21-127 Homework 10

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Section J

Tuesday 10^{th} April, 2018

Complete the following problems. Fully justify each response.

1. Let X be a set, and let $\sim_1$ and $\sim_2$ be two equivalence relations on X. Define a relation $\sim$ by $x \sim y$ if and only if $x \sim_1 y \wedge x \sim_2 y$. Prove that this is an equivalence relation. Describe the equivalence classes $[x]_\sim$ in terms of $[x]_{\sim_1}$ and $[x]_{\sim_2}$.

2. Let P be a poset with element set $\mathbb{N}$, and order $\preceq$, where $a \preceq b$ if and only if $a|b$. Formally prove that this is a poset.

3. Let P be a poset with partial order $\leq$. We define an element m of P to be *minimal* if there does not exist any $x \in P$ with $x < m$.

   (a) How is a minimal element different from a minimum element (as defined in class)?

   (b) Give an example of a poset that has at least one minimal element, but no minimum element.

   (c) Prove that if a poset has a minimum element, then it has only one minimal element.

   (Note: This entire problem could be repeated verbatim for a maximal element as well)

4. Let X be a set, and let P be the poset whose elements are $\mathcal{P}(X)$, and $A \leq B$ if and only if $A \subseteq B$.

Let $\{A_1, A_2, \ldots, A_n\}$ be elements of P. Prove that $\bigwedge_{i=1}^{n} A_i = \bigcap_{i=1}^{n} A_i$, and $\bigvee_{i=1}^{n} A_i = \bigcup_{i=1}^{n} A_i$.

5. Let $(X, \preceq)$ be a lattice, and let $x, y \in X$. Prove the following:

   (a) $x \vee y = y \vee x$

   (b) $x \wedge y = y \wedge x$

   (c) $x \vee (x \wedge y) = x$

   (d) $x \wedge (x \vee y) = x$

6. Give an example of a lattice that is not distributive.