commutativity: $A \cdot B = B \cdot A$ for all $A, B \in \mathcal{C}$.

4. Distributivity: $A \cdot (B + C) = A \cdot B + A \cdot C$ for all $A, B, C \in \mathcal{C}$.

5. Order arithmetic: For all $A, B \in \mathcal{C}$, if $A, B \geq A_0$ then $A \cdot B \geq A_0$.

Exercise 12. Let $\mathcal{A} \subseteq \mathcal{C}$ be a set of cuts. Prove that there exists $S \in \mathcal{C}$ that satisfies the definition of a supremum.