IMPERIAL

Hamlyn Centre for Robotic Surgery

HARMS Lab
Human-centred Automation, Robotics and Monitoring in Surgery

Week 8
Jacobian and Velocities

MRes in Medical Robotics and Instrumentation

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Contents of the lecture

- Revision of main concepts from last week
- Velocity Representation
- Linear and Angular Velocities
- Interlink Velocity Propagation
- Use of Jacobian for velocities mapping

- John J Craig, Introduction to robotics – mechanics and control, 3rd Edition, Pearson Education International, London, ISBN 0-13-123629-6.
- Reza N Jazar, Theory of applied robotics, 2nd Edition, Springer, ISBN 978-1-4419-1749-2.

Revision: Inverse Kinematics (IK) problem

- Inverse kinematics determine the joint configurations of a robot model to achieve a desired end-effect position

We are looking for these

$$
\begin{tikzcd}
\{0\} \arrow[r] \arrow[r, bend left, "{}^{0}T_{1}=?"] & \{1\} \arrow[r] \arrow[r, bend left, "{}^{1}T_{2}=?"] & \{2\} \arrow[r] \arrow[r, bend left, "{}^{2}T_{3}=?"] & \{3\} \arrow[r] \arrow[r, bend left, "{}^{3}T_{4}=?"] & \{4\} \arrow[r] \arrow[r, bend left, "{}^{4}T_{5}=?"] & \{5\} \arrow[r] \arrow[r, bend left, "{}^{5}T_{6}=?"] & \{6\} \arrow[llllll, bend left, "{}^{0}T_{6}"]
\end{tikzcd}
$$

We know this

The problem of inverse kinematics is **nonlinear**;
the change of the output is not proportional to the change of the input

Revision: Algebraic solution of Inverse Kinematics

Inverse kinematics problem to solve

- End-effector link 3 is located at (x_a, y_a) with an orientation of β relative to frame {0}
- Problem: Find θ_i, $(i=1, 2, 3)$?

$$ {}^0T_3 = \begin{bmatrix} c\beta & -s\beta & 0 & x_a \\ s\beta & c\beta & 0 & y_a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^0T_3 = \begin{bmatrix} c\theta_{123} & -s\theta_{123} & 0 & l_1 c\theta_1 + l_2 c\theta_{12} \\ s\theta_{123} & c\theta_{123} & 0 & l_1 s\theta_1 + l_2 s\theta_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

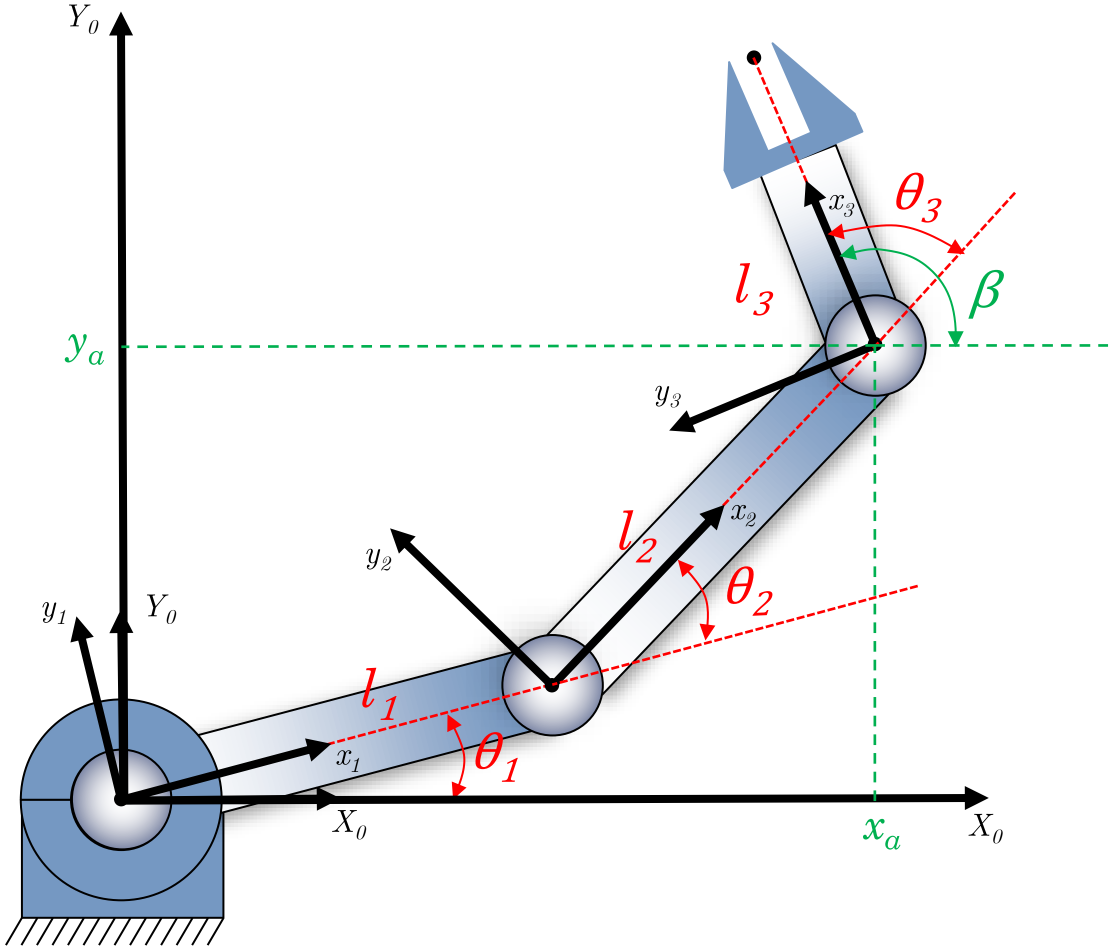
Expressed in task/cartesian space

Expressed in joint space

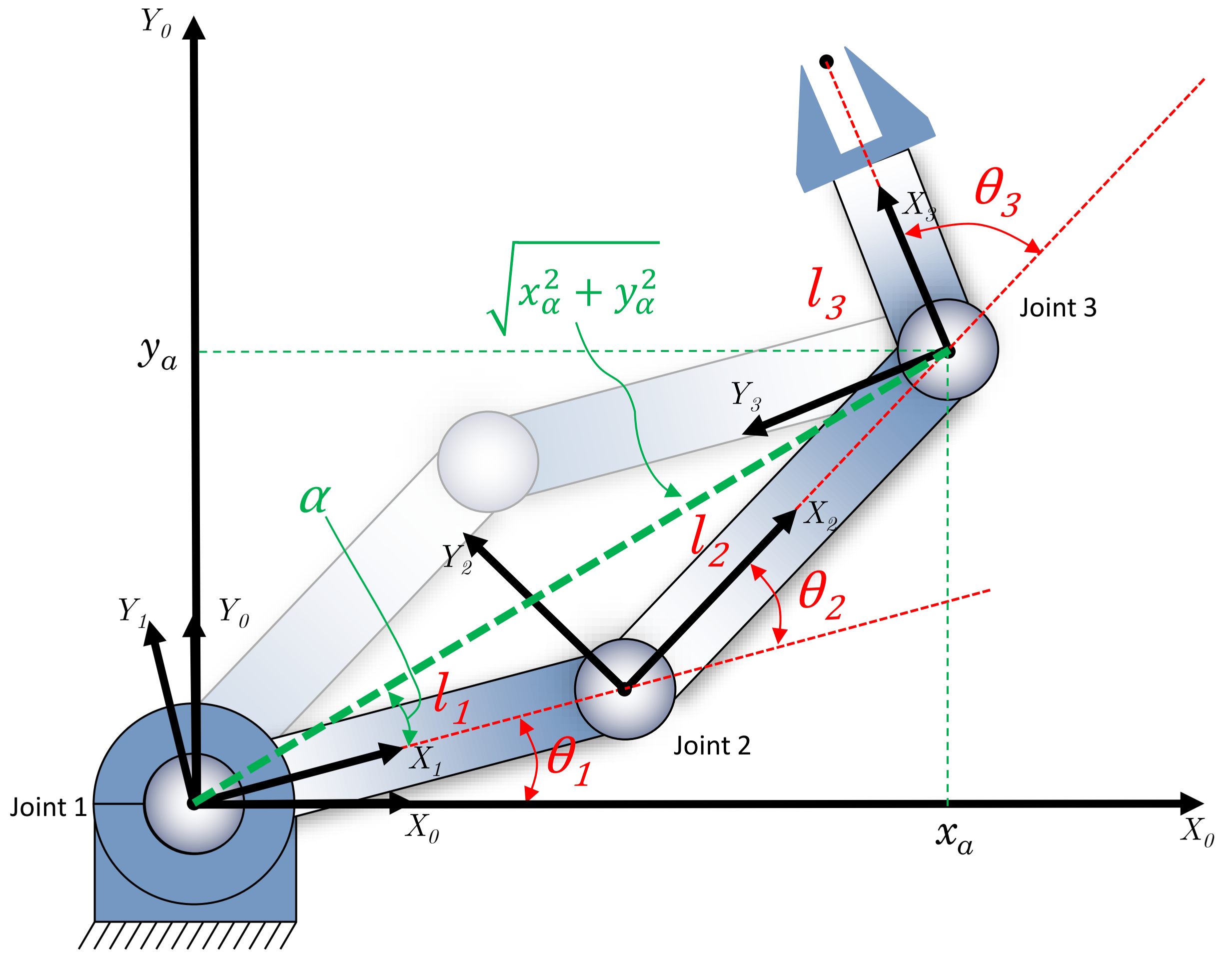
Equate above:

$$ \begin{cases} x_a = l_1 c\theta_1 + l_2 c\theta_{12} \\ y_a = l_1 s\theta_1 + l_2 s\theta_{12} \\ c\beta = c\theta_{123} \\ s\beta = s\theta_{123} \end{cases} $$

Kinematic equations (non-linear)



Revision: Geometric solution of Inverse Kinematics



$$x_a^2 + y_a^2 = l_1^2 + l_2^2 - 2l_1l_2c(180-\theta_2)$$
$$= l_1^2 + l_2^2 + 2l_1l_2c\theta_2$$

Cosine rule

$$\boxed{\theta_2 = \pm\cos^{-1}\left[(x_a^2+y_a^2)-(l_1^2+l_2^2)\right]/2l_1l_2}$$

Condition: $\left|\left[(x_a^2+y_a^2)-(l_1^2+l_2^2)\right]/2l_1l_2\right| \le 1$

Which basically means that $\sqrt{x_a^2+y_a^2} \le l_1 + l_2$

$$l_2^2 = x_a^2 + y_a^2 + l_1^2 - 2l_1\sqrt{x_a^2+y_a^2}\,c\alpha$$

Cosine rule in α

$$c\alpha = \frac{x_a^2+y_a^2+l_1^2-l_2^2}{2l_1\sqrt{x_a^2+y_a^2}}$$

$$\boxed{\theta_1 = \operatorname{atan2}(y_a, x_a) \pm \alpha}$$

$\theta_1 + \theta_2 + \theta_3 = \beta$, solve for θ_3

Revision: Newton-Raphson Method

- Solve (find the roots of) $p(\theta) = a$ or $p(\theta) - a = 0$

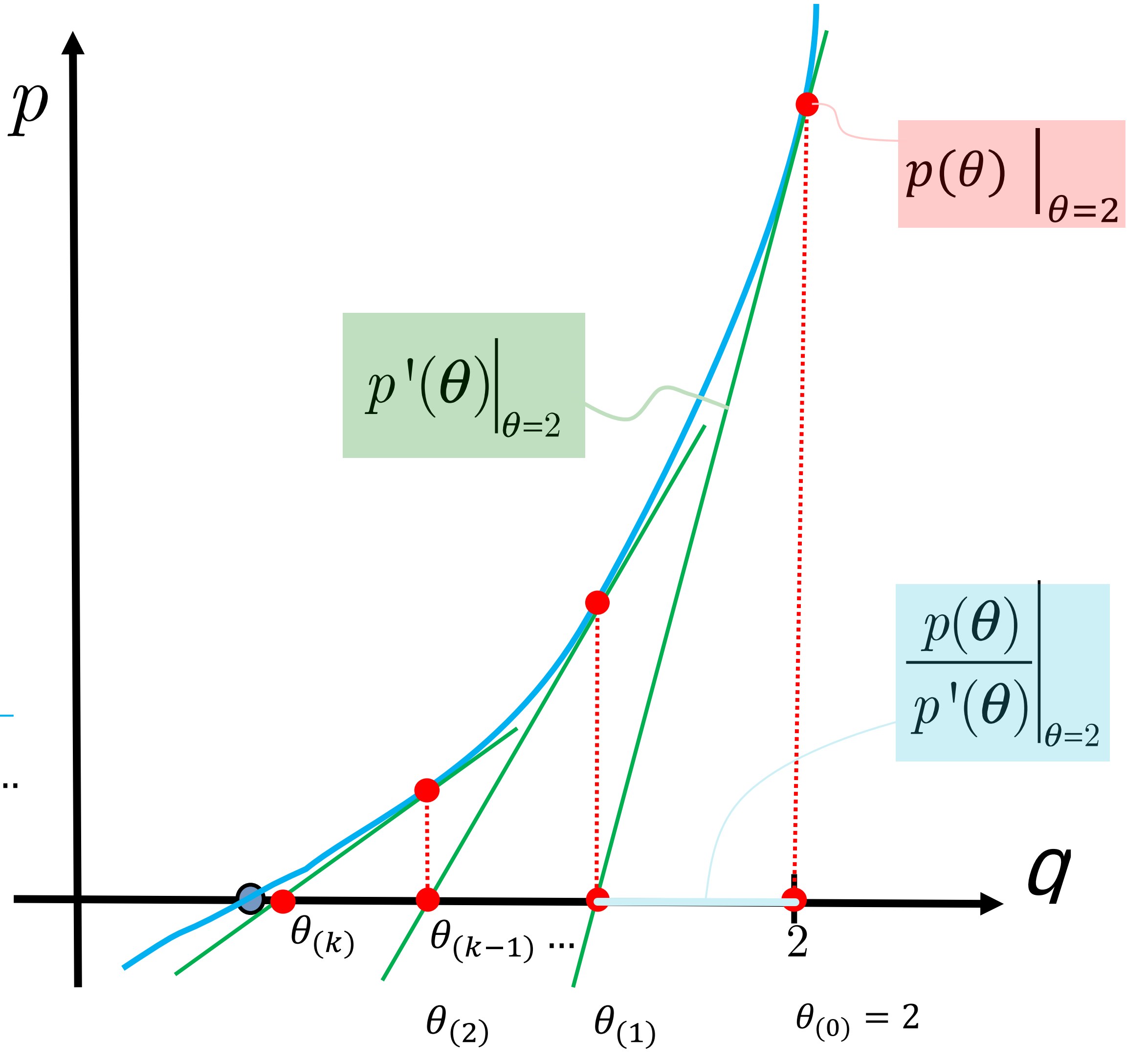
Iterative solution steps

1. Start with an initial value of $\theta_{(0)}$ (**a –hopefully good – guess**)
2. Evaluate p for this guess $p(\theta_{(0)})$
3. Iteratively refine the initial guess of θ values, to approach the true solution for p

$\theta_{(k)}$ represents the result at iteration step k.

$$\overbrace{\theta_{(0)} \to p(\theta_{(0)}) \to p'(\theta_{(0)})}^{k=0,\ \theta_0 \text{ is a guess}} \to \overbrace{\theta_{(1)} \to p(\theta_{(1)}) \to p'(\theta_{(1)})}^{k=1} \to \overbrace{\theta_{(2)} \to p(\theta_{(2)}), \ldots}^{k=2}$$

$$\theta_{(k)} = \theta_{(k-1)} + p'\left(\theta_{(k-1)}\right)^{-1}\left(a - p\left(\theta_{(k-1)}\right)\right)$$



Revision: Iterative numerical solution of Inverse Kinematics

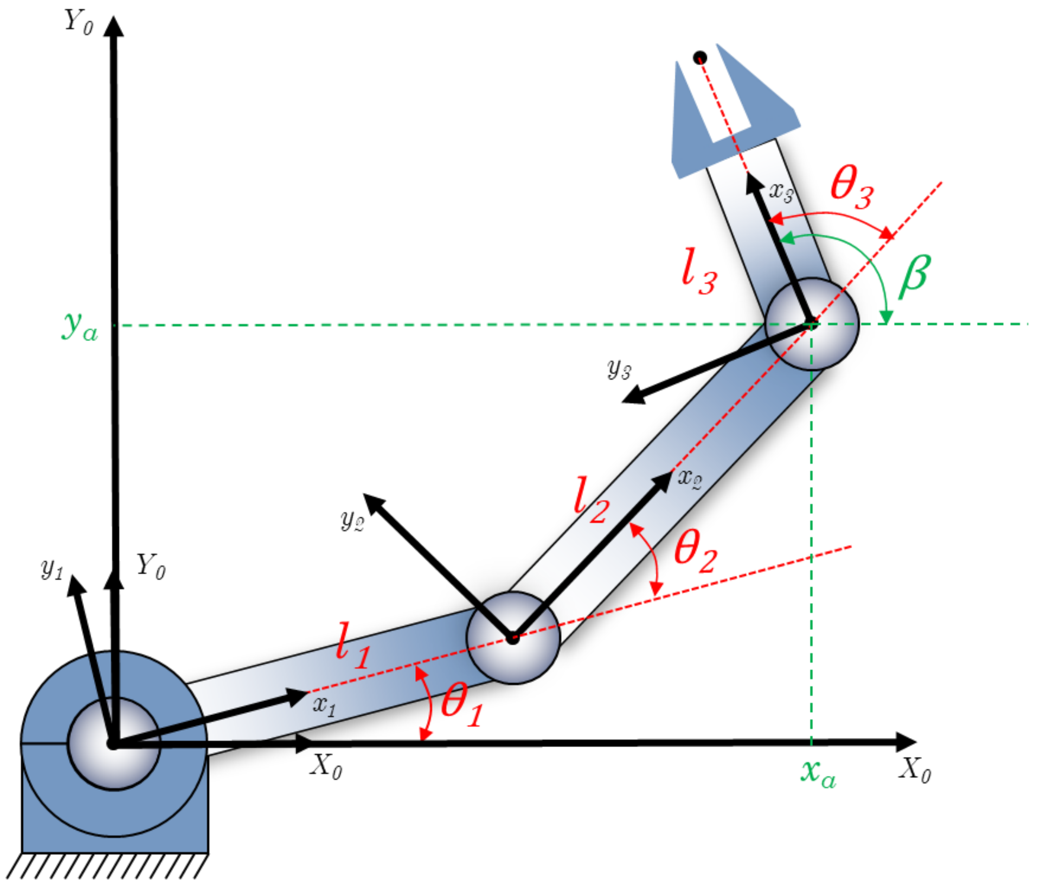
- For a given pose $\mathbf{P}$ of the end effector, we can use the general vector function below to represent the joint kinematics:

$$\mathbf{P} = \mathbf{Q}(\theta_1, \theta_2, \ldots, \theta_N) = \mathbf{Q}(\vec{\theta})$$

- The inverse kinematics problem then would be:

Given $\mathbf{P} = \mathbf{Q}(\vec{\theta})$, find $\vec{q}$

e.g., here, given $\mathbf{P} = \begin{bmatrix} x_a \\ y_a \\ \beta \end{bmatrix} = \mathbf{Q}(\vec{\theta}_1, \vec{\theta}_2, \vec{\theta}_3)$, find $\vec{\theta}_1, \vec{\theta}_2, \vec{\theta}_3$



$$\theta_{(k)} = \theta_{(k-1)} + p'\left(\theta_{(k-1)}\right)^{-1}\left(a - p\left(\theta_{(k-1)}\right)\right)$$

$$\vec{\theta}_{(k)} = \vec{\theta}_{(k-1)} + \mathbf{J}^{-1}\left(\vec{\theta}_{(k-1)}\right)\left(\mathbf{P} - \mathbf{Q}\left(\vec{\theta}_{(k-1)}\right)\right)$$

where $\mathbf{J}\left(\vec{\theta}_{k-1}\right) = \dfrac{\partial \mathbf{Q}\left(\vec{\theta}_{(k-1)}\right)}{\partial \vec{\theta}_{(k-1)}}$

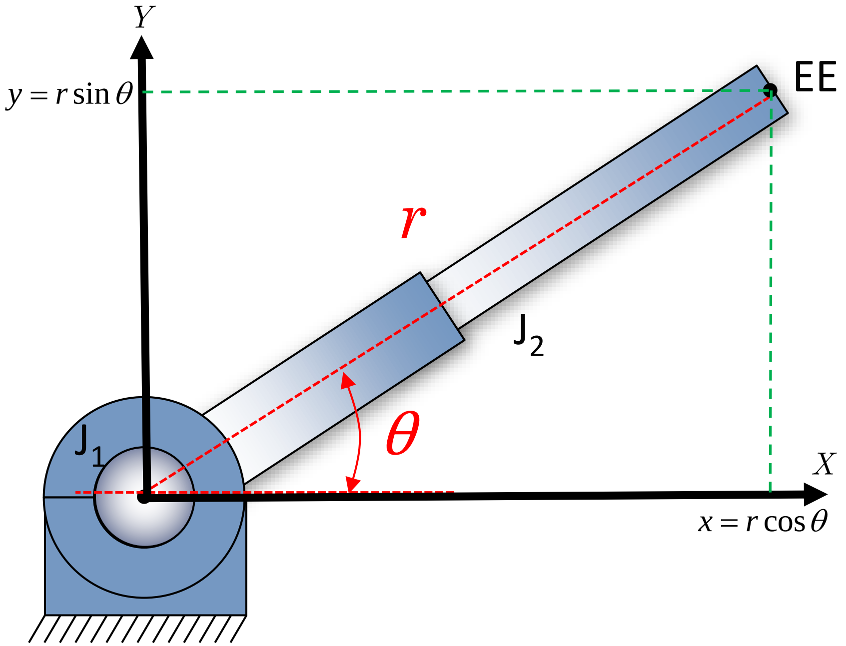
Revision: General expression of manipulator Jacobian matrix

$$\mathbf{J}(\vec{\theta}) = \frac{\partial \mathbf{Q}(\vec{\theta})}{\partial \vec{\theta}} = \underbrace{\begin{bmatrix} \partial q_1(\vec{\theta})/\partial \theta_1 & \partial q_1(\vec{\theta})/\partial \theta_2 & \cdots & \partial q_1(\vec{\theta})/\partial \theta_n \\ \partial q_2(\vec{\theta})/\partial \theta_1 & \partial q_2(\vec{\theta})/\partial \theta_2 & \cdots & \partial q_2(\vec{\theta})/\partial \theta_n \\ \cdots & \cdots & \cdots & \cdots \\ \partial q_N(\vec{\theta})/\partial \theta_1 & \partial q_N(\vec{\theta})/\partial \theta_2 & \cdots & \partial q_N(\vec{\theta})/\partial \theta_n \end{bmatrix}}_{n \text{ columns, number of 1 DoF joints}} \left.\vphantom{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}\right\} N \text{ rows, task space dimension}$$

$$\vec{\theta}_{(k)} = \vec{\theta}_{(k-1)} + \mathbf{J}^{-1}\left(\vec{\theta}_{(k-1)}\right)\left(\mathbf{P} - \mathbf{Q}\left(\vec{\theta}_{(k-1)}\right)\right) \Rightarrow \underbrace{\vec{\theta}_{(k)} - \vec{\theta}_{(k-1)}}_{\text{Joint space}} = \mathbf{J}^{-1}\left(\vec{\theta}_{(k-1)}\right)\left(\underbrace{\mathbf{P} - \mathbf{Q}\left(\vec{\theta}_{(k-1)}\right)}_{\text{Task space}}\right)$$

The ***Jacobian*** defines a mapping between small (differential) changes in joint space, and how they create small (differential) changes in cartesian space, and vice versa

Revision: Jacobian derivation for a simple manipulator



- Polar manipulator: J_1 revolute, J_2 prismatic
- Joint space size: n= 2, as two 1-DoF joints
- Task space dimension: N=2, two translations, ignore rotation
- Kinematic equations: $x = r\cos\theta$, $y = r\sin\theta$

In terms of earlier slides on iterative solution of IK

- Vector of joint variables: $\vec{\theta}=[\theta, r]$
- Pose vector: $\mathbf{P}=[x, y]^T$
- Vector of kinematic equations: $\mathbf{Q}=[r\cos\theta, r\sin\theta]^{\mathrm{T}}$

- End-effector coordinates are $(x, y) = (r\cos\theta, r\sin\theta)$
- The rate of change (velocities) of x and y, using the chain rule to differentiate with respect to time t, is:

$$\frac{dx}{dt} = \frac{\partial(r\cos\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\cos\theta)}{\partial\theta}\frac{d\theta}{dt} \Rightarrow \dot{x} = \cos\theta\dot{r} - r\sin\theta\dot{\theta}$$

$$\frac{dy}{dt} = \frac{\partial(r\sin\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\sin\theta)}{\partial\theta}\frac{d\theta}{dt} \Rightarrow \dot{y} = \sin\theta\dot{r} + r\cos\theta\dot{\theta}$$

$$\begin{bmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{bmatrix} = \begin{bmatrix}\frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial r}\\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial r}\end{bmatrix}\begin{bmatrix}\frac{d\theta}{dt}\\ \frac{dr}{dt}\end{bmatrix}$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}-r\sin\theta & \cos\theta\\ r\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}\dot{\theta}\\ \dot{r}\end{bmatrix} \Rightarrow \begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \mathbf{J}\begin{bmatrix}\dot{\theta}\\ \dot{r}\end{bmatrix}$$

Chain rule for functions of two independent variables

$$z = F(x, y)$$

$$dz = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

$$\frac{dz}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$$

- We can also use the inverse Jacobian to find the joint velocities for a given cartesian velocity of the EE

$$\mathbf{J}^{-1}\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \dot{r}\end{bmatrix} \quad \text{where} \quad \mathbf{J}^{-1} = \begin{bmatrix}\frac{-\sin\theta}{r} & \frac{\cos\theta}{r}\\ \cos\theta & \sin\theta\end{bmatrix}$$

The ***Jacobian*** also defines a mapping between joint velocities and end effector velocity

Today’s material – Velocities and Jacobian

- So far, we have been dealing with static robotic manipulators –
  - Basically, robot configurations frozen in time
- We need to consider linear and angular velocity concepts –
  - Robot configuration changing in time
  - By considering infinitesimally small steps of time
- We will see how to calculate joint/link velocities
  - Analytical approach
  - Differential approach, i.e., using the Jacobian

Velocity representation

- The velocity of a position vector (e.g., ${}^{A}Q$) is the linear velocity of the point in space represented by this vector (e.g., Q).
- The velocity of Q relative to frame $\{A\}$ can be expressed by the derivative of Q relative to $\{A\}$:

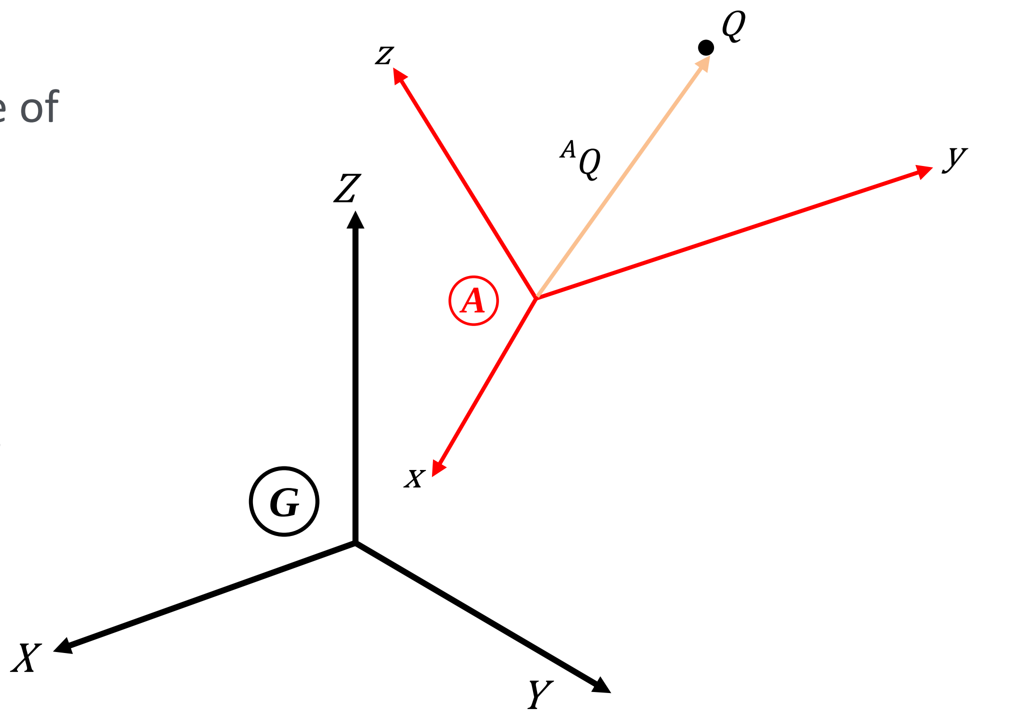
$$ {}^{A}V(Q) = \frac{d}{dt}\,{}^{A}Q = \lim_{\Delta t \to 0} \frac{{}^{A}Q(t+\Delta t) - {}^{A}Q(t)}{\Delta t} $$

- The calculated velocity is <u>written in terms of the frame of differentiation</u>, which is $\{A\}$ in this case.
- However, the velocity vector can be expressed/described in terms of any other frame, e.g., $\{G\}$:

$$ {}^{G}({}^{A}V(Q)) = \frac{{}^{G}d}{dt}\,{}^{A}Q $$

- If the calculated velocity is described in terms of the frame of differentiation, $\{A\}$ in this case, for simplicity:

$$ {}^{A}({}^{A}V(Q)) \equiv {}^{A}V(Q) $$



Velocity representation: change in reference frame

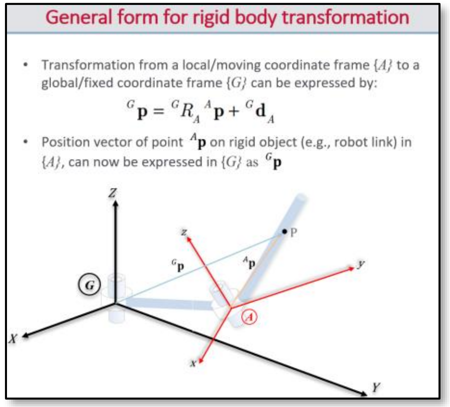
- In the case that **$\{A\}$ is stationary** relative to $\{G\}$:

From Lecture on Kinematics and Transformations

$${}^{G}Q = {}^{G}R_{A}\,{}^{A}Q + {}^{G}P_{AORG}$$

$${}^{G}\left({}^{A}V(Q)\right) = \frac{d}{dt}\,{}^{G}R_{A}\,{}^{A}Q + \cancelto{=0}{\frac{d}{dt}\,{}^{G}P_{AORG}}$$

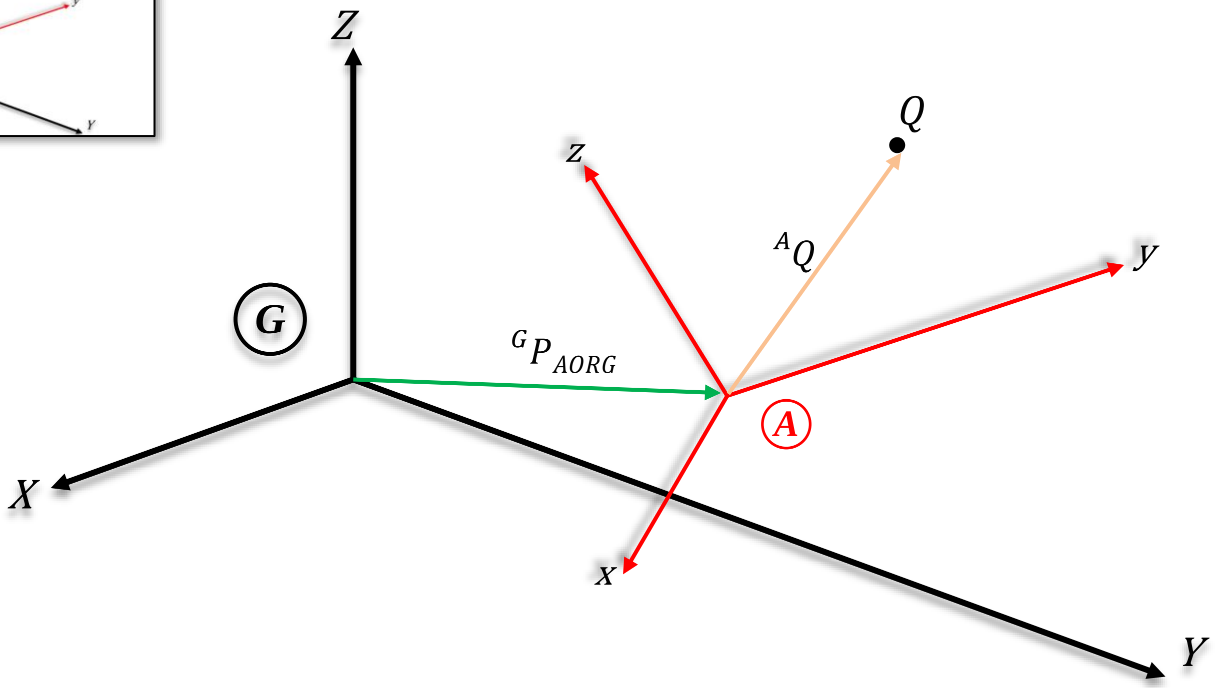
$$\boxed{{}^{G}\left({}^{A}V(Q)\right) = {}^{G}R_{A}\,{}^{A}V(Q)}$$


General form for rigid body transformation

- Transformation from a local/moving coordinate frame $\{A\}$ to a global/fixed coordinate frame $\{G\}$ can be expressed by:

$${}^{G}\mathbf{p} = {}^{G}R_{A}\,{}^{A}\mathbf{p} + {}^{G}\mathbf{d}_{A}$$

- Position vector of point ${}^{A}\mathbf{p}$ on rigid object (e.g., robot link) in $\{A\}$, can now be expressed in $\{G\}$ as ${}^{G}\mathbf{p}$



Linear Velocity when Local Frame is Moving (not rotating)

- In the case that **$\{A\}$ is also moving** with a linear velocity V and <u>fixed orientation</u> relative to $\{G\}$:

$$^{A}V(Q) = \frac{d}{dt}\,{}^{A}Q = \lim_{\Delta t \to 0} \frac{^{A}Q(t+\Delta t) - {}^{A}Q(t)}{\Delta t}$$

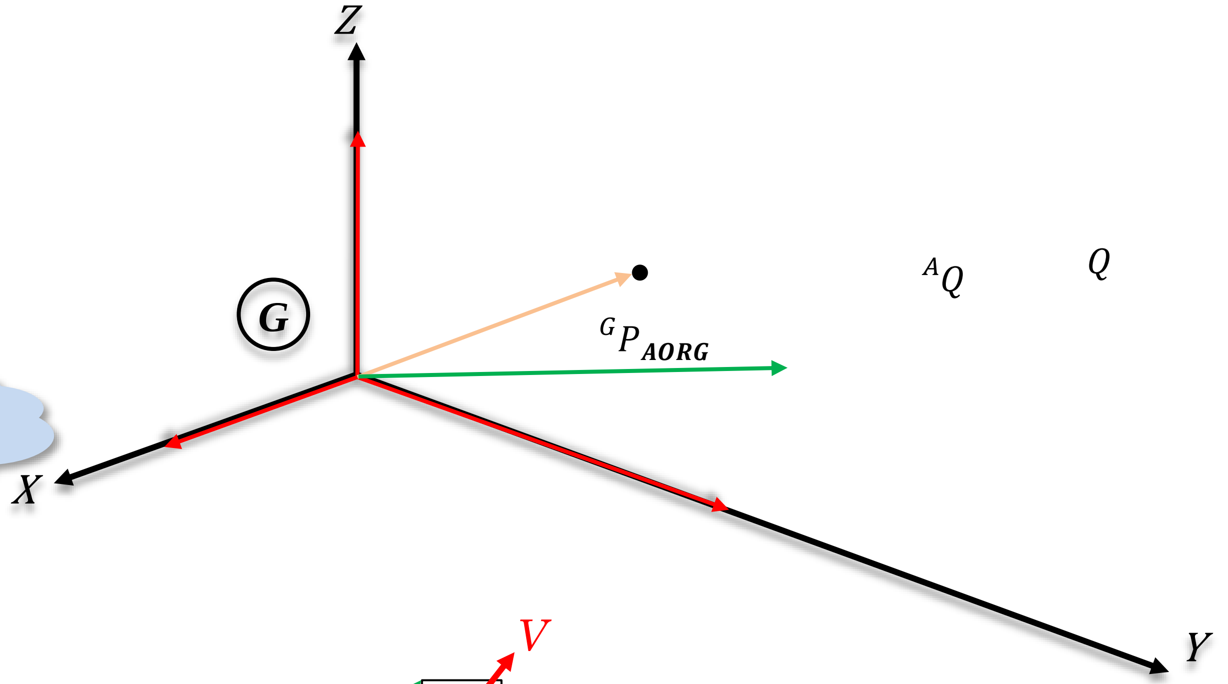
$$^{G}V_{AORG} = \frac{d}{dt}\,{}^{G}P_{AORG} = \lim_{\Delta t \to 0} \frac{^{G}P_{AORG}(t+\Delta t) - {}^{G}P_{AORG}(t)}{\Delta t}$$

- Adding the two velocity vectors in $\{G\}$

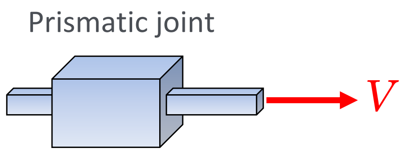
$$^{G}V(Q) = {}^{G}({}^{A}V(Q)) + {}^{G}V_{AORG}$$

We can only add vectors if expressed in the same frame, here G

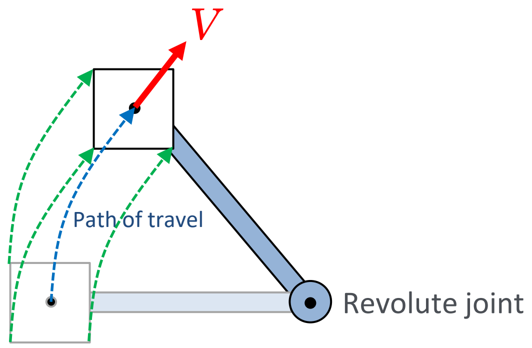
$$\boxed{^{G}V(Q) = {}^{G}R_{A}\,{}^{A}V(Q) + {}^{G}V_{AORG}}$$



NOTE: this is valid only when the relative orientation of *{A}* and *{G}* remains constant during motion i.e., during **rectilinear or curvilinear** motion



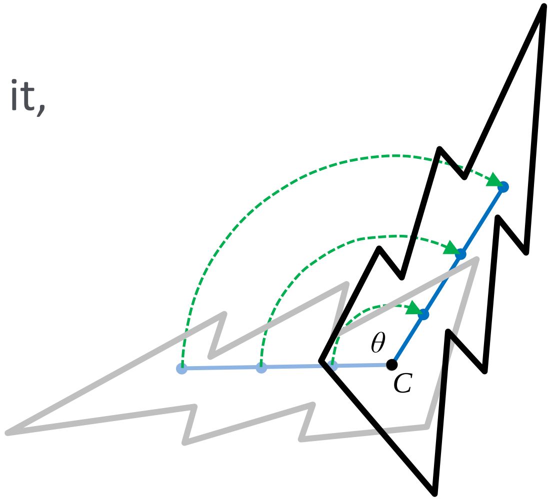
Rectilinear motion



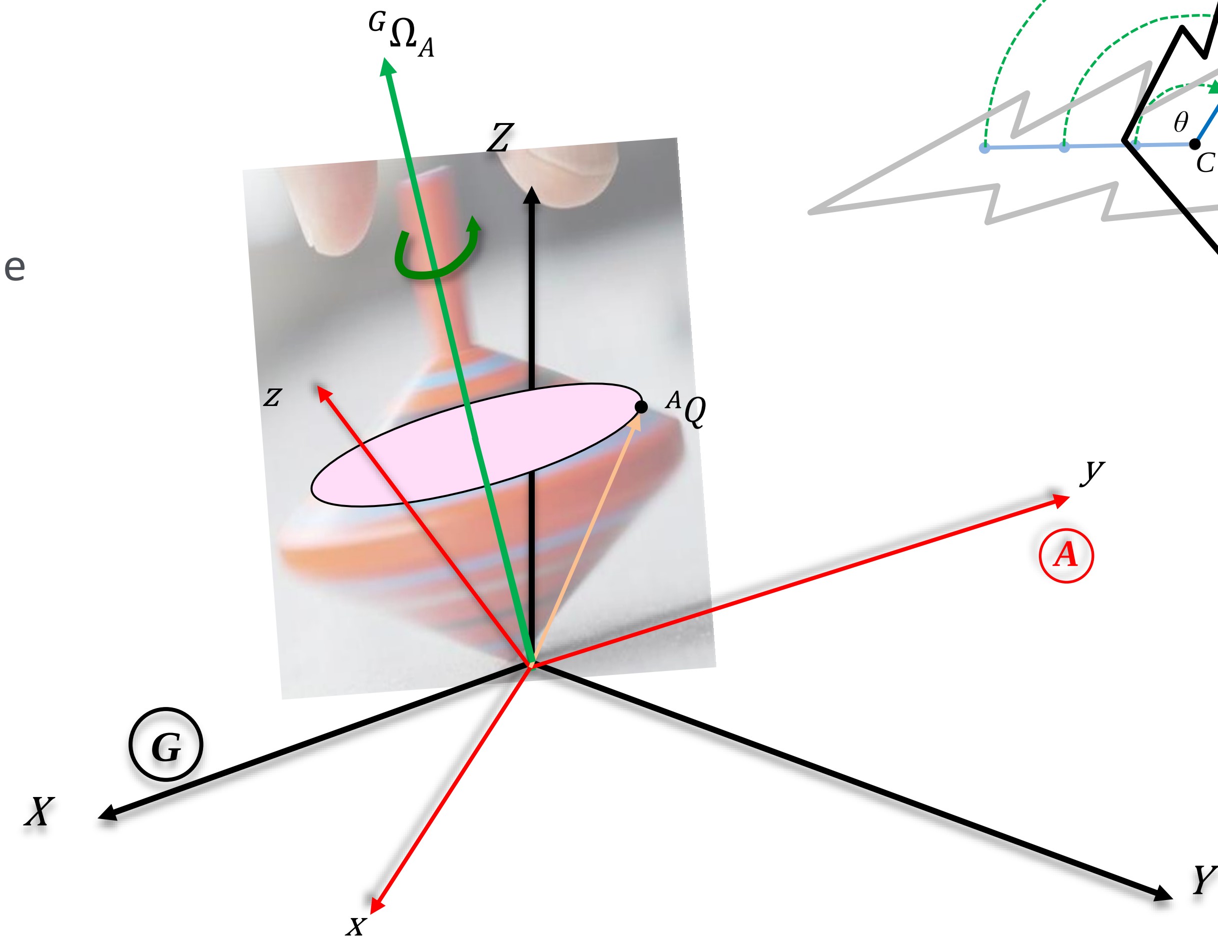
Curvilinear motion

Angular (rotational) velocity: Rotating Local Frame

- Angular (rotational) velocity describes the rotational motion of a body, and a frame attached to it, around a centre of rotation (e.g., centre of mass)

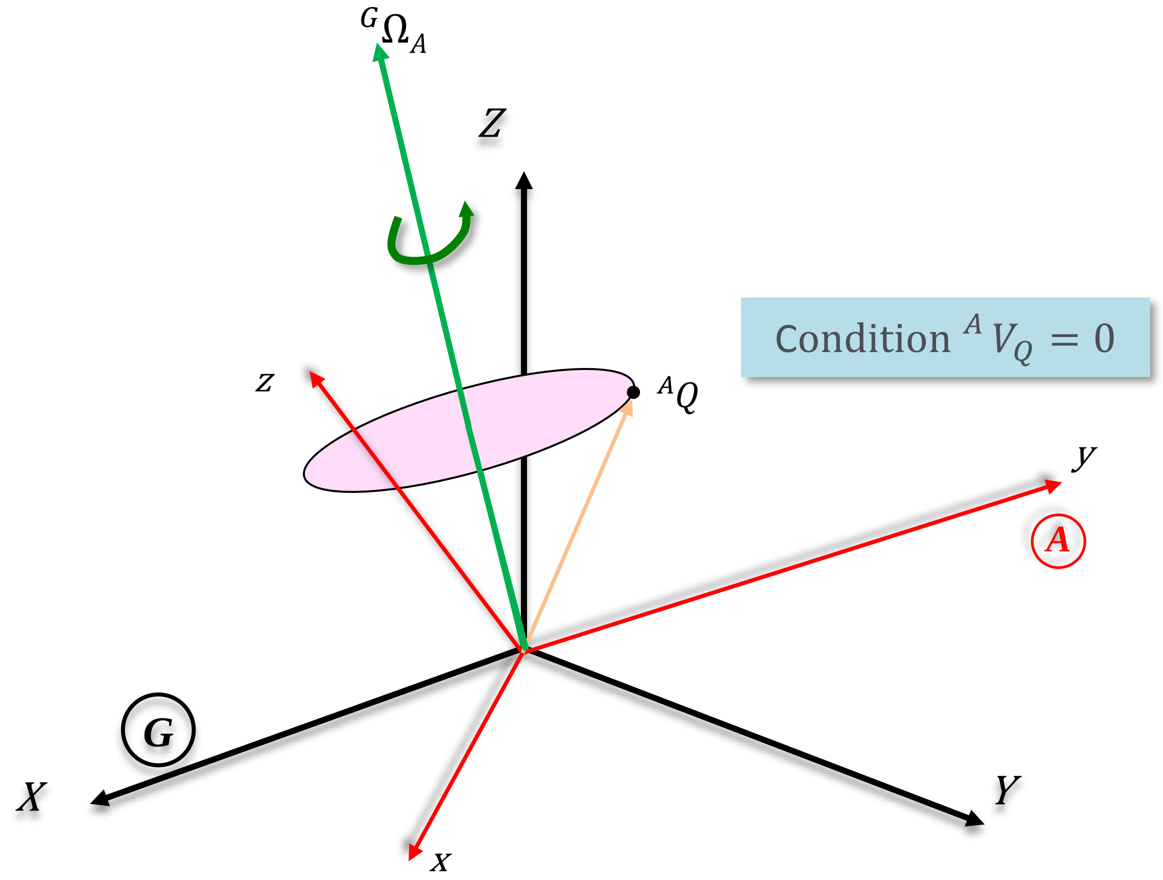


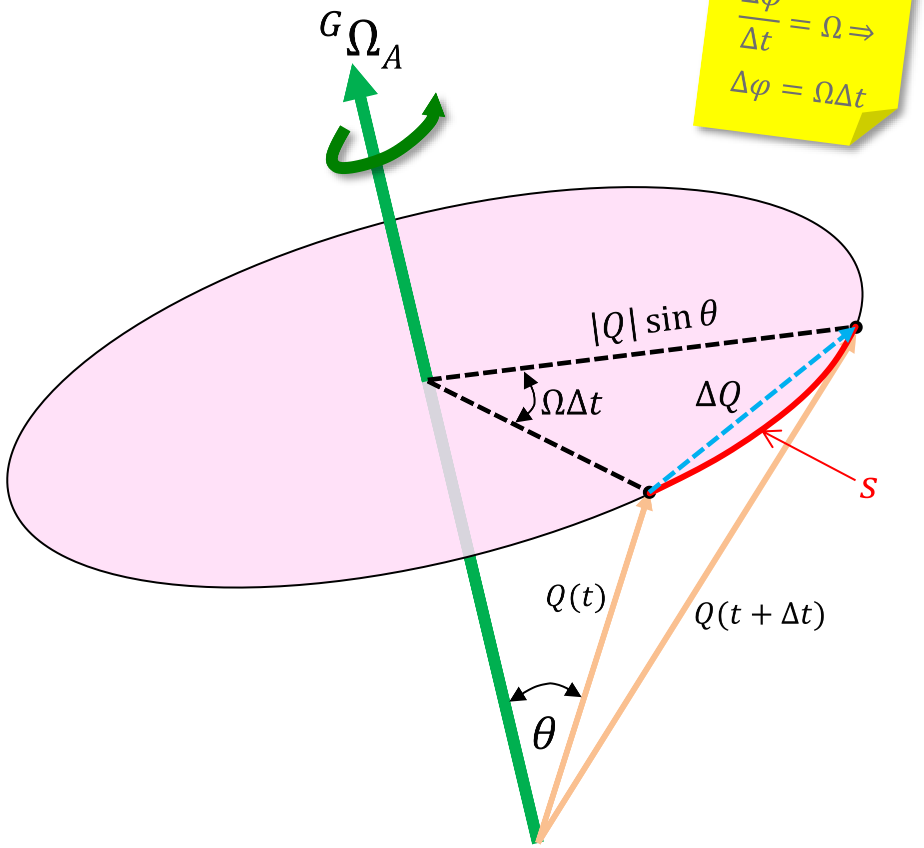
- Frames $\{A\}$ and $\{G\}$ are at same origin, and have zero linear relative velocity
- The rotation of frame $\{A\}$ relative to $\{G\}$ is given by the **angular velocity vector** ${}^{G}\boldsymbol{\Omega}_{A}$
- Physically, the **direction** of vector ${}^{G}\Omega_{A}$ represents the instantaneous axis of rotation of $\{A\}$ relative to $\{G\}$
- The **magnitude** of vector ${}^{G}\Omega_{A}$ indicates the angular speed of rotation



- Assuming linear velocity ${}^{A}V_{Q} = 0$, when $\{A\}$ rotates about ${}^{G}\Omega_{A}$, point Q is tracing a circular path in $\{G\}$

Solving for the velocity of *Q* for Rotating Local Frame





- Arc length $s = \left(\left|{}^{G}Q\right|\sin\theta\right)\left(\left|{}^{G}\Omega_A\right|\Delta t\right)$ When Δt is infinitesimally small: $s = \left|\Delta Q\right|$
- Therefore $\left|\Delta Q\right| = \left(\left|{}^{G}Q\right|\sin\theta\right)\left(\left|{}^{G}\Omega_A\right|\Delta t\right) = \left|{}^{G}Q\right|\left|{}^{G}\Omega_A\right|\sin\theta\Delta t \Rightarrow \left|\Delta Q/\Delta t\right| = \left|{}^{G}Q\right|\left|{}^{G}\Omega_A\right|\sin\theta$

$$\boxed{{}^{G}V(Q) = {}^{G}\Omega_A \times {}^{G}Q}$$

${}^{G}V(Q) =$ linear velocity
${}^{G}\Omega_A =$ angular velocity

Velocity of Q for Rotating Local Frame and Moving Point Q

- In the previous case that Q was fixed in $\{A\}$:

$$^{G}V(Q) = {}^{G}\Omega_{A} \times {}^{G}Q$$

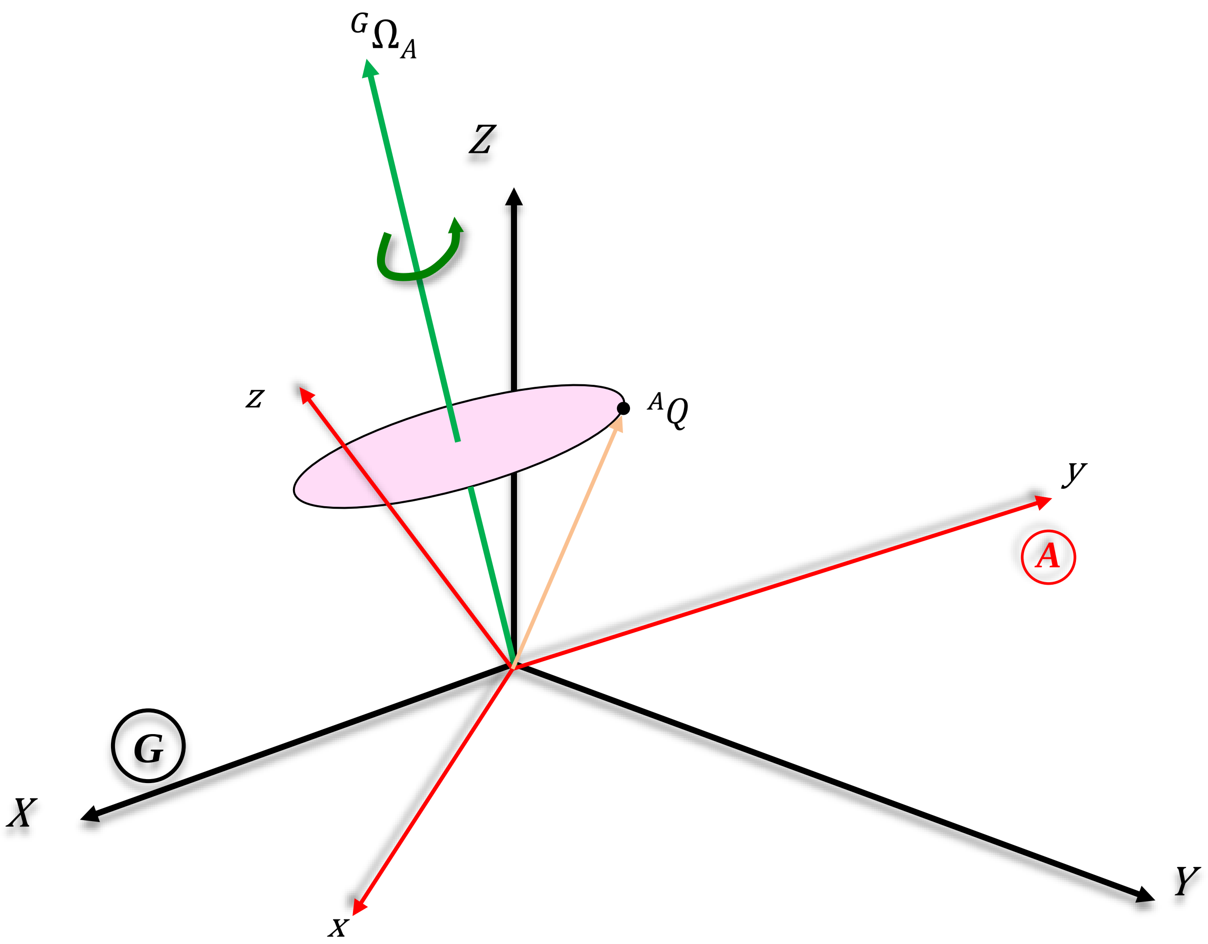
- What if Q is also moving in $\{A\}$, with linear velocity ^{A}V?

- Simply add velocity components:

$$^{G}V(Q) = \underbrace{{}^{G}\left({}^{A}V(Q)\right)}_{\text{Linear velocity of point } Q \text{ in } \{G\}, \text{ due to its linear velocity in } \{A\}} + \underbrace{{}^{G}\Omega_{A} \times {}^{G}Q}_{\text{Linear velocity of point } Q \text{ in } \{G\}, \text{ due to angular velocity of } \{A\}}$$

- Or to remove dual superscript notation and directly express ^{A}V in {G}, use rotation matrix:

$$\boxed{^{G}V(Q) = {}^{G}R_{A}\,{}^{A}V(Q) + {}^{G}\Omega_{A} \times {}^{G}R_{A}\,{}^{A}Q}$$

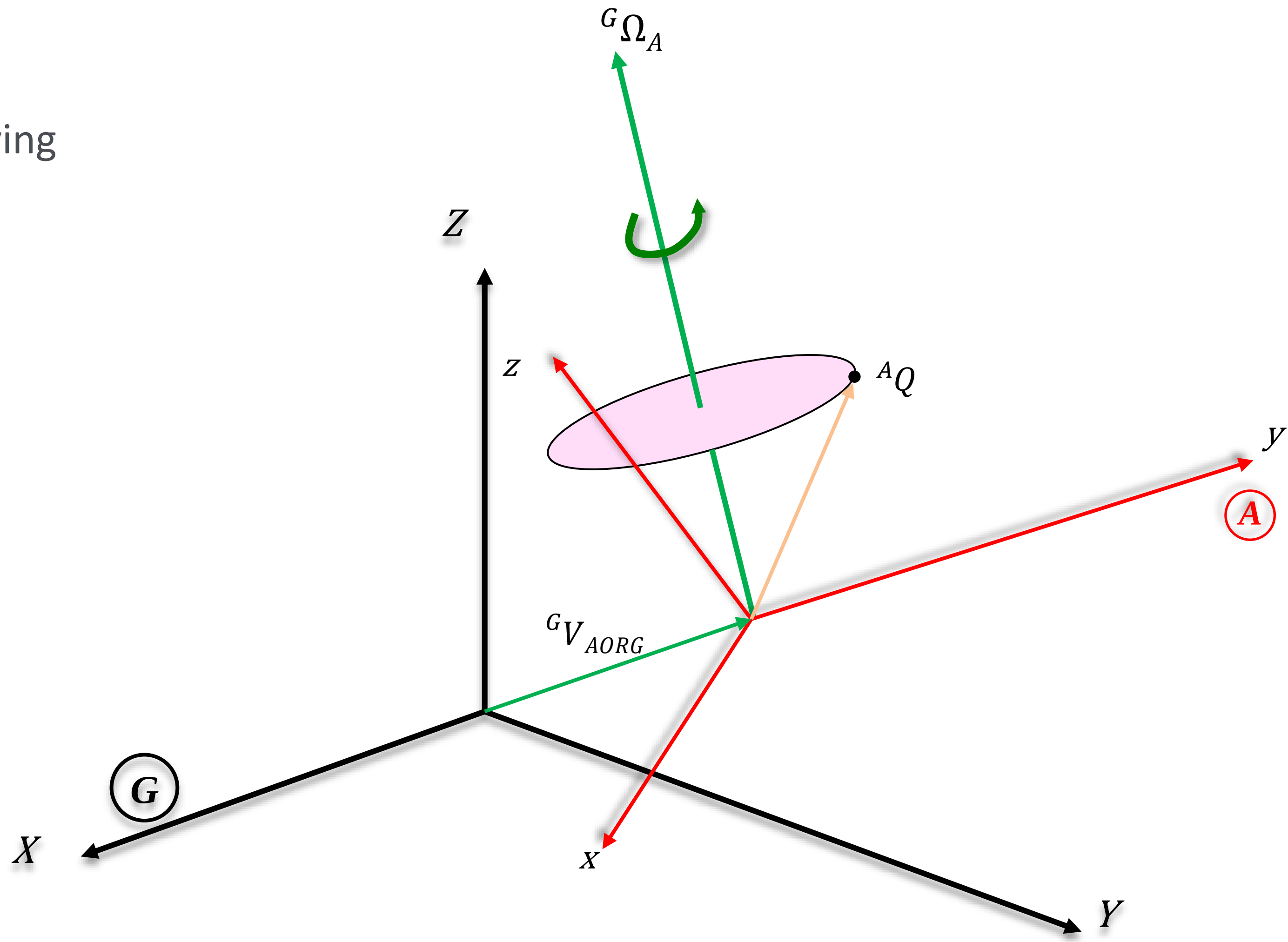


Velocity of Q in simultaneous Rotating and Moving Local Frame and Moving Point

- Now let's make it even more complicated
- Point Q is moving in $\{A\}$, while $\{A\}$ is rotating and moving with a velocity of ${}^{G}V_{AORG}$
- Again, simply add velocity components

$$\boxed{{}^{G}V(Q) = \underbrace{{}^{G}V_{AORG}}_{\text{Linear velocity of } \{A\} \text{ relative to } \{G\}} + \underbrace{{}^{G}R_{A}\,{}^{A}V(Q)}_{\text{Linear velocity of point } {}^{A}Q \text{ in } \{G\}, \text{ due to its linear velocity in } \{A\}} + \underbrace{{}^{G}\Omega_{A} \times {}^{G}R_{A}\,{}^{A}Q}_{\text{Linear velocity of point } {}^{A}Q \text{ in } \{G\}, \text{ due to angular velocity of } \{A\}}}$$

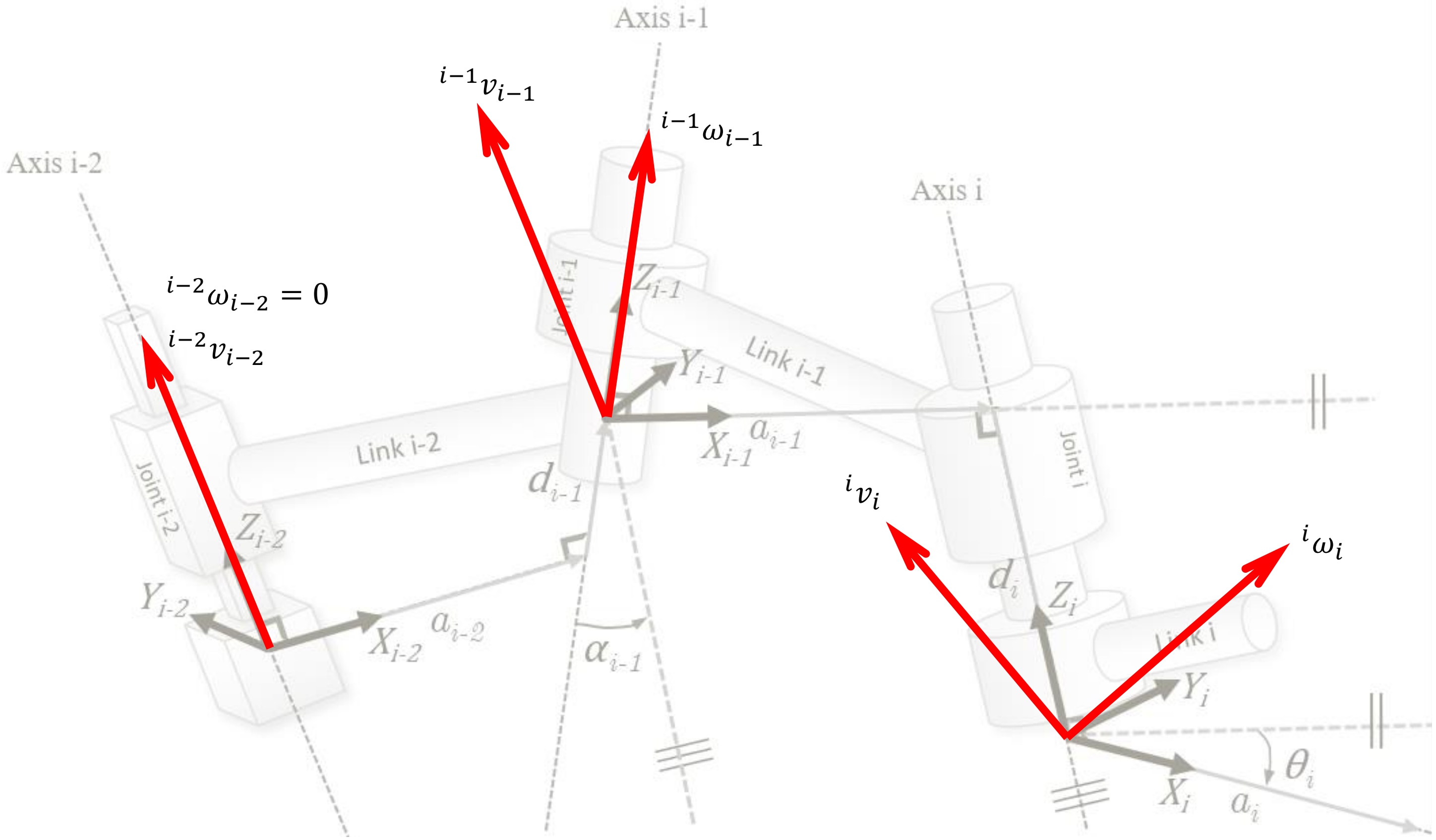
The above equation is the general expression for the derivative of a vector in a moving frame as seen from a stationary frame



Interlink Velocity Propagation

- Problem: Calculate the **linear and angular velocities** of the links of a robotic manipulator
- A manipulator is a chain of bodies; hence velocities **propagate** from the previous to the next links in the chain
- The velocity of **link i** is the velocity of **link i-1** plus any additional velocities contributed by **joint i**

> - "Velocity of a link" means **the linear velocity of the origin of the link frame** and **the angular velocity of the link**.
> - We express link velocities **with respect to the link frame itself**, rather than the base frame
> - Remember: Velocities can only be added when vectors are expressed in the same frame



Interlink Angular Velocity Propagation

- The angular velocity of link i is that of i-1, plus a new component caused by angular velocity of joint i. This can be expressed in frame $\{i\text{-}1\}$ as:

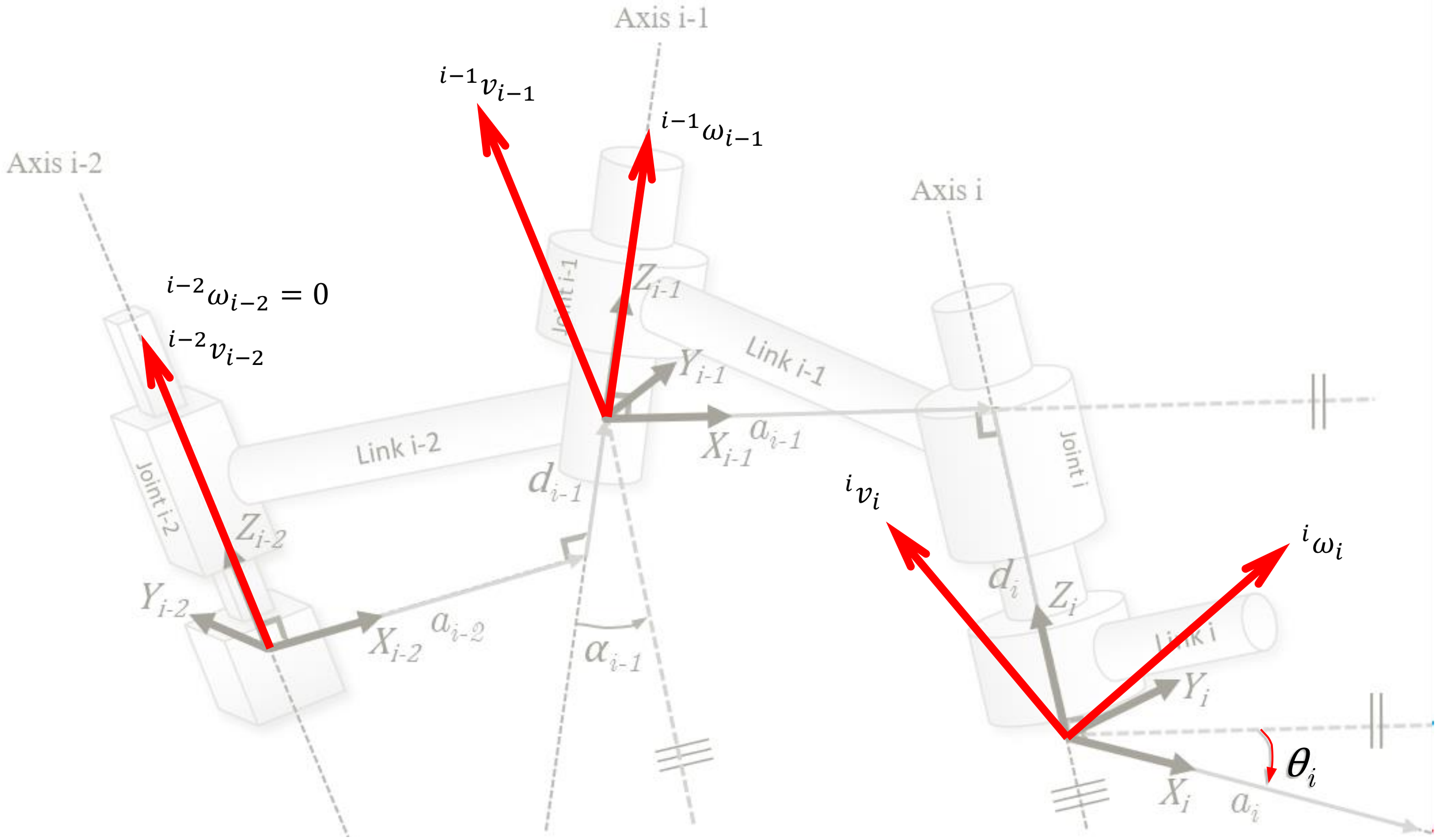
$$ {}^{i-1}\omega_i = {}^{i-1}\omega_{i-1} + \boxed{{}^{i-1}R_i} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix} $$

Derived from the link transformation matrix using the DH table of parameters (Lecture on FK, General form of DH transformation)

$$ {}^{i-1}T_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1} d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1} d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

- Pre-multiplying both sides by ${}^{i}R_{i-1}$ allows us to express the angular velocity of link i in its own frame:

$$ {}^{i}\omega_i = {}^{i}R_{i-1}\, {}^{i-1}\omega_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix} $$



Interlink Linear Velocity Propagation

- Similarly, the linear velocity of the origin of frame $\{i\}$ is the same as that of the origin of $\{i\text{-}1\}$, **plus** a new component caused by angular velocity of link i-1.

- Based on the general expression derived earlier

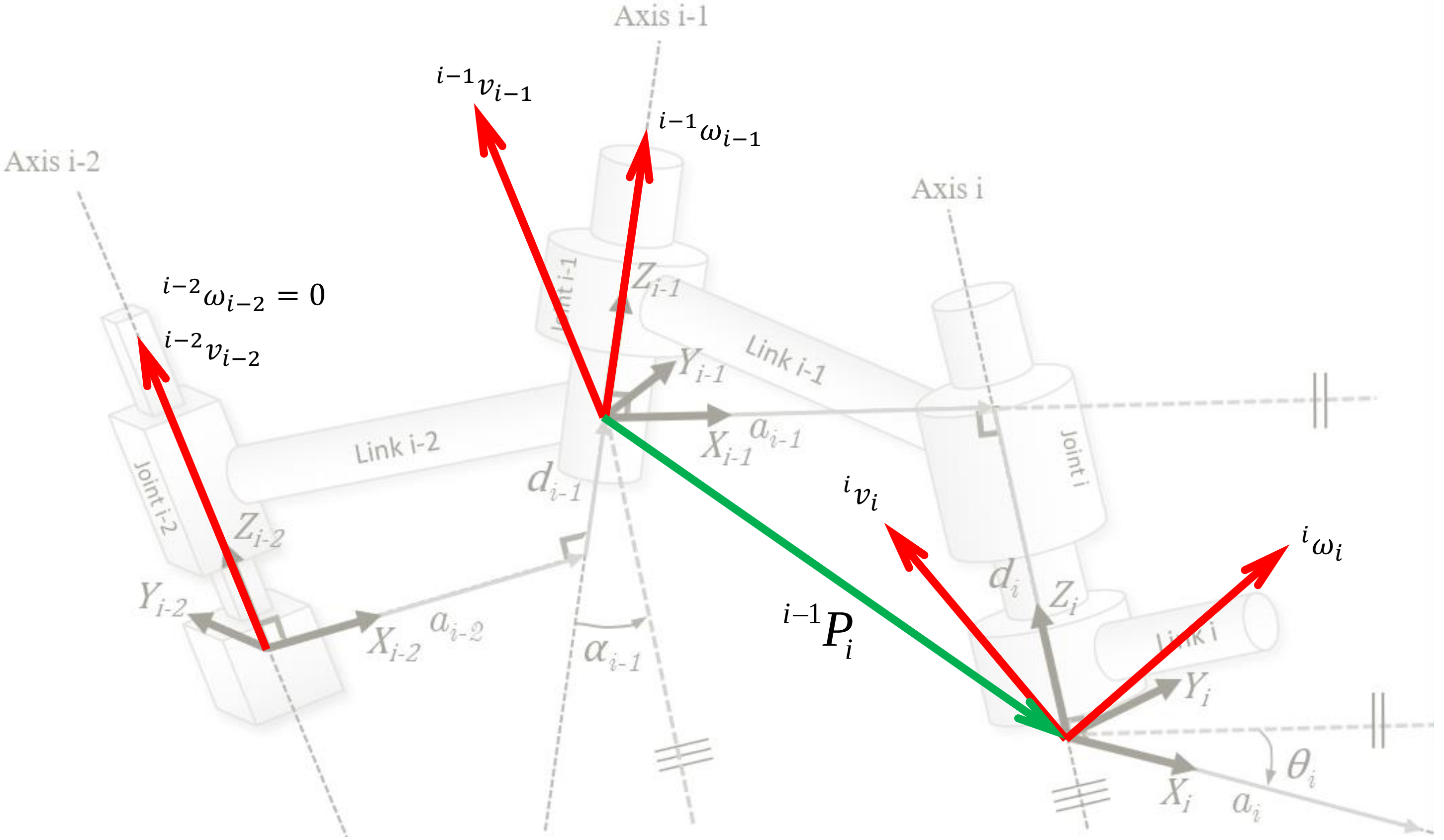
$$^{G}V(Q) = {}^{G}V_{AORG} + {}^{G}R_A\,{}^{A}V(Q) + {}^{G}\Omega_A \times {}^{G}R_A\,{}^{A}Q$$

- and considering that in this case $^{i-1}P_i$ is constant in $\{i\text{-}1\}$

$$^{i-1}v_i = {}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}P_i$$

- Pre-multiplying both sides by $^{i}R_{i-1}$

$$\boxed{^{i}v_i = {}^{i}R_{i-1}\left({}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)}$$



Link to Link Velocity Propagation Summary

- The derived equations can be applied successively to calculate ${}^N\omega_N$ and ${}^N v_N$, the angular and linear velocities of the last link, or any other link
- Can be used directly and applied iteratively by computer code
- If velocities are required relative to the base frame, simply multiply by ${}^0R_N$

$$\boxed{{}^i\omega_i = {}^iR_{i-1}\,{}^{i-1}\omega_{i-1} + \begin{bmatrix}0\\0\\\dot{\theta}_i\end{bmatrix}}$$

$$\boxed{{}^iv_i = {}^iR_{i-1}\left({}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)}$$

- If joint i is **prismatic**, the above equations become:

$$\boxed{{}^i\omega_i = {}^iR_{i-1}\,{}^{i-1}\omega_{i-1}}$$

$$\boxed{{}^iv_i = {}^iR_{i-1}\left({}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right) + \begin{bmatrix}0\\0\\\dot{d}_i\end{bmatrix}}$$

Example of deriving the analytical expression for last link velocities

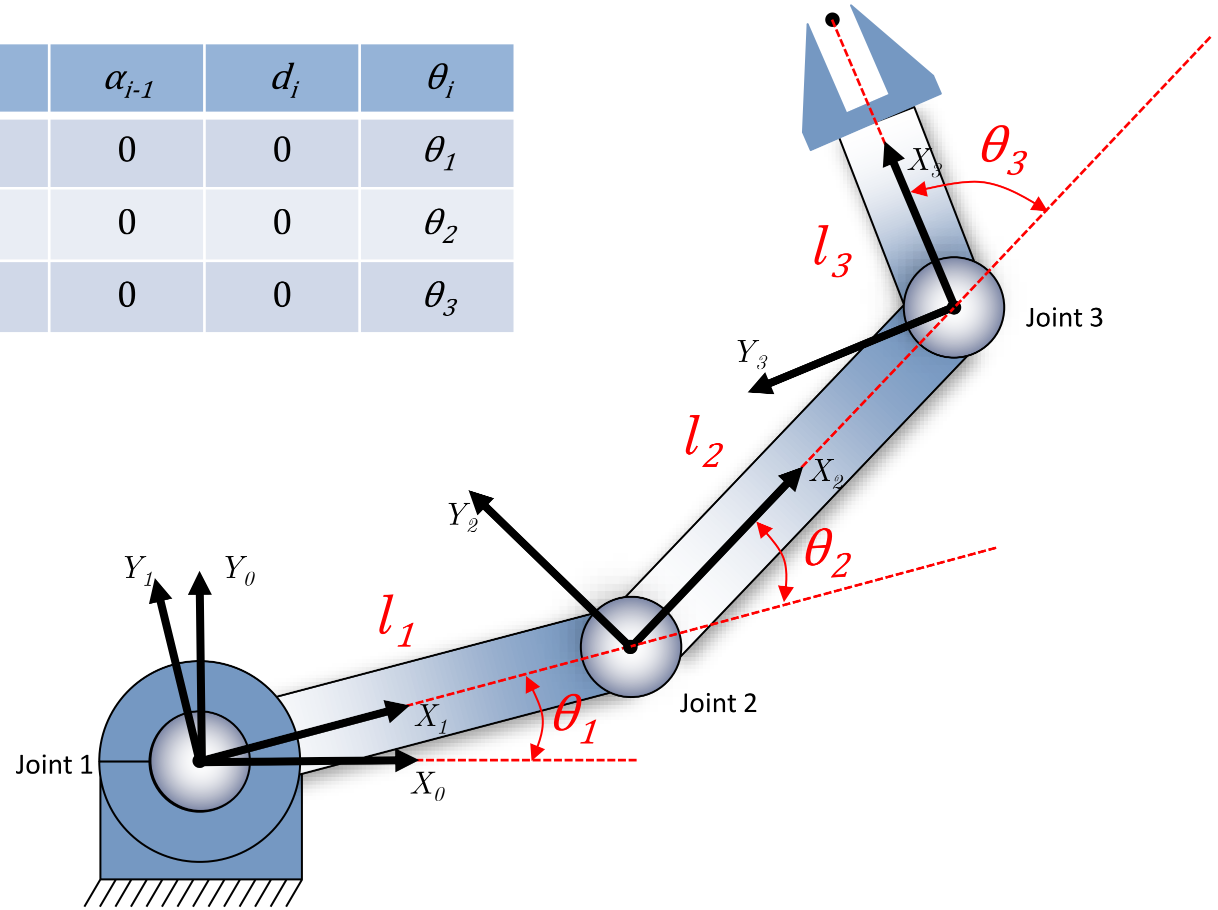
$$^0T_1 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^1T_2 = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^2T_3 = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & l_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^0T_3 = {}^0T_1\,{}^1T_2\,{}^2T_3$$

Frame	a_{i-1}	α_{i-1}	d_i	θ_i
1	0	0	0	θ_1
2	l_1	0	0	θ_2
3	l_2	0	0	θ_3

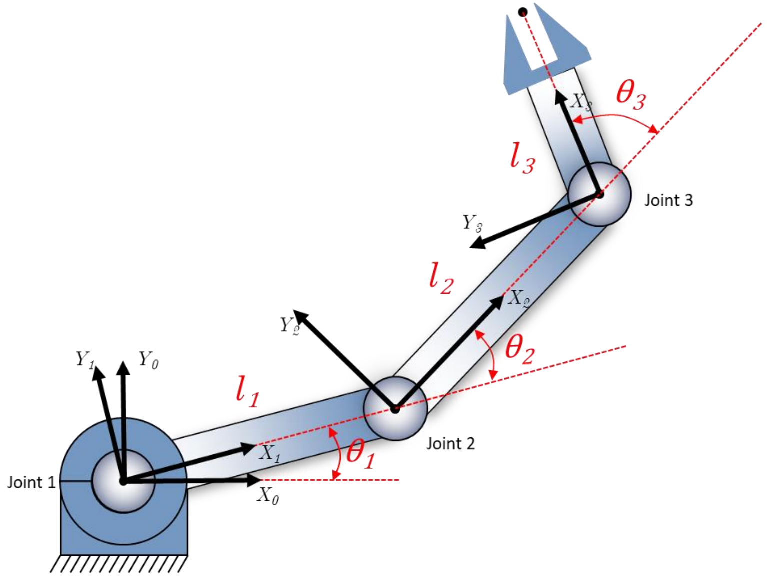


Example of deriving the analytical expression for last link velocities

- Calculate link velocities for the manipulator

$$\boxed{{}^{i}\omega_i = {}^{i}R_{i-1}\,{}^{i-1}\omega_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_i \end{bmatrix}}$$

$$\boxed{{}^{i}v_i = {}^{i}R_{i-1}\left({}^{i-1}v_{i-1} + {}^{i-1}\omega_{i-1} \times {}^{i-1}P_i\right)}$$



$${}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \qquad {}^{1}v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$${}^{1}R_2^{-1}$$

$${}^{2}\omega_2 = {}^{2}R_1\,{}^{1}\omega_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 \\ -s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$${}^{2}v_2 = {}^{2}R_1\left({}^{1}v_1 + {}^{1}\omega_1 \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} c\theta_2 & s\theta_2 & 0 \\ -s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\dot{\theta}_1 s\theta_2 \\ l_1\dot{\theta}_1 c\theta_2 \\ 0 \end{bmatrix}$$

$${}^{1}\omega_1 \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} = \begin{vmatrix} i & j & k \\ 0 & 0 & \dot{\theta}_1 \\ l_1 & 0 & 0 \end{vmatrix}_{\text{DETERMINANT}}$$

$${}^{3}\omega_3 = {}^{3}R_2\,{}^{2}\omega_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix}$$

$${}^{3}v_3 = {}^{3}R_2\left({}^{2}v_2 + {}^{2}\omega_2 \times \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} l_1\dot{\theta}_1 s\theta_2 \\ l_1\dot{\theta}_1 c\theta_2 + l_2\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ 0 \end{bmatrix}$$

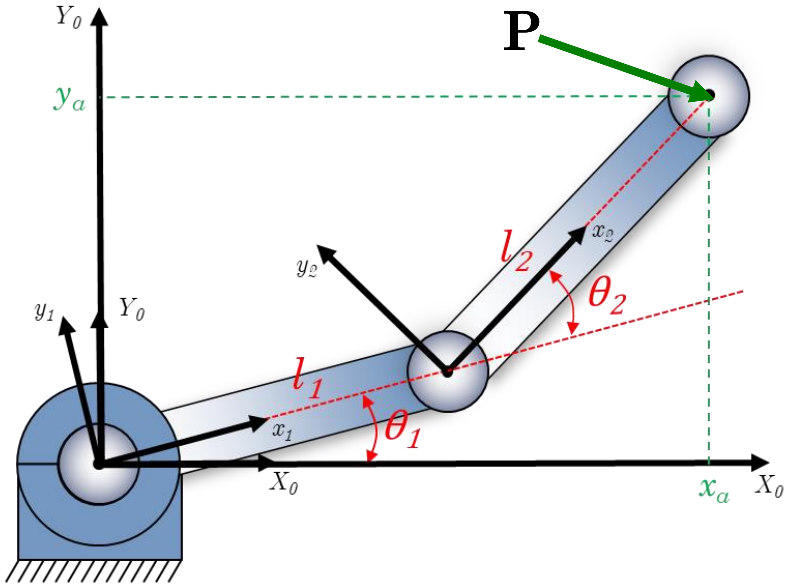
Example using the Jacobian-derived velocities

Kinematics Equations
$$\begin{cases} x_a = l_1 c\theta_1 + l_2 c\theta_{12} \\ y_a = l_1 s\theta_1 + l_2 s\theta_{12} \end{cases}$$

$$\mathbf{P} = \begin{bmatrix} x_a \\ y_a \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} l_1 c\theta_1 + l_2 c\theta_{12} \\ l_1 s\theta_1 + l_2 s\theta_{12} \end{bmatrix}$$

$$\mathbf{J}(\vec{\theta}) = \left[\frac{\partial \mathbf{Q}(\vec{\theta})}{\partial \vec{\theta}} \right] = \begin{bmatrix} -l_1 s\theta_1 - l_2 s\theta_{12} & -l_2 s\theta_{12} \\ l_1 c\theta_1 + l_2 c\theta_{12} & l_2 c\theta_{12} \end{bmatrix}$$

$$\dot{\mathbf{P}} = \mathbf{J}(\vec{\theta})\dot{\vec{\theta}} \Rightarrow \begin{bmatrix} \dot{x}_a \\ \dot{y}_a \end{bmatrix} = \begin{bmatrix} -l_1 s\theta_1 - l_2 s\theta_{12} & -l_2 s\theta_{12} \\ l_1 c\theta_1 + l_2 c\theta_{12} & l_2 c\theta_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$



- If for example we want to limit the end-effector $\mathbf{P}$ to move at **1m/s** horizontally, the required joint speed can be found as:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s\theta_1 - l_2 s\theta_{12} & -l_2 s\theta_{12} \\ l_1 c\theta_1 + l_2 c\theta_{12} & l_2 c\theta_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -l_1 s\theta_1 - l_2 s\theta_{12} & -l_2 s\theta_{12} \\ l_1 c\theta_1 + l_2 c\theta_{12} & l_2 c\theta_{12} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Changing a Jacobian's Frame of Reference

- Given a Jacobian in frame {A}: $$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = {}^{A}\mathbf{J}(\vec{\theta})\dot{\vec{\theta}}$$
- We might be interested in deriving an expressions of the Jacobian in another frame, e.g., {G}
- As discussed previously, a 6x1 Cartesian velocity vector in $\{A\}$ can be described relative to $\{G\}$ by the transformation:

$$\underbrace{\begin{bmatrix} {}^{G}v \\ {}^{G}\omega \end{bmatrix}}_{6\text{x}1} = \underbrace{\begin{bmatrix} {}^{G}R_{A} & 0 \\ 0 & {}^{G}R_{A} \end{bmatrix}}_{6\text{x}6} \underbrace{\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix}}_{6\text{x}1}$$

Hence, we can write: $$\begin{bmatrix} {}^{G}v \\ {}^{G}\omega \end{bmatrix} = \begin{bmatrix} {}^{G}R_{A} & 0 \\ 0 & {}^{G}R_{A} \end{bmatrix} {}^{A}\mathbf{J}(\vec{\theta})\dot{\vec{\theta}}$$

- Therefore, changing the frame of reference of a Jacobian can be accomplished by the following relationship:

$$\boxed{{}^{G}\mathbf{J}(\vec{\theta}) = \begin{bmatrix} {}^{G}R_{A} & 0 \\ 0 & {}^{G}R_{A} \end{bmatrix} {}^{A}\mathbf{J}(\vec{\theta})}$$

Conclusions

- Velocity Representation
- Linear and Angular Velocities
- Interlink Velocity Propagation
- Analytical and Jacobian Velocities
- Changing of Jacobian Frame