I. PTR-BCR

Binary variables: $x_1, x_2, \ldots, x_n \in \{0,1\}$, and let $X = |x| := \sum_{i=1}^{n} x_i$

A. PTR-BCR-1

Let's begin with the equation for the quadratization of P. (In the formulas, summations for $i, i' = 1, 2, \ldots, n$ are understood unless stated otherwise.)
(Theorem 4.3, [1])

$$\begin{aligned} g(x,y) &= -2\left[n - \frac{1}{2} - l\right]^{-} + E''(l) \\ &= \frac{l(l+1)}{2} + \sum_{\substack{i=1: \\ i\ odd}}^{n-2} 2y_i\left[i - \frac{1}{2} - l\right]^{-} \\ &= \sum_i x_i + \sum_{1\le i<j\le n} x_i x_j + \min_y \sum_{\substack{i=1: \\ i\ odd}}^{n-2} 2y_i\left(i - \frac{1}{2} - l\right) \end{aligned} \tag{1}$$

Using the relationship $l = \sum_{j=1}^{n} x_j$ we get

$$g(x,y) = \sum_i x_i + \sum_{1\le i<j\le n} x_i x_j + \min_y \sum_{\substack{i=1: \\ i\ odd}}^{n-2} 2y_i\left(i - \frac{1}{2} - \sum_j x_j\right) \tag{2}$$

Finally, after rearranging we end with the equation

$$g(x,y) = \sum_i x_i + \sum_{2i-1}(2(2i-1)-1)y_{2i-1} + \sum_{1\le i<j\le n} x_i x_j - \sum_{2i-1}\sum_j y_{2i-1}x_j \tag{3}$$

Dictionary:

$x_{i,j} \mapsto b_{i,j}$,

$y_{2i-1} \mapsto b_{a_{2i-1}}$

B. PTR-BCR-2

To begin, we start with the following equation: (In the formulas, summations for $i, i' = 1, 2, \dots, n$ and $j, j' = 0, 1, \dots, m$ are understood.)
(Theorem 10, [2])

$$g(x,y) = \frac{1}{2}\Big(|x| - 2|y| - (N-2)y_1\Big)\Big(|x| - 2|y| - (N-2)y_1 - 1\Big) \tag{4}$$

Substituting for x, y, and N using $\sum_i x_i$, $\sum_j y_j$, and $n - 2m$, respectively, we get

$$g(x,y) = \frac{1}{2}\Big(\sum_i x_i - 2\sum_j y_j - (n-2m-2)y_1\Big)\Big(\sum_i x_i - 2\sum_j y_j - (n-2m-2)y_1 - 1\Big) \tag{5}$$

After expanding and simplifying, the equation becomes

$$\begin{aligned} g(x,y) &= -\frac{1}{2}\sum_i x_i + \sum_j y_j + \frac{1}{2}(n-2m-2)y_1 \\ &+\frac{1}{2}\sum_{i,i'} x_i x_{i'} - 2\sum_i\sum_j x_i y_j - (n-1m-2)\sum_i x_i y_1 \\ +2\sum_{j,j'} y_j y_{j'} &+ 2(n-2m-2)\sum_{j,j'} y_j y_1 + \frac{1}{2}((n-2m-2)y_1)^2 \end{aligned} \tag{6}$$

Because $y_1 \in \{0,1\}$, we have $y_1^2 = y_1$, so we can join the two terms,

$$\frac{1}{2}(n-2m-2)y_1 + \frac{1}{2}((n-2m-2)y_1)^2 = \frac{1}{2}(-3n+6m+n^2-4mn+4m^2+2)y_1 \tag{7}$$

Thus we are left with

$$\begin{aligned} g(x,y) = {} & \underbrace{-\frac{1}{2}}_{\alpha^{b}} \sum_i x_i + \underbrace{1}_{\alpha^{b_{a,1}}} \sum_j y_j + \underbrace{\frac{1}{2}(-3n + 6m + n^2 - 4mn + 4m^2 + 2)}_{\alpha^{b_{a,2}}} y_1 \\ & + \underbrace{\frac{1}{2}}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \underbrace{-2}_{\alpha^{bb_{a,1}}} \sum_i \sum_j x_i y_j + \underbrace{-(n-2m-2)}_{\alpha^{bb_{a,2}}} \sum_i x_i y_1 \\ & + \underbrace{2}_{\alpha^{b_{a,1}b_{a,1}}} \sum_{j,j'} y_j y_{j'} + \underbrace{2(n-2m-2)}_{\alpha^{b_{a,1}b_{a,1}}} \sum_{j,j'} y_j y_1 \end{aligned} \tag{8}$$

The coefficients are

$$\alpha^{b} = -\frac{1}{2} \tag{9a}$$

$$\alpha^{b_{a,1}} = 1 \tag{9b}$$

$$\alpha^{b_{a,2}} = \frac{1}{2}(-3n + 6m + n^2 - 4mn + 4m^2 + 2) \tag{9c}$$

$$\alpha^{bb} = \frac{1}{2} \tag{9d}$$

$$\alpha^{bb_{a,1}} = -2 \tag{9e}$$

$$\alpha^{bb_{a,2}} = -(n-2m-2) \tag{9f}$$

$$\alpha^{b_{a,1}b_{a,1}} = 2 \tag{9g}$$

$$\alpha^{b_{a,1}b_{a,2}} = 2(n-2m-2) \tag{9h}$$

Dictionary:

$y_1 \mapsto b_{a_m}$,

$x_i \mapsto b_i$,

$y_j \mapsto b_{a_i}$

C. PTR-BCR-3

We begin with the following equation: (In the formulas, summations for $i, i' = 1, 2, \ldots, n$ and $j, j' = 0, 1, \ldots, k-1$ are understood.)
(Theorem 4, [2])

$$g(x,y) = \Big(K + X - \sum_j 2^j y_j\Big)^2 \tag{10}$$

Knowing that $X = \sum_{i=1}^n x_i$ and $K = 2^k - n$, we substitute to get

$$g(x,y) = \Big(2^k - n + \sum_i x_i - \sum_j 2^j y_j\Big)^2 \tag{11}$$

Expanding the square and collecting like terms we end with

$$\begin{aligned} g(x,y) =& \underbrace{(2^k-n)^2}_{\alpha} + \underbrace{2(2^k-n)}_{\alpha^b} \sum_i x_i + \underbrace{-2(2^k-n)}_{\alpha^{b_a}} \sum_j 2^j y_j \\ & + \underbrace{1}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \sum_i \sum_j \underbrace{2^{j-1}}_{\alpha^{bb_a}} x_i y_j + \sum_{j,j'} \underbrace{2^{j+j'}}_{\alpha^{b_a b_a}} y_j y_{j'} \end{aligned} \tag{12}$$

The coefficients are

$$\alpha = (2^k - n)^2 \tag{13a}$$

$$\alpha^b = 2(2^k - n) \tag{13b}$$

$$\alpha^{b_a} = -2(2^k - n) \tag{13c}$$

$$\alpha^{bb} = 1 \tag{13d}$$

$$\alpha^{bb_a} = 2^{j-1} \tag{13e}$$

$$\alpha^{b_a b_a} = 2^{j+j'} \tag{13f}$$

Dictionary:

$k \mapsto m$,

$n \mapsto k$,

$x_i \mapsto b_i$,

$x_{i'} \mapsto b_j$,

$y_j \mapsto b_{a_i}$,

$y_{j'} \mapsto b_{a_j}$

D. PTR-BCR-4

We start with the following equation for the quadratization of PTR-BCR-4: (In the formulas, summations for $i, i' = 1, 2, \ldots, n$ and $j, j' = 0, 1, \ldots, l-1$ are understood.)

(Theorem 9, [2])

$$g(x,y) = \frac{1}{2}\Big(|x| - 2^l - n - \sum_j 2^j y_j\Big)\Big(|x| - 2^l - n - \sum_j 2^j y_j - 1\Big) \tag{14}$$

After substituting for X, we end with

$$g(x,y) = \frac{1}{2}\Big(-2^l - n + \sum_i x_i - \sum_j 2^j y_j\Big)\Big(-2^l - n + \sum_i x_i - \sum_j 2^j y_j - 1\Big) \tag{15}$$

Dictionary: $l \mapsto m+1$,

$n \mapsto k$,

$x_i \mapsto b_i$,

$y_j \mapsto b_{a_i}$,

E. PTR-BCR-5

With $log(n)$ auxiliary variables we have the equation: (In the formulas, summations for $i, i' = 1, 2, \ldots, n$ and $j, j' = 0, 1, \ldots, l-1$ are understood.)

(Remark 5 from Theorem 9, [2])

$$g'(x,y) = \Big(|x| - \sum_j 2^j y_j\Big)^2 \tag{16}$$

We can express x using the formula $|x| = \sum_i x_i$

$$g'(x,y) = \Big(\sum_i x_i - \sum_j 2^j y_j\Big)^2 \tag{17}$$

By expanding and collecting like terms the equation finally becomes

$$g'(x,y) = \underbrace{1}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \sum_{i,j} \underbrace{2^{j-1}}_{\alpha^{bb_a}} x_i y_j + \sum_{j,j'} \underbrace{2^{j+j'}}_{\alpha^{b_a b_a}} y_j y_{j'} \tag{18}$$

The coefficients are

$$\alpha^{bb} = 1 \tag{19a}$$

$$\alpha^{bb_a} = 2^{j-1} \tag{19b}$$

$$\alpha^{b_a b_a} = 2^{j+j'} \tag{19c}$$

Dictionary:

$l \mapsto m+1$,

$x_i \mapsto b_i$,

$y_j \mapsto b_{a_i}$

[1] Martin Anthony, Endre Boros, Yves Crama, and Aritanan Gruber, "Quadratization of symmetric pseudo-Boolean functions," (2014), arXiv:1404.6535.

[2] Endre Boros, Yves Crama, and Elisabeth Rodríguez-Heck, "Compact quadratizations for pseudo-boolean functions," in *unpublished* (2018).LastBibItem