1 SFR-BCR

Binary variables: $x_1, x_2, \ldots, x_n \in \{0, 1\}$, and let $X := \sum_{i=1}^{n} x_i$, $2 \leq k \in \mathbb{N}$, $l = \lceil \log k \rceil$

1.1 SFR-BCR-1,2

We have the $l+1$ auxiliary variables $y_0, y_1, \ldots, y_{l-1}, z \in \{0, 1\}$. (In the formulas, summations for $i, i' = 1, 2, \ldots, n$ and $j, j' = 0, 1, \ldots, l-1$ are understood.)

We have the function (See Observation 1. from [22])

eq:Ak

$$A_k(X, y, z) = X - (k - 2^l)z - (k+1)(1-z) - \sum_j 2^j y_j. \tag{1}$$

After substituting for X, and collecting the terms, the equation for SFR-BCR-1 becomes

eq:Ak1

$$A_k(X, y, z) = -(1+k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \tag{2}$$

Similarly for SFR-BCR-2 we have

eq:Ak2

$$A_{n-k}(n - X, y, z) = -(1-k) + \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l). \tag{3}$$

For the two cases (top/bottom = SFR-BCR-1/SFR-BCR-2), by recognizing their similarities we can simplify the equation to

eq:Ak12

$$\left.\begin{array}{l} A_k(X, y, z) \\ A_{n-k}(n - X, y, z) \end{array}\right\} = -(1 \pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1 + 2^l)z. \tag{4}$$

Then the squares are

$$\begin{aligned} \left.\begin{array}{l} A_k(X, y, z)^2 \\ A_{n-k}(n - X, y, z)^2 \end{array}\right\} = {} & (1 \pm k)^2 \mp 2(1 \pm k) \sum_i x_i + 2(1 \pm k) \sum_j 2^j y_j - 2(1 \pm k)(1 + 2^l)z \\ & + \sum_i \sum_{i'} x_i x_{i'} \mp 2 \sum_i \sum_j 2^j x_i y_j \pm 2(1 + 2^l) \sum_i z x_i \\ & + \sum_j \sum_{j'} 2^{j+j'} y_j y_{j'} - 2(1 + 2^l) \sum_j z 2^j y_j \\ & + (1 + 2^l)^2 z^2. \end{aligned} \tag{5}$$

eq:Ak12sq

Because $z \in \{0, 1\}$, we have $z^2 = z$, so we can join the two terms,

$$-2(1 \pm k)(1 + 2^l)z + (1 + 2^l)^2 z^2 = (1 + 2^l)(2^l \mp 2k - 1)z, \tag{6}$$

so we end up with

$$\left.\begin{array}{l} A_k(X,y,z)^2 \\ A_{n-k}(n-X,y,z)^2 \end{array}\right\} =$$

$$\underbrace{(1\pm k)^2}_{\alpha} + \underbrace{\mp 2(1\pm k)}_{\alpha^b} \sum_i x_i + \sum_j \underbrace{(1\pm k)2^{j+1}}_{\alpha^{b_{a,1}}} y_j + \underbrace{(1+2^l)(2^l \mp 2k-1)}_{\alpha^{b_{a,2}}} z$$
$$+ \underbrace{1}_{\alpha^{bb}} \sum_{i,i'} x_i x_{i'} + \sum_{i,j} \underbrace{\mp 2^{j+1}}_{\alpha^{bb_{a,1}}} x_i y_j + \underbrace{\pm 2(1+2^l)}_{\alpha^{bb_{a,2}}} \sum_i x_i z$$
$$+ \sum_{j,j'} \underbrace{2^{j+j'}}_{\alpha^{b_a b_{a,1}}} y_j y_{j'} + \sum_j \underbrace{-(1+2^l)2^{j+1}}_{\alpha^{b_a b_{a,2}}} y_j z. \quad \text{eq:alpha12} \quad (7)$$

The coefficients are eq:BCR12alpha

$$\alpha = (1\pm k)^2, \quad (8a)$$
$$\alpha^b = \mp 2(1\pm k), \quad (8b)$$
$$\alpha^{b_{a,1}} = (1\pm k)2^{j+1}, \quad (8c)$$
$$\alpha^{b_{a,2}} = (1+2^l)(2^l \mp 2k - 1), \quad (8d)$$
$$\alpha^{bb} = 1, \quad (8e)$$
$$\alpha^{bb_{a,1}} = \mp 2^{j+1}, \quad (8f)$$
$$\alpha^{bb_{a,2}} = \pm 2(1+2^l), \quad (8g)$$
$$\alpha^{b_a b_{a,1}} = 2^{j+j'}, \quad (8h)$$
$$\alpha^{b_a b_{a,2}} = -(1+2^l)2^{j+1}. \quad (8i)$$

Dictionary:
$l \mapsto m-1$, ($l+1=m$ auxiliary variables),
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$, ($b_{\mathrm{a},j}$ would be a better choice, a is a label, j is an index, they should be on the same level. Also the indexing of the $\alpha^{\cdots}$ coefficients could be made more expressive.)
$z \mapsto b_{a_m}$, ($b_{\mathrm{a},m}$ would be better)
$k \mapsto c$.
Note that, in this case, the j indices of the auxiliary bits have to be shifted, since they are ranging from 1, not 0. (This is not the case in the next subsection.)

1.2 SFR-BCR-3,4

We have now the l auxiliary variables $y_1, y_2, \ldots, y_{l-1}, z \in \{0,1\}$. (In the formulas, summations for $i,i' = 1,2,\ldots,n$ and $j,j' = 1,2,\ldots,l-1$ are understood.)

We have the functions

$$A'_k(X,y,z) = X - (k-2^l)z - (k+1)(1-z) - \sum_j 2^j y_j. \quad \text{eq:Akp} \quad (9)$$

Note that, compared to (1), the difference is only in the range of index j of the sum in the last term. For the two cases (top/bottom = SFR-BCR-3/SFR-BCR-4), after substituting and collecting the terms,

eq:Akp34

$$\left.\begin{array}{l} A'_k(X,y,z) \\ A'_{n-k}(n-X,y,z) \end{array}\right\} = -(1 \pm k) \pm \sum_i x_i - \sum_j 2^j y_j + (1+2^l)z. \tag{10}$$

(Again, although not written out explicitly, the difference is in the range of j of the summation, c.f., (4))

We can obtain the $\alpha^{\cdots}$ coefficients for BCR-3,4 from those of BCR-1,2. Instead of taking the squares, $A_k(X,y,z)^2$ and $A_{n-k}(n-X,y,z)^2$, for BCR-3,4 we have to take $\frac{1}{2}A'_k(X,y,z)\big(A'_k(X,y,z)-1\big) = \frac{1}{2}\big(A'_k(X,y,z)^2 - A'_k(X,y,z)\big)$ and $\frac{1}{2}A'_{n-k}(n-X,y,z)\big(A'_{n-k}(n-X,y,z)-1\big) = \frac{1}{2}\big(A'_{n-k}(n-X,y,z)^2 - A'_{n-k}(n-X,y,z)\big)$, so, to get the new $\alpha^{\cdots}$ coefficients, we have to substract the corresponding coefficients of $A'_k(X,y,z)$ and $A'_{n-k}(n-X,y,z)$ (these can be read off from (10)) from the old ones (8), and divide by 2. (And not to forget that the summations for j run over a different range.)

eq:BCR34alpha

$$\alpha = \frac{1}{2}\Big((1\pm k)^2 - -(1\pm k)\Big) = \frac{1}{2}(k^2 \pm 3k + 2), \tag{11a}$$

$$\alpha^b = \frac{1}{2}\Big(\mp 2(1\pm k) - \pm 1\Big) = -k \mp \frac{3}{2}, \tag{11b}$$

$$\alpha^{b_{a,1}} = \frac{1}{2}\Big((1\pm k)2^{j+1} - -2^j\Big) = (3\pm k)2^{j-1}, \tag{11c}$$

$$\alpha^{b_{a,2}} = \frac{1}{2}\Big((1+2^l)(2^l \mp 2k - 1) - (1+2^l)\Big) = (1+2^l)(2^{l-1} \mp k - 1), \tag{11d}$$

$$\alpha^{bb} = \frac{1}{2}(1) = \frac{1}{2}, \tag{11e}$$

$$\alpha^{bb_{a,1}} = \frac{1}{2}\big(\mp 2^{j+1}\big) = \mp 2^j, \tag{11f}$$

$$\alpha^{bb_{a,2}} = \frac{1}{2}\big(\pm 2(1+2^l)\big) = \pm(1+2^l), \tag{11g}$$

$$\alpha^{b_a b_{a,1}} = \frac{1}{2}\big(2^{j+j'}\big) = 2^{j+j'-1}, \tag{11h}$$

$$\alpha^{b_a b_{a,2}} = \frac{1}{2}\big(-(1+2^l)2^{j+1}\big) = -(1+2^l)2^j. \tag{11i}$$

Dictionary:
$l \mapsto m$, ($l = m$ auxiliary variables),
$x_i \mapsto b_i$,
$y_j \mapsto b_{a_j}$, ($b_{\mathrm{a},j}$ would be a better choice, a is a label, j is an index, they should be on the same level. Also the indexing of the $\alpha^{\cdots}$ coefficients could be made more expressive.)
$z \mapsto b_{a_m}$, ($b_{\mathrm{a},m}$ would be better)
$k \mapsto c$.

1.3 SFR-BCR-5,6

1.4 SFR-ABCG-2

We begin with the following representation of the parity function.
(Theorem 4.6, Anthony et al. [14])

$$\prod(x) = \sum_{j=1}^{n} x_j + 2\sum_{i=1}^{n-1}(-1)^{i-1}\left[i - \sum_{j=1}^{n} x_j\right]^{-} \tag{12}$$

Adding $E(l) = l(l-1) + 2\sum_{i=1}^{n-1}[i-l]^{-}$ where $l = \sum_{j=1}^{n} x_j$, we get the quadratization

$$\begin{aligned} g(x,y) &= 2\sum_{i<j} x_i x_j + \sum_{j=1}^{n} x_j + 4\sum_{\substack{i=1:\\ i\ odd}}^{n-1} y_i\Big(i - \sum_{j=1}^{n} x_j\Big) \\ &= \sum_{i} x_i + 2\sum_{ij} x_i x_j + 4\sum_{2i-1}^{n-1} y_i\Big(2i - 1 - \sum_{j} x_j\Big) \end{aligned} \tag{13}$$

Dictionary:
$x_{i,j} \mapsto b_{i,j}$,
$y_i \mapsto b_{a_i}$

1.5 SFR-ABCG-3

We begin with the complement of the previous function from SFR-ABCG-2:
(Theorem 4.6, Anthony et al. [14])

$$\overline{\prod}(x) = 1 - \sum_{j=1}^{n} x_j + 2\sum_{i=1}^{n-1}(-1)^{i}\left[i - \sum_{j=1}^{n} x_j\right]^{-} \tag{14}$$

Adding $E(l) = l(l-1) + 2\sum_{i=1}^{n-1}[i-l]^{-}$ where $l = \sum_{j=1}^{n} x_j$, we get the quadratization

$$\begin{aligned} g(x,y)' &= 1 + 2\sum_{i<j} x_i x_j - \sum_{i}^{n} x_i + 4\sum_{\substack{i=2:\\ i\ even}}^{n-1} y_i\Big(i - \sum_{j=1}^{n} x_j\Big) \\ &= 1 + 2\sum_{ij} x_i x_j - \sum_{i} x_i + 4\sum_{2i}^{n-1} y_i\Big(i - \sum_{j} x_j\Big) \end{aligned} \tag{15}$$

Dictionary:
$x_{i,j} \mapsto b_{i,j}$,
$y_i \mapsto b_{a_i}$

1.6 SFR-BCR-7,8

1.7 SFR-BCR-9