

# Volume 3: List of Multi-run Quadratizations

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## DECOMPOSITION OF A MONOMIAL

$$b_1 b_2 b_3 \dots b_k = \min (b_1 b_2 \dots b_{k_1}, b_{k_1+1} b_{k_1+2} \dots b_{k_2}, b_{k_2+1} b_{k_2+2} \dots b_{k_3}, \dots, b_{k_n+1} b_{k_n+2} \dots b_k) \quad (1)$$

$$b_1 b_2 b_3 \dots b_k = \min (b_1, b_2, b_3, \dots, b_k) \quad (\text{Example of Eq. 1: Linearization of a degree-}k \text{ monomial}). \quad (2)$$

$$b_1 b_2 b_3 b_4 = \min (b_1 b_2, b_3 b_4) \quad (\text{Example of Eq. 1: Quadratization of a degree-4 monomial}). \quad (3)$$

$$b_1 b_2 b_3 b_4 b_5 b_6 b_7 b_8 : \quad (4)$$

$$\longrightarrow 3b_a + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 - 2b_a(b_1 + b_2 + b_3 + b_4) \quad (5)$$

$$\longrightarrow 3b_a + b_5 b_6 + b_5 b_7 + b_5 b_8 + b_6 b_7 + b_6 b_8 + b_7 b_8 - 2b_a(b_5 + b_6 + b_7 + b_8) \quad (6)$$

## DECOMPOSITION OF BINOMIALS OF DEGREE- $k$ TERMS

$$b_1 b_2 b_3 b_4 + b_3 b_4 b_5 b_6 = \min (b_2 b_3 + b_3 b_6, b_1 b_4 + b_4 b_5, b_1 b_2 + b_5 b_6 - b_3 - b_4 + 2) \quad (k, n) = (4, 6). \quad (7)$$

$$b_1 b_2 b_3 b_4 + b_3 b_4 b_5 b_6 = \min_{b_a} (b_2 b_3 + b_a(1 - b_2 - b_3 + 2b_4) + b_3 b_4, b_1 b_2 + b_5 b_6 + b_5 b_a) \quad (k, n) = (4, 6). \quad (8)$$

$$b_1 b_2 b_3 b_4 + b_4 b_5 b_6 b_7 : \quad (k, n) = (4, 7). \quad (9)$$

$$\longrightarrow b_2 b_3 + b_5 b_6 + b_a(1 - b_5 - b_6 + b_7) \quad (10)$$

$$\longrightarrow b_1 b_4 + b_4 + b_a \quad (11)$$

$$\longrightarrow b_5 b_6 + b_1 + b_a(1 - b_5 - b_6 + b_7) \quad (12)$$

$$b_1 b_2 b_3 b_4 b_5 + b_3 b_4 b_5 b_6 b_7 : \quad (k, n) = (5, 7). \quad (13)$$

$$\longrightarrow b_2 b_5 + b_5 b_6 + b_5 b_7 + b_6 b_7 + b_a(b_5 + b_6 + b_7 - 1) - b_5 - b_6 - b_7 + 1 \quad (14)$$

$$\longrightarrow b_1 b_3 + b_3 b_7 + b_a(1 + b_5 - b_7) - b_5 + 1 \quad (15)$$

$$\longrightarrow b_1 b_4 + b_4 b_6 - b_5 b_6 + b_5 b_a - b_5 + b_6 + 1 \quad (16)$$

$$b_1 b_2 b_3 b_4 b_5 b_6 + b_2 b_3 b_4 b_5 b_6 b_7 : \quad (k, n) = (6, 7). \quad (17)$$

$$\longrightarrow 2b_3 b_6 \quad (18)$$

$$\longrightarrow 2b_4 b_5 - b_5 b_6 + b_5 \quad (19)$$

$$\longrightarrow b_1 b_4 - b_2 b_5 + b_2 b_6 + b_2 b_7 + b_5 b_7 - b_6 b_7 - b_5 - b_6 + 2 \quad (20)$$

$$\longrightarrow b_1 b_2 - b_1 b_5 + b_1 b_7 + b_2 b_3 + b_3 b_6 - b_3 b_7 - b_4 b_5 - b_5 b_6 - b_3 + b_5 + 2 \quad (21)$$

$$b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 : \quad (k, n) = (6, 7). \quad (22)$$

$$\longrightarrow b_5b_6 + b_5b_7 - b_5b_8 + b_6b_8 - b_7b_8 + b_8 \quad (23)$$

$$\longrightarrow b_1b_4 + b_a(b_4 - b_7) + b_7 \quad (24)$$

$$\longrightarrow b_2b_3 + b_2b_7 - b_5b_6 - b_7b_a + b_5 + b_7 \quad (25)$$

$$\longrightarrow b_2b_3 + b_7b_a + b_3 \quad (26)$$

$$b_1b_2b_3b_4b_5 + b_4b_5b_6b_7b_8 : \quad (k, n) = (5, 8). \quad (27)$$

$$\longrightarrow b_1b_2 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad (28)$$

$$\longrightarrow b_3b_5 + b_8b_a + b_5 \quad (29)$$

$$\longrightarrow b_4b_7 + b_4 \quad (30)$$

$$\longrightarrow b_7b_8 + b_a(1 + b_6 - b_7 - b_8) + b_3 \quad (31)$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (6, 8). \quad (32)$$

$$\longrightarrow b_1b_6 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad (33)$$

$$\longrightarrow b_2b_5 + b_4b_5 + b_4b_a \quad (34)$$

$$\longrightarrow b_3b_4 + b_3b_7 - b_a + 1 \quad (35)$$

$$\longrightarrow b_2b_4 + b_7b_8 \quad (36)$$

$$\longrightarrow b_3b_4 + b_4 \quad (37)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 : \quad (k, n) = (7, 8). \quad (38)$$

$$\longrightarrow b_6b_7 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad (39)$$

$$\longrightarrow b_2b_3 + b_3b_4 \quad (40)$$

$$\longrightarrow b_1b_4 + b_4b_8 - b_6b_a + b_6 \quad (41)$$

$$\longrightarrow b_2b_3 + b_2 \quad (42)$$

$$\longrightarrow b_1b_5 + b_3b_5 + b_6b_a \quad (43)$$

$$b_1b_2b_3b_4 + b_5b_6b_7b_8 : \quad (k, n) = (4, 8). \quad (44)$$

$$\longrightarrow b_1b_2 + b_6b_8 + b_a(1 - b_6 + b_7 - b_8) \quad (45)$$

$$\longrightarrow b_3b_4 + b_6b_8 + 2b_8b_a \quad (46)$$

$$\longrightarrow b_2b_3 + b_5b_7 + b_a(1 - b_6 + b_7) \quad (47)$$

$$\longrightarrow b_1b_4 + b_5b_7 - b_6b_8 + b_7b_a + b_6 \quad (48)$$

$$b_1b_2b_3b_4b_5 + b_6b_7b_8b_9b_{10} : \quad (k, n) = (5, 10). \quad (49)$$

$$\longrightarrow b_2b_3 + b_6b_9 + b_9b_a \quad (50)$$

$$\longrightarrow b_1b_4 + b_8b_{10} + b_9b_a \quad (51)$$

$$\longrightarrow b_3b_5 + b_7b_{10} + b_1b_a + b_9b_a \quad (52)$$

$$\longrightarrow b_4b_5 + b_6b_9 + b_9b_a \quad (53)$$

$$\longrightarrow b_1b_2 + b_7b_9 + b_9b_a \quad (54)$$

$$\longrightarrow b_2b_5 + b_6b_8 + b_9b_a \quad (55)$$

$$\longrightarrow b_2b_3 + b_8b_{10} + b_9b_a \quad (56)$$

$$\longrightarrow b_1b_4 + b_6b_7 + b_9b_a \quad (57)$$

$$b_1b_2b_3b_4b_5b_6 + b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (6, 10). \quad (58)$$

$$\longrightarrow b_5b_6 + b_5b_7 + b_a(1 - b_{10}) \quad (59)$$

$$\longrightarrow b_1b_2 + b_8b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (60)$$

$$\longrightarrow b_3b_4 + b_a(1 + b_7 - b_9 - b_{10}) + b_{10} \quad (61)$$

$$\longrightarrow b_3b_6 - b_5b_{10} + b_6b_7 + b_a(1 - b_9) + b_{10} \quad (62)$$

$$\longrightarrow b_1b_2 + b_7b_{10} + b_a(1 - b_9 - b_{10}) + b_{10} \quad (63)$$

$$\longrightarrow b_3b_4 + b_8b_9 + b_a(1 - b_9 - b_{10} - b_2) - b_2 + b_9 + b_{10} + 1 \quad (64)$$

$$b_1b_2b_3b_4b_5b_6b_7 + b_4b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (7, 10). \quad (65)$$

$$\longrightarrow b_4b_7 + b_6b_7 + b_a(1 - b_4 - b_7 + b_{10}) \quad (66)$$

$$\longrightarrow b_2b_5 + b_5b_9 + b_a \quad (67)$$

$$\longrightarrow b_1b_4 + b_4b_8 + b_a \quad (68)$$

$$\longrightarrow b_1b_3 + b_6b_{10} \quad (69)$$

$$\longrightarrow b_3b_6 + b_6b_9 + b_a \quad (70)$$

$$\longrightarrow b_2b_3 + b_8b_{10} \quad (71)$$

$$\longrightarrow b_1b_4 + b_9 \quad (72)$$

$$b_1b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (8, 10). \quad (73)$$

$$\longrightarrow b_2b_4 + b_4b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (74)$$

$$\longrightarrow b_1b_7 + b_7b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (75)$$

$$\longrightarrow b_5b_8 + b_6b_8 - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (76)$$

$$\longrightarrow b_3b_6 + b_a(b_3 - b_{10}) + b_{10} \quad (77)$$

$$\longrightarrow b_1b_5 + b_5b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (78)$$

$$\longrightarrow b_6b_9 - b_a(b_9 + b_{10}) + b_6 + b_9 + 1 \quad (79)$$

$$\longrightarrow b_1b_2 - b_9b_a + b_{10} + 1 \quad (80)$$

$$b_1b_2b_3b_4b_5b_6b_7b_8b_9 + b_2b_3b_4b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (9, 10). \quad (81)$$

$$\longrightarrow b_1b_6 + b_6b_{10} + b_9b_a \quad (82)$$

$$\longrightarrow b_4b_7 + b_7b_8 \quad (83)$$

$$\longrightarrow b_4b_9 + b_a(b_9 - b_4) + b_4 \quad (84)$$

$$\longrightarrow b_2b_3 + b_3b_8 + b_9b_a \quad (85)$$

$$\longrightarrow b_1b_5 + b_2b_5 + b_9b_a \quad (86)$$

$$\longrightarrow b_2b_8 - b_6b_7 + b_8b_{10} + b_9b_a + 1 \quad (87)$$

$$\longrightarrow b_2b_{10} + b_2 \quad (88)$$

**DECOMPOSITION OF DEGREE- $k$ , EXACT- $k$ -OF- $n$  TRINOMIALS**

$$\begin{aligned}
 & b_1b_2b_3b_4 + b_2b_3b_4b_5 + b_3b_4b_5b_6 : & (k, n) = (4, 6). \quad (89) \\
 \longrightarrow & 3b_3b_4 + b_3b_5 + b_4b_5 - b_3 - b_4 - b_5 + 1 & 41/64 \text{ (64\%)} \quad (90) \\
 \longrightarrow & b_1b_4 + b_3b_5 + b_4b_5 & 56/64 \text{ (88\%)} \quad (91) \\
 \longrightarrow & b_1b_2 + b_2b_6 + b_3b_5 + b_5b_6 + b_2 - b_3 - b_4 - b_5 + 2 & 64/64 \text{ (100\%)} \quad (92)
 \end{aligned}$$

$$\begin{aligned}
 & b_1b_2b_3b_4 + b_3b_4b_5b_6 + b_5b_6b_7b_8 : & (k, n) = (4, 8). \quad (93) \\
 \longrightarrow & b_1b_4 + 2b_5b_6 & 159/256 \text{ (62\%)} \quad (94) \\
 \longrightarrow & b_2b_3 + b_3b_5 + b_7b_8 & 225/256 \text{ (88\%)} \quad (95) \\
 \longrightarrow & b_1b_4 + b_3b_4 - b_5b_7 + b_6b_7 + b_7b_8 - b_6 + 1 & 244/256 \text{ (95.3\%)} \quad (96) \\
 \longrightarrow & b_2b_3 + b_6b_8 + b_6 & 253/256 \text{ (98.8\%)} \quad (97) \\
 \longrightarrow & b_2b_3 + b_5b_7 + b_5 & 256/256 \text{ (100\%)} \quad (98)
 \end{aligned}$$

$$\begin{aligned}
 & b_1b_2b_3b_4b_5 + b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7 : & (k, n) = (5, 7). \quad (99) \\
 \longrightarrow & b_1b_5 + 2b_5b_6 & (100) \\
 \longrightarrow & b_2b_4 + b_2b_7 + b_3b_4 + b_6b_7 + b_a(b_6 + b_7 - 1) - b_6 - b_7 + 1 & (101) \\
 \longrightarrow & b_1b_3 - b_2b_3 - b_2b_4 - b_2b_6 + b_a(-b_2 + b_4 - b_5 + b_6 - 1) & (102) \\
 & + b_3b_5 + b_3b_7 + b_4b_5 + b_5b_7 + b_6b_7 + b_3 - b_4 - b_5 - b_6 - 2b_7 + 5 & (103) \\
 \longrightarrow & b_2b_3 + b_2b_6 - b_4b_5 + b_5b_6 + b_6b_7 + b_a(2b_6 + b_7) - b_6 + 1 & (104) \\
 \longrightarrow & b_1b_4 + b_4b_5 - b_5b_7 + b_a(b_5 - 2b_6 - 1) - b_2 + b_4 + 2b_6 + 2 & (105)
 \end{aligned}$$

$$\begin{aligned}
 & b_1b_2b_3b_4b_5b_6 + b_2b_3b_4b_5b_6b_7 + b_3b_4b_5b_6b_7b_8 : & (k, n) = (6, 8). \quad (106) \\
 \longrightarrow & b_1b_3 + b_3b_5 + b_3b_8 + b_a(1 + b_6 - b_7) & (107) \\
 \longrightarrow & b_2b_6 + b_6b_7 + b_a(-b_6 + b_7) + b_6 & (108) \\
 \longrightarrow & b_1b_5 - b_3b_4 + b_4b_5 + b_5b_6 + b_4 & (109) \\
 \longrightarrow & -b_1b_3 + b_1b_6 - b_1b_7 + b_2b_4 - b_3b_7 - b_3b_8 + b_4b_5 + b_4b_6 - b_4b_7 + b_4b_8 & (110) \\
 & + b_5b_8 - b_6b_8 + b_7b_8 + b_a(b_2 - b_4 + b_7 + b_8) + b_1 - b_5 - b_6 + 3 & (111) \\
 \longrightarrow & b_1b_4 + b_6b_7 + b_7b_8 & (112) \\
 \longrightarrow & b_2b_7 + b_7b_8 + b_8b_a + b_2 & (113)
 \end{aligned}$$

$$\begin{aligned}
 & b_1b_2b_3b_4b_5b_6b_7b_8 + b_2b_3b_4b_5b_6b_7b_8b_9 + b_3b_4b_5b_6b_7b_8b_9b_{10} : & (k, n) = (8, 10). \quad (114) \\
 \longrightarrow & b_3b_5 + b_5b_9 + b_5b_{10} & (115) \\
 \longrightarrow & b_1b_4 + b_4b_7 + b_4b_9 & (116) \\
 \longrightarrow & b_1b_6 + b_2b_6 - b_5b_6 + b_6b_{10} + b_6 & (117) \\
 \longrightarrow & b_4b_8 + 2b_8 & (118) \\
 \longrightarrow & b_1b_7 + b_2b_7 - b_a(b_4 + b_6) + b_7b_9 + b_4 + 1 & (119) \\
 \longrightarrow & b_2b_3 + b_3b_5 + b_3 & (120) \\
 \longrightarrow & b_1b_2 + b_2b_9 - b_5b_6 - b_5b_a + b_6b_9 + 2 & (121) \\
 \longrightarrow & b_2b_7 + b_2 + b_{10} & (122)
 \end{aligned}$$

$$b_1b_2b_3b_4b_5b_6 + b_3b_4b_5b_6b_7b_8 + b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (6, 10). \quad (123)$$

$$\longrightarrow b_2b_5 + b_5b_6 + b_5b_9 - b_9b_{11} + b_9 \quad (124)$$

$$\longrightarrow b_1b_4 + b_4b_6 + b_6b_{10} - b_9b_{11} + b_9 - 2b_{11} + 2 \quad (125)$$

$$\longrightarrow b_3b_4 + b_3b_7 + b_9b_{10} - b_9b_{11} + b_9 \quad (126)$$

$$\longrightarrow b_2b_3 + b_8b_{11} - b_9b_{11} + b_8 + b_9 \quad (127)$$

$$\longrightarrow b_1b_6 + b_6b_7 - b_9b_{11} + b_{10}b_{11} + b_9 \quad (128)$$

$$\longrightarrow b_1b_3 + b_7b_{10} - b_9b_{11} + b_7 + b_9 \quad (129)$$

$$\longrightarrow b_2b_4 + b_4b_7 - b_9b_{11} + 2b_{10}b_{11} + b_9 \quad (130)$$

$$\longrightarrow b_2b_6 + b_6b_7 + b_6b_8 - b_9b_{11} + b_9 \quad (131)$$

$$\longrightarrow b_1b_4 - b_2b_5 - b_3b_{10} + b_5b_8 - b_7b_{11} + b_8b_{10} - b_9b_{11} + b_{10}b_{11} + b_9 + 2 \quad 1020/1024 \text{ (99.6\%)} \quad (132)$$

$$b_1b_2b_3b_4 + b_4b_5b_6b_7 + b_7b_8b_9b_{10} : \quad (k, n) = (4, 10). \quad (133)$$

$$\longrightarrow b_3b_4 + b_4b_5 + b_8b_{10} + b_a(1 - b_9) \quad (134)$$

$$\longrightarrow b_1b_2 + b_a(1 + b_3 - b_9) + b_5b_6 + b_7b_9 \quad (135)$$

$$\longrightarrow b_1b_3 + b_6b_7 + b_7b_{10} + b_a \quad (136)$$

$$\longrightarrow b_4b_6 + b_7b_9 + b_a(1 - b_9) + b_4 \quad (137)$$

$$\longrightarrow b_2b_3 + b_5b_7 + b_8b_9 + b_a(1 - b_9) \quad (138)$$

$$\longrightarrow b_2b_3 + b_8b_{10} + b_a(1 - b_9) + b_6 - b_9 + 1 \quad (139)$$

$$\longrightarrow b_1b_2 + b_5b_7 + b_8b_{10} + b_a(1 - b_9) - b_3 + 1 \quad (140)$$

$$\longrightarrow b_5b_6 + b_a(1 - b_9) + b_3 + b_9 \quad (141)$$

$$\longrightarrow b_7b_8 + b_2 + b_7 \quad 1023/1024 \text{ (99.9\%)} \quad (142)$$

$$b_1b_2b_3b_4b_5 + b_3b_4b_5b_6 + b_4b_5b_6b_7b_8 : \quad (k, n) = (5, 8). \quad (143)$$

$$\longrightarrow b_2b_4 + b_4b_6 + b_4b_7 + b_a(b_7 + b_8) \quad (144)$$

$$\longrightarrow b_1b_3 + b_3b_6 + b_6b_7 \quad (145)$$

$$\longrightarrow b_3b_5 + b_4b_5 - b_6b_8 + b_5 + b_8 + b_9(1 + b_7) \quad (146)$$

$$\longrightarrow b_2b_5 + b_6b_8 + b_6 \quad (147)$$

$$\longrightarrow b_1b_3 + b_3 + b_8 \quad (148)$$

## DECOMPOSITION OF DEGREE- $k$ , EXACT- $k$ -OF- $n$ QUADRINOMIALS

$$b_1b_2b_3b_4b_5b_6b_7 + b_2b_3b_4b_5b_6b_7b_8 + b_3b_4b_5b_6b_7b_8b_9 + b_4b_5b_6b_7b_8b_9b_{10} : \quad (k, n) = (7, 10). \quad (149)$$

$$\longrightarrow b_1b_2 + b_2b_5 + b_5b_7 + b_5b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (150)$$

$$\longrightarrow b_5b_6 + b_6b_8 + b_6b_9 + b_6b_{10} - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (151)$$

$$\longrightarrow b_1b_7 + b_5b_7 + b_6b_7 + b_7b_9 - b_a(b_9 + b_{10}) + b_9 + b_{10} \quad (152)$$

$$\longrightarrow b_4b_5 + b_4b_8 + b_4b_{10} + b_a(b_4 - b_9 - b_{10}) + b_9 + b_{10} \quad (153)$$

$$\longrightarrow b_1b_3 + b_3b_4 + b_3b_9 - b_6b_7 + b_6b_{10} + b_a(1 - b_9 - b_{10}) - b_6 + b_{10} + 2 \quad (154)$$

$$\longrightarrow b_2b_3 + b_2b_8 + b_3b_9 - b_5b_7 + b_6b_9 - b_a(1 + b_{10}) - b_7 + b_9 + b_{10} + 3 \quad (155)$$

$$\longrightarrow b_2b_3 - b_6b_7 + b_6b_8 + b_7b_8 - b_a(b_9 + b_{10}) + b_8 + b_9 + b_{10} + 1 \quad (156)$$

$$\longrightarrow -b_1b_2 + b_1b_4 + b_2b_5 - b_4b_9 + b_5b_8 + b_5b_9 - b_a(b_9 + b_{10}) - b_4 + b_5 - b_7 + 2b_9 + b_{10} + 2 \quad (157)$$

$$\longrightarrow b_2b_8 + b_8b_{10} + b_a(b_8 - b_9 - b_{10}) + b_1 + b_9 + b_{10} \quad 1021/1024 \text{ (99.7\%)} \quad (158)$$

