Due: Thursday, 4th April, 18:00 (AEDT)

Submission is through inspera. Your assignment will be automatically submitted at the above due date. If you manually submit before this time, you can reopen your submission and continue until the deadline.

If you need to make a submission after the deadline, please use this link to request an extension: https://www.cse.unsw.edu.au/ cs9020/extension_request.html. Unless you are granted Special Consideration, a lateness penalty of 5% of raw mark per 24 hours or part thereof for a maximum of 5 days will apply. You can request an extension up to 5 days after the deadline.

Answers are expected to be provided either:

- In the text box provided using plain text, including unicode characters and/or the built-in formula editor (diagrams can be drawn using the built-in drawing tool); or
- as a pdf (e.g. using LaTeX) – each question should be submitted on its own pdf, with at most one pdf per question.

Handwritten solutions will be accepted if unavoidable, but we don't recommend this approach as the assessments are designed to familiarise you with typesetting mathematics in preparation for the final exam and for future courses.

Discussion of assignment material with others is permitted, but the work submitted *must* be your own in line with the University's plagiarism policy.

Problem 1 (18 marks)

Consider the following two algorithms that naïvely compute the sum and product of two $n \times n$ matrices.

```
sum(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = A[i,j] + B[i,j]
    end for
  end for
  return C
```

```
product(A,B):
  for i ∈ [0,n):
    for j ∈ [0,n):
      C[i,j] = add{A[i,k] ∗ B[k,j] : k ∈ [0,n)}
    end for
  end for
  return C
```

Assuming that adding and multiplying matrix elements can be carried out in $O(1)$ time, and add will add the elements of a set S in $O(|S|)$ time:

a) Give an asymptotic upper bound, in terms of n, for the running time of sum. 5 marks

b) Give an asymptotic upper bound, in terms of n, for the running time of product. 5 marks

When n is even, we can define a recursive procedure for multiplying two $n \times n$ matrices as follows. First, break the matrices into smaller submatrices:

$$A = \begin{pmatrix} S & T \\ U & V \end{pmatrix} \qquad B = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

where S, T, U, V, W, X, Y, Z are $\frac{n}{2} \times \frac{n}{2}$ matrices. Then it is possible to show:

$$AB = \begin{pmatrix} SW + TY & SX + TZ \\ UW + VY & UX + VZ \end{pmatrix}$$

where $SW + TY$, $SX + TZ$, etc. are sums of products of the smaller matrices. If n is a power of 2, each smaller product (SW, TY, etc) can be computed recursively, until the product of 1×1 matrices needs to be computed – which is nothing more than a simple multiplication, taking $O(1)$ time.

Assume n is a power of 2, and let $T(n)$ be the worst-case running time for computing the product of two $n \times n$ matrices using this method.

c) With justification, give a recurrence equation for $T(n)$. 6 marks

d) Find an asymptotic upper bound for $T(n)$. 2 marks

Problem 2 (18 marks)

a) Consider the following proposition $P(n)$

$$1 + 3 + \ldots + (2n - 1) = n^2$$

Use induction to show $P(n)$ is true for all $n > 0$. 6 marks

b) Prove by induction that following proposition $Q(n)$ holds for all $n > 0$:

$$n^3 = \sum_{i=0}^{n-1} \left(n^2 - n + 1 + 2i \right)$$

6 marks

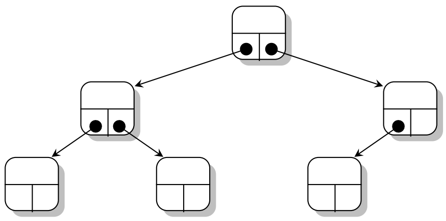
c) Using answers from $a)$ and $b)$, show the following proposition $R(n)$ holds for all $n > 0$

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i \right)^2$$

6 marks

Problem 3 (30 marks)

A *binary tree* is a data structure where each node is linked to at most two successor nodes:



If we include empty binary trees (trees with no nodes) as part of the definition, then we can simplify the description of the data structure. Rather than saying a node has 0, 1, or 2 successor nodes, we can instead say that a node has exactly two *children*, where a child is a binary tree. That is, we can abstractly define the structure of a binary tree as follows:

- (B): An empty tree, τ
- (R): An ordered pair $(T_{\text{left}}, T_{\text{right}})$ where T_{left} and T_{right} are trees.

So, for example, the above tree would be defined as the tree T where:

$$T = (T_1, T_2), \text{ where}$$
$$T_1 = (T_3, T_4) \text{ and } T_2 = (T_5, \tau), \text{ where}$$
$$T_3 = T_4 = T_5 = (\tau, \tau)$$

That is,

$$T = \Big(\big((\tau, \tau), (\tau, \tau) \big), \big((\tau, \tau), \tau \big) \Big)$$

A *leaf* in a binary tree is a node that has no successors (i.e. it is of the form (τ, τ)). A *fully-internal* node in a binary tree is a node that has exactly two successors (i.e. it is of the form (T_1, T_2) where $T_1, T_2 \neq \tau$). The example above has 3 leaves (T_3, T_4, and T_5) and 2 fully-internal nodes (T and T_1). For technical reasons (that will become apparent) we assume that an empty tree has 0 leaves and -1 fully-internal nodes.

a) Based on the recursive definition above, recursively define a function count(T) that counts the number of nodes in a binary tree T. 6 marks

b) Based on the recursive definition above, recursively define a function leaves(T) that counts the number of leaves in a binary tree T 6 marks

c) Based on the recursive definition above, recursively define a function internal(T) that counts the number of fully-internal nodes in a binary tree T. 6 marks

d) If T is a binary tree, let $P(\mathsf{T})$ be the proposition that $\mathsf{leaves}(\mathsf{T}) = \mathsf{internal}(\mathsf{T}) + 1$. Prove that $P(\mathsf{T})$ holds for all binary trees T. Your proof should be based on your answers given in (b) and (c). 12 marks

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Problem 4 (9 marks)

Show that the asymptotic growth of the following recurrence relation is $O(n)$:

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + n$$

(Hint: The master theorem will not help you in this case) (Partial marks may be given if you successfully show a more relaxed asymptotic growth such as $O(n^2)$.) 9 marks

Advice on how to do the assignment

Collaboration is encouraged, but all submitted work must be done individually without consulting someone else's solutions in accordance with the University's "Academic Dishonesty and Plagiarism" policies.

- Assignments are to be submitted in inspera.
- When giving answers to questions, we always would like you to prove/explain/motivate your answers. You are being assessed on your understanding and ability.
- Be careful with giving multiple or alternative answers. If you give multiple answers, then we will give you marks only for your worst answer, as this indicates how well you understood the question.
- Some of the questions are very easy (with the help of external resources). You may make use of external material provided it is properly referenced[1] – however, answers that depend too heavily on external resources may not receive full marks if you have not adequately demonstrated ability/understanding.
- Questions have been given an indicative difficulty level:

  PASS   CREDIT   DISTINCTION   HIGH DISTINCTION

  This should be taken as a *guide* only. Partial marks are available in all questions, and achievable by students of all abilities.

[1] Proper referencing means sufficient information for a marker to access the material. Results from the lectures or textbook can be used without proof, but should still be referenced.