1 SFR-BCR

Binary variables: $x_1, x_2, \dots, x_n \in \{0, 1\}$, and let $X := \sum_{i=1}^{n} x_i$, $2 \le k \in \mathbb{N}$, $l = \lceil \log k \rceil$

1.1 SFR-BCR-1,2

We have the $l+1$ auxiliary variables $y_0, y_1, \dots, y_{l-1}, z \in \{0, 1\}$. (In the formulas, summations for $i, i' = 1, 2, \dots, n$ and $j, j' = 0, 1, \dots, l-1$ are understood.)

We have the function (See Observation 1. from [22])

eq:Ak

$$A_k(X, y, z) = X - (k - 2^l)z - (k+1)(1-z) - \sum_j 2^j y_j. \tag{1}$$

After substituting for X, and collecting the terms, the equation for SFR-BCR-1 becomes

eq:Ak1

$$A_k(X, y, z) = -(1+k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \tag{2}$$

Similarly for SFR-BCR-2 we have

eq:Ak2

$$A_{n-k}(n - X, y, z) = -(1-k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \tag{3}$$

For the two cases (top/bottom = SFR-BCR-1/SFR-BCR-2), by recognizing their similarities we can simplify the equation to

eq:Ak12

$$\left.\begin{matrix} A_k(X, y, z) \\ A_{n-k}(n - X, y, z) \end{matrix}\right\} = -(1 \pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l)z. \tag{4}$$

Then the squares are

$$\begin{aligned} \left.\begin{matrix} A_k(X, y, z)^2 \\ A_{n-k}(n - X, y, z)^2 \end{matrix}\right\} = (1 \pm k)^2 \mp 2(1 \pm k)\sum_i x_i + 2(1 \pm k)\sum_j 2^j y_j - 2(1 \pm k)(1 + 2^l)z \\ + \sum_i \sum_{i'} x_i x_{i'} \mp 2 \sum_i \sum_j 2^j x_i y_j \pm 2(1 + 2^l)\sum_i z x_i \\ + \sum_j \sum_{j'} 2^{j+j'} y_j y_{j'} - 2(1 + 2^l)\sum_j z 2^j y_j \\ + (1 + 2^l)^2 z^2. \end{aligned} \tag{5}$$

eq:Ak12sq

Because $z \in \{0, 1\}$, we have $z^2 = z$, so we can join the two terms,

$$-2(1 \pm k)(1 + 2^l)z + (1 + 2^l)^2 z^2 = (1 + 2^l)(2^l \mp 2k - 1)z, \tag{6}$$

so we end up with

$$
\begin{aligned}
&\left.\begin{array}{l} A_k(X,y,z)^2 \\ A_{n-k}(n-X,y,z)^2 \end{array}\right\} = \\
&\underbrace{(1\pm k)^2}_{\alpha} + \underbrace{\mp 2(1\pm k)}_{\alpha^b}\sum_i x_i + \sum_j \underbrace{(1\pm k)2^{j+1}}_{\alpha^{b_{a,1}}} y_j + \underbrace{(1+2^l)(2^l \mp 2k-1)}_{\alpha^{b_{a,2}}} z \\
&+ \underbrace{1}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \sum_{i,j} \underbrace{\mp 2^{j+1}}_{\alpha^{bb_{a,1}}} x_i y_j + \underbrace{\pm 2(1+2^l)}_{\alpha^{bb_{a,2}}} \sum_i x_i z \\
&+ \sum_{j,j'} \underbrace{2^{j+j'}}_{\alpha^{b_a b_{a,1}}} y_j y_{j'} + \sum_j \underbrace{-(1+2^l)2^{j+1}}_{\alpha^{b_a b_{a,2}}} y_j z.
\end{aligned}
\tag{7}
$$

eq:alpha12

The coefficients are eq:BCR12alpha

$$
\begin{aligned}
\alpha &= (1\pm k)^2, && \text{(8a)} \\
\alpha^b &= \mp 2(1\pm k), && \text{(8b)} \\
\alpha^{b_{a,1}} &= (1\pm k)2^{j+1}, && \text{(8c)} \\
\alpha^{b_{a,2}} &= (1+2^l)(2^l \mp 2k-1), && \text{(8d)} \\
\alpha^{bb} &= 1, && \text{(8e)} \\
\alpha^{bb_{a,1}} &= \mp 2^{j+1}, && \text{(8f)} \\
\alpha^{bb_{a,2}} &= \pm 2(1+2^l), && \text{(8g)} \\
\alpha^{b_a b_{a,1}} &= 2^{j+j'}, && \text{(8h)} \\
\alpha^{b_a b_{a,2}} &= -(1+2^l)2^{j+1}. && \text{(8i)}
\end{aligned}
$$

Dictionary:
$l \mapsto m-1$, ($l+1=m$ auxiliary variables),
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$, ($b_{\mathrm{a},j}$ would be a better choice, a is a label, j is an index, they should be on the same level. Also the indexing of the $\alpha^{\cdots}$ coefficients could be made more expressive.)
$z \mapsto b_{a_m}$, ($b_{\mathrm{a},m}$ would be better)
$k \mapsto c$.
Note that, in this case, the j indices of the auxiliary bits have to be shifted, since they are ranging from 1, not 0. (This is not the case in the next subsection.)

1.2 SFR-BCR-3,4

We have now the l auxiliary variables $y_1, y_2, \ldots, y_{l-1}, z \in \{0,1\}$. (In the formulas, summations for $i,i' = 1,2,\ldots,n$ and $j,j'=1,2,\ldots,l-1$ are understood.)

We have the functions

$$A'_k(X,y,z) = X - (k-2^l)z - (k+1)(1-z) - \sum_j 2^j y_j. \tag{9}$$

eq:Akp

Note that, compared to (1), the difference is only in the range of index j of the sum in the last term. For the two cases (top/bottom = SFR-BCR-3/SFR-BCR-4), after substituting and collecting the terms,

eq:Akp34

$$\left.\begin{array}{l} A'_k(X,y,z) \\ A'_{n-k}(n-X,y,z) \end{array}\right\} = -(1\pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1+2^l)z. \tag{10}$$

(Again, although not written out explicitly, the difference is in the range of j of the summation, c.f., (4))

We can obtain the $\alpha^{\cdots}$ coefficients for BCR-3,4 from those of BCR-1,2. Instead of taking the squares, $A_k(X,y,z)^2$ and $A_{n-k}(n-X,y,z)^2$, for BCR-3,4 we have to take $\frac{1}{2}A'_k(X,y,z)\big(A'_k(X,y,z)-1\big) = \frac{1}{2}\big(A'_k(X,y,z)^2 - A'_k(X,y,z)\big)$ and $\frac{1}{2}A'_{n-k}(n-X,y,z)\big(A'_{n-k}(n-X,y,z)-1\big) = \frac{1}{2}\big(A'_{n-k}(n-X,y,z)^2 - A'_{n-k}(n-X,y,z)\big)$, so, to get the new $\alpha^{\cdots}$ coefficients, we have to substract the corresponding coefficients of $A'_k(X,y,z)$ and $A'_{n-k}(n-X,y,z)$ (these can be read off from (10)) from the old ones (8), and divide by 2. (And not to forget that the summations for j run over a different range.)

eq:BCR34alpha

$$\alpha = \frac{1}{2}\Big((1\pm k)^2 - -(1\pm k)\Big) = \frac{1}{2}(k^2 \pm 3k + 2), \tag{11a}$$

$$\alpha^{b} = \frac{1}{2}\Big(\mp 2(1\pm k) - \pm 1\Big) = -k \mp \frac{3}{2}, \tag{11b}$$

$$\alpha^{b_{a,1}} = \frac{1}{2}\Big((1\pm k)2^{j+1} - -2^j\Big) = (3\pm k)2^{j-1}, \tag{11c}$$

$$\alpha^{b_{a,2}} = \frac{1}{2}\Big((1+2^l)(2^l \mp 2k - 1) - (1+2^l)\Big) = (1+2^l)(2^{l-1} \mp k - 1), \tag{11d}$$

$$\alpha^{bb} = \frac{1}{2}(1) = \frac{1}{2}, \tag{11e}$$

$$\alpha^{bb_{a,1}} = \frac{1}{2}\big(\mp 2^{j+1}\big) = \mp 2^j, \tag{11f}$$

$$\alpha^{bb_{a,2}} = \frac{1}{2}\big(\pm 2(1+2^l)\big) = \pm(1+2^l), \tag{11g}$$

$$\alpha^{b_a b_{a,1}} = \frac{1}{2}\big(2^{j+j'}\big) = 2^{j+j'-1}, \tag{11h}$$

$$\alpha^{b_a b_{a,2}} = \frac{1}{2}\big(-(1+2^l)2^{j+1}\big) = -(1+2^l)2^j. \tag{11i}$$

Dictionary:
$l \mapsto m$, ($l = m$ auxiliary variables),
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$, ($b_{\mathrm{a},j}$ would be a better choice, a is a label, j is an index, they should be on the same level. Also the indexing of the $\alpha^{\cdots}$ coefficients could be made more expressive.)
$z \mapsto b_{a_m}$, ($b_{\mathrm{a},m}$ would be better)
$k \mapsto c$.

1.3 SFR-BCR-5

We begin with the quadratization of f:
(Theorem 6, Boros et al. [23])

$$\begin{aligned} g(x,y,z) &= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} r(il+j)y_i z_j + 2M\left(1-\sum_{i=0}^{l-1} y_i\right)^2 + 2M\left(1-\sum_{j=0}^{l-1} z_j\right)^2 \\ &+ 2M\left(|x| - \left(l\sum_{i=0}^{l-1} iy_i + \sum_{j=0}^{l-1} jz_j\right)\right)^2 \end{aligned} \tag{12}$$

After rearranging and substituting $|x| = \sum_i x_i$ we get the equation

$$\begin{aligned} g(x,y,z) &= \sum_{i,j=1}^{l-1} r((i-1)l+(j-1))y_i z_j + 2M\Bigg[\left(1-\sum_{i=0}^{l-1} y_i\right)^2 + \left(1-\sum_{j=0}^{l-1} z_j\right)^2 \\ &+ \left(\sum_i x_i - \left(l\sum_{i=1}^{l-1}(i-1)y_i + \sum_{j=1}^{l-1}(j-1)z_j\right)\right)^2\Bigg] \end{aligned} \tag{13}$$

Dictionary:
$l \mapsto m+1$,
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$,
$z \mapsto b_{a_{c+i}}$,
$\lambda \mapsto 2M$
$r(x) \mapsto f(x)$

1.4 SFR-BCR-6

Lets start with the following quadratization of f:
(Theorem 10, Boros et al. [22])

$$\begin{aligned} g(x,y,z) &= \sum_{i=1}^{l-1}\sum_{j=1}^{l-1} a_{i,j}\cdot y_i\cdot z_j + M + M\cdot(X-Y-1)\cdot(X-Y+1) \\ &+ M\cdot\sum_{i=1}^{l-2}(1-y_i)\cdot y_{i+1} + M\cdot\sum_{j=1}^{l-2}(1-z_j)\cdot z_{j+1} \end{aligned} \tag{14}$$

After factoring out M

$$
\begin{aligned}
g(x,y,z) &= \sum_{i,j}^{l-1} a_{ij} y_i z_j + M\Big(1 + \Big((X-Y-1)(X-Y+1)\Big) \\
&+ \sum_{i=1}^{l-2} (1-y_i) y_{i+1} + \sum_{j=1}^{l-2} (1-z_j) z_{j+1}\Big)
\end{aligned} \tag{15}
$$

Substituting for X, Y, and m using $\sum_i w_i x_i$, $l\left(\sum_{j=1}^{l-1} y_j\right) + \sum_{j=1}^{l-1} z_j$, and $l-1$, respectively, we get

$$
\begin{aligned}
g(x,y,z) &= \sum_{i,j}^{m} a_{ij} y_i z_j + M\Big[1 + \Big(\sum_i w_i x_i - (m-1)\sum_{j=1}^{m} y_j + \sum_{j=1}^{m} z_j - 1\Big) \\
&\Big(\sum_i w_i x_i - (m-1)\sum_{j=1}^{m} y_j + \sum_{j=1}^{m} z_j + 1\Big) + \sum_{i=1}^{m-1} (1-y_i) y_{i+1} + \sum_{j=1}^{m-1} (1-z_j) z_{j+1}\Big]
\end{aligned} \tag{16}
$$

Dictionary:
$l \mapsto m+1$,
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$,
$z \mapsto b_{a_{c+i}}$,
$\lambda \mapsto M$

1.5 SFR-ABCG-2

We begin with the following representation of the parity function.
(Theorem 4.6, Anthony et al. [14])

$$
\prod(x) = \sum_{j=1}^{n} x_j + 2\sum_{i=1}^{n-1} (-1)^{i-1} \left[i - \sum_{j=1}^{n} x_j\right]^{-} \tag{17}
$$

Adding $E(l) = l(l-1) + 2\sum_{i=1}^{n-1} [i-l]^{-}$ where $l = \sum_{j=1}^{n} x_j$, we get the quadratization

$$
\begin{aligned}
g(x,y) &= 2\sum_{i<j} x_i x_j + \sum_{j=1}^{n} x_j + 4\sum_{\substack{i=1: \\ i\ odd}}^{n-1} y_i\Big(i - \sum_{j=1}^{n} x_j\Big) \\
&= \sum_i x_i + 2\sum_{ij} x_i x_j + 4\sum_{2i-1}^{n-1} y_i\Big(2i - 1 - \sum_j x_j\Big)
\end{aligned} \tag{18}
$$

Dictionary:
$x_{i,j} \mapsto b_{i,j}$,

$y_i \mapsto b_{a_i}$

1.6 SFR-ABCG-3

We begin with the complement of the previous function from SFR-ABCG-2:
(Theorem 4.6, Anthony et al. [14])

$$\overline{\prod}(x) = 1 - \sum_{j=1}^{n} x_j + 2\sum_{i=1}^{n-1}(-1)^i \left[i - \sum_{j=1}^{n} x_j\right]^- \tag{19}$$

Adding $E(l) = l(l-1) + 2\sum_{i=1}^{n-1}[i-l]^-$ where $l = \sum_{j=1}^{n} x_j$, we get the quadratization

$$\begin{aligned} g(x,y)' &= 1 + 2\sum_{i<j} x_i x_j - \sum_{i}^{n} x_i + 4\sum_{\substack{i=2:\\ i\ even}}^{n-1} y_i\Big(i - \sum_{j=1}^{n} x_j\Big) \\ &= 1 + 2\sum_{ij} x_i x_j - \sum_{i} x_i + 4\sum_{2i}^{n-1} y_i\Big(i - \sum_{j} x_j\Big) \end{aligned} \tag{20}$$

Dictionary:
$x_{i,j} \mapsto b_{i,j}$,
$y_i \mapsto b_{a_i}$

1.7 SFR-BCR-7

Beginning with the quadratization of f:
(Theorem 9, Boros et al. [23])

$$\begin{aligned} g(x,y,z) &= \sum_{i=0}^{l-1}\sum_{j=0}^{l-1} r(il+j)y_i z_j + 2M\left(1 - \sum_{i=0}^{l-1} y_i\right)^2 + 2M\left(1 - \sum_{j=0}^{l-1} z_j\right)^2 \\ &\quad + 2M\left(|x| - \left(l\sum_{i=0}^{l-1} iy_i + \sum_{j=0}^{l-1} jz_j\right)\right)^2 \end{aligned} \tag{21}$$

Finally, we end with the equation

$$g(x,y) = \sum_{i} x_i + \sum_{2i-1}(2(2i-1)-1)y_{2i-1} + \sum_{1\le i<j\le n} x_i x_j - \sum_{2i-1}\sum_{j} y_{2i-1}x_j \tag{22}$$

1.8 SFR-BCR-8

Take the quadratization of the exact k-out-of-n function $f_{=k}$:
(Theorem 7, Boros et al. [23])

$$G_k(x, y, z) = \frac{1}{2} A_k(x, y, z)(A_k(x, y, z) - 1) \tag{23}$$

Where

$$A_k(X, y, z) = |x| - (k - 2^l)z - (k + 1)(1 - z) - \sum_{j}^{l-1} 2^j y_j. \tag{24}$$

Finally, we end with the equation

$$g(x, y) = \sum_i x_i + \sum_{2i-1} (2(2i - 1) - 1)y_{2i-1} + \sum_{1 \le i < j \le n} x_i x_j - \sum_{2i-1} \sum_j y_{2i-1} x_j \tag{25}$$

1.9 SFR-BCR-9

Take the quadratization of the at least k-out-of-n function $f_{\ge k}$:
(Theorem 8, Boros et al. [23])

$$G_k(x, y, z) = \frac{1}{2} A_k(x, y, z)(A_k(x, y, z) - 1) + (1 - z) \tag{26}$$

Where

$$A_k(X, y, z) = |x| - (k - 2^l)z - (k + 1)(1 - z) - \sum_{j}^{l-1} 2^j y_j. \tag{27}$$

Finally, we end with the equation

$$g(x, y) = \sum_i x_i + \sum_{2i-1} (2(2i - 1) - 1)y_{2i-1} + \sum_{1 \le i < j \le n} x_i x_j - \sum_{2i-1} \sum_j y_{2i-1} x_j \tag{28}$$