

# Co-dependence of Extreme Events in High Frequency FX Returns

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## Abstract

In this paper, we investigate extreme events in high frequency, multivariate FX returns within a purposely built framework. We generalize univariate tests and concepts to multidimensional settings and employ these novel techniques for parametric and nonparametric analysis. In particular, we investigate and quantify the co-dependence of cross-sectional and intertemporal extreme events. We find evidence of the cubic law of extreme returns, their increasing and asymmetric dependence and of the scaling property of extreme risk in joint symmetric tails.

**Keywords:** High Frequency Returns, Distributional Characteristics, Multidimensional Risk, Dependence in Risk, Extreme Risk Assessment, Multidimensional Value at Risk.

**JEL classification:** C14; C51.

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## 1. Introduction

Elliptical distribution of asset returns is a convenient assumption in finance that allows for numerous applications including risk management, asset and option pricing and portfolio decisions. In particular, the standard practice with many risky assets is to assume that the density is multinormal with perhaps a time-varying covariance matrix (see Diebold et al., 1999). However, Leon et al. (2009), *inter alia*, find that joint normality is not supported by empirical evidence. Moreover, correlation – the standard measure of dependence in the multivariate context – is only a measure of linear dependence and suffers from a number of limitations (see Embrechts et al., 2002). For example, Patton (2004) argues that the dependence between assets is stronger during market downturns than during market upturns and here we find that the probability of extreme co-events varies over time. These deficiencies are compounded in the covariance measure – an essential input in many financial applications including hedging, portfolio selection and systematic risk.

While there is a natural interest in the joint density of asset returns, in some cases a more focused approach is required. Since the density's interior characterizes small day-to-day disturbances, it may be of substantially less concern to financial institutions, managers and regulators than the tail behaviour. For this reason, attention has recently shifted to exceedence measures that account for the expected magnitude of large movements in the underlying variables of interest such as stock returns, interest or exchange rates and changes in energy prices and GDP (see, for example, Longin and Solnik, 2001; Butler and Joaquin, 2002; Bae et al., 2003). In particular, Berkowitz (2001) proposes a censored likelihood test, in which the observations not falling into the negative tail of the distribution

are truncated. While in the univariate setting, the tails of financial processes have been studied at length (see, for example, Adler et al., 1998), the literature on multivariate tail analysis studies is still in its infancy. Employing results from the multivariate Extreme Value Theory (EVT), Hsing et al. (2004) fit specific copulas on the bivariate densities (see also Jondeau and Rockinger 2006; Brodin and Kluppelberg, 2010; Ning, 2010). Then, employing coefficients inherent in these copulas, they examine the dependence between returns in the tails. However, recent attempts to generalise this framework to the multivariate case turned out to be technically and computationally demanding (see, for example, Aas et al., 2009; see also Diebold, et al., 2000 for some commonsense caveats to uncritical use of EVT).

In the theoretical part of this study, we develop novel techniques tailored to multidimensional data. The proposed techniques employ a natural generalization of the VaR concept. Multidimensional Value at Risk (MVaR) is the region of the intersection of univariate VaRs with a nominal probability mass under a given density function. MVaR is defined by a single cut-off value and a directional vector. Despite its conceptual simplicity, MVaR is a versatile framework that allows for simple testing of the tails as well as the overall accuracy of a multidimensional density forecast (MDF). Moreover, it is straightforward in this framework to examine extreme co-events and to evaluate dependence in risk.

In the empirical part of this paper, we illustrate the forecast evaluation technique by investigating two of the most widely used elliptical distribution functions to approximate

the empirical joint distribution of (high frequency) FX returns overall and in the tails. Using a rich set of synchronized returns, we examine further the co-dependence among FX returns in the tails. We analyse and quantify both, cross-sectional and intertemporal dependence of the extreme returns. Our empirical study covers several return frequencies providing a rich picture of FX (extreme) returns. Further, to the best of our knowledge, this is the first study to investigate and present clear evidence of the scaling law for multivariate extreme returns. This is important because the tails of a fat-tailed distribution are invariant under addition although the distribution as a whole may vary according to temporal aggregation (Feller 1971). For instance, if daily returns are *i.i.d.* t-distributed, then the central limit law implies that weekly returns converge to the normal distribution. However, the tails of the weekly return distribution behave like the tails of the daily returns with the same tail index.

The outline of the remainder of this paper is as follows. In Section 2, we discuss the concept of MVaR, its accuracy evaluation and dependence in risk. This section contains also a short discussion of the policy and practical significance of the MVaR framework. Section 3 summarises the high frequency FX dataset employed and presents the results of our empirical studies. Finally, Section 4 summarises the main findings and offers some concluding remarks.

## **2. Theoretical Framework**

In this section, we develop a formal framework for investigating distributional characteristics and co-dependence of multidimensional variables.

### 2.1. Joint Density Tails

In analogy to univariate density tails, a joint density tail (JDT) is defined as an unbounded region of the Euclidean space that is marked off by cut-off values. A parsimonious definition of the JDT  $O(d, \nu)$  in the  $N$ -dimensional space requires only one cut-off value  $\nu \in R$  and a directional vector  $d \in R^N$ ,

$$O(d, \nu) := \{y \in R^N : y_i / d_i \geq \nu, \forall d_i \neq 0\} \quad (1)$$

Figure 1 illustrates a JDT in the two-dimensional space.

[Figure 1]

It follows directly from the definition (1) that  $O(d, \nu)$  is an intersection of univariate tails,

$$O(d, \nu) = \bigcap_{i: d_i \neq 0} O(d_i u^i, \nu)$$

where  $u^i$  is a vector in the canonical basis of the Euclidean space that points in direction  $i = 1, \dots, N$  and  $O(d_i u^i, \nu)$  is a half-hyperspace in  $R^N$ .

For an  $N$ -dimensional probability density function (PDF)  $f$ , the probability mass of the tail  $O(d, \nu)$  under  $f$  is computed as,

$$\begin{aligned}
z^d(v, f) &= \int_{O(d, v)} f(\tau) d\tau \\
&= \left| \int_{d_N v}^{d_N^\infty} \dots \int_{d_k v}^{d_k^\infty} \dots \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f(\tau_1, \dots, \tau_N) d\tau_1 \dots d\tau_N \right|
\end{aligned} \tag{2}$$

where we first integrate over the entire real line for each of the ordered variables  $\tau_1, \dots, \tau_{k-1}$  with  $d_1 = \dots = d_{k-1} = 0$ .<sup>1</sup> Given the parametric line  $v \cdot d$  in  $R^N$  in direction  $d \in R^N$ , the following projection  $x^d$  of the point  $x \in R^N$  on  $v \cdot d$  is defined as,

$$x^d = v^d(x) \cdot d, \text{ where } v^d(x) = \min_{d_i \neq 0} \{x_i / d_i\} \tag{3}$$

Importantly for our purposes, the projection  $x^d$  of a point  $x$ , that lies inside (outside) of the JDT  $O(d, v)$ ,  $d \neq 0$ , and  $v \in R$ , remains inside (outside) this JDT as shown in Figure 2. Formally,

$$x \in O(d, v) \Leftrightarrow x^d \in O(d, v) \tag{4}$$

[Figure 2]

A proof of property (4) is included in the Appendix. As we will see in what follows, the above property of projection (3) allows for a systematic accuracy evaluation of MDFs over the tails and also overall.

Note that there is an infinite number of directional vectors  $d$  and hence MVaRs. Statistically, there may be no reason to prefer one over the others. Economically, however, the directional vector implicitly defines the proportion of the assets in the portfolio. For example, the symmetric directional vector  $d = -(1, \dots, 1)$  captures negative events for a portfolio with equally weighted assets. This vector is of main relevance for an investor holding such a portfolio.

## 2.2. *Multidimensional Value at Risk*

The (unidimensional) Value at Risk (VaR) is now one of the most widely used measures of tail risk among practitioners, largely due to its adoption by the Basel Committee on Banking Regulation (1996) for the assessment of the risk of the proprietary trading books of banks and its use in setting risk capital requirements (see Gouriéroux and Jasiak, 2010). For the unidimensional continuous CDF  $F_t$  (PDF  $f_t$ ), the VaR at the coverage level  $1 - \alpha$  is the quantile  $v_\alpha$  for which  $F_t(v_\alpha) = \alpha$ . From the VaR definition follows that the probability mass under  $f_t$  of the interval  $\{y \in R : y \leq v_\alpha\}$  is equal to the nominal level  $\alpha$ .

In analogy to VaR, the multidimensional Value at Risk in direction  $d$  at the nominal level  $\alpha$  ( $MVaR_\alpha^d$ ) is defined as the cut-off value  $v^d(f, \alpha) \in R$  such that the probability mass (2) under  $f$  of the JDT  $O(d, v^d(f, \alpha))$  is equal to the nominal level  $\alpha$ . We will refer to  $v^d(f, \alpha)$  as the  $MVaR_\alpha^d$ -value. As this value uniquely defines the corresponding JDT, we shall often interpret  $MVaR_\alpha^d$  as a subset of  $R^N$ . The boundary of the latter subset in direction  $i = 1, \dots, N$ , where  $d_i \neq 0$ , is defined by the value  $v^d(f, \alpha) \cdot d_i$ . Then,  $x \in R^N$

is an extreme observation, when  $x$  exceeds (violates or falls into)  $MVaR_\alpha^d$ , i.e. when  $x \in MVaR_\alpha^d$ . The following equivalent condition in terms of the projection (3) is proven in the Appendix,

$$x \in MVaR_\alpha^d \Leftrightarrow \nu^d(x) \geq \nu^d(f, \alpha) \quad (5)$$

Due to the decomposition of JDTs into univariate tails,  $MVaR_\alpha^d$  can be seen as an intersection of univariate VaRs as illustrated in Figure 3.

[Figure 3]

In practice, the relevant MVaR inference can be obtained from the  $z_t^d$ -scores. Specifically, from (5) follows that for a continuous PDF  $f$ , a directional vector  $d \in R^N$ ,  $d \neq 0$ , a nominal significance level  $\alpha \in (0,1)$  and an observation  $x \in R^N$ ,

$$x \in MVaR_\alpha^d \Leftrightarrow z^d(\nu^d(x), f) \leq \alpha \quad (6)$$

where  $z_t^d$ -score is the probability mass (2) under  $f$  of the corresponding JDT  $O(d, \nu^d(x))$ . Therefore, we can identify observations lying in the extreme tails by simply inspecting their z-scores.



An important application of this result is the assessment of (time-varying)  $MVaR_\alpha^d$ , computed from density forecasts  $\{\widehat{f}_{t-1}\}_{t=1}^T$ , based on a sequence of multidimensional observations  $\{x_t\}_{t=1}^T$ . Under the correct forecasting model, the proportion of the corresponding  $z_t^d$ -scores with values less than  $\alpha$  should approach the nominal significance level  $\alpha$  for a sufficiently large sample. We refer to this procedure as unconditional accuracy. On the other hand, the conditional accuracy requires that the  $MVaR_\alpha^d$  violations should be serially uncorrelated. To assess both types of accuracy, we resort to the unconditional accuracy test of Kupiec (1995) and the conditional accuracy test of Christoffersen (1998). Although both tests are designed for testing the (univariate) VaR accuracy, they still apply for our purposes because the score computation effectively converts a MDF into a univariate score variable.

Moreover, given a series of MDF forecasts  $\{\widehat{f}_{t-1}\}_{t=1}^T$  and a series of extreme events, i.e., observations that exceed  $MVaR_\alpha^d$ , the proposition below (proven in the Appendix), allows for testing the hypothesis that  $\{\widehat{f}_{t-1}\}_{t=1}^T$  are correct forecasts over this tail.

*Proposition 1:*

If the sequence of MDFs  $\{\widehat{f}_{t-1}\}_{t=1}^T$  is the true DGP for the sequence of observations  $\{x_t\}_{t=1}^T$ ,  $x_t = (x_{1,t}, \dots, x_{N,t}) \in MVaR_\alpha^d$ , then the sequence of normalized z-scores  $\{z_t^d / \alpha\}_{t=1}^T$ , where  $z_t^d = z^d(v^d(x_t), \widehat{f}_{t-1})$ , is *i.i.d.*  $U[0,1]$ .

This procedure effectively transforms a MDF  $\hat{f}_{t-1}$  over the MVaR into a unidimensional variable. The null hypothesis of the tail accuracy of a density model can, then, be tested by the standard tests of uniformity (see Noceti et al., 2003) and independence (see Brock et al., 1991). Note that by setting  $\alpha=1$ ,  $MVaR_\alpha^d$  covers the entire domain of the density, which allows for the overall accuracy testing of the MDF (for further insights into density forecast evaluation see the survey of Corradi and Swanson, 2006).

### 2.3. *Dependence in Risk*

Poon et al. (2004) and Hartmann et al. (2010) employ tail dependence coefficients to examine extreme events in the equity and FX markets, respectively. However, these coefficients, while theoretically robust, measure only asymptotic dependence between marginals of a joint distribution (see Heffernan, 2001 for a directory of tail dependence coefficients). By contrast, the dependence measures, which we propose below as natural extensions of the MVaR framework, can capture dependence between more complicated events (e.g., between two disjoint sets of marginals) with vanishing or non-vanishing probabilities.

The first one is the relative log difference in the probability of the  $MVaR_\alpha^d$ -event, given that  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$  has occurred. Specifically, for the multidimensional random variable  $R$  with the joint PDF  $f$ ,

$$p_{\tilde{\alpha}}^\alpha(f, d, \tilde{d}) := \Pr_f(R \in MVaR_\alpha^d \mid R \in MVaR_{\tilde{\alpha}}^{\tilde{d}}) = \frac{\Pr_f(MVaR_\alpha^d \cap MVaR_{\tilde{\alpha}}^{\tilde{d}})}{\Pr_f(MVaR_{\tilde{\alpha}}^{\tilde{d}})}$$

is the conditional probability of the  $MVaR_\alpha^d$ -event, given the occurrence of  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$ . By the definition of statistical independence, it holds in the special case  $\alpha = \tilde{\alpha}$  that,

$$p_\alpha^\alpha(f, d, \tilde{d}) = \Pr_f(MVaR_\alpha^d) = \alpha$$

when the events  $MVaR_\alpha^d$  and  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$  are independent. Therefore, for  $\alpha > 0$ , we can express the degree of risk dependence between these events by the relative log difference of the nominal and conditional probability,

$$\gamma_\alpha(f, d, \tilde{d}) = \gamma_\alpha(f, \tilde{d}, d) := \frac{\ln \alpha - \ln p_\alpha^\alpha(f, d, \tilde{d})}{\ln \alpha + \ln p_\alpha^\alpha(f, d, \tilde{d})} \quad (7)$$

This measure is normalized to lie in  $[-1, 1]$  and its positive (negative) values indicate that the occurrence of  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$  increases (decreases) the probability of  $MVaR_\alpha^d$ . In particular,  $p_\alpha^\alpha(f, d, \tilde{d}) = 1$  (0) and  $\gamma_\alpha(f, d, \tilde{d}) = 1$  (-1) for perfect positive (negative) dependence, while  $p_\alpha^\alpha(f, d, \tilde{d}) = \alpha$  and  $\gamma_\alpha(f, d, \tilde{d}) = 0$  when  $MVaR_\alpha^d$  and  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$  are independent<sup>2</sup>. Similar dependence measures are discussed, e.g., in Coles et al. (1999), Hartmann et al. (2004) and Dias and Embrechts (2010).

The second risk dependence measure, conditional MVaR (CMVaR), is similar to CoVaR in Adrian and Brunnermeier (2009) and is defined as the relative change in the  $MVaR_\alpha^d$ -value when conditioned on the  $MVaR_\alpha^{\tilde{d}}$ -event,

$$CMVaR_\alpha^{d,\tilde{d}} = (\nu^d(f | MVaR_\alpha^{\tilde{d}}, \alpha) - \nu^d(f, \alpha)) / |\nu^d(f, \alpha)| \quad (8)$$

where  $\nu^d(f, \alpha) \neq 0$  is the unconditional MVaR value computed with respect to the PDF  $f$  while  $\nu^d(f | MVaR_\alpha^{\tilde{d}}, \alpha)$  is the conditional MVaR value computed with respect to the conditional PDF  $f | MVaR_\alpha^{\tilde{d}}$ , i.e., with respect to the (normalized) density  $f$  over  $MVaR_\alpha^{\tilde{d}}$ . This measure indicates the relative change in the  $MVaR_\alpha^d$ -value, when conditioned on the occurrence of  $MVaR_\alpha^{\tilde{d}}$ . If the latter event has no impact on  $MVaR_\alpha^d$ , then  $CMVaR_\alpha^{d,\tilde{d}}$  is equal to zero. On the other hand, positive (negative) values of  $CMVaR_\alpha^{d,\tilde{d}}$  indicate the magnitude, by which conditioning increases (decreases) the conditioned extreme event  $MVaR_\alpha^{\tilde{d}}$  with nominal probability  $\alpha$ .

We note here that measure (8) is also useful for examining systemic risk and contagion. For example, if  $\nu^d(f, \alpha)$  measures the unconditional risk of a system, then  $CMVaR_\alpha^{d,\tilde{d}}$  may capture the exposure of this system to an institution, represented by  $\tilde{d}$ , experiencing the extreme event  $MVaR_\alpha^{\tilde{d}}$ .

The risk dependence measures can be computed either from a theoretical density function  $f$  or from observations that define the empirical distribution  $f_E$ . It turns out that the inference is simplified in the latter case. First, multidimensional integration in the calculation of the  $MVaR_\alpha^d$ -value from the PDF  $f$  is replaced by a simpler task of computing the  $(1-\alpha)$ -quantile for projections (3) of  $f_E$  on the directed line  $v \cdot d$ .<sup>3</sup> For example, in order to compute  $\Pr_{f_E}(R \in MVaR_\alpha^d \mid R \in MVaR_{\tilde{\alpha}}^{\tilde{d}})$ , we first select the  $MVaR_\alpha^d$  ( $MVaR_{\tilde{\alpha}}^{\tilde{d}}$ ) to contain the proportion  $\alpha$  ( $\tilde{\alpha}$ ) of observations in  $f_E$  with the largest projections on the line  $v \cdot d$  ( $v \cdot \tilde{d}$ ). Then, we compute the empirical conditional probability from the number of observations in the intersection  $MVaR_\alpha^d \cap MVaR_{\tilde{\alpha}}^{\tilde{d}}$  over the number of observations in  $MVaR_{\tilde{\alpha}}^{\tilde{d}}$ . We can compute  $CMVaR_\alpha^{d,\tilde{d}}$  in a similar manner. Importantly, these computations can be performed efficiently in higher dimensions and for large samples. Therefore,  $\gamma_\alpha(f, d, \tilde{d})$  and  $CMVaR_\alpha^{d,\tilde{d}}$  are convenient and flexible non-parametric tools for analyzing dependence in multidimensional, high frequency data.

#### 2.4. Some Applications of MVaR Framework

The recent financial crisis brought to the forefront of attention systemic risk. Due to the interconnectivity of the financial institutions, a shock faced by one institution in the form of an extreme event, increases the probability other financial institutions experiencing similar extreme events, (see Nijskens and Wagner, 2011). Recently, there has been increasing concern among researchers, practitioners and regulators over the evaluation of models of (financial) risk (see, for example, the report on Global Risks 2012 by the World Economic Forum). Moreover, while it is important to have an aggregate measure

of the total risk, often it is also important to know the direct dependence on, and inter-relationships of, the specific sources of risk. These developments accentuate the need for modelling and evaluation techniques that are flexible and yet powerful (Lopez and Saidenberg, 2000). However, while the literature on aggregating the multiple sources of risk is gaining momentum (see, for example, Rosenberg and Schuermann, 2006), there appears to be little research into the joint distribution and evaluation of such risks. By focusing on the joint distribution, the MVaR framework measures not only the risk inherent to each source but also the co-dependence of these risks.

Important innovations in the derivatives markets include basket and rainbow options whose payoffs depend on the value of a basket of assets. Pricing basket options is difficult as the underlying portfolio is a weighted sum of the constituent asset prices. There are basically two approaches to address this issue. The first is by modeling the (unidimensional) distribution of the basket value (e.g., Borovkova et al., 2007). The second approach, which is perhaps more intuitive, is by focusing directly on the joint density of the basket's constituent assets. For example, Huang and Guo (2009) price basket (Bermudan) options as a function of the value of the option in each state of the basket's constituent assets times the joint probability of the assets being in that state. They estimate the joint probability of each state using copulas. By offering a simple and versatile method for an efficient evaluation of joint density estimates, MVaR framework is readily adapted to assess risk-neutral probabilities that can then be used for multidimensional option pricing.

### 3. Empirical Results

In this section, we present the results of our empirical studies that employ the techniques discussed in Section 2. The data was provided by Bank of America and covers the period 2 January 2001 to 29 December, 2006. It consists of synchronized 1-minute exchange rates for three currencies, EUR, GBP and CHF against the USD (a total of 1,006,544 observations). The FX market operates continuously from 10.00pm GMT on Sunday to 10.00pm GMT on Friday, so there are a total of 1440 observations in each 24-hour window while the market is open.

Summary statistics for the three log returns at 8 and 64 minute frequencies are reported in Table 1. For all three series, daily log returns are leptokurtic but the departure from the normal kurtosis diminishes for lower frequencies. The ARCH(4) test for up to fourth order serial correlation in squared returns shows that all three currency return series display significant volatility clustering. All mean returns are close to zero and there is a strong positive correlation between each pairs of the series that increases as the frequency decreases.<sup>4</sup>

[Table 1 and Figure 4]

In the first part of the experiment, we test the accuracy of two parametric distributions – the multinormal (MN) and the multivariate-t distribution (MT) – over multivariate tails. Both specifications are time-varying. The details of the dynamic estimation of the parametric distributions via a multivariate GARCH model are given in the Appendix.

While the covariance matrix for these distributions was forecasted one step-ahead by MV-GARCH, the parameters of the MV-GARCH model and the degrees of freedom of the MT distribution for the different return frequencies were ML-estimated from the entire sample, given the sample statistics. We investigate the multivariate tails in directions proportional to the standard deviations  $\sigma_{\epsilon}$ ,  $\sigma_{\xi}$  and  $\sigma_F$  of the respective sample returns. For the directional vectors  $(\sigma_{\epsilon}, \sigma_{\xi}, \sigma_F)$  and  $-(\sigma_{\epsilon}, \sigma_{\xi}, \sigma_F)$ , we will refer to the corresponding JDTs as symmetric tails, while  $(\sigma_{\epsilon}, \sigma_{\xi}, -\sigma_F)$  defines an asymmetric tail. Generally, symmetric tails correspond to price movements in the same direction, while asymmetric tails involve at least one pair of variables with movements in opposite directions.

First, we focus on the goodness-of-fit test of the MN and MT specifications over JDTs that are defined by the selected directional vectors and nominal probabilities. To this end, we compute from the observations in the relevant tails the normalized z-scores and the corresponding p-values of the uniformity test. The results of the experiment for returns at 8 minutes frequency are reported in Table 2. Similar results were obtained at other frequencies, for which a sufficient number of tail observations was available. While we strongly reject multinormality for all tested JDTs, there is some support for the multivariate-t distribution in symmetric tails with probability mass 5 percent or less. As all p-values for these tails are larger than 5 percent, we would not reject the null that the corresponding tail observations were drawn from the multivariate-t at this significance level.



[Table 2]

Regarding the MVaR accuracy of the two models, recall that  $MVaR_\alpha^d$  is defined by a real number  $\nu^d(f, \alpha)$ , such that the probability mass (2) under  $f$  over the JDT  $O(d, \nu^d(f, \alpha))$  is equal to the nominal significance level  $\alpha$ . The unconditional and conditional accuracy of  $MVaR_\alpha^d$  is measured by the frequency and serial independence, respectively, of observations that fall inside this tail. The corresponding Kupiec and Christoffersen tests, reported in Table 2, imply that the MN distribution is rejected (un)conditionally for all nominal levels (tails). In particular, the observed frequency of observations in the positive symmetric one percent JDT is 2.5 percent, which implies that the empirical distributions have considerably thicker symmetric tails than the multinormal. For the MT, on the other hand, we obtain, generally, a fairly accurate fit in the symmetric tails for  $\alpha$  less than 10 percent, which is reflected in the large values of the Kupiec statistic. However, the conditional accuracy is rejected at all levels  $\alpha$ , indicating a significant serial correlation of extreme co-events in spite of the MV-GARCH modelling. As the overall MDF accuracy necessarily requires accuracy in the tails, we may also conclude that both parametric models do not perform well in approximating the density over the entire domain.

The (cumulative) distribution of asset returns has frequently been assumed to be Gaussian, Lévy or a truncated Lévy distribution, where the tails become “approximately exponential”. In contrast, Gopikrishnan et al. (1998) find that the asset return distribution exhibit a strong cubic power-law behaviour which differs from all three previous models:

unlike the Gaussian or the truncated Lévy distribution, it has diverging higher moments, and unlike the Gaussian or Lévy it is not a stable distribution. Our results in Table 2 together with our estimate of 2.75 for the degrees of freedom (i.e., relatively thick tails), can be seen as a generalization of the cubic law for extreme returns to the multivariate context. However, our tentative evidence of the cubic law is limited to symmetric tails with probability mass not exceeding 5 percent.

Turning now to the MVaR risk dependence, we compute the tail correlation (see Longin and Solnik, 2001), the risk dependence coefficient  $\gamma_\alpha(f, d, \tilde{d})$  and the CMVaRs for different nominal levels  $\alpha$  and directional vectors  $d$  and  $\tilde{d}$ , where the latter vectors are proportional to the relevant standard deviations. Table 3 presents the results for the cross-sectional dependence between the EUR/USD and GBP/USD exchange rates,<sup>5</sup> while Table 4 presents the results for two consecutive returns in EUR/USD and hence, can be interpreted as capturing the inter-temporal dependence.

[Tables 3 and 4]

In Table 3, we observe that the co-dependence between EUR/USD and GBP/USD decreases in  $\alpha$ , at least in the symmetric tails. This trend is present in both, the dependence coefficients and the CMVaRs. Therefore, the more extreme the return in one currency, the stronger its impact on the return in the other currency. Moreover, the reported CMVaRs suggest that the extreme GBP/USD returns have a stronger impact on the EUR/USD returns than vice versa (although we cannot measure causality in our

framework). Interestingly, the co-dependence seems to become weaker for higher frequencies. On the other hand, the dependence is negative and increases in  $\alpha$  in the mixed tail (defined by the vectors  $d = (\sigma_{\epsilon}, 0)$  and  $\tilde{d} = (0, -\sigma_{\epsilon})$ ). Hence, a positive (negative) extreme event in GBP/USD decreases the probability of a negative (positive) extreme event in EUR/USD. Furthermore, we did not observe any consistent patterns for the tail correlations, except in the positive symmetric tail, when it decreased in  $\alpha$ . This finding further highlights the shortcomings of correlation as a dependence measure.

For the intertemporal dependence for EUR/USD exchange rate, i.e., the co-dependence between two consecutive EUR/USD returns<sup>6</sup>, the co-dependence in Table 4 appears to decrease in  $\alpha$  and it is stronger for higher frequencies. The overall significant intertemporal dependence is a hallmark of volatility clustering and indicates that clustering becomes stronger for more extreme returns.

The patterns emerging from Tables 3 and 4 have been confirmed for other bi-dimensional tails and frequencies. It appears that the relationship between the CMVaR and the risk dependence coefficient on the one hand, and alphas on the other, is relatively consistent and stable across the three tails and in both cross-section and intertemporal frameworks.

Figure 5 illustrates succinctly the co-dependence between the pairs of exchange rates at 5 percent nominal significance level. It shows, in particular, that the co-dependence, as measured by  $\gamma_{\alpha}(f, d, \tilde{d})$ , is strongest between EUR/USD and CHF/USD although it is also pronounced for the other pairs.

Whilst the latter measure does not depend, by definition, on the order of directional vectors, i.e.,  $\gamma_\alpha(f, d, \tilde{d}) = \gamma_\alpha(f_E, \tilde{d}, d)$ , the conditional MVaRs may depend on this order. Indeed, different conditioning variables appear to lead to considerable asymmetries of extreme events. For example, the fact that the negative 5 percent EUR/USD (CHF/USD) MVaR has been exceeded increases the (conditional) MVaR for CHF/USD (EUR/USD) by 176 percent (187 percent). Note further that Figure 5 does not indicate any significant asymmetries between the positive and the negative tails.

[Figure 5]

Finally, we investigate the scaling properties of MVaR. The scaling law relates the MVaR at frequency  $\Delta t$  to the size of the time interval  $\Delta t$ ,

$$MVaR(\Delta t) \approx MVaR(1) \cdot (\Delta t)^\kappa$$

where  $\kappa$  is the scaling exponent. Figure 6 shows a log-log plot of MVaR estimates for the three exchange rates against the frequency (in minutes) and the fitted straight lines. The fits appear to be highly accurate in the symmetric tails (for the MVaRs in directions  $(\sigma_\epsilon, \sigma_\xi, \sigma_F)$  and  $-(\sigma_\epsilon, \sigma_\xi, \sigma_F)$ ) but not satisfactory in the asymmetric tails (directions  $(\sigma_\epsilon, -\sigma_\xi, \sigma_F)$  and  $(-\sigma_\epsilon, \sigma_\xi, -\sigma_F)$ ). The scaling exponents that we observed were not  $1/2$ , as implied by the Brownian motion, but consistently larger. Moreover, these differ

markedly from the estimates of around 0.42 in Hauksson et al. (2001) for the univariate VaRs.

[Figure 6]

As the exchange rates are strongly correlated at all frequencies, the results might appear to follow from the scaling of (univariate) volatility (Andersen et al., 2001). In order to verify the robustness of the scaling law, we plotted in Figure 7 the MVaR estimates for the two (uncorrelated) consecutive EUR/USD returns and fitted straight lines through the data points. Again, we found an accurate fit in the symmetric tail  $(\sigma_t, \sigma_{t+1})$  that deteriorates considerably in the asymmetric tail  $(\sigma_t, -\sigma_{t+1})$ . Therefore, we may conclude that the behaviour of asymmetric tails is significantly different from that of the symmetric tails and hence, considerable care should be exerted when employed e.g., in hedging.

[Figure 7]

The practical implications of these findings seem to be that, at least for symmetric tails, we can estimate the multidimensional risk at high frequencies and then, scale this estimate up to match the desired return time interval, which is generally daily, weekly or monthly.

#### 4. Conclusion

We develop a formal framework for investigating the distributional characteristics of multivariate variables with a particular focus on the tails. We extend important

unidimensional risk concepts to the multivariate settings and employ these to examine risk dependence for a rich set of high frequency FX data.

Our investigation into the tails of high frequency multidimensional returns reveals interesting phenomena, such as asymmetry of dependence in the positive and negative tails, cubic law of extreme multidimensional returns and risk scaling in the symmetric tails. Generally, MVaR framework seems to be a versatile approach, which could be applied also in portfolio decisions and to study systemic risk. We intend to pursue these avenues in future research.

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## Appendix

- *Proof of property (4):*  $x \in O(d, v) \Leftrightarrow x^d \in O(d, v)$

$$\begin{aligned} \Rightarrow: \quad & x \in O(d, v) \Rightarrow x_i / d_i \geq v \quad \forall d_i \neq 0 \Rightarrow \min_{i: d_i \neq 0} \{x_i / d_i\} = v^d(x) \geq v \\ & \Rightarrow v^d(x) d_i / d_i = x_i^d / d_i \geq v \quad \forall d_i \neq 0 \Rightarrow x^d \in O(d, v). \end{aligned}$$

$$\begin{aligned} \Leftarrow: \quad & x^d \in O(d, v) \Rightarrow x_i^d / d_i \geq v \Rightarrow v^d(x) d_i / d_i \geq v \quad \forall d_i \neq 0 \\ & \Rightarrow \min_{i: d_i \neq 0} \{x_i / d_i\} \geq v \Rightarrow x_i / d_i \geq v \quad \forall i: d_i \neq 0 \Rightarrow x \in O(d, v). \end{aligned}$$

- *Proof of property (5):*  $x \in MVaR_\alpha^d \Leftrightarrow v^d(x) \geq v^d(f, \alpha)$

$$\begin{aligned} \Rightarrow: \quad & x \in MVaR_\alpha^d \Rightarrow x_i / d_i \geq v^d(f, \alpha) \quad \forall i: d_i \neq 0 \Rightarrow \min_{i: d_i \neq 0} \{x_i / d_i\} = v^d(x) \geq v^d(f, \alpha) \\ \Leftarrow: \quad & v^d(x) = \min_{i: d_i \neq 0} \{x_i / d_i\} \geq v^d(f, \alpha) \Rightarrow x_i / d_i \geq v^d(f, \alpha) \quad \forall i: d_i \neq 0 \Rightarrow x \in MVaR_\alpha^d \end{aligned}$$

- *Proof of Proposition 1:*

From property (5), definition (2) and the fact that  $\Pr_f(MVaR_\alpha^d) = \alpha$ , it follows for  $\alpha > 0$ ,

$$v^d(x) \geq v^d(f, \alpha) \Leftrightarrow z^d(v^d(x), f) \leq z^d(v^d(f, \alpha)) = \alpha.$$

The last property and the property (5) imply,

$$\Pr_f(z^d(v^d(X), f) \leq \alpha) = \Pr_f(v^d(X) \geq v^d(f, \alpha)) = \Pr_f(MVaR_\alpha^d) = \alpha,$$

which defines the z-score  $z^d(v^d(X), f)$  as the  $U[0,1]$  random variable. The uniformity implies that the random variable  $z^d(v^d(X), f)$ , when conditioned on  $z^d(v^d(X), f) \leq \alpha$ , is  $U[0, \alpha]$ . Then, the normalized variable  $z^d(v^d(X), f) / \alpha$ , which is conditioned on  $z^d(v^d(X), f) \leq \alpha$ , is again  $U[0,1]$ .

Finally, the uniformity of the normalized z-score  $z_t^d / \alpha$  holds for any date t and density forecast  $\hat{f}_{t-1}$ . Hence, it is independent of the distribution of the normalized z-scores at any other date  $s \neq t$ .

- *Multivariate GARCH model and Estimation Method*

To obtain forecasts of the time-varying three-dimensional covariance matrix we employ the simplified GARCH (S-GARCH) model of Harris et al. (2007). Our choice can be explained on the basis of the ability of this model to handle t-distributed residuals and its ease of estimation. However, we experimented with other, widely used multivariate GARCH models such as BEKK and DCC but the estimation did not always converge.

For our parametric specifications (MN and MT distribution), covariance matrix together with the degrees of freedom and means fully define the PDF. The S-GARCH involves the estimation of only univariate GARCH models, both for the individual return series and for the sum and difference of each pair of currencies. The covariance between each pair of return series is then imputed from these conditional variance estimates. First, the conditional variances are estimated using univariate GARCH(1, 1) models:

$$r_{i,t} = \mu_i + \varepsilon_{i,t}, \quad i = GBP, EUR, CHF \quad (A1)$$

$$\sigma_{ii,t} = \alpha_{ii,0} + \alpha_{ii,1}\sigma_{ii,t-1} + \alpha_{ii,2}\varepsilon_{i,t-1}^2, \quad i = GBP, EUR, CHF \quad (A2)$$

Then six auxiliary variables,  $r_{+,t} = r_{i,t} + r_{j,t}$  and  $r_{-,t} = r_{i,t} - r_{j,t}$ ,  $i, j = GBP, EUR, CHF, i \neq j$  are constructed, and univariate GARCH(1, 1) models used to estimate the conditional variances of these.

$$r_{l,t} = \mu_l + \varepsilon_{l,t}, \quad l = +, - \quad (A3)$$

$$\sigma_{ll,t} = \alpha_{ll,0} + \alpha_{ll,1}\sigma_{ll,t-1} + \alpha_{ll,2}\varepsilon_{l,t-1}^2, \quad l = +, - \quad (A4)$$

where the residuals  $\varepsilon_i$   $i = GBP, EUR, CHF, +, -,$  are either normal or t-distributed. The conditional covariance between each pair of currencies is then imputed using the identity

$$\sigma_{ij,t} \equiv (1/4)(\sigma_{+,t}^2 - \sigma_{-,t}^2) \quad (A5)$$

Like many other multivariate GARCH models, the S-GARCH does not guarantee that the conditional variance-covariance matrix is positive semi-definite. However, for all three pairs, the estimated correlation coefficients were found to be between  $-1$  and  $+1$  for all observations. We ML-estimate the S-GARCH parameters (and the degrees of freedom parameter for the t-distributed residuals) using the entire sample. We then use these estimates to obtain the one step-ahead forecast of the covariance matrix.

**Table 1: Summary Statistics for the Synchronized Returns**

	<b>EUR/USD</b>	<b>GBP/USD</b>	<b>CHF/USD</b>
<b>Mean (%)</b>	0.000/0.000	0.000/0.000	0.000/0.000
<b>Stand Dev</b>	0.044/0.122	0.041/0.113	0.049/0.135
<b>Skewness</b>	0.166/0.005	0.382/0.175	0.331/-0.094
<b>Kurtosis</b>	43.307/12.235	24.048/8.330	25.971/9.474
<b>p-val. (B-J)</b>	0/0	0/0	0/0
<b>ARCH(4)</b>	3547.641/992.510	4517/364/1073.646	4982.815/1298.691
<b>Correlation</b>			
	<b>EUR/USD</b>	<b>GBP/USD</b>	<b>CHF/USD</b>
<b>EUR/USD</b>	1.000	0.686/0.766	0.839/0.922
<b>GBP/USD</b>		1.000	0.634/0.745
<b>CHF/USD</b>			1.000

Notes: The table reports the mean, standard deviation, skewness, excess kurtosis, Bera-Jarque statistic and the correlation matrix for the synchronized log returns for EUR/USD GBP/USD and CHF/USD exchange rates at 8 and 64 minutes frequency, for the sample period from 2 January 2004 to 6 September 2006 ( $1006544/2^k$  observations at  $2^k$ -minutes frequency)

**Table 2: Directional accuracy of MN and MT distributions**

Nominal Probability	$d = -(\sigma_{\epsilon}, \sigma_{\xi}, \sigma_F)$	$d = (\sigma_{\epsilon}, \sigma_{\xi}, \sigma_F)$	$d = (\sigma_{\epsilon}, \sigma_{\xi}, -\sigma_F)$
<b>MN</b>			
$\alpha = 0.01$	0 / .023 (0/0)	0 / .025 (0/0)	0 / .008 (0/0)
$\alpha = 0.025$	0 / .032 (0/0)	0 / .036 (0/0)	0 / .029 (0/0)
$\alpha = 0.05$	0 / .042 (0/0)	0 / .045 (0/0)	0 / .081 (0/0)
$\alpha = 0.10$	0 / .077 (0/0)	0 / .077 (0/0)	0 / .167 (0/0)
<b>MT</b>			
$\alpha = 0.01$	.104 / .009 (.09/0)	.172 / .15 (.58/0)	.016 / .007 (0/0.2)
$\alpha = 0.025$	.126 / .025 (.31/0)	.131 / .025 (.47/0)	.007 / .029 (0/0)
$\alpha = 0.05$	.067 / .049 (.05/0)	.053 / .047 (0/0)	.002 / .055 (0/0)
$\alpha = 0.10$	0 / .087 (0/0)	0 / .087 (0/0)	0 / .013 (0/0)

Notes: The table reports the p-values of the  $\chi^2$ -test of uniformity for the normalized z-scores, the exception rates (i.e., the proportion of times the forecasted MVaR is exceeded) and the p-values of the Kupiec's statistic/Christoffersen likelihood ratio test statistic (in parentheses) for the multinormal (MN) and the multivariate-t distributions (MT). The p-values were computed based on log returns at 8 minutes frequency for EUR/USD GBP/USD and CHF/USD exchange rates. Data was obtained from Bank of America from 02 January 2004 to 06 September 2006 (1006544/ $2^k$  observations at  $2^k$ -minutes frequency). Mean and covariance matrix were computed from the respective samples. The degrees of freedom for MT, 2.75 at 8 minutes frequency, were ML estimated, given the empirical means and covariances.

**Table 3: Cross-sectional Risk Dependence between EUR/USD and GBP/USD**

Nominal Probability	$d = (\sigma_{\epsilon}, 0)$ $\tilde{d} = (0, \sigma_{\xi})$			$d = (-\sigma_{\epsilon}, 0)$ $\tilde{d} = (0, -\sigma_{\xi})$			$d = (\sigma_{\epsilon}, 0)$ $\tilde{d} = (0, -\sigma_{\xi})$		
	JDT correl.	Dep. Coeff.	CMVaR	JDT correl.	Dep. Coeff.	CMVaR	JDT correl.	Dep. Coeff.	CMVaR
<b>Frequency: 8 minutes</b>									
$\alpha = 0.01$	0.79	0.70	3.49/3.29	0.55	0.69	3.33/2.31	0.53	-0.11	-0.655/-0.748
$\alpha = 0.025$	0.76	0.66	2.25/1.85	0.58	0.64	2.18/1.93	-0.21	-0.25	-0.726/-0.932
$\alpha = 0.05$	0.73	0.61	1.87/1.67	0.59	0.61	1.82/1.69	-0.11	-0.28	-0.872/-0.932
$\alpha = 0.10$	0.70	0.56	1.75/1.53	0.57	0.55	1.73/1.55	-0.27	-0.32	-0.919/-0.921
<b>Frequency: 64 minutes</b>									
$\alpha = 0.01$	0.77	0.76	2.78/2.55	0.56	0.76	2.16/1.97	NA	NA	NA
$\alpha = 0.025$	0.76	0.73	2.11/1.85	0.66	0.72	1.94/1.75	0.43	-0.24	-0.76/-1.02
$\alpha = 0.05$	0.76	0.68	2.05/1.87	0.63	0.66	2.01/1.66	0.43	-0.37	-0.90/-1.12
$\alpha = 0.10$	0.75	0.63	1.92/1.63	0.66	0.63	1.85/1.67	-0.25	-0.39	-1.13/-1.23

Notes: The table reports tail correlation, dependence coefficient (7) and conditional MVaRs  $CMVaR_{\alpha}^{d, \tilde{d}} / CMVaR_{\alpha}^{\tilde{d}, d}$  (8) for nominal significance levels 1 percent, 2.5 percent, 5 percent, 10 percent and different directional vectors.

**Sample** consists of synchronized log returns for EUR/USD and GBP/USD exchange rates. Observations were obtained from Bank of America for the period from 02 January 2004 to 06 September 2006 (1006544/2<sup>k</sup> observations at 2<sup>k</sup>-minutes frequency).

**Table 4: Intertemporal Risk Dependence for EUR/USD**

Nominal Probability	$d = (\sigma_t, 0)$ $\tilde{d} = (0, \sigma_{t+1})$			$d = (-\sigma_t, 0)$ $\tilde{d} = (0, -\sigma_{t+1})$			$d = (\sigma_t, 0)$ $\tilde{d} = (0, -\sigma_{t+1})$		
	JDT correl.	Dep. Coeff.	CMVaR	JDT correl.	Dep. Coeff.	CMVaR	JDT correl.	Dep. Coeff.	CMVaR
<b>Frequency: 8 minutes</b>									
$\alpha = 0.01$	0.24	0.29	1.19/1.31	0.23	0.28	1.24/1.31	-0.09	0.23	1.664/1.006
$\alpha = 0.025$	0.09	0.21	0.91/0.87	0.21	0.22	0.83/0.85	-0.13	0.20	0.805/0.633
$\alpha = 0.05$	0.15	0.16	0.57/0.65	0.25	0.16	0.52/0.60	-0.16	0.18	0.622/0.478
$\alpha = 0.10$	0.17	0.09	0.37/0.45	0.18	0.09	0.37/0.41	-0.19	0.14	0.563/0.490
<b>Frequency: 64 minutes</b>									
$\alpha = 0.01$	-0.57	0.17	0.69/0.60	0.51	0.14	1.07/0.31	-0.87	0.11	1.12/2.04
$\alpha = 0.025$	0.13	0.17	0.53/0.47	-0.01	0.14	0.48/0.71	-0.91	0.11	0.36/0.36
$\alpha = 0.05$	0.01	0.20	0.73/0.67	0.01	0.16	0.51/0.50	-0.55	0.13	0.43/0.37
$\alpha = 0.10$	0.04	0.13	0.49/0.57	0.09	0.12	0.43/0.46	-0.46	0.10	0.41/0.30

Notes: The table reports tail correlation, dependence coefficient (7) and conditional MVaRs  $CMVaR_{\alpha}^{d,\tilde{d}} / CMVaR_{\alpha}^{\tilde{d},d}$  (8) for nominal significance levels 1 percent, 2.5 percent, 5 percent, 10 percent and different directional vectors.

**Sample:** Each bivariate data point consists of log returns for EUR/USD exchange rate at time t and at time t+1. Observations were obtained from Bank of America for the period from 02 January 2004 to 06 September 2006 (1006544/2<sup>k</sup> observations at 2<sup>k</sup>-minutes frequency).



Figure 1: Directed line  $v \cdot d$  and a JDT  $O(d, v_\alpha)$  in  $\mathbb{R}^2$ .

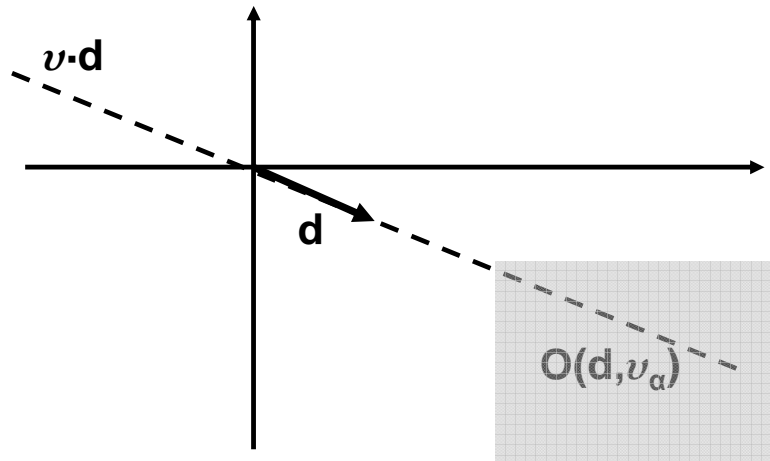
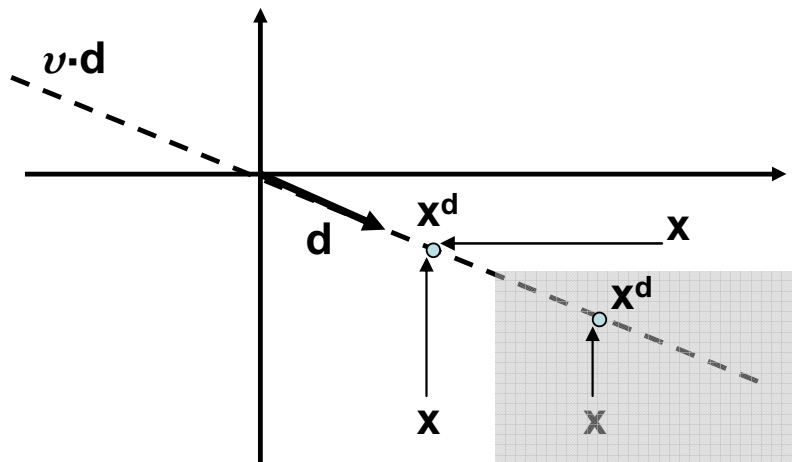
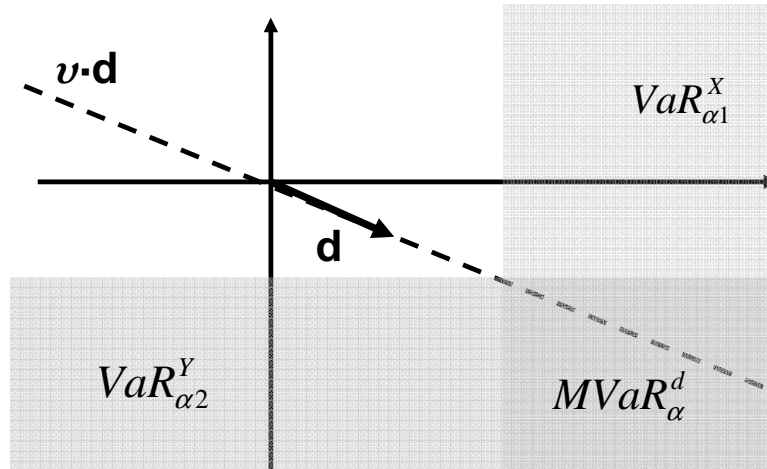


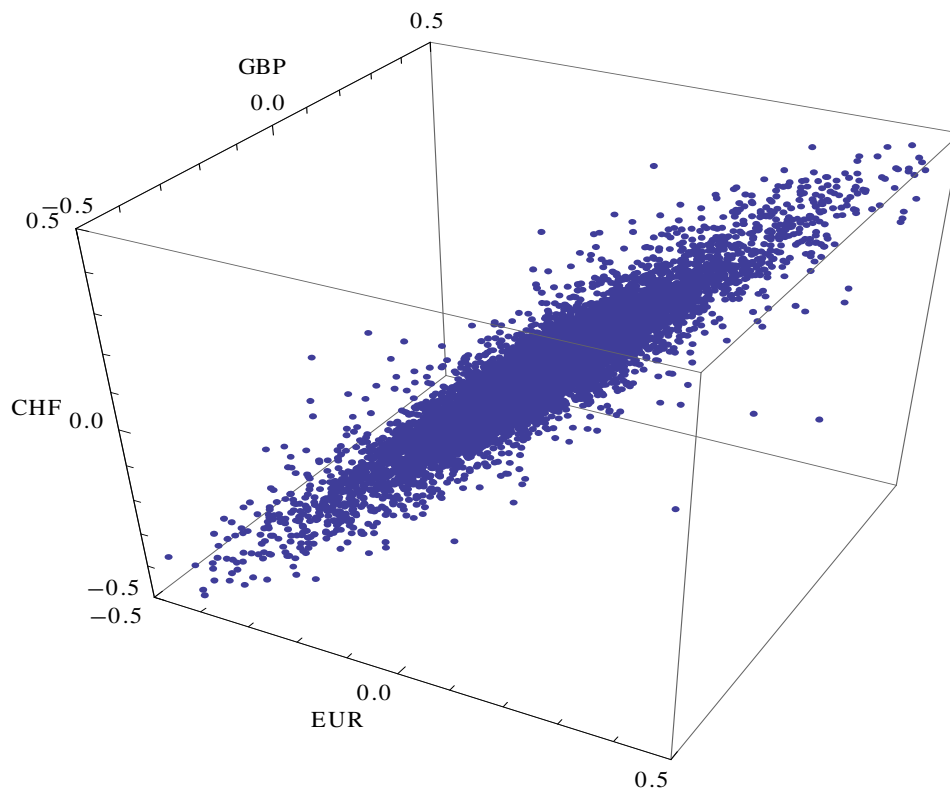
Figure 2: Projections on the directed line  $v \cdot d$  in  $\mathbb{R}^2$ .



**Figure 3:**  $MVaR_{\alpha}^d$  as an intersection of unidimensional VaRs.

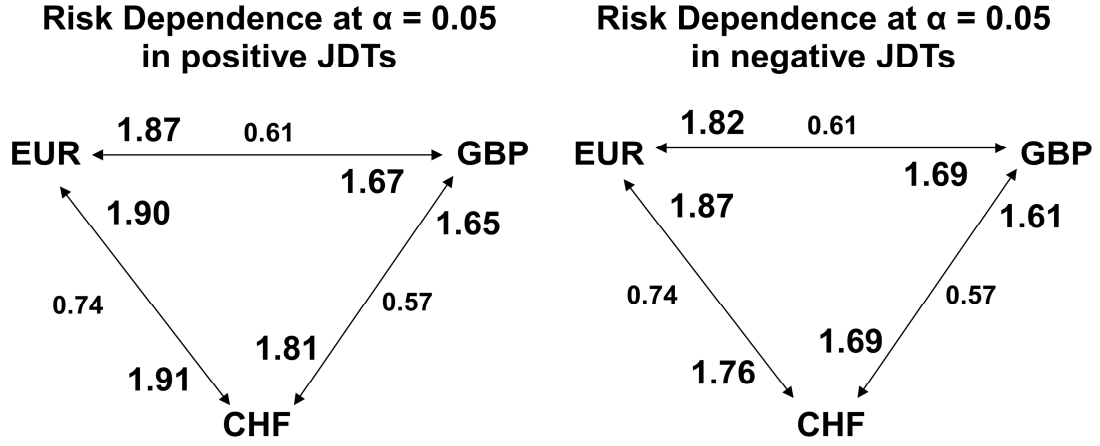


**Figure 4:** Joint log returns at 64 minutes frequency



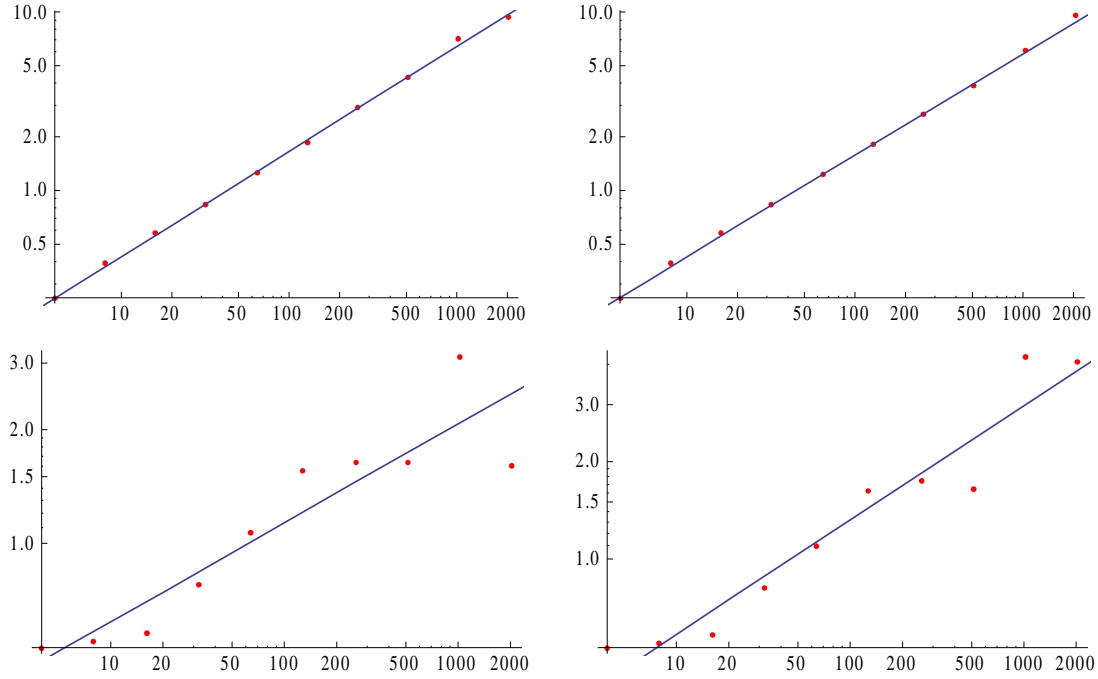
Notes: Joint log returns for EUR/USD, GBP/USD and CHF/USD exchange rates at 64 minute frequency for the sample period from 2 January 2004 to 6 September 2006 ( $1006544/2^k$  observations at  $2^k$ -minutes frequency)

**Figure 5: Conditional MVaRs and dependence coefficients for EUR/USD, GBP/USD and CHF/USD exchange rates.**



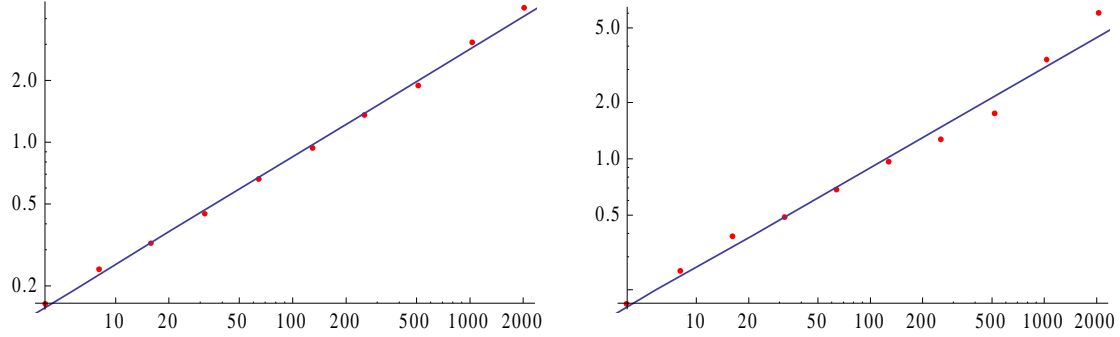
Notes: Conditional MVaRs and dependence coefficients for the 8-minutes log returns on (EUR/USD, GBP/USD, CHF/USD) exchange rates as computed in the symmetric positive and negative JDTs (directional vectors  $(\sigma_{\text{€}}, \sigma_{\text{£}}, \sigma_{\text{F}})$  and  $-(\sigma_{\text{€}}, \sigma_{\text{£}}, \sigma_{\text{F}})$ , respectively). The number in the middle of each double-arrow line is the co-dependence coefficient between the corresponding exchange rates. The number close to a currency on a double-arrow line is this currency's conditional  $\alpha$ -MVaR, given the occurrence of the  $\alpha$ -MVaR for the currency on the opposite side of the same line. Observations were obtained from Bank of America for the period from 02 January 2004 to 06 September 2006 (125818 observations at 8 minutes frequency).

**Figure 6: MVaR Scaling for EUR/USD, GBP/USD and CHF/USD returns**



Notes: 5 percent-MVaR (y-axis, times  $10^{-4}$ ) for the EUR/USD, GBP/USD, CHF/USD exchange rates log returns at different frequencies (x-axis, in minutes) computed for the directional vectors  $(\sigma_{\epsilon}, \sigma_{\pounds}, \sigma_F)$ ,  $-(\sigma_{\epsilon}, \sigma_{\pounds}, \sigma_F)$ ,  $(\sigma_{\epsilon}, -\sigma_{\pounds}, \sigma_F)$  and  $(-\sigma_{\epsilon}, \sigma_{\pounds}, -\sigma_F)$  (from top left to bottom right). The respective scaling parameters (slopes) are 0.587, 0.5701, 0.2618 and 0.3537. Data was obtained from Bank of America for the period from 02 January 2004 to 06 September 2006 ( $1006544/2^k$  observations at  $2^k$ -minutes frequency).

**Figure 7: MVaR Scaling for two consecutive EUR/USD returns**



Notes: 5 percent-MVaR (y-axis, times  $10^{-3}$ ) for two consecutive EUR/USD exchange rate log returns at different frequencies (x-axis, in minutes) computed for the directional vectors  $(\sigma_t, \sigma_{t+1})$  and  $(\sigma_t, -\sigma_{t+1})$  (from left to right). The respective scaling parameters (slopes) are 0.526 and 0.534. Data was obtained from Bank of America for the period from 02 January 2004 to 06 September 2006 ( $1006544/2^k$  observations at  $2^k$ -minutes frequency).

## Notes

<sup>1</sup> Note that the absolute value is necessary as the integration of the non-negative density can take place “from right to left” in some directions resulting in a negative value.’

<sup>2</sup> An alternative measure is the degree of risk dependence between events expressed as the relative change in conditional probability given by  $\gamma'_\alpha(f, d, \tilde{d}) = \gamma'_\alpha(f, \tilde{d}, d) := (p_\alpha(f, d, \tilde{d}) - \alpha) / \alpha$ .

<sup>3</sup> Note that the observation  $x_1$  is more extreme (along the direction  $d$ ) than the observation  $x_2$  if and only if  $\nu^d(x_1) > \nu^d(x_2)$ .

<sup>4</sup> These conclusions were confirmed for other frequencies (not shown in Table 1 but available upon request) and are in line with previously reported stylised facts.

<sup>5</sup> We also employed CHF/USD in the cross-sectional risk dependence analysis and found that the additional conditioning variable did not qualitatively change the results. To preserve space, we do not report these results but they are available upon request.

<sup>6</sup> We experimented also with GBP/USD and CHF/USD consecutive returns and found similar qualitative results. To preserve space, we do not report these results but they are available upon request.