Sub-division gadget of BDLT

Nike Dattani*

Harvard-Smithsonian Center for Astrophysics

It is stated in Oliveira-Terhal that a 5-local Hamiltonian AB can be transformed into:

$$H_{OT} = \Delta\left(\frac{1 - z_a}{2}\right) + \frac{A^2}{2} + \frac{B^2}{2} + \sqrt{\frac{\Delta}{2}}(B - A)x_a, \tag{1}$$

which has the same low-lying eigenspectrum as the original 5-local Hamiltonian to within $\mathcal{O}(\epsilon)$ as long as:

$$\Delta = \frac{\left(\max\left(\|A\|, \|B\|\right)\Omega(\sqrt{2})\right)^6}{\epsilon^2}. \tag{2}$$

When $\|A\|, \|B\| \leq 1$, this becomes:

$$\Delta = \epsilon^{-2}. \tag{3}$$

The same result is obtained in Bravyi-DiVincenzo-Loss-Terhal, except the proof is given by showing that for the S they choose, we have:

$$e^S He^{-S} = \begin{pmatrix} AB & 0 \\ 0 & G \end{pmatrix} \tag{4}$$

$$Pe^S He^{-S} P = AB, \tag{5}$$

as long as $\Delta = \epsilon^{-2}$.

In general the unitary transformation $e^S He^{-S}$ will not preserve the eigenspectrum of H (and hence $Pe^S He^{-S} P$ won't preserve the eigenspectrum of AB) because unitary transformations preserve eigenvalues but not eigenvectors. The eigenvectors of $Pe^S He^{-S} P$ need to be multiplied by e^{-S} to recover the eigenvectors of AB. However it's possible that Eq. 4 holds true to within ϵ if $\Delta = \epsilon^{-2}$, meaning we do not need to multiply by e^{-S} .

* n.dattani@cfa.harvard.edu

Let's calculate Eq. 3 of 10.1103/PhysRevLett.101.070503 explicitly:

$$H_0 = \Delta\left(\frac{1-z_a}{2}\right) \tag{6}$$

$$V = \sqrt{\frac{\Delta J}{2}}(B-A)x_a + \frac{J}{2}\left(A^2+B^2\right) \tag{7}$$

$$S = -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a \tag{8}$$

$$H = H_0 + V \tag{9}$$

$$[S,H] = [S, H_0 + V_1 + V_2] \tag{10}$$

$$= [S,H_0] + [S,V_1] + [S,V_2] \tag{11}$$

$$[S,H_0] = \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \Delta\left(\frac{1-z_a}{2}\right)\right] \tag{12}$$

$$= \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \frac{\Delta}{2}\right] + \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \frac{-\Delta z_a}{2}\right] \tag{13}$$

$$= \left[\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \frac{\Delta z_a}{2}\right] \tag{14}$$

$$= \mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\frac{\Delta z_a}{2} - \frac{\Delta z_a}{2}\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a \tag{15}$$

$$= \mathrm{i}\sqrt{\frac{J\Delta}{2^3}}(B-A)y_a z_a - z_a(B-A)y_a \tag{16}$$

$$= \mathrm{i}\sqrt{\frac{J\Delta}{2^3}}(B-A)y_a z_a - (B-A)z_a y_a \tag{17}$$

$$= \mathrm{i}\sqrt{\frac{J\Delta}{2^3}}(B-A)\left[y_a, z_a\right] \tag{18}$$

$$= \mathrm{i}\sqrt{\frac{J\Delta}{2^3}}(B-A)2\mathrm{i}x_a \tag{19}$$

$$= -\sqrt{\frac{J\Delta}{2}}(B-A)x_a \tag{20}$$

$$
\begin{aligned}
[S,V_1] &= \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \sqrt{\frac{\Delta J}{2}}(B-A)x_a\right] && (21)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\sqrt{\frac{\Delta J}{2}}(B-A)x_a - \sqrt{\Delta J/2}(B-A)x_a\left(-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\right) && (22)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}\sqrt{\frac{\Delta J}{2}}\left((B-A)y_a(B-A)x_a - (B-A)x_a(B-A)y_a\right) && (23)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}\sqrt{\frac{\Delta J}{2}}\left((B-A)^2y_ax_a - (B-A)^2x_ay_a\right) && (24)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}\sqrt{\frac{\Delta J}{2}}(B-A)^2[y_a,x_a] && (25)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}\sqrt{\frac{\Delta J}{2}}(B-A)^2(-2\mathrm{i}z_a) && (26)\\
&= -J(B-A)^2z_a && (27)\\
&= -J(B^2-BA+A^2)z_a, \text{ but } BAz_a \text{ is } (k+1)\text{-local} && (28)\\
[S,V_2] &= \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, \frac{J}{2}\left(A^2+B^2\right)\right] && (29)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\frac{J}{2}\left(A^2+B^2\right) - \frac{J}{2}\left(A^2+B^2\right)\left(-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\right) && (30)\\
&= -\mathrm{i}\sqrt{\frac{J^3}{2^3\Delta}}\left((B-A)\left(A^2+B^2\right)y_a - \left(A^2+B^2\right)(B-A)y_a\right) && (31)\\
&= -\mathrm{i}\sqrt{\frac{J^3}{2^3\Delta}}(B-A)\left(A^2+B^2\right)(y_a-y_a) && (32)\\
&= 0 && (33)\\
[S,H] &= -\sqrt{\frac{J\Delta}{2}}(B-A)x_a - J(B-A)^2z_a && (34)\\
[S,[S,H]] &= \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -\sqrt{\frac{J\Delta}{2}}(B-A)x_a - J(B-A)^2z_a\right] && (35)\\
&= \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -\sqrt{\frac{J\Delta}{2}}(B-A)x_a\right] + \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -J(B-A)^2z_a\right] && (36)\\
\left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -\sqrt{\frac{J\Delta}{2}}(B-A)x_a\right] &= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\left(-\sqrt{\frac{J\Delta}{2}}(B-A)x_a\right) - \left(-\sqrt{\frac{J\Delta}{2}}(B-A)x_a\right)\left(-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\right) && (37)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}\left(-\sqrt{\frac{J\Delta}{2}}\right)\left((B-A)y_a(B-A)x_a - (B-A)x_a(B-A)y_a\right) && (38)\\
&= \mathrm{i}\frac{J(B-A)^2}{2}\left(y_ax_a - x_ay_a\right) && (39)\\
&= \mathrm{i}\frac{J(B-A)^2}{2}[y_a,x_a] && (40)\\
&= \mathrm{i}\frac{J(B-A)^2}{2}\left(-2\mathrm{i}z_a\right) && (41)\\
&= J(B-A)^2z_a && (42)\\
\left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -J(B-A)^2z_a\right] &= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\left(-J(B-A)^2z_a\right) - \left(-J(B-A)^2z_a\right)\left(-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a\right) && (43)\\
&= -\mathrm{i}\sqrt{\frac{J}{2\Delta}}(-J)\left((B-A)y_a(B-A)^2z_a - \left((B-A)^2z_a\right)(B-A)y_a\right) && (44)\\
&= \mathrm{i}\sqrt{\frac{J^3}{2\Delta}}(B-A)^3\left(y_az_a - z_ay_a\right) && (45)\\
&= \mathrm{i}\sqrt{\frac{J^3}{2\Delta}}(B-A)^3[y_a,z_a] && (46)\\
&= \mathrm{i}\sqrt{\frac{J^3}{2\Delta}}(B-A)^3\left(2\mathrm{i}x_a\right) && (47)\\
&= -\sqrt{\frac{2J^3}{\Delta}}(B-A)^3x_a, \text{ contains } A^2Bx_a \text{ and } B^2Ax_a \text{ which both act on } k+1 \text{ qubits.} && (48)
\end{aligned}
$$

$$[S,[S,H]] = \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -\sqrt{\frac{J\Delta}{2}}(B-A)x_a\right] + \left[-\mathrm{i}\sqrt{\frac{J}{2\Delta}}(B-A)y_a, -J(B-A)^2 z_a\right] \tag{49}$$

$$= J(B-A)^2 z_a - \sqrt{\frac{2J^3}{\Delta}}(B-A)^3 x_a \tag{50}$$

$$= J(B-A)^2\left(z_a - \sqrt{\frac{2J}{\Delta}}(B-A)x_a\right) \tag{51}$$

$$[S,H] + \frac{1}{2}[S,[S,H]] = -\sqrt{\frac{J\Delta}{2}}(B-A)x_a - J(B-A)^2 z_a + \frac{J(B-A)^2}{2}\left(z_a - \sqrt{\frac{2J}{\Delta}}(B-A)x_a\right) \tag{52}$$

$$= -\sqrt{\frac{J\Delta}{2}}(B-A)x_a + \left(\frac{J(B-A)^2}{2} - J(B-A)^2\right)z_a - \frac{J(B-A)^2}{2}\left(\sqrt{\frac{2J}{\Delta}}(B-A)x_a\right) \tag{53}$$

$$= -\sqrt{\frac{J\Delta}{2}}(B-A)x_a - \frac{J(B-A)^2}{2}z_a - \frac{J}{2}\sqrt{\frac{2J}{\Delta}}(B-A)^3 x_a \tag{54}$$

$$= -\sqrt{\frac{J\Delta}{2}}(B-A)x_a - \frac{J(B-A)^2}{2}z_a - \sqrt{\frac{J^3}{2\Delta}}(B-A)^3 x_a \tag{55}$$

$$= \sqrt{\frac{J}{2}}(B-A)\left(\Delta x_a + \sqrt{\frac{J}{2}}(B-A)z_a + \sqrt{\frac{J^2}{\Delta}}(B-A)^2 x_a\right) \tag{56}$$

$$= \sqrt{\frac{J}{2}}(B-A)\left(\Delta x_a + \sqrt{\frac{J}{2}}(B-A)z_a\right) + \mathcal{O}\left(\sqrt{\frac{J^3}{\Delta}}\right) \tag{57}$$

$$H + [S,H] + \frac{1}{2}[S,[S,H]] = \Delta\left(\frac{1-z_a}{2}\right) + \sqrt{\frac{\Delta J}{2}}(B-A)x_a + \frac{J}{2}\left(A^2+B^2\right) + \sqrt{\frac{J}{2}}(B-A)\left(\Delta x_a + \sqrt{\frac{J}{2}}(B-A)z_a\right) + \mathcal{O}\left(\sqrt{\frac{J^3}{\Delta}}\right) \tag{58}$$

$$= \Delta\left(\frac{1-z_a}{2}\right) + \frac{J}{2}\left(A^2+B^2\right) + \sqrt{\frac{J}{2}}(B-A)\left(2\Delta x_a + \sqrt{\frac{J}{2}}(B-A)z_a\right) + \mathcal{O}\left(\sqrt{\frac{J^3}{\Delta}}\right) \tag{59}$$

$$= \Delta\left(\frac{1-z_a}{2}\right) + \frac{J}{2}\left(A^2+B^2\right) + \sqrt{2J}(B-A)\Delta x_a + \frac{J}{2}(B-A)^2 z_a + \mathcal{O}\left(\sqrt{\frac{J^3}{\Delta}}\right) \tag{60}$$

$$= \Delta\left(\frac{1-z_a}{2}\right) + \frac{J}{2}\left(A^2+B^2\right) + \sqrt{2J}(B-A)\Delta x_a + \frac{J}{2}(B^2 - BA + A^2)z_a + \mathcal{O}\left(\sqrt{\frac{J^3}{\Delta}}\right) \tag{61}$$