GIFFEN GOODS: A DUALITY THEOREM

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September 2010

Abstract

We show that if two goods whose Indirect Utility Function $V(p,q)$ exhibits the giffen property for good 1 in some subdomain $\mathcal{G}(p,q)$ of the positive quadrant, and if $U(x,y)$ is a Direct Utility Function given by $U(x,y) = -V(x,y)$ and therefore having the same convex contours as V, then U also exhibits the giffen property for good 2 rather than for good 1, in the corresponding region $\mathcal{G}(x,y)$ of the positive (x,y) quadrant. The converse is also true.

1 Introduction

A good is said to exhibit "Giffen behaviour" if its quantity demanded rises in response to an increase in its own price. Finding theoretically plausible direct utility functions that give rise to giffen behaviour is known to be a problematic exercise — see, for example, Wold and Jureen (1953), Spiegel (1994), Weber (1997), Weber (2001), Moffatt (2002), Sørenson (2006). Even if such a direct utility function is found, and is expressible in closed form, proving giffenity analytically is likely to be impossible, because of the difficulties of finding the Marshallian demands. Such problems do not arise when working with the indirect utility function, since Marshallian demands can be found directly via Roy's identity (Moffatt (2010), for example).

In this paper, we prove that an indirect utility function that exhibits giffen characteristics can be converted into a direct utility function that also exhibits giffen characteristics, by a straightforward sign reversal. By this means, we provide a procedure for the construction of a family of direct utility functions which have the giffen property, and for which this property can be analytically proved.

2 Giffenity for the Indirect Utility Function

Consider two goods x and y with unit prices p and q, and normalised budget constraint

$$px + qy \leq 1\,. \tag{1}$$

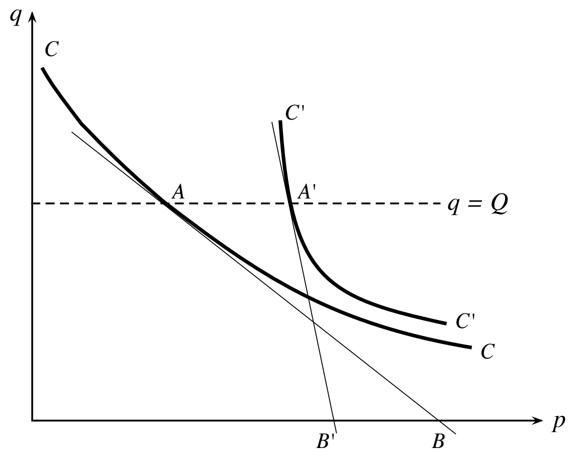

Figure 1: The curves C and C' are contours $V = c$ and $V = c'$ of the indirect utility function $V(p, q)$. The line $q = Q$ intersects C at A, C' at A$'$. AB is tangent to C and A$'$B$'$ is tangent to C'. As A moves to the right towards A$'$, B moves to the left towards B$'$, a symptom of giffenity in good 1.

Let $V(p, q)$ be the indirect utility function (IUF) satisfying the negative gradient conditions

$$V_p \equiv \frac{\partial V}{\partial p} < 0\,, \quad V_q \equiv \frac{\partial V}{\partial q} < 0\,, \tag{2}$$

the contours of V being convex to the origin (see Eq.(6) below). A clear intuitive proof that quasi-convexity is a necessary property of any IUF is provided by Suen (1992).

Let A(P, Q) be a point on the contour $C : V(p, q) = c$, (so that $c = V(P, Q)$) (see Fig. 1). On this contour,

$$dV \equiv V_p dp + V_q dq = 0\,, \tag{3}$$

and so the gradient of the tangent to the contour at A is $m(P, Q)$, where

$$m(p, q) = \left[\frac{dq}{dp}\right]_{V=cst.} = -V_p/V_q\,. \tag{4}$$

Clearly $-\infty < m < 0$. The convexity condition may be expressed by the fact that m increases as the point (p, q) moves along the contour in the direction of increasing p, i.e.

$$\frac{dm}{dp} \equiv \frac{\partial m}{\partial p} + \frac{\partial m}{\partial q}\left[\frac{dq}{dp}\right]_{V=cst} \equiv \frac{\partial m}{\partial p} - \frac{V_p}{V_q}\frac{\partial m}{\partial q} \;>\; 0\,. \tag{5}$$

From (4), this then yields the convexity condition in the form

$$C(p, q) \equiv \frac{2V_pV_qV_{pq} - V_p^2V_{qq} - V_q^2V_{pp}}{V_q^3} \;>\; 0\,. \tag{6}$$

This convexity condition is assumed satisfied at all points of the positive quadrant.

Now the equation of the tangent at A is

$$q - Q = m(P, Q)(p - P)\,, \tag{7}$$

and this intersects the p-axis at the point B$(g, 0)$, where

$$g(P, Q) = P - \frac{Q}{m(P, Q)} = P + Q\frac{V_Q}{V_P} = \frac{PV_P + QV_Q}{V_P},\,. \tag{8}$$

The reciprocal of (8) is, by Roy's Identity, the marshallian demand for good 1. Hence, as we have previously observed (Moffatt, 2010), the condition for giffenity of good 1 at the point A is that $g(P, Q)$ should be a decreasing function of P for constant Q, i.e.that, as A moves to the right across adjacent contours, B should move to the left on the p-axis, as illustrated in Fig. 1. Thus the condition for giffenity for good 1 at the point A is that the function $G_1(P, Q)$ defined by

$$G_1(P, Q) \equiv \frac{\partial g}{\partial P} = \frac{V_P^2 + Q(V_P V_{PQ} - V_Q V_{PP})}{V_P^2}\,, \tag{9}$$

should be negative:

$$G_1(P, Q) < 0\,. \tag{10}$$

We shall suppose that this inequality is satisfied at each point of some 'Giffen-subdomain' $\mathcal{G}(p, q)$ of the positive (p, q) quadrant.

3 Giffenity for a related Direct Utility Function

We may now think of V as being a function of (x, y) rather than (p, q), and define a Direct Utility Function (DUF) $U(x, y)$ by

$$U = -V, \tag{11}$$

so that, by virtue of Eqs. (2) and (11), as required for a DUF,

$$U_x = -V_x > 0\,, \quad U_y = -V_y > 0\,. \tag{12}$$

The relationship (11) implies that U is constant on curves on which V is constant; i.e. the contours of U are the same as the contours of V, and are therefore convex to the origin. We have also that

$$U_{xx} = -V_{xx}, \quad U_{xy} = -V_{xy}, \quad U_{yy} = -V_{yy}\,, \tag{13}$$

and the convexity condition (6) readily translates to

$$C \equiv \frac{2U_x U_y U_{xy} - U_x^2 U_{yy} - U_y^2 U_{xx}}{U_y^3} > 0\,. \tag{14}$$

Similarly, the condition (10) translates to

$$G_1(x,y) = \frac{\partial g}{\partial x} = \frac{{U_x}^2 + y(U_x U_{xy} - U_y U_{xx})}{{U_x}^2} < 0 \quad \text{in } \mathcal{G}(x,y)\,. \tag{15}$$

Both these conditions are properties of the contour map, which remains unchanged.

We may consider again the tangent at the point A(X,Y) of the curve $U(x,y) = \hat{c}\,(= -c)$, meeting the x-axis at the point B$(g,0)$, where now (cf Eq. (8))

$$g = X + Y\frac{U_Y}{U_X}\,. \tag{16}$$

If this tangent is the budget line

$$px + qy = 1\ (= pX + qY)\,, \tag{17}$$

with gradient $m_b = -p/q$, then this gradient is equal to $-U_X/U_Y$ at the point (X,Y), i.e.

$$p\,U_Y - q\,U_X = 0\,. \tag{18}$$

Consider now what happens if we increase the price q of good 2 keeping p fixed; this merely changes the gradient m_b while keeping $g = 1/p$ fixed. Thus

$$dg \equiv \frac{\partial g}{\partial X}dX + \frac{\partial g}{\partial Y}dY = 0\,. \tag{19}$$

Also, from Eq.(17), the point of contact (X,Y) of the budget line varies according to

$$pdX + qdY + Y dq = 0\,. \tag{20}$$

Hence, eliminating dX, Eqs. (19) and (20) give

$$\frac{dY}{dq} = \frac{Y\partial g/\partial X}{p\partial g/\partial Y - q\partial g/\partial X}\,. \tag{21}$$

Now, from Eq.(16), we have

$$\frac{\partial g}{\partial X} = 1 + Y\frac{U_X U_{XY} - U_Y U_{XX}}{U_X^2} = G_1(X,Y)\,, \tag{22}$$

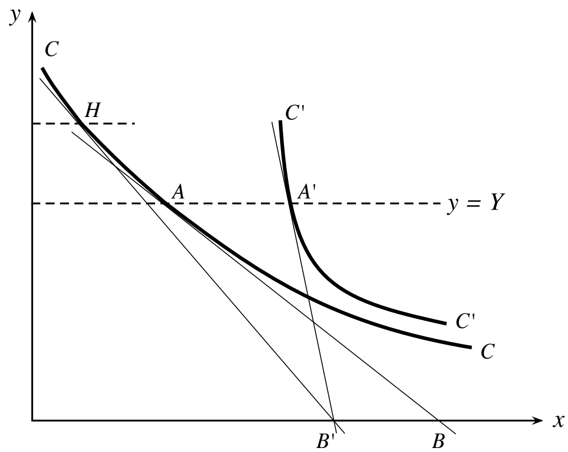

Figure 2: The same as Fig.1, but now in the (x, y)–plane. B′H is now the tangent from B′ to C, and it is evident, as a result of the convexity of C, that H is above A on C, i.e. $y_H > y_A$

and, using Eq. (14),

$$\frac{\partial g}{\partial Y} = \frac{U_Y}{U_X} + Y\frac{U_X U_{YY} - U_Y U_{XY}}{U_X^2} = \frac{U_Y(U_X^2 G_1(X,Y) - YU_Y^2 C(X,Y))}{U_X^3}\,. \tag{23}$$

Thus Eq. (21) gives

$$\frac{dY}{dq} = \frac{YU_X^3 G_1(X,Y)}{-pYU_Y^3 C + U_X^2(pU_Y - qU_X)G_1(X,Y)} = \frac{-U_X^3 G_1(X,Y)}{pU_Y^3 C(X,Y)} = \frac{(m(X,Y))^3 G_1(X,Y)}{pC(X,Y)}\,, \tag{24}$$

using Eq.(18). It follows immediately (using Eq. (14) and $p > 0, m < 0$) that

$$G_1 < 0 \quad \Leftrightarrow \quad dY/dq > 0\,. \tag{25}$$

Thus, under standard utility function conditions, giffenity in the IUF for good 1 in a region $\mathcal{G}(p,q)$ of the (p,q) quadrant is a necessary and sufficient condition for giffenity in the associated DUF (defined by Eq. (11)) for good 2 in the corresponding region $\mathcal{G}(x,y)$ of the (x,y) quadrant. This completes the proof of the theorem stated in the abstract of this paper.

The result has a simple geometrical interpretation – see Fig. 2, which shows two adjacent contours C and C' with tangents AB, A′B′ with $Y_A = Y_{A'}$; the condition $G_1 < 0$ means that B′ is to the left of B, as shown. As B moves to the left along the x-axis towards B′, the point of contact Q, say, of the tangent from B to C moves up the curve from A by virtue of the convexity condition, and reaches the point H when B reaches B′. Hence X_Q decreases and Y_Q increases. When B reaches B′ therefore,

$X_H < X_A$ and $Y_H > Y_A = Y_{A'}$. This geometrical argument, which implicitly uses both conditions $C > 0$ and $G_1 < 0$, indeed confirms giffenity in good 2. The argument is reversible, confirming that the conditions $Y_H > Y_{A'}$ and $C > 0$ together imply that $G_1 < 0$.

4 Example

In Moffatt (2010), we constructed the following IUF and verified that it exhibits a subdomain of giffenity:

$$V(p,q) = \frac{-p - q + 2\lambda(1-\lambda) + S(p,q)}{2(1-\lambda^2)}\,, \quad 0 < \lambda < 1 \tag{26}$$

where

$$S(p,q) = \sqrt{[p + q - 2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[pq - (1-\lambda)^2]}\,. \tag{27}$$

Figure 3 shows an indifference map of the IUF (26) with $\lambda = 0.9$. The key feature of this function is that the indifference curves become more curved as they become further from the origin, and the parameter λ represents the rate at which the curvature increases.

In order to demonstrate that the case depicted in Figure 3 is similar to that of Figure 1, a horizontal line is drawn at $q = 0.13$. Tangents of two adjacent contours at this value of q are drawn. As explained in §2, the fact that these two tangents intersect above the axis $q = 0$ is indicative of giffenity in good 1.

Now consider the direct utility function that is obtained by reverising the sign of (26),

$$U(x,y) = \frac{x + y - 2\lambda(1-\lambda) - S(x,y)}{2(1-\lambda^2)}\,, \quad 0 < \lambda < 1 \tag{28}$$

where

$$S(x,y) = \sqrt{[x + y - 2\lambda(1-\lambda)]^2 - 4(1-\lambda^2)[xy - (1-\lambda)^2]}\,. \tag{29}$$

The indifference map of (28), again with $\lambda = 0.9$, is drawn in Figure 4. Clearly the indifference curves themselves are the same as those shown in Figure 3. Here, we demonstrate that we have a case similar to that depicted in Figure 2 above. Tangents are drawn on two adjacent contours, representing budget lines before and after a rise in the price q of good 2. That the budget line on the left has a higher point of tangency than the one on the right is indicative of giffenity of good 2.

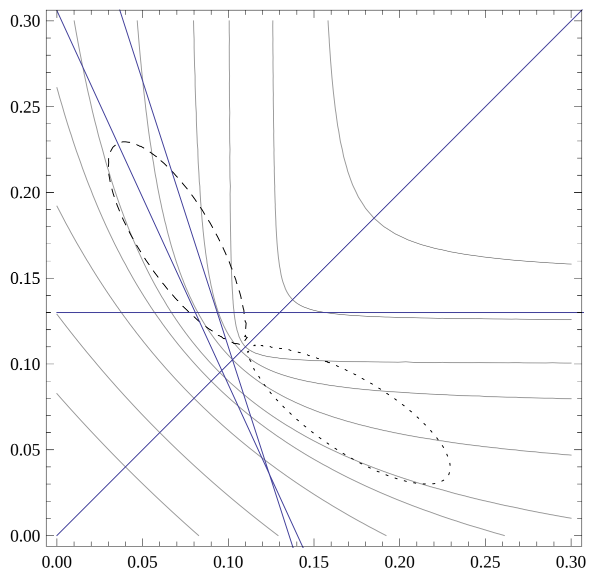

Figure 3: Indifference map of the IUF (26) with $\lambda = 0.9$. A horizontal line is drawn at $q = 0.13$. Tangents of two adjacent contours at this value of q are drawn. These two tangents intersect above the axis $q = 0$, indicating giffenity in good 1. The region surrounded by the dashed loop is the region of giffenity for good 1; the region surrounded by the dotted loop is the same for good 2.

Note that, in agreement with the theorem stated in the abstract and proved in section 3, there is a duality between the giffenity of good 1 demonstrated in the context of Figure 3, and the giffenity of good 2 seen in the corresponding region of Figure 4.

Finally, note that the regions of giffenity are shown in figures 3 and 4. Those surrounded by a dashed loop are the regions of giffenity for good 1; those with a dotted loop are the same for good 2. Since the utility functions (26) and (28) are symmetric in the two arguments, it is of course the case that if one good is giffen, the other must also be giffen at a different set of price combinations.

5 Conclusion

The central feature of this paper is the proof of the result that if (and only if) a two-good IUF exibits giffen behaviour in one of the two goods, a direct utility function that exhibits giffen behaviour in the *other* good may be obtained by simply reversing the sign of the IUF.

One reason for the practical usefulness of this result is that it is not generally possible to derive Marshallian demands explicitly from a direct utility function. The theorem may be invoked by reversing the sign of the direct utility function and then applying Roy's identity. If the resulting Marshallian demands exhibit giffenity, it is verified that

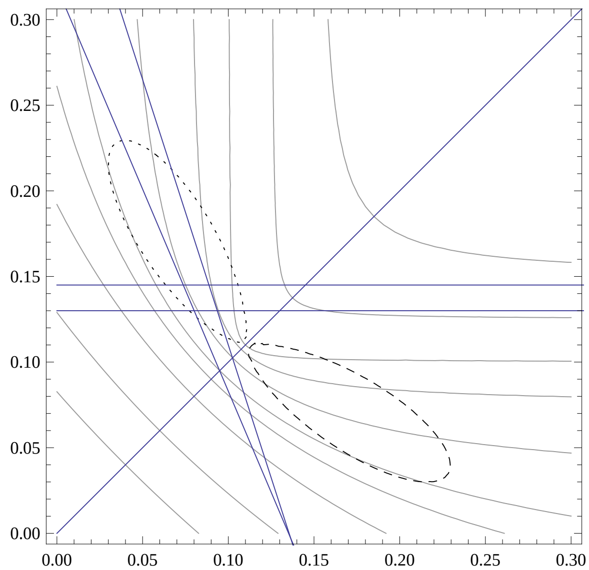

Figure 4: Indifference map of the Direct Utility Function (28) with $\lambda = 0.9$. Tangents are drawn on two adjacent contours, representing budget lines before and after a rise in the price q of good 2. The vertical positions of the points of tangency are represented by the two horizontal lines. The budget line on the left gives rise to a higher point of tangency than the one on the right, indicating giffenity of good 2. The region surrounded by the dotted loop is the region of giffenity for good 2; the region surrounded by the dashed loop is the same for good 1.

the Marshallian demands that *would have* been derived from the direct utility function, had it been possible to do so, would have exhibited giffenity.

One final point of qualification is that for the above reasoning to be applied, it would be necessary for the direct utility function to be quasi-concave, since there is a logical requirement for the IUF to be quasi-convex. Under some circumstances, for example under mixture aversion (see (Butler and Moffatt, 2000)), the assumption of quasi-concavity of the direct utility function is deliberately relaxed, and giffen behaviour is a consequence. Under such circumstances, the theorem established in this paper may not be invoked.

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