21-127 Homework 8

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Section J

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Complete the following problems. Fully justify each response.

NOTE: due to the Spring Break, this homework set is a bit longer than is typical. You only need to turn in those problems marked with (*).

1. (*) Let X be a finite set, and suppose there is a surjection $f : X \to Y$. Prove that $|X| \geq |Y|$.

   Proof. If f is surjective, then $\forall y \in Y$, $\exists x \in X$ such that $f(x) = y$. Since f is a function, each x contributes at most one to y. Let $|B| = n$. There must be at least n xs, so $|X| \geq n = |Y|$. ∎

2. (*) Let

$$X_2 = \{n \mid 1 \leq n \leq 200, n = k^2 \ \exists k \in \mathbb{Z}\},$$

$$X_3 = \{n \mid 1 \leq n \leq 200, n = k^3 \ \exists k \in \mathbb{Z}\},$$

   and

$$X_4 = \{n \mid 1 \leq n \leq 200, n = k^4 \ \exists k \in \mathbb{Z}\}.$$

   Determine $|X_2 \cup X_3 \cup X_4|$.

3. (*) Let $X = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \{0,1\} \forall i\} = \{0,1\}^n$. These are sometimes called bitstrings of length n.

   (a) Show that there is a bijection between X and $\{f : [n] \to \{0,1\}\}$, the set of functions from $[n]$ to $\{0,1\}$

   (b) Show that there is a bijection between X and $\mathcal{P}([n])$.

(c) Determine $|X|$.

4. (*) Let X and Y be finite sets. Define $X^Y = \{f : Y \to X\}$, the set of functions from Y to X. Prove that $|X^Y| = |X|^{|Y|}$.

5. (*) Let $n, k \in \mathbb{N}$ with $n \geq k$. Prove, by counting in 2 ways, that $k\binom{n}{k} = (n-k+1)\binom{n}{k-1}$.

   Proof. Suppose we wish to select from a group of n people a committee of k people with a president (so there are $k-1$ members plus one president on the committee). There are two ways to do this. First, we could select the k people for the committee from the n total people, and then select the president from the committee of k people so we have $\binom{k}{1}\binom{n}{k} = k\binom{n}{k}$.

   On the other hand, we could first select the committee with $k-1$ members from the n total people. We then choose the president from the remaining group of people, which is now $n-k+1$ in size. Using this selection technique we get $\binom{n-k+1}{1}\binom{n}{k-1} = (n-k+1)\binom{n}{k-1}$.

   Thus, since both sides of the equality count the same set, they are equal. ■

6. (*) How many subsets of [20] contain a multiple of 4? Prove that your answer is correct.

   Proof. There are 2^{20} possible subsets of $\{1, 2, 3, \ldots 20\}$. There are 2^5 possible subsets of $\{4, 8, 12, 16, 20\}$. Thus, there are $2^5 = 32$ subsets of [20] that contain a multiple of 4. ■

7. (*) Let $f : X \to Y$ be a bijection. Prove that X is countably infinite if and only if Y is countably infinite.

8. (*) Let X be a finite set. Show that $\mathbb{N}^X = \{f : X \to \mathbb{N}\}$ is countably infinite.