

# All-pay auctions with interdependent valuations: The highly competitive case<sup>1</sup>

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## **Abstract**

We analyze symmetric, two-player all-pay auctions with interdependent valuations and general discrete signal structures. We extend the previous literature by being able to analyze auctions in which an increase in a bidder's posterior expected value for winning the auction is likely to be accompanied by a corresponding increase for the other bidder. Such environments are “highly competitive” in the sense that the bidder's higher valuation also signals that the other bidder has an incentive to bid aggressively. We present a construction which computes all symmetric equilibria, and show how, in highly competitive environments, the search problem this construction faces can be complex. In equilibrium, randomization can take place over disjoint ranges of bids, with equilibrium supports having a potentially rich structure.

**JEL Classifications:** D44; D82; D72.

**Keywords:** contests, all-pay auctions, mixed strategies.

# 1 Introduction

Consider an emerging industry, in which potential entrants are considering irrevocable research and development investments to develop a new product. It is believed the entrant who develops the best product will dominate. Suppose an entrant privately observes information which suggests that demand in the industry will be high, so the successful firm will obtain high profits. Higher profits are good news for the entrant – provided it is the entrant who will obtain them! If other entrants are likely to observe similar information, then the entrant will expect to face fierce competition in the research and development stage; this is bad news. Because high signals are both good and bad news, determining what equilibrium behavior will look like is a potentially difficult problem.

Much existing research in all-pay auction models has ruled out this case by restricting the signal structure such that “higher” signals are unambiguously good news. One approach is to assume that signals are statistically independent; see e.g. Amann and Leininger (1996). In a setting with affiliated values and continuous strategy and signal spaces Krishna and Morgan (1997) adopt an assumption which rules out the case that signals are “too affiliated.” The result is that bidding strategies and equilibrium payoffs are monotonically increasing in the private signal. More recently, Siegel (2014) analyzes a setting with finite sets of signals, in which valuations need not be affiliated. Using a discrete analogue of the assumption of Krishna and Morgan, he shows that equilibrium bidding strategies are monotonic in a stochastic sense, and provides a constructive algorithm for finding equilibrium.

Siegel notes that “it would be valuable to extend the analysis to ... non-monotonic equilibria.” A main contribution of this paper is to provide exactly that extension. This exercise is valuable because applications of the all-pay contest model may involve information structures that do not satisfy the assumptions required to ensure monotonic equilibria. If, for example, the entrants in the emerging industry base their judgments in part on public information, and/or use similar methodologies, their posterior assessments of the value of winning are likely to be highly correlated. As a technical contribution, by relaxing monotonicity assumptions, we can better understand what they imply in terms of the simplicity of both the structure of equilibria and the problem of identifying equilibrium strategies.

In this paper we consider the symmetric, two-bidder all-pay auction in an environment with a finite set of signals, and a continuous strategy space. Each bidder privately observes a signal, and then they simultaneously choose their bids. The highest bid wins the prize, and they both pay their bids. The signals may have an arbitrary correlation structure, and the posterior valuations of the prize by the bidders can depend in an arbitrary way on the signals; we make only a technical assumption that both bidders can receive the same signal with positive probability.

We establish the existence of symmetric equilibria, and provide a construction which finds all

families of symmetric equilibria. The construction provided by Siegel (2014) is a special case in our environment. The algorithm identifies which subsets of signals can simultaneously include the same bid in the supports of their mixed strategies, which we call admissible sets. Whether a subset of signals is admissible is independent of the bid level, but the order in which admissible sets can appear in equilibrium is constrained by equilibrium conditions. These equilibrium considerations induce a directed graph over admissible sets. When the Krishna-Morgan-Siegel monotonicity condition does not hold, this graph can become complex, and therefore the number of paths the construction must consider can grow rapidly.

Because of the restrictions on the structure of information and values in previous analysis of all-pay auctions, little is known about some settings which have received considerable attention in the winner-pay auction literature. One prominent example is a pure common-values environment in which each signal is the common value plus independent uniform noise. Athey (2001) has noted that the single crossing property fails in all-pay auctions with this information structure. Nevertheless, it has been used extensively in the experimental literature on the winner's curse; see e.g. Casari et al. (2007). We apply our construction to this setting, and show that the supports of equilibrium strategies are not connected intervals for all signals; there are gaps in the supports. We also apply our construction to other settings drawn from the literature.

This paper forms part of the burgeoning literature on all-pay auctions. In a complete information environment, Baye et al. (1996) characterizes the set of equilibria. Siegel (2009) and Siegel (2010) have recently greatly enhanced the understanding in complete information environments by allowing for a variety of asymmetries among bidders. In addition, Tullock contests, in which the highest bidder does not win with certainty, are the subject of a quickly growing literature; see e.g. Tullock (1980) and Cornes and Hartley (2005).

Introducing incomplete information into both all-pay auctions and Tullock contests substantially complicates analysis. The literature regarding incomplete information predominantly focuses on all-pay auctions.<sup>1</sup> Much of the analysis focuses on the case of independent signals. Morath and Münster (2008) demonstrate that all-pay auctions generate lower expected revenue if the vector of valuations is common knowledge relative to the case in which each bidder's valuation is private information. Amann and Leininger (1996) examine the two player case under independent private values in which valuations need not be drawn from the same distribution. Kirkegaard (2007) considers comparative statics in the case of ex-ante heterogeneous bidders and finds that total effort expenditure may increase when one bidder gets weaker. Kirkegaard (2010) also considers such ex-ante asymmetry and also allows for the presence of handicapped bidders, as well as advantaged bidders whose bids are costlessly increased by an additive bonus. In all the environments

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<sup>1</sup>There are some notable exceptions. For example, the case of common value Tullock contests with asymmetric information has been studied in Wärneryd (2003) and Wärneryd (2008).

considered, equilibrium is monotonic.

When signals are not independent, in general a monotonicity assumption is imposed on the signal structure which guarantees that equilibria will be monotonic. In addition to the already-mentioned analyses of Krishna and Morgan (1997) and Siegel (2014), Harstad (1995) and Lizzeri and Persico (2000) both adopt this strategy. An important exception is Lu and Parreiras (2014), who extend Amann and Leininger (1996) by allowing for correlated signals and interdependent valuations in an environment with continuous type and strategy spaces. They provide a necessary and sufficient condition for the existence of monotonic equilibria. They also compute a non-monotonic equilibrium in a setting with quadratic valuations.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 proves existence of an equilibrium, and presents an algorithm that constructs all families of equilibria. Section 4 analyzes some selected cases which illustrate what equilibria can look like, and illustrates that the challenge of finding equilibria can increase rapidly outside the set of environments satisfying the Krishna-Morgan-Siegel assumption. Section 5 concludes with a discussion. For conciseness in the exposition, we relegate proofs to appendices. An implementation of the algorithm in Python is available on request from the authors.

## 2 Model

We consider symmetric, two-bidder all-pay auctions with a single, indivisible prize. Each bidder receives a private signal regarding her value of the prize from a finite set  $S$ , with typical element  $s_k$ . After receiving their signals, they simultaneously submit bids, which can be any nonnegative real number. Each bid is irrevocably sunk, but only the bidder with the higher of the bids wins the prize. In the event of a tie the winner is determined by a fair randomization, although in equilibrium ties are a zero-probability event.

A bidder's valuation of the prize may depend on both her and her opponent's signal in any arbitrary way. Let  $V_{k,l} > 0$  denote the posterior expected value of the prize to a bidder, conditional on receiving signal  $s_k$  while the other bidder receives  $s_l$ . Given that a bidder receives signal  $s_k$ , write the conditional probability that the other bidder receives signal  $s_l$  as  $h_{l|k}$ .

The values  $V$  and conditional probabilities  $h$  are sufficient to define the payoff functions for the model. In several examples, it will be convenient, for the purpose of exposition, to motivate or calculate  $h$  and  $V$  in terms of an underlying state of the world. When this is employed,  $\Omega$  will denote the set of states of the world, with typical elements written using lowercase  $\omega$  with subscripts.

We consider symmetric Bayes-Nash equilibria of this game. A behavior strategy  $\Pi$  assigns to each signal  $s_k$  a corresponding probability distribution  $\Pi_k$  over bids. For any Lebesgue-measurable

subset  $B$  of bids,  $\Pi_k : B \rightarrow \mathbb{R}_+$  denotes the probability that the bidder will choose a bid in  $B$ . We can then write the expected payoff to a bidder of a bid  $b$ , conditional on receiving a signal  $s_k$ , assuming the opponent bids according to  $\Pi$ , as

$$u_k(b|\Pi) = \sum_{s_l \in S} h_{l|k} V_{k,l} [\Pi_l([0, b)) + \Pi_l(\{b\})] - b. \quad (1)$$

For notational compactness, we define the quantity  $\psi_{l|k} \equiv h_{l|k} V_{k,l}$ . In the case where a behavior strategy  $\Pi$  has a density for signal  $s_k$  at bid  $b$ , we will write  $\pi_k(b)$ .

Siegel (2014) provides a procedure for computing equilibria in two-bidder all-pay auctions (symmetric or asymmetric), under the assumption that for every  $s_l \in S$ ,  $\psi_{l|k}$  is strictly increasing in  $s_k$ . This assumption is a discrete analogue of the sufficient condition for existence of symmetric and monotonic equilibrium identified in Krishna and Morgan (1997). We will refer to this assumption as the KMS (monotonicity) assumption. When the KMS condition holds, the unique equilibrium is (stochastically) monotonic. That is, for any two signals  $s_k > s_l$ , if bid  $b$  is played with positive probability in equilibrium by type  $s_l$  and bid  $b'$  is played with positive probability in equilibrium by type  $s_k$ , then  $b' \geq b$ . Siegel (2014) shows that the equilibrium involves piecewise uniform randomization (with possibly a mass point); for every signal  $s_k$ , the support of the mixed strategy is a connected interval.

### 3 Constructing an equilibrium

In this setting, a symmetric equilibrium always exists.

**Theorem 1.** *Suppose that  $h_{k|k} > 0$  for all  $s_k \in S$ . There exists a symmetric equilibrium, in which there are no atoms in the distribution of bids for any signal.*

*Proof.* See Appendix A. □

Neither the KMS monotonicity assumption nor affiliation, is required for existence.<sup>2</sup> However, when the KMS assumption does not hold, the structure of equilibria can become quite complex. In what follows, we develop an algorithm for constructing all equilibria of the game. This algorithm can be viewed as an extension of the construction presented by Siegel (2014).

Given an equilibrium  $\Pi$ , define  $\theta_k(\Pi)$  to be the expected payoff of a bidder with signal  $s_k$ . Because there are no atoms in the distribution of bids,  $u_k(b|\Pi)$  is a continuous function in  $b$ , and the set of best responses  $\{b : u_k(b|\Pi) = \theta_k(\Pi)\}$  is a closed set, which can be written as the union

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<sup>2</sup>The fact that equilibria are without atoms is a consequence both of having only two bidders, and of symmetry. Baye et al. (1996) and Siegel (2014) exhibit equilibria with atoms in cases with asymmetries between bidders, and with more than two bidders.

of a collection of disjoint closed intervals. Let the set of endpoints of those closed intervals be  $E_k(\Pi)$ , and write  $E(\Pi) = \cup_k E_k(\Pi)$ . These endpoints can be ordered as  $e_0 > e_1 > \dots > e_j > \dots > e_J = 0$ .<sup>3</sup> Note that higher bids have lower indices, as our algorithm, like that of Siegel (2014), starts at higher bids and works downwards.

Let  $\mathcal{I}_j$  denote the open interval  $(e_{j+1}, e_j)$ . By construction, on any interval  $\mathcal{I}_j$  we can partition the signals  $S$  into  $A_j = \{s_k \in S : u_k(b|\Pi) = \theta_k(\Pi) \ \forall b \in \mathcal{I}_j\}$ , and  $A_j^c = \{s_k \in S : u_k(b|\Pi) < \theta_k(\Pi) \ \forall b \in \mathcal{I}_j\}$ . We refer to  $A_j$  as the **active set** associated with  $\mathcal{I}_j$ , in the sense that the constraint  $u_k(b|\Pi) \leq \theta_k(\Pi)$  required by equilibrium is binding or active for signals in  $A_j$ .

In any equilibrium, a necessary condition for a subset  $A \subseteq S$  to be an active set associated with some interval is for the system

$$\sum_{s_l \in S} \psi_{l|k} \pi_l(b) = 1 \quad \forall s_k \in A, \forall b \in \mathcal{I} \quad (2)$$

to have a solution  $\pi_k(b) \geq 0$  for all  $s_k \in A$ , where  $\pi_k(b) = 0$  for all  $s_k \in A^c$ . If this condition is satisfied, we say that  $A$  is an **admissible active set**, and that such a solution **supports** the admissible active set.

Whether an active set  $A$  is admissible does not depend on the location or size of  $\mathcal{I}$ . The bid  $b$  enters into equations (2) only as a parameter on the densities. Suppressing for the moment the dependence of  $\pi$  on  $b$ , the set of densities which support an admissible active set is determined by a linear system of equations whose solutions are computationally straightforward to determine. If the solution is unique, then there is one supporting  $\pi(b)$  for the active set, which consists of a constant density. If the set of solutions has positive dimension, then any function  $\pi(b)$  satisfying the equations (2) for all  $b$  supports the admissible active set.

The dimension of solutions of (2) is  $|A| - r$ , where  $r$  is the rank of the linear system. In the case of a positive-dimensional set of solutions, we can select  $|A| - r$  signals from  $A$ , and express the densities of the remaining  $r$  signals as linear functions of the densities of those  $|A| - r$  signals. Furthermore, this implies that the total probability mass of the  $r$  nonbasic signals over the interval can be written as linear functions of the total probability mass of the  $|A| - r$  basic signals.

Therefore, the total probability mass for each signal associated with an admissible active set  $A$  can be summarized by  $|A| - r + 1$  variables: one variable for the length of the associated interval  $\mathcal{I}$ , and  $|A| - r$  variables which select among supporting solutions for the active set. We write  $M(A)$  to denote the set of solutions for probability masses expended by all signals corresponding to an active set  $A$ .

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<sup>3</sup>A bid of zero must always be in the support of at least one signal. To see this, observe that there must always be some lowest bid submitted with positive probability. Because there are no atoms, the lowest bid wins with probability zero. If the lowest bid is strictly positive, this earns a negative expected payoff, whereas a bid of zero guarantees zero payoff.

The algorithm below will compute equilibria using  $M(A)$  by interval. In the case where (2) admits multiple solutions, randomization over the interval by a signal need not be uniform. However, solutions for which  $\pi_k(b)$  is constant for all  $b$  on the interval always exist. We therefore immediately have this result:

**Corollary 2.** *There exists a symmetric equilibrium, in which there are no atoms in the distribution of bids for any signal, and randomization is piecewise uniform for all signals.*

Assuming we always take a uniform solution for each interval, the derivative of the payoff function  $u'(b|\Pi)$  is well-defined and constant on each interval for all signals.

The admissibility of an active set does not place any constraints on the length of the associated interval, or the location in the space of bids at which the associated interval could appear in equilibrium. This is because whether an active set  $A$  is admissible does not depend on the behavior of  $u_k(b|\Pi)$  for signals  $s_k \notin A$ . However, the order in which active sets can appear in the sequence  $\{A_j\}$  induced by  $\Pi$  is constrained by the fact that  $\Pi$  is an equilibrium.

Consider two intervals  $\mathcal{I}_j$  and  $\mathcal{I}_{j+1}$ . Suppose that  $s_k \in \mathcal{I}_j$  but  $s_k \notin \mathcal{I}_{j+1}$ ; that is,  $s_k$  “drops out” in passing from  $\mathcal{I}_j$  to  $\mathcal{I}_{j+1}$ . In order for this to occur in an equilibrium, it must be that the payoff  $s_k$  receives from bids in  $\mathcal{I}_{j+1}$  is less than the payoff it receives from bids in  $\mathcal{I}_j$ . Therefore, it must be that  $u'_k > 0$  on  $\mathcal{I}_{j+1}$ .<sup>4</sup>

Similarly, suppose that  $s_k \notin \mathcal{I}_j$  but  $s_k \in \mathcal{I}_{j+1}$ ; that is,  $s_k$  “enters” in passing from  $\mathcal{I}_j$  to  $\mathcal{I}_{j+1}$ . In order for this to be consistent with equilibrium, it must be that the payoff  $s_k$  receives from bids in  $\mathcal{I}_j$  is less than the payoff it receives from bids in  $\mathcal{I}_{j+1}$ . Therefore, it must be that  $u'_k < 0$  on  $\mathcal{I}_j$ .

We can capture these observations by defining a partial ordering  $\succ$  over admissible active sets.

**Definition 3.** *We say that an admissible active set  $A'$  may succeed  $A$  in an equilibrium,  $A \succ A'$ , if and only if, for all  $s_k$ ,*

- *If  $s_k \in A$  and  $s_k \notin A'$ , then  $u'_k > 0$  on  $A'$ .*
- *If  $s_k \notin A$  and  $s_k \in A'$ , then  $u'_k < 0$  on  $A$ .*

The partial ordering  $\succ$  induces a directed graph on the admissible active sets. Each admissible active set is a node in the graph, and there is a directed edge from  $A$  to  $A'$  if and only if  $A \succ A'$ . Any symmetric equilibrium divides the bid space into a collection of intervals  $\{\mathcal{I}_j\}$  with corresponding admissible active sets  $\{A_j\}$ , which corresponds to a path through the graph. However, paths through the graph of  $\succ$  need not correspond to an equilibrium. Determining whether a given candidate path corresponds to an equilibrium can be done by solving a system of linear equations.

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<sup>4</sup>The strict inequality follows because admissible active sets permit the edge case of a signal  $s_k$  having  $\pi_k(b) = 0$  as the solution on the interval.



Let  $\{A_j\}$  be the list of admissible active sets encountered (in order) on some path through the graph of  $\succ$ . We wish to construct an equilibrium  $\Pi$  which induces this sequence of active sets, or to verify no such corresponding equilibrium exists. An equilibrium  $\Pi$  must satisfy the following sets of conditions. First, the total probability mass expended by each signal must be one:

$$\sum_{j: s_k \in A_j} \mu_{jk} = 1 \quad \forall s_k \in S \quad (3)$$

where  $\mu_{jk}$  is the total probability mass expended on  $\mathcal{I}_j$  by a bidder with signal  $s_k$ . Second, the expected payoff to all bids in the support of a given signals behavior strategy must be the same. Fix a signal  $s_k$ , and suppose that  $s_k \in A_j$  and  $s_k \in A_{j+n}$  for some  $n > 2$ , but  $s_k \notin A_{j'}$  for any  $j' \in \{A_{j+1}, \dots, A_{j+n-1}\}$ . The net payoff change for a bidder with signal  $s_k$  over the bids  $\cup_{j'=j+1}^{j'+n-1} \mathcal{I}_{j'}$  must be zero:

$$\sum_{j'=j+1}^{j'+n-1} \sum_{s_l \in S} \psi_{l|k} \mu_{j'l} = 0. \quad (4)$$

Finally, for each active set  $\{A_j\}$  corresponding to  $\Pi$  we require that the masses expended  $\mu_{jk}$  must be in the set  $M(A_j)$ , which is defined as above by a linear set of equations.

Therefore, the feasibility problem of whether a particular candidate path  $\{A_j\}$  is consistent with an equilibrium reduces to a system of linear equations. In addition, note that the solutions for any given admissible active set  $A$  can be pre-computed once, as they do not depend on the bid under consideration. Except in degenerate cases, these will be parameterized only by the length of the corresponding interval. Therefore, the feasibility problem given a path  $\{A_j\}$  is computationally easy.

As examples below will illustrate, the value of the KMS monotonicity assumption is that it uniquely pins down the path  $\{A_j\}$  induced by equilibrium. In games which violate the KMS assumption, the graph of  $\succ$  can become complex, with many candidate paths to consider.

## 4 Examples

### 4.1 Example 1: Common-values with highly correlated signals

Common-value auctions with conditionally independent signals have received considerable attention in the experimental literature on the winner's curse, for example, in Casari et al. (2007). In the typical setup, the common value of the good is drawn from a uniform distribution. Each bidder receives a signal which is the common value plus a conditionally independent noise term, drawn from a uniform distribution with mean zero. Wang (1991) theoretically analyzed the discrete ana-

logue of this model for first-price auctions and shows that in the symmetric equilibrium types mix on continuous and non-overlapping intervals. We will now show in the all-pay auction case, the continuous intervals on which types randomize overlap.

There are  $K$  possible states of the world,  $\Omega = \{\omega_1, \dots, \omega_K\}$ , with  $\omega_k = \frac{k}{K}$ . These correspond to the common value of the prize. There are  $K + 1$  possible signals,  $S = \{s_0, s_1, \dots, s_K\}$ . Conditional on state  $\omega_k$  being realized, each bidder independently receives either signal  $s_{k-1}$  or  $s_k$ , with equal probability. We will refer to signals  $\{s_1, \dots, s_{K-1}\}$  as *interior signals*. For interior signals, the conditional probability structure satisfies  $h_{k-1|k} = h_{k+1|k} = \frac{1}{4}$  and  $h_{k|k} = \frac{1}{2}$ . For the end cases, we have  $h_{1|0} = h_{0|0} = h_{K-1|K} = h_{K|K} = h_{K|K} = \frac{1}{2}$ . All other conditional probabilities are zero.

Turning to conditional expected values, it is useful to define  $\Delta = \frac{1}{2K}$ . Then, for any pair of interior signals  $s_{k-1}$  and  $s_k$ , we have  $V_{k,k} = \frac{k+\frac{1}{2}}{K} = (2k+1)\Delta$  and  $V_{k-1,k} = \frac{k}{K} = 2k\Delta$ . Changing either signal from  $s_{k-1}$  to  $s_k$  increases the posterior expected value by  $\Delta$ . The end case signals  $s_0$  and  $s_K$  are special because each reveals the value with certainty, and so  $V_{0,0} = V_{0,1} = \frac{1}{K} = 2\Delta$  and  $V_{K-1,K} = V_{K,K} = 1 = 2K\Delta$ .

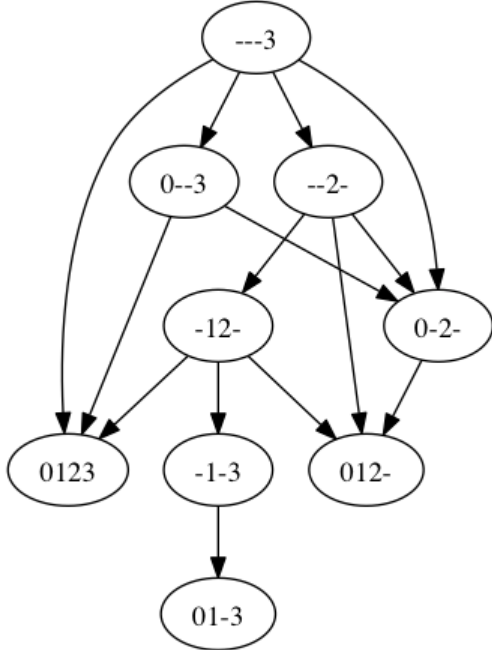
We first present the calculation of the equilibria of this game for the case of  $K = 3$  as an illustration of the operation of the constructive procedure. There are 15 possible active sets; of these, 12 are admissible. The graph induced by  $\succ$  over these admissible active sets is shown in Figure 1a. Table 1 lists the admissible active sets and the corresponding required densities. It also tabulates the slopes of the payoff function for signals not active on a given interval; the signs of these determine the edges in the graph of Figure 1a. Of note is the case when all signals are active. Here, the required indifference conditions for any three signals imply the fourth. Therefore, there are multiple solutions on this interval.

In this game, there is one family of equilibria, corresponding to one path through this graph. This path is summarized in Table 2. We now show how to characterize the family of equilibrium behavior strategies corresponding to this path of active sets. Because of the degeneracy in solutions for the active set  $\{s_0, s_1, s_2, s_3\}$ , the mass expended on interval  $\mathcal{I}_3$  is characterized by two free parameters, the length  $L_3$  of the interval and the mass  $\mu_{30}$  expended by signal  $s_0$ . The mass expended by the other signals can be expressed as linear expressions of these two quantities; for example,  $\mu_{33} = 5L_0 - \mu_{30}$ .

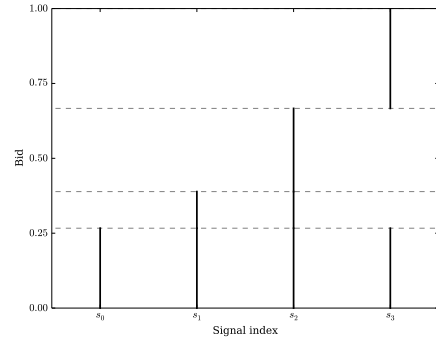
There are then five unknowns to be computed: the four interval lengths  $L_0, L_1, L_2$ , and  $L_3$ , and the mass  $\mu_{30}$  expended on interval  $L_3$  by signal  $s_0$ . The integrate-to-one conditions for each signal

Active	Densities				$u'$			
	$s_0$	$s_1$	$s_2$	$s_3$	$s_0$	$s_1$	$s_2$	$s_3$
$s_3$	0	0	0	2	-1	-1	$-\frac{1}{2}$	
$s_2$	0	0	$\frac{12}{5}$	0	-1	$-\frac{3}{5}$		$+\frac{1}{5}$
$s_1$	0	4	0	0	$-\frac{1}{3}$		$-\frac{1}{3}$	-1
$s_1 s_3$	0	4	0	2	$-\frac{1}{3}$		$+\frac{1}{6}$	
$s_1 s_2$	0	$\frac{36}{11}$	$\frac{12}{11}$	0	$-\frac{5}{11}$			$-\frac{5}{11}$
$s_0$	6	0	0	0		$-\frac{1}{2}$	-1	-1
$s_0 s_3$	6	0	0	2		$-\frac{1}{2}$	$-\frac{1}{2}$	
$s_0 s_2$	6	0	$\frac{12}{5}$	0		$-\frac{1}{10}$		$+\frac{1}{5}$
$s_0 s_1$	3	3	0	0			$-\frac{1}{2}$	-1
$s_0 s_1 s_3$	3	3	0	2		0		
$s_0 s_1 s_2$	5	1	2	0			0	
$s_0 s_1 s_2 s_3$	$p$	$6 - p$	$p - 3$	$5 - p$				

Table 1: Summary of density solutions for admissible active sets in the common-values with highly correlated signals model with  $K = 3$ . There is a one-dimensional family of solutions for the active set with all signals, parameterized here by  $p \in [3, 5]$ . The columns  $u'$  list the slopes of the payoff functions for signals not in the corresponding active set.



(a) Admissible active set paths



(b) Schematic of equilibrium support. Bars indicate intervals of bids on which each signal places positive mass. Horizontal dashed indicates bids at which the admissible active set changes.

Figure 1: Visualization of admissible active sets and equilibrium in the common-values with highly correlated signals model with  $K = 3$ .

	Active	Densities				$u'$			
		$s_0$	$s_1$	$s_2$	$s_3$	$s_0$	$s_1$	$s_2$	$s_3$
$\mathcal{I}_0$	$s_3$	0	0	0	2	-1	-1	$-\frac{1}{2}$	
$\mathcal{I}_1$	$s_2$	0	0	$\frac{12}{5}$	0	-1	$-\frac{3}{5}$		$+\frac{1}{5}$
$\mathcal{I}_2$	$s_1 s_2$	0	$\frac{36}{11}$	$\frac{12}{11}$	0	$-\frac{5}{11}$			$-\frac{5}{11}$
$\mathcal{I}_3$	$s_0 s_1 s_2 s_3$	$p$	$6 - p$	$p - 3$	$5 - p$				

Table 2: Sequence of active sets corresponding to equilibrium in the common-values with highly correlated signals model with  $K = 3$ .

are, respectively,

$$\begin{aligned}
\mu_{30} &= 1 \\
\frac{36}{11}L_2 + 6L_3 - \mu_{30} &= 1 \\
\frac{12}{5}L_1 + \frac{12}{11}L_2 - 3L_3 + \mu_{30} &= 1 \\
2L_0 + 5L_3 - \mu_{30} &= 1.
\end{aligned}$$

In addition, signal  $s_3$  must be indifferent across its two disjoint regions of activity:

$$\frac{1}{5}L_1 + \frac{5}{11}L_2 = 0.$$

These equations have a unique solution, given by  $\mu_{30} = 1$ ,  $L_0 = \frac{1}{3}$ ,  $L_1 = \frac{5}{18}$ ,  $L_2 = \frac{11}{90}$ ,  $L_3 = \frac{4}{15}$ . Because all signals are active at a bid of zero, it follows immediately that the equilibrium payoff for each signal is zero. Figure 1b illustrates the support of the equilibrium. For each signal, the bar indicates regions of bids on which that signal is active; the signal with the highest posterior estimation of the value,  $s_3$ , is active only on the highest and the lowest interval of bids, but not at intermediate bids.

The support shown in Figure 1b is unique, but the densities on the lowest interval of bids are not uniquely determined. Any density function  $\pi_{30}(b)$  on  $[0, \frac{4}{15}]$  subject to the constraints that  $3 \leq \pi_{30}(b) \leq 5$  for all  $b$  and  $\int_0^{\frac{4}{15}} \pi_{30}(b)db = 1$  results in an equilibrium. That is, the order of active sets and the lengths of the corresponding intervals are uniquely determined, and behavior on all but the lowest interval is also uniquely determined. However, while the total probability mass on the lowest interval is the same for all equilibria, there are infinitely many ways in which that total probability mass can be distributed on  $\mathcal{I}_3$ .

The number of paths through the graph of  $\succ$  to be considered grows rapidly. For  $K = 8$ , for

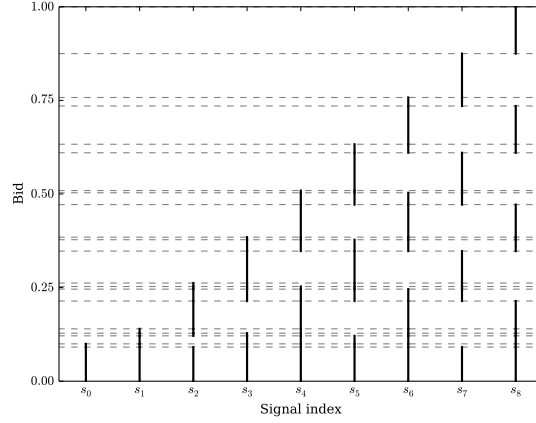


Figure 2: Schematic of equilibrium support in the common-values with highly correlated signals model with  $K = 8$ . Bars indicate intervals of bids on which each signal places positive mass. Horizontal dashed lines indicate bids at which the admissible active set changes.

example, there are 909,238 possible paths through the graph; the graph is too large to be visualized in a figure in any practical way. Nevertheless, main qualitative results from the  $K = 3$  can be established to hold more generally. Figure 2 displays the unique equilibrium support for the case of  $K = 8$ . The presence of gaps in the equilibrium support for most signals, the diagonal “banding” of the component intervals in the supports from top right to bottom left, and the non-uniqueness of equilibrium behavior in the lowest interval, all remain.

We now formalize some properties of the equilibrium structure which are independent of  $K$ . First note that if  $\pi_k = 0$  on some interval, then the strategic calculations for signals below  $s_k$  and those above  $s_k$  can be considered in isolation. We define an **isolated group** of signals as a set of signals  $\{s_k, \dots, s_{k'}\}$  which are all active for  $k' > k$ , but where  $s_{k-1}$  and  $s_{k'+1}$  are inactive.

**Proposition 4.** *The isolated groups in any admissible active set fall into these categories: (i) singleton signals; (ii) even-parity groups of adjacent interior signals; (iii) all signals from  $s_0$  to  $s_k$  for any  $0 \leq k \leq K$ .*

*Proof.* See Appendix B. □

The fact that odd-parity isolated groups of interior signals cannot be part of an admissible active set implies that there will always be gaps in equilibrium supports for any  $K$ . If in moving from  $\mathcal{I}_j$  to  $\mathcal{I}_{j+1}$  signal  $s_k$  goes from active to inactive, then it must be that both  $s_{k-1}$  and  $s_{k+1}$  are active in  $\mathcal{I}_{j+1}$ .

**Proposition 5.** *If a signal  $s_k \notin A$  and  $u'_k > 0$  on  $A$ , then  $s_{k-1} \in A$  and  $s_{k+1} \in A$ .*

*Proof.* See Appendix B. □

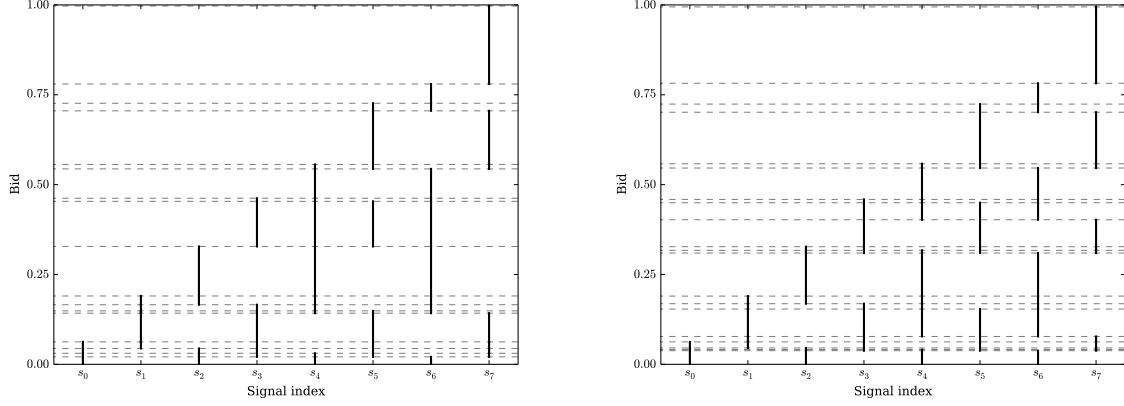


Figure 3: Schematics of the two equilibrium supports in the correlated private-values model when  $K = 8$ . Bars indicate intervals of bids on which each signal places positive mass. Horizontal dashed indicates bids at which the admissible active set changes.

Finally, for any  $K$ , the set of all signals is always an admissible active set, and there are always many solutions for the randomization on a corresponding interval.

**Proposition 6.** *The active set consisting of all signals is always admissible, and is always supported by a multiplicity of solutions. The set of admissible densities converges to a point as  $K \rightarrow \infty$ , with  $\pi_0 = \ln 2$ .*

*Proof.* See Appendix B. □

## 4.2 Example 2: Correlated private values and multiplicity of equilibrium supports

We can relax the assumption of pure common values from Example 1 and retain the main qualitative features of the equilibrium support in those examples. Consider now a private-values variation on Example 1. Retain as before that there are  $K$  states of nature and  $K + 1$  signals, and that conditional on state  $\omega_k$ , each bidder independently receives either signal  $s_{k-1}$  or  $s_k$  with equal probability. However, instead of the state determining the common value, now let the bidders have private values, determined by their signals, with  $V_{k,l} = \frac{k+1}{K+1}$ .

For brevity we do not replicate the analogues to the characterization of equilibrium properties done for Example 1, other than to note that in this game, the set of all signals can never be an admissible active set. Figure 3 depicts the equilibrium supports for the case of  $K = 8$ , which can be compared with those in Figure 2 from Example 1.

In this game, there are multiple equilibrium supports in this game. Each admissible active set is supported by a unique profile of densities, and therefore each equilibrium support corresponds to

exactly one equilibrium behavior strategy, with piecewise-uniform randomization. Also, because all signals cannot be active at the same bid, some signals necessarily must receive positive payoffs in expectation in equilibrium. In the case of these equilibria, the odd-indexed signals have positive equilibrium payoffs, while the even-indexed ones receive zero in equilibrium.

The assumption that each state of the world can generate only two signals, and therefore that the conditional probabilities  $h_{l|k}$  are zero for  $l \notin \{k-1, k, k+1\}$ , is useful for analytical tractability. This can be relaxed without changing the qualitative features of the supports shown in Figures 2 and 3.

### 4.3 Example 3: Equilibrium with non-monotonic maximum bids

The strategic interplay between the information a signal brings about the value of the prize to a bidder, versus the information it brings about the likely valuations of the other bidder, drives the complexity of the equilibrium problem in the examples so far. In all the cases seen, however, the magnitude of the two effects is roughly the same, and results in the maximum bid submitted by a signal to be monotonically increasing in the posterior value conditional on that signal. We now exhibit an example where an increase in signal contains more bad news about the strength of competition, than good news about the value of the prize, at least over some ranges of signals.

Let there be  $K$  signals, where  $K$  is even, and assume private values, with  $V_{k,l} = \frac{k}{K}$ . For any  $1 \leq j \leq K/2$ , let  $h_{2j-1|2j-1} = h_{2j|2j} = p_s$  and  $h_{2j-1|2j} = h_{2j|2j-1} = p_d$ , with  $p_s, p_d > 0$  and  $p_s + p_d \leq 1$ . For any other pairs of signals  $s_k$  and  $s_l$ ,  $h_{l|k} = \frac{1-p_s-p_d}{K-2}$ .

Note that when  $p_s + p_d$  is close to unity, the implication of an increase from an even-indexed signal  $s_{2j}$  to the next higher signal  $s_{2j+1}$  is such that the value of the prize to the bidder increases, but the expectation of the value of the object to the other bidder increases even more. Figure 4 shows the unique equilibrium of the game, for the case where  $K = 4$  and  $p_s = p_d = \frac{49}{100}$ . The maximum bid submitted by bidders in equilibrium is not monotonic in their posterior value. When comparing the situation faced by, for example, a bidder with signal  $s_4$  versus  $s_5$ , while  $s_4$  has a lower value for the prize than  $s_5$ ,  $s_4$  knows the other bidder's value is no higher than her own, while  $s_5$  knows the other bidder's value is no lower than her own. As a result,  $s_5$  bids less aggressively in equilibrium.

### 4.4 Example 4: The KMS assumption and equilibrium selection

Baye et al. (1996) showed that when two bidders have a commonly-known and identical value  $v$  for the object, in equilibrium the bidders randomize uniformly on the interval  $[0, v]$ . In this model,  $v$  can either be thought of as a value which is known with certainty by both bidders, or that bidders have access to the same prior information and form the same expected value based on that

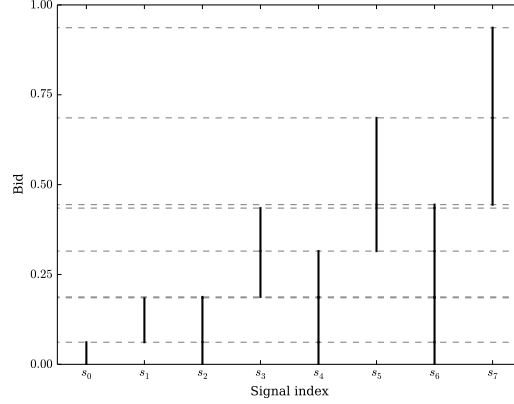


Figure 4: The support of the unique equilibrium in a private-values setting with  $K = 8$  signals, where increasing from an even-indexed to an odd-indexed signal implies a larger upward shift in the expected signal of the other bidder. Bars indicate intervals of bids on which each signal places positive mass. Horizontal dashing indicates bids at which the admissible active set changes.

information. In this example we explore the relationship between these two cases, by considering a highly-competitive case where the values are known with almost-certainty, compared to a case where only very noisy signals are received by the bidders.

**Example 4a.** Suppose there are  $K$  possible realizations of the common value  $\{v_1, \dots, v_K\}$ . There are  $K$  signals  $\{s_1, \dots, s_K\}$ . When the true state is  $v_k$ , the signal  $s_k$  is received by a bidder with probability  $p_c$ . All other signals are equally likely, and occur with probability  $p_w$ . We assume  $p_c > p_w$  so that signal  $s_k$  can be thought of as being the “correct” signal given  $v_k$ , and by necessity  $p_c + (M - 1)p_w = 1$ .

Straightforward calculations show that if both bidders receive the same signal  $s_k$ , then

$$\psi_{k|k} = h_{k|k} V_{k,k} = v_k p_c^2 + \left[ \sum_{m \neq k} v_m \right] p_w^2,$$

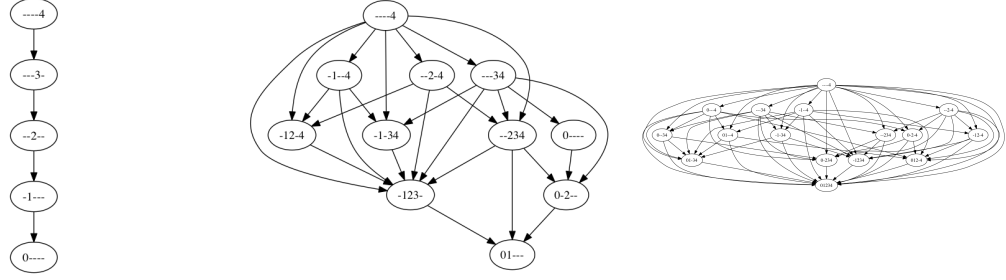
and if bidders receive different signals  $s_k \neq s_l$ ,

$$\psi_{k|l} = h_{k|l} V_{k,l} = (v_k + v_l) p_c p_w + \left[ \sum_{m \notin \{k,l\}} v_m \right] p_w^2.$$

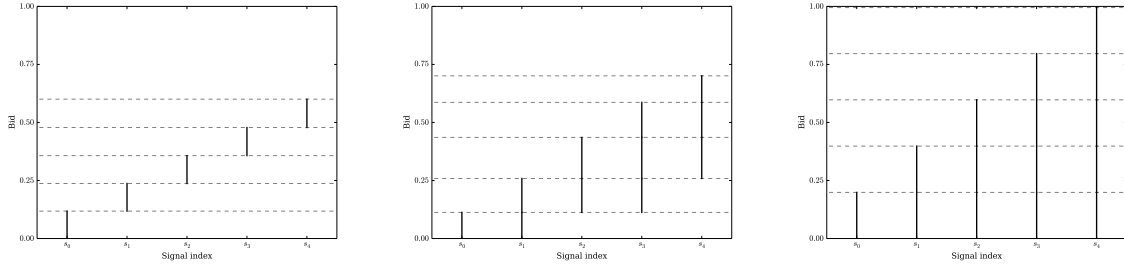
The KMS monotonicity assumption holds if and only if  $\frac{p_w}{p_c} > \frac{v_k}{v_{k+1}}$  for all  $k < M$ ; that is, when the accuracy of the signal is relatively poor.

**Proposition 7.** *In Example 4a, for any set of values  $\{v_k\}$  and any  $p_c \geq p_w$ , there is a unique*





(a) Graph of admissible active sets, for  $p_c = 0.21$ ,  $p_c = 0.45$ , and  $p_c = 0.99$ .



(b) Graph of equilibrium supports, for  $p_c = 0.21$ ,  $p_c = 0.45$ , and  $p_c = 0.99$ . Bars indicate intervals of bids on which each signal places positive mass. Horizontal dashed indicates bids at which the admissible active set changes.

Figure 5: Complexity of the graph of admissible active sets and structure of the equilibrium of Example 4a, as a function of the accuracy of the signal.

*equilibrium, in which (i) each signal randomizes over a connected support; (ii) on each interval, all signals between  $s_l$  and  $s_k$  are active, for some  $1 \leq l \leq k \leq K$ .*

*Proof.* See Appendix B. □

Equilibrium in this setting has an interesting, and perhaps counterintuitive, implication about behavior when signals are very precise, versus when they are very imprecise. As an example, consider the case where there are five possible values and five possible signals. Figure 5 depicts the cases of accurate and inaccurate signals. For each setting, the graph of admissible active sets is shown, as well as the support of the unique equilibrium. In the case of inaccurate signals, for which the KMS condition is satisfied, the graph of admissible active sets is simple, with a unique path consisting of singleton active signals. This illustrates how the algorithm of Siegel (2014) is a special case of ours, as when the KMS condition is satisfied, the admissible set graph always has this simple form. When signals are accurate, however, the graph of admissible sets is complex.

When the quality of the signal is good, i.e.,  $p_w \rightarrow 0$ , then there is a unique equilibrium of the game, which converges to the Baye et al. (1996) equilibrium. However, the same is not true when the signal becomes uninformative. In the limit where  $p_w \rightarrow p_c$ , the equilibrium is in fact *separating*; bidders who receive a “higher” signal outbid those who receive a “lower” one, with probability one.

We note that this limiting separating equilibrium is, in fact, an equilibrium of the game with  $p_w = p_c$ . Although the signal is completely uninformative in this case, it can serve as a coordination device. All that is required for equilibrium in this case is for a bidder to believe that he is facing a uniform distribution of bids; because the signal of the other bidder is not payoff-relevant, how that distribution of bids is realized as a function of the other bidder’s signal is not important. Therefore, any bidding strategy such that the ex-ante distribution of bids by a bidder is uniform is an equilibrium of this game.

This selection is counterintuitive in that, in the limiting case of payoff-irrelevant signals, the simplest behavior would be to ignore the signal. However, for this game, and indeed for any similar game where the KMS assumption is satisfied when signals are very noisy, the selected limiting equilibrium will be one in which bidders adopt a fully-separating equilibrium. There are perturbations of the Baye et al. (1996) model with very noisy signals, which do not satisfy the KMS assumption and do select the signal-independent equilibrium.

**Example 4b.** Modify Example 4a by assuming that, conditional on value  $v_k$ , with probability  $1 - \varepsilon$  both bidders receive the signal  $s_k$ , for some fixed  $\varepsilon > 0$ . Otherwise, the signal received is uniformly distributed over the signals, with the randomization being realized independently for each bidder.

In the case where both bidders receive the same signal  $s_k$ ,

$$\psi_{k|k} = h_{k|k} V_{k,k} = (1 - \varepsilon)v_k + \frac{\varepsilon}{M} \bar{V}.$$

If bidders receive different signals  $s_k \neq s_l$ , the case of uninformative signals must have occurred, and so immediately  $\psi_{k|l} = h_{k|l} V_{k,l} = \frac{\varepsilon}{M} \bar{V}$ . With this structure, the KMS monotonicity assumption does not hold for any value of  $\varepsilon$ , because for any triple of signals  $s_{k-1}$ ,  $s_k$ , and  $s_{k+1}$ ,  $\psi_{k+1|k} = \psi_{k-1|k}$ , with  $\psi_{k|k}$  differing from both. As a result, even when the probability of the “correct” signal is only slightly greater than an incorrect one, but the incidence of the correct signal is correlated between bidders, the equilibrium corresponding to Baye et al. (1996) is selected.

**Proposition 8.** *In Example 4b, the following are true of admissible active sets:*

1. Any subset of actions forms an admissible action set;

2. For each action set, there is a unique vector of densities that is consistent with equilibrium;
3. For each action set, the payoff derivative of inactive signals is always negative.

There is a unique equilibrium, in which each signal  $s_k$  randomizes over a connected interval  $[0, B_k]$ , with  $B_k$  strictly increasing in  $k$ .

*Proof.* See Appendix B. □

## 5 Conclusion

This paper examines symmetric, two-bidder all-pay auctions with interdependent values and a general signal structure. We assume a finite set of signals, and a continuous strategy space. Our contribution is to relax monotonicity assumptions on signals, to allow for cases in which an increase in a bidder’s signal can be good news about the value of the prize while also being bad news about the expected level of competitiveness due to a higher valuation by the other bidder. We show symmetric equilibria exist in these settings, and provide a constructive method which precisely calculates equilibrium parameters for any example.

We therefore provide answers to the discussion in the conclusion of Siegel (2014) regarding the challenge of identifying equilibria when the monotonicity condition is not satisfied. Even for examples with valuation and signal structures drawn from the literature, construction of the equilibrium is challenging, and equilibria can display an exquisitely rich and complex structure. We show how using only “local” information, in the form of admissible active sets, can help to provide some qualitative understanding of features of possible equilibria. However, in the absence of a monotonicity condition, constructing an equilibrium involves considering “global” information, in the form of solving a system of linear equations over a path of admissible active sets. The systems of equations corresponding to different paths do not bear any straightforward relationship to each other in general. The complexity of the graphs that the construction traverses capture the possible complexity in the equilibrium problem, and, correspondingly, the simplicity that the monotonicity condition used by Siegel (2014) brings.

An extension of these techniques should also apply to asymmetric all-pay auctions with two players. The complication will be that in the asymmetric case, mass points at a bid of zero will be possible. The examples in this paper already show that the symmetric case is quite complex to deal with already; in asymmetric cases, the number of possible active sets, and the paths to be explored through their graph, may be even larger.

## A Proof of Theorem 1

In this section, we give the argument for the existence of a symmetric equilibrium in which there are no atoms in the distribution of bids. The argument is similar to that of Govindan and Wilson (2010) who consider the case of a continuum of signals.

Let  $G$  denote the all-pay auction game. What prohibits the immediate application of the Fan-Glicksberg fixed-point theorem to establish existence of an equilibrium is that the payoff function is discontinuous in the case of strategies which place a mass point on some bid for some signal.

Therefore, we define a sequence of perturbed games  $\{G^j\}_{j=1}^\infty$ . These are obtained by supposing that, instead of the higher bid winning with probability one, a bidder may now win with positive probability even when his bid is slightly inferior to the other bidder's. Specifically, in game  $G^j$ , we add, independent of each bidder's bid, a uniform random variable on  $[-1/j, +1/j]$ . It is these perturbed bids that are used to determine which bidder wins the auction. The cost for each bidder remains their unperturbed bids.

Let  $\tilde{\Pi}_k^j(b; \Pi)$  denote the probability that bid  $b$  wins, in game  $G^j$ , conditional on a bidder receiving signal  $s_k$ , assuming the "intended" distribution of bids by the other bidder is given by the behavior strategy  $\Pi$ . Then the expected payoff to a bidder with signal  $s_k$  who bids  $b$  against a behavior strategy  $\Pi$  is given by

$$u^j(s_k, b, \Pi) = \sum_{s_l \in S} \psi_{l|k} \tilde{\Pi}_k^j(b; \Pi_k) - b.$$

Because of the addition of the continuous random noise, this payoff function is continuous. Therefore, the Fan-Glicksberg fixed-point theorem guarantees the existence of a symmetric equilibrium  $\Pi^j$  in this game. For each equilibrium  $\Pi^j$  let  $\theta^j = (\theta_k^j)_{s_k \in S}$  denote the corresponding vector of equilibrium payoffs for each signal.

Next, because the set of behavior strategies is compact, there must be some subsequence of the equilibria  $\{\Pi^j\}$  which converges to some limit; call that limit  $\Pi^*$ . Further, because the vector of equilibrium payoffs is in a compact set, we can, if necessary, consider a further subsequence such that the vector of equilibrium payoffs converges to some limit; call that limit  $\theta^*$ . We will argue that  $\Pi^*$  is in fact an equilibrium of the original all-pay auction game  $G$ .

**Claim.** *If  $h_{k|k} > 0$  for all  $s_k \in S$ , the limiting behavior strategy  $\Pi^*$  cannot have atoms.*

*Proof.* The argument is by contradiction. Suppose that some signal  $s_k$  places a mass point at some bid  $b^*$  in  $\Pi^*$ . Let  $m = \lim_{\delta \downarrow 0} [\Pi_k^*(b^* + \delta) - \Pi_k^*(b^* - \delta)]$ . Because  $\{\tilde{\Pi}_k^j(b; \Pi_k^j)\} \rightarrow \Pi_k^*(b)$ , for any  $\varepsilon > 0$ , there exists some  $J(\varepsilon)$  such that, for all  $j > J(\varepsilon)$ ,  $\tilde{\Pi}_k^j(b^* + \varepsilon; \Pi^j) - \tilde{\Pi}_k^j(b^* - \varepsilon; \Pi^j) \geq m$ .

Let  $S_k$  denote the support of  $\Pi_k^j$ , and define  $\underline{\beta}^j = \inf\{b : b \in S_k \cap [b^* - \varepsilon, b^* + \varepsilon]\}$  and

$\bar{\beta}^j = \sup\{b : b \in S_k \cap [b^* - \varepsilon, b^* + \varepsilon]\}$ . The difference in payoff in bidding  $\bar{\beta}^j$  versus  $\underline{\beta}^j$  is

$$\begin{aligned} & \sum_{s_l \in S} \psi_{l|k} [\tilde{\Pi}^j(\bar{\beta}^j; \Pi_l) - \tilde{\Pi}^j(\underline{\beta}^j; \Pi_l)] - (\bar{\beta}^j - \underline{\beta}^j) \\ & \geq \psi_{k|k} [\tilde{\Pi}^j(\bar{\beta}^j; \Pi_k) - \tilde{\Pi}^j(\underline{\beta}^j; \Pi_k)] - 2\varepsilon \\ & \geq \psi_{k|k} m - 2\varepsilon, \end{aligned}$$

which is positive because  $\varepsilon$  can be taken to be arbitrarily small. This contradicts that both  $\underline{\beta}^j$  and  $\bar{\beta}^j$  can be in the support of  $\Pi^j$ , and therefore  $\Pi^j$  cannot be an equilibrium of the game  $G^j$  as assumed.  $\square$

**Claim.** *The limiting  $\Pi^*$  is an equilibrium of the original auction game.*

*Proof.* Because  $\Pi^*$  does not have any mass points, the payoff functions  $u_k(b; \pi^*)$  are continuous in  $b$ . Further, the functions  $u_k^j(b; \pi^j)$  converge to  $u_k(b; \pi^*)$  (taking a subsequence if necessary).

First, suppose that there exists some signal  $s_k$  and some bid  $b$  which obtains a payoff strictly higher than  $V_k^*$ . By continuity, there exists some  $J$  such that, for all games with  $j > J$ , that same bid  $b$  must also obtain a higher payoff than  $\theta_k^j$ . This contradicts that  $\Pi^j$  is an equilibrium of  $G^j$ .

Next, consider a signal  $s_k$  and a bid  $b$  which is strictly suboptimal for  $s_k$  in  $\Pi^*$ . Because  $\Pi^*$  does not have mass points, there must be an open neighborhood around  $b$  such that all bids in the neighborhood are also strictly suboptimal. Because the equilibrium value functions converge, it follows that there exists some  $J$  such that for all games  $G^j$  with  $j > J$ , all bids in some neighborhood of  $b$  are suboptimal. Therefore, the corresponding equilibria  $\Pi^j$  must place zero probability mass on the interval, and therefore the limiting equilibrium  $\Pi^*$  must also place zero probability mass on the interval.  $\square$

## B Other Proofs

### Proofs for Section 4.1

#### Proof of Proposition 4

The following facts are useful in several of the arguments below. Both can be shown by straightforward induction arguments.

**Fact (Fact EVEN).** *Suppose  $\{s_k, \dots, s_{k+2j-1}\}$  are interior signals which comprise an isolated group within an admissible active set, for any  $1 \leq k < K$  and any  $j > 0$ . Then,*

$$\frac{k}{2}(\pi_{k-1} + \pi_k) = \frac{k+2j}{2}(\pi_{k+2j-1} + \pi_{k+2j}).$$

**Fact (Fact ODD).** Suppose  $\{s_k, \dots, s_{k+2j}\}$  are interior signals which comprise an isolated group within an admissible active set, for any  $1 \leq k < K$  and any  $j \geq 0$ . Then,

$$\frac{k}{2}(\pi_{k-1} + \pi_k) = \frac{1}{\Delta} - \frac{k+2j+1}{2}(\pi_{k+2j} + \pi_{k+2j+1}).$$

**Claim.** No isolated group of an odd number of interior signals can be part of an admissible active set.

*Proof.* Suppose  $\{s_k, s_{k+2j}\}$  is an isolated group of signals which are part of an admissible active set, for any  $j \geq 1$ . For all signals  $s_{k+2i}$ ,  $i < j$ , applying Fact EVEN to the indifference condition for the signal implies

$$k\pi_k + (k+2i+1)(\pi_{k+2i} + \pi_{k+2i+1}) = 2\Delta^{-1}.$$

For all signals  $s_{k+2i-1}$ ,  $i < j$ , applying Fact ODD to the indifference condition for the signal implies

$$k\pi_k + (k+2i)(\pi_{k+2i-1} + \pi_{k+2i}) = 0.$$

In matrix form, the system of equations required to satisfy all the associated indifference conditions can be written  $C\pi = d$ , with

$$C = \begin{bmatrix} 2k+1 & k+1 & 0 & 0 & \dots & 0 \\ k & -(k+2) & -(k+2) & 0 & \dots & 0 \\ k & 0 & k+3 & k+3 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ k & 0 & \dots & 0 & -(k+2j) & -(k+2j) \\ k & 0 & \dots & \dots & 0 & k+2j+1 \end{bmatrix}, d = \begin{bmatrix} 2\Delta^{-1} \\ 0 \\ 2\Delta^{-1} \\ \vdots \\ 0 \\ 2\Delta^{-1} \end{bmatrix}.$$

By Farkas' Lemma, this system has no nonnegative solution for  $\pi$  if and only if there exists some vector  $y$  such that  $C^T y \geq 0$  and  $d^T y < 0$ . The matrix  $C^T$  does not have full rank, so there will be multiple solutions to  $C^T y = 0$ . We can write  $C^T y = 0$  row-wise as

$$\begin{aligned} (2k+1)y_1 + k \sum_{m=2}^{2j+1} y_m &= 0 \\ (k+2i-1)y_{2i-1} - (k+2i)y_{2i} &= 0 \quad \forall 1 \leq i \leq j \\ -(k+2i)y_{2i} + (k+2i+1)y_{2i+1} &= 0 \quad \forall 1 \leq i < j \end{aligned}$$

The equations generated by the last two lines imply

$$y_n = \left( \frac{k+2j+1}{k+n} \right) y_{2j+1} \quad \forall 2 \leq n \leq 2j$$

To obtain  $d^T y < 0$  we only require that the sum of the odd-indexed entries in  $y$  be negative. We have

$$\begin{aligned} \sum_{i=1}^j y_{2i-1} &= -\frac{k}{2k+1} \sum_{i=1}^j y_{2i} + \frac{k+1}{2k+1} \sum_{i=1}^j y_{2i+1} \\ &= -\frac{k}{2k+1} \sum_{i=1}^j \frac{k+2j+1}{k+2i} y_{2j+1} + \frac{k+1}{2k+1} \sum_{i=1}^j \frac{k+2j+1}{k+2i+1} y_{2j+1} \\ &= \frac{k+2j+1}{2k+1} y_{2j+1} \sum_{i=1}^j \left[ -\frac{k}{k+2i} + \frac{k+1}{k+2i+1} \right]. \end{aligned}$$

As the term in square brackets is positive, the sign of  $d^T y$  is determined by the sign of  $y_{2j+1}$ . Because  $y_{2j+1}$  can be chosen freely because  $C^T y$  does not have full rank, choosing  $y_{2j+1}$  negative generates the required condition. Therefore, the original system does not have a solution with nonnegative densities, and the claim follows.  $\square$

**Claim.** Any isolated group of all signals between  $s_0$  and  $s_m$  can be part of an admissible active set, for any  $0 \leq m \leq K$ . For any such group, for all  $k \leq m$ , the supporting solution satisfies  $\pi_{k-1} + \pi_k = \frac{1}{k\Delta}$ .

*Proof.* For signal  $s_0$  the indifference condition can be written  $\Delta(\pi_0 + \pi_1) = 1$ . Now consider a signal  $k \leq m$ . We proceed by induction. We have shown the claim holds for  $k = 0$ . So assume it is true for  $k - 1$ , and consider signal  $s_k$ . Taking the indifference condition for  $s_k$  with  $k < K$  and applying the induction hypothesis,

$$\begin{aligned} \frac{k}{2} (\pi_{k-1} + \pi_k) + \frac{k+1}{2} (\pi_k + \pi_{k+1}) &= \frac{1}{\Delta} \\ \frac{k}{2} \times \frac{1}{k\Delta} + \frac{k+1}{2} (\pi_k + \pi_{k+1}) &= \frac{1}{\Delta} \\ \pi_k + \pi_{k+1} &= \frac{1}{(k+1)\Delta}, \end{aligned}$$

which establishes the result for  $k$  and completes the induction step. For  $s_K$  the solution can be confirmed by direct calculation.

Now consider signal  $s_{m-1}$ . We have that  $\pi_{m-1} + \pi_m = \frac{1}{m\Delta}$ . This, and the indifference condition for signal  $s_m$  constitute a  $2 \times 2$  linear system of equations. The unique solution of this system is

$\pi_m = \frac{1}{\Delta(m+1)}$  and  $\pi_{m-1} = \frac{1}{\Delta m} - \frac{1}{\Delta(m+1)}$ . Note that both of these densities are positive. Further, given that for any signal  $s_k$  with  $k < m$ ,  $\pi_k = \frac{1}{(m+1)\Delta} - \pi_{k+1}$  all the densities in this group are strictly positive.  $\square$

**Claim.** *No isolated set of all signals from  $s_k$  to  $s_K$  with  $0 < k < K$  may be simultaneously part of an admissible active set.*

*Proof.* By direct calculation,  $s_{K-1}$  and  $s_K$  cannot comprise an isolated group within an admissible active set. Now suppose that all signals from  $s_k$  to  $s_K$  do comprise such an isolated group, with  $0 < k < K - 1$ . Consider signal  $s_{K-1}$ . If  $k + K$  is even, Fact EVEN tells us that

$$\frac{k}{2}\pi_k = \frac{K}{2}(\pi_{K-1} + \pi_K).$$

Since  $\pi_{K-1} + \pi_K = 2$ ,  $\pi_k = \frac{1}{2k\Delta}$ . By Fact ODD, we know that

$$\frac{k}{2}\pi_k = \frac{1}{\Delta} - \left(\frac{k+1}{2}\right)(\pi_k + \pi_{k+1}).$$

Solving for  $\pi_{k+1}$  leaves us with  $\pi_{k+1} = \frac{(2k-1)}{2k\Delta(k+1)}$ . By Fact EVEN we know that

$$\frac{k}{2}\pi_k = \frac{k+2}{2}(\pi_{k+1} + \pi_{k+2}).$$

Solving for  $\pi_{k+2}$  leaves us with  $\pi_{k+2} = \frac{1}{2(k+2)\Delta} - \frac{(2k-1)}{2k\Delta(k+1)}$ , which is only positive if  $k < 1$ .

If  $k + K$  is odd then Fact ODD tells us that

$$\frac{k}{2}\pi_k = \frac{1}{\Delta} - \left(\frac{K}{2}\right)(\pi_{K-1} + \pi_K).$$

Solving for  $\pi_{k+1}$  as above yields  $\pi_{k+1} = \frac{-1}{k(k+1)\Delta} < 0$ .  $\square$

## Proof of Proposition 5

First, observe that the highest density for any signal  $s_k$  in any admissible active set occurs when  $s_k$  is an isolated signal. So, it is enough for us to consider two cases:  $s_{k-1} \notin A$  and  $s_{k+1} \in A$  as an isolated signal, and  $s_{k+1} \notin A$  and  $s_{k-1} \in A$  as an isolated signal.

First suppose  $s_{k-1} \notin A$  and  $s_{k+1} \in A$  is an isolated signal. Then the indifference condition for  $s_{k+1}$  reduces to  $\pi_{k+1} = \frac{2}{2k+3}\Delta^{-1}$ . Then direct calculation gives that  $u'_k = \frac{1}{2} \times \frac{2k+2}{2k+3} - 1 < 0$ , which is a contradiction. Next suppose  $s_{k+1} \notin A$  and  $s_{k-1} \in A$  is an isolated signal. Then the indifference condition for  $s_{k-1}$  reduces to  $\pi_{k-1} = \frac{2}{2k-1}\frac{1}{\Delta}$ . Then direct calculation gives that  $u'_k = \frac{k}{2k-1} - 1 < 0$ , which is a contradiction.



Therefore, both  $s_{k-1}$  and  $s_{k+1}$  must be active, as claimed. When both are active,

$$u'_k = h_{k-1|k} V_{k-1,k} \pi_{k-1} + h_{k+1|k} V_{k,k+1} \pi_{k+1} - 1 = \frac{1}{2} \times \frac{8k(k-1) - 2}{4k(k-1) - 3} - 1,$$

which is positive for  $k > 1$ .

## Proof of Proposition 6

When all signals comprise an active set, the required conditions for indifference lead to a degeneracy; if the conditions for  $K$  of the signals are satisfied, then the indifference condition for the other signal is automatically satisfied as well. In this case an earlier claim has shown that  $\pi_{k-1} + \pi_k = \frac{1}{k\Delta}$  for all  $k$ . This implies that there are only  $K$  linearly independent equations; the indifference equation for  $s_K$  is implied by the indifference equations for  $s_0 \dots s_{K-1}$  being satisfied.

We parameterize the one-dimensional set of solutions to the indifference conditions by  $\pi_0$ . Given  $\pi_0$ , the recursive relationship can be unrolled to show that  $\pi_k = \frac{1}{k\Delta} - \pi_{k-1}$  and  $\pi_k = \pi_{k-2} - \frac{1}{k(k-1)\Delta}$ . For any  $k$ , we can then state that

$$\begin{aligned} \pi_{2k} &= \pi_0 - \frac{1}{\Delta} \sum_{j=1}^k \frac{1}{2j(2j-1)} \\ \pi_{2k+1} &= \pi_1 - \frac{1}{\Delta} \sum_{j=1}^k \frac{1}{(2j+1)2j}. \end{aligned}$$

Remembering that  $\pi_0 + \pi_1 = \frac{1}{\Delta}$ , in order for all of these densities  $\pi_{2k}$  and  $\pi_{2k+1}$  to be nonnegative for all  $k$ , it must be that

$$\sum_{j=1}^{k/2} \frac{1}{2j(2j-1)} \leq \pi_0 \leq 1 - \sum_{j=1}^{k/2} \frac{1}{(2j+1)(2j)}.$$

As  $k \rightarrow \infty$ , the series on the left converges monotonically from below to  $\ln 2$ , and the series on the right converges monotonically from above to  $1 - \ln 2$ . Therefore, for any finite  $K$ , there is a range of  $\pi_0$  consistent with equilibrium on this interval, with the size of that range decreasing as  $K$  increases. In other words, for a finely discretized signal space, randomization on this interval is “almost” uniform.

## Proofs for Section 4.4

### Proof of Proposition 7

Consider two signals  $s_k$  and  $s_l$  with  $k > l$ . Direct calculation shows that

$$u'_k - u'_l = (v_k p_c - v_l p_w)(p_c - p_w)\pi_k + (v_k p_w - v_l p_c)(p_c - p_w)\pi_l + (v_k - v_l)p_w(p_c - p_w) \sum_{m \notin \{k, l\}} \pi_m.$$

The first and third terms are always positive. The middle term is positive if  $\frac{p_w}{p_c} > \frac{v_l}{v_k}$ , that is, if signals are not too accurate relative to the difference in values. If this inequality holds for  $l = k - 1$  and all  $k$ , then the KMS monotonicity condition is satisfied, and therefore admissible active sets can only be singleton signals.

Suppose that  $s_k$  is active and  $s_l$  is inactive. Then it is immediate that  $u'_k - u'_l > 0$ , and therefore  $u'_l < 0$ . This means that between two intervals  $\mathcal{I}_j$  and  $\mathcal{I}_{j+1}$ , no signal can transition from active to inactive if another higher signal is active on  $\mathcal{I}_{j+1}$ . Given an equilibrium, let  $L_j$  be the lowest signal active in interval  $\mathcal{I}_j$ . It follows directly that  $L_j$  is nonincreasing in  $j$ .

Next, suppose both  $s_k$  and  $s_l$  are active. Then it must be that

$$(v_k p_c - v_l p_w)\pi_k + (v_k p_w - v_l p_c)\pi_l < 0,$$

which implies that

$$\frac{\pi_k}{\pi_l} < \frac{v_l p_c - v_k p_w}{v_k p_c - v_l p_w} < 1,$$

and therefore  $\pi_k < \pi_l$ .

Now consider the case where  $s_k$  was active in  $\mathcal{I}_j$  but is inactive in  $\mathcal{I}_{j+1}$ . Then  $u'_k > 0$  on this interval. We claim that for intervals  $j' \geq j + 1$ ,

$$(v_k p_w - v_{L_{j'}} p_c)(p_c - p_w)\pi_{L_{j'}} + (v_k - v_{L_{j'}})p_w(p_c - p_w) \sum_{m \notin \{k, L_{j'}\}} \pi_m > 0.$$

This must be true by definition on  $j' = j + 1$  because  $s_k$  exits. Because  $L_j$  is nonincreasing, both terms in the sum must be nondecreasing. Therefore, once a signal  $s_k$  exits, it never returns to activity, and therefore the support of signal  $s_k$ 's strategy is connected.

Finally, it must be that active sets must have all signals between  $s_l$  and  $s_k$  for some  $l \leq k$ . Suppose not, and  $s_l$  and  $s_k$  are active for some  $l < k$  but some signal  $s_n$  with  $l < n < k$  is not. Because signals must exit in decreasing order of their indices, the number of intervals on which  $s_n$  is active must be fewer than  $s_l$ . However, on each such interval,  $\pi_n < \pi_l$ . Because  $\pi_l$  integrates to one over all intervals on which  $s_l$  is active, then  $\pi_n$  must integrate to a value strictly less than one,

which is a contradiction.

### Proof of Proposition 8

We can write the payoff derivative conditional on a signal  $s_k$  as

$$u'_k(b) = \frac{\varepsilon \bar{V}}{M} \sum_{m \neq k} \pi_m + (1 - \varepsilon) v_k \pi_k - 1.$$

Active sets consisting of a single action are always admissible, so we consider the case when the active set is not a singleton. Suppose  $s_k$  and  $s_l \neq s_k$  are both active. Then,  $u'_k = u'_l$  implies

$$\frac{\varepsilon \bar{V}}{M} \sum_{m \neq k} \pi_m + (1 - \varepsilon) v_k \pi_k - 1 = \frac{\varepsilon \bar{V}}{M} \sum_{m \neq l} \pi_m + (1 - \varepsilon) v_l \pi_l - 1,$$

which implies  $v_k \pi_k = v_l \pi_l$ . This fact, combined with  $u'_k = 0$ , implies that

$$\frac{\varepsilon \bar{V}}{M} \sum_{m \neq k} \frac{v_k}{v_m} \pi_k + (1 - \varepsilon) v_k \pi_k = 1.$$

The left side is linear in  $\pi_k$  and positive when  $\pi_k$  is positive; therefore, a solution exists and is unique. Substituting the solution for  $\pi_k$  back into the expression for other  $\pi_l$  is guaranteed to give positive solutions for  $\pi_l$ .

Suppose  $s_k$  is a signal in the active set, and  $s_l$  is an inactive signal. Then  $u'_k - u'_l = v_k \pi_k - v_l \pi_l = v_k \pi_k > 0$ . Because  $u'_k = 0$ , it follows that  $u'_l < 0$ .

Along any path induced by  $\succ$ , a signal can only leave the active set in a transition to a new active set where that signal has a positive payoff derivative. In light of the previous claim, this can never occur; therefore each signal  $s_k$  randomizes over some connected interval  $[0, B_k]$ , as in the construction the signal never becomes inactive again after becoming active. To show that  $B_k$  is increasing in  $k$ , we proceed by contradiction. Suppose that  $B_k \geq B_{k+1}$  for some  $k$ . On every active set below  $B_{k+1}$ , both signals are active. On each, the densities for these two signals must satisfy  $\pi_{k+1} = \frac{v_k}{v_{k+1}} \pi_k$ , so therefore  $\pi_{k+1} < \pi_k$ . By construction, the total probability mass for signal  $s_k$  on  $[0, B_k]$  is equal to one. However, on each active set in that interval,  $\pi_{k+1} < \pi_k$ , so the total probability mass for signal  $s_{k+1}$  is strictly less than one, which establishes the contradiction.

## References

- Erwin Amann and Wolfgang Leininger. Asymmetric all-pay auctions with incomplete information: The two-player case. *Games and Economic Behavior*, 14(1):1–18, 1996.
- Susan Athey. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. *Econometrica*, 69(4):861–889, 2001.
- Michael R. Baye, Dan Kovenock, and Casper G. de Vries. The all-pay auction with complete information. *Economic Theory*, 8:291–305, 1996.
- Marco Casari, John C. Ham, and John H. Kagel. Selection bias, demographic effects, and ability effects in common value auction experiments. *American Economic Review*, 97(4):1278–1304, 2007.
- Richard Cornes and Roger Hartley. Asymmetric contests with general technologies. *Economic Theory*, 26(4):923–946, 2005.
- Srihari Govindan and Robert Wilson. Existence of equilibria in all-pay auctions. Stanford Graduate School of Business Research Paper Series, No. 2058, 2010.
- Ronald M. Harstad. Privately informed seekers of an uncertain rent. *Public Choice*, 83(1):81–93, 1995.
- René Kirkegaard. Comparative statics and welfare in heterogeneous contests: Bribes, caps, and performance thresholds. Working paper 702, Brock University Department of Economics, 2007.
- René Kirkegaard. Favoritism in asymmetric contests: head starts and handicaps. In *Proceedings of the Behavioral and Quantitative Game Theory Conference on Future Directions*, page 45. ACM, 2010.
- Vijay Krishna and John Morgan. An analysis of the war of attrition and the all-pay auction. *Journal of Economic Theory*, 72:343–362, 1997.
- Alessandro Lizzeri and Nicola Persico. Uniqueness and existence of equilibrium in auctions with a reserve price. *Games and Economic Behavior*, 30(1):83–114, 2000.
- Jingfeng Lu and Sérgio O. Parreiras. Asymmetric all-pay auctions, monotone and non-monotone equilibrium. Working Paper, 2014.
- Florian Morath and Johannes Münster. Private versus complete information in auctions. *Economics Letters*, 101(3):214–216, 2008.

- Ron Siegel. All-pay contests. *Econometrica*, 77(1):71–92, 2009.
- Ron Siegel. Asymmetric contests with conditional investments. *American Economic Review*, 100(5):2230–60, 2010.
- Ron Siegel. Asymmetric all-pay auctions with interdependent valuations. *Journal of Economic Theory*, 2014.
- Gordon Tullock. Efficient rent seeking. In James M. Buchanan, Robert D. Tollison, and Gordon Tullock, editors, *Toward a theory of the rent seeking society*. Texas A&M University Press, College Station, TX, 1980.
- Ruqu Wang. Common-value auctions with discrete private information. *Journal of Economic Theory*, 54(2):429–447, 1991.
- Karl Wärneryd. Information in conflicts. *Journal of Economic Theory*, 110(1):121–136, 2003.
- Karl Wärneryd. Multi-player contests with asymmetric information. *Economic Theory*, 51(2): 277–287, 2008.