



# COMP9020

Foundations of Computer Science

## Lecture 4: Set Theory

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# Outline

Recap of Key Definitions

Set Equality

Laws of Set Operations

Derived Laws

Two Useful Results

Feedback

# Outline

## Recap of Key Definitions

Set Equality

Laws of Set Operations

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# Defining Sets

- ① Explicitly list elements
- ② Take a subset of an existing set by restricting the elements
- ③ Build up from existing sets using Set Operations

# Set Operations

## Definition

$A \cup B$  – **union** ( $a$  or  $b$ ):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$  – **intersection** ( $a$  and  $b$ ):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

$A^c$  – **complement** (with respect to a universal set  $\mathcal{U}$ ):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that  $A, B$  are **disjoint** if  $A \cap B = \emptyset$

# Set Operations

Other set operations

## Definition

$A \setminus B$  – **set difference**, relative complement ( $a$  but not  $b$ ):

$$A \setminus B = A \cap B^c$$

$A \oplus B$  – **symmetric difference** ( $a$  and not  $b$  or  $b$  and not  $a$ ; also known as  $a$  or  $b$  exclusively;  $a$  xor  $b$ ):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

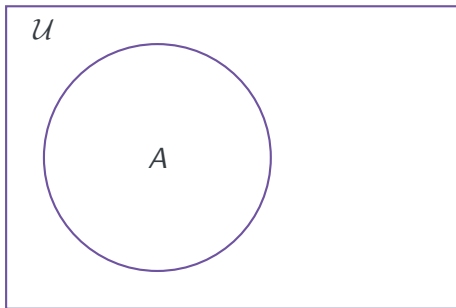
# Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



# Venn Diagrams

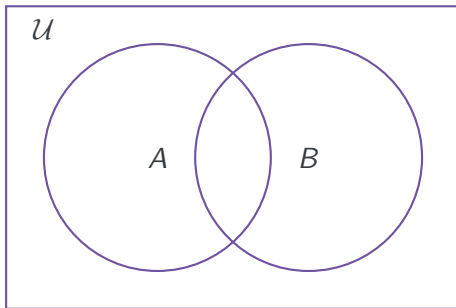
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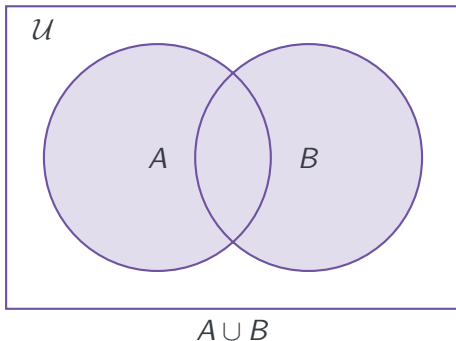
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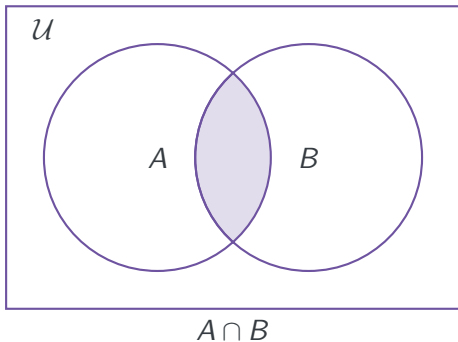
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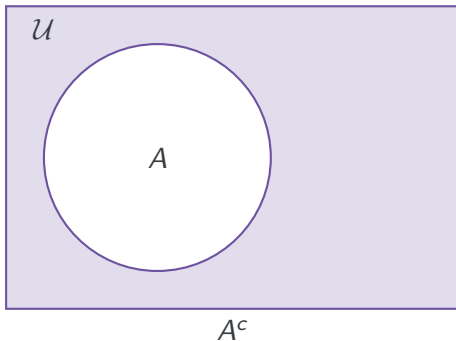
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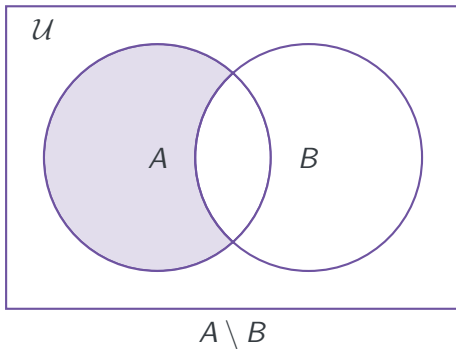
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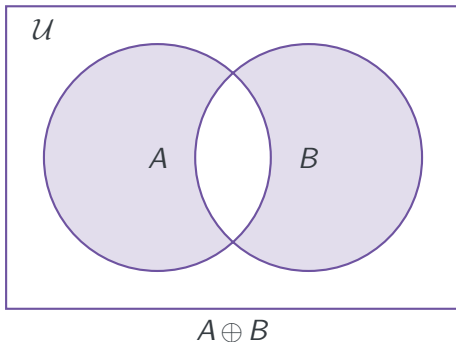
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A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



# Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



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**Set Equality**

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# Set Equality

Two sets are **equal** ( $A = B$ ) if they contain the same elements

To show equality:

- Examine all the elements
- Show  $A \subseteq B$  and  $B \subseteq A$
- Use the Laws of Set Operations

## Important!

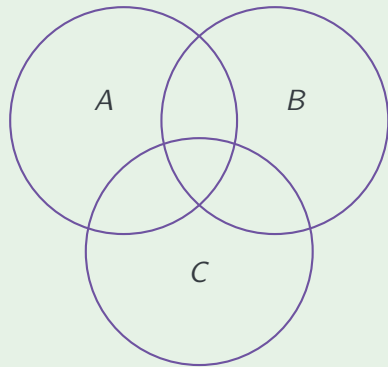
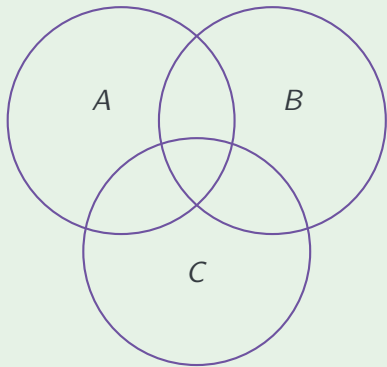
Venn diagrams can help visualize, but are **not** rigorous.



# Example

## Example

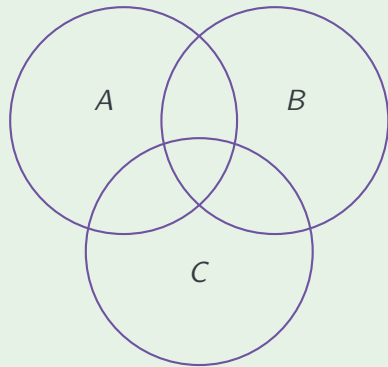
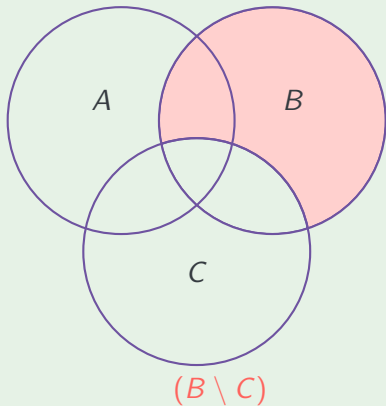
$$A \setminus (B \setminus C) \stackrel{?}{=} (A \setminus B) \setminus C$$



# Example

## Example

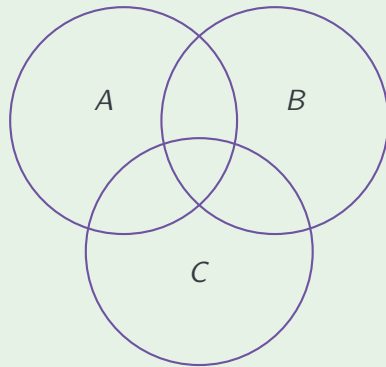
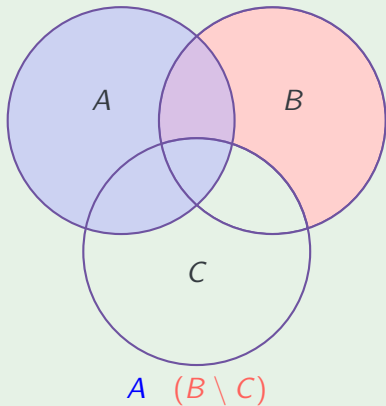
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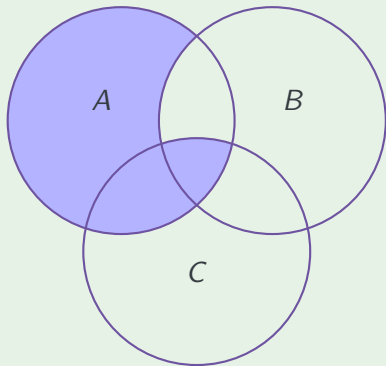
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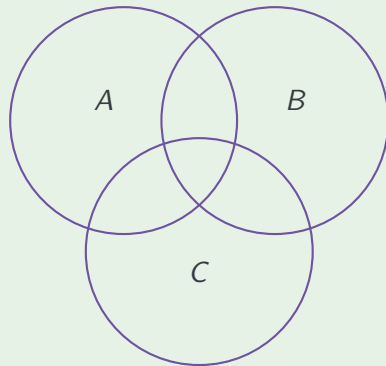
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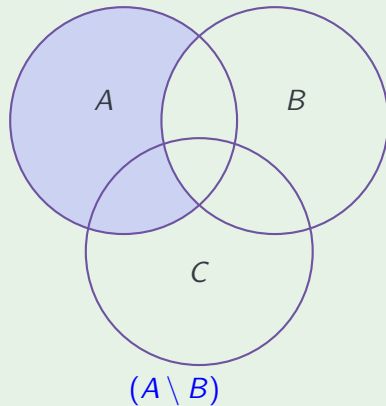
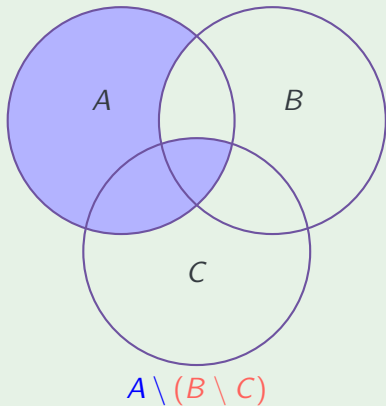
$A \setminus (B \setminus C)$



# Example

## Example

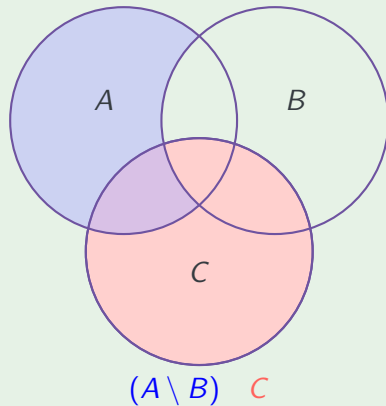
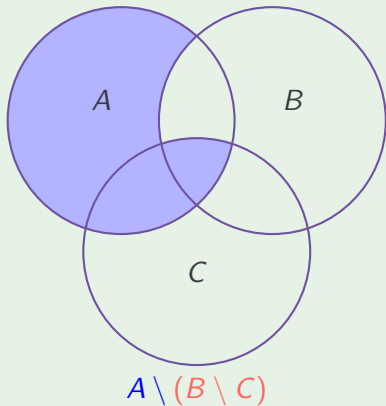
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# Example

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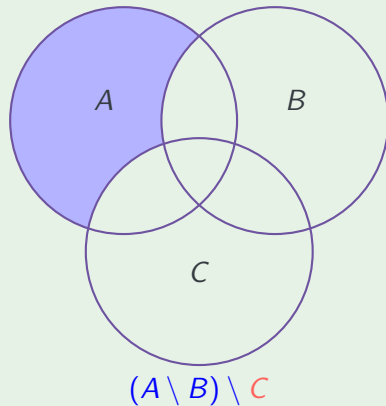
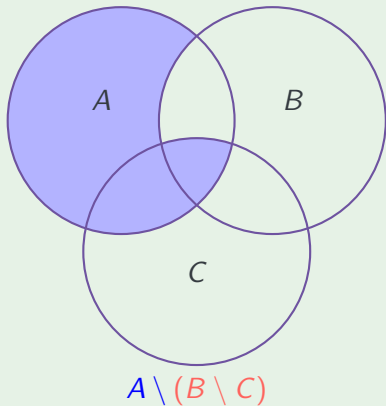
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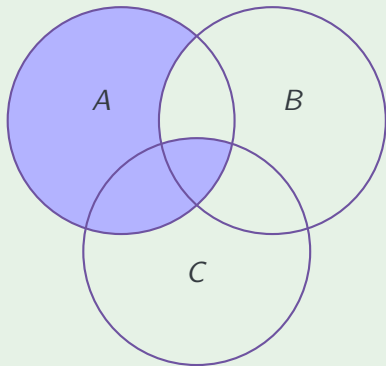
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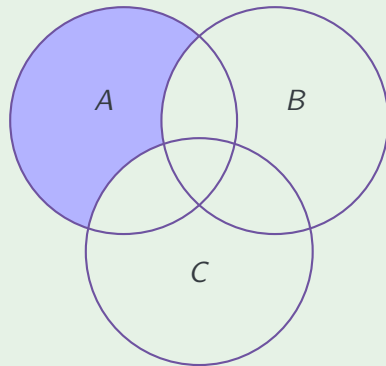
# Example

## Example

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$



$$A \setminus (B \setminus C)$$



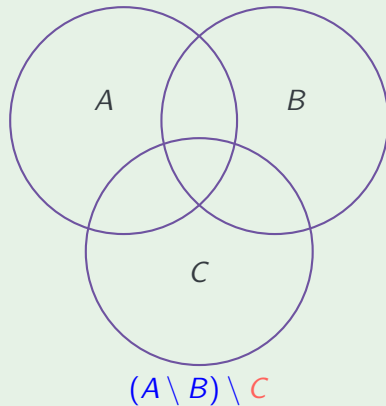
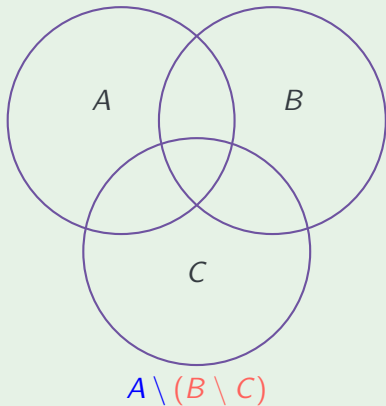
$$(A \setminus B) \setminus C$$



# Example

## Example

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$



# Examples

## Example

Show  $\{3, 2, 1\} = (0, 4)$ .

# Examples

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Show  $\{3, 2, 1\} = (0, 4)$ .

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$ .

# Examples

## Example

Show  $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

# Examples

## Example

Show  $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\begin{aligned}\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} &= \{-2, -1, 0, 1, 2\} \\ &= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}\end{aligned}$$

# Examples

## Example

Show  $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

# Examples

## Example

Show  $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all  $n \in \mathbb{Z}$ , if  $n^2 > 5$  then  $|n| > 2$ ; and
- For all  $n \in \mathbb{Z}$ , if  $|n| > 2$  then  $n^2 > 5$ .

That is, show:

For all  $n \in \mathbb{Z}$ :  $n^2 > 5$  if, and only if  $|n| > 2$

# Outline

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# Laws of Set Operations

For all sets  $A, B, C$ :

Commutativity

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associativity

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Distribution

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Identity

$$A \cup \emptyset = A$$

$$A \cap \mathcal{U} = A$$

Complementation

$$A \cup (A^c) = \mathcal{U}$$

$$A \cap (A^c) = \emptyset$$

# Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

## Example

Commutativity

$$A \cup B = B \cup A$$

# Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

## Example

Commutativity

$$A \cup B = B \cup A$$

Therefore:  $(C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$

## Example

### Example

Show that for all sets  $A \cap (B \cap C) = C \cap (B \cap A)$ :

# Example

## Example

Show that for all sets  $A \cap (B \cap C) = C \cap (B \cap A)$ :

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && [\text{Associativity}] \\ &= C \cap (A \cap B) && [\text{Commutativity}] \\ &= C \cap (B \cap A) && [\text{Commutativity}] \end{aligned}$$

## Important!

(Aim to) limit each step to a non-overlapping applications of a single rule

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## Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence

$$A \cap A = A$$

$$A \cup A = A$$

Double complementation

$$(A^c)^c = A$$

Annihilation

$$A \cap \emptyset = \emptyset$$

$$A \cup \mathcal{U} = \mathcal{U}$$

de Morgan's Laws

$$(A \cap B)^c = A^c \cup B^c$$

$$(A \cup B)^c = A^c \cap B^c$$

### Example (Idempotence of $\cup$ )

$$A = A \cup \emptyset$$

(Identity)



### Example (Idempotence of $\cup$ )

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \end{aligned}$$

### Example (Idempotence of $\cup$ )

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \end{aligned}$$

### Example (Idempotence of $\cup$ )

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \end{aligned}$$

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## Two useful results

### Definition

If  $A$  is a set defined using  $\cap$ ,  $\cup$ ,  $\emptyset$  and  $\mathcal{U}$ , then  $\text{dual}(A)$  is the expression obtained by replacing  $\cap$  with  $\cup$  (and vice-versa) and  $\emptyset$  with  $\mathcal{U}$  (and vice-versa).

### Theorem (Principle of Duality)

*If you can prove  $A_1 = A_2$  using the Laws of Set Operations then you can prove  $\text{dual}(A_1) = \text{dual}(A_2)$*

### Example

Absorption law:  $A \cup (A \cap B) = A$

Dual:  $A \cap (A \cup B) = A$

## Application (Idempotence of $\cap$ )

Recall Idempotence of  $\cup$ :

$$\begin{aligned} A &= A \cup \emptyset && \text{(Identity)} \\ &= A \cup (A \cap A^c) && \text{(Complementation)} \\ &= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\ &= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\ &= (A \cup A) && \text{(Identity)} \end{aligned}$$

## Application (Idempotence of $\cap$ )

Invoke the dual laws!

$$\begin{aligned} A &= A \cap \mathcal{U} && \text{(Identity)} \\ &= A \cap (A \cup A^c) && \text{(Complementation)} \\ &= (A \cap A) \cup (A \cap A^c) && \text{(Distributivity)} \\ &= (A \cap A) \cup \emptyset && \text{(Complementation)} \\ &= (A \cap A) && \text{(Identity)} \end{aligned}$$



## Two useful results

### Theorem (Uniqueness of complement)

*$A \cap B = \emptyset$  and  $A \cup B = \mathcal{U}$  if, and only if,  $B = A^c$ .*

## Two useful results

### Theorem (Uniqueness of complement)

$A \cap B = \emptyset$  and  $A \cup B = \mathcal{U}$  if, and only if,  $B = A^c$ .

### Proof (Only if).

$$\begin{aligned} B &= B \cap \mathcal{U} && \text{(Identity)} \\ &= B \cap (A \cup A^c) && \text{(Complement)} \\ &= (B \cap A) \cup (B \cap A^c) && \text{(Distributivity)} \\ &= (A \cap B) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= \emptyset \cup (A^c \cap B) && \text{(Given)} \\ &= (A \cap A^c) \cup (A^c \cap B) && \text{(Complement)} \\ &= (A^c \cap A) \cup (A^c \cap B) && \text{(Commutativity)} \\ &= A^c \cap (A \cup B) && \text{(Distributivity)} \\ &= A^c \cap \mathcal{U} && \text{(Given)} \\ &= A^c && \text{(Identity)} \end{aligned}$$



### Application (Double complement)

Take  $A = X^c$  and  $B = X$ :

$$\begin{aligned} X^c \cap X &= X \cap X^c && \text{(Commutativity)} \\ &= \emptyset && \text{(Identity)} \end{aligned}$$

$$X^c \cup X = \mathcal{U} \quad \text{(Principle of duality)}$$

By the uniqueness of complement,  $(X^c)^c = X$ .

# Exercises

## Exercises

Show the following for all sets  $A$ ,  $B$ ,  $C$ :

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

## Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$

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# Weekly Feedback

We would appreciate any comments/suggestions/requests you have on this week's lectures.



<https://forms.office.com/r/aHRCGANHiB>