1

a) $A=(2,3,1)$ $B=(4,3,2)$ $C=(1,1,1)$

$\vec{AB}=(2,0,1)$

$\vec{AC}=(-1,-2,0)$

$$\vec{AC}\times\vec{AB}=\left|\begin{array}{ccc|cc} \hat{i} & \hat{j} & \hat{k} & \hat{i} & \hat{j} \\ -1 & -2 & 0 & -1 & -2 \\ 2 & 0 & 1 & 2 & 0 \end{array}\right|$$

$$= -2\hat{i}+0\hat{j}+0\hat{k}-(-4\hat{k}+0\hat{i}-\hat{j}) = -2\hat{i}+\hat{j}+4\hat{k}$$

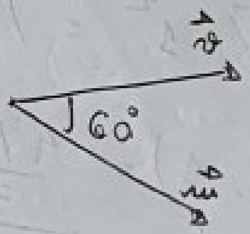
$\Rightarrow \overrightarrow{AC\times AB}=(-2,1,4)$

$\|\overrightarrow{AC\times AB}\|=\sqrt{(-2)^2+1^2+4^2}=\sqrt{4+1+16}=\sqrt{21}$

$A=\dfrac{\|\overrightarrow{AC}\times\overrightarrow{AB}\|}{2}=\dfrac{\sqrt{21}}{2}$

b) $\vec{u}\perp\vec{v}$ $\|\vec{u}\|=2$; $\|\vec{v}\|=4$; $\|\vec{w}\|=3$

$\vec{u}\perp\vec{w}$ $\langle u, v\times w\rangle=?$



$\cos\alpha=\dfrac{\langle u, v\times w\rangle}{\|u\|.\|v\times w\|}$

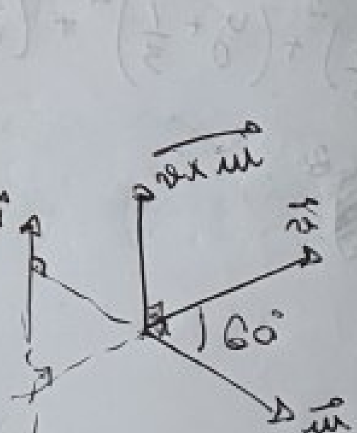
α é o ângulo entre u e $v\times w$, que é $0°$ ou $180°$.

$\langle u, v\times w\rangle=\|v\times w\|.\|u\|.\cos 0°$

Considere $\alpha=0°$.

$\langle u, v\times w\rangle=+2.\|v\times w\|$

Agora, de $\vec{v}$ e $\vec{w}$.



$\text{sen}\,60°=\dfrac{\|v\times w\|}{\|v\|.\|w\|}=\dfrac{\|v\times w\|}{4.3}$

$\Rightarrow \dfrac{\sqrt{3}}{2}=\dfrac{\|v\times w\|}{12} \Rightarrow \|v\times w\|=6\sqrt{3}.$

Assim, $\langle u, v \times w \rangle = 2 \cdot \|v \times w\| = 2 \cdot 6\sqrt{3}$

$$\Rightarrow \boxed{\langle u, v \times w \rangle = 12\sqrt{3}}$$

c) A mesma da prova da T5.

d) $r: \begin{cases} P_r = (1, -2, 2) \\ \vec{v_r} = (2, 3, 1) \end{cases}$ $\quad s: \begin{cases} P_s = (-2, -1, 2) \\ \vec{v_s} = (1, -2, 3) \end{cases}$

$\vec{v_r}$ e $\vec{v_s}$ não são paralelos pois não existe $\lambda \in \mathbb{R}$ tal que $\vec{v_r} = \lambda \vec{v_s}$.

Agora, $\overrightarrow{P_rP_s} = (-3, +1, 0)$

$$\langle \vec{v_r}, \vec{v_s} \times \overrightarrow{P_rP_s} \rangle = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 3 \\ -3 & 1 & 0 \end{vmatrix} \begin{matrix} 2 & 3 \\ 1 & -2 \\ -3 & 1 \end{matrix} = 0 - 27 + 1 - (6 + 6 + 0)$$

$$= -26 - 12 = -38.$$

Como $\langle v_r, v_s \times P_rP_s \rangle = -38 \neq 0$, então, estes vetores não são coplanares, consequentemente, r e s também não são coplanares. Dessa forma, r e s são reversas.

(3) α forma 30° com $x=4$

α forma 60° com plano xy

$\|\vec{\eta}_\alpha\| = 8$ Seja $\vec{\eta}_\alpha = (a,b,c)$

Plano $x=4$, $\vec{\eta}_x = (1,0,0)$

$$\cos 30^\circ = \frac{\langle \eta_\alpha, \eta_x \rangle}{\|\eta_\alpha\| . \|\eta_x\|} = \frac{\langle (a,b,c),(1,0,0)\rangle}{8.\sqrt{1^2+0^2+0^2}} = \frac{a}{8}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{a}{8} \Rightarrow \boxed{a = 4\sqrt{3}}$$

Plano xy, $\vec{\eta}_{xy} = (0,0,1)$

$$\cos 60^\circ = \frac{\langle \eta_\alpha, \eta_{xy} \rangle}{\|\eta_\alpha\| \|\eta_{xy}\|} = \frac{\langle (a,b,c),(0,0,1)\rangle}{8 \cdot \sqrt{0^2+0^2+1^2}} = \frac{c}{8}$$

$$\Rightarrow \frac{1}{2} = \frac{c}{8} \Rightarrow \boxed{c = 4}$$

Daí, $\vec{\eta}_\alpha = (4\sqrt{3}, b, 4)$

$$\|\vec{\eta}_\alpha\| = \sqrt{(4\sqrt{3})^2 + b^2 + 4^2} = \sqrt{16.3 + b^2 + 16} = 8$$

$$\Rightarrow b^2 + 64 = 64 \Rightarrow b = 0.$$

Assim, $\vec{\eta}_\alpha = (4\sqrt{3}, 0, 4)$. Como $A = (2,3,2) \in \alpha$.

Equação Geral de α.

$$4\sqrt{3}(x-2) + 0(y-3) + 4(z-2) = 0 \Rightarrow$$

$$\Rightarrow 4\sqrt{3}x + 4z - 8\sqrt{3} - 8 = 0 \Rightarrow \alpha: 4\sqrt{3}x + 4z = 8 + 8\sqrt{3}$$

③ $S: x^2+y^2+z^2-4x-2y+2z=-2$

$\Rightarrow x^2-4x+4+y^2-2y+1+z^2+2z+1=-2+4+1+1$

$\Rightarrow (x-2)^2+(y-1)^2+(z+1)^2=4$

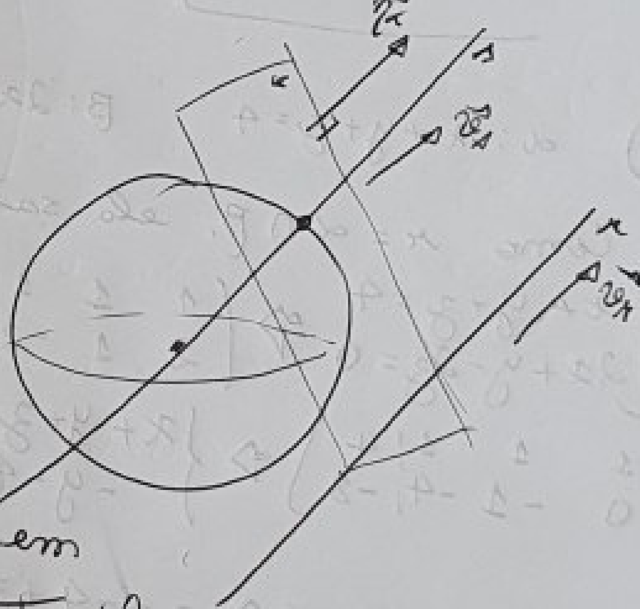
$O=(2,1,-1)\ ;\ r=2$

$$r: \begin{cases} P_r=(1,-2,4) \\ \vec{v_r}=(2,2,1) \end{cases}$$

Seja α o plano que estamos buscando.

Como $\alpha \perp r$, $\vec{n_\alpha} \parallel \vec{v_r}$. Considerando $\lambda=1$, temos: $\vec{n_\alpha}=(2,2,1)$

Considere uma reta s que passe por $O=(2,1,-1)$ e tem direção $\vec{n_\alpha}=(2,2,1)$



$$s: \begin{cases} x=2+2t \\ y=1+2t \\ z=-1+t \end{cases},\ t\in\mathbb{R}.$$

A reta s toca a esfera S em dois pontos, vamos encontrá-los.

Existe $t_0\in\mathbb{R}$, tal que, quando $t=t_0$, o ponto $P_0=(x_0,y_0,z_0)=(2+2t_0,\ 1+2t_0,\ -1+t_0)\in s\cap S$.

Daí, $(x_0-2)^2+(y_0-1)^2+(z_0+1)^2=4$

$\Rightarrow (2+2t_0-2)^2+(1+2t_0-1)^2+(-1+t_0+1)^2=4$

$\Rightarrow (2t_0)^2+(2t_0)^2+t_0^2=4 \Rightarrow 4t_0^2+4t_0^2+t_0^2=4$

$\Rightarrow 9t_0^2=4 \Rightarrow t_0^2=\frac{4}{9} \Rightarrow t_0=\pm\frac{2}{3}$

Utilizando $t_0 = \frac{2}{3}$, temos:

$$P_0 = \left(2+\frac{4}{3}, 1+\frac{4}{3}, -1+\frac{2}{3}\right) = \left(\frac{10}{3}, \frac{7}{3}, -\frac{1}{3}\right)$$

Ou seja, P_0 é o ponto de tangência de α em S.

Equação Geral de α:

$\vec{\eta}_\alpha = (2, 2, 1)$

$$\alpha: 2\left(x-\frac{10}{3}\right) + 2\left(y-\frac{7}{3}\right) + 1\left(z-\frac{1}{3}\right) = 0$$

$$2x+2y+z = \frac{20}{3} + \frac{14}{3} + \frac{1}{3} = \frac{35}{3}$$

$$\boxed{\alpha: 2x+2y+z = \frac{35}{3}}$$

(4) $\alpha: x+y+z=4$ $\quad \beta: 2x+y-2z=0$

a) Como $r = \alpha \cap \beta$, ela satisfaz as equações dos dois planos.

$$\begin{cases} x+y+z=4 \\ 2x+y-2z=0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 1 & -2 & 0 \end{array}\right) L_2 \to L_2 - 2L_1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -1 & -4 & -8 \end{array}\right) \Rightarrow \begin{cases} x+y+z=4 \quad \text{(I)} \\ -y-4z=-8 \end{cases} \Rightarrow -y = -8+4z$$

$$\underline{y = 8-4z}$$

(I) $x+8-4z+z=4 \Rightarrow \underline{x=-4+3z}$

$$r: \begin{cases} x=-4+3z \\ y=8-4z \end{cases} \xrightarrow{z=t} r: \begin{cases} x=-4+3t \\ y=8-4t, \ t \in \mathbb{R} \\ z=t \end{cases}$$

b) Quando $t=1$, $O=(-1,4,1)$.

Como a esfera S é tangente à reta s, ao calcularmos a distância do centro de S até s teremos o raio da esfera (Fórmula dada na prova).

$P_s=(2,0,2)$

$\vec{v_s}=(1,1,1)$

$$d(O,s)=\frac{\|\overrightarrow{OP_s}\times\vec{v_s}\|}{\|\vec{v_s}\|}$$

$\overrightarrow{OP_s}=(3,-4,1)$

$$\overrightarrow{OP_s}\times\vec{v_s}=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\3&-4&1\\1&1&1\end{vmatrix}\begin{matrix}\vec{i}&\vec{j}\\3&-4\\1&1\end{matrix}=-4\vec{i}+\vec{j}+3\vec{k}-(-4\vec{k}+\vec{i}+3\vec{j})$$

$$=-5\vec{i}-2\vec{j}+7\vec{k}=(-5,-2,7)$$

$\|\overrightarrow{OP_s\times v_s}\|=\sqrt{(-5)^2+(-2)^2+7^2}=\sqrt{25+4+49}=\sqrt{78}$

$\|\vec{v_s}\|=\sqrt{1^2+1^2+1^2}=\sqrt{3}$

Daí, $d(O,s)=\dfrac{\sqrt{78}}{\sqrt{3}}=\sqrt{\dfrac{78}{3}}=\sqrt{26}$. Logo, $r=\sqrt{26}$.

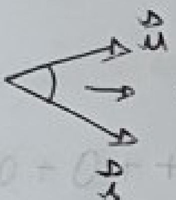
Equação da Esfera:

$$\boxed{S:(x+1)^2+(y-4)^2+(z-1)^2=26.}$$

Dessa forma.

$$\langle u, v\times w\rangle = 3\cdot 5\cdot 1 = 15 \Rightarrow \langle u, v\times w\rangle = 15$$

1c) $\vec{u} = (k, 2, 1)$ $\quad \vec{v} = (1, 1, -2)$



agudo (menor do que $90°$).

$$\langle \vec{u}, \vec{i}\rangle = \langle (k,2,1), (1,0,0)\rangle = k > 0.$$

$$\vec{u\times v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ k & 2 & 1 \\ 1 & 1 & -2 \end{vmatrix}\begin{matrix} \hat{i} & \hat{j} \\ k & 2 \\ 1 & 1 \end{matrix} = -2\hat{i} + \hat{j} + k\cdot\hat{k} - (2\hat{k} + \hat{i} - 2k\hat{j})$$

$$= -3\hat{i} + (1+2k)\hat{j} + (k-2)\hat{k}$$

$$\Rightarrow \vec{u\times v} = (-3, 1+2k, k-2)$$

$$\|\vec{u\times v}\| = \sqrt{(-3)^2 + (1+2k)^2 + (k-2)^2} = \sqrt{9+1+4k+4k^2+k^2-4k+4}$$

$$= \sqrt{5k^2+14}.$$

Agora, $\dfrac{\|\vec{u\times v}\|}{2} = \sqrt{57} \Rightarrow \dfrac{\sqrt{5k^2+14}}{2} = \sqrt{57}$

$$\Rightarrow \frac{5k^2+14}{4} = 57 \Rightarrow 5k^2 + 14 = 228 \Rightarrow 5k^2 = 214$$

$\Rightarrow k^2 = \dfrac{214}{5} \Rightarrow k = \pm\sqrt{\dfrac{214}{5}}$. Como $k > 0$, então

$$\boxed{k = +\sqrt{\frac{214}{5}}}$$