21-127 Homework 11

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Section J

Wednesday 18^{th} April, 2018

Complete the following problems. Fully justify each response.

1. Let $(X, 0, 1, +, \cdot)$ be a field, where X is a finite set. Prove that there is no ordering $\leq$ on X under which X is an ordered field.

   Proof. If X is finite, then $|X| = b$. We know that $0 \leq 1$, and by axiom F1 that $0 \neq 1 \Rightarrow 0 < 1$. By extension, it holds that $a < a + 1$ by OF2 for some $a \in X$. However, since X is finite, that means that there exists one element that cannot be compared with anything; the maximum of X. Consider the field ordering

   $$0 < 1 < 1 + 1 < 1 + 1 + 1 < 1 + 1 + 1 + \dots$$

   Since X is finite, the final element in the ordering is the maximum. Let this max value be b. Thus, $b < b + 1$, but notice that $b + 1 \notin X$ since the set is finite. So to be ordered, it must hold that $b \leq a$ for some $a \in X$. But we have shown that $b \neq a$ by F1. And $b \not< a$ because by definition b is the maximum value in X. Thus, X cannot be an ordered field since $b + 1 \notin X$ and b cannot be compared to any element. ■

2. Let $(X, 0, 1, +, \cdot, \leq)$ be an ordered field. Prove each of the following basic ordered field properties, from axioms.

   (a) For all $x \in X$, $x^2 > 0$.

Proof. There are two cases, $x > 0$ or $x < 0$.
Case 1: $x > 0$. So,

$$x > 0$$

$$x \cdot x > x \cdot 0$$

$$x^2 > 0$$

Case 2: $x < 0$. So,

$$-x > 0$$

$$-x \cdot -x > 0 \cdot -x$$

$$x^2 > 0$$

and we are done. ■

(b) For all $w, x, y, z \in X$, if $w \leq x$ and $y \leq z$, then $w + y \leq x + z$.

Proof. If $w \leq x$ and $y \leq z$, then we have $z - y \geq 0$ and $x - w \geq 0$. Rearranging, we get $w - x \leq 0$ and $z - y \geq 0$. Thus, it follows that

$$z - y \geq w - x$$

$$z \geq w - x + y$$

$$z + x \geq w + y$$

Thus, $w + y \leq x + z$. and we are done. ■

(c) For all $x, y, z \in X$, if $x \geq 0$ and $y \leq z$, then $xy \leq xz$.

Proof. Since $x \geq 0$ and $y \leq z$ then $y - z \leq 0$. So,

$$x \cdot (y - z) \leq 0$$

$$xy - xz \leq 0$$

$$xy \leq xz$$

and we are done. ■

(d) For all $x, y, z \in X$, if $x \leq 0$ and $y \leq z$, then $xy \geq xz$.

Proof. Since $x \leq 0$ and $y \leq z$, then $y - z \leq 0$. Then since $x \leq 0$, we have two negative numbers and thus

$$x \cdot (y - z) \geq 0 \cdot x$$

$$xy - xz \geq 0$$

$$xy \geq xz$$

and we are done. ■

3. Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^n$. Prove that $\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$.

   Proof. Recall the triangle inequality: $\|a+b\| = \|a\|+\|b\|$. Let $a = \vec{x}-\vec{y}$ and let $b = \vec{y} - \vec{z}$. Then we have

$$\|\vec{x} - \vec{y} + \vec{y} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$

$$\|\vec{x} - \vec{z}\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\|$$

   and we are done.

   ■

4. Let $x_n = \frac{n+2}{n+1}$. Prove that x_n converges to 1.

   Proof. Let $\epsilon > 0$ and let $N \in \mathbb{N}$ with $N > \frac{1}{\epsilon} - 1$. Let $n \geq N$. Then $|\frac{n+2}{n+1} - 1| = \frac{1}{n+1} < \frac{1}{N+1} < \frac{1}{\frac{1}{\epsilon}-1+1} = \epsilon$. Therefore, by definition, $(\frac{n+2}{n+1}) \to 1$. ■

5. Let (x_n) and (y_n) be sequences of real numbers, with $(x_n) \to a$ and $(y_n) \to b$. Let $z_n = x_n y_n$ for all $n \in \mathbb{N}$. Prove that $(z_n) \to ab$.

   Proof. Let $\epsilon > 0$. Since every convergent sequence of real numbers must be bounded, there exists some $M > 0$, $N_1 \in \mathbb{N}$, where

$$\forall n \geq N_1, |x_n| < M$$

   In addition, since (x_n) and (y_n) both converge, then there exists $N_2, N_3 \in \mathbb{N}$ such that

$$\forall n \geq N_2, |x_n - a| < \frac{\epsilon}{2|b|}$$

$$\forall n \geq N_3, |y_n - b| < \frac{\epsilon}{2M}$$

So $\forall n \geq N$, $N = \max\{N_1, N_2, N_3\}$.

Now,

$$\begin{aligned}
|(x_n - y_n) - ab| &= |x_n y_n - x_n b + x_n b - ab| \\
&\leq |x_n y_n - x_n b| + |x_n b - ab| \\
&\leq |x_n(y_n - b)| + |b(x_n - a)|
\end{aligned}$$

Substituting, we have

$$M \cdot \frac{\epsilon}{|2M|} + |b| \cdot \frac{\epsilon}{|2b|} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, $|x_n y_n - ab| < \epsilon$. Thus, we conclude $x_n y_n \to ab$. ■

6. Prove that if (x_n) is a monotonically decreasing sequence, having a lower bound, then (x_n) converges.

Proof. First, we must show that x_n has a greatest lower bound; that it has an infimum.

Consider the set bounded between two values, given by $X = \{x \in \mathbb{R} \mid m < x < n\}$, where $m, n \in \mathbb{R}$. So, we have that $m \leq x \; \forall x \in X$. Hence, m is by defintion a lower bound. Now we show that for some b, if $m < b$ then $\exists a \in X$ so $a < b$, to prove m is the greatest lower bound. There are two cases, either $m < b < n$ or $n \leq b$. In both cases, we need to choose values $a \in X$ such that $a < b$.

In the first case, we set $a = \frac{b+m}{2}$. In the second case, set $a = \frac{n+m}{2}$. Notice, in both cases, $a < b$ holds, so the properties of infimum are fufilled. Therefore, we conclude that the infimum of X is m.

Now, we know that any arbitrary bounded set has an greatest lower bound.

Next, we show that x_n converges.

Since $\mathbb{R}$ is complete, $\{x_n \mid n \in \mathbb{N}\}$ has a greatest lower bound, say a. Fix $\epsilon > 0$. Note that by defintion, $a + \epsilon$ is not a lower bound for

x_n since a is the greatest lower bound. Thus, there must be some x_n having $x_N < a + \epsilon$. For any $n \geq N$, we therefore have

$$a - \epsilon < a \leq x_n \leq x_N < a + \epsilon$$

Thus, $a - \epsilon < x_n < a + \epsilon$, so $|x_n - a| < \epsilon$, so $(x_n) \to a$. Therefore, we conclude that the monotonically decreasing bounded sequence x_n converges. ■