More Reductions

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I. THEOREM 4.1 OF ANTHONY-BOROS-CRAMA-GRUBER (ABCG)

$$f\left(b_1, b_2, \ldots, b_n\right) \rightarrow -\alpha_0 - \alpha_0 \sum_i b_i + 2a_2 \sum_{ij} b_i b_j + 2 \sum_{2i-1} \left(\alpha_{2i-1} - \min\left(\alpha_{2j-1}\right)\right) b_{a_{2i-1}} \left(2i - \frac{3}{2} - \sum_j b_j\right) + \tag{1}$$

$$2 \sum_{2i} \left(\alpha_{2i} - \min\left(\alpha_{2j}\right)\right) b_{a_{2i}} \left(2i - \frac{1}{2} - \sum_j b_j\right) \tag{2}$$

$$\tag{3}$$

$$\alpha_i = -4 \sum_{j=0}^{i} \left(-1\right)^{i-j} f(j) - f(i-1) + 3f(i) \tag{4}$$

II. UNPUBLISHED WORK OF ALEXANDER FIX

Any symmetric fuction can be quadratized with $n-1$ auxiliaries. Add a multiple of $E(\sum b_r)$ to each term of Corollary 2.4 of the above paper.

III. "A RELATED PAPER BY THE PRESENT AUTHORS [1] GIVES A COMPLETE CHARACTERIZATION OF ALL THE QUADRATIZATIONS OF NEGATIVE MONOMIALS INVOLVING ONE AUXILIARY VARIABLE"

IV. COROLLARY 4.4 OF ABCG

Need to conver $[]^-$ into something more readable.

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V. COROLLARY 4.5 OF ABCG

VI. THEOREM 5.6 OF ABCG

VII. THEOREM 1.1 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $n/2 \le m \le n$

=

$$f(b_1, b_2, \ldots, b_n) \to \left(\sum_i b_i - \left(m - 2^{\lceil \log m \rceil}\right) b_{a_{\lceil \log m \rceil + 1}} - (m+1)\left(1 - b_{a_{\lceil \log m \rceil + 1}}\right) - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{5}$$

$$= \left(\sum_i b_i - \left(m - 2^{\lceil \log m \rceil}\right) b_{a_{\lceil \log m \rceil + 1}} - (m+1) + (m+1) b_{a_{\lceil \log m \rceil + 1}} - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{6}$$

$$= \left(-(m+1) + \sum_i b_i - \left(2m - 2^{\lceil \log m \rceil} + 1\right) b_{a_{\lceil \log m \rceil + 1}} - \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{7}$$

$$= (m+1)^2 - 2(m+1)\sum_i b_i + 2(m+1)\left(2m - 2^{\lceil \log m \rceil} + 1\right) b_{a_{\lceil \log m \rceil + 1}} + 2(m+1) \sum_i^{\lceil \log m \rceil} 2^{i-1} b_{a_i} \tag{8}$$

$$+ \sum_{ij} b_i b_j - 2 \sum_i \left(2m - 2^{\lceil \log m \rceil} + 1\right) b_i b_{a_{\lceil \log m \rceil + 1}} - 2 \sum_i \sum_j^{\lceil \log m \rceil} 2^{i-1} b_i b_{a_j} + \sum_{i,j}^{\lceil \log m \rceil} 2^{i+j-2} b_{a_i} b_{a_j} \tag{9}$$

$$= \alpha^I + \alpha^b \sum_i b_i + \alpha^{b_a,1} \sum_i^{\lceil \log m \rceil} b_{a_i} + \alpha^{b_a,2} b_{a_{\lceil \log m \rceil + 1}} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_a,1} \sum_i \sum_j^{\lceil \log m \rceil} b_i b_{a_j} \tag{10}$$

$$+ \alpha^{bb_a,2} \sum_i b_i b_{a_m} + \alpha^{b_a b_a} \sum_{i,j}^{m-1} b_{a_i} b_{a_j} \tag{11}$$

$$= \alpha + \alpha^b \sum_i b_i + \alpha^{b_a,1} \sum_i^{m-1} b_{a_i} + \alpha^{b_a,2} b_{a_m} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_a,1} \sum_i \sum_j^{m-1} b_i b_{a_j} + \alpha^{bb_a,2} \sum_i b_i b_{a_m} + \alpha^{b_a b_a} \sum_{i,j}^{m-1} b_{a_i} b_{a_j} \tag{12}$$

VIII. THEOREM 1.2 OF BOROS-CRAMA-RODRIGUEZHECTOR (BCR)

For any symmetric k-local function that is non-zero (they actually say =1) only if $\sum b_i = m$, if $0 \le m \le n/2$.

$$f(b_1, b_2, \ldots, b_n) \to \left(n - \sum_i b_i - \left(n - m - 2^{\lceil \log(n-m) \rceil}\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - (n-m+1)\left(1 - b_{a_{\lceil \log(n-m) \rceil + 1}}\right) - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{13}$$

$$= \left(n - \sum_i b_i - \left(n - m - 2^{\lceil \log(n-m) \rceil}\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - (n-m+1) + (n-m+1) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{14}$$

$$= \left((m-1) - \sum_i b_i - \left(2(n-m) - 2^{\lceil \log(n-m) \rceil} + 1\right) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i}\right)^2 \tag{15}$$

The number of auxiliary varibales is $\lceil \log(n-m) \rceil + 1$.

$$\begin{pmatrix} \alpha^I & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} \\ \end{pmatrix}. \tag{16}$$

IX. COROLLARY 1 OF BCR

transformation not explicitly given, but the function can be more general than in Theorem 1, but requires a factor of μ more variables.

X. THEOREM 2.1 OF BCR

Once again requires typing out a nasty function [should really be done in mathematica rather than by hand)

$$f\left(b_1, b_2, \ldots, b_n\right) \to \frac{1}{2}\left(\sum_i b_i - \left(m-2^c\right) b_{a_{c+1}} - (m+1)((1-b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i}\right)\left(\sum_i b_i - \left(m-2^c\right) b_{a_{c+1}} - (m+1)((1-b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i} - 1\right) \tag{17}$$

XI. THEOREM 2.2 OF BCR

$$f\left(b_1, b_2, \ldots, b_n\right) \to \frac{1}{2}\left((c-1) - \sum_i b_i - \left(2(n-c) - 2^{m-1} + 1\right) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i}\right)\left((c-1) - \sum_i b_i - \left(2(n-c) - 2^{m-1} + 1\right) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} - 1\right) \tag{18}$$

XII. THEOREM 7 OF BCR (PTR-BCR)

$$b_1b_2\cdots b_k \to \frac{1}{2}\left(\sum_i b_i - 2\sum_i^{\lceil\frac{k}{4}\rceil-1} b_{a_i} - \left(n-\lceil\frac{k}{4}\rceil\right) b_{a_{\lceil\frac{k}{4}\rceil}}\right)\left(\sum_i b_i - 2\sum_i^{\lceil\frac{k}{4}\rceil-1} b_{a_i} - \left(n-\lceil\frac{k}{4}\rceil\right) b_{a_{\lceil\frac{k}{4}\rceil}} - 1\right) \tag{19}$$

$$= \frac{1}{2}\Bigg(\sum_{ij} b_ib_j - 4\sum_i\sum_j^{\lceil\frac{k}{4}\rceil-1} b_ib_{a_j} - 2\left(n-\lceil\frac{k}{4}\rceil\right)\sum_i b_ib_{a_{\lceil\frac{k}{4}\rceil}} - \sum_i b_i + 4\sum_{ij}^{\lceil\frac{k}{4}\rceil-1} b_{a_i}b_{a_j} + 4\left(n-\lceil\frac{k}{4}\rceil\right)\sum_i^{\lceil\frac{k}{4}\rceil-1} b_{a_i}b_{a_{\lceil\frac{k}{4}\rceil}} + 2\sum_i^{\lceil\frac{k}{4}\rceil-1} b_{a_i} + \left(n-\lceil\frac{k}{4}\rceil\right)^2 b_{a_{\lceil\frac{k}{4}\rceil}} + \Big(\tag{20}$$

$$= \alpha + \alpha^b\sum_i b_i + \alpha^{b_{a_1}}\sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_{\lceil\frac{k}{4}\rceil}} + \alpha^{bb}\sum_{ij} b_ib_j + \alpha^{bb_{a_1}}\sum_i\sum_j^{\lceil\frac{k}{4}\rceil-1} b_ib_{a_j} + \alpha^{bb_{a_2}}\sum_i b_ib_{a_{\lceil\frac{k}{4}\rceil}} + \alpha^{b_{a_1}b_{a_1}}\sum_{ij}^{\lceil\frac{k}{4}\rceil-1} b_{a_i}b_{a_j} + \alpha^{b_{a_1}b_{a_2}}\sum_{ij}^{\lceil\frac{k}{4}\rceil-1} b_{a_i}b_{a_{\lceil\frac{k}{4}\rceil}} \tag{21}$$

$$= \alpha + \alpha^b\sum_i b_i + \alpha^{b_{a_1}}\sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_c} + \alpha^{bb}\sum_{ij} b_ib_j + \alpha^{bb_{a_1}}\sum_i\sum_j^{c-1} b_ib_{a_j} + \alpha^{bb_{a_2}}\sum_i b_ib_{a_c} + \alpha^{b_{a_1}b_{a_1}}\sum_{ij}^{c-1} b_{a_i}b_{a_j} + \alpha^{b_{a_1}b_{a_2}}\sum_{ij}^{c-1} b_{a_i}b_{a_c} \tag{22}$$

$$b_1b_2\cdots b_k \to \alpha^b\sum_i b_i + \alpha^{b_{a_1}}\sum_i b_{a_i} + \alpha^{b_{a_2}} b_{a_c} + \alpha^{bb}\sum_{ij} b_ib_j + \alpha^{bb_{a_1}}\sum_i\sum_j^{c-1} b_ib_{a_j} + \tag{23}$$

$$\alpha^{bb_{a_2}}\sum_i b_ib_{a_c} + \alpha^{b_{a_1}b_{a_1}}\sum_{ij}^{c-1} b_{a_i}b_{a_j} + \alpha^{b_{a_1}b_{a_2}}\sum_{ij}^{c-1} b_{a_i}b_{a_c} \tag{24}$$

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_ab_a} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & -1 \\ 1 & -2 \\ \frac{1}{2}(n-c+n^2-2cn+c^2) & -(n-c) \end{pmatrix}. \tag{25}$$

XIII. THEOREM 9 OF BCR

For the symmetric function which is a function of the sum of all n variables, for some huge integer λ such that $\lambda > \max(f)$, we have:

$$f\left(\sum b_i\right) \to \sum_{ij}^{\sqrt{n+1}} f\left((i-1)\left(\lceil\sqrt{n+1}\rceil+1\right)+(j-1)\right) b_{a_i} b_{a_{\sqrt{n+1}+j}} + \tag{26}$$

$$+\lambda\left(\left(1-\sum_i^{\sqrt{n+1}} b_{a_i}\right)^2+\left(1-\sum_i^{\sqrt{n+1}} b_{a_{\sqrt{n+1}+i}}\right)^2+\right. \tag{27}$$

$$\left.+\left(\sum_i b_i-\left(\left(\lceil\sqrt{n+1}\rceil+1\right)\sum_i^{\sqrt{n+1}}(i-1)y_{a_i}+\sum_i^{\sqrt{n+1}}(i-1)b_{a_{\sqrt{n+1}+i}}\right)\right)^2\right) \tag{28}$$

$$=\sum_{ij}^{c} f\left((i-1)(c+1)+(j-1)\right) b_{a_i} b_{a_{c+j}} + \lambda\left(\left(1-\sum_i^c b_{a_i}\right)^2+\left(1-\sum_i^c b_{a_{c+i}}\right)^2+\right. \tag{29}$$

$$\left.\left(\sum_i b_i-\left((c+1)\sum_i^c(i-1)y_{a_i}+\sum_i^c(i-1)b_{a_{c+i}}\right)\right)^2+\left(\sum_i b_i-\left((c+1)\sum_i^c(i-1)y_{a_i}+\sum_i^c(i-1)b_{a_{c+i}}\right)\right)^2\right) \tag{30}$$

$$\sum_{ij}^{m} f\left((i-1)(m+1)+(j-1)\right) b_{a_i} b_{a_{c+j}} + \lambda\left(\left(1-\sum_i^m b_{a_i}\right)^2+\left(1-\sum_i^m b_{a_{c+i}}\right)^2+\right. \tag{31}$$

$$\left.\left(\sum_i b_i-\left((m+1)\sum_i^m(i-1)y_{a_i}+\sum_i^m(i-1)b_{a_{c+i}}\right)\right)^2+\left(\sum_i b_i-\left((m+1)\sum_i^m(i-1)y_{a_i}+\sum_i^m(i-1)b_{a_{c+i}}\right)\right)^2\right) \tag{32}$$

XIV. SOME MORE BY ABCG IN DIFFERENT PAPER

https://orbi.uliege.be/bitstream/2268/184526/1/Quadratization_Revision%20April2016.pdf