Master of Science in Analytics

MSCA 37016 – Advanced Linear Algebra for Machine Learning

Instructions:

- Mark the question number and your final answer clearly (use a textbox.)
- Remember to show and explain your work (*If you can't explain it, you don't understand it.*)
- Please submit your solution through Canvas.

(6 points) Question 1:

1) 5% - Solve $A\hat{x} = b$ by least squares (i.e., calculate $\hat{x}$).

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix};\ b = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

2) 1% - Verify your answer using Python.

(2 point) Question 2:

$$T(x, y, z) = (z * x + 1, z, y)$$

1) 2% - Is this a linear transformation? Explain your answer.

(7 points) Question 3:

Suppose T is the linear transformation from R^2 to R^2

$$T(x, y) = (2x, 3y)$$

1) 2% - Find the matrix representation of the given linear transformation based on the standard basis.
2) 1% - Is T invertible? Explain your answer.
3) 2% - Find the Kernel of T
4) 2% - Find the Range of T

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Question 1:

① $\vec{b} = \vec{e} + \vec{p} = \vec{e} + A\hat{x}$ $\qquad A\hat{x} = \vec{p}$ $\qquad A^TA\hat{x} = A^Tb$

$\hat{x} = (A^TA)^{-1}A^Tb$ $\qquad \because$ A has linearly independent columns A^TA is invertible

$A^TA\hat{x} = A^Tb$

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\hat{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 2 & 1 \end{array}\right] \quad L_1 - 2L_2 \Rightarrow L_2 \left[\begin{array}{cc|c} 2 & 1 & -1 \\ 0 & -3 & -3 \end{array}\right]$$

$-3y = -3 \Rightarrow y = 1$

$2x + y = -1 \Rightarrow x = -1$

$$\boxed{\hat{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

or $\quad (A^TA)^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \dfrac{1}{4-1}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \dfrac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$\hat{x} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Question 2:

$\boxed{\text{No}}$, this is not a linear transformation because it does not preserve linear combination $T(cv) = cT(v)$.

If $v = (0,0,0)$ and $c = 3$

$$T(v) = (1,0,0)$$
$$T(cv) = (1,0,0)$$
$$cT(v) = (3,00)$$
$$T(cv) \neq cT(v)$$

Question 3

① $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \qquad T(e_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$$\boxed{\begin{aligned} A &= [\, T(e_1) \mid T(e_2) \,] \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \end{aligned}}$$

② T is invertible because the matrix representation of the transformation has linearly independent columns and therefore is invertible.

③ Because A is invertible, the nullspace of A contains only the null vector.

$$\boxed{\ker(T) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}$$

④ Since A is a basis for R^2 (rank = 2)

$$\boxed{\text{Range}(T) = R^2}$$

In [3]:
```
import numpy as np
from scipy import linalg as la
```

In [2]:
```
A=np.array([[1,0],[0,1],[1,1]])
b=np.array([-1,1,0])
```

In [6]:
```
AtA = np.dot(A.transpose(),A)
Atb = np.dot(A.transpose(),b)
AtA_inverse = np.linalg.inv(AtA)
```

In [7]:
```
xhat = np.dot(AtA_inverse,Atb)
print('x by least squares =',xhat)
```

```
x by least squares = [-1.  1.]
```

In [ ]: