Semantic Web

— to Artificial Intelligence

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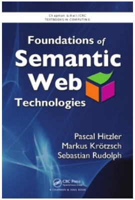
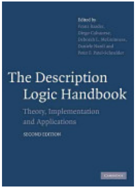
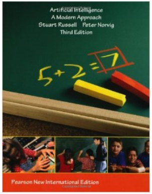
General Information

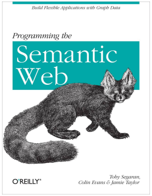
- This lecture is inspired by the following courses :
  - Knowledge Representation for the Semantic Web
    Pascal Hitzler, Wright State University, US
  - Foundations of Semantic Web Technologies
    Sebastian Rudolph, TU Dresden, Germany
  - Knowledge Graph Analysis
    Jens Lehmann, University of Bonn, Germany
  - Introduction à l'intelligence artificielle
    Laurent Simon, Polytech Paris Sud

- Reference books :









General Information

- Course materials : `www.lri.fr/~ma/M2DK`
- Organizational matters :
  - Initial plans :
    - Part 1 : Semantic Web Standards (RDF, RDFS, OWL, SPARQL)
    - Part 2 : Semantics and Reasoning for Semantic Web
    - Projects : Querying Semantic Data, Ontology Reasoning
  - Grading : average(Projects, Exam)

Reasoning Techniques for Expressive DLs

Tableau-based algorithms (1990–1995)

- first systems employing such algorithms : Kris and Crack
- Highly optimized systems : FaCT, Race, DLP

Alternative approaches (2000–2005)

- resolution-based approaches (use an optimized translation of DLs into first-order predicate logic and then apply appropriate first-order resolution provers)
- automata-based approaches (often more convenient for showing ExpTime complexity upper-bounds than tableau-based approaches)

Tableau method

Tableau method: most popular approach to reasoning in expressive DLs

- implemented in state-of-the-art DL reasoners

Tableau algorithms are used to **decide satisfiability**.

- can also be used for other reasoning tasks (e.g. instance checking) that can be reduced to satisfiability

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Idea: to determine whether a given (concept or KB) Ψ is satisfiable, **try to construct a (representation of a) model** of Ψ

- if we succeed, then we have shown that Ψ is satisfiable
- if we fail despite having considered all possibilities, then we have proven that Ψ is unsatisfiable

$\mathcal{ALC}$-concepts

Recall that $\mathcal{ALC}$-concepts are built using the following constructors:

$$\top \quad \bot \quad \neg \quad \sqcup \quad \sqcap \quad \forall R.C \quad \exists R.C$$

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We say that an $\mathcal{ALC}$-concept C is in **negation normal form (NNF)** if the symbol $\neg$ only appears directly in front of atomic concepts.

- in NNF: $A \sqcap \neg B$, $\exists R.\neg A$, $\neg A \sqcup \neg B$
- not in NNF: $\neg(A \sqcap B)$, $\exists R.\neg(\forall S.B)$, $A \sqcup \neg\forall R.B$, $\neg\top$

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Fact. Every $\mathcal{ALC}$-concept C can be transformed into an equivalent concept in NNF in linear time by applying the following rewriting rules:

$$\begin{array}{lll} \neg\top \rightsquigarrow \bot & \neg(C \sqcap D) \rightsquigarrow \neg C \sqcup \neg D & \neg(\forall R.C) \rightsquigarrow \exists R.\neg C \\ \neg\bot \rightsquigarrow \top & \neg(C \sqcup D) \rightsquigarrow \neg C \sqcap \neg D & \neg(\exists R.C) \rightsquigarrow \forall R.\neg C \end{array}$$

Note: say C and D are equivalent if the empty TBox entails $C \equiv D$.

Satisfiability of $\mathcal{ALC}$-concepts via tableau

We start by giving a tableau algorithm for deciding satisfiability of $\mathcal{ALC}$-concepts in NNF w.r.t. the empty TBox.

Procedure for testing satisfiability of C_0:

- We work with a set S of ABoxes
- Initially, S contains a single ABox $\{C_0(a_0)\}$

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- Stop applying rules when either:
  - every $\mathcal{A} \in S$ contains a clash, i.e. an assertion $\bot(b)$ or a pair of assertions $\{B(b), \neg B(b)\}$
  - some $\mathcal{A} \in S$ is clash-free and complete: no rule can be applied to $\mathcal{A}$

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- Return "yes" if some $\mathcal{A} \in S$ is clash-free, else "no".

Tableau rules for $\mathcal{ALC}$

$\sqcap$-**rule:** if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$

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$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$ and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$
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$\exists$-rule: if $\{\exists R.C(a)\} \in \mathcal{A}$ and there is no b with $\{R(a,b), C(b)\} \subseteq \mathcal{A}$
then **pick a new individual name** d and
replace $\mathcal{A}$ with $\mathcal{A} \cup \{R(a,d), C(d)\}$

Tableau example: only $\sqcap$ and $\sqcup$

Let's use the tableau procedure to test satisfiability of

$$C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$$

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Apply $\sqcap$-rule to $\mathcal{A}_0$:
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Apply $\sqcup$-rule to $\mathcal{A}_2'$:
get $S = \{\, \mathcal{A}_1', \mathcal{A}_3, \mathcal{A}_4 \,\}$ where $\mathcal{A}_3 = \mathcal{A}_2' \cup \{\neg B(a_0)\}$, $\mathcal{A}_4 = \mathcal{A}_2' \cup \{D(a_0)\}$

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$\mathcal{A}_3$ contains clash $\{B(a_0), \neg B(a_0)\}$!

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Apply $\sqcup$-rule to $\mathcal{A}_0'$:
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Apply $\sqcap$-rule to $\mathcal{A}_1$:
get $S = \{\, \mathcal{A}_1', \mathcal{A}_2 \,\}$ where $\mathcal{A}_1' = \mathcal{A}_1 \cup \{(\neg B \sqcup D)(a_0), \neg A(a_0)\}$

Apply $\sqcap$-rule to $\mathcal{A}_2$:
get $S = \{\, \mathcal{A}_1', \mathcal{A}_2' \,\}$ where $\mathcal{A}_2' = \mathcal{A}_2 \cup \{(\neg B \sqcup D)(a_0), \neg A(a_0)\}$

Apply $\sqcup$-rule to $\mathcal{A}_2'$:
get $S = \{\, \mathcal{A}_1', \mathcal{A}_3, \mathcal{A}_4 \,\}$ where $\mathcal{A}_3 = \mathcal{A}_2' \cup \{\neg B(a_0)\}$, $\mathcal{A}_4 = \mathcal{A}_2' \cup \{D(a_0)\}$

$\mathcal{A}_4$ *is complete, so we can stop.*

Tableau example: only $\sqcap$ and $\sqcup$

Let's use the tableau procedure to test satisfiability of

$$C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ ((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \}$.

Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}'_0 \}$ where $\mathcal{A}'_0 = \mathcal{A}_0 \cup \{(A \sqcup B)(a_0), ((\neg B \sqcup D) \sqcap \neg A)(a_0)\}$.

Apply $\sqcup$-rule to $\mathcal{A}'_0$:
get $S = \{ \mathcal{A}_1, \mathcal{A}_2 \}$ where $\mathcal{A}_1 = \mathcal{A}'_0 \cup \{A(a_0)\}$ and $\mathcal{A}_2 = \mathcal{A}'_0 \cup \{B(a_0)\}$.

Apply $\sqcap$-rule to $\mathcal{A}_1$:
get $S = \{ \mathcal{A}'_1, \mathcal{A}_2 \}$ where $\mathcal{A}'_1 = \mathcal{A}_1 \cup \{(\neg B \sqcup D)(a_0), \neg A(a_0)\}$

Apply $\sqcap$-rule to $\mathcal{A}_2$:
get $S = \{ \mathcal{A}'_1, \mathcal{A}'_2 \}$ where $\mathcal{A}'_2 = \mathcal{A}_2 \cup \{(\neg B \sqcup D)(a_0), \neg A(a_0)\}$

Apply $\sqcup$-rule to $\mathcal{A}'_2$:
get $S = \{ \mathcal{A}'_1, \mathcal{A}_3, \mathcal{A}_4 \}$ where $\mathcal{A}_3 = \mathcal{A}'_2 \cup \{\neg B(a_0)\}$, $\mathcal{A}_4 = \mathcal{A}'_2 \cup \{D(a_0)\}$

$\mathcal{A}_4$ contains no clash $\Rightarrow$ C_0 is satisfiable

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0) \qquad \textbf{$\sqcap$-rule}$$
$$(A \sqcup B)\,(a_0)$$

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$ | $B\,(a_0)$ **$\sqcup$-rule**

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$	$B\,(a_0)$
$\sqcap$-rule $(\neg B \sqcup D)\,(a_0)$	
$\neg A\,(a_0)$	

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$	$B\,(a_0)$
$(\neg B \sqcup D)\,(a_0)$	$(\neg B \sqcup D)\,(a_0)$
$\neg A\,(a_0)$	$\neg A\,(a_0)$

$\sqcap$-rule

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$ | $B\,(a_0)$

$(\neg B \sqcup D)\,(a_0)$ | $(\neg B \sqcup D)\,(a_0)$

$\neg A\,(a_0)$ | $\neg A\,(a_0)$

$\neg B\,(a_0)$ | $D\,(a_0)$ **$\sqcup$-rule**

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$	$B\,(a_0)$
$(\neg B \sqcup D)\,(a_0)$	$(\neg B \sqcup D)\,(a_0)$
$\neg A\,(a_0)$	$\neg A\,(a_0)$

Under the right branch:

$\neg B\,(a_0)$	$D\,(a_0)$

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$$\boxed{A\,(a_0)}$$

$$(\neg B \sqcup D)\,(a_0)$$

$$\boxed{\neg A\,(a_0)}$$

✗

$$\boxed{B\,(a_0)}$$

$$(\neg B \sqcup D)\,(a_0)$$

$$\neg A\,(a_0)$$

$$\boxed{\neg B\,(a_0)}$$

✗

$$D\,(a_0)$$

✔

Previous example in a picture

$\mathcal{A}'_1$

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$

$$(A \sqcup B)\,(a_0)$$

$$A\,(a_0)$$

$$(\neg B \sqcup D)\,(a_0)$$

$$\neg A\,(a_0)$$

✗

$$B\,(a_0)$$

$$(\neg B \sqcup D)\,(a_0)$$

$$\neg A\,(a_0)$$

$$\neg B\,(a_0)$$

✗

$$D\,(a_0)$$

✔

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$		$B\,(a_0)$
$(\neg B \sqcup D)\,(a_0)$		$(\neg B \sqcup D)\,(a_0)$
$\neg A\,(a_0)$		$\neg A\,(a_0)$
✗	$\neg B\,(a_0)$	$D\,(a_0)$
	✗	✔

$\mathcal{A}_3$

Previous example in a picture

$$(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)\,(a_0)$$
$$(A \sqcup B)\,(a_0)$$

$A\,(a_0)$	$B\,(a_0)$
$(\neg B \sqcup D)\,(a_0)$	$(\neg B \sqcup D)\,(a_0)$
$\neg A\,(a_0)$	$\neg A\,(a_0)$
✗	

$\neg B\,(a_0)$	$D\,(a_0)$
✗	✓
	$\mathcal{A}_4$

Tableau example: only $\sqcap$ and $\sqcup$

In our example, we had the complete and clash-free ABox $\mathcal{A}_4$:

$$((A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A))(a_0) \quad (A \sqcup B)(a_0)$$
$$((\neg B \sqcup D) \sqcap \neg A)(a_0) \quad B(a_0) \quad (\neg B \sqcup D)(a_0) \quad \neg A(a_0) \quad D(a_0)$$

Can build from $\mathcal{A}_4$ the interpretation $\mathcal{I}$ with:

- $\Delta^{\mathcal{I}} = \{a_0\}$ — use individuals from $\mathcal{A}_4$
- $A^{\mathcal{I}} = \emptyset$ — since $\mathcal{A}_4$ does not contain $A(a_0)$
- $B^{\mathcal{I}} = D^{\mathcal{I}} = \{a_0\}$ — since $\mathcal{A}_4$ contains $B(a_0)$ and $D(a_0)$

We can verify that $(A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)^{\mathcal{I}} = \{a_0\}$.

- $\mathcal{I}$ witnesses the satisfiability of $C_0 = (A \sqcup B) \sqcap ((\neg B \sqcup D) \sqcap \neg A)$

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}$.

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}$.

Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\}$.

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Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\}$.

Apply $\exists$-rule to $\mathcal{A}_0'$:
get $S = \{ \mathcal{A}_0'' \}$ where $\mathcal{A}_0'' = \mathcal{A}_0' \cup \{R(a_0, a_1), A(a_1)\}$.

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}$.

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Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{\, \mathcal{A}_0 \,\}$ where $\mathcal{A}_0 = \{\, (\, \exists R.A \sqcap \forall R.\neg A \,)(a_0) \,\}$.

Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{\, \mathcal{A}_0' \,\}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\}$.

Apply $\exists$-rule to $\mathcal{A}_0'$:
get $S = \{\, \mathcal{A}_0'' \,\}$ where $\mathcal{A}_0'' = \mathcal{A}_0' \cup \{R(a_0, a_1), A(a_1)\}$.

Apply $\forall$-rule to $\mathcal{A}_0''$:
get $S = \{\, \mathcal{A}_0''' \,\}$ where $\mathcal{A}_0''' = \mathcal{A}_0'' \cup \{\neg A(a_1)\}$.

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}$.

Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\}$.

Apply $\exists$-rule to $\mathcal{A}_0'$:
get $S = \{ \mathcal{A}_0'' \}$ where $\mathcal{A}_0'' = \mathcal{A}_0' \cup \{R(a_0, a_1), A(a_1)\}$.

Apply $\forall$-rule to $\mathcal{A}_0''$:
get $S = \{ \mathcal{A}_0''' \}$ where $\mathcal{A}_0''' = \mathcal{A}_0'' \cup \{\neg A(a_1)\}$.

$\mathcal{A}_0'''$ contains clash $\{A(a_1), \neg A(a_1)\}$!

Tableau example: $\forall$ and $\exists$

Let's use the tableau procedure to test satisfiability of

$$C_0 = \exists R.A \sqcap \forall R.\neg A$$

Start with $S = \{ \mathcal{A}_0 \}$ where $\mathcal{A}_0 = \{ (\exists R.A \sqcap \forall R.\neg A)(a_0) \}$.

Apply $\sqcap$-rule to $\mathcal{A}_0$:
get $S = \{ \mathcal{A}_0' \}$ where $\mathcal{A}_0' = \mathcal{A}_0 \cup \{(\exists R.A)(a_0), (\forall R.\neg A)(a_0)\}$.

Apply $\exists$-rule to $\mathcal{A}_0'$:
get $S = \{ \mathcal{A}_0'' \}$ where $\mathcal{A}_0'' = \mathcal{A}_0' \cup \{R(a_0, a_1), A(a_1)\}$.

Apply $\forall$-rule to $\mathcal{A}_0''$:
get $S = \{ \mathcal{A}_0''' \}$ where $\mathcal{A}_0''' = \mathcal{A}_0'' \cup \{\neg A(a_1)\}$.

The only set in S contains a clash $\Rightarrow$ C_0 is unsatisfiable

Previous example in a picture

$$(\,\exists R.A \sqcap \forall R.\neg A\,)\,(a_0)$$

Previous example in a picture

$$(\,\exists R.A \sqcap \forall R.\neg A\,)\,(a_0)$$

$$(\,\exists R.A\,)\,(a_0) \qquad \textbf{$\sqcap$-rule}$$

$$(\,\forall R.\neg A\,)\,(a_0)$$

Previous example in a picture

$$(\,\exists R.A \sqcap \forall R.\neg A\,)\,(a_0)$$

$$(\,\exists R.A\,)\,(a_0)$$

$$(\,\forall R.\neg A\,)\,(a_0)$$

$$R\,(a_0, a_1) \qquad \textbf{$\exists$-rule}$$

$$A\,(a_1)$$

Previous example in a picture

$$(\,\exists R.A \sqcap \forall R.\neg A\,)\,(a_0)$$

$$(\,\exists R.A\,)\,(a_0)$$

$$(\,\forall R.\neg A\,)\,(a_0)$$

$$R\,(a_0, a_1)$$

$$A\,(a_1) \qquad \textbf{$\forall$-rule}$$

$$\neg A\,(a_1)$$

Previous example in a picture

$$(\,\exists R.A \sqcap \forall R.\neg A\,)\,(a_0)$$

$$(\,\exists R.A\,)\,(a_0)$$

$$(\,\forall R.\neg A\,)\,(a_0)$$

$$R\,(a_0, a_1)$$

$$\boxed{\begin{array}{c} A\,(a_1) \\ \neg A\,(a_1) \end{array}}$$

✘

Tableau example: $\forall$ and $\exists$

Suppose that we consider a slightly different concept

$$C_0 = \exists R.A \sqcap \forall R.\neg B$$

Now the tableau algorithm yields the following complete, clash-free ABox:

$(\exists R.A \sqcap \forall R.\neg B)(a_0) \quad (\exists R.A)(a_0) \quad (\forall R.\neg B)(a_0) \quad R(a_0, a_1) \quad A(a_1) \quad \neg B(a_1)$

Tableau example: $\forall$ and $\exists$

Suppose that we consider a slightly different concept

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Now the tableau algorithm yields the following complete, clash-free ABox:

$$(\exists R.A \sqcap \forall R.\neg B)(a_0) \quad (\exists R.A)(a_0) \quad (\forall R.\neg B)(a_0) \quad R(a_0, a_1) \quad A(a_1) \quad \neg B(a_1)$$

Corresponding interpretation $\mathcal{I}$:

- $\Delta^{\mathcal{I}} = \{a_0, a_1\}$
- $A^{\mathcal{I}} = \{a_1\}$
- $B^{\mathcal{I}} = \emptyset$
- $R^{\mathcal{I}} = \{(a_0, a_1)\}$

Can check that $\mathcal{I}$ is such that $C_0^{\mathcal{I}} = \{a_0\}$.

Properties of the tableau algorithm

Let's call our tableau algorithm `CSat` (for concept satisfiability).

To show that `CSat` is a decision procedure, we must show:

Termination: The algorithm `CSat` always terminates.

Soundness: `CSat` outputs "yes" on input $C_0 \Rightarrow C_0$ is satisfiable.

Completeness: C_0 satisfiable $\Rightarrow$ `CSat` will output "yes".

Preliminary definitions

The size of a concept C, denoted $|C|$, is defined as the total number of occurrences of atomic concepts, atomic roles, and constructors in C.

For example : $|A \sqcap B| = 3, |\neg(A \sqcap B)| = 4, |\neg\neg A| = 3, |\exists R.A| = 3$

Preliminary definitions

Subconcepts of a concept:

$$\begin{aligned}
\mathsf{sub}(A) &= \{A\} \\
\mathsf{sub}(\neg C) &= \{\neg C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(\exists R.C) &= \{\exists R.C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(\forall R.C) &= \{\forall R.C\} \cup \mathsf{sub}(C) \\
\mathsf{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2) \\
\mathsf{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)
\end{aligned}$$

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\mathsf{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)
\end{aligned}
$$

Role depth of a concept:

$$
\begin{aligned}
\mathsf{depth}(A) &= 0 \\
\mathsf{depth}(\neg C) &= \mathsf{depth}(C) \\
\mathsf{depth}(\exists R.C) &= \mathsf{depth}(\forall R.C) = \mathsf{depth}(C) + 1 \\
\mathsf{depth}(C_1 \sqcup C_2) &= \mathsf{depth}(C_1 \sqcap C_2) = \mathsf{max}(\mathsf{depth}(C_1), \mathsf{depth}(C_2))
\end{aligned}
$$

Preliminary definitions

Subconcepts of a concept: $|\mathsf{sub}(C)| \leq |C|$

?

$$
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\mathsf{sub}(A) &= \{A\} \\
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\mathsf{sub}(C_1 \sqcup C_2) &= \{C_1 \sqcup C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2) \\
\mathsf{sub}(C_1 \sqcap C_2) &= \{C_1 \sqcap C_2\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)
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$$

1 1 1

Role depth of a concept: $\mathsf{depth}(C) \leq |C|$

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$$

Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

Termination of $\mathtt{CSat}$

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We observe that for every ABox $\mathcal{A}$ generated by the procedure:

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1. if $D(b) \in \mathcal{A}$, then $D \in \mathsf{sub}(C_0)$

Termination of CSat

Suppose we run CSat starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \mathsf{sub}(C_0)$
   - $\mathcal{A}$ contains at most $|C_0|$ concept assertions per individual

So, there can only be a finite number of rule applications per individual

Next to show that the number of newly introduced individuals in a chain of rule

Termination of CSat

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2. the set of role assertions in $\mathcal{A}$ forms a tree

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2. the set of role assertions in $\mathcal{A}$ forms a tree
3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\text{depth}(D) \leq \text{depth}(C_0) - k$

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3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\text{depth}(D) \leq \text{depth}(C_0) - k$
   - each individual in $\mathcal{A}$ is at distance $\leq$ depth(C_0) from a_0

Termination of $\mathtt{CSat}$

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3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\mathsf{depth}(D) \leq \mathsf{depth}(C_0) - k$
   - each individual in $\mathcal{A}$ is at distance $\leq \mathsf{depth}(C_0)$ from a_0
4. for every individual b in $\mathcal{A}$, there are at most $|C_0|$ individuals c such that $R(b, c) \in \mathcal{A}$ for some R (at most one per existential concept)

Termination of $\mathtt{CSat}$

Suppose we run $\mathtt{CSat}$ starting from $S = \{\{C_0(a_0)\}\}$.

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3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\mathsf{depth}(D) \leq \mathsf{depth}(C_0) - k$
   - each individual in $\mathcal{A}$ is at distance $\leq$ $\mathsf{depth}(C_0)$ from a_0
4. for every individual b in $\mathcal{A}$, there are at most $|C_0|$ individuals c such that $R(b, c) \in \mathcal{A}$ for some R (at most one per existential concept)

Thus: **bound on the size of ABoxes** generated by the procedure.

Termination of $\mathtt{CSat}$

Suppose we run $\mathtt{CSat}$ starting from $S = \{\{C_0(a_0)\}\}$.

We observe that for every ABox $\mathcal{A}$ generated by the procedure:

1. if $D(b) \in \mathcal{A}$, then $D \in \mathsf{sub}(C_0)$
   - $\mathcal{A}$ contains at most $|C_0|$ concept assertions per individual
2. the set of role assertions in $\mathcal{A}$ forms a tree
3. if $D(b) \in \mathcal{A}$ and the unique path from a_0 to b has length k, then $\mathsf{depth}(D) \leq \mathsf{depth}(C_0) - k$
   - each individual in $\mathcal{A}$ is at distance $\leq \mathsf{depth}(C_0)$ from a_0
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Thus: **bound on the size of ABoxes** generated by the procedure.

The tableau procedure **only adds assertions** to ABoxes
$\Rightarrow$ **eventually all ABoxes will contain a clash or will be complete**

Soundness and Completeness :

$\boxed{\text{CSat returns “yes” on input } C_0. \iff C_0 \text{ is satisfiable.}}$

CSat returns “yes” on input C_0. $\Longrightarrow$ C_0 is satisfiable. (Soundness)
(I.e., C_0 is unsatisfiable. $\Longrightarrow$ CSat returns “no” on input C_0. That is, the positive answers of the algorithm are correct : if the algorithm says “consistent”, then the input concept is indeed satisfiable.)

CSat returns “yes” on input C_0. $\Longleftarrow$ C_0 is satisfiable. (Completeness)
(I.e., C_0 is unsatisfiable. $\Longleftarrow$ CSat returns “no” on input C_0. That is, the negative answers of the algorithm is correct : if the algorithm says “no”, the input concept is unsatisfiable.)

Soundness of $\texttt{CSat}$ (1)

Suppose that $\texttt{CSat}$ returns “yes” on input C_0.

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- $\Delta^\mathcal{I} = \{a \mid a \text{ is an individual in } \mathcal{A}\}$
- $A^\mathcal{I} = \{a \mid A(a) \in \mathcal{A}\}$
- $R^\mathcal{I} = \{(a, b) \mid R(a, b) \in \mathcal{A}\}$

Claim: $\mathcal{I}$ is such that $C_0^\mathcal{I} \neq \emptyset$

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Claim: $\mathcal{I}$ is such that $C_0^{\mathcal{I}} \neq \emptyset$

To show the claim, we prove by induction on the size of concepts that:

$$D(b) \in \mathcal{A} \quad \Rightarrow \quad b \in D^{\mathcal{I}}$$

Soundness of $\mathtt{CSat}$ (2)

Base case: $D = A$ or $D = \neg A$

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Induction step: show statement holds for D with $|D| = k + 1$

Again, many cases to consider:

- $D = E \sqcap F$: since $\mathcal{A}$ is complete, it must contain both $E(b)$ and $F(b)$. Applying the IH, we get $b \in E^{\mathcal{I}}$ and $b \in F^{\mathcal{I}}$, hence $b \in (E \sqcap F)^{\mathcal{I}}$

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- $D = \exists R.E$: since $\mathcal{A}$ is complete, there is some c such that $R(b, c) \in \mathcal{A}$ and $E(c) \in \mathcal{A}$. Then $(b, c) \in R^{\mathcal{I}}$ and by the IH, we get $c \in E^{\mathcal{I}}$, so $b \in (\exists R.E)^{\mathcal{I}}$

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- $D = E \sqcup F$: left as practice
- $D = \forall R.E$: left as practice

Completeness of CSat

Suppose that the concept C_0 is satisfiable.

Then the ABox $\{C_0(a_0)\}$ must be satisfiable too.

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We observe that the tableau rules are **satisfiability-preserving**:

- If an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}'$ is the result of applying a rule to $\mathcal{A}$, then $\mathcal{A}'$ is also satisfiable.
- If an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}_1$ and $\mathcal{A}_2$ are obtained when applying a rule to $\mathcal{A}$, then either $\mathcal{A}_1$ is satisfiable or $\mathcal{A}_2$ is satisfiable.

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- If an ABox $\mathcal{A}$ is satisfiable and $\mathcal{A}_1$ and $\mathcal{A}_2$ are obtained when applying a rule to $\mathcal{A}$, then either $\mathcal{A}_1$ is satisfiable or $\mathcal{A}_2$ is satisfiable.

We start with a satisfiable ABox and the rules are satisfiability-preserving, so eventually we will reach a complete, satisfiable (thus: clash-free) ABox.

Complexity of $\mathtt{CSat}$

Bad news: our algorithm may require exponential time and space...

To see why, consider what happens if we run $\mathtt{CSat}$ on the concept

$$\prod_{0 \leq i < n} \underbrace{\forall R. \ldots \forall R.}_{i \text{ times}} (\exists R.B \sqcap \exists R.\neg B)$$

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Good news: can **modify the procedure so it runs in polynomial space**

- instead of a set of ABoxes, keep only one ABox in memory at a time
  - when apply the $\sqcup$-rule, first examine $\mathcal{A}_1$, then afterwards examine $\mathcal{A}_2$
  - remember that second disjunct stills needs to be checked
- explore the children of an individual one at a time
  - possible because no interaction between the different "branches"
  - store which $\exists R.C$ concepts have been tested, which are left to do
- this allows us to keep at most $|C_0|$ individuals in memory at a time

Complexity of $\mathcal{ALC}$ concept satisfability

Hierarchy of complexity classes

PTIME $\subseteq$ NP $\subseteq$ ... $\subseteq$ PSPACE $\subseteq$ EXPTIME $\subseteq$ NEXPTIME $\subseteq$ ... $\subseteq$ EXPSPACE ...

(it is believed that all inclusions are strict)

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Theorem: $\mathcal{ALC}$ concept satisfiability (no TBox) is PSPACE-complete.

- Membership in PSPACE shown using modified tableau procedure
- Hardness for PSPACE shown by giving a reduction from some known PSPACE-hard problem (e.g. QBF validity)

Homework

Exercise 1: Concept satisfiability via tableau

Use the tableau algorithm to decide which of the following concepts is satisfiable:

1. $\exists R.(A \sqcap B) \sqcap \forall R.(\neg A \sqcup D)$
2. $(\exists R.\exists S.A) \sqcap (\forall R.\forall S.\neg A)$
3. $\exists R.B \sqcap \forall R.(\forall R.A) \sqcap \forall R.\neg A$

If a concept is found to be satisfiable, use the result to construct an interpretation in which the concept is non-empty.

Exercise 2: Negation normal form

Consider the following recursive algorithm for putting a concept in NNF.

Algorithm NNF

Input: $\mathcal{ALC}$-concept C

If $C = A$ or $C = \neg A$ (with A atomic concept), then output C

If $C = D_1 \sqcap D_2$, then output $\mathsf{NNF}(D_1) \sqcap \mathsf{NNF}(D_2)$

If $C = D_1 \sqcup D_2$, then output $\mathsf{NNF}(D_1) \sqcup \mathsf{NNF}(D_2)$

If $C = \exists R.D$, then output $\exists R.\mathsf{NNF}(D)$

If $C = \forall R.D$, then output $\forall R.\mathsf{NNF}(D)$

If $C = \neg(D_1 \sqcap D_2)$, then output $\mathsf{NNF}(\neg D_1) \sqcup \mathsf{NNF}(\neg D_2)$

If $C = \neg(D_1 \sqcup D_2)$, then output $\mathsf{NNF}(\neg D_1) \sqcap \mathsf{NNF}(\neg D_2)$

If $C = \neg\exists R.D$, then output $\forall R.\mathsf{NNF}(\neg D)$

If $C = \neg\forall R.D$, then output $\exists R.\mathsf{NNF}(\neg D)$

If $C = \neg(\neg D)$, then output $\mathsf{NNF}(\neg D)$

(we use $\mathsf{NNF}(E)$ to denote output of NNF on input E)

Exercise 2: Negation normal form

Show that the algorithm NNF has the following properties:

1. NNF terminates on every input.
2. NNF(C) is in negation normal form.
3. NNF(C) is equivalent to C.

Hint: you may want to use the size of C, which is denoted $|C|$ and defined as the total number of occurrences of atomic concepts, atomic roles, and constructors in C.

Example: $|A \sqcap B| = 3$, $|\neg(A \sqcap B)| = 4$, $|\neg\neg A| = 3$, $|\exists R.A| = 3$

Exercise 3: Adding acyclic terminologies (1)

In this exercise, we see how to test concept satisfiability w.r.t. a special kind of TBox called an acyclic terminology.

A **terminology** is a set of concept definitions

$$A \equiv C$$

where A is an atomic concept and C a (possibly complex) concept, and *no two concept definitions have the same atomic concept on the left*.

A concept definition $A \equiv C$ corresponds to the pair of inclusions $A \sqsubseteq C$ and $C \sqsubseteq A$, and so it is satisfied by $\mathcal{I}$ if $A^{\mathcal{I}} = C^{\mathcal{I}}$.

Exercise 3: Adding acyclic terminologies (2)

Given a terminology $\mathcal{T}$ and concept C, the **unfolding of C w.r.t. $\mathcal{T}$** is obtained by applying the following operation as long as possible:

- replace any atomic concept A such that $A \equiv D \in \mathcal{T}$ by D

Example: the unfolding of $A \sqcap \exists R.B$ w.r.t. the terminology

$$\{A \equiv \forall S.\forall R.B \quad B \equiv E \sqcap F \quad F \equiv \neg G\}$$

is the concept $\forall S.\forall R.(E \sqcap \neg G) \sqcap \exists R.(E \sqcap \neg G)$.

Note: for some terminologies, the process may never stop!

- take $C = A$ and $\mathcal{T} = \{A \equiv \exists R.A\}$

A terminology is called **acyclic** if the **unfolding procedure always halts**.

Solutions to the exercises 2 and 3

Exercise 2: Negation normal form

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If $C = \forall R.D$, then output $\forall R.\mathsf{NNF}(D)$

If $C = \neg(D_1 \sqcap D_2)$, then output $\mathsf{NNF}(\neg D_1) \sqcup \mathsf{NNF}(\neg D_2)$

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Solution to exercise 2: Termination

The proof is by induction on the size of the input concept.

Base case: input concept C with $|C| \leq 2$

Then we must have $C = A$ or $C = \neg A$, so `NNF` terminates on input C

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Induction hypothesis (IH): `NNF` terminates on all inputs C with $|C| \leq k$

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The proof is by induction on the size of the input concept.

Base case: input concept C with $|C| \leq 2$

Then we must have $C = A$ or $C = \neg A$, so NNF terminates on input C

Induction hypothesis (IH): NNF terminates on all inputs C with $|C| \leq k$

Induction step: show NNF terminates on input C with $|C| = k + 1$

Let C be a concept with $|C| = k + 1$. There are several cases to consider:

- $C = D_1 \sqcap D_2$: call NNF on D_1 and D_2, both calls terminate since $|D_1| < |C|$ and $|D_2| < |C|$, hence $|D_1| \leq k$ and $|D_2| \leq k$
- $C = \neg\exists R.D$: call NNF on $\neg D$, and the call terminates since $|\neg D| \leq k$
- $C = \ldots$

Solution to exercise 2: NNF

The proof is by induction on the size of the input concept C.

Base case: input concept C with $|C| \leq 2$

We must have $C = A$ or $C = \neg A$. In both cases, `NNF` will output C, which is in NNF.

Solution to exercise 2: NNF

The proof is by induction on the size of the input concept C.

Base case: input concept C with $|C| \leq 2$

We must have $C = A$ or $C = \neg A$. In both cases, `NNF` will output C, which is in NNF.

Induction hypothesis (IH): $\texttt{NNF}(C)$ is in NNF, whenever $|C| \leq k$

Solution to exercise 2: NNF

The proof is by induction on the size of the input concept C.

Base case: input concept C with $|C| \leq 2$

We must have $C = A$ or $C = \neg A$. In both cases, $\mathtt{NNF}$ will output C, which is in NNF.

Induction hypothesis (IH): $\mathtt{NNF}(C)$ is in NNF, whenever $|C| \leq k$

Induction step: show $\mathtt{NNF}(C)$ is in NNF when $|C| = k + 1$

Let C be a concept with $|C| = k + 1$. There are several cases to consider:

- $C = D_1 \sqcap D_2$: by the IH, both $\mathtt{NNF}(D_1)$ and $\mathtt{NNF}(D_2)$ are in NNF, hence the output $\mathtt{NNF}(D_1) \sqcap \mathtt{NNF}(D_2)$ must be too.
- $C = \neg \exists R.D$: by the IH, $\mathtt{NNF}(\neg D)$ is in NNF. Thus, $\forall R.\mathtt{NNF}(\neg D)$ must also be in NNF.
- $C = \ldots$

Solution to exercise 2: Equivalence

Base case: input concept C with $|C| \leq 2$

Easy since `NNF` will output C, and C is clearly equivalent to itself.

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Easy since NNF will output C, and C is clearly equivalent to itself.

Induction hypothesis (IH): $\mathtt{NNF}(C)$ is equivalent to C when $|C| \leq k$

Induction step: show $\mathtt{NNF}(C)$ is equivalent to C when $|C| = k + 1$

Let C be a concept with $|C| = k + 1$. There are several cases to consider:

- $C = \neg\exists R.D$: by the IH, $\mathtt{NNF}(\neg D)$ is equivalent to $\neg D$. Let $\mathcal{I}$ be an interpretation. We have:

$$\begin{aligned}
C^{\mathcal{I}} &= (\neg\exists R.D)^{\mathcal{I}} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \text{ there is no } e \text{ with } (d, e) \in R^{\mathcal{I}} \text{ and } e \in D^{\mathcal{I}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \text{ for every } e \text{ with } (d, e) \in R^{\mathcal{I}} \text{ we have } e \in (\neg D)^{\mathcal{I}}\} \\
&= (\forall R.\neg D)^{\mathcal{I}} \\
&= (\forall R.\mathtt{NNF}(\neg D))^{\mathcal{I}} \\
&= (\mathtt{NNF}(C))^{\mathcal{I}}
\end{aligned}$$

- $C = \ldots$

Exercise 3: Adding acyclic terminologies (1)

In this exercise, we see how to test concept satisfiability w.r.t. a special kind of TBox called an acyclic terminology.

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where A is an atomic concept and C a (possibly complex) concept, and *no two concept definitions have the same atomic concept on the left*.

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Exercise 3: Adding acyclic terminologies (2)

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is the concept $\forall S.\forall R.(E \sqcap \neg G) \sqcap \exists R.(E \sqcap \neg G)$.

Note: for some terminologies, the process may never stop!

- take $C = A$ and $\mathcal{T} = \{A \equiv \exists R.A\}$

A terminology is called **acyclic** if the **unfolding procedure always halts**.

Exercise 3: Adding acyclic terminologies (3)

We can test satisfiability of C w.r.t. acyclic terminology $\mathcal{T}$ as follows:

Step 1 Compute the unfolding U of C w.r.t. $\mathcal{T}$.

Step 2 Put U into negation normal form.

Step 3 Run `CSat` on U, and output the same answer as `CSat`.

First task: Use the above procedure to test satisfiability of the concepts:

1. $B \sqcap \forall R.\neg G$
2. $A \sqcap \forall R.\exists R.G$

with respect to the following acyclic terminology:

$$\{A \equiv \exists R.B \sqcap \exists S.\neg B \quad B \equiv \exists R.D \sqcap \exists R.E \quad G \equiv D \sqcap E\}$$

Second task: explain why the above procedure is a decision procedure for $\mathcal{ALC}$ concept satisfiability w.r.t. acyclic terminologies.

(note: I am not asking for a formal proof, just a clear explanation)

Solution to exercise 3: first task (1)

For the first concept, we first unfold w.r.t. $\mathcal{T}$ to get:

$$(\exists R.D \sqcap \exists R.E) \sqcap \forall R.\neg(D \sqcap E)$$

and then put the latter concept into NNF:

$$(\exists R.D \sqcap \exists R.E) \sqcap \forall R.(\neg D \sqcup \neg E)$$

The tableau algorithm returns "yes" on the preceding concept, so the **first concept is satisfiable w.r.t.** $\mathcal{T}$.

Solution to exercise 3: first task (2)

For the second concept, we first unfold w.r.t. $\mathcal{T}$ to get:

$$\exists R.(\exists R.D \sqcap \exists R.E) \sqcap \exists S.(\neg(\exists R.D \sqcap \exists R.E)) \sqcap \forall R.\exists R.(D \sqcap E)$$

and then put the latter concept into NNF:

$$\exists R.(\exists R.D \sqcap \exists R.E) \sqcap \exists S.(\forall R.\neg D \sqcup \forall R.\neg E) \sqcap \forall R.\exists R.(D \sqcap E)$$

The tableau algorithm returns "yes" on the preceding concept, so the **second concept is also satisfiable w.r.t.** $\mathcal{T}$.

Solution to exercise 3: second task

Suppose we run the procedure on concept C and acyclic terminology $\mathcal{T}$.

Termination: Step 1 terminates because $\mathcal{T}$ is acyclic. Steps 2 and 3 terminate because we know that $\mathtt{NNF}$ and $\mathtt{CSat}$ both terminate.

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Soundness and Completeness: Let C_1 be the concept after Step 1 and C_2 the concept after Step 2. Then we have:

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Soundness and Completeness: Let C_1 be the concept after Step 1 and C_2 the concept after Step 2. Then we have:

$$\mathtt{CSat} \text{ returns "yes" on } C_2 \quad \Leftrightarrow \quad C_2 \text{ is satisfiable}$$

Soundness and completeness of $\mathtt{CSat}$ (previous lecture)

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$$\begin{aligned}\mathtt{CS}a\mathtt{t}\text{ returns "yes" on } C_2 \quad &\Leftrightarrow \quad C_2 \text{ is satisfiable}\\ &\Leftrightarrow \quad C_1 \text{ is satisfiable}\end{aligned}$$

NNF preserves equivalence (previous lecture)

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Soundness and Completeness: Let C_1 be the concept after Step 1 and C_2 the concept after Step 2. Then we have:

$$\begin{aligned} \mathtt{CSat} \text{ returns "yes" on } C_2 \quad &\Leftrightarrow \quad C_2 \text{ is satisfiable} \\ &\Leftrightarrow \quad C_1 \text{ is satisfiable} \\ &\Leftrightarrow \quad C \text{ is satisfiable w.r.t. } \mathcal{T} \end{aligned}$$

C_1 satisfiable $\Rightarrow$ take $\mathcal{I}$ with $C_1^{\mathcal{I}} \neq \emptyset$, then reinterpret the defined concepts according to $\mathcal{T}$ in order to get model of $\mathcal{T}$ with C non-empty

Solution to exercise 3:   second task

Suppose we run the procedure on concept C and acyclic terminology $\mathcal{T}$.

Termination: Step 1 terminates because $\mathcal{T}$ is acyclic. Steps 2 and 3 terminate because we know that NNF and CSat both terminate.

Soundness and Completeness: Let C_1 be the concept after Step 1 and C_2 the concept after Step 2. Then we have:

$$\begin{array}{lcl} \texttt{CSat} \text{ returns "yes" on } C_2 & \Leftrightarrow & C_2 \text{ is satisfiable} \\ & \Leftrightarrow & C_1 \text{ is satisfiable} \\ & \Leftrightarrow & C \text{ is satisfiable w.r.t. } \mathcal{T} \end{array}$$

C satisfiable w.r.t. $\mathcal{T} \Rightarrow$ take model $\mathcal{I}$ of $\mathcal{T}$ with $C^{\mathcal{I}} \neq \emptyset$, then because of the definitions in $\mathcal{T}$, must have $C_1^{\mathcal{I}} \neq \emptyset$

Back to tableau algorithms

Extension to KB satisfiability

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Idea: if we have $C \sqsubseteq D$, then every element must satisfy either $\neg C$ or D

Concretely, we might try adding the following rule:

TBox rule: if a is in $\mathcal{A}$, $C \sqsubseteq D \in \mathcal{T}$, & $(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))\,(a) \notin \mathcal{A}$

then replace $\mathcal{A}$ with $\mathcal{A} \cup \{(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a)\}$

Examples: KB satisfiability

Let's try the modified procedure on the following KBs

1. $(\mathcal{T}, \{A(a)\})$
2. $(\mathcal{T}, \{R(c, a), B(a)\})$

where $\mathcal{T}$ is the following TBox:

$$\{ \quad A \sqsubseteq \exists R.B \quad B \sqsubseteq D \quad \exists R.D \sqsubseteq \neg A \quad \}$$

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Seems we have a problem... How can we ensure termination?

Blocking

Basic idea: if two individuals "look the same", then it is unnecessary to explore both of them

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Formally, give individuals a, b in $\mathcal{A}$, we say that b **blocks** a if:

- $\{C \mid C(a) \in \mathcal{A}\} \subseteq \{C \mid C(b) \in \mathcal{A}\}$
- b was present in $\mathcal{A}$ before a was introduced

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Say that individual a **is blocked** (in $\mathcal{A}$) if some b blocks a.

Modify rules so that they **only apply to unblocked individuals**.

Tableau rules for KBs

$\sqcap$-rule: if $(C_1 \sqcap C_2)(a) \in \mathcal{A}$, a **is not blocked**, and $\{C_1(a), C_2(a)\} \not\subseteq \mathcal{A}$,
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a), C_2(a)\}$

$\sqcup$-rule: if $(C_1 \sqcup C_2)(a) \in \mathcal{A}$, a **is not blocked**, and $\{C_1(a), C_2(a)\} \cap \mathcal{A} = \emptyset$,
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C_1(a))\}$ **and** $\mathcal{A} \cup \{C_2(a))\}$

$\forall$-rule: if $\{\forall R.C(a), R(a,b)\} \in \mathcal{A}$, a **is not blocked**, and $C(b) \notin \mathcal{A}$,
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{C(b))\}$

$\exists$-rule: if $\{\exists R.C(a)\} \in \mathcal{A}$, a **is not blocked**, and no $\{R(a,b), C(b)\} \subseteq \mathcal{A}$,
then **pick a new individual name** d and replace $\mathcal{A}$ with
$\mathcal{A} \cup \{R(a,d), C(d)\}$

$\sqsubseteq$-rule: if a appears in $\mathcal{A}$ and a **is not blocked**, $C \sqsubseteq D \in \mathcal{T}$, and
$(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a) \notin \mathcal{A}$,
then replace $\mathcal{A}$ with $\mathcal{A} \cup \{(\mathtt{NNF}(\neg C) \sqcup \mathtt{NNF}(D))(a)\}$

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$$(\neg F \sqcup \exists S.F)\,(a_0)$$

$$\begin{array}{c|l}
\neg F\,(a_0) & (\exists S.F)\,(a_0) \\
 & S(a_0, a_1) \\
✗ & F(a_1) \\
 & \\
 & a_1 \text{ is blocked by } a_0
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$\neg F(a_0)$	$(\exists S.F)(a_0)$
✘	$S(a_0, a_1)$
	$F(a_1)$
	a_1 is blocked by a_0

We obtain a complete and clash-free ABox $\Rightarrow$ the KB is satisfiable !

Another blocking example

Consider the following TBox

$$\mathcal{T} = \{A \sqsubseteq \exists R.A, A \sqsubseteq B, \exists R.B \sqsubseteq D\}$$

and suppose we want to test whether $\mathcal{T} \models A \sqsubseteq D$.

We can do this by running the algorithm on the KB $(\mathcal{T}, \{(A \sqcap \neg D)(a_0)\})$.

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Result: the KB is unsatisfiable $\Rightarrow \mathcal{T} \models A \sqsubseteq D$

Observation: an individual can be blocked, then later become unblocked

Properties of the tableau algorithm

Let's call our new tableau algorithm `KBSat` (for KB satisfiability).

Termination: The algorithm `KBSat` always terminates.

- similar to before: bound the size of generated ABoxes

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So: KBSat is a decision procedure for KB satisfiability.

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The `KBSat` algorithm generates ABoxes of at most exponential size

But even if with the tricks from earlier, need exponential space.

- *single branch may be exponentially long*

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Theorem: $\mathcal{ALC}$ KB satisfiability is EXPTIME-complete.

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This is most likely not optimal, as we can show the following:

Theorem: $\mathcal{ALC}$ KB satisfiability is EXPTIME-complete.

This result means **no polynomial-time algorithm can every be found**.

Optimizations

Despite high worst-case complexity, **tableau algorithms** for $\mathcal{ALC}$ and other expressive DLs **can work well in practice**.

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Absorption (1)

When $\mathcal{T} = \{C_i \sqsubseteq D_i \mid 1 \leq i \leq n\}$, we get n disjunctions per individual:

$$(\mathsf{NNF}(\neg C_1) \sqcup \mathsf{NNF}(D_1))(a), \ldots, (\mathsf{NNF}(\neg C_n) \sqcup \mathsf{NNF}(D_n))(a)$$

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So for inclusions with atomic left-hand side, can replace $\sqsubseteq$-rule by:

$\sqsubseteq^{at}$-rule: if $A(a) \in \mathcal{A}$, a is not blocked, $A \sqsubseteq D \in \mathcal{T}$ (with A atomic), and $D(a) \notin \mathcal{A}$, then replace $\mathcal{A}$ with $\mathcal{A} \cup \{D(a)\}$

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Good news: we've lowered the number of disjunctions!

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Second observation: can transform some inclusions with complex concept on left into equivalent inclusions with atomic left-hand side

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Absorption technique:

1. preprocess the TBox by replacing inclusions with equivalent inclusions with atomic concept on left, whenever possible
2. when running tableau algorithm
   - use new $\sqsubseteq^{at}$-rule for inclusions $A \sqsubseteq D$ with A atomic
   - use regular $\sqsubseteq$-rule for the other TBox inclusions

Example: Absorption

Let's use absorption on the KB $(\mathcal{T}, \{A(a)\})$ from earlier with:

$$\{ \quad A \sqsubseteq \exists R.B \quad B \sqsubseteq D \quad \exists R.D \sqsubseteq \neg A \quad \}$$

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Result: completely avoid disjunction, algorithm terminates much faster!

Optimizations for classification

Classification: find all pairs of atomic concepts A, B with $\mathcal{T} \models A \sqsubseteq B$

Naïve approach: test satisfiability of $A \sqcap \neg B$ w.r.t. $\mathcal{T}$ for all pairs A, B

- *but $\mathcal{T}$ may contain hundreds or thousands of atomic concepts....*

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Some ideas:

- some subsumptions are obvious
  - $A \sqsubseteq A$ and subsumptions that are explicitly stated in $\mathcal{T}$
- can use simple reasoning to obtain new (non-)subsumptions
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models B \sqsubseteq D$, then $\mathcal{T} \models A \sqsubseteq D$
  - if know $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \not\models A \sqsubseteq D$, then $\mathcal{T} \not\models B \sqsubseteq D$

Homework

Exercise 4: KB satisfiability via tableau

Consider the following TBox (from a previous homework):

$$\mathcal{T} = \{A \sqsubseteq \forall R.B, B \sqsubseteq \neg F, E \sqsubseteq G, A \sqsubseteq D \sqcup E, D \sqsubseteq \exists R.F, \exists R.\neg B \sqsubseteq G\}$$

Use the $\mathtt{KBSat}$ algorithm to decide whether:

1. $\mathcal{T} \models A \sqsubseteq E$
2. $\mathcal{T} \models E \sqsubseteq G$
3. $\mathcal{T} \models E \sqsubseteq F$
4. $\mathcal{T} \models A \sqsubseteq G$
5. $\mathcal{T} \models D \sqsubseteq G$
6. $\mathcal{T} \models G \sqsubseteq F$

Please explain your steps. You are encouraged to use the optimizations introduced in the course.