More Reductions

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I. THEOREM 4.1 OF ANTHONY-BOROS-CRAMA-GRUBER (ABCG)

Any symmetric fuction can be quadratized with $n-2$ auxiliaries, where α_i comes from Corollary 2.3:

$$f\left(b_1,b_2,\ldots,b_n\right)\rightarrow-\alpha_0-\alpha_0\sum_i b_i+2a_2\sum_{ij}b_ib_j+ \tag{1}$$

$$2\sum_{2i-1}\left(\alpha_{2i-1}-\min\left(\alpha_{2j-1}\right)\right)b_{a_{2i-1}}\left(2i-\frac{3}{2}-\sum_j b_j\right)+ \tag{2}$$

$$2\sum_{2i}\left(\alpha_{2i}-\min\left(\alpha_{2j}\right)\right)b_{a_{2i}}\left(2i-\frac{1}{2}-\sum_j b_j\right) \tag{3}$$

$$\alpha_i=-4\sum_{j=0}^{i}\left(-1\right)^{i-j}f(j)-f(i-1)+3f(i) \tag{4}$$

Alternatively:

$$f\left(b_1,b_2,\ldots,b_n\right)\rightarrow-\alpha_0-\alpha_0\sum_i b_i+a_2\sum_{ij}b_ib_j+2\sum_i\left(\alpha_i-c\right)b_{a_i}\left(2i-\frac{1}{2}-\sum_j b_j\right) \tag{5}$$

$$c=\begin{cases}\min\left(\alpha_{2j}\right) & ,i\in\text{even}\\ \min\left(\alpha_{2j-1}\right) & ,i\in\text{odd}\end{cases} \tag{6}$$

$$a_2=\text{has to be obtained from page 12 of the paper.} \tag{7}$$

They say this is linear in the auxiliary variables, but it doesn't seem to be, because we have $b_{a_i}b_j$ terms where b_{a_i} are auxiliaries.

Pro: quadratization symmetric in all non-auxiliary variables, which isn't true for all quadratizations of symmetric functions. Reproduces the full spectrum.

Con: all quadratic terms of the non-auxiliary variables, are non-submodular. Also very complicated and uses more auxiliaris than simlper methods.

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II. UNPUBLISHED WORK OF ALEXANDER FIX

Any symmetric fuction can be quadratized with $n-1$ auxiliaries. Add a multiple of $E(\sum b_r)$ to each term of Corollary 2.4 of the above paper.

III. ASYMMETRIC REDUCTION FOR NEGATIVE MONOMIALS OF ARBITRARY k

$$-b_1 b_2 \dots b_k \rightarrow (k-1) b_k b_a - \sum_i b_i (b_a + b_k - 1) \tag{8}$$

$$-b_1 b_2 \dots b_k \rightarrow -\sum_i b_i - \sum_i b_i b_k - \sum_i b_i b_a + (k-1) b_k b_a \tag{9}$$

Pro: only 1 auxiliary to quadratize k degree term. Only one non-submodular term (and it's quadratic). Reproduces the full spectrum.

Con: Turns symmetric into non-symmetric (but only the b_k is asymmetric).

IV. "A RELATED PAPER BY THE PRESENT AUTHORS [1] GIVES A COMPLETE CHARACTERIZATION OF ALL THE QUADRATIZATIONS OF NEGATIVE MONOMIALS INVOLVING ONE AUXILIARY VARIABLE"

V. ANOTHER REDUCTION FOR NEGATIVE MONOMIALS OF ARBITRARY k

$$-b_1 b_2 \dots b_k \rightarrow 2 b_a \left(k - \frac{1}{2} - \sum_i b_i \right) \tag{10}$$

$$-b_1 b_2 \dots b_k \rightarrow (2k-1)\, b_a - 2 \sum_i b_i b_a \tag{11}$$

Pro: only 1 auxiliary to quadratize k degree term. Only one non-submodular term (and it's linear). Symmetric with respect to all non-auxiliary variables. Reproduces the full spectrum.

Con: Coefficients of quadratic terms are twice the size of in the "standard" quadratization for negative monomials, and roughly twice the size for the linear term.

VI. ABCG VERSION OF ISHIKAWA:

$$b_1 b_2 \ldots b_k \to \sum_i b_i + \sum_{ij} b_i b_j + \sum_{2i-1} b_{a_{2i-1}} \left(4i - 3 - \sum_j b_j \right) \tag{12}$$

$$b_1 b_2 \ldots b_k \to \sum_i b_i + (4i-3) \sum_{2i-1} b_{a_{2i-1}} + \sum_{ij} b_i b_j - \sum_{2i-1,j} b_j b_{a_{2i-1}} \tag{13}$$

Pro. Same number of auxiliaries as Ishikawa. Reproduces the full spectrum.

Con. Only works for odd k, but when k is even we can use Ishikawa, so no big loss.

VII. ANOTHER ABCG VERSION OF ISHIKAWA:

$$b_1 b_2 \ldots b_k \to \prod_{i=1}^{k-1} b_i - \prod_{i=1}^{k-1} b_i (1 - b_k) \tag{14}$$

Now quadratize the first term using Ishikawa, and use a negative monomial method for the second term.

VIII. COROLLARY 4.4 OF ABCG

Need to conver $[]^-$ into something more readable.

IX. COROLLARY 4.5 OF ABCG

X. QUADRATIZATION OF "PARITY" FUNCTION ON PAGE 17 OF ABCG (THEOREM 4.6)

For any k-local function that is non-zero (they actually say =1) only if $\sum b_i = 2m - 1$, we call it the "partity function" an it can be quadratized as follows:

$$f\left(b_1, b_2, \ldots, b_n\right) \to \sum_i b_i + 2 \sum_{ij} b_i b_j + 4 \sum_{2i-1}^{n-1} b_{a_i} \left(2i - 1 - \sum_j b_j \right) \tag{15}$$

$m = \lfloor n/2 \rfloor$auxiliary variables.

XI. QUADRATIZATION OF "PARITY" FUNCTION ON PAGE 17 OF ABCG (THEOREM 4.6) WITH FEWER VARIABLES

For the complement of the parity function we quadratize as follows:

$$f\left(b_1, b_2, \ldots, b_n\right) \rightarrow 1 + 2\sum_{ij} b_i b_j - \sum_i b_i + 4\sum_{2i}^{n-1} b_{a_i}\left(i - \sum_j^n b_j\right) \tag{16}$$

$m = \lfloor \frac{n-1}{2} \rfloor$auxiliary variables.

XII. THEOREM 5.6 OF ABCG

XIII. THEOREM 1.1 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $n/2 \leq m \leq n$

$$f\left(b_1, b_2, \ldots, b_n\right) \rightarrow \left(\sum_i b_i - \left(m - 2^{\lceil \log m \rceil}\right) b_{a_{\lceil \log m \rceil + 1}} - (m+1)\left(1 - b_{a_{\lceil \log m \rceil + 1}}\right) - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{17}$$

$$= \left(\sum_i b_i - \left(m - 2^{\lceil \log m \rceil}\right) b_{a_{\lceil \log m \rceil + 1}} - (m+1) + (m+1) b_{a_{\lceil \log m \rceil + 1}} - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{18}$$

$$= \left(-(m+1) + \sum_i b_i - \left(2m - 2^{\lceil \log m \rceil} + 1\right) b_{a_{\lceil \log m \rceil + 1}} - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{19}$$

$$= (m+1)^2 - 2(m+1)\sum_i b_i + 2(m+1)\left(2m - 2^{\lceil \log m \rceil} + 1\right) b_{a_{\lceil \log m \rceil + 1}} + 2(m+1)\sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i} \tag{20}$$

$$+ \sum_{ij} b_i b_j - 2\sum_i \left(2m - 2^{\lceil \log m \rceil} + 1\right) b_i b_{a_{\lceil \log m \rceil + 1}} - 2\sum_i \sum_j^{\lceil \log m \rceil} 2^{i-1} b_i b_{a_j} + \sum_{i,j}^{\lceil \log m \rceil} 2^{i+j-2} b_{a_i} b_{a_j} \tag{21}$$

$$= \alpha^I + \alpha^b \sum_i b_i + \alpha^{b_{a,1}} \sum_i^{\lceil \log m \rceil} b_{a_i} + \alpha^{b_{a,2}} b_{a_{\lceil \log m \rceil + 1}} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_{a,1}} \sum_i \sum_j^{\lceil \log m \rceil} b_i b_{a_j} \tag{22}$$

$$+ \alpha^{bb_{a,2}} \sum_i b_i b_{a_{\lceil \log m \rceil + 1}} + \alpha^{b_a b_a} \sum_{i,j}^{\lceil \log m \rceil} b_{a_i} b_{a_j} \tag{23}$$

The number of auxiliary varibales is $\lceil \log m \rceil + 1$.

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (m+1)^2 & 1 \\ -2(m+1) & -2^i \\ 2(m+1) & -2\left(2m - 2^{\lceil \log m \rceil} + 1\right) \\ 2(m+1)\left(2m - 2^{\lceil \log m \rceil} + 1\right) & 2^{i+j-2} \end{pmatrix}. \tag{24}$$

XIV. THEOREM 1.2 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $0 \le m \le n/2$.

$$f\left(b_1, b_2, \ldots, b_n\right) \to \left(n - \sum_i b_i - \left(n - m - 2^{\lceil \log(n-m) \rceil}\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - (n - m + 1)\left(1 - b_{a_{\lceil \log(n-m) \rceil + 1}}\right) - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{25}$$

$$= \left(n - \sum_i b_i - \left(n - m - 2^{\lceil \log(n-m) \rceil}\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - (n - m + 1) + (n - m + 1) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{26}$$

$$= \left((m-1) - \sum_i b_i - \left(2(n-m) - 2^{\lceil \log(n-m) \rceil} + 1\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{27}$$

The number of auxiliary varibales is $\lceil \log(n-m) \rceil + 1$.

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}. \tag{28}$$

XV. COROLLARY 1 OF BCR

transformation not explicitly given, but the function can be more general than in Theorem 1, but requires a factor of μ more variables.

XVI. THEOREM 2.1 OF BCR

Once again requires typing out a nasty function [should really be done in mathematica rather than by hand)

$$f\left(b_1, b_2, \ldots, b_n\right) \to \frac{1}{2}\left(\sum_i b_i - \left(m - 2^c\right) b_{a_{c+1}} - (m+1)((1 - b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i}\right)\left(\sum_i b_i - \left(m - 2^c\right) b_{a_{c+1}} - (m+1)((1 - b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i} - 1\right) \tag{29}$$

XVII. THEOREM 2.2 OF BCR

$$f\left(b_1, b_2, \ldots, b_n\right) \to \frac{1}{2}\left((m-1) - \sum_i b_i - \left(2(n-m) - 2^{\lceil \log(n-m) \rceil} + 1\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)\left((m-1) - \sum_i b_i - \left(2(n-m) - 2^{\lceil \log(n-m) \rceil} + \right.\right. \tag{30}$$

XVIII. THEOREM 4 OF BCR

This is a special case of Theorem 1, for the specific function $f = b_1 b_2 \ldots b_k$. For som p such that $k \leq 2^p$, we have:

$$b_1 b_2 \ldots b_k \to \left(2^p - k + \sum_i b_i - \sum_i 2^{i-1} b_{a_i}\right)^2 \tag{31}$$

$$= \left(2^p - k\right)^2 + 2\left(2^p - k\right) \sum_i b_i - 2\left(2^p - k\right) \sum_i 2^{i-1} b_{a_i} + \sum_{ij} b_i b_j - \sum_{ij} 2^{j-1} b_i b_{a_j} + \sum_{ij} 2^{i+-2} b_{a_i} b_{a_j} \tag{32}$$

$$= \alpha^I + \alpha^b \sum_i b_i + \alpha^{b_{a_i}} \sum_i 2^{i-1} b_{a_i} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_a} \sum_{ij} b_i b_{a_j} + \alpha^{b_{a_i} b_{a_j}} b_{a_i} b_{a_j} \tag{33}$$

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_a} \\ \alpha^{b_a} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} \left(2^p - k\right)^2 & 1 \\ 2\left(2^p - k\right) & 2^{j-1} \\ -2\left(2^p - k\right) & 2^{i+-2} \end{pmatrix}. \tag{34}$$

Pro: only requires $\log k$ auxiliaries.

Con: All terms non-submodular except for the term linear in auxiliaries.

XIX. THEOREM 5 OF BCR

(already written up by Richard)

Pro: only requires $\log k$-1 auxiliaries.

XX. THEOREM 7 OF BCR

$$
\begin{aligned}
b_1 b_2 \cdots b_k &\rightarrow \frac{1}{2}\left(\sum_i b_i - 2\sum_i^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} - \left(n - \lceil \frac{k}{4} \rceil\right) b_{a_{\lceil \frac{k}{4}\rceil}}\right)\left(\sum_i b_i - 2\sum_i^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} - \left(n - \lceil \frac{k}{4} \rceil\right) b_{a_{\lceil \frac{k}{4}\rceil}} - 1\right) && (35)\\
&= \frac{1}{2}\left(\sum_{ij} b_i b_j - 4\sum_i \sum_j^{\lceil \frac{k}{4}\rceil - 1} b_i b_{a_j} - 2\left(n - \lceil \frac{k}{4} \rceil\right)\sum_i b_i b_{a_{\lceil \frac{k}{4}\rceil}} - \sum_i b_i + 4\sum_{ij}^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} b_{a_j} + 4\left(n - \lceil \frac{k}{4} \rceil\right)\sum_i^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} b_{a_{\lceil \frac{k}{4}\rceil}} + 2\sum_i^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} + \left(n - \lceil \frac{k}{4} \rceil\right)^2 b_{a_{\lceil \frac{k}{4}\rceil}} + \left(\right.\right. && (36)\\
&= \alpha + \alpha^b \sum_i b_i + \alpha^{b_{a_1}} \sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_{\lceil \frac{k}{4}\rceil}} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_{a_1}} \sum_i \sum_j^{\lceil \frac{k}{4}\rceil - 1} b_i b_{a_j} + \alpha^{bb_{a_2}} \sum_i b_i b_{a_{\lceil \frac{k}{4}\rceil}} + \alpha^{b_{a_1} b_{a_1}} \sum_{ij}^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} b_{a_j} + \alpha^{b_{a_1} b_{a_2}} \sum_{ij}^{\lceil \frac{k}{4}\rceil - 1} b_{a_i} b_{a_{\lceil \frac{k}{4}\rceil}} && (37)\\
&= \alpha + \alpha^b \sum_i b_i + \alpha^{b_{a_1}} \sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_c} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_{a_1}} \sum_i \sum_j^{c-1} b_i b_{a_j} + \alpha^{bb_{a_2}} \sum_i b_i b_{a_c} + \alpha^{b_{a_1} b_{a_1}} \sum_{ij}^{c-1} b_{a_i} b_{a_j} + \alpha^{b_{a_1} b_{a_2}} \sum_{ij}^{c-1} b_{a_i} b_{a_c} && (38)
\end{aligned}
$$

$$
\begin{aligned}
b_1 b_2 \cdots b_k \rightarrow \alpha^b \sum_i b_i + \alpha^{b_{a_1}} \sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_c} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_{a_1}} \sum_i \sum_j^{c-1} b_i b_{a_j} + & \quad (39)\\
\alpha^{bb_{a_2}} \sum_i b_i b_{a_c} + \alpha^{b_{a_1} b_{a_1}} \sum_{ij}^{c-1} b_{a_i} b_{a_j} + \alpha^{b_{a_1} b_{a_2}} \sum_{ij}^{c-1} b_{a_i} b_{a_c} & \quad (40)
\end{aligned}
$$

$$
\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & -1 \\ 1 & -2 \\ \frac{1}{2}\left(n - c + n^2 - 2cn + c^2\right) & -(n-c) \end{pmatrix}. \tag{41}
$$

XXI. THEOREM 9 OF BCR

For the symmetric function which is a function of the sum of all n variables, for some huge integer λ such that $\lambda > \max(f)$, we have:

$$f\left(\sum b_i\right) \to \sum_{ij}^{\sqrt{n+1}} f\left((i-1)\left(\lceil\sqrt{n+1}\rceil+1\right)+(j-1)\right) b_{a_i} b_{a_{\sqrt{n+1}+j}} + \tag{42}$$

$$+\lambda\left(\left(1-\sum_i^{\sqrt{n+1}} b_{a_i}\right)^2+\left(1-\sum_i^{\sqrt{n+1}} b_{a_{\sqrt{n+1}+i}}\right)^2+\right. \tag{43}$$

$$\left.+\left(\sum_i b_i-\left(\left(\lceil\sqrt{n+1}\rceil+1\right)\sum_i^{\sqrt{n+1}}(i-1)y_{a_i}+\sum_i^{\sqrt{n+1}}(i-1)b_{a_{\sqrt{n+1}+i}}\right)\right)^2\right) \tag{44}$$

$$=\sum_{ij}^{c} f\left((i-1)(c+1)+(j-1)\right) b_{a_i} b_{a_{c+j}} + \lambda\left(\left(1-\sum_i^c b_{a_i}\right)^2+\left(1-\sum_i^c b_{a_{c+i}}\right)^2+\right. \tag{45}$$

$$\left(\sum_i b_i-\left((c+1)\sum_i^c(i-1)y_{a_i}+\sum_i^c(i-1)b_{a_{c+i}}\right)\right)^2+\left(\sum_i b_i-\left((c+1)\sum_i^c(i-1)y_{a_i}+\sum_i^c(i-1)b_{a_{c+i}}\right)\right)^2\Bigg) \tag{46}$$

XXII. THEOREM 10 OF BCR

Works on a generalization of $f\left(\sum b_i\right)$ but instead we have a weighted sum.

$$f\left(\sum w_i b_i\right) \to \sum_{ij}^{c} \alpha_{ij} b_{a_i} b_{a_{c+i}} + \lambda\left(1+\left(\sum_i w_i b_i-(c-1)\sum_i^c b_{a_i}+\sum_i^c b_{a_{c+i}}\right)^2+\sum_i^{c-1}\left(1-b_{a_i}\right)b_{a_{i+1}}+\sum_i^{c-1}\left(1-b_{a_{i+c}}\right)b_{a_{i+c+}}\right. \tag{47}$$

$$\sum_i^{\alpha}\sum_j^{\beta}\alpha_{ij}=f\left(\alpha(c+1)+\beta\right) \tag{48}$$

Uses only $2c$ auxiliary variables, where $\max\left(\sum w_i b_i\right) < (c+1)^2$