Optimal Compatibility in Systems Markets*

Sang-Hyun Kim†

Jay Pil Choi‡

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Abstract

We analyze private and social incentives for standardization to ensure market-wide system compatibility in a two-dimensional spatial competition model. It is shown that there is a fundamental conflict of interest between consumers and producers over the standardization decision. Consumers prefer standardization with full compatibility because it offers more variety that confers better match with their ideal specifications. However, firms are likely to choose the minimum compatibility to maximize product differentiation and soften competition. This is in sharp contrast to the previous literature that shows the alignment of private and social incentives for compatibility. We also characterize the free-entry equilibria under the maximum and the minimum compatibility. With free entry, more firms enter without standardization, but the number of available system variety is less than the one under standardization.

JEL Classification: D43, L13, L22

Key Words: Compatibility, System Competition, Standardization, Spatial Competition Model, Free Entry

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†School of Economics and the ESRC Centre for Competition Policy, University of East Anglia, Norwich Research Park, Norwich, NR4 7TJ, UK. Email: Sang-Hyun.Kim@uea.ac.uk.

‡School of Economics, University of New South Wales, Sydney, NSW 2052, Australia and Department of Economics, Michigan State University, 110 Marshall-Adams Hall, East Lansing, MI 48824 -1038. E-mail: choijay@msu.edu.

1 Introduction

This paper reexamines the incentives for firms to achieve standardization to ensure market-wide system compatibility in a two-dimensional spatial competition model. The issue of compatibility choice has been extensively studied. The literature on this has addressed two main questions: Do firms have incentives to achieve compatibility across components made by different producers? Is the market compatibility choice socially optimal? Major contributors on this subject, in particular Matutes and Regibeau (1988) and Economides (1989), answered positively to both questions, demonstrating the alignment of private and social incentives for compatibility in the absence of network effects.

We address the same questions in a more general framework that allows more than 2 varieties for each component and show that their results are limited to the special case of two firms in each market and not robust to changes in the number of firms. More specifically, we show that there is a fundamental conflict of interest between consumers and producers over the standardization decision. Consumers prefer standardization with full compatibility because it offers more variety that confers better match with their ideal specifications. However, firms are likely to choose the minimum compatibility to maximize product differentiation and soften competition. We also characterize the free-entry equilibria under the maximum and the minimum compatibility. With free entry, more firms enter without standardization, but the number of available system variety is less than the one under standardization.

To analyze the desirability of compatibility across different producers of each component products, we adopt the framework of Matutes and Regibeau (1988) to facilitate the comparison of the results. However, we modify it to maintain symmetry across system products in a more general case with more than 2 varieties for each component.[1] More

[1] Economides (1989) analyzed essentially the same problem we are considering. His analysis, however, was logically inconsistent because he implicitly assumed two topologically different manifolds, sphere and torus, when he compared two different regimes. We are not the first one to point this out. To quote Farrell *et al.* (1998), "Economides (1989) states results generalizing Matutes and Regibeau's to the case of general n, but Matutes and Regibeau (1991) argue that Economides' results were in error (p. 146)." Monroe (1993) shows by example that the comparison of industry profits across regimes is ambiguous for $n > 4$ and depends on firms' locations in product space."

precisely, we consider a system good market which consists of two differentiated component goods, called A and B, such as hardware and software.[2] We search for symmetric equilibria with n producers in each of the two differentiated component-markets. The distribution of consumers' preference is modeled as a uniform distribution on a *torus*, the Cartesian product of two circles. With a general number of firms in each market, we need to deal with a large number of compatibility possibilities across the two components and location choices for each system variety. In this paper, we focus on two polar cases. *Full compatibility* is the case where any component in the market A is compatible with any component in market B. For instance, this can be an outcome of industry-wide standardization efforts that ensures interoperability between any components that adhere to the industry standard. The other case we consider is the minimum level of compatibility in which one particular variety of component A is compatible with only one variety of component B, and vice versa. We call this regime *pairwise compatibility* under which no mix-and-match is possible. This would arise if each firm has a proprietary technology and firms in market A form an exclusive partnership with partner firms in market B.

We show that for all $n \geq 3$, firms prefer pairwise compatibility to full compatibility. In contrast, social welfare is maximized under full compatibility. This implies that the alignment of private and social incentives towards full compatibility in Matutes and Regibeau is a special result that applies to only $n = 2$, and not robust to changes in the number of firms. The logic behind our results is simple and can be explained in geometric terms. First, under full compatibility there are n^2 systems available while there are only n systems available under pairwise compatibility. As a result, it is more costly for an individual consumer to change his choice under pairwise compatibility because the second best alternative tends to be farther away from his ideal specification in comparison to the full compatibility regime. Second, the measure of marginal consumers, those who are indifferent between two alternatives, is usually smaller under pairwise compatibility. This implies that the marginal gains from a price cut, or equivalently the firms' price cutting incentives, are smaller. Taken together, firms have incentives to engage in exclusive partnerships that result in pairwise

[2] See Katz and Shapiro (1994) for a discussion of economic issues in systems markets.

compatibility in order to reduce the intensity of price competition. On the other hand, as well-understood in the literature, full compatibility is desirable for the consumers and for the whole economy since it reduces consumers' "transportation costs" by increasing variety of systems available in the market.

We also extend our analysis to allow for free entry. We assume that given the number of entrants in each component market, they are located symmetrically in the profit-maximizing fashion. The equilibrium number of firms in each compatibility regime is determined by the zero profit condition. Our previous result that firm profits are higher under pairwise compatibility compared to full compatibility implies more entry of firms in the regime of pairwise compatibility. Nonetheless, we can show that more system variety is available under full compatibility due to consumers' ability to mix-and-match. As a result, social welfare is higher under full compatibility; more variety is available with less fixed costs of entry.

Our paper mainly contributes to the literature on spatial competition models of compatibility, but also relates to other branches of research including products bundling (e.g. Matutes and Regibeau (1992), Chen (1997), Denicolo (2000) and Nalebuff (2004)), exclusive dealing (e.g. Besanko and Perry (1994)) and vertical organization of industry in general (Farrell *et al.* (1998)). For instance, if we reinterpret two component products as a final good and retailing service, respectively, our model can be considered as an analysis of incentives for exclusive dealing contracts.

The rest of the paper is organized as follows. In section 2, we present the model of differentiated system products in a two dimensional product space. In section 3, we analyze market equilibrium in two regimes, incompatibility without any mix-and-match possibility and full compatibility enabled by market-wide standardization. We derive the market equilibrium in each regime and analyze incentives to achieve standardization in the market. We show that under full compatibility the market equilibrium in each component market can be analyzed in isolation of the other component market. As a result, the market equilibrium replicates the one in the classical circular city model. We also characterize the equilibrium of price competition under any symmetric configuration of product differentiation, which allows the comparison of equilibrium profits under full compatibility and under pairwise compatibility. The analysis reveals a fundamental conflict of interest between the consumers and the producers over the standardization decision. In section 4, we extend our analysis to investigate the free-entry equilibria. Section 5 contains a discussion about an alternative market structure examined in Matutes and Regibeau (1988) as a robustness check of our main results. Concluding remarks follow.

2 The Model

Consider a market for system goods that comprise two component products, A and B. We assume that these two component products can generate value only when they are combined together. To analyze incentives to achieve compatibility between the two components, we adopt a variation of Matutes and Regibeau (1988) that accommodates more than 2 varieties in each component market while maintaining symmetry among varieties. There are n (≥ 2) firms in each component market. Let us denote the set of firms that produce component A by I_A, and similarly for B by I_B. Each firm produces only one component, that is, $I_A \cap I_B = \phi$ and the total number of firms is $2n$.[3] To maintain symmetry among system goods, we adopt a torus to represent the product space. Consumers have heterogenous preferences over the characteristics of each component. Each consumer's preference is summarized by her location which represents her ideal variety (x_A, x_B) in the product space. The torus is constructed by the following equivalence relations over $\mathbb{R}^2$.

$$(x_A, x_B) \sim (x_A + 1, x_B) \sim (x_A, x_B + 1) \text{ for any } (x_A, x_B) \in \mathbb{R}^2$$

As shown in Figure 1, the torus is isomorphic to Cartesian product of two circles, i.e. $S^1 \times S^1$, thus is a natural two-dimensional extension of the circular city model *à la* Salop (1979).

[3] Matutes and Regibeau (1988) mostly consider a situation in which each firm produces both components of a system even though they also discuss the case where each firm produces only one component as in our paper. In section 4, we also consider the case where each firm produces both components as in Matutes and Regibeau to check the robustness of our results.

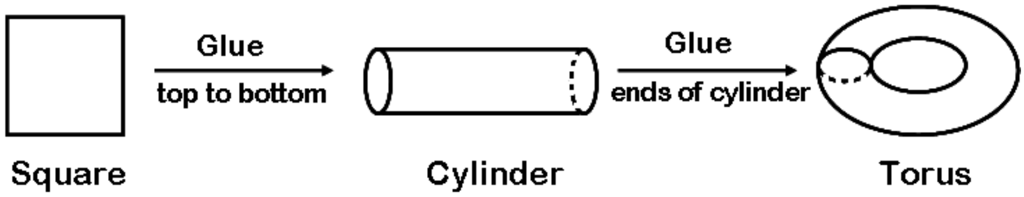

Figure 1: Transforming a square into a torus

As each firm produces only one component, firms producing component A and component B need to make coalitions to build a compatible system. With $n(>2)$ manufacturers for each component, we can have a variety of coalition structures, most of which are analytically intractable. Instead, we consider two polar cases. In the first, there is *full compatibility*; any component A is compatible with any arbitrary component B to make a feasible system. Under full compatibility, there are n^2 systems in the market with the maximum number of system goods available to consumers. This case arises if all component manufacturers participate in a standard-setting organization and establish industry-wide standards that would allow "mix-and-match" between two components from any producers. Without full compatibility, we can imagine a plethora of possibilities to the extent the component products are compatible with each other. We consider the other extreme case of compatibility, which we call *pairwise compatibility*; a firm in I_A forms an exclusive coalition with a single firm in I_B to make their products compatible with each other, while incompatible with all the others.[4] Therefore, only n system goods, the least varieties, are available in the market. As in Salop (1979), we assume that all component producers are symmetrically located, and focus on symmetric equilibria in which firms independently set the price of their own components taking the compatibility configurations and location choices as given.

Consumers are uniformly distributed on the torus. A consumer, who is at (x_A^o, x_B^o) and

[4] This is similar to the case which Economides (1989) calls *incompatibility*. The difference is that he considers the case of n integrated system producers, whereas we consider the case of $2n$ independent component producers in this paper. We adopt this assumption to focus purely on the competitiveness of oligopoly in a two-dimensional characteristic space. It turns out that this assumption does not play a critical role because competition effect dominates ownership effect for $n \geq 4$. We discuss about this in detail in section 4.

purchases system (i,j) where $i \in I_A$ and $j \in I_B$, derives the net utility of

$$v - t[(x_A^o - x_i)^2 + (x_B^o - x_j)^2] - p_i - p_j,$$

where v is the reservation value of the ideal system common to all consumers, $t > 0$ is a "transportation cost" parameter that represents the degree of product differentiation, x_i is the location of firm i on the coordinate A, p_i is the price of component A produced by firm i, and x_j and p_j are defined similarly for component B.[5] Each consumer buys at most one unit of the system good that provides the highest net utility. We assume that v is sufficiently large, thus every consumer makes a purchase in any equilibrium. Each firm's marginal cost is normalized to zero.

3 Market Equilibrium: Full Compatibility vs. Pairwise Compatibility

In this section, we derive market equilibrium under full compatibility and pairwise compatibility to analyze incentives to achieve industry-wide standardization.

3.1 Equilibrium under Full Compatibility

Under full compatibility, competition takes place at the component level. As a consequence, the symmetric equilibrium is identical to that of the one-dimensional circular city model. This can be easily shown by considering an individual consumer's utility maximization problem:

$$\begin{aligned} & \max_{i \in I_A,\ j \in I_B} \left\{ v - t[(x_A^o - x_i)^2 + (x_B^o - x_j)^2] - p_i - p_j \right\} \\ = \quad & v - \min_{i \in I_A} \{ t(x_A^o - x_i)^2 + p_i \} - \min_{j \in I_B} \{ t(x_B^o - x_j)^2 + p_j \} \end{aligned}$$

[5] We adopt a quadratic transportation cost in our analysis as in d'Aspremont *et al.* (1979). However, any specification of transportation costs which is a monotonic transformation of the Euclidean distance generates the same qualitative results.

With full compatibility that allows every component in A to be combined with any component B, the choice of each component can be made independently of the other. Furthermore, each marginal distribution of consumers' preference is uniform on a circle of length 1, since the preferences for A and B are jointly uniform on the torus. Therefore, provided that the firms in I_A are equidistant from each other along the (say, horizontal) coordinate x_A and those in I_B along the (vertical) coordinate x_B, there exist a symmetric equilibrium where all firms in the market set their price same and share the market equally. Since the market B is identical to that of A, below we just consider the market of component A (horizontal circle).

Suppose firm i located at the origin charges p_i, and all the other firms in I_A charge the identical price p. A consumer located at x is indifferent between purchasing from firm i and purchasing from i's closest neighbor if $p_i - tx^2 = p + t\left(\frac{1}{n} - x\right)^2$. So, firm i's demand can be written as

$$D_i^{FC}(p_i, p) = 2x = \frac{1}{n} - \frac{n(p_i - p)}{t},$$

and the profit be

$$\pi_i^{FC}(p_i, p) = p_i D_i^{FC}(p_i, p) = p_i \left[\frac{1}{n} - \frac{n(p_i - p)}{t}\right].$$

Differentiating with respect to p_i and using the symmetry condition that $p_i = p$, we derive the following proposition.

Proposition 1 *Under full compatibility, the symmetric equilibrium price and profit are given by $p^{FC}(n) = t/n^2$ and $\pi^{FC}(n) = t/n^3$, respectively.*

3.2 Equilibrium under Pairwise Compatibility

Suppose now that each firm in I_A forms an exclusive coalition with a partner firm in I_B to produce a compatible system. Denote the set of the coalitions in the market by I_C of which generic element is (i, j) for $i \in I_A$ and $j \in I_B$. With a slight abuse of notation, we use the same notation to denote the set of the systems available in the market. With

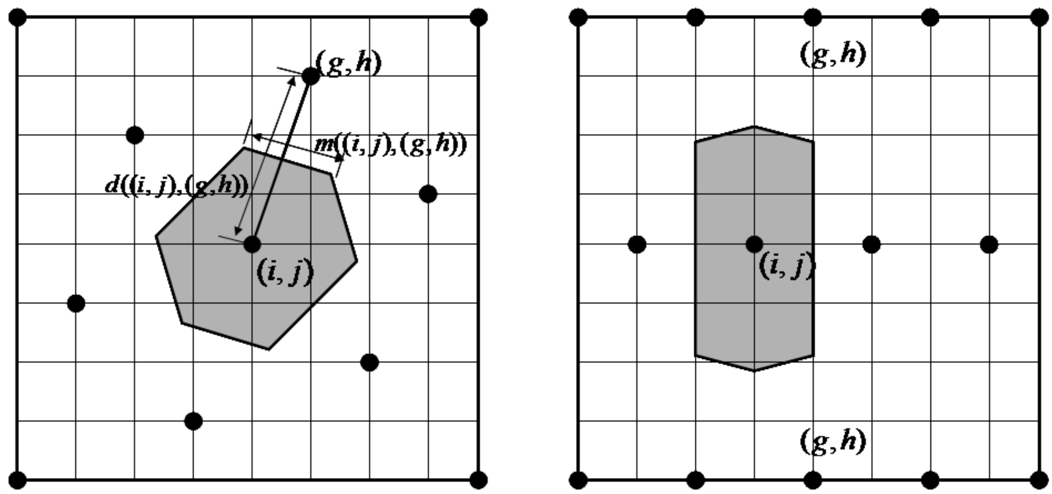

Figure 2: Two symmetric formations with $n = 8$

pairwise compatibility, there are n systems available which are symmetrically located on the torus. Formally, we define a *symmetric formation* as a distribution of feasible systems on the torus, with which given the same system prices, the shape and size of market areas are identical across the systems. Let $\mathcal{F}_n$ be the set of all symmetric formations given n, and F_n be a generic element of it.

Figure 2 shows two examples of symmetric formations for $n = 8$. Note that all dots at the corners of the square in the left panel represent the same system because $(x_A, x_B) \sim (x_A + 1, x_B) \sim (x_A, x_B + 1)$. By the same token, the dots on the top boundary in the right panel represent the same systems that the corresponding dots on the bottom boundary represent. The shaded areas represent the consumers purchasing one unit of goods from (i, j), i.e., (i, j)'s *market areas.*

Let $N_{(i,j)}$ be the set of the *neighboring systems* of system $(i, j) \in I_C$, where the neighboring systems are defined as follows. In a symmetric equilibrium, $(g, h) \in I_C$ is a neighboring system of (i, j) if there exists a set of consumers who are indifferent between (g, h) and (i, j) and purchase either of the systems. In other words, the neighboring systems are direct competitors who share *market boundaries.* As in the circular city model where a firm has two direct competitors, in this torus city model, a system competes directly against neighboring systems.

To characterize the equilibrium price and profit for any symmetric formation F_n, we introduce a few more definitions. Let $m((i,j),(g,h);F_n)$ denote the length of the equilibrium market boundary between system (i,j) and its neighboring system (g,h), and let $d((i,j),(g,h);F_n)$ be the Euclidean distance between these two systems on the torus. Using these notations, we define the *m-to-d ratio,* $\mu(F_n)$ as

$$\mu(F_n) = \sum_{(g,h)\in N_{(i,j)}} \frac{m((i,j),(g,h);F_n)}{d((i,j),(g,h);F_n)}$$

for an arbitrary $(i,j) \in I_C$, which is the sum of the ratios of two orthogonal segments (see the left panel of Figure 2). The following proposition shows how this ratio relates to the equilibrium prices and profits.

Proposition 2 *Given a symmetric formation F_n, the unique symmetric equilibrium price and the corresponding profit are given by $p^{PC}(F_n) = 2t/[n\mu(F_n)]$ and $\pi^{PC}(F_n) = 2t/[n^2\mu(F_n)]$, respectively.*

Proof. See the Appendix. ■

First of all, Proposition 2 shows that the equilibrium price depends on the location formation of competing systems. More specifically, it shows that the intensity of price competition in the market is completely characterized by and inversely related to the *m-to-d ratio* $\mu(F_n)$ and the number of firms n. Intuitively, the market boundary $m((i,j),(g,h);F_n)$ represents the measure of marginal consumers who would respond to a small price change, meaning that a higher $m((i,j),(g,h);F_n)$ would lead the firms to engage in a more severe price competition. On the other hand, the distance $d((i,j),(g,h);F_n)$ captures the extent of product differentiation between the two competing systems (i,j) and (g,h). The farther the two systems located, the less substitutable they become. Proposition 2 essentially states that the (symmetric) equilibrium profits increase as the location configuration induces a shorter market boundaries and a longer distance from each other. This result generalizes the *principle of maximum differentiation* in one dimensional location model and is consistent with Irmen and Thisse (1998) who show that firms seek for the formation which generates the smallest market boundary.[6]

The result can be illustrated with the examples in Figure 2. The systems in the left panel are distributed more evenly over the space, while those in the right panel are concentrated on two horizontal lines.[7] Thus, we expect that the price competition will be less severe, therefore the equilibrium profit will be higher with the formation on the left panel. Using the proposition, we confirm this intuition; the equilibrium profit associated with the formation on the left panel is $t/112$, which is about 35% higher than $t/152$, the profit with the formation on the right panel.

It is also noteworthy that m-to-d ratio itself has little to do with how big or small n is. That is, because the length of a market boundary is scaled by the associated distance between the systems, the ratio does not necessarily shrink down to zero as n grows to infinity. Instead, the ratio depends crucially on the shape of the market area; when the systems are distributed more evenly over the product space, the shape of each market area becomes more *round*, as we can see again in Figure 2. So, alternatively, we can predict that the equilibrium profit will be higher when the shape of equilibrium market areas is more round.[8] We explore this issue further in Section 4.

3.3 Comparison between the Regimes

In this section, we compare equilibrium profits under pairwise compatibility and full compatibility. Matutes and Regibeau (1988) show that the profits under the two different regimes are the same when $n = 2$.[9] However, we show that the profit under pairwise compatibility is always higher than that under full compatibility when $n \geq 3$.

[6] They state that "the lower the density of marginal consumers, the lower is the elasticity. Accordingly, as the consumer distribution is uniform, demand has minimal elasticity when the corresponding hyperplane has minimal surface area."

[7] Note again that the top and the bottom boundaries represent the same line in the torus.

[8] If, hypothetically, the shape of the market area is completely round (i.e. a disk), its m-to-d ratio would be its circumference over its diameter, which must be approximately 3.141592.

[9] The main analysis of Matutes and Regibeau (1988) focuses on two integrated firms that offer both components, but they point out that the profits are independent of the compatibility regime if the two intergred firms are replaced by four independent single component producers. In section 5, we consider the case of integrated firms that offer both components as in Matutues and Regibeau (1988).

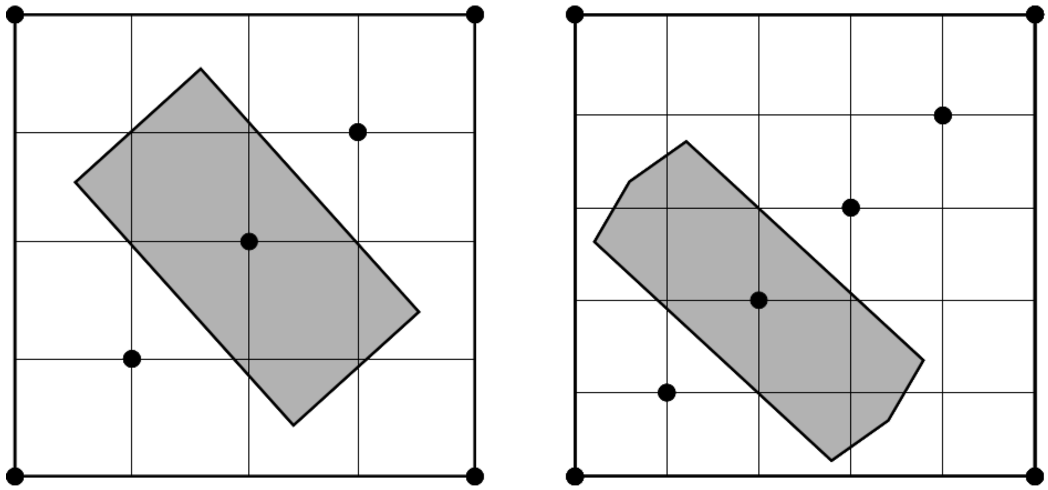

Figure 3: 1-jump formations when $n = 4$ (left) and 5 (right)

To compare the profits under the two different compatibility regimes, we consider a simple symmetric formation under pairwise compatibility, which we call *1-jump formation,* denoted by J_n^1.[10] In this formation, all the systems in I_C equidistantly lie on the 45 degree line. Figure 3 shows two examples of such formations. Notice that with this formation, the equilibrium market area can have only two possible shapes depending on n; a rectangle when n is an even number, and a hexagon when n is an odd number.

Let us first consider the case where n is an even number, so the market area is a rectangle (left panel of Figure 3). Thanks to Proposition 2, we can easily figure our the symmetric equilibrium profits by calculating the m-to-d ratios. The length of a longish market boundary is $\sqrt{2}/2$, and the associated distance between the two neighboring systems is $\sqrt{2}/n$. The ratio is therefore $n/2$. The ratio for a shorter market boundary, on the other hand, is $\left(\sqrt{2}/n\right) / \left(\sqrt{2}/2\right) = 2/n$. Therefore, when n is an even number, the m-to-d ratio of 1-jump formation is

$$\mu(J_n^1 | n \text{ is even}) = \frac{n}{2} + \frac{n}{2} + \frac{2}{n} + \frac{2}{n} = \frac{n^2 + 4}{n}.$$

[10]We explain a more general k-jump formation in the next section.

So, the symmetric equilibrium profit is given by

$$\begin{aligned}\pi^{PC}(J_n^1|n \text{ is even}) &= \frac{2t}{n^2}\left(\frac{n^2+4}{n}\right)^{-1} \\ &= \frac{2t}{n\left(n^2+4\right)} \geq \frac{t}{n^3} = \pi^{FC}(n), \forall n \geq 2.\end{aligned}$$

The inequality above holds as equality only when $n = 2$, which means that the profit equivalence result in Matutes and Regibeau (1988) is an exception rather than a rule.

Next, consider the case where n is an odd number, so the market area is a hexagon (right panel of Figure 3). The length of a longish market boundary is $\sqrt{2}(n^2-1)/2n^2$, and the associated distance is $\sqrt{2}/n$. The length of each of the remaining four market boundaries and the associated distance are $\sqrt{(n^2+1)/2n^4}$ and $\sqrt{(n^2+1)/2n^2}$, respectively. Thus, the m-to-d ratio is

$$\mu(J_n^1|n \text{ is odd}) = 2 \cdot \frac{\sqrt{2}(n^2-1)}{2n^2}\frac{n}{\sqrt{2}} + 4 \cdot \sqrt{\frac{n^2+1}{2n^4}} \cdot \sqrt{\frac{2n^2}{n^2+1}} = \frac{n^2+3}{n},$$

and the equilibrium profit is given by

$$\begin{aligned}\pi^{PC}(J_n^1|n \text{ is odd}) &= \frac{2t}{n^2}\left(\frac{n^2+3}{n}\right)^{-1} \\ &= \frac{2t}{n\left(n^2+3\right)} > \frac{t}{n^3}, \forall n \geq 3.\end{aligned}$$

The discussion in this subsection is summarized in the following proposition.

Proposition 3 *For any $n \geq 3$, there exists at least one symmetric formation under pairwise compatibility, which allows higher equilibrium profits than those under full compatibility.*

We can understand this result through the lens of Proposition 2 which states that the equilibrium profit is inversely related to the m-to-d ratio. Under full compatibility, the competition is at the component level, and firm i has two direct competitors with each of whom the firm shares a market boundary of length 1. Since the distance between two systems is $1/n$, the *m-to-d ratio under full compatibility* is $2n$. Note that this specific m-to-d

ratio can be generated under pairwise compatibility as well by what can be called 0-jump or no jump formation in which all feasible systems are restricted to lie on a horizontal line. However, firms can do better than this under pairwise compatibility by scattering their systems more evenly over the characteristic space. The above considered formation is a configuration which shortens the market boundaries and differentiates the systems more, which in turn softens price competition and increases the profits.

The discussion so far suggests that pairwise compatibility which provides the smallest number of varieties is most likely to be *the profit-maximizing compatibility configuration* among *all* possible compatibility regimes including the ones not considered in this paper. This is because as more systems are added to the market, a system is likely to encounter more directly competing systems at nearer locations. In contrast, full compatibility is *the welfare-maximizing form of compatibility* when the number of firms is fixed. Given the assumption that every consumer purchases a system good, the social welfare is completely determined by the total "transportation costs" which is minimized when the number of available systems is maximized.

4 Free Entry Equilibria

In the previous section, we analyzed the private incentives to achieve compatibility assuming the number of firms in the industry is fixed. In this section, we consider free entry equilibria in which the number of firms is endogenously determined by zero profit condition. Since the equilibrium profit depends on the locational configuration, it is inevitable to search for the profit-maximizing formations for each n. As one can imagine, however, finding the profit-maximizing formation among all symmetric formations is by no means easy. So, instead of exactly characterizing the maximum of equilibrium profits for each n, we construct an upper and a lower bounds of m-to-d ratios for an arbitrary n, and show that for any fixed entry cost, the equilibrium number of firms under pairwise compatibility cannot be smaller than that under full compatibility, but the number of available systems must be smaller under pairwise compatibility.

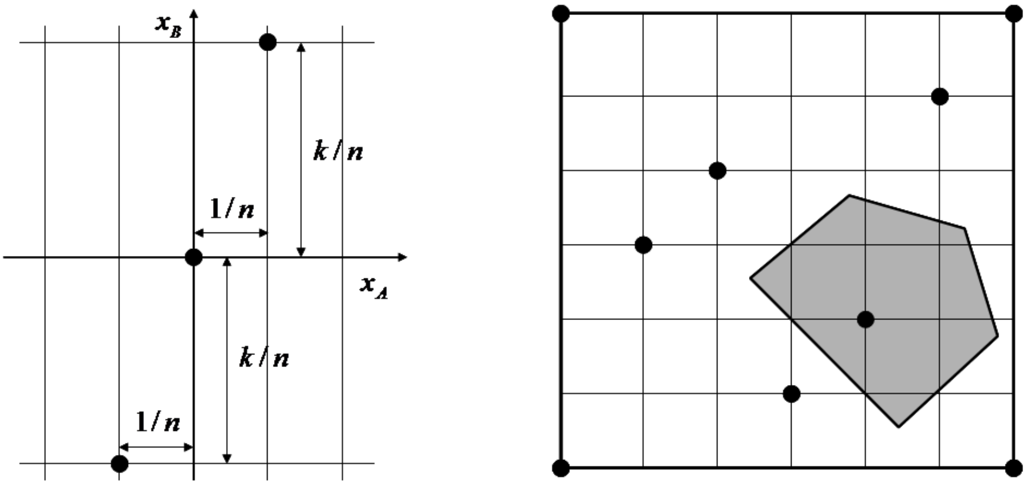

Figure 4: k-jump formation (left); a symmetric but not regularly symmetric formation (right)

To do so, we first introduce a class of symmetric formations that are not only simple and tractable, but also profit-maximizing sometimes. Given n, *k-jump formation,* denoted by J_n^k, is the formation in which every system in I_C lies equidistantly on the line of slope k (and intercept zero) for some integer k (see the left panel of Figure 4).[11] It is noteworthy that all the examples considered before were k-jump formations with some n and k. For instance, the formation depicted on the left panel of Figure 2 is 3-jump formation, while the one on the right panel is 4-jump formation. A k-jump formation is well defined for any n and k, and is tractable because it is repetitive.

Note that there are formations which are symmetric but not k-jump. The right panel of Figure 4 provides an example. As we can see, the system in the shaded area is not located at the center of it. Because it is located more closely to one market boundary than the other, we expect that the profit will be enhanced if the system is moved slightly toward the center of the shaded market area. It means that the formation is not the profit-maximizing one.

[11] When k is a rational number, a formation can be defined in a similar way. We consider rational number k in the proof of Lemma 1. If k is an irrational number, the line with slope of k never goes back to an integer point. Thus, there does not exist a regularly symmetric formation in which systems are located on a line with slope of an irrational number.

To go more into this point, we introduce another definition that helps categorize the shapes of market areas. Suppose all the firms charge the same price while the market is fully covered. A *regularly symmetric formation* is a symmetric formation in which (i) the number of the market boundaries of each system is an even number, and (ii) for any parallel market boundaries of a system, the distances from a market boundary to the system is the same with the distance from the other boundary. In other words, a regularly symmetric formation has each system locate at the center of its market area. Then, it is apparent that the formation depicted in the right panel of Figure 4 is an example of a symmetric but *not* regularly symmetric formation.

In what follows, we restrict our focus on regularly symmetric formations for two reasons. First, because the irregular formations are, as the name suggests, irregular, full characterization of all possible irregular formations is often intractble for large n. More importantly, although it is difficult to show it formally, it is intuitively apparent that if a formation lets each system locate more closely to one market boundary to the others, the formation is not the profit-maximizing one. Therefore, we strongly believe that the set of profit-maximizing formations is a subset of that of regularly symmetric formations. Below, we focus on regularly symmetric formations, and let F_n^* denote the profit-maximizing formation among regularly symmetric ones.

The following lemma states some useful facts about k-jump formations.

Lemma 1 *For any natural number k, the followings are true:*

(i) k-jump formation is regularly symmetric.

(ii) With k-jump formation, the shape of an equilibrium market area is either a rectangle or a hexagon.

(iii) k-jump formation is equivalent to $(an+k)$-jump formation and $(an-k)$-jump formation for any integer a.

(iv) If n is a prime number, for any regularly symmetric formation, there exists a natural number k such that k-jump formation replicates the regularly symmetric formation.

Proof. See the Appendix ■

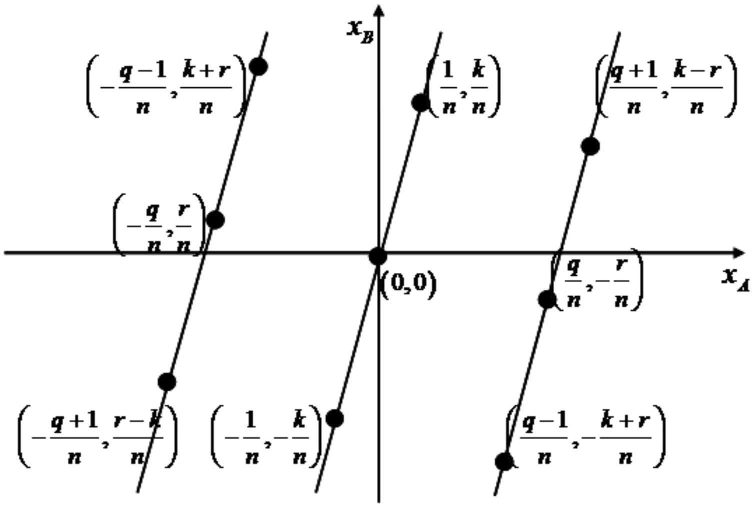

Figure 5: The candidates for neighboring systems of the system at the origin in k-jump formation for $k \geq 2$; q is the quotient of the division of n by k, and r is the remainder, i.e. $n = qk + r$.

Note that there exist regularly symmetric formations that are not k-jump. For instance, consider a formation where each system is located at the nodes of a square grid. This formation is well defined for any square number n, i.e., $n = a^2$ for some integer $a \geq 2$, and is not a k-jump formation because there does not exist a single line on which all feasible systems are located. This, together with the lemma, implies that the set of k-jump formations is a subset of that of regularly symmetric formations in general, and the two sets coincide when n is a prime number. So, k-jump formation does not help us characterize a lower bound of m-to-d ratio, but can still provide an upper bound of it.

Searching for a lower bound, we check m-to-d ratios of polygons, and confirm the intuition developed in the previous section; as the shape of a polygon (i.e. the market area) becomes more round, the induced m-to-d ratio becomes smaller. At the extreme, the m-to-d ratio of a circle is $\pi \approx 3.141592$. However, because identical disks cannot tile a plane without truncations, we find 3.141592 is too loose as a lower bound. It turns out that the polygon which tiles a plane and generates the smallest m-to-d ratio is a regular hexagon, and that when the market is shaped like a honeycomb, the m-to-d ratio is $2\sqrt{3} \approx 3.464$.

Lemma 2 *For any $n \geq 2$, the m-to-d ratio of the profit-maximizing formation $\mu(F_n^*)$ must be in* $[2\sqrt{3},\, 3\sqrt{13})$.

Proof. See the Appendix. ■

We now consider an entry game with the following timing structure. The compatibility regime (full or pairwise) is determined before firms' entry decision, and the potential entrants regard that as a *market norm*. This assumption is introduced so as to compare the long run welfare implications of the two compatibility regimes. Given the compatibility regime, firms decide whether to enter or stay out of the market. Entry entails fixed cost C which is common to both component producers. After the entry decision, which companies entered becomes public knowledge. Knowing the identities of the active firms, they coordinate their positions on the torus in order to maximize their profits. Given the formation, each firm independently sets the price of its component.

Let n_{FC} be the number of firms in each component market in the free-entry equilibrium under full compatibility and n_{PC} be the number under pairwise compatibility. Setting aside the integer constraint, the free-entry conditions can be written as

$$\pi^{FC}(n_{FC}) = C; \quad \pi^{PC}(F^*_{n_{PC}}) = C.$$

For the case of full compatibility, the equilibrium number of firms in each component is given by $n_{FC} = (t/C)^{1/3}$, so the number of feasible systems is $n_{FC}^2 = (t/C)^{2/3}$. Using the free-entry condition $\pi^{PC}(F^*_{n_{PC}}) = C$ and Proposition 2, we derive the following relationship between n_{FC} and n_{PC}.

$$n_{FC} = \left(\frac{t}{C}\right)^{1/3} = \left(\frac{t}{\pi^{PC}(F^*_{n_{PC}})}\right)^{1/3} = \left(\frac{n_{PC}^2\mu(F^*_{n_{PC}})}{2}\right)^{1/3} < n_{PC}^{2/3} \cdot \left(\frac{3\sqrt{13}}{2}\right)^{1/3},$$

where the inequality comes from the upper bound of $\mu(F^*_{n_{PC}})$ in Lemma 2. Therefore, $n_{FC} < n_{PC}$ if $n_{PC}^{2/3} \cdot \left(3\sqrt{13}/2\right)^{1/3} < n_{PC}$, or equivalently

$$n_{PC} > \frac{3\sqrt{13}}{2}.$$

In words, the number of firms under full compatibility is smaller than that under pairwise compatibility when C is small enough that n_{PC} is larger than $3\sqrt{13}/2 \approx 5.408$. In the appendix, we show that the ranking remains for greater C as well.

In a similar manner, the number of systems available to the consumers under the two regimes can be compared as follows:

$$n_{FC}^2 = \left(\frac{n_{PC}^2 \mu(F_{n_{PC}}^*)}{2} \right)^{2/3} \geq n_{PC}^{4/3} \cdot \left(\frac{2\sqrt{3}}{2} \right)^{2/3},$$

where Lemma 2 is used again to derive the inequality above. Since $n_{PC}^{4/3} \cdot \left(2\sqrt{3}/2\right)^{2/3} = n_{PC}^{4/3} \cdot 3^{1/3} > n_{PC}$ for all $n_{PC} \geq 2$, the number of feasible systems under full compatibility is always larger than that under pairwise compatibility even when firms are allowed to enter the market freely. The discussion so far is summarized in the following proposition.

Proposition 4 *For any fixed cost C, a greater variety of systems is provided to the consumers under full compatibility, while more firms enter into the market under pairwise compatibility, i.e. $(n_{FC})^2 > n_{PC} \geq n_{FC}$.*

Proof. See the Appendix. ■

The proposition essentially states that full compatibility yields higher social welfare because full compatibility provides a greater variety to the consumers, while it incurs less entry costs to the entering firms, where the social welfare is defined as

$$\begin{aligned} & (\text{Consumers' aggregate utility}) + 2n \times (\text{An individual firm's profit}) \\ = \; & [v - (\text{Aggregate transportation cost}) - 2p] + 2n \times [p/n - C] \\ = \; & v - (\text{Aggregate transportation cost}) - 2nC. \end{aligned}$$

Therefore, we conclude that the social incentive for full compatibility remains in free-entry equilibrium.

5 Alternative Market Structure with Integrated Producers

In Section 3, we showed that pairwise compatibility yields higher equilibrium profits than full compatibility when each firm produces only one component. In this section, we consider an alternative market structure in which a single firm produces both A and B components, and show that once again incompatibility between components that are produced by different companies induces higher equilibrium profits compared to the case of full compatibility, if the number of firms n is greater than or equal to 4.

Matutes and Regibeau (1988) analyze compatibility incentives for two different market structures. In one case, as in the model considered so far, each component producer is assumed to be an independent entity, and have to form an alliance with a partner company to make their products compatible. In the other case, a component A producer is vertically integrated with a component B producer from the beginning. The integrated system providers set their component prices to maximize the profits from both components, internalizing the effect of a price change in one component on the sales of the other. Incompatibility among different systems prevails if the integrated firms decide to make their component products incompatible with their rivals'. Their model predicts that if the market is served by two vertically integrated producers, the companies would prefer (full) compatibility because the price competition will be less intensive when the firms cannot fully appropriate the benefit of a price cut; under (full) compatibility, a reduction in the price of component A produced by firm 1 will increase the market share not only of the system produced by firm 1 but also of the system of component A produced by firm 1 and component B produced by its rival. In contrast, under incompatibility the benefit of a price cut is fully captured by the system producer, and consequently, the price competition is more severe, and the equilibrium profits are lower.[12]

It turns out however their result is not generalized beyond the case of $n = 2$. Here, we show that the profit ordering is reversed for $n \geq 4$. Under full compatibility, we know

[12] Analyzing the case of four independent component producers, Matutes and Regibeau (1988) state that "with such an industry structure, there is no difference in the degree of internalization of complementarity between compatibility and incompatibility equilibria, since the firms always ignore these effects."

that the equilibrium profit for an integrated firm is twice of the profit for a component producer because the competition is at the component level, and the profit of each company comes from selling two components. So, $\pi^C(n) = 2\pi^{FC}(n)$ where $\pi^C(n)$ is the profit for an integrated firm under compatibility. On the contrary, an integrated firm's profit under incompatibility is the same as a component producer's profit under pairwise compatibility, i.e. $\pi^{IN}(F_n^*) = \pi^{PC}(F_n^*)$ where $\pi^{IN}(F_n^*)$ is the profit for an integrated firm under incompatibility. This can be easily seen by comparing the first order conditions for component producers and integrated producers. The first order conditions are mathematically equivalent, which implies that the equilibrium component price under pairwise compatibility is the same as the equilibrium system price under incompatibility with integrated producers. As a result, the system price under pairwise compatibility is twice as high as the system price under incompatibility with integrated producers. The reason is that under pairwise compatibility each component producer ignores the effect of its price cut on the partner firm's demand, which results in double marginalization of system prices. However, this is beneficial to producers in competitive environments because it relaxes price competition as in Bonanno and Vickers (1988).

Note however the effect of imperfect appropriation prevailing under compatibility is not affected by the number of firms n. This contrasts sharply with the fact that the benefit of reducing m-to-d ratio increases as the number of firms increases. Therefore, for sufficiently large n, the competition-relaxing effect of imperfect appropriation is dominated by the effect of product differentiation, and the firms end up prefering incompatibility which allows greater space for differentiation. Specifically, the ratio of the equilibrium profit under incompatibility to that under compatibility $\pi^{IN}(F_n^*)/\pi^C(n) = \pi^{PC}(F_n^*)/2\pi^{FC}(n)$ is larger than $n/3\sqrt{13}$ according to Lemma 2, which implies that the ratio is greater than 1 for $n > 10$. See also the left panel of Figure 6 which shows that the ratio of the maximum profit attained by k-jump formation to the equilibrium profit under full compatibility is increasing in n. Under the alternative market structure of integrated firms, it turns out that incompatibility induces higher equilibrium profits if $n \geq 4$ even after taking into account

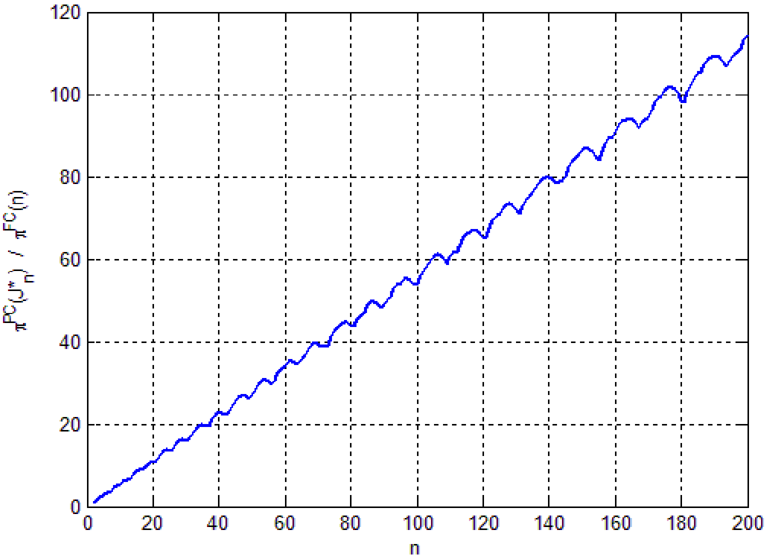

Figure 6: The ratio of the equilibrium profits under pairwise and full compatibility

intensified competition due to vertical integration.[13]

6 Concluding Remarks

In this paper, we examine private and social incentives for compatibility in a two-dimensional spatial competition model of system markets. We show that there is a fundamental discrepancy between private and social incentive towards compatibility among different vendors. Consumers find full compatibility more attractive because it allows more variety of systems, and they can easily find a system close to their ideal specification on average. However, the availability of more variety under full compatibility implies intensified competition and a lower profit for every firm in the market. If the number of firms in each component market is fixed, the firms prefer pairwise compatibility to full compatibility. However,

[13]To complete the statement, we need to consider the case where $4 \leq n \leq 10$, and show that $\pi^{PC}(F_n^*) \geq 2\pi^{FC}(n)$ as follows.

$\pi^{PC}(J_4^2) = t/28 > t/32 = 2\pi^{FC}(4)$; $\pi^{PC}(J_5^2) = t/50 > 2t/125 = 2\pi^{FC}(5)$; $\pi^{PC}(J_6^2) = t/66 > t/108 = 2\pi^{FC}(6)$; $\pi^{PC}(J_7^2) = t/98 > 2t/343 = 2\pi^{FC}(7)$; $\pi^{PC}(J_8^3) = t/112 > t/256 = 2\pi^{FC}(8)$; $\pi^{PC}(J_9^3) = t/144 > 2t/729 = 2\pi^{FC}(9)$; $\pi^{PC}(J_{10}^4) = t/190 > t/500 = 2\pi^{FC}(10)$.

social welfare is higher under full compatibility. We also perform a long-run analysis in which the number of firms is endogenously determined by the zero profit condition. We show that more firms enter under pairwise compatibility due to higher profits compared to full compatibility. Nonetheless, the number of available systems is still higher under full compatibility. This implies that full compatibility is the optimal regime in the long-run as well as in the short run because it provides more variety with less entry fixed costs.

We have not explicitly considered any costs involved in achieving a particular form of compatibility. If there are differences in such costs across regimes, the attractiveness of each regime would change in a predictable way. However, it is not clear *a priori* which type of compatibility would be more costly. In cases where the cost of forming an exclusive coalition (e.g., transaction costs involved in signing a contract) is non-negligibly higher than the cost of establishing standards that would ensure interoperability across all manufacturers, the firms may find full compatibility more attractive. However, in other cases the cost of achieving full compatibility may be higher than that of pairwise compatibility.[14]

One shortcoming of our paper is that we consider only two possible compatibility regimes: pairwise and full compatibility. For the case of pairwise compatibility, we have not considered each firm's incentive to deviate and form another type of coalition. Suppose that firm i in market A made an exclusive coalition with firm j in market B. However, firm i may have incentives to approach h and offer to build a new system (i, h) by making their components compatible. If firm h accepts the offer, now there are two feasible systems in the market whose component A is made by firm i. Even though the firm i may earn a smaller profit from system (i, j), the loss may be more than made up by an additional source of revenue from system (i, h). A full analysis of endogenous formation of coalitions that account for externalities among coaltions in our model would be an important extension.[15]

[14] In the context of exclusive dealing, Besanko and Perry (1994) argues that exclusive dealing– pairwise compatibility in the context of our model–reduces the retailing cost such as the costs of inventory and store space.

[15] See Bloch (1996) and Yi (1997) for such an analysis.

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Appendix

Proof of Proposition 2. We first characterize symmetric first order conditions for maximization in terms of m-to-d ratio. Then we show that the FOC should be satisfied in equilibrium.

Suppose that given a symmetric formation F_n, firm i charges p_i while the other firms charge p. When (g,h) is an element of $N_{(i,j)}$, a consumer located at (x_A, x_B) is indifferent between system (i,j) and (g,h) if

$$t\left(x_A - x_i\right)^2 + t\left(x_B - x_j\right)^2 + p_i + p = t\left(x_A - x_g\right)^2 + t\left(x_B - x_h\right)^2 + 2p$$

or equivalently,

$$2t\left(x_g - x_i\right)x_A + 2t\left(x_h - x_j\right)x_B = p - p_i + t\left(x_g^2 - x_i^2\right) + t(x_h^2 - x_j^2).$$

Note that the above formula describes the market boundary that is orthogonal to the line connecting (i,j) and (g,h). To see how much the market boundary moves in the direction of coordinate A as a response to a price change in p_i, we first fix x_B. Then, it is clear that a marginal price increase in p_i moves the market boundary along coordinate A as much as $\partial x_A/\partial p_i = -1/\left[2t\left(x_g - x_i\right)\right]$. Now, let us define a new coordinate which is orthogonal to the market boundary, and see how much the boundary moves along this new coordinate. Letting $x_{(i,j),(g,h)}$ be the projection of x_A onto the new coordinate, the following is immediate from the Pythagorean theorem.

$$\frac{\partial x_{(i,j),(g,h)}}{\partial x_A} = \frac{x_g - x_i}{\sqrt{\left(x_g - x_i\right)^2 + \left(x_h - x_j\right)^2}} = \frac{x_g - x_i}{d((i,j),(g,h))}$$

Therefore, a marginal price increase in p_i moves the market boundary toward its orthogonal direction by

$$\frac{\partial x_{(i,j),(g,h)}}{\partial p_i} = \frac{\partial x_{(i,j),(g,h)}}{\partial x_A}\frac{\partial x_A}{\partial p_i} = -\frac{1}{2td((i,j),(g,h))}.$$

Next, consider the response of the demand $D_{(i,j)}(p_i, p)$ to a price change Δp_i:

$$\Delta D_{(i,j)}(p_i, p) \approx \sum_{(g,h) \in N_{(i,j)}} \left[m\left(p_i, p; (i,j), (g,h)\right) \cdot \Delta x_{(i,j),(g,h)} + \Delta m\left(p_i, p; (i,j), (g,h)\right) \cdot \Delta x_{(i,j),(g,h)} \right]$$

where $m\left(p_i, p; (i,j), (g,h)\right)$ is the length of the market boundary given the prices (p_i, p), $\Delta x_{(i,j),(g,h)}$ is the extent to which the market boundary moves toward its orthogonal direction, and $\Delta m\left(p_i, p; (i,j), (g,h)\right)$ is the corresponding change in the market boundary. Notice that when the price change is small, the second term in the square bracket is second order, and vanishes faster than the first-order term in the limit. So by dividing by Δp_i and taking limit on both sides, we obtain the following formula.

$$\frac{\partial D_{(i,j)}(p_i, p)}{\partial p_i} = \sum_{(g,h) \in N_{(i,j)}} m\left(p_i, p; (i,j), (g,h)\right) \cdot \frac{\partial x_{(i,j),(g,h)}}{\partial p_i} = - \sum_{(g,h) \in N_{(i,j)}} \frac{m\left(p_i, p; (i,j), (g,h)\right)}{2td((i,j),(g,h))} \tag{1}$$

The first order condition for firm i is

$$p_i \frac{\partial D_{(i,j)}(p_i, p)}{\partial p_i} + D_{(i,j)}(p_i, p) = 0$$

which can be rewritten after imposing symmetry $p_i = p = p^{PC}$ as

$$p^{PC} = - \left. \frac{D_{(i,j)}(p_i, p)}{\partial D_{(i,j)}(p_i, p)/\partial p_i} \right|_{p_i = p = p^{PC}} = \frac{1}{n} \left(\sum_{(g,h) \in N_{(i,j)}} \frac{m((i,j),(g,h))}{2td((i,j),(g,h))} \right)^{-1} = \frac{2t}{n\mu(F_n)}.$$

The corresponding equilibrium profit is $\pi^{PC}(F_n) = 2t/[n^2 \mu(F_n)]$.

We next turn to the issue of equilibrium existence. If firm i sets $p_i = 0$, its demand $D_{(i,j)}(p_i, p)$ must be positive because p cannot be negative. By continuity of the demand function, for small p_i the profit is positive. Once again, by the continuity and boundedness of the demand along with the fact that the profit is zero when $p_i \geq v$, the profit function has non-zero maximum with the argmax $p_i(p) \in (0, v)$. Because all firms' profit functions are identical and continuous, the best response correspondences are continuous and symmetric.

Therefore, there exists at least one pure strategy symmetric equilibrium.

In the remainder of the proof, we show that there does not exist a local maximum at which the profit function is not differentiable. This implies that the strategy profile characterized above is the unique symmetric equilibrium because it is the only symmetric solution of the first order conditions.

To prove the claim, we show that the demand function is convex in p_i. To this end, we first prove that the market area is convex, i.e. if consumers located at (x_A, x_B) and (x'_A, x'_B) purchase system (i, j), then a consumer located at $(\alpha x_A + (1-\alpha)x'_A, \alpha x_B + (1-\alpha)x'_B)$ for any $\alpha \in [0, 1]$ also purchases the same system. Suppose both consumers at (x_A, x_B) and (x'_A, x'_B) buy system (i, j). For all $(g, h) \in I_C \backslash \{(i, j)\}$, the following inequalities are true.

$$\begin{aligned} v - t[(x_A - x_i)^2 + (x_B - x_j)^2] - p_i - p &\geq v - t[(x_A - x_g)^2 + (x_B - x_h)^2] - 2p \\ v - t[(x'_A - x_i)^2 + (x'_B - x_j)^2] - p_i - p &\geq v - t[(x'_A - x_g)^2 + (x'_B - x_h)^2] - 2p \end{aligned}$$

Rearranging terms, the conditions can be rewritten as

$$(p - p_i)/t \geq (x_i - x_g)(2x_A - x_g - x_i) + (x_j - x_h)(2x_B - x_h - x_j)$$

$$(p - p_i)/t \geq (x_i - x_g)(2x'_A - x_g - x_i) + (x_j - x_h)(2x'_B - x_h - x_j).$$

Summing up the above inequalities after multiplying α and $(1 - \alpha)$ respectively, we have

$$(p - p_i)/t \geq (x_i - x_g) \left[2\left(\alpha x_A + (1-\alpha)x'_A\right) - x_g - x_i\right] + (x_j - x_h) \left[2\left(\alpha x_B + (1-\alpha)x'_B\right) - x_h - x_j\right],$$

which implies that (i, j) is the best choice for any consumer located on the segment connecting (x_A, x_B) and (x'_A, x'_B). Therefore, the total length of the market boundaries is monotonically decreasing as p_i increases. This, in turn, implies $\partial D_{(i,j)}(p_i, p)/\partial p_i$ is increasing (see equation (1)) in p_i, i.e. the demand function is convex in p_i. Note that there does not exist a non-differentiable local maximum of $p_i D(p_i)$ if $D(p_i)$ is a convex function. Therefore, in the unique symmetric equilibrium, the first order condition derived above should hold. ■

Proof of Lemma 1. (i) By definition, there is a line of slope k on which all the systems in I_C lie. Let us call the line L. In $\mathbb{R}^2$ representation, infinitely many L's lie equidistantly from each other by the rule of $(x_A, x_B) \sim (x_A + 1, x_B) \sim (x_A, x_B + 1)$. Furthermore, since the systems are distributed regularly on each line, if a system finds a competiting system on one side, it must find another on the opposite side. This means that if we draw a line connecting any two systems and further, we would encounter a third system, and the distance between the first and the second is the same as the one between the second and the third. See Figure 5 for a concrete picture.

(ii) By result (i), the shape of a market area generated by k-jump formation should be either rectangle, hexagon, octagon, or $2a$-gon for $a \geq 5$. It is not possible that a market area is shaped as an octagon. To see this, suppose that there exists (n, k) such that given n, k-jump formation generates an octagon-shaped market area. Since the market area is convex as shown in the proof of Proposition 2, every interior angle of the octagon must be smaller than $180°$. Thus, three or more vertices are required to complete one $360°$. In addition, in tiling the torus ($\mathbb{R}^2$ plane), each vertex of the octagon should participate only once, and all together eight vertices should make three complete $360°$ because the sum of the interior angles is $180 \times (8-2) = 1080 = 360 \times 3$. However, if we distribute the eight vertices into three disjoint sets, there is always a set which has only two or less elements. This contradicts the condition that each interior angle is smaller than $180°$. By the same token, when $a \geq 5$, $2a$ vertices cannot be distributed into $a-1$ disjoint sets without allowing at least one set has two or less elements.

(iii) The equivalence between k-jump and $(an+k)$-jump is immediate from $(x_A, x_B) \sim (x_A, x_B + 1)$. To see the second equivalence, assume system (i, j) is located at the origin. Moving along line of slope k to the right, the first system we encounter is at $(1/n, (an-k)/n)$, and the second one is $(2/n, 2(an-k)/n)$ and so on. Note, however, that by $(x_A, x_B) \sim (x_A, x_B + 1)$,

$$\left(\frac{b}{n}, \frac{b(an-k)}{n}\right) = \left(\frac{b}{n}, ba - \frac{bk}{n}\right) \sim \left(\frac{b}{n}, -\frac{bk}{n}\right)$$

for any integer b. Thus, k-jump formation is also equivalent to $(-k)$-jump formation.

(iv) First, we claim that if n is a prime number, for any regularly symmetric formation, there exists a line on which every system in I_C lies equidistantly. Given the formation, draw a line starting from system (i, j) toward one of its neighboring systems (g, h). By the definition of regular symmetry, the distance from (g, h) to the third system on the line (the one next (g, h)) is the same with the distance between (i, j) and (g, h). Eventually, this line should meet (i, j) again since the number of the firms is finite. To prove the claim by contradition, suppose there are some systems that are not on the line. Since all systems are identical in a regularly symmetric formation, there must exist a parallel line on which the same number of systems are located. Therefore, the number of these parallel lines q is a divisor of n, i.e. $qw = n$ for some natural number $w \neq 1$, the number of systems on each of those lines. This contradicts the assumption that n is a prime number.

By the same token, there does not exist a line that connects only n' systems where $n' < n$. Therefore, no two systems are located on a horizontal line or a vertical line unless every system is located on such a line. In other words, the systems are located at (a, b) where $a, b \in \{1/n, 2/n, \cdots, 1\}$, and for any locations of two different systems (a, b) and (a', b'), $a \neq a'$ and $b \neq b'$. Suppose we draw a line from the origin to a location of a system $(1/n, b)$. Because the formation is regularly symmetric, there must be a system located at $(2/n, 2b)$ and on. It implies the formation is b-jump formation. ■

Proof of Lemma 2. By the definition of regularly symmetic formations, the number of edges of a market area must be an even number. Because we showed in the proof of (ii) in Lemma 1 that identical $2a$-gons cannot tile a plane when $a \geq 4$, a market area generated by a regularly symmetric formation must be shaped either of rectangle or of hexagon.

We show now that a polygon's m-to-d ratio is minimized when it is regular, i.e. when the lengths of its edges are all the same, and so are the interior angles. To see this, let us take a rectangle as an example. Denote the lengths of two different edges by x and y. Then, m-to-d ratio is simply

$$\mu(F_n) = 2\left(x/y + y/x\right),$$

which is minimized when $x = y$. In other words, m-to-d ratio is minimized when the rectangle is a square, and in that case $\mu(F_n^*) = 4$. Consider a hexagon. Since all the edges of the hexagon are symmetric in the objective function (the m-to-d ratio expressed in terms of the lengths of the edges), the first order conditions for minimization yield a symmetric solution. The candidates for the minimum m-to-d ratio which are drawn from the regular polygons are as follows.

$$\mu(F_n^*) \geq \begin{cases} 4 & \text{if } F_n^* \text{ generates rectangles} \\ 2\sqrt{3} \approx 3.464 & \text{if } F_n^* \text{ generates hexagons} \end{cases}$$

Therefore, the m-to-d ratio for the profit-maximizing formation is always larger than or equal to that of a honeycomb, $2\sqrt{3}$.

Next, we characterize an upper bound of $\mu(F_n^*)$ using a k-jump formation. Note that when n is larger than or equal to 4, in any k-jump formation with $k \geq 2$, eight systems nearby the focused system are the potential neighbor systems. (See Figure 5.) Since the number of the market boundaries is four or six by (ii) in Lemma 1, at least two systems turn out to be too far to form the market boundaries.

Let $k_o = \lceil \sqrt{n} \rceil$, the largest integer that is smaller than or equal to $\sqrt{n}$, and consider k_o-jump formation. Suppose that system (i, j) is located at the origin. We then refer to the system at $(1/n, k_o/n)$ as *the first system* and the system at $(2/n, 2k_o/n)$ as *the second system*, and so on. Then, since $k_o^2 \leq n$, or equivalently $(k_o + 1)k_o/n \leq 1 + k_o/n$, for the $(k_o + 1)^{\text{st}}$ system from the origin which is at $((k_o + 1)/n, (k_o + 1)k_o/n)$, the maximum distance from the horizontal axis is k_o/n because $((k_o + 1)/n, 1 + k_o/n) \sim ((k_o + 1)/n, k_o/n)$. Note that potentially $(k_o - 1)^{\text{st}}$, k_o^{th}, and $(k_o + 1)^{\text{st}}$ system can participate to build one side of the market boundaries around (i, j). (See Figure 5. In this case, $q = k_o$ and $r = n - k_o^2$.) Thus, the distance between (i, j) and a neighboring system of (i, j) cannot be larger than $\sqrt{(k_o + 1)^2 + k_o^2}/n$, which is the distance between the origin and $(k_o + 1)^{\text{st}}$ system when $k_o = \sqrt{n}$. At the same time, it cannot be smaller than k_o/n, which is the distance from the origin to k_o^{th} system when $k_o = \sqrt{n}$.

For any rectangle, the longest distance between two edges is equal to the length of the longest edge. For any hexagon, the longest distance between two edges is longer than or equal to the longest edge by the triangle inequality. In other words, every market boundary is shorter than or equal to the longest distance between two neighboring systems. Thus, each ratio of a market boundary to the associated distance must be smaller or equal to $\sqrt{(k_o+1)^2+k_o^2}/n$ over k_o/n:

$$\mu(F_n^*) \leq \mu(J_n^{k_o}) < 6 \cdot \frac{\sqrt{(k_o+1)^2+k_o^2}/n}{k_o/n} = \frac{6\sqrt{2k_o^2+2k_o+1}}{k_o} \leq 3\sqrt{13}$$

where the last inequality comes from the fact that $6\sqrt{2k_o^2+2k_o+1}/k_o$ is decreasing in k_o, and is $3\sqrt{13}$ when $k_o = 2$.

For the cases of $n = 2$ and 3, one can easily confirm that $\mu(J_n^1) = 4 < 3\sqrt{13}$. ■

Proof of Proposition 4. We first figure out the maximized profit $\pi^{PC}(F_n^*)$ for $n \leq 5$. Using (i), (iii) and (iv) in lemma 1, we can figure out the maximized profits by considering k-jump formation for some $k \leq n/2$ when n is a prime number.

When n is 4, there is only one regularly symmteric formation (or its equivalent formations) other than k-jump formations. To see this, consider a regularly symmetric formation where there is no straight line that connects every system. Since the systems are distributed on the torus in a symmetric manner, there must exist two parallel lines on each of which two systems are located equidistantly. Consider a line that intersects with the system at the origin, labelled as (i, j), and another system, labelled as (g, h). The line starting from the origin must go back to a point, say (a, b) where a and b are integers. Then, the location of system (g, h) is $(a/2, b/2)$. By the equivalence relation $(x_A, x_B) \sim (x_A, x_B + 1) \sim (x_A + 1, x_B)$, $(a/2, b/2)$ must be equivalent with either $(0, 1/2)$, $(1/2, 0)$ or $(1/2, 1/2)$.

In short, when $n = 4$, if a regularly symmetric formation is not a k-jump formation, it should be the one where the systems are located on the nodes of a square grid. The m-to-d ratio generated by the formation is 4, so the corresponding equilibrium profit is $2t/(4^2 \times 4) = t/32$ which is smaller than $t/28$, the profit induced by 2-jump formation

$\pi^{PC}(J_4^2)$. Therefore, the maximum profits for $n \leq 5$ are given by

$$\pi^{PC}(F_2^*) = \pi^{PC}(J_2^1) = t/8; \quad \pi^{PC}(F_3^*) = \pi^{PC}(J_3^1) = t/18;$$
$$\pi^{PC}(F_4^*) = \pi^{PC}(J_4^2) = t/28; \quad \pi^{PC}(F_5^*) = \pi^{PC}(J_5^2) = t/50.$$

Since $\pi^{PC}(F_6^*) \geq \pi^{PC}(J_6^2) = t/66$, the fixed cost C considered here is larger than $t/66$. When C is larger than $t/8$, each component market is monopolized. The number of firms entering into the market for given C is summarized as below.

$$\begin{array}{ll}
n_{FC} = n_{PC} = 2 & \text{if } C \in [t/8, t/18) \\
n_{FC} = 2;\ n_{PC} = 3 & \text{if } C \in [t/18, t/27) \\
n_{FC} = n_{PC} = 3 & \text{if } C \in [t/27, t/28) \\
n_{FC} = 3;\ n_{PC} = 4 & \text{if } C \in [t/28, t/50) \\
n_{FC} = 3;\ n_{PC} \geq 5 & \text{if } C \in [t/50, t/64) \\
n_{FC} = 4;\ n_{PC} \geq 5 & \text{if } C \in [t/64, t/66)
\end{array}$$

Together with the discussion in the main text, this table completes the proof. ■