# 21-127 Homework 5

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Section J

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Complete the following problems. Fully justify each response.

1. Let $X, Y, Z$ be sets, with $X, Y \subseteq Z$. Prove that

$$[(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y) = Z\backslash Y.$$

*Proof.* ($\subseteq$) If we let $z \in [(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y)$ then $z \in [(Z\backslash X) \cap (Z\backslash Y)]$ or $z \in (X\backslash Y)$ Then, ($z \in Z$ and $z \notin X$ and $z \notin Y$) or ($z \in X$ and $z \notin Y$). In the second part, when ($z \in X$ and $z \notin Y$), since $X \subseteq Z$, and $z \in X$, we know that $z \in Z$. In the first part, when ($z \in Z$ and $z \notin X$ and $z \notin Y$), we know $z \in Z$ and $z \notin Y$. Hence, we know that $z \in Z$ and $z \notin Y$ in both cases. Therefore, $z \in Z\backslash Y$ by definition, since $z \in Z$ and $z \notin Y$. Thus, we conclude, $[(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y) \subseteq Z\backslash Y$

($\supseteq$) If we let $z \in Z\backslash Y$, then $z \in Z$ and $z \notin Y$. Since $X \subseteq Z$, and $z \in Z$, then $z \in X$ or $z \notin X$. In the first case, when $z \in X$, we can say $z \in X\backslash Y$, because $z \in X$ and $z \notin Y$ In the second case, when $z \notin X$, and we know $z \notin Y$, we can say $z \notin X \cup Y$. Since $z \in Z$, then $z \in Z\backslash(X \cup Y)$. So, combining the two cases, we have $z \in Z\backslash(X \cup Y)$ or $z \in X\backslash Y$, so $z \in [Z\backslash(X \cup Y)] \cup (X\backslash Y)$. By DeMorgan's laws, we can expand this to $z \in [(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y)$. We have shown that $(Z\backslash Y) \subseteq [(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y)$.

Since we have shown two sides of containment, we can conclude that $[(Z\backslash X) \cap (Z\backslash Y)] \cup (X\backslash Y) = Z\backslash Y$. ■

2. Let $X$ be a set. Prove that $X \times \emptyset = \emptyset$.

   *Proof.* By definition, the Cartesian Product is defined as $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$. Thus, when we consider $Y = \emptyset$, it is impossible to create an ordered pair $(x, y)$ with $x \in X, y \in Y$, because by definition of the empty set, there are no elements in $Y$. Thus, we cannot create such an ordered pair and $X \times \emptyset = \emptyset$. ■

3. Let $X, Y, Z$ be sets. Is it true that $X \times (Y \times Z) = (X \times Y) \times Z$? Explain your answer with a proof or a counterexample.

   False. Consider the following counterexample: Let $X = \{1\}, Y = \{2\}, Z = \{3\}$. We calculate $X \times (Y \times Z)$ as $X \times \{(2, 3)\} = \{(1, (2, 3))\}$. Now, calculating $(X \times Y) \times Z$ as $\{(1, 2)\} \times Z = \{(3, (1, 2))\}$. Thus, $\{(1, (2, 3))\} \neq \{(3, (1, 2))\}$ and therefore $X \times (Y \times Z) \neq (X \times Y) \times Z$.

4. For each of the following subsets $G$ of $X \times Y$, determine if the subset represents the graph of a function from $X \to Y$. If so, specify the function.

   (a) $X = \mathbb{R}$, $Y = \mathbb{R}$, $G = \{(x, x + 1) \mid x \in \mathbb{R}\}$.
   Yes, $f : X \to Y$, defined by $f(x) = x + 1$

   (b) $X = \mathbb{R}$, $y = \mathbb{R}$, $G = \{(x^2, x) \mid x \in \mathbb{R}\}$.
   No. $f : X \to Y$, defined by $f(x) = \sqrt{x}$ violates the condidion of existence. Changing to domain to be only positive reals would fix.

   (c) $X = \mathbb{R}^+$, $y = \mathbb{R}^+$, $G = \{(x^2, x) \mid x \in \mathbb{R}^+\}$.
   Yes, $f : X \to Y$, defined by $f(x) = \sqrt{x}$

   (d) $X = \mathbb{Q}$, $y = \mathbb{Q}$, $G = \{(x, y \mid x, y \in \mathbb{Q}$ and $xy = 1\}$.
   No. $f : X \to Y$, defined by $f(x) = \frac{1}{x}$ is undefined when $x = 0$. You could remove 0 from the domain to fix this issue.

5. Which of the following function specifications are well-defined? If one is not well-defined, determine a modification to the specification that would rectify the issue.

   (a) $g : \mathbb{Q} \to \mathbb{Q}$ defined by $g(x)(x + 1) = 2$.

Does not exist at -1, and thus violates the condition of totality. We can fix by redefining:

$$g(x) = \begin{cases} \frac{2}{x+1} & x \neq -1 \\ 2 & x = -1 \end{cases}$$

(b) $f : \mathbb{Q} \to \mathbb{R}$ defined by $f(x)(x + \pi) = 1$.
Well Defined.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \sqrt{x}$.
No, violates the condition of existence. We can fix by redefining the domain: $h : \mathbb{R}^+ \to \mathbb{R}^+$

(d) $\ell : \mathbb{C} \to \mathbb{C}$ defined by $\ell(x) = \sqrt{x}$.
Well-defined.

6. Let $f, g, h, \ell : \mathbb{R} \to \mathbb{R}$ be functions with the following specifications:

$$f(x) = x + 2; \quad g(x) = x^2; \quad h(x) = \frac{1}{x^2 + 1}; \quad \ell(x) = -x.$$

Write a specification, via a single equation, for each of the following:

(a) $f \circ g = x^2 + 2$.

(b) $g \circ f = (x + 2)^2$.

(c) $f \circ (g \circ (h \circ \ell)) = (\frac{1}{-x^2+1})^2 + 2$.

(d) $(f \circ g) \circ (h \circ \ell) = (\frac{1}{-x^2+1})^2 + 2$.