

# 1 SFR-BCR

Binary variables:  $x_1, x_2, \dots, x_n \in \{0, 1\}$ , and let  $X := \sum_{i=1}^n x_i$ ,  $2 \leq k \in \mathbb{N}$ ,  $l = \lceil \log k \rceil$

## 1.1 SFR-BCR-1,2

We have the  $l + 1$  auxiliary variables  $y_0, y_1, \dots, y_{l-1}, z \in \{0, 1\}$ . (In the formulas, summations for  $i, i' = 1, 2, \dots, n$  and  $j, j' = 0, 1, \dots, l - 1$  are understood.)

We have the functions

$$A_k(X, y, z) = X - (k - 2^l)z - (k + 1)(1 - z) - \sum_j 2^j y_j. \quad \text{eq: Ak (1)}$$

After substituting for  $X$ , and collecting the terms, the equation for SFR-BCR-1 becomes

$$A_k(X, y, z) = -(1 + k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \quad \text{eq: Ak1 (2)}$$

Similarly for SFR-BCR-2 we have

$$A_{n-k}(n - X, y, z) = -(1 - k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \quad \text{eq: Ak2 (3)}$$

For the two cases (top/bottom = SFR-BCR-1/SFR-BCR-2), by recognizing their similarities we can simplify the equation to

$$\left. \begin{array}{l} A_k(X, y, z) \\ A_{n-k}(n - X, y, z) \end{array} \right\} = -(1 \pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l)z. \quad \text{eq: Ak12 (4)}$$

Then the squares are

$$\left. \begin{array}{l} A_k(X, y, z)^2 \\ A_{n-k}(n - X, y, z)^2 \end{array} \right\} = (1 \pm k)^2 \mp 2(1 \pm k) \sum_i x_i + 2(1 \pm k) \sum_j 2^j y_j - 2(1 \pm k)(1 + 2^l)z \\ + \sum_i \sum_{i'} x_i x_{i'} \mp 2 \sum_i \sum_j 2^j x_i y_j \pm 2(1 + 2^l) \sum_i z x_i \\ + \sum_j \sum_{j'} 2^{j+j'} y_j y_{j'} - 2(1 + 2^l) \sum_j z 2^j y_j \\ + (1 + 2^l)^2 z^2. \quad \text{eq: Ak12sq (5)}$$

Because  $z \in \{0, 1\}$ , we have  $z^2 = z$ , so we can join the two terms,

$$- 2(1 \pm k)(1 + 2^l)z + (1 + 2^l)^2 z^2 = (1 + 2^l)(2^l \mp 2k - 1)z, \quad (6)$$

so we end up with

$$\begin{aligned}
& \left. \begin{aligned} & A_k(X, y, z)^2 \\ & A_{n-k}(n - X, y, z)^2 \end{aligned} \right\} = \\
& \underbrace{(1 \pm k)^2}_{\alpha} + \underbrace{\mp 2(1 \pm k)}_{\alpha^b} \sum_i x_i + \sum_j \underbrace{(1 \pm k)2^{j+1}}_{\alpha^{b_{a,1}}} y_j + \underbrace{(1 + 2^l)(2^l \mp 2k - 1)}_{\alpha^{b_{a,2}}} z \\
& + \underbrace{1}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \sum_{i,j} \underbrace{\mp 2^{j+1}}_{\alpha^{bb_{a,1}}} x_i y_j + \underbrace{\pm 2(1 + 2^l)}_{\alpha^{bb_{a,2}}} \sum_i x_i z \\
& + \sum_{j,j'} \underbrace{2^{j+j'}}_{\alpha^{b_a b_{a,1}}} y_j y_{j'} + \sum_j \underbrace{-(1 + 2^l)2^{j+1}}_{\alpha^{b_a b_{a,2}}} y_j z.
\end{aligned}$$

eq:alpha12  
(7)

The coefficients are | eq:BCR12alpha

$$\alpha = (1 \pm k)^2, \quad (8a)$$

$$\alpha^b = \mp 2(1 \pm k), \quad (8b)$$

$$\alpha^{b_{a,1}} = (1 \pm k)2^{j+1}, \quad (8c)$$

$$\alpha^{b_{a,2}} = (1 + 2^l)(2^l \mp 2k - 1), \quad (8d)$$

$$\alpha^{bb} = 1, \quad (8e)$$

$$\alpha^{bb_{a,1}} = \mp 2^{j+1}, \quad (8f)$$

$$\alpha^{bb_{a,2}} = \pm 2(1 + 2^l), \quad (8g)$$

$$\alpha^{b_a b_{a,1}} = 2^{j+j'}, \quad (8h)$$

$$\alpha^{b_a b_{a,2}} = -(1 + 2^l)2^{j+1}. \quad (8i)$$

Dictionary:

$l \mapsto m - 1$ , ( $l + 1 = m$  auxiliary variables),

$x_i \mapsto b_i$ ,

$y_j \mapsto b_{a_j}$ , ( $b_{a,j}$  would be a better choice,  $a$  is a label,  $j$  is an index, they should be on the same level. Also the indexing of the  $\alpha^{\dots}$  coefficients could be made more expressive.)

$z \mapsto b_{a_m}$ , ( $b_{a,m}$  would be better)

$k \mapsto c$ .

Note that, in this case, the  $j$  indices of the auxiliary bits have to be shifted, since they are ranging from 1, not 0. (This is not the case in the next subsection.)

## 1.2 SFR-BCR-3,4

We have now the  $l$  auxiliary variables  $y_1, y_2, \dots, y_{l-1}, z \in \{0, 1\}$ . (In the formulas, summations for  $i, i' = 1, 2, \dots, n$  and  $j, j' = 1, 2, \dots, l - 1$  are understood.)

We have the functions

$$A'_k(X, y, z) = X - (k - 2^l)z - (k + 1)(1 - z) - \sum_j 2^j y_j. \quad (9)$$

eq:Akp

Note that, compared to (1), the difference is only in the range of index  $j$  of the sum in the last term. For the two cases (top/bottom = SFR-BCR-3/SFR-BCR-4), after substituting and collecting the terms,

$$\left. \begin{array}{l} A'_k(X, y, z) \\ A'_{n-k}(n - X, y, z) \end{array} \right\} = -(1 \pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l)z. \quad \text{eq: Akr34} \quad (10)$$

(Again, although not written out explicitly, the difference is in the range of  $j$  of the summation, c.f., (4))

We can obtain the  $\alpha^{\dots}$  coefficients for BCR-3,4 from those of BCR-1,2. Instead of taking the squares,  $A_k(X, y, z)^2$  and  $A_{n-k}(n - X, y, z)^2$ , for BCR-3,4 we have to take  $\frac{1}{2}A'_k(X, y, z)(A'_k(X, y, z) - 1) = \frac{1}{2}(A'_k(X, y, z)^2 - A'_k(X, y, z))$  and  $\frac{1}{2}A'_{n-k}(n - X, y, z)(A'_{n-k}(n - X, y, z) - 1) = \frac{1}{2}(A'_{n-k}(n - X, y, z)^2 - A'_{n-k}(n - X, y, z))$ , so, to get the new  $\alpha^{\dots}$  coefficients, we have to subtract the corresponding coefficients of  $A'_k(X, y, z)$  and  $A'_{n-k}(n - X, y, z)$  (these can be read off from (10)) from the old ones (8), and divide by 2. (And not to forget that the summations for  $j$  run over a different range). eq: BCR34alpha

$$\alpha = \frac{1}{2}((1 \pm k)^2 - (1 \pm k)) = \frac{1}{2}(k^2 \pm 3k + 2), \quad (11a)$$

$$\alpha^b = \frac{1}{2}(\mp 2(1 \pm k) - \pm 1) = -k \mp \frac{3}{2}, \quad (11b)$$

$$\alpha^{b_{a,1}} = \frac{1}{2}((1 \pm k)2^{j+1} - 2^j) = (3 \pm k)2^{j-1}, \quad (11c)$$

$$\alpha^{b_{a,2}} = \frac{1}{2}((1 + 2^l)(2^l \mp 2k - 1) - (1 + 2^l)) = (1 + 2^l)(2^{l-1} \mp k - 1), \quad (11d)$$

$$\alpha^{bb} = \frac{1}{2}(1) = \frac{1}{2}, \quad (11e)$$

$$\alpha^{bb_{a,1}} = \frac{1}{2}(\mp 2^{j+1}) = \mp 2^j, \quad (11f)$$

$$\alpha^{bb_{a,2}} = \frac{1}{2}(\pm 2(1 + 2^l)) = \pm(1 + 2^l), \quad (11g)$$

$$\alpha^{b_a b_{a,1}} = \frac{1}{2}(2^{j+j'}) = 2^{j+j'-1}, \quad (11h)$$

$$\alpha^{b_a b_{a,2}} = \frac{1}{2}(-(1 + 2^l)2^{j+1}) = -(1 + 2^l)2^j. \quad (11i)$$

Dictionary:

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$z \mapsto b_{a,m}$ , ( $b_{a,m}$  would be better)

$k \mapsto c$ .