

# Quadratization in Discrete Optimization and Quantum Mechanics

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A collaborative, evolving, open review paper on  $k$ -local to 2-local transformations (quadratizations) in classical computing, quantum annealing, and universal adiabatic quantum computing.

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When optimizing discrete functions, it is often easier when the function is quadratic than if it is of higher degree. But notice that the cubic and quadratic functions:

$$b_1b_2 + b_2b_3 + b_3b_4 - 4b_1b_2b_3 \quad (\text{cubic}), \quad (1)$$

$$b_1b_2 + b_2b_3 + b_3b_4 + 4b_1 - 4b_1b_2 - 4b_1b_3 \quad (\text{quadratic}), \quad (2)$$

where each  $b_i$  can either be 0 or 1, both never go below the value of -1, and all minima have the form  $(b_1, b_2, b_3, b_4) = (1, 1, 1, b_4)$ . Therefore if we are interested in the ground state of a discrete function of degree  $k$ , we may optimize either function and get exactly the same result. **Part I** gives more than 15 different ways to do this.

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The binary variables  $b_i$  can be either of the eigenvalues of the matrix  $b$  below, which is related to the Pauli  $z$  matrix by  $z = 2b - \mathbb{1}$ . Other Pauli matrices are listed below:

$$b \equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \mathbb{1} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3)$$

Any Hermitian  $2 \times 2$  matrix can be written as a linear combination of the Pauli matrices  $z, x, y, \mathbb{1}$ , so we can therefore describe the Hamiltonian of any number of spin- $1/2$  particles by a function of Pauli matrices acting on each particle, for instance:

$$z_1x_2 + x_2y_3 + z_3z_4 - 4y_1y_2z_3 \quad (\text{cubic}), \quad (4)$$

where the coefficients tell us about the strengths of couplings between these particles. The Schrödinger equation tells us that the eigenvalues of the Hamiltonian are the allowed energy levels and their eigenvectors (wavefunctions) are the corresponding physical states. More generally these do not have to be spins but can be any type of qubits, and we can encode the solution to *any* problem in the ground state of a Hamiltonian, then solve the problem by finding the lowest energy state of the physical system (this is called adiabatic quantum computing).

Two-body physical interactions occur more naturally than many-body interactions so **Parts II-III** give more than 10 different ways to quadratize general Hamiltonians (some of these methods do not even require the  $s_i$  to be  $2 \times 2$  matrices, meaning that we can have types of qudits that are not qubits).

The optimization problems of Eqs. (1)-(2) are specific cases of the type in Eq. (4) where only  $z$  matrices are present.

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# Part I

## Diagonal Hamiltonians (pseudo-Boolean functions)

### I. METHODS THAT INTRODUCE ZERO AUXILIARY VARIABLES

#### A. Deduction Reduction (Deduc-reduc)

##### Summary

We look for *deductions* (e.g.  $b_1 b_2 = 0$ ) that must hold true at the global minimum. These can be found by *a priori* knowledge of the given problem, or by enumerating solutions of a small subset of the variables. We can then substitute high-order terms using the low-order terms of the deduction, and add on a penalty term to preserve the ground states [? ].

##### Cost

- 0 auxiliary variables needed.
- For a particular value of  $m$ , we have  $\binom{n}{m}$  different  $m$ -variable subsets of the  $n$  variable problem, and  $\binom{n}{m} 2^m$  evaluations of the objective function to find all possible  $m$ -variable deductions, whereas  $2^n$  evaluations is enough to solve the entire problem. We therefore choose  $m \lll n$ .

##### Pros

- No auxiliary variables needed.

##### Cons

- When deductions cannot be determined naturally (as in the Ramsey number determination problem, see Example ??), deductions need to be found by ‘brute force’, which scales exponentially with respect to  $m$ . For highly connected systems (systems with a large number of non-zero  $\alpha_{ij}$  coefficients), the value of  $m$  required to find even one deduction can be prohibitively large.

##### Example

Consider the Hamiltonian:

$$H_{4\text{-local}} = b_1 b_2 (10 + b_3 + b_3 b_4) + b_1 (b_3 - 3) + b_2 (b_3 - 2b_3 - b_4 - 1) + F(x_3, x_4, x_5, \dots, x_N) \quad (5)$$

where  $F$  is any polynomial in  $b_i$  for  $i \geq 3$ .

Since the coefficient of  $b_1 b_2$  is 10, which is greater than the sum of all of the other coefficients involving  $b_1$  or  $b_2$ , ***it must be the case that  $b_1 b_2 = 0$*** . Specifically, for the 4 assignments of  $(b_3, b_4)$ , we see that  $b_1 b_2 = 0$  at every minimum of  $H_{4\text{-local}} - F$ .

Using deduc-reduc we have:

$$H_{2\text{-local}} = 12b_1 b_2 + b_1 (b_3 - 3) + b_2 (b_3 - 2b_3 - b_4 - 1) + F(x_3, x_4, x_5, \dots, x_N) \quad (6)$$

which has the same global minima as  $H_{4\text{-local}}$  but one fewer quartic and one fewer cubic term.

##### Bibliography

- Original paper, with more implementation details, and application to integer factorization: [? ].

## B. ELC Reduction (Ishikawa 2014)

### Summary

An Excludable Local Configuration (ELC) is a partial assignment of variables that make it impossible to achieve the minimum. We can therefore add a term that corresponds to the energy of this ELC without changing the solution to the minimization problem. In practice we can eliminate all monomials with a variable in which a variable is set to 0, and reduce any variable set to 1. Given a general Hamiltonian we can try to find ELCs by enumerating solutions of a small subset of variables in the problem [? ].

### Cost

- 0 auxiliary variables needed.
- For a particular value of  $m$ , we have  $\binom{n}{m}$  different  $m$ -variable subsets of the  $n$  variable problem, and  $\binom{n}{m}2^m$  evaluations of the objective function to find all possible  $m$ -variable deductions, whereas  $2^n$  evaluations is enough to solve the entire problem. We therefore choose  $m \lll n$ .
- Approximate methods exist which have been shown to be much faster and give good approximations to the global minimum [? ].

### Pros

- No auxiliary variables needed.

### Cons

- No known way to find ELCs except by ‘brute force’, which scales exponentially with respect to  $m$ .
- ELCs do not always exist.

### Example

Consider the Hamiltonian:

$$H_{3\text{-local}} = b_1b_2 + b_2b_3 + b_3b_4 - 4b_1b_2b_3. \quad (7)$$

If  $b_1b_2b_3 = 0$ , no assignment of our variables will we be able to reach a lower energy than if  $b_1b_2b_3 = 1$ . Hence this gives us *twelve* ELCs, and one example is  $(b_1, b_2, b_3) = (1, 0, 0)$  which we can use to form the polynomial:

$$H_{2\text{-local}} = H_{3\text{-local}} + 4b_1(1 - b_2)(1 - b_3) \quad (8)$$

$$= b_1b_2 + b_2b_3 + b_3b_4 + 4b_1 - 4b_1b_2 - 4b_1b_3. \quad (9)$$

In both cases Eqs. (7) and (9), the only global minima occur when  $b_1b_2b_3 = 1$ .

### Bibliography

- Original paper and application to computerized image denoising: [? ].

### C. Groebner Bases

#### *Summary*

Given a set of polynomials, a Groebner basis is another set of polynomials that have exactly the same zeros. The advantage of a Groebner basis is it has nicer algebraic properties than the original equations, in particular they tend to have smaller degree polynomials. The algorithms for calculating Groebner bases are generalizations of Euclid's algorithm for the polynomial greatest common divisor.

Work has been done in the field of 'Boolean Groebner bases', but while the variables are Boolean the coefficients of the functions are in  $\mathbb{F}_2$  rather than  $\mathbb{Q}$ .

#### *Cost*

- 0 auxiliary variables needed.
- $\mathcal{O}(2^{2^n})$  in general,  $\mathcal{O}(d^{n^2})$  if the zeros of the equations form a set of discrete points, where  $d$  is the degree of the polynomial and  $n$  is the number of variables [? ].

#### *Pros*

- No auxiliary variables needed.
- General method, which can be used for other rings, fields or types of variables.

#### *Cons*

- Best algorithms for finding Groebner bases scale double exponentially in  $n$ .
- Only works for Hamiltonians whose minimization corresponds to solving systems of discrete equations, as the method only preserves roots, not minima.

#### *Example*

Consider the following pair of equations:

$$b_1 b_2 b_3 b_4 + b_1 b_3 + b_2 b_4 - b_3 = b_1 + b_1 b_2 + b_3 - 2 = 0. \quad (10)$$

Feeding these to Mathematica's `GroebnerBasis` function, along with the binarizing  $b_1(b_1 - 1) = \dots = b_4(b_4 - 1) = 0$  constraints, gives a Groebner basis:

$$\{b_4 b_3 - b_4, b_2 + b_3 - 1, b_1 - 1\}. \quad (11)$$

From this we can immediately read off the solutions  $b_1 = 1$ ,  $b_2 = 1 - b_3$  and reduce the problem to  $b_3 b_4 - b_4 = 0$ . Solving this gives a final solution set of:  $(b_1, b_2, b_3, b_4) = (1, 0, 1, 0), (1, 0, 1, 1), (1, 1, 0, 0)$ , which should be the same as the original 4-local problem.

#### *Bibliography*

- Reduction and embedding of factorizations of all bi-primes less than 200,000: [? ].

### D. Application of ELM (Present Work)

#### Summary

We use the formula from [?] for representing any function of three binary variables:

$$f(b_1, b_2, b_3) = (f(1, 1, 1) + f(1, 0, 0) - f(1, 1, 0) - f(1, 0, 1) - f(0, 1, 1) - f(0, 0, 0) + \quad (12)$$

$$f(0, 0, 1) + f(0, 1, 0)) b_1 b_2 b_3 + (f(0, 1, 1) + f(0, 0, 0) - f(0, 0, 1) - f(0, 1, 0)) b_2 b_3 + \quad (13)$$

$$(f(1, 0, 1) + f(0, 0, 0) - f(0, 0, 1) - f(1, 0, 0)) b_1 b_3 + \quad (14)$$

$$(f(1, 1, 0) + f(0, 0, 0) - f(1, 0, 0) - f(0, 1, 0)) b_1 b_2 + (f(0, 1, 0) - f(0, 0, 0)) b_2 \quad (15)$$

$$+ (f(1, 0, 0) - f(0, 0, 0)) b_1 + (f(0, 0, 1) - f(0, 0, 0)) b_3 + f(0, 0, 0). \quad (16)$$

If the cubic term is zero, then the function becomes quadratic. We can use ELM (Energy Landscape Manipulation) to change the energy landscape *without* changing the ground state [?]. In this case, we apply ELM in order to make the cubic term zero.

#### Cost

- No auxiliary variables.
- May require many evaluations of the cubic function varying coefficients in order to find the right ELM coefficients.

#### Pros

- Can be generalized to arbitrary  $k$ -local functions, but the ELM constraints may become harder to achieve.
- Can quadratize an entire cubic function (or a cubic part of a more general function) with no auxiliary qubits.
- Can reproduce the full spectrum.

#### Cons

- May not always be possible.
- May require a local search to find appropriate deductions.

#### Example

In order to reduce the number of constraints required for the cubic term to be zero, we will assume that Deduc-Reduc told us that  $(1 - b_1)(1 - b_2) + (1 - b_2)(1 - b_3) + (1 - b_1)(1 - b_3) = 0$  when the overall function is minimized, which means the ground state only occurs when at least two variables are 1. This does not allow us to assign the linear terms, but assigns all quadratic terms to have  $b_i b_j = 1$ . This also suggests that  $b_1 b_2 b_3$  can also be reduced to a linear term, but we do not know whether it is  $b_1$ ,  $b_2$ , or  $b_3$ . So we have the following constraint on the cubic term, after setting all  $f(b_1, b_2, b_3)$  to zero if there is not at least two 1's:

$$f(1, 1, 1) - f(1, 1, 0) - f(1, 0, 1) - f(0, 1, 1) = 0. \quad (17)$$

We now show an example of a function for which ELM has the power to enforce this constraint:

## E. Split Reduction (Okada, Tanburn, Dattani, 2015)

### *Summary*

It has been shown in [?] that, if multiple runs of a minimization algorithm is permitted, it is possible to reduce a lot of the problem by conditioning on the most connected variables. We call each of these operations a *split*.

### *Cost*

Exponential in the number of splits, as the number of problems to solve doubles with every split.

### *Pros*

- This method can be applied to any problem and can be very effective on problems with a few very connected variables.

### *Cons*

- Exponential cost in the worst case.

### *Example*

Consider the simple objective function

$$H = 1 + b_1b_2b_5 + b_1b_6b_7b_8 + b_3b_4b_8 - b_1b_3b_4. \quad (18)$$

In order to quadratize  $H$ , we first have to choose a variable to split over. In this case  $b_1$  is the obvious choice since it is present in the most terms and contributes to the quartic term.

We then obtain two different problems:

$$H_0 = 1 + b_3b_4b_8 \quad (19)$$

$$H_1 = 1 + b_2b_5 + b_6b_7b_8 + b_3b_4b_8 - b_3b_4. \quad (20)$$

At this point, we could split  $H_0$  again and solve it entirely, or use a variable we saved in the previous split to quadratize our only problem.

To solve  $H_1$ , we can split again on  $b_8$ , resulting in problems:

$$H_{1,0} = 1 + b_2b_5 + b_6b_7 \quad (21)$$

$$H_{1,1} = 1 + b_2b_5 + b_3b_4. \quad (22)$$

Now both of these problems are quadratic. Hence we have reduced our original, hard problem into 3 easy problems, requiring only 2 extra (much easier) runs of our minimization algorithm, and without needing any auxiliary variables.

### *Bibliography*

- Original paper and application to Ramsey number determination: [? ].

## II. METHODS THAT INTRODUCE AUXILIARY VARIABLES TO QUADRATIZE A SINGLE NEGATIVE TERM (NEGATIVE TERM REDUCTIONS, NTR)

### A. NTR-KZFD (Kolmogorov & Zabih, 2004; Freedman & Drineas, 2005)

#### *Summary*

For a negative term  $-b_1 b_2 \dots b_k$ , introduce a single auxiliary variable  $b_a$  and make the substitution:

$$-b_1 b_2 \dots b_k \rightarrow (k-1)b_a - \sum_i b_i b_a. \quad (23)$$

#### *Cost*

- 1 auxiliary variable for each  $k$ -local term.

#### *Pros*

- All resulting quadratic terms are submodular (have negative coefficients).
- Can reduce arbitrary order terms with only 1 auxiliary.
- Reproduces the full spectrum.

#### *Cons*

- Only works for negative terms.

#### *Example*

$$H_{6\text{-local}} = -2b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6, \quad (24)$$

has a unique minimum energy of -1 when all  $b_i = 1$ .

$$H_{2\text{-local}} = 2(5b_a - b_1 b_a - b_2 b_a - b_3 b_a - b_4 b_a - b_5 b_a - b_6 b_a) + b_5 b_6 \quad (25)$$

has the same unique minimum energy, and it occurs at the same place (all  $b_i = 1$ ), with  $b_a = 1$ .

#### *Alternate Forms*

$$-b_1 b_2 \dots b_k = \min_{b_a} \left( (k-1 - \sum_i b_i) b_a \right) \quad (26)$$

$$\rightarrow \left( (k-1 - \sum_i b_i) b_a \right). \quad (27)$$

#### *Alternate Names*

- "Standard quadratization" of negative monomials [? ].
- $s_k(b, b_a)$  [? ]

#### *Bibliography*

- 2004: Kolmogorov and Zabih presented this for cubic terms [? ].
- 2005: Generalized to arbitrary order by Freedman and Drineas [? ].
- Discussion: [? ], [? ].



## B. NTR-ABCG (Anthony, Boros, Crama, Gruber, 2014)

### Summary

For a negative term  $-b_1b_2\dots b_k$ , introduce a single auxiliary variable  $b_a$  and make the substitution:

$$-b_1b_2\dots b_k \rightarrow -\sum_i b_i - \sum_i b_i b_k - \sum_i b_i b_a + (k-1)b_k b_a. \quad (28)$$

### Cost

- 1 auxiliary variable for each  $k$ -local term.
- 1 non-submodular term for each  $k$ -local term (and it is quadratic).

### Pros

- Can reduce arbitrary order terms with only 1 auxiliary.
- Reproduces the full spectrum.

### Cons

- Only works for negative terms.
- Turns a symmetric term into a non-symmetric term (but only  $b_k$  is asymmetric).

### Example

$$H_{6\text{-local}} = -2b_1b_2b_3b_4b_5b_6 + b_5b_6, \quad (29)$$

has a unique minimum energy of -1 when all  $b_i = 1$ .

$H_{2\text{-local}}$  has the same unique minimum energy, and it occurs at the same place (all  $b_i = 1$ ), with  $b_a = 1$ .

### Alternate Forms

$$-b_1b_2\dots b_k \rightarrow (k-1)b_k b_a - \sum_i b_i (b_a + b_k - 1) \quad (30)$$

$$= (k-2)b_a - \sum_i^{k-1} b_i (b_a + b_k - 1) \quad (31)$$

### Alternate Names

- "Extended standard quadratization" of negative monomials [? ].
- $s_k(b, b_a)^+ [? ]$

### Bibliography

- 2014, First presentation: [? ? ].
- Further discussion: [? ].

### C. NTR-ABCG-2 (Anthony, Boros, Crama, Gruber, 2016)

#### *Summary*

For a negative term  $-b_1 b_2 \dots b_k$ , introduce a single auxiliary variable  $b_a$  and make the substitution:

$$-b_1 b_2 \dots b_k \rightarrow (2k - 1) b_a - 2 \sum_i b_i b_a \quad (32)$$

#### *Cost*

- 1 auxiliary variable for each  $k$ -local term.
- 1 non-submodular term for each  $k$ -local term (and it is linear).

#### *Pros*

- Can reduce arbitrary order terms with only 1 auxiliary.
- Reproduces the full spectrum.
- The non-submodular term is linear as opposed to NTR-ABCG-1 whose non-submodular term is quadratic.
- Symmetric with respect to all non-auxiliary variables.

#### *Cons*

- Only works for negative terms.
- Turns a symmetric term into a non-symmetric term (but only  $b_k$  is asymmetric).
- Coefficients of quadratic terms are twice the size of their size in NTR-KZFD or NTR-ABCG-1, and roughly twice the size for the linear term.

#### *Example*

$$H_{6\text{-local}} = -2b_1 b_2 b_3 b_4 b_5 b_6 + b_5 b_6, \quad (33)$$

has a unique minimum energy of -1 when all  $b_i = 1$ .

$H_{2\text{-local}}$  has the same unique minimum energy, and it occurs at the same place (all  $b_i = 1$ ), with  $b_a = 1$ .

#### *Bibliography*

- Discussion: [? ].

## D. Asymmetric Cubic Reduction

### *Summary*

$$-b_1b_2b_3 \rightarrow b_a(-b_1 + b_2 + b_3) - b_1b_2 - b_1b_3 + b_1 \quad (34)$$

### *Cost*

1 auxiliary variable per negative cubic term.

### *Pros*

- Asymmetric which allows more flexibility in cancelling with other quadratics.

### *Cons*

- Only works for negative cubic monomials.

### *Example*

$$-b_1b_2b_3 + b_1b_3 - b_2 = \min_{b_a} (b_a - b_1b_a - b_3b_a + b_2b_a + 2b_1b_3) - b_2 \quad (35)$$

### *Alternate Forms*

$$-b_1b_2b_3 = \min_{b_a} b_a - b_1 + b_2 + b_3 - b_1b_2 - b_1b_3 + b_1 \quad (36)$$

### *Bibliography*

- Discussion: [?] , [?] .

### III. METHODS THAT INTRODUCE AUXILIARY VARIABLES TO QUADRATIZE A SINGLE POSITIVE TERM (POSITIVE TERM REDUCTIONS, PTR)

#### A. Positive Term Reduction

##### *Summary*

By considering the negated literals  $\bar{b}_i = 1 - b_i$ , we recursively apply NTR-KZFD to  $b_1 b_2 \dots b_k = -\bar{b}_1 b_2 \dots b_k + b_2 b_3 \dots b_k$ . The final identity is:

$$b_1 b_2 \dots b_k \rightarrow \left( \sum_{i=1}^{k-2} b_{a_i} (k - i - 1 + b_i - \sum_{j=i+1}^k b_j) \right) + b_{k-1} b_k \quad (37)$$

##### *Cost*

- $k - 2$  auxiliary variables for each  $k$ -local term.

##### *Pros*

- Works for positive monomials.

##### *Cons*

- $k - 1$  non-submodular quadratic terms.

##### *Example*

$$b_1 b_2 b_3 b_4 \rightarrow b_{a_1} (2 + b_1 - b_2 - b_3 - b_4) + b_{a_2} (1 + b_2 - b_3 - b_4) + b_3 b_4 \quad (38)$$

##### *Bibliography*

- Summary: [?] .

## B. PTR-Ishikawa (Ishikawa, 2011)

### Summary

This method re-writes a positive monomial using symmetric polynomials, so all possible quadratic terms are produced and they are all non-submodular:

$$b_1 \dots b_k \rightarrow \left( \sum_{i=1}^{n_k} b_{a_i} \left( c_{i,d} \left( - \sum_{j=1}^k b_j + 2i \right) - 1 \right) + \sum_{i < j} b_i b_j \right) \quad (39)$$

where  $n_k = \lfloor \frac{k-1}{2} \rfloor$  and  $c_{i,k} = \begin{cases} 1, & i = n_d \text{ and } k \text{ is odd,} \\ 2, & \text{else.} \end{cases}$

### Cost

- $\lfloor \frac{k-1}{2} \rfloor$  auxiliary variables for each  $k$ -order term
- $\mathcal{O}(kt)$  for a  $k$ -local Hamiltonian with  $t$  terms.

### Pros

- Works for positive monomials.
- About half as many auxiliary variables for each  $k$ -order term as the previous method.
- Reproduces the full spectrum.

### Cons

- $\mathcal{O}(k^2)$  quadratic terms are created, which may make chimerization more costly.
- $\frac{k(k-1)}{2}$  non-submodular terms.
- Worse than the previous method for quartics, with respect to submodularity.

### Example

$$b_1 b_2 b_3 b_4 \rightarrow (3 - 2b_1 - 2b_2 - 2b_3 - 2b_4) b_a + b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4 \quad (40)$$

### Alternate Forms

For even  $k$ , and equivalent expression is given in [? ]:

$$b_1 b_2 \dots b_k \rightarrow \sum_i b_i + \sum_{ij} b_i b_j + \sum_{2i} b_{a_{2i}} \left( 4i - 2 - \sum_j b_j \right) \quad (41)$$

$$\rightarrow \sum_i b_i + 2 \sum_{2i} b_{a_{2i}} (2i - 1) + \sum_{ij} b_i b_j - \sum_{2i,j} b_j b_{a_{2i}} \quad (42)$$

### Alternate Names

- "Ishikawa's Symmetric Reduction" [? ].
- "Ishikawa Reduction"
- "Ishikawa"

### Bibliography

- Original paper and application to image denoising: [? ].
- Equivalent way of writing it for even  $k$ , shown in [? ].

### C. PTR-BCR-1 (Boros, Crama, and Rodríguez-Heck, 2018)

#### *Summary*

This is very similar to the alternative form of Ishikawa Reduction, but works for odd values of  $k$ , and is different from Ishikawa Reduction:

$$b_1 b_2 \dots b_k \rightarrow \sum_i b_i + \sum_{2i-1} (4i-3) b_{a_{2i-1}} + \sum_{ij} b_i b_j - \sum_{2i-1, j} b_j b_{a_{2i-1}} \quad (43)$$

#### *Cost*

- Same number of auxiliaries as Ishikawa Reduction.

#### *Pros*

- Same as for Ishikawa Reduction.

#### *Cons*

- Same as for Ishikawa Reduction.
- Only works for odd  $k$ , but for even  $k$  we have an analogous method which is equivalent to Ishikawa Reduction.

#### *Alternate Forms*

$$b_1 b_2 \dots b_k \rightarrow \sum_i b_i + \sum_{ij} b_i b_j + \sum_{2i-1} b_{a_{2i-1}} \left( 4i-3 - \sum_j b_j \right) \quad (44)$$

#### *Bibliography*

- Original paper: [? ].

### D. PTR-BCR-2 (Boros, Crama, and Rodríguez-Heck, 2018)

#### Summary

Let  $m = \lceil \frac{k}{4} \rceil$ ,

$$\begin{aligned}
 b_1 b_2 \dots b_k \rightarrow & \alpha^b \sum_i b_i + \alpha^{b_{a,1}} \sum_i b_{a_i} + \alpha^{b_{a,2}} b_{a_m} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_{a,1}} \sum_i \sum_j^{m-1} b_i b_{a_j} + \\
 & \alpha^{bb_{a,2}} \sum_i b_i b_{a_m} + \alpha^{b_{a,1} b_{a,1}} \sum_{ij}^{m-1} b_{a_i} b_{a_j} + \alpha^{b_{a,1} b_{a,2}} \sum_i^{m-1} b_{a_i} b_{a_m},
 \end{aligned} \tag{45}$$

where:

$$\begin{pmatrix} \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_{a,1} b_{a,1}} \\ \alpha^{bb} & \alpha^{b_{a,1} b_{a,2}} \end{pmatrix} = \begin{pmatrix} -1/2 & -1 \\ 1 & -2 \\ \frac{1}{2}(n-m+n^2-2mn+m^2) & -(n-m) \\ 1/2 & 4(n-m) \end{pmatrix}. \tag{46}$$

#### Cost

$\lceil \frac{k}{4} \rceil$  auxiliary qubits per positive monomial.

#### Pros

- Smallest number of auxiliary coefficients that scales linearly with  $k$ .
- Smaller coefficients than the logarithmic reduction.

#### Cons

- Introduces many non-submodular terms.

#### Example

We quadratize a quartic term with only 1 auxiliary (half as many as in PTR-Ishikawa):

$$b_1 b_2 b_3 b_4 \rightarrow \frac{1}{2} (b_1 + b_2 + b_3 + b_4 - 2b_{a_1}) (b_1 + b_2 + b_3 + b_4 - 2b_{a_1} - 1) \tag{47}$$

#### Bibliography

- Original paper: [? ].

### E. PTR-BCR-3 (Boros, Crama, and Rodríguez-Heck, 2018)

#### Summary

Pick  $m$  such that  $k < 2^{m+1}$ ,

$$b_1 b_2 \dots b_k \rightarrow \alpha + \alpha^b \sum_i b_i + \alpha^{b_{a_i}} \sum_i 2^{i-1} b_{a_i} + \alpha^{bb} \sum_{ij} b_i b_j + \alpha^{bb_a} \sum_{ij} b_i b_{a_j} + \alpha^{b_{a_i} b_{a_j}} b_{a_i} b_{a_j}, \quad (48)$$

where,

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_a} \\ \alpha^{b_a} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (2^m - k)^2 & 1 \\ 2(2^m - k) & 2^{j-1} \\ -2(2^m - k) & 2^{i+-2} \end{pmatrix}. \quad (49)$$

#### Cost

$\lceil \log k \rceil$  auxiliary qubits per positive monomial.

#### Pros

- Logarithmic number of auxiliary variables.

#### Cons

- Introduces all terms non-submodular except for the term linear in auxiliaries.

#### Example

$$b_1 b_2 b_3 b_4 \rightarrow \frac{1}{2} (4 + b_1 + b_2 + b_3 + b_4 - b_{a_1} - 2b_{a_2}) (3 + b_1 + b_2 + b_3 + b_4 - b_{a_1} - 2b_{a_2}) \quad (50)$$

#### Alternate Forms

$$b_1 b_2 \dots b_k \rightarrow \left( 2^m - k + \sum_i b_i - \sum_i 2^{i-1} b_{a_i} \right)^2 \quad (51)$$

#### Bibliography

- Original paper (Theorem 4, special case of Theorem 1): [? ].



## F. PTR-BCR-4 (Boros, Crama, and Rodríguez-Heck, 2018)

### Summary

Pick  $m$  such that  $k \leq 2^{m+1}$ ,

$$b_1 \dots b_k \rightarrow \frac{1}{2} \left( 2^{m+1} - k + \sum_i b_i - \sum_i^m 2^i b_{a_i} \right) \left( 2^{m+1} - k + \sum_i b_i - \sum_i^m 2^i b_{a_i} - 1 \right). \quad (52)$$

### Cost

$\lceil \log k \rceil - 1$  auxiliary qubits per positive monomial.

### Pros

- Logarithmic number of auxiliary variables.

### Cons

- Introduces many non-submodular terms.

### Example

$$b_1 b_2 b_3 b_4 \rightarrow \frac{1}{2} (b_1 + b_2 + b_3 + b_4 - 2b_a) (b_1 + b_2 + b_3 + b_4 - 2b_a - 1) \quad (53)$$

### Alternate Forms

Let  $X = \sum b_i$  and  $N = 2^{m+1} - k$ ,

$$b_1 \dots b_k = \min_{b'_1, \dots, b'_n} \frac{1}{2} \left( N + X - \sum_{i=1}^n 2^i b'_i \right) \left( N + X - \sum_{i=1}^n 2^i b'_i - 1 \right). \quad (54)$$

### Bibliography

- Original paper (Theorem 5): [? ].

### G. PTR-KZ (Kolmogorov & Zabih, 2004)

#### *Summary*

This method can be used to re-write positive or negative cubic terms in terms of 6 quadratic terms. The identity is given by:

$$b_1 b_2 b_3 \rightarrow 1 - (b_a + b_1 + b_2 + b_3) + b_a (b_1 + b_2 + b_3) + b_1 b_2 + b_1 b_3 + b_2 b_3 \quad (55)$$

#### *Cost*

- 1 auxiliary variable per positive or negative cubic term.

#### *Pros*

- Works on positive or negative monomials.
- Reproduces the full spectrum.

#### *Cons*

- Introduces all 6 possible non-submodular quadratic terms.

#### *Alternate Names*

- "Reduction by Minimum Selection" [? ? ].

#### *Bibliography*

- Original paper: [? ].

## H. Reduction by Minimum Selection (in terms of $z$ )

### *Summary*

The formula is almost the same as in the version PTR-KZ in terms of  $b$ , but with a factor of 2, a change of sign for the linear terms, and a slight change in the constant term:

$$\pm z_1 z_2 z_3 \rightarrow 3 \pm (z_1 + z_2 + z_3 + z_a) + 2z_a (z_1 + z_2 + z_3) + z_1 z_2 + z_1 z_3 + z_2 z_3. \quad (56)$$

### *Cost*

- 1 auxiliary variable per positive or negative cubic term.

### *Pros*

- Works on positive or negative monomials.
- Reproduces the full spectrum.

### *Cons*

- Introduces all 6 possible non-submodular quadratic terms.

### *Bibliography*

- 31 March 2016 published by Chancellor, Zohren, and Warburton without the constant term: [? ].
- 8 April 2016 published independently by Leib, Zoller, and Lechner without the constant term [? ? ].

## I. Asymmetric Reduction

### *Summary*

Similar to other methods of reducing one term, this method can reduce a positive cubic monomial into quadratic terms using only one auxiliary variable, while introducing fewer non-submodular terms than the symmetric version.

The identity is given by:

$$b_1 b_2 b_3 \rightarrow b_a - b_2 b_a - b_3 b_a + b_1 b_a + b_2 b_3 \quad (57)$$

$$\rightarrow b_a - b_1 b_a - b_3 b_a + b_2 b_a + b_1 b_3 \quad (58)$$

$$\rightarrow b_a - b_1 b_a - b_2 b_a + b_3 b_a + b_1 b_2. \quad (59)$$

$$(60)$$

### *Cost*

1 auxiliary variable per positive cubic term.

### *Pros*

- Works on positive monomials.
- Fewer non-submodular terms than Ishikawa Reduction.

### *Cons*

- Only been shown to work for cubics.

### *Example*

$$b_1 b_2 b_3 + b_1 b_3 - b_2 \rightarrow (b_a - b_1 b_a - b_3 b_a + b_2 b_a + 2b_1 b_3) - b_2 \quad (61)$$

### *Bibliography*

- Original paper and application to computer vision: [? ].

## J. Symmetry Based Mappings

### Summary

Auxilliary qubits can be made to “count” the number of logical qubits in the 1 configuration. By applying single qubit terms to the auxilliary qubits, the spectrum of *any* permutation symmetric Hamiltonian can be reproduced.

### Cost

- For a  $k$  local coupler requires  $k$  auxilliary qubits.

### Pros

- Natural flux qubit implementation [? ].
- Single gadget can reproduce any permutation symmetric spectrum.
- High degree of symmetry means this method is natural for some kinds of quantum simulations [? ].

### Cons

- Requires coupling between all logical bits and from all logical bits to all auxilliary bits.
- Requires single body terms of increasing strength as  $k$  is increased.

### Example

A 4 qubit gadget guarantees that the number of auxillary bits in the  $-1$  state is equal to the number of logical bits in the  $1$  state

$$H_{4\text{-count}} = 4 \sum_{i=2}^4 \sum_{j=1}^{i-1} b_i b_j + 4 \sum_{i=1}^4 \sum_{j=1}^4 b_i b_{a_j} - 15 \sum_{i=1}^4 b_i - 8 \sum_{i=1}^4 b_{a_i} + (5b_{a_1} + b_{a_2} - 3b_{a_3} - 7b_{a_4}) + 26 \quad (62)$$

This gadget can be expressed more naturally in terms of  $z$ :

$$H_{4\text{-count}} = \sum_{i=2}^4 \sum_{j=1}^{i-1} z_i z_j - \frac{1}{2} \sum_{i=1}^4 z_i + \sum_{i=1}^4 \sum_{j=1}^4 z_i z_{a_j} + \frac{1}{2} (5z_{a_1} + z_{a_2} - 3z_{a_3} - 7z_{a_4}). \quad (63)$$

To replicate the spectrum of  $b_1 b_2 b_3 b_4$ , we add

$$H_{2\text{-local}} = -b_{a_4} + \lambda H_{4\text{-count}}. \quad (64)$$

where  $\lambda$  is a large number.

For the spectrum of

$$\begin{aligned} z_1 z_2 z_3 z_4 &= 16 b_1 b_2 b_3 b_4 - 8 (b_1 b_2 b_3 + b_1 b_2 b_4 + b_1 b_3 b_4 + b_2 b_3 b_4) + \\ &4 (b_1 b_2 + b_1 b_3 + b_1 b_4 + b_2 b_3 + b_2 b_4 + b_3 b_4) - 2 (b_1 + b_2 + b_3 + b_4) + 1, \end{aligned} \quad (65)$$

we implement,

$$H_{2\text{-local}} = 2b_{a_1} - 2b_{a_2} + 2b_{a_3} - 2b_{a_4} + \lambda H_{4\text{-count}}, \quad (66)$$

### Bibliography

- Paper on flux qubit implementation: [? ]
- Paper on max-k-sat mapping: [? ]
- Talk including use in quantum simulation: [? ]

## K. Bit flipping (Ishikawa, 2011)

### Summary

For any variable  $b$ , we can consider the negation  $\bar{b} = 1 - b$ . The process of exchanging  $b$  for  $\bar{b}$  is called *flipping*. Using bit-flipping, an arbitrary function in  $n$  variables can be represented using at most  $2^{(n-2)}(n-3) + 1$  variables, though this is a gross overestimate.

Can be used in many different ways:

1. Flipping positive terms and using **II A**, recursively;
2. For  $\alpha < 0$ , we can reduce  $\alpha \bar{b}_1 \bar{b}_2 \dots \bar{b}_k$  very efficiently to submodular form using **II A**. A generalized version exists for arbitrary combinations of flips in the monomial which makes reduction entirely submodular [?];
3. When we have quadratized we can minimize the number of non-submodular terms by flipping.
4. We can make use of both  $b_i$  and  $\bar{b}_i$  in the same Hamiltonian by adding on a sufficiently large penalty term:  $\lambda(b_i + \bar{b}_i - 1)^2 = \lambda(1 + 2b_i \bar{b}_i - b_i - \bar{b}_i)$ . This is similar to the ideas in reduction by substitution or deduc-reduc. In this way, given a quadratic in  $n$  variables we can make sure it only has at most  $n$  nonsubmodular terms if we are willing to use the extra  $n$  negation variables as well (so we have  $2n$  variables in total).

### Cost

- None, as replacing  $b_i$  with it's negation  $\bar{b}_i$  costs nothing except a trivial symbolic expansion.

### Pros

- Cheap and effective way of improving submodularity.
- Can be used to combine terms in clever ways, making other methods more efficient.

### Cons

- Unless the form of the Hamiltonian is known, spotting these 'factorizations' using negations is difficult.
- We need an auxiliary variable for each  $b_i$  for which we also want to use  $\bar{b}_i$  in the same Hamiltonian.

### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (67)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (68)$$

The first expression is highly non-submodular while the second is entirely submodular.

### Bibliography

- Original paper: [?].

#### IV. METHODS THAT QUADRATIZE MULTIPLE TERMS WITH THE SAME AUXILIARIES (CASE 1: SYMMETRIC FUNCTION REDUCTIONS, SFR)

A symmetric function is one where if we switch any of the variable names (for example  $b_1 \rightarrow b_5 \rightarrow b_8 \rightarrow b_1$ ), the function's output is unaffected.

##### A. SFR-ABCG-1 (Anthony, Boros, Crama, Gruber, 2014)

###### *Summary*

Any  $n$ -variable symmetric function  $f(b_1, b_2, \dots, b_n) \equiv f(b)$  can be quadratized with  $n - 2$  auxiliaries:

$$f(b) \rightarrow -\alpha_0 - \alpha_0 \sum_i b_i + a_2 \sum_{ij} b_i b_j + 2 \sum_i (\alpha_i - c) b_{a_i} \left( 2i - \frac{1}{2} - \sum_j b_j \right) \quad (69)$$

$$c = \begin{cases} \min(\alpha_{2j}) & , i \in \text{even} \\ \min(\alpha_{2j-1}) & , i \in \text{odd} \end{cases} \quad (70)$$

$$a_2 = \text{Determined from Page 12 of [? ]} \quad (71)$$

###### *Cost*

- $n - 2$  auxiliaries for any  $n$ -variable symmetric function.
- $n^2$  non-submodular quadratic terms (of the non-auxiliary variables).
- $n - 2$  non-submodular linear terms (of the auxiliary variables).

###### *Pros*

- Quadratization is symmetric in all non-auxiliary variables (this is not always true, for example some of the methods in the NTR section).
- Reproduces the full spectrum.
- When there's a large number of terms, there's fewer auxiliary variables than quadratizing each positive monomial separately.

###### *Cons*

- Only works on a specific class of functions, although the quadratizations of arbitrary functions can be related to the quadratizations of symmetric functions on a larger number of variables.
- All quadratic terms of the non-auxiliary variables are non-sub-modular.
- All linear terms of the auxiliaries are non-submodular.
- Not meant so much to be practical, but rather an easy proof of an upper bound on the number of needed auxiliaries.

###### *Bibliography*

- 2014, original paper (Theorem 4.1, with  $\alpha_i$  from Corollary 2.3): [? ].

## B. SFR-BCR-1 (Boros, Crama, Rodríguez-Hector, 2018)

### Summary

Any  $n$ -variable symmetric function  $f(b_1, b_2, \dots, b_n) \equiv f(b)$  that is non-zero only when  $\sum b_i = c$  where  $n/2 \leq c \leq n$ , can be quadratized with  $m = \lceil \log_2 c \rceil + 1$  auxiliary variables:

$$f(b) \rightarrow \alpha + \alpha^b \sum_i b_i + \alpha^{b_{a,1}} \sum_i^{m-1} b_{a_i} + \alpha^{b_{a,2}} b_{a_m} + \alpha^{bb} \sum_{i,j} b_i b_j + \alpha^{bb_{a,1}} \sum_i^{m-1} b_i b_{a_j} + \alpha^{bb_{a,2}} \sum_i b_i b_{a_m} + \alpha^{b_a b_a} \sum_{i,j}^{m-1} b_{a_i} b_{a_j}, \quad (72)$$

where:

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (c+1)^2 & 1 \\ -2(c+1) & -2^i \\ 2(c+1) & -2(2m - 2^{m-1} + 1) \\ 2(c+1)(2c - 2^{m-1} + 1) & 2^{i+j-2} \end{pmatrix}. \quad (73)$$

### Cost

- $m = \lceil \log_2 c \rceil + 1$  auxiliary variables.
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

### Pros

- Small number of auxiliary terms

### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (74)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (75)$$

The first expression is highly non-submodular while the second is entirely submodular.

### Alternate Forms

$$f(b_1, b_2, \dots, b_n) \rightarrow \left( -(c+1) + \sum_i b_i - (2c - 2^{m-1} + 1) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} \right)^2 \quad (76)$$

### Bibliography

- 2018, original paper (Theorem 1): [? ].



### C. SFR-BCR-2 (Boros, Crama, Rodríguez-Hector, 2018)

#### Summary

Any  $n$ -variable symmetric function  $f(b_1, b_2, \dots, b_n) \equiv f(b)$  that is non-zero only when  $\sum b_i = c$  where  $0 \leq c \leq n/2$ , can be quadratized with  $m = \lceil \log_2(n - c) \rceil + 1$  auxiliary variables:

$$f(b) \rightarrow \left( (m-1) - \sum_i b_i - (2(n-m) - 2^{\lceil \log(n-m) \rceil} + 1) b_{a_{\lceil \log(n-m) \rceil + 1}} - \sum_i^{\lceil \log(n-m) \rceil} 2^{i-1} b_{a_i} \right)^2 \quad (77)$$

where:

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (c+1)^2 & 1 \\ -2(c+1) & -2^i \\ 2(c+1) & -2(2m - 2^{m-1} + 1) \\ 2(c+1)(2c - 2^{m-1} + 1) & 2^{i+j-2} \end{pmatrix}. \quad (78)$$

#### Cost

- $m = \lceil \log_2(n - c) \rceil + 1$  auxiliary variables.
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

#### Pros

- Small number of auxiliary terms

#### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

#### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (79)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (80)$$

The first expression is highly non-submodular while the second is entirely submodular.

#### Bibliography

- 2018, original paper (Theorem 1): [? ].

### D. SFR-BCR-3 (Boros, Crama, Rodríguez-Hector, 2018)

#### Summary

Any  $n$ -variable symmetric function  $f(b_1, b_2, \dots, b_n) \equiv f(b)$  that is non-zero only when  $\sum b_i = c$  where  $n/2 \leq c \leq n$ , can be quadratized with  $m = \lceil \log_2 c \rceil + 1$  auxiliary variables  $f \rightarrow$ :

$$\frac{1}{2} \left( \sum_i b_i - (m - 2^c) b_{a_{c+1}} - (m + 1)((1 - b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i}) \right) \left( \sum_i b_i - (m - 2^c) b_{a_{c+1}} - (m + 1)((1 - b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i} - 1) \right) \quad (81)$$

where:

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (c+1)^2 & 1 \\ -2(c+1) & -2^i \\ 2(c+1) & -2(2m - 2^{m-1} + 1) \\ 2(c+1)(2c - 2^{m-1} + 1) & 2^{i+j-2} \end{pmatrix}. \quad (82)$$

#### Cost

- $m = \lceil \log_2 c \rceil + 1$  auxiliary variables.
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

#### Pros

- Small number of auxiliary terms

#### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

#### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (83)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (84)$$

The first expression is highly non-submodular while the second is entirely submodular.

#### Alternate Forms

$$f(b_1, b_2, \dots, b_n) \rightarrow \left( \frac{\frac{1}{2} \left( \sum_i b_i - (m - 2^c) b_{a_{c+1}} - (m + 1)((1 - b_{a_{c+1}}) - \sum_i^c 2^{i-1} b_{a_i}) \right)}{2} \right) \quad (85)$$

#### Bibliography

- 2018, original paper (Theorem 2): [? ].

### E. SFR-BCR-4 (Boros, Crama, Rodríguez-Hector, 2018)

#### Summary

Any  $n$ -variable symmetric function  $f(b_1, b_2, \dots, b_n) \equiv f(b)$  that is non-zero only when  $\sum b_i = c$  where  $0 \leq c \leq n/2$ , can be quadratized with  $m = \lceil \log_2(n - c) \rceil + 1$  auxiliary variables  $f \rightarrow$ :

$$\frac{1}{2} \left( (c-1) - \sum_i b_i - (2(n-c) - 2^{m-1} + 1) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} \right) \left( (c-1) - \sum_i b_i - (2(n-c) - 2^{m-1} + 1) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} - 1 \right) \quad (86)$$

where:

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_{a,2}b_{a,1}} \end{pmatrix} = \begin{pmatrix} (c+1)^2 & 1 \\ -2(c+1) & -2^i \\ 2(c+1) & -2(2m - 2^{m-1} + 1) \\ 2(c+1)(2c - 2^{m-1} + 1) & 2^{i+j-2} \end{pmatrix}. \quad (87)$$

#### Cost

- $m = \lceil \log_2(n - c) \rceil + 1$  auxiliary variables.
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

#### Pros

- Small number of auxiliary terms

#### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

#### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (88)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (89)$$

The first expression is highly non-submodular while the second is entirely submodular.

#### Alternate Forms

$$f(b_1, b_2, \dots, b_n) \rightarrow \left( \frac{1}{2} \left( (c-1) - \sum_i b_i - (2(n-c) - 2^{m-1} + 1) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} \right) \right) \quad (90)$$

#### Bibliography

- 2018, original paper (Theorem 2): [? ].

## F. SFR-BCR-5 (Boros, Crama, Rodríguez-Hector, 2018)

### Summary

For an  $n$ -variable symmetric function that is a function of the sum of all variables  $f(b_1, b_2, \dots, b_n) = f(\sum b_i)$ , for some large value of  $\lambda > \max(f)$ , and  $\lceil \sqrt{n+1} \rceil$ :

$$f(\sum b_i) \rightarrow \sum_{i,j}^m f((i-1)(m+1) + (j-1)) b_{a_i} b_{a_{c+j}} + \lambda \left( \left( 1 - \sum_i^m b_{a_i} \right)^2 + \left( 1 - \sum_i^m b_{a_{c+i}} \right)^2 + \left( \sum_i b_i - \left( (m+1) \sum_i^m (i-1) y_{a_i} + \sum_i^m (i-1) b_{a_{c+i}} \right) \right)^2 + \left( \sum_i b_i - \left( (m+1) \sum_i^m (i-1) y_{a_i} + \sum_i^m (i-1) b_{a_{c+i}} \right) \right)^2 \right) \quad (91)$$

where:

$$\begin{pmatrix} \alpha & \alpha^{bb} \\ \alpha^b & \alpha^{bb_{a,1}} \\ \alpha^{b_{a,1}} & \alpha^{bb_{a,2}} \\ \alpha^{b_{a,2}} & \alpha^{b_a b_a} \end{pmatrix} = \begin{pmatrix} (c+1)^2 & 1 \\ -2(c+1) & -2^i \\ 2(c+1) & -2(2m - 2^{m-1} + 1) \\ 2(c+1)(2c - 2^{m-1} + 1) & 2^{i+j-2} \end{pmatrix}. \quad (92)$$

### Cost

- $m = \lceil \sqrt{n+1} \rceil$  auxiliary variables.
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

### Pros

- Small number of auxiliary terms

### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (93)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (94)$$

The first expression is highly non-submodular while the second is entirely submodular.

### Alternate Forms

$$f(b_1, b_2, \dots, b_n) \rightarrow \left( \frac{1}{2} \left( (c-1) - \sum_i b_i - (2(n-c) - 2^{m-1} + 1) b_{a_m} - \sum_i^{m-1} 2^{i-1} b_{a_i} \right) \right) \quad (95)$$

### Bibliography

- 2018, original paper (Theorem 9): [? ].

### G. SFR-BCR-6 (Boros, Crama, Rodríguez-Hector, 2018)

#### Summary

For an  $n$ -variable symmetric function that is a function of a *weighted* sum of all variables  $f(b_1, b_2, \dots, b_n) = f(\sum w_i b_i)$ , for some large value of  $\lambda > \max(f)$ , and  $\max(f(\sum w_i b_i)) < (m+1)^2$ :

$$f(\sum w_i b_i) \rightarrow \sum_{ij}^m \alpha_{ij} b_{a_i} b_{a_{m+i}} + \lambda \left( 1 + \left( \sum_i w_i b_i - (m-1) \sum_i^m b_{a_i} + \sum_i^m b_{a_{c+i}} \right)^2 + \sum_i^{m-1} (1 - b_{a_i}) b_{a_{i+1}} + \sum_i^{m-1} (1 - b_{a_{i+m}}) b_{a_{i+m+1}} \right) \quad (96)$$

where:

$$\sum_i^\alpha \sum_j^\beta \alpha_{ij} = f(\alpha(m+1) + \beta) \quad (97)$$

#### Cost

- $2m$  auxiliary variables, where  $m > \sqrt{\max(f(\sum w_i b_i))} - 1$ .
- $n^2 + m^2$  non-submodular quadratic terms (all possible quadratic terms involving only non-auxiliary or only auxiliary variables).
- $m$  non-submodular linear terms (all possible linear terms involving auxiliaries).

#### Pros

- Small number of auxiliary terms

#### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms.

#### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (98)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (99)$$

The first expression is highly non-submodular while the second is entirely submodular.

#### Alternate Forms

$$f(b_1, b_2, \dots, b_n) \rightarrow \binom{\frac{1}{2}((c-1) - \sum_i b_i - (2(n-c) - 2^{m-1} + 1)b_{a_m} - \sum_i^{m-1} 2^{i-1}b_{a_i})}{2} \quad (100)$$

#### Bibliography

- 2018, original paper (Theorem 10): [? ].

## H. SFR-ABCG-2 (Anthony, Boros, Crama, Gruber, 2014)

### Summary

For any  $n$ -variable,  $k$ -local function that is non-zero only if  $\sum b_i = 2m - 1$ , we call it the "partity function" and it can be quadratized as follows:

$$f(b_1, b_2, \dots, b_n) \rightarrow \sum_i b_i + 2 \sum_{i,j} b_i b_j + 4 \sum_{2i-1}^{n-1} b_{a_i} \left( 2i - 1 - \sum_j b_j \right). \quad (101)$$

### Cost

- $m = 2\lfloor n/2 \rfloor$  auxiliary variables.
- $\lfloor 1.5n \rfloor$  non-submodular linear terms.
- $n^2$  non-submodular quadratic terms.

### Pros

- Smaller number of auxiliary variables than the most naive methods.

### Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms (everything is non-submodular except for  $0.5n^2$  quadratic terms involving the auxiliaries with the non-auxiliaries).
- non-submodular terms can be rather large compared to the submodular terms (about  $4n$  times as big).

### Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (102)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (103)$$

The first expression is highly non-submodular while the second is entirely submodular.

### Bibliography

- 2014, original paper (Theorem 4.6): [? ? ].

# I. SFR-ABCG-3 (Anthony, Boros, Crama, Gruber, 2014)

## Summary

The complement of the parity function can be quadratized as follow:

$$f(b_1, b_2, \dots, b_n) \rightarrow 1 + 2 \sum_{ij} b_i b_j - \sum_i b_i + 4 \sum_{2i}^{n-1} b_{a_i} \left( i - \sum_j^n b_j \right) \quad (104)$$

## Cost

- $m = 2 \lfloor \frac{n-1}{2} \rfloor$  auxiliary variables.
- $\lfloor 0.5n \rfloor$  non-submodular linear terms.
- $n^2$  non-submodular quadratic terms.

## Pros

- Smaller number of auxiliary variables than the most naive methods.
- Fewer non-submodular linear terms than in the analogous quadratization for its complement (the parity function).

## Cons

- Only works for a special class of functions
- Introduces many linear and even more quadratic non-submodular terms (everything is non-submodular except for  $0.5n^2$  quadratic terms involving the auxiliaries with the non-auxiliaries, and all  $n$  linear terms involving only the non-auxiliaries).
- non-submodular terms can be rather large compared to the submodular terms (about  $4n$  times as big).

## Example

By bit-flipping  $b_2$  and  $b_4$ , i.e. substituting  $b_2 = 1 - \bar{b}_2$  and  $b_4 = 1 - \bar{b}_4$ , we see that:

$$H = 3b_1b_2 + b_2b_3 + 2b_1b_4 - 4b_2b_4 \quad (105)$$

$$= -3b_1\bar{b}_2 - \bar{b}_2b_3 - 2b_1\bar{b}_4 - \bar{b}_2\bar{b}_4 + 5b_1 + b_3 + 4\bar{b}_2 + 4\bar{b}_4 - 4. \quad (106)$$

The first expression is highly non-submodular while the second is entirely submodular.

## Bibliography

- 2014, original paper (Theorem 4.6): [? ? ].

### J. Lower bounds for SFRs (Anthony, Boros, Crama, Gruber, 2014)

- There exist symmetric functions on  $n$  variables for which no quadratization can be done without at least  $\Omega(\sqrt{n})$  auxiliary variables (Theorem 5.3 from [? ?]).
- There exist symmetric functions on  $n$  variables for which no quadratization linear in the auxiliaries can be done without at least  $\Omega\left(\frac{n}{\log_2(n)}\right)$  auxiliary variables (Theorem 5.5 from [? ?]).
- The parity function on  $n$  variables cannot be quadratized without quadratic terms involving the auxiliary variables, unless there is at least  $\sqrt{n/4 - 1} + 1 = \Omega(\sqrt{n})$  auxiliary variables (Theorem 5.6 from [? ?]).



## V. METHODS THAT QUADRATIZE MULTIPLE TERMS WITH THE SAME AUXILIARIES (CASE 2: ARBITRARY FUNCTIONS)

### A. Reduction by Substitution (Rosenberg 1975)

#### *Summary*

Pick a variable pair  $(b_i, b_j)$  and substitute  $b_i b_j$  with a new auxiliary variable  $b_{a_{ij}}$ . Enforce equality in the ground states by adding some scalar multiple of the penalty  $P = b_i b_j - 2b_i b_{a_{ij}} - 2b_j b_{a_{ij}} + 3b_{a_{ij}}$  or similar. Since  $P > 0$  if and only if  $b_{a_{ij}} \neq b_i b_j$ , the minimum of the new  $(k-1)$ -local function will satisfy  $b_{a_{ij}} = b_i b_j$ , which means that at the minimum, we have precisely the original function. Repeat  $(k-2)$  times for each  $k$ -local term and the resulting function will be 2-local.

#### *Cost*

- 1 auxiliary variable per reduction.
- At most  $kt$  auxiliary variables for a  $k$ -local Hamiltonian of  $t$  terms, but usually fewer.

#### *Pros*

- Variable can be used across the entire Hamiltonian, reducing many terms at once.
- Very easy to implement.
- Reproduces not only the ground state, but the full spectrum.

#### *Cons*

- Inefficient for single terms as it introduces many auxiliary variables compared to Ishikawa reduction, for example.
- Introduces quadratic terms with large positive coefficients, making them highly non-submodular.
- Determining optimal substitutions can be expensive.

#### *Example*

We pick the pair  $(b_1, b_2)$  and combine.

$$b_1 b_2 b_3 + b_1 b_2 b_4 \mapsto b_3 b_a + b_4 b_a + b_1 b_2 - 2b_1 b_a - 2b_2 b_a + 3b_a \quad (107)$$

#### *Bibliography*

- Original paper: [? ]
- Re-discovered in the context of diagonal quantum Hamiltonians: [? ].
- Used in: [? ? ].

## B. FGBZ Reduction (Fix-Gruber-Boros-Zabih, 2011)

### Summary

Here we consider a set  $C$  of variables which occur in multiple monomials throughout the Hamiltonian. Each application 'rips out' this common component from each term [? ].

Let  $\mathcal{H}$  be a set of monomials, where  $C \subseteq H$  for each  $H \in \mathcal{H}$  and each monomial  $H$  has a weight  $\alpha_H$ . The algorithm comes in 2 parts: when all  $\alpha_H > 0$  and when all  $\alpha_H < 0$ . Combining the 2 gives the final method:

1.  $\alpha_H > 0$

$$\sum_{H \in \mathcal{H}} \alpha_H \prod_{j \in H} b_j = \min_{b_a} \left( \sum_{H \in \mathcal{H}} \alpha_H \right) b_a \prod_{j \in C} b_j + \sum_{H \in \mathcal{H}} \alpha_H (1 - b_a) \prod_{j \in H \setminus C} b_j \quad (108)$$

2.  $\alpha_H < 0$

$$\sum_{H \in \mathcal{H}} \alpha_H \prod_{j \in H} b_j = \min_{b_a} \sum_{H \in \mathcal{H}} \alpha_H \left( 1 - \prod_{j \in C} b_j - \prod_{j \in H \setminus C} b_j \right) b_a \quad (109)$$

### Cost

- One auxiliary variable per application.
- In combination with II A, it can be used to make an algorithm which can reduce  $t$  positive monomials of degree  $d$  in  $n$  variables using  $n + t(d - 1)$  auxiliary variables in the worst case.

### Pros

- Can reduce the connectivity of a Hamiltonian, as it breaks interactions between variables.

### Cons

- $\alpha_H > 0$  method converts positive terms into negative ones of same order rather than reducing them, though these can then be reduced more easily.
- $\alpha_H < 0$  method only works for  $|C| > 1$ , and cannot quadratize cubic terms.

### Example

First let  $C = b_1$  and use the positive weight version:

$$b_1 b_2 b_3 + b_1 b_2 b_4 \mapsto 2b_{a_1} b_1 + (1 - b_{a_1}) b_2 b_3 + (1 - b_{a_1}) b_2 b_4 \quad (110)$$

$$= 2b_{a_1} b_1 + b_2 b_3 + b_2 b_4 - b_{a_1} b_2 b_3 - b_{a_1} b_2 b_4 \quad (111)$$

now we can use II A:

$$-b_{a_1} b_2 b_3 - b_{a_1} b_2 b_4 \mapsto 2b_{a_2} - b_{a_1} b_{a_2} - b_{a_2} b_2 - b_{a_2} b_3 + 2b_{a_2} - b_{a_1} b_{a_2} - b_{a_2} b_2 - b_{a_2} b_4 \quad (112)$$

$$= 4b_{a_2} - 2b_{a_1} b_{a_2} - 2b_{a_2} b_2 - b_{a_2} b_3 - b_{a_2} b_4. \quad (113)$$

### Bibliography

- Original paper and application to image denoising: [? ].

### C. Flag Based SAT Mapping

#### Summary

This method uses gadgets to produce separate 3-SAT clauses which allow variables which ‘flag’ the state of pairs of other variables.

#### Cost

- Varies, but generally higher than other methods.

#### Pros

- Highly general and therefore conducive to proofs.

#### Cons

- Designed for generality rather than efficiency.

#### Example

To create a system which maps  $b_1 b_2 b_3$ , we use the following gadget (note that this is given in terms of  $z$  in the original work and translated to  $b$  here):

$$H_{3\text{-SAT}}(b_1, b_2, b_3) = 2 \sum_{i=1}^3 b_i b_{a_i} + 2 \sum_{i < j}^3 b_{a_i} b_{a_j} - 4 \sum_{i=1}^3 b_{a_i} - 2 \sum_{i=1}^3 b_i + \frac{23}{2}. \quad (114)$$

Implementing  $\lambda H_{3\text{-SAT}}$ , creates a situation where  $b_3$  is a ‘flag’ for  $b_1$  and  $b_2$  in other words  $b_3$  is constrained to be 1 in the low energy manifold if  $b_1 = 0$  and  $b_2 = 0$ . It follows from the universality of 3 – SAT that these ‘flag’ clauses can be combined to map any spin Hamiltonian. To do this, we also need anti-ferromagnetic couplings to express the ‘negated’ variable, to do this, we define,

$$H_{\text{AF}}(b_1, b_{-1}) = 2 b_1 b_{-1} - b_i - b_{-1} + 1. \quad (115)$$

As an explicit example, consider reproducing the spectrum of  $z_1 z_2 z_3 = (2 b_1 - 1)(2 b_2 - 1)(2 b_3 - 1)$ . In this case we need to assign a higher energy to the  $(1, 1, 1)$ ,  $(0, 0, 1)$ ,  $(0, 1, 0)$ , and  $(1, 0, 0)$  states. A flag ( $b_f$ ) which is forced into a higher energy state if these conditions are satisfied can be constructed from two instances of  $H_{3\text{-SAT}}$  and an auxilliary qubit, combining these leads to

$$\begin{aligned} H_{2\text{-local}}(b_1, b_2, b_3) = & \lambda \left[ \sum_{i=1}^3 H_{\text{AF}}(b_i, b_{-i}) + \sum_{i=1}^8 H_{\text{AF}}(b_{a_i}, b_{-a_i}) + H_{\text{AF}}(b_f, b_{-f}) \right. \\ & + H_{3\text{-SAT}}(b_{-1}, b_{-2}, b_{-a_1}) + H_{3\text{-SAT}}(b_{a_1}, b_{-3}, b_f) + H_{3\text{-SAT}}(b_1, b_2, b_{-a_2}) + H_{3\text{-SAT}}(b_{a_2}, b_3, b_{-f}) \\ & + H_{3\text{-SAT}}(b_1, b_2, b_{-a_3}) + H_{3\text{-SAT}}(b_{a_3}, b_{-3}, b_f) + H_{3\text{-SAT}}(b_{-1}, b_{-2}, b_{-a_4}) + H_{3\text{-SAT}}(b_{a_4}, b_3, b_{-f}) \\ & + H_{3\text{-SAT}}(b_1, b_{-2}, b_{-a_5}) + H_{3\text{-SAT}}(b_{a_5}, b_3, b_f) + H_{3\text{-SAT}}(b_{-1}, b_2, b_{-a_6}) + H_{3\text{-SAT}}(b_{a_6}, b_{-3}, b_{-f}) \\ & \left. + H_{3\text{-SAT}}(b_1, b_2, b_{-a_7}) + H_{3\text{-SAT}}(b_{a_7}, b_{-3}, b_f) + H_{3\text{-SAT}}(b_{-1}, b_{-2}, b_{-a_8}) + H_{3\text{-SAT}}(b_{a_8}, b_3, b_{-f}) \right] \\ & - 2 b_f + 1. \end{aligned}$$

The terms on the first line are all to enforce the relationship between a variable and its logical inverse. Each of the next four lines assigns a value to the flag variable  $b_f$  for a state and it’s logical inverse, for instance the leftmost two terms of the second line

enforce that  $b_f = 0$  if  $(b_1, b_2, b_3) = (0, 0, 0)$ , while the right two terms enforce that  $b_f = 1$  if  $(b_1, b_2, b_3) = (1, 1, 1)$ . Because there are  $2^3 = 8$  possible bitstrings for  $(b_1, b_2, b_3)$ , and each term to enforce a flag state requires two instances of  $H_{3\text{-SAT}}$  (and two auxilliary bits), a total of 16 instances are required as well as 16 auxilliary bits. Finally the last line of this equation actually implements the penalty by adding an energy difference between the two states of  $b_f$ . Note that the total number of bits required to express this term is six for the logical bits and their inverses, 16 for the auxilliary qubits, and two for the flag and its inverse. The total number of bits to map this three local Hamiltonian is therefore 24.

### ***Bibliography***

- Paper showing the universality of the Ising spin models: [? ].

**D. Lower bounds for arbitrary functions (Anthony, Boros, Crama, Gruber, 2015)**

- There exist functions on  $n$  variables for which no quadratization can be done without at least  $\frac{2^{n/2}}{8} = \Omega(\sqrt{n})$  auxiliary variables (Theorem 5.3 from [?]).
- There exist symmetric functions on  $n$  variables for which no quadratization linear in the auxiliaries can be done without at least  $\Omega\left(\frac{2^n}{n}\right)$  auxiliary variables (Theorem 5.5 from [?]).

## VI. STRATEGIES FOR COMBINING METHODS

### A. SCM-BCR (Boros, Crama, and Rodríguez-Heck, 2018)

#### *Summary*

Split a  $k$ -local monomial with odd  $k$  into a  $(k - 1)$ -local term (with even degree) and a new odd  $k$ -local term which has negative coefficient:

$$b_1 b_2 \cdots b_k \rightarrow \prod_{i=1}^{k-1} b_i - \prod_{i=1}^{k-1} b_i (1 - b_k) \quad (116)$$

We can use any of the PTR methods for even  $k$  on the first term, and we can use any of the NTR methods on the second term. Can be generalized to split into different-degree factors when seeking an "optimum" quadratization. Can be generalized into more splits.

#### *Cost*

- Depends on the methods used for the PTR and NTR procedures.

#### *Pros*

- Very flexible.

#### *Cons*

- First turns one term into two terms, so might not be preferred when we wish to minimize the number of terms.

#### *Bibliography*

- Original paper: [? ].

## B. Decomposition into symmetric and anti-symmetric parts

### *Summary*

Split a any function  $f$  into a symmetric part and anti-symmetric part:

$$f(b_1, b_2, \dots, b_n) = f_{\text{symmetric}} + f_{\text{anti-symmetric}}, \quad (117)$$

$$f_{\text{symmetric}} \equiv \frac{1}{2} (f(b_1, b_2, \dots, b_n) + f(1 - b_1, 1 - b_2, \dots, 1 - b_n)) \quad (118)$$

$$f_{\text{anti-symmetric}} \equiv \frac{1}{2} (f(b_1, b_2, \dots, b_n) - f(1 - b_1, 1 - b_2, \dots, 1 - b_n)) \quad (119)$$

We can now use any of the methods described only for symmetric functions, on the symmetric part, and use the (perhaps less powerful) general methods on the anti-symmetric part.

### *Cost*

- Depends on the methods used.

### *Pros*

- Allows non-symmetric functions to benefit from techniques designed only for symmetric functions.

### *Cons*

- May result in more terms than simply quadratizing the non-symmetric function directly.

### *Bibliography*

- Discussed in: [? ].

## Part II

### Hamiltonians quadratic in $z$ and linear in $x$ (Transverse Field Ising Hamiltonians)

The Ising Hamiltonian with a transverse field in the  $x$  direction is possible to implement in hardware:

$$H = \sum_i \left( \alpha_i^{(z)} z_i + \alpha_i^{(x)} x_i \right) + \sum_{ij} \left( \alpha_{ij}^{(zz)} z_i z_j \right). \quad (120)$$

#### C. ZZZ-TI-CBBK: Transverse Ising from ZZZ, by Cao, Babbush, Biamonte, and Kais (2015)

There is only one reduction in the literature for reducing a Hamiltonian term to the transverse Ising Hamiltonian, and it works on 3-local  $zzz$  terms, by introducing an auxiliary qubit with label  $a$ :

$$\begin{aligned} \alpha z_i z_j z_k &\rightarrow \alpha^I + \alpha_i^z z_i + \alpha_j^z z_j + \alpha_k^z z_k + \alpha_a^z z_a + \alpha_a^x x_a + \alpha_{ia}^{zz} z_i z_a + \alpha_{ja}^{zz} z_j z_a + \alpha_{ka}^{zz} z_k z_a \\ \alpha^I &= \frac{1}{2} \left( \Delta + \left( \frac{\alpha}{6} \right)^{2/5} \Delta^{3/5} \right) \\ \alpha_i^z &= -\frac{1}{2} \left( \left( \frac{7\alpha}{6} + \left( \frac{\alpha}{6} \right)^{3/5} \Delta^{2/5} \right) - \left( \frac{\alpha \Delta^4}{6} \right)^{1/5} \right) \\ \alpha_j^z &= \alpha_i^{(z)} \\ \alpha_k^z &= \alpha_i^{(z)} \\ \alpha_a^z &= \frac{1}{2} \left( \Delta - \left( \frac{\alpha}{6} \right)^{2/5} \Delta^{3/5} \right) \\ \alpha_a^x &= \left( \frac{\alpha \Delta^4}{6} \right)^{1/5} \\ \alpha_{ia}^{zz} &= -\frac{1}{2} \left( \left( \frac{7\alpha}{6} + \left( \frac{\alpha}{6} \right)^{3/5} \Delta^{2/5} \right) + \left( \frac{\alpha \Delta^4}{6} \right)^{1/5} \right) \\ \alpha_{ja}^{zz} &= \alpha_{ia}^{(zz)} \\ \alpha_{ka}^{zz} &= \alpha_{ja}^{(zz)} \end{aligned} \quad (121)$$

Including all coefficients and factorizing, we get:

$$\alpha z_i z_j z_k \rightarrow \left( \Delta + \frac{\alpha \Delta^{4/5}}{6} (z_i + z_j + z_k) \right) \left( \frac{1 - z_a}{2} \right) + \frac{\alpha \Delta^{4/5}}{6} x_a \quad (122)$$

$$+ \left( \left( \frac{\alpha}{6} \right)^{2/5} \Delta^{3/5} - \left( \frac{7\alpha}{6} + \left( \frac{\alpha}{6} \right)^{3/5} \Delta^{2/5} \right) (z_i + z_j + z_k) \right) \left( \frac{1 + z_a}{2} \right) \quad (123)$$

The low-lying spectrum (eigenvalues *and* eigenvectors) of the right side of Eq. (121) will match those of the left side to within a spectral error of  $\epsilon$  as long as  $\Delta = \mathcal{O}(\epsilon^{-5})$ .

#### Cost

- 1 auxiliary qubit
- 8 auxiliary terms not proportional to  $\mathbb{1}$ .



## Part III

### General Quantum Hamiltonians

#### VII. NON-PERTURBATIVE GADGETS

##### A. NP-OY (Ocko & Yoshida, 2011)

###### Summary

For the 8-body Hamiltonian:

$$H_{8\text{-body}} = -J \sum_{ij} (x_{ij3}x_{ij+1,2}x_{ij4}x_{ij+1,4}x_{i+1j1}x_{i+1j+1,1}x_{i+1j3}x_{ij+1,2} + z_{ij1}z_{ij2}z_{ij3}z_{ij4} + z_{i-1j4}z_{ij1} + z_{ij2}z_{ij-1,3} + z_{ij4}z_{i+1j1} + z_{ij3}z_{ij+1,2}), \quad (124)$$

we define auxiliary qubits labeled by  $a_{ijk}$ , two auxiliaries for each pair  $ij$ : labeled  $a_{ij1}$  and  $a_{ij2}$ . Then the 8-body Hamiltonian has the same low-lying eigenspace as the 4-body Hamiltonian:

$$\begin{aligned} H_{4\text{-body}} = & - \sum_{ij} \alpha (z_{ij1}z_{ij2}z_{ij3}z_{ij4} + z_{i,j,-1,4}z_{ij1} + z_{ij2}z_{i,j-1,3} + z_{ij4}z_{i+1,j,1} + z_{ij3}z_{i,j+1,2} \\ & (1 - z_{a_{ij1}} + z_{a_{ij2}} + z_{a_{ij1}}z_{a_{ij2}}) (z_{a_{i,j+1,1}} + z_{a_{i,j+1,2}} + z_{a_{i,j+1,1}}z_{a_{i,j+1,2}} - 1) + \\ & (1 + z_{a_{ij1}} - z_{a_{ij2}} + z_{a_{ij1}}z_{a_{ij2}}) (1 - z_{a_{i+1,j,1}} - z_{a_{i+1,j,2}} - z_{a_{i+1,j,1}}z_{a_{i+1,j,2}})) + \\ & \frac{U}{2} (z_{a_{ij1}} + z_{a_{ij2}} + z_{a_{ij1}}z_{a_{ij2}} - 1) + \\ & \frac{t}{2} ((x_{a_{ij2}} + z_{a_{ij1}}x_{a_{ij2}})x_{ij3}x_{ij4} + (x_{a_{ij1}}x_{a_{ij2}} + y_{a_{ij1}}y_{a_{ij2}})x_{i,j+1,2}x_{i,j+1,4} + \\ & (x_{a_{ij2}} - z_{a_{ij1}}x_{a_{ij2}})x_{i+1,j+1,1}x_{i+1,j+1,2} + (x_{a_{ij1}}x_{a_{ij2}} - y_{a_{ij1}}y_{a_{ij2}})x_{i+1,j,1}x_{i+1,j,3})). \end{aligned} \quad (125)$$

Now by defining the following ququits (spin-3/2 particles, or 4-level systems):

$$s_{ijk i' j' k'}^{zz} = z_{ijk} z_{i' j' k'} \quad (126)$$

$$s_{a_{ij}1}^{zz} = (1 - z_{a_{1ij}} + z_{a_{2ij}} + z_{a_{1ij}}z_{a_{2ij}}) \quad (127)$$

$$s_{a_{ij}2}^{zz} = (z_{a_{1ij}} + z_{a_{2ij}} + z_{a_{1ij}}z_{a_{2ij}} - 1) \quad (128)$$

$$s_{a_{ij}3}^{zz} = (1 + z_{a_{1ij}} - z_{a_{2ij}} + z_{a_{1ij}}z_{a_{2ij}}) \quad (129)$$

$$s_{a_{ij}1}^{xz} = (x_{a_{2ij}} + z_{a_{1ij}}x_{a_{2ij}}) \quad (130)$$

$$s_{a_{ij}2}^{xz} = (x_{a_{2ij}} - z_{a_{1ij}}x_{a_{2ij}}) \quad (131)$$

$$s_{ijk i' j' k'}^{xx} = x_{ijk} x_{i' j' k'} \quad (132)$$

$$s_{a_{ij}1}^{xy} = (x_{a_{1ij}}x_{a_{2ij}} + y_{a_{1ij}}y_{a_{2ij}}) \quad (133)$$

$$s_{a_{ij}2}^{xy} = (x_{a_{1ij}}x_{a_{2ij}} - y_{a_{1ij}}y_{a_{2ij}}) \quad (134)$$

### NP-OY (Ocko & Yoshida, 2011) [Continued]

We can write the 4-body Hamiltonian on qubits as a 2-body Hamiltonian on ququits:

$$H_{2\text{-body}} = - \sum_{ij} \left( \alpha \left( s_{ij1ij2}^{zz} s_{ij3ij4}^{zz} + s_{ij-1,4ij1}^{zz} + s_{ij2ij-1,3}^{zz} + s_{ij4i+1j1}^{zz} + s_{ij3ij+1,2}^{zz} + s_{a_{ij}1}^{zz} s_{a_{ij+1}2}^{zz} - s_{a_{ij}3}^{zz} s_{a_{i+1j}3}^{zz} \right) \right. \\ \left. + \frac{U}{2} s_{a_{ij},1}^{zz} + \frac{t}{2} \left( s_{a_{ij}1}^{xz} s_{ij3ij4}^{xx} + s_{a_{ij}1}^{xy} s_{ij+1,2ij+1,4}^{xx} + s_{a_{ij}2}^{xz} s_{i+1,j+1,1i+1,j+1,2}^{xx} + s_{a_{ij}2}^{xy} s_{i+1j1,i+1,j3}^{xx} \right) \right). \quad (135)$$

The low-lying eigenspace of  $H_{2\text{-body}}$  is *exactly* the same as for  $H_{4\text{-local}}$ .

#### Cost

- 2 auxiliary ququits for each pair  $ij$ .
- 6 more total terms (6 terms in the 8-body version becomes 12 terms: 11 of them 2-body and 1 of them 1-body).

#### Pros

- Non-perturbative. No prohibitive control precision requirement.
- Only two auxiliaries required for each pair  $ij$ .
- 8-body to 2-body transformation can be accomplished in 1 step, rather than a 1B1 gadget which would take 6 steps or an SD +  $(3 \rightarrow 2)$  gadget combination which would take 4 steps.

#### Cons

- Increase in dimension from working with only 2-level systems (spin-1/2 particles or  $2 \times 2$  matrices) to working with 4-level systems (spin-3/2 particles).
- Until now, only derived for a very specific Hamiltonian form.
- This approach may become more demanding for Hamiltonians that are more than 8-local.

#### Bibliography

- Original paper: [? ].

## B. NP-SJ (Subasi & Jarzynski, 2016)

### Summary

Determine the  $k$ -local term,  $H_{k\text{-local}}$ , whose degree we wish to reduce, and factor it into two commuting factors:  $H_{k'\text{-local}}H_{(k-k')\text{-local}}$ , where  $k'$  can be as low as 0. Separate all terms that are at most  $(k-1)$ -local into ones that commute with one of these factors (it does not matter which one, but without loss of generality we assume it to be the  $(k-k')$ -local one) and ones that anti-commute with it:

$$H_{<k\text{-local}}^{\text{commuting}} + H_{<k\text{-local}}^{\text{anti-commuting}} + \alpha H_{k'\text{-local}} H_{(k-k')\text{-local}} \quad (136)$$

Introduce one auxiliary qubit labeled by  $a$  and the Hamiltonian:

$$\alpha x_a H_{k'\text{-local}} + H_{<k\text{-local}}^{\text{commuting}} + z_a H_{<k\text{-local}}^{\text{anti-commuting}} \quad (137)$$

no longer contains  $H_{k\text{-local}}$  but  $H_{<k\text{-local}}^{\text{anti-commuting}}$  is now one degree higher.

### Cost

- 1 auxiliary qubit to reduce  $k$ -local term to  $(k' + 1)$ -local where  $k'$  can even be 0-local, meaning the  $k$ -local term is reduced to a 1-local one.
- Raises the  $k$ -locality of  $H_{<k\text{-local}}^{\text{anti-commuting}}$  by 1 during each application. It can become  $(> k)$ -local!

### Pros

- Non-perturbative
- Can linearize a term of arbitrary degree in one step.
- Requires very few auxiliary qubits.

### Cons

- Can introduce many new non-local terms as an expense for reducing only one  $k$ -local term.
- If the portion of the Hamiltonian that does not commute with the  $(k-k')$ -local term has terms of degree  $k-1$  (which can happen if  $k' = 0$ ) they will all become  $k$ -local, so there is no guarantee that this method reduces  $k$ -locality.
- If any terms were more than 1-local, this method will not fully quadratize the Hamiltonian (it must be combined with other methods).
- It only works when the Hamiltonian's terms of degree at most  $k-1$  all either commute or anti-commute with the  $k$ -local term to be eliminated.

### Example

$$4z_5 - 3x_1 + 2z_1y_2x_5 + 9x_1x_2x_3x_4 - x_1y_2z_3x_5 \rightarrow 9x_{a_1} + 4z_{a_4}z_5 - 3z_{a_3}x_1 - z_{a_3}x_{a_4}a_3 + 2x_{a_3}x_5 \quad (138)$$

### Bibliography

- Original paper, and description of the choices of terms and factors used for the given example [? ].