Notes of Mathematics Statistics

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1 Probability Theory

1.1 Probability Spaces and Random Elements

Exercise 1.1. $\mathbb{C} = \{(-\infty, t], t \in \mathbb{R}\}$ *, prove* $\sigma(\mathbb{C}) = \mathcal{B}(\mathbb{R})$

Proof:
To prove $\sigma(\mathbb{C}) = \mathcal{B}(\mathbb{R})$, we have to prove

1. $\sigma(\mathbb{C}) \subseteq \mathcal{B}(\mathbb{R})$
2. $\sigma(\mathbb{C}) \supseteq \mathcal{B}(\mathbb{R})$

By definition of Borel σ-field, $\mathcal{B}$ is generated by the open set. Thus, the first thing we have to do before proving 1. and 2. is to argue whether $\{t\} \in \mathcal{B}(\mathbb{R})$.

We construct a open interval $(t - \frac{1}{n}, t + \frac{1}{n})$ $\forall t \in \mathbb{R}$, We have

$$\cap_{n=1}^{\infty}(t - \tfrac{1}{n}, t + \tfrac{1}{n}) = t$$

Thus, $\{t\} \in \mathcal{B}(\mathbb{R})$.

Next, we construct $\sigma(\mathbb{C})$,

Let $A_i = (-\infty, t_i], t_i \in \mathbb{R}, i = 1, 2, ...$ and $B_j = \mathbb{R}\backslash(-\infty, t_j], t_j \in \mathbb{R}, j = 1, 2, ...$
$\sigma(\mathbb{C}) = \{\cup_{i \in I} A_i; i \subseteq I = \{1, 2, ...\}\} \cup \{\cup_{j \in J} B_j; j \subseteq J = \{1, 2, ...\}\}$.

By the definition of σ-field, $\sigma(\mathbb{C})$ should satisfied conditions below:

i $\emptyset \in \sigma(\mathbb{C})$

ii If $C \in \sigma(\mathbb{C})$, then $C^c \in \sigma(\mathbb{C})$

iii If $C_i \in \sigma(\mathbb{C}) \forall i = 1, 2, ...$,then $\cup C_i \in \sigma(\mathbb{C})$

(1) $\sigma(\mathbb{C}) \subseteq \mathcal{B}(\mathbb{R})$

Let $C \in \sigma(\mathbb{C})$, since $\{A_i\} \subseteq \mathbb{R}$ and $\{B_j\} \subseteq \mathbb{R}$, $C \subseteq \mathbb{R}$,$\sigma(\mathbb{C}) \subseteq \mathcal{B}(\mathbb{R})$

(2) $\sigma(\mathbb{C}) \supseteq \mathcal{B}(\mathbb{R})$

Since, $(-\infty, t]^c$ is a open set $\forall t \in \mathbb{R}$, by the definition (ii) of σ-field, if $(-\infty, t] \in \mathcal{F}$ then $(-\infty, t]^c \in \mathcal{F}$.And by the definition (iii) of σ-field, we can show

$$\sigma(\mathbb{C}) \supseteq \mathcal{B}(\mathbb{R})$$

Consider (1) and (2), we have $\sigma(\mathbb{C}) = \mathcal{B}(\mathbb{R})$.

Exercise 1.2. $\mathbb{C} = \{A_1 \times A_2, A_1 \in B(\mathbb{R}); A_2 \in B(\mathbb{R})\}$. *Is* $\mathbb{C}$ *a* σ*-field ?*

Proof:
Assume $\mathbb{C}$ is a σ-field, $\mathbb{C}$ satisfy

i $\emptyset, \Omega \in \mathbb{C}$

ii If $C \in \mathbb{C}$, then $C^c \in \mathbb{C}$

iii If $C_i \in \mathbb{C} \forall i = 1, 2, ...$,then $\cup C_i \in \mathbb{C}$

In other words, if $\mathbb{C}$ cannot satisfy (i),(ii) or (iii), then we reject the assumption.

Let $C_1 = (a_i, a_2] \times (b_1, b_2]$ and $C_2 = (a_2, a_3] \times (b_2, b_3]$, $a_1 < a_2 < a_3$,$b_1 < b_2 < b_3$
$a_i, b_i \in \mathbb{R}$,$\forall i = 1, 2, 3$

If $C_1 \in \mathbb{C}$ and $C_2 \in \mathbb{C}$, then $C_1 \cup C_2 \in \mathbb{C}$ by (iii).
We can rewrite $C_1 \cup C_2 = A \times B$,$A, B \subseteq \mathbb{R}$,$(a_1, a_3] \subseteq A$ and $(b_1, b_3] \subseteq B$.
$\exists\ a' \in A$ where $a_1 < a' < a_2$ and $\exists\ b' \in B$ where $b_2 < b' < b_3$.
$a' \times b\ \in A \times B$, however, $a' \times b\ \notin C_1 \times C_2$. This implies $C_1 \cup C_2 \notin \mathbb{C}$.

Exercise 1.3. $\{A_n\}_{n=1}^{\infty}$: *sequence of event in* $\mathcal{F}$,$A_{n+1} \subset A_n$*,* $\forall n \in \mathbb{N}$
show $v(\cap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} v(A_n)$*.*

Proof:
Suppose $v(A_n)$ is finite (i.e. $v(A_n) \leq \infty$).

$v(A_1 \backslash \cap_{n=1}^{\infty} A_n) = v(A_1) - v(\cap_{n=2}^{\infty} A_n)$.

$A_1 \backslash \cap_{n=1}^{\infty} A_n = A_1 \cap (\cap_{n=2}^{\infty} A_n)^c = A_1 \cap (\cup_{n=2}^{\infty} A_n^c) = \cup_{n=2}^{\infty}(A_1 \cap A_n^c) = \cup_{n=2}^{\infty}(A_1 \backslash A_n)$

Thus, $v(A_1 \backslash \cap_{n=1}^{\infty} A_n) = v(\cup_{n=2}^{\infty}(A_1 \backslash A_n)) = v(A_1) - \lim_{n\to\infty} v(A_n)$.
We have $v(A_1) - v(\cap_{n=2}^{\infty} A_n) = v(A_1) - \lim_{n\to\infty} v(A_n)$, $v(\cap_{n=2}^{\infty} A_n) = \lim_{n\to\infty} v(A_n)$

Exercise 1.4. *Let* $f : (\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$*,* f *is a measurable function*
$f^+(\omega) = \begin{cases} f(x), iff(x) > 0 \\ 0, iff(x) \leq 0 \end{cases}$ *, show* $f^+(\omega)$ *is mble from* $(\Omega, \mathcal{F}) \to (\bar{\mathbb{R}}, \bar{\mathcal{B}})$

This question is about measurable function, the function f is