Numerical Analysis for the Price of Option Under Black-Scholes Formula & Heston model

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Background & Motivation:

During the last decades, several alternatives have been proposed to improve volatility modeling in the context of derivatives pricing. One such approach is to model volatility as a stochastic quantity.Stochastic volatility models, on the other hand, allow for variation in both the asset's price and its price volatility, or standard deviation. Steven Heston proposed one of the most widely used stochastic volatility models in 1993.

The Heston model extends the Black-Scholes model by adding a stochastic process for stock volatility. For each stochastic volatility model, it has a unique *characteristic function* that describes the probability density function of that model.

<u>Basic Heston Model</u>:

$$dS_t = \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t^S$$
$$d\nu_t = \kappa(\theta - \nu_t) \, dt + \xi \sqrt{\nu_t} \, dW_t^\nu$$

<u>Characteristic Function</u>:

$$C_0 = S_0 . \Pi_1 - \mathrm{e}^{-rT} K . \Pi_2$$

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{\mathrm{e}^{-i.w.ln(K)} . \Psi_{lnS_T}(w - i)}{i.w.\Psi_{lnS_T}(-i)} \right] \mathrm{d}w$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left[\frac{\mathrm{e}^{-i.w.ln(K)} . \Psi_{lnS_T}(w)}{i.w} \right] \mathrm{d}w$$

$$\Psi_{lnS_T}(w) = \mathrm{e}^{[C(t,w).\bar{V} + D(t,w).V_0 + i.w.ln(S_0.\mathrm{e}^{rt})]}$$

$$C(t,w) = a.\left[r_{-}.t - \frac{2}{\eta^2}.ln\left(\frac{1-g.e^{-ht}}{1-g}\right)\right]$$

$$D(t,w) = r_{-}.\frac{1-e^{-ht}}{1-g.e^{-ht}}$$

$$r_{\pm} = \frac{\beta \pm h}{\eta^2}; h = \sqrt{\beta^2 - 4.\alpha.\gamma}$$

$$g = \frac{r_{-}}{r_{+}}$$

$$\alpha = -\frac{w^2}{2} - \frac{iw}{2}; \beta = \alpha - \rho.\eta.i.w; \gamma = \frac{\eta^2}{2}$$

Problem Statement:

Where St is the asset price. μ is the rate of return of the asset.√ νt is the volatility (standard deviation) of the asset price. ξ is the volatility of volatility. θ is the long-term price variance. κ is the rate of reversion to the long-term price variance. dW_t^S is the Brownian motion of the asset price. dWνt is the Brownian motion of the asset's price variance; ρ is the correlation coefficient for dW_t^S and dW_t^ν.

Our goal is to find Π1 and Π2 by given V0, $\bar{V}$, α, η, ρ (after calibrated) by using different <u>composite numerical integration methods</u> in Math 104B.

Methods:

- <u>Framework</u>:
  In order to find the price of the call option, we select 15 datasets as our sample to build our simulation model.
  Also, after calibrating (MLE), we obtain the parameters, we want to calculate the value of Π1 and Π2.(below)

Parameter Estimates	Value
V_0	$6.47*10^{(-5)}$
$\tilde{V}$	$6.47*10^{(-5)}$
α	$6.57*10^{(-3)}$
η	$5.09*10^{(-4)}$
ρ	$-1.98*10^{(-3)}$

After obtaining those parameters, we decide to use *Composite Midpoint Method*, *Composite Trapezoidal Method*, *Composite Simpson's method* to calculate the prices of the option.

We compare those numerical integration methods to decide which one is the best in our calculations of the project.

In Math 104A project, we also make an option pricing of Black-Scholes Model, so we decide to compare two models with the actual value of call option it provided.

Data of Interest:

Expiration	Stock Price	Strike Price	Actual Call
24/365	425.73	395	36.75
24/365	425.73	400	27.88
24/365	425.73	405	25
24/365	425.73	410	17.5
24/365	425.73	415	15.88
87/265	425.73	395	33
87/265	425.73	400	28.5
87/265	425.73	405	24.13
87/265	425.73	410	20.38
87/265	425.73	415	16.13
115/365	425.73	380	47.25
115/365	425.73	390	38.13
115/365	425.73	400	13.29
115/365	425.73	410	21.19
115/365	425.73	420	13.88

Results:

We use *Composite Midpoint Method*, *Composite Trapezoidal Method*, *Composite Simpson's method* and *the Black-Scholes Moel* to calculate the option prices under differerent time to maturity t (when t = 24/365, 87/365,), differernt strike prices K (K1 = 395, K2 = 400, K3 = 405, K4 = 410, and K5 = 415).

Example result: when t = 24/365:

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Results are presented in this form: [Approximated value, Absolute error]
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K1=395
Actual value: 36.75
Composite Mid Point:  [32.62214985273221, 4.127850147267793]
Composite Trapezoidal:  [32.58363664766216, 4.166363352337839]
Composite Simpsons:  [32.62216098733876, 4.127839012661241]
Black-Schole model:  [30.746622116003437, 6.0033778839965635]
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K2=400
Actual value: 27.88
Composite Mid Point:  [26.939224726199427, 0.9407752738005719]
Composite Trapezoidal:  [26.94129950026047, 0.9387004997395287]
Composite Simpsons:  [26.939261624799883, 0.9407383752001159]
Black-Schole model:  [25.7468325225351, 2.133167477464898]
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K3=405
Actual value: 25.0
Composite Mid Point:  [23.168593352506946, 1.8314066474930542]
Composite Trapezoidal:  [23.19654454480485, 1.8034554551951487]
Composite Simpsons:  [23.16860514687562, 1.8313948531243796]
Black-Schole model:  [20.747042929066822, 4.252957070933178]
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K4=410
Actual value: 17.5
Composite Mid Point:  [19.644126125030084, 2.144126125030084]
Composite Trapezoidal:  [19.657495280005094, 2.1574952800050937]
Composite Simpsons:  [19.644116911943627, 2.1441169119436267]
Black-Schole model:  [15.747253335598487, 1.7527466644015135]
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K5=415
Actual value: 15.88
Composite Mid Point:  [14.292574859760066, 1.5874251402399349]
Composite Trapezoidal:  [14.283445905792007, 1.5965540942079937]
Composite Simpsons:  [14.292567284340805, 1.5874327156591956]
Black-Schole model:  [10.747463742130208, 5.132536257869793]
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(The result for t = 87/365 is included in the code file.)

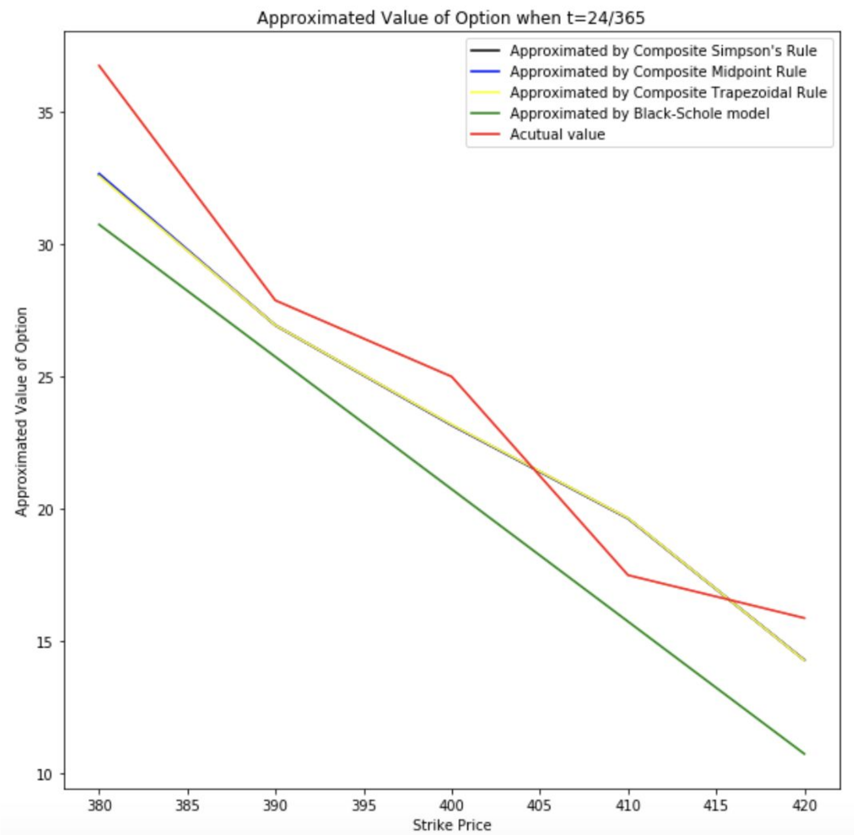
According to our results, we find that the Composite Trapezoidal Method has less error than other methods in our project, which violates the O(h^4) error term in the book.We consider that this error due to some reasons.

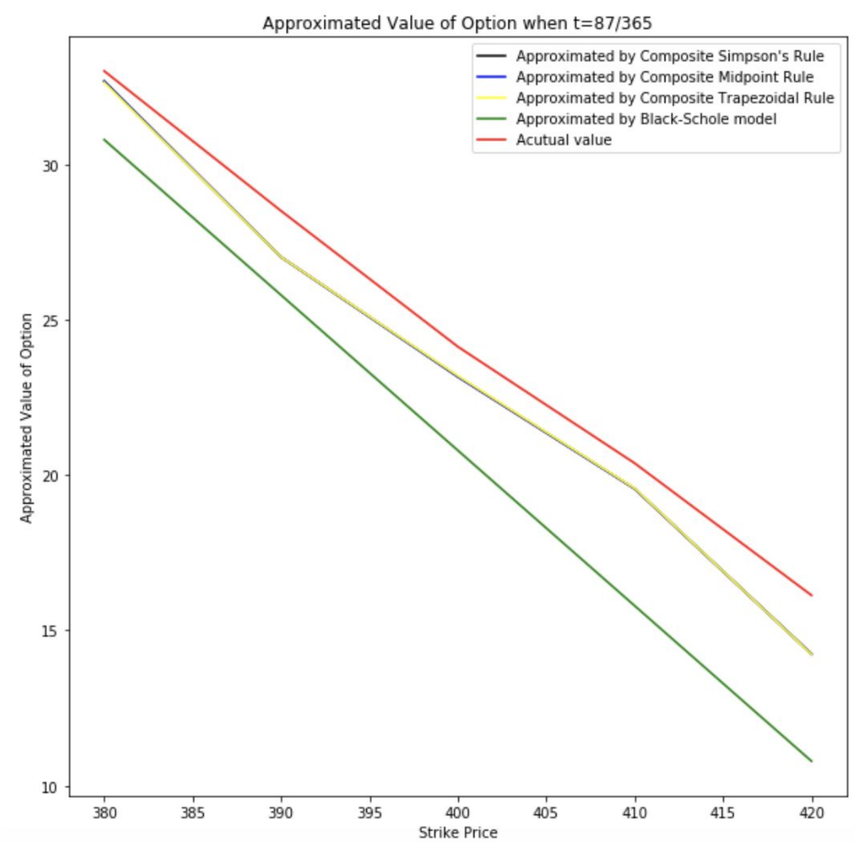
By inputting K, St, r, σ, T into the function, we can also calculate the option price of the Black-Scholes Model below. Here, we decide to use Composite Trapezoidal Method as examples to compare the error with the Black-Scholes model.

A	E	F	G	H
Expiration	Heston Model	Black-Scholes Model	Heston Error	Black-Scholes Error
24/365	32.58	30.75	4.17	6
24/365	26.94	25.75	0.94	2.13
24/365	23.2	20.75	1.8	4.25
24/365	19.66	15.75	2.16	1.75
24/365	14.28	10.75	1.6	5.13
87/265	32.63	30.79	0.37	2.21
87/265	27.02	25.79	1.48	2.71
87/265	23.2	20.79	0.93	3.34
87/265	19.57	15.79	0.81	4.59
87/265	14.23	10.79	1.9	5.34
115/365	47.85	45.81	0.6	1.44
115/365	39.1	35.81	0.97	2.32
115/365	27.05	25.81	2.33	3.57
115/365	19.54	15.81	1.65	5.38
115/365	7.6	5.81	6.28	8.07

Conclusion:

According to the table above, in most cases, **the Heston model has less error than Black-Scholes model. According to the plots below, it clearly shows that the Heston model is better than the Black-Scholes model.**


Approximated Value of Option when t=24/365
Approximated by Composite Simpson's Rule
Approximated by Composite Midpoint Rule
Approximated by Composite Trapezoidal Rule
Approximated by Black-Schole model
Acutual value
Approximated Value of Option
Strike Price


Approximated Value of Option when t=87/365
Approximated by Composite Simpson's Rule
Approximated by Composite Midpoint Rule
Approximated by Composite Trapezoidal Rule
Approximated by Black-Schole model
Acutual value
Approximated Value of Option
Strike Price

Now we use another set of data to **test** if our conclusion made earlier is credible or not.
When time to maturity t = 115/365, strike prices are K1 = 380, K2 = 390, K3 = 400, K4 = 410, K5 = 420:
From the plot below, as we can see, our conclusion is credible.