Volume 3: List of Multi-run Quadratizations

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(Dated: 23rd September 2019)

DECOMPOSITION OF A MONOMIAL

$$b_1b_2b_3\ldots b_k = \min\left(b_1b_2\ldots b_{k_1}, b_{k_1+1}b_{k_1+2}\ldots b_{k_2}, b_{k_2+1}b_{k_2+2}\ldots b_{k_3}, ..., b_{k_n+1}b_{k_n+2}...b_k\right) \tag{1}$$

$$b_1b_2b_3\ldots b_k = \min\left(b_1, b_2, b_3, \ldots, b_k\right) \qquad \text{(Example of Eq. 1: Linearization of a degree-k monomial).} \tag{2}$$

$$b_1b_2b_3b_4 = \min\left(b_1b_2, b_3b_4\right) \qquad \text{(Example of Eq. 1: Quadratization of a degree-4 monomial).} \tag{3}$$

$$b_1b_2b_3b_4b_5b_6b_7b_8 : \tag{4}$$
$$\longrightarrow\ 3b_a + b_1b_2 + b_1b_3 + b_1b_4 + b_2b_3 + b_2b_4 + b_3b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \tag{5}$$
$$\longrightarrow\ 3b_a + b_5b_6 + b_5b_7 + b_5b_8 + b_6b_7 + b_6b_8 + b_7b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \tag{6}$$

DECOMPOSITION OF BINOMIALS OF DEGREE-k TERMS

$$b_1b_2b_3b_4 + b_3b_4b_5b_6 = \min\left(b_2b_3 + b_3b_6, b_1b_4 + b_4b_5, b_1b_2 + b_5b_6 - b_3 - b_4 + 2\right) \qquad (k,n) = (4,6). \tag{7}$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 : \qquad (k,n) = (4,7). \tag{8}$$
$$\longrightarrow\ b_2b_3 + b_5b_6 + b_a(1 - b_5 - b_6 + b_7) \tag{9}$$
$$\longrightarrow\ b_1b_4 + b_4 + b_a \tag{10}$$
$$\longrightarrow\ b_5b_6 + b_1 + b_a(1 - b_5 - b_6 + b_7) \tag{11}$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6b_7 : \qquad (k,n) = (5,7). \tag{12}$$
$$\longrightarrow\ b_2b_5 + b_5b_6 + b_5b_7 + b_6b_7 + b_a(b_5 + b_6 + b_7 - 1) - b_5 - b_6 - b_7 + 1 \tag{13}$$
$$\longrightarrow\ b_1b_3 + b_3b_7 + b_a(1 + b_5 - b_7) - b_5 + 1 \tag{14}$$
$$\longrightarrow\ b_1b_4 + b_4b_6 - b_5b_6 + b_5b_a - b_5 + b_6 + 1 \tag{15}$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \qquad (k,n) = (6,7). \tag{16}$$
$$\longrightarrow\ 2b_3b_6 \tag{17}$$
$$\longrightarrow\ 2b_4b_5 - b_5b_6 + b_5 \tag{18}$$
$$\longrightarrow\ b_1b_4 - b_2b_5 + b_2b_6 + b_2b_7 + b_5b_7 - b_6b_7 - b_5 - b_6 + 2 \tag{19}$$
$$\longrightarrow\ b_1b_2 - b_1b_5 + b_1b_7 + b_2b_3 + b_3b_6 - b_3b_7 - b_4b_5 - b_5b_6 - b_3 + b_5 + 2 \tag{20}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 : \qquad\qquad (k,n) = (6,7). \tag{21}$$
$$\longrightarrow b_5 b_6 + b_5 b_7 - b_5 b_8 + b_6 b_8 - b_7 b_8 + b_8 \tag{22}$$
$$\longrightarrow b_1 b_4 + b_a (b_4 - b_7) + b_7 \tag{23}$$
$$\longrightarrow b_2 b_3 + b_2 b_7 - b_5 b_6 - b_7 b_a + b_5 + b_7 \tag{24}$$
$$\longrightarrow b_2 b_3 + b_7 b_a + b_3 \tag{25}$$

$$b_1 b_2 b_3 b_4 b_5 + b_4 b_5 b_6 b_7 b_8 : \qquad\qquad (k,n) = (5,8). \tag{26}$$
$$\longrightarrow b_1 b_2 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) \tag{27}$$
$$\longrightarrow b_3 b_5 + b_8 b_a + b_5 \tag{28}$$
$$\longrightarrow b_4 b_7 + b_4 \tag{29}$$
$$\longrightarrow b_7 b_8 + b_a (1 + b_6 - b_7 - b_8) + b_3 \tag{30}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 + b_3 b_4 b_5 b_6 b_7 b_8 : \qquad\qquad (k,n) = (6,8). \tag{31}$$
$$\longrightarrow b_1 b_6 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) \tag{32}$$
$$\longrightarrow b_2 b_5 + b_4 b_5 + b_4 b_a \tag{33}$$
$$\longrightarrow b_3 b_4 + b_3 b_7 - b_a + 1 \tag{34}$$
$$\longrightarrow b_2 b_4 + b_7 b_8 \tag{35}$$
$$\longrightarrow b_3 b_4 + b_4 \tag{36}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 : \qquad\qquad (k,n) = (7,8). \tag{37}$$
$$\longrightarrow b_6 b_7 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) \tag{38}$$
$$\longrightarrow b_2 b_3 + b_3 b_4 \tag{39}$$
$$\longrightarrow b_1 b_4 + b_4 b_8 - b_6 b_a + b_6 \tag{40}$$
$$\longrightarrow b_2 b_3 + b_2 \tag{41}$$
$$\longrightarrow b_1 b_5 + b_3 b_5 + b_6 b_a \tag{42}$$

$$b_1 b_2 b_3 b_4 + b_5 b_6 b_7 b_8 : \qquad\qquad (k,n) = (4,8). \tag{43}$$
$$\longrightarrow b_1 b_2 + b_6 b_8 + b_a (1 - b_6 + b_7 - b_8) \tag{44}$$
$$\longrightarrow b_3 b_4 + b_6 b_8 + 2 b_8 b_a \tag{45}$$
$$\longrightarrow b_2 b_3 + b_5 b_7 + b_a (1 - b_6 + b_7) \tag{46}$$
$$\longrightarrow b_1 b_4 + b_5 b_7 - b_6 b_8 + b_7 b_a + b_6 \tag{47}$$

$$b_1 b_2 b_3 b_4 b_5 + b_6 b_7 b_8 b_9 b_{10} : \qquad\qquad (k,n) = (5,10). \tag{48}$$
$$\longrightarrow b_2 b_3 + b_6 b_9 + b_9 b_a \tag{49}$$
$$\longrightarrow b_1 b_4 + b_8 b_{10} + b_9 b_a \tag{50}$$
$$\longrightarrow b_3 b_5 + b_7 b_{10} + b_1 b_a + b_9 b_a \tag{51}$$
$$\longrightarrow b_4 b_5 + b_6 b_9 + b_9 b_a \tag{52}$$
$$\longrightarrow b_1 b_2 + b_7 b_9 + b_9 b_a \tag{53}$$
$$\longrightarrow b_2 b_5 + b_6 b_8 + b_9 b_a \tag{54}$$
$$\longrightarrow b_2 b_3 + b_8 b_{10} + b_9 b_a \tag{55}$$
$$\longrightarrow b_1 b_4 + b_6 b_7 + b_9 b_a \tag{56}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6 b_7 b_8 b_9 b_{10} : \qquad\qquad (k,n) = (6,10). \tag{57}$$
$$\longrightarrow b_5 b_6 + b_5 b_7 + b_a(1 - b_{10}) \tag{58}$$
$$\longrightarrow b_1 b_2 + b_8 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{59}$$
$$\longrightarrow b_3 b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10} \tag{60}$$
$$\longrightarrow b_3 b_6 - b_5 b_{10} + b_6 b_7 + b_a(1 - b_9) + b_{10} \tag{61}$$
$$\longrightarrow b_1 b_2 + b_7 b_{10} + b_a(1 - b_9 - b_{10}) + b_{10} \tag{62}$$
$$\longrightarrow b_3 b_4 + b_8 b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1 \tag{63}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 + b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : \qquad\qquad (k,n) = (7,10). \tag{64}$$
$$\longrightarrow b_4 b_7 + b_6 b_7 + b_a(1 - b_4 - b_7 + b_{10}) \tag{65}$$
$$\longrightarrow b_2 b_5 + b_5 b_9 + b_a \tag{66}$$
$$\longrightarrow b_1 b_4 + b_4 b_8 + b_a \tag{67}$$
$$\longrightarrow b_1 b_3 + b_6 b_{10} \tag{68}$$
$$\longrightarrow b_3 b_6 + b_6 b_9 + b_a \tag{69}$$
$$\longrightarrow b_2 b_3 + b_8 b_{10} \tag{70}$$
$$\longrightarrow b_1 b_4 + b_9 \tag{71}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 + b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : \qquad\qquad (k,n) = (8,10). \tag{72}$$
$$\longrightarrow b_2 b_4 + b_4 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{73}$$
$$\longrightarrow b_1 b_7 + b_7 b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{74}$$
$$\longrightarrow b_5 b_8 + b_6 b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{75}$$
$$\longrightarrow b_3 b_6 + b_a(b_3 - b_{10}) + b_{10} \tag{76}$$
$$\longrightarrow b_1 b_5 + b_5 b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \tag{77}$$
$$\longrightarrow b_6 b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1 \tag{78}$$
$$\longrightarrow b_1 b_2 - b_9 b_a + b_{10} + 1 \tag{79}$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 + b_2 b_3 b_4 b_5 b_6 b_7 b_8 b_9 b_{10} : \qquad\qquad (k,n) = (9,10). \tag{80}$$
$$\longrightarrow b_1 b_6 + b_6 b_{10} + b_9 b_a \tag{81}$$
$$\longrightarrow b_4 b_7 + b_7 b_8 \tag{82}$$
$$\longrightarrow b_4 b_9 + b_a(b_9 - b_4) + b_4 \tag{83}$$
$$\longrightarrow b_2 b_3 + b_3 b_8 + b_9 b_a \tag{84}$$
$$\longrightarrow b_1 b_5 + b_2 b_5 + b_9 b_a \tag{85}$$
$$\longrightarrow b_2 b_8 - b_6 b_7 + b_8 b_{10} + b_9 b_a + 1 \tag{86}$$
$$\longrightarrow b_2 b_{10} + b_2 \tag{87}$$

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n TRINOMIALS

$b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6$: $\qquad\qquad (k,n) = (4,6)$. (88)

$\longrightarrow 3b_3b_4 + b_3b_5 + b_4b_5 - b_3 - b_4 - b_5 + 1 \qquad\qquad 41/64(\approx 64\%)$ (89)

$\longrightarrow b_1b_4 + b_3b_5 + b_4b_5 \qquad\qquad 56/64(\approx 88\%)$ (90)

$\longrightarrow b_1b_2 + b_2b_6 + b_3b_5 + b_5b_6 + b_2 - b_3 - b_4 - b_5 + 2 \qquad\qquad 64/64(100\%)$ (91)

$b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8$: $\qquad\qquad (k,n) = (4,8)$. (92)

$\longrightarrow b_1b_4 + 2b_5b_6 \qquad\qquad 159/256(\approx 62\%)$ (93)

$\longrightarrow b_2b_3 + b_3b_5 + b_7b_8 \qquad\qquad 225/256(\approx 88\%)$ (94)

$\longrightarrow b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1 \qquad\qquad 244/256(\approx 95.3\%)$ (95)

$\longrightarrow b_2b_3 + b_6b_8 + b_6 \qquad\qquad 253/256(\approx 98.8\%)$ (96)

$\longrightarrow b_2b_3 + b_5b_7 + b_5 \qquad\qquad 256/256(100\%)$ (97)

$b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7$: $\qquad\qquad (k,n) = (5,7)$. (98)

$\longrightarrow b_1b_5 + 2b_5b_6 \qquad\qquad$ (99)

$\longrightarrow b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1 \qquad\qquad$ (100)

$\longrightarrow b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1) \qquad\qquad$ (101)

$\quad + b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5 \qquad\qquad$ (102)

$\longrightarrow b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1 \qquad\qquad$ (103)

$\longrightarrow b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2 \qquad\qquad$ (104)

$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8$: $\qquad\qquad (k,n) = (6,8)$. (105)

$\longrightarrow b_1b_3 + b_3b_5 + b_3b_8 + b_a\left(1 + b_6 - b_7\right) \qquad\qquad$ (106)

$\longrightarrow b_2b_6 + b_6b_7 + b_a\left(-b_6 + b_7\right) + b_6 \qquad\qquad$ (107)

$\longrightarrow b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4 \qquad\qquad$ (108)

$\longrightarrow -b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8 \qquad\qquad$ (109)

$\quad + b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3 \qquad\qquad$ (110)

$\longrightarrow b_1b_4 + b_6b_7 + b_7b_8 \qquad\qquad$ (111)

$\longrightarrow b_2b_7 + b_7b_8 + b_8b_a + b_2 \qquad\qquad$ (112)

$b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10}$: $\qquad\qquad (k,n) = (8,10)$. (113)

$\longrightarrow b_3b_5 + b_5b_9 + b_5b_{10} \qquad\qquad$ (114)

$\longrightarrow b_1b_4 + b_4b_7 + b_4b_9 \qquad\qquad$ (115)

$\longrightarrow b_1b_6 + b_2b_6 - b_5b_6 + b_6b_{10} + b_6 \qquad\qquad$ (116)

$\longrightarrow b_4b_8 + 2b_8 \qquad\qquad$ (117)

$\longrightarrow b_1b_7 + b_2b_7 - b_a(b_4 + b_6) + b_7b_9 + b_4 + 1 \qquad\qquad$ (118)

$\longrightarrow b_2b_3 + b_3b_5 + b_3 \qquad\qquad$ (119)

$\longrightarrow b_1b_2 + b_2b_9 - b_5b_6 - b_5b_a + b_6b_9 + 2 \qquad\qquad$ (120)

$\longrightarrow b_2b_7 + b_2 + b_{10} \qquad\qquad$ (121)

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10} : \qquad\qquad (k,n) = (6,10). \quad (122)$$
$$\longrightarrow b_2b_5 + b_5b_6 + b_5b_9 - b_9b_{11} + b_9 \qquad\qquad (123)$$
$$\longrightarrow b_1b_4 + b_4b_6 + b_6b_{10} - b_9b_{11} + b_9 - 2b_{11} + 2 \qquad\qquad (124)$$
$$\longrightarrow b_3b_4 + b_3b_7 + b_9b_{10} - b_9b_{11} + b_9 \qquad\qquad (125)$$
$$\longrightarrow b_2b_3 + b_8b_{11} - b_9b_{11} + b_8 + b_9 \qquad\qquad (126)$$
$$\longrightarrow b_1b_6 + b_6b_7 - b_9b_{11} + b_{10}b_{11} + b_9 \qquad\qquad (127)$$
$$\longrightarrow b_1b_3 + b_7b_{10} - b_9b_{11} + b_7 + b_9 \qquad\qquad (128)$$
$$\longrightarrow b_2b_4 + b_4b_7 - b_9b_{11} + 2b_{10}b_{11} + b_9 \qquad\qquad (129)$$
$$\longrightarrow b_2b_6 + b_6b_7 + b_6b_8 - b_9b_{11} + b_9 \qquad\qquad (130)$$
$$\longrightarrow b_1b_4 - b_2b_5 - b_3b_{10} + b_5b_8 - b_7b_{11} + b_8b_{10} - b_9b_{11} + b_{10}b_{11} + b_9 + 2 \qquad 1020/1024(\approx 99.6\%) \quad (131)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10} : \qquad\qquad (k,n) = (4,10). \quad (132)$$
$$\longrightarrow b_3b_4 + b_4b_5 + b_8b_{10} + b_a(1 - b_9) \qquad\qquad (133)$$
$$\longrightarrow b_1b_2 + b_a(1 + b_3 - b_9) + b_5b_6 + b_7b_9 \qquad\qquad (134)$$
$$\longrightarrow b_1b_3 + b_6b_7 + b_7b_{10} + b_a \qquad\qquad (135)$$
$$\longrightarrow b_4b_6 + b_7b_9 + b_a(1 - b_9) + b_4 \qquad\qquad (136)$$
$$\longrightarrow b_2b_3 + b_5b_7 + b_8b_9 + b_a(1 - b_9) \qquad\qquad (137)$$
$$\longrightarrow b_2b_3 + b_8b_{10} + b_a(1 - b_9) + b_6 - b_9 + 1 \qquad\qquad (138)$$
$$\longrightarrow b_1b_2 + b_5b_7 + b_8b_{10} + b_a(1 - b_9) - b_3 + 1 \qquad\qquad (139)$$
$$\longrightarrow b_5b_6 + b_a(1 - b_9) + b_3 + b_9 \qquad\qquad (140)$$
$$\longrightarrow b_7b_8 + b_2 + b_7 \qquad\qquad 1023/1024(\approx 99.9\%) \quad (141)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : \qquad\qquad (k,n) = (5,8). \quad (142)$$
$$\longrightarrow b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8) \qquad\qquad (143)$$
$$\longrightarrow b_1b_3 + b_3b_6 + b_6b_7 \qquad\qquad (144)$$
$$\longrightarrow b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7) \qquad\qquad (145)$$
$$\longrightarrow b_2b_5 + b_6b_8 + b_6 \qquad\qquad (146)$$
$$\longrightarrow b_1b_3 + b_3 + b_8 \qquad\qquad (147)$$

DECOMPOSITION OF DEGREE-k, EXACT-k-OF-n QUADRINOMIALS

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} : \qquad (k,n) = (7,10). \quad (148)$$
$$\longrightarrow b_1b_2 + b_2b_5 + b_5b_7 + b_5b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \qquad\qquad (149)$$
$$\longrightarrow b_5b_6 + b_6b_8 + b_6b_9 + b_6b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \qquad\qquad (150)$$
$$\longrightarrow b_1b_7 + b_5b_7 + b_6b_7 + b_7b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \qquad\qquad (151)$$
$$\longrightarrow b_4b_5 + b_4b_8 + b_4b_{10} + b_a(b_4 - b_9 - b_{10}) + b_9 + b_{10} \qquad\qquad (152)$$
$$\longrightarrow b_1b_3 + b_3b_4 + b_3b_9 - b_6b_7 + b_6b_{10} + b_a(1 - b_9 - b_{10}) - b_6 + b_{10} + 2 \qquad\qquad (153)$$
$$\longrightarrow b_2b_3 + b_2b_8 + b_3b_9 - b_5b_7 + b_6b_9 - b_a(1 + b_{10}) - b_7 + b_9 + b_{10} + 3 \qquad\qquad (154)$$
$$\longrightarrow b_2b_3 - b_6b_7 + b_6b_8 + b_7b_8 - b_a(b_9 + b_{10}) + b_8 + b_9 + b_{10} + 1 \qquad\qquad (155)$$
$$\longrightarrow -b_1b_2 + b_1b_4 + b_2b_5 - b_4b_9 + b_5b_8 + b_5b_9 - b_a(b_9 + b_{10}) - b_4 + b_5 - b_7 + 2b_9 + b_{10} + 2 \qquad\qquad (156)$$
$$\longrightarrow b_2b_8 + b_8b_{10} + b_a(b_8 - b_9 - b_{10}) + b_1 + b_9 + b_{10} \qquad\qquad 1021/1024(\approx 99.7\%) \quad (157)$$