Bayesian Stochastic Search for the Best Predictors: Nowcasting GDP Growth

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Abstract

We propose a Bayesian framework for nowcasting GDP growth in real time. Using vintage data on macroeconomic announcements we set up a state space system connecting latent GDP growth rates to agencies' releases of GDP and other economic indicators. We propose a Gibbs sampling scheme to filter out daily GDP growth rates using all available macroeconomic information. The sample draws from the resulting posterior distribution, thereby allowing us to simulate backcasting, nowcasting, and forecasting densities. A stochastic search variable selection procedure yields a data-driven way of selecting the relevant GDP predictors out of a potentially large set of economic indicators.

Keywords: Nowcasting; Real-time econometrics; Kalman filter smoother; Bayesian stochastic search

JEL classification: C11, C32, C53, E27

1 Introduction

Measuring the current state of real economic activity is a challenging task. This is particularly true for GDP, as it is one of the most important measures of real activity, and yet is typically announced only on a quarterly basis. These quarterly announcements are based on data collections and aggregations, which reflect the past but are not necessarily based on accurate contemporaneous information. Therefore, there is a significant time delay between the data collection period and the official announcements. Correspondingly, GDP estimates are not timely and are often revised as soon as information on (past) economic activity becomes richer and more precise. This lack of timely information hinders policy makers such as central banks in their decision-making. Accordingly, in both economic policy and financial practice there is a need for (real time) 'nowcasts' and 'backcasts' using contemporaneous observable information to predict current and recent unobservable GDP growth.

In this paper, we propose a Bayesian framework for producing real-time estimates of GDP growth, exploiting the information flow from vintage data on macroeconomic releases. Following Evans (2005) and Aruoba et al. (2009), we treat daily GDP growth as a latent variable which cumulates to monthly or quarterly GDP growth. Statistical inference on the latent process is updated whenever new information from corresponding (macroeconomic) predictors arrives or whenever statistical agencies publish estimates of past GDP figures. We estimate the underlying model using Gibbs sampling with a Kalman filter smoother. To address the inherent problem of possible over-parameterization in a high-dimensional state space system, we include a stochastic search variable selection (SSVS) procedure. The latter provides a data-driven way of selecting the *relevant* predictors of GDP growth out of a large set of economic indicators, while keeping the model parsimonious.

GDP growth is computed based on disaggregated information which cannot be collected on a timely basis. Accordingly, the precision of a GDP estimate increases over time when data collection advances, the underlying information set increases and problems due to incomplete and missing data vanish. This raises a natural trade-off between timeliness and the accuracy of estimates. Therefore, statistical agencies typically update estimates of GDP growth, starting with an early preliminary (and typically noisy) estimate which is revised several times thereafter. Likewise, agencies also release news on related macroeconomic figures which, in turn, provide information for estimates of past, contemporaneous and future GDP growth rates. Therefore, GDP growth can be most naturally seen as a latent variable whose realizations at single points in time can be re-estimated whenever the information set is updated. Accordingly, 'nowcasting' is associated with 'predictions' of the current state of the variable, whereas 'backcasting' refers to the estimation of past GDP growth using the most recently updated information set.

Extant literature in this area has moved in different directions. Liu and Hall (2001), Evans (2005) and Aruoba et al. (2009), among others, use state space models and filter out the latent variable using the Kalman filter. Bridge equations are, for example, used by Parigi and Schlitzer (1995) and Baffigi et al. (2004). Here, quarterly national account figures are related to quarterly aggregates of monthly data on economic activity. Giannone et al. (2008) extend this approach by considering monthly predictors as latent (common) factors which are extracted from a (potentially large) set of monthly variables. These factors are then used in a vector autoregression (VAR) model to forecast monthly and quarterly data. See also Angelini et al. (2011) for out-of-sample forecasting evaluations of this approach. Alternatively, Clements and Galvão (2010) and Andreou et al. (2013) use Mixed Data Sampling (MIDAS) techniques, as proposed by Ghysels et al. (2006) and Ghysels et al. (2007), to forecast US output growth. Being closely related to distributed lag models, MIDAS regressions allow regressors being sampled at different frequencies. Finally, Schumacher and Breitung (2008) use a factor model of a large set of economic indicators and extract common factors using principal components.

In this paper, we contribute to this string of the literature by modelling daily GDP growth as a latent variable in a state space framework. We use vintage data containing news releases of major macroeconomic variables including information about announcement dates and corresponding reporting lags (RL). Unobservable daily GDP growth is modelled as a latent variable which is updated by (noisy) estimates provided by the U.S. Bureau of Economic Analysis (BEA). Moreover, it is potentially connected to (noisy) estimates of other macroeconomic figures. Parametrizing these relationships allows expressing the underlying framework in the form of a time-varying state space system. This enables estimating unknown parameters (linking daily GDP growth to observable information) and filtering out the latent GDP growth rate.

Unlike extant studies, such as Evans (2005) or Aruoba et al. (2009), we do not estimate the model using maximum likelihood (ML) but propose a Gibbs sampling algorithm building on a Kalman filter smoother. In this sense, this paper also contributes to the literature on Bayesian Vector Autoregressive models based on mixed frequencies. Estimating the state space model using Bayesian techniques has several advantages: Firstly, in contrast to ML or expectation maximization (EM) estimators, we can provide exact finite-sample inference. Given that macroeconomic news releases occur at best on a monthly frequency, sample sizes of news announcements – even if observed over a longer time span – are ultimately small, making finite-sample inference particularly important. Secondly, employing the Kalman filter smoother allows a straightforward computation of latent GDP growth, and thus derivations of backcasts, nowcasts and forecasts at any point in time. Utilizing Markov chain Monte Carlo (MCMC) random draws, we can easily simulate the full predictive nowcast density, thereby yielding valuable information on forecasting accuracy.

A further contribution of this paper is to suggest a data-driven way of selecting the set of underlying economic predictors in real-time nowcasts. Most studies select predictors based on subjective criteria (e.g., Evans 2005 or Baffigi et al. 2004), or tackle the high dimensionality induced by a large set of possible variables by Bayesian shrinkage techniques (e.g., De Mol et al. 2008). Here, we address the problem of possible over-parametrization in the state space system by employing a SSVS procedure. Besides guaranteeing model parsimony, this procedure provides valuable insights into the informativeness of related macroeconomic figures for GDP growth.

Using the proposed Bayesian framework, we analyse the impact of the utilized observable information set on the backcasting, nowcasting and forecasting density. By applying twenty-three series of macroeconomic vintage announcements and three sets of GDP estimates covering the period from January 2, 1985, to December 31, 2009, we demonstrate that the uncertainties of backcasts, nowcasts and forecasts can be straightforwardly evaluated.

The remainder of the paper is organized as follows: Section 2 introduces the model. Section 3 illustrates the designed MCMC procedure and the implement of the SSVS approach. Section 4 presents the empirical results, while Section 5 concludes.

2 A Nowcasting State Space Model

Denote t as the time index of the current day. Assuming that the present quarter started n days ago, define g_{t-n} as the real GDP up to and excluding the first day of a quarter, i.e., the level of real GDP just when the previous quarter is completed. Correspondingly, denote the real GDP level measured up to and excluding day t by g_t. Finally, s_t^Q is the cumulated growth from the first day of the present quarter up to (but excluding) day t,

$$s_t^Q = \ln g_t - \ln g_{t-n}.$$

Similarly, we have $s_{t+1}^Q = \ln g_{t+1} - \ln g_{t-n}$ and s_t^d defines the daily contribution to GDP growth on day t in the current quarter:

$$s_t^d = s_{t+1}^Q - s_t^Q = \ln g_{t+1} - \ln g_t.$$

To provide statistical inference of s_t^d as the information set evolves, we specify a state space model similarly to Evans (2005). Denote $\lambda_t^{M/Q}$ as a dummy variable taking the value one whenever t is the first day of the reference month/quarter and zero otherwise. Then, cumulated monthly (M) or quarterly (Q) growth from the first day of the month/quarter up to (but excluding) day $t+1$ is computed as

$$s_{t+1}^{M/Q} = \left(1 - \lambda_t^{M/Q}\right) s_t^{M/Q} + s_t^d. \tag{1}$$

Below we use m and q to indicate growth for a complete month or quarter, respectively. In particular, $m_{t,1}$ is denoted as the growth over the most recent (completed) month with t as the reference day and the subscript '1' indicating that the RL is less than one month. Similarly, $q_{t,1}$ denotes the growth over the most recent (past) completed quarter with RL being less than one quarter. Likewise, $q_{t,2}$ indicates that the RL is more than one quarter but less than two quarters. Accordingly, we have

$$m_{t+1,1} = \iota_t^M \left(s_t^M + s_t^d\right) + \left(1 - \iota_t^M\right) m_{t,1}, \tag{2}$$

and

$$q_{t+1,1} = \iota_t^Q \left(s_t^Q + s_t^d\right) + \left(1 - \iota_t^Q\right) q_{t,1}, \tag{3}$$

where $\iota_t^{M/Q}$ denotes a dummy variable being one if t is the last day of the month/quarter and zero otherwise. Hence, whenever the days t and $t+1$ are in the same month/quarter, the most recently completed monthly/quarterly growth is the same. Only in cases where t is the last day of a period (and thus $t+1$ is the first day of the next period), $m_{t+1,1}$ or $q_{t+1,1}$ would then indicate the growth in the most recent completed month or quarter (with reference point $t+1$) equal to $s_t^{M/Q} + s_t^d$. Likewise, we obtain

$$m_{t+1,2} = \iota_t^M m_{t,1} + \left(1 - \iota_t^M\right) m_{t,2}, \quad q_{t+1,2} = \iota_t^Q q_{t,1} + \left(1 - \iota_t^Q\right) q_{t,2}. \tag{4}$$

As in Evans (2005) we allow for dynamics in daily GDP growth by expressing s_{t+1}^d as a function of lagged monthly growth,

$$s_{t+1}^d = \phi_1 \left(1 - \iota_t^M\right) m_{t,1} + \phi_1 \iota_t^M \left(s_t^M + s_t^d\right) + \phi_2 m_{t,2} + \phi_3 m_{t,3} + v_t, \tag{5}$$

where $v_t \overset{i.i.d}{\sim} N\left(0, \sigma_v^2\right)$, and the coefficients $\phi_i, i = 1, 2, 3$, jointly satisfy the underlying stationarity conditions.

As shown by Evans (2005), the specification in (5) implies an AR(p) process for monthly growth m_t. Note that we allow daily dynamics to depend only on the most recent completed monthly growth rates. Hence, forecasts of daily growth s_{t+h}^d are the same for any value of h provided $t+h$ and t are in the same period. Allowing for daily GDP growth dynamics would be theoretically possible, yet, very hard to identify.

Summarizing the above relations (1)–(5) in matrix form we obtain

$$\underbrace{\begin{pmatrix} s_{t+1}^Q \\ q_{t+1,1} \\ q_{t+1,2} \\ s_{t+1}^M \\ m_{t+1,1} \\ m_{t+1,2} \\ m_{t+1,3} \\ s_{t+1}^d \end{pmatrix}}_{\mathbb{Z}_{t+1}} = \underbrace{\begin{pmatrix} 1-\lambda_t^Q & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \iota_t^Q & 1-\iota_t^Q & 0 & 0 & 0 & 0 & 0 & \iota_t^Q \\ 0 & \iota_t^Q & 1-\iota_t^Q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\lambda_t^M & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \iota_t^M & 1-\iota_t^M & 0 & 0 & \iota_t^M \\ 0 & 0 & 0 & 0 & \iota_t^M & 1-\iota_t^M & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \iota_t^M & 1-\iota_t^M & 0 \\ 0 & 0 & 0 & \phi_1 \iota_t^M & \phi_1\left(1-\iota_t^M\right) & \phi_2 & \phi_3 & \phi_1 \iota_t^M \end{pmatrix}}_{A_t} \underbrace{\begin{pmatrix} s_t^Q \\ q_{t,1} \\ q_{t,2} \\ s_t^M \\ m_{t,1} \\ m_{t,2} \\ m_{t,3} \\ s_t^d \end{pmatrix}}_{\mathbb{Z}_t} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_t \end{pmatrix}}_{V_t},$$

and thus

$$\mathbb{Z}_{t+1} = A_t \mathbb{Z}_t + V_t \tag{6}$$

reflecting period-to-period dynamics in period-specific cumulated (latent) GDP growth rates.

Since the variables contained in $\mathbb{Z}_t$ are unobservable, they have to be linked to observable BEA announcements of growth rates. We denote the BEA estimate of most recent completed quarter's growth ($q_{t,1}$) as $\widetilde{y}_t^1$, where the sub-index '1' indicates the BEA's *first* estimate of $q_{t,1}$ at day t.

We assume that BEA estimates of $q_{t,1}$ are subject to error and thus are given by

$$\widetilde{y}_t^i = q_{t,1} + e_t^i, \quad i = \{1, 2, 3\}, \tag{7}$$

where e_t^i denotes the BEA's estimation errors associated with the first ('advanced', $i = 1$), second ('preliminary', $i = 2$), and third ('final', $i = 3$) estimate. If the RL is greater than one quarter but less than two quarters, $\widetilde{y}_t^i$ is then

$$\widetilde{y}_t^i = q_{t,2} + e_t^i, \quad i = \{1, 2, 3\}. \tag{8}$$

Note that Evans (2005) imposes a restriction on the variance of e_t^3, assuming the BEA's 'final' estimate of GDP growth being equivalent to the latent growth $q_{t,l}$, $l = 1, 2$, and thus error-free. Here, we relax this assumption and explicitly estimate the variance of e_t^3. We illustrate the effect of such a restriction with applications to a sub-sample.

Since the BEA does not provide daily estimates of GDP growth, we observe their estimates only on one specific day in each quarter. On all other days, the BEA's estimates are unobservable and thus are recorded as missing values (N/A). For instance, by denoting the actual BEA's announcement of their 'advanced' estimate of last quarter's GDP growth as y_t^1, we have

$$y_t^1 = \begin{cases} \widetilde{y}_t^1 & \text{, if BEA releases an 'advanced' estimate of GDP growth on day } t, \\ N/A & \text{, otherwise.} \end{cases}$$

Then, $\widetilde{y}_t^1$ can be interpreted as a *hypothetical* BEA estimate which, however, is only observable once per quarter. We call these hypothetical BEA estimates as *potential* releases of the BEA estimates. Defining a dummy variable $D^d_{i,t}$, indicating the GDP growth announcement date, we obtain

$$y^i_t = D^d_{i,t}\widetilde{y}^i_t, \quad i = 1, 2, 3 \tag{9}$$

where

$$D^d_{i,t} = \begin{cases} 1 & \text{, if BEA releases an estimate of GDP growth on day } t, \\ N/A & \text{, otherwise.} \end{cases}$$

Denote $D^Q_{1,t} = 1$ if the RL is less than one quarter and zero otherwise. Likewise, we define $D^Q_{2,t} = 1$ if the RL is more than one quarter but less than two quarters, and zero otherwise. Then, the corresponding relations in (7) and (8) can be summarized as:

$$\widetilde{y}^i_t = D^Q_{1,t}q_{t,1} + D^Q_{2,t}q_{t,2} + e^i_t \quad i = 1, 2, 3,$$

and in matrix form as

$$\begin{pmatrix} \widetilde{y}^3_t \\ \widetilde{y}^2_t \\ \widetilde{y}^1_t \end{pmatrix} = \begin{pmatrix} 0 & D^Q_{1,t} & D^Q_{2,t} \\ 0 & D^Q_{1,t} & D^Q_{2,t} \\ 0 & D^Q_{1,t} & D^Q_{2,t} \end{pmatrix} \begin{pmatrix} s^Q_t \\ q_{t,1} \\ q_{t,2} \end{pmatrix} + \begin{pmatrix} e^3_t \\ e^2_t \\ e^1_t \end{pmatrix}, \tag{10}$$

where the first column of the matrix is artificially included to make the relationship fit into the corresponding system below.

Together with equation (9), we then have

$$\begin{pmatrix} y^3_t \\ y^2_t \\ y^1_t \end{pmatrix} = \begin{pmatrix} D^d_{3,t} & 0 & 0 \\ 0 & D^d_{2,t} & 0 \\ 0 & 0 & D^d_{1,t} \end{pmatrix} \begin{pmatrix} \widetilde{y}^3_t \\ \widetilde{y}^2_t \\ \widetilde{y}^1_t \end{pmatrix}, \tag{11}$$

where the dummy variables $D^d_{2,t}$ and $D^d_{3,t}$ denote the GDP announcement days of the 'preliminary' and 'final' releases, respectively.

Finally, we assume that latent monthly growth rate $m_{t,1}$ is also linked to the announcements of related macroeconomic figures. Adapting the notation above, we denote the potential release of a macroeconomic variable z^i, $i = 1, \ldots, n$ as $\widetilde{z}^i_t$. Following Evans (2005), we assume that GDP growth contributes linearly to the variations in the macroeconomic variables, i.e.,

$$\widetilde{z}^i_t = \beta_i \left(D^M_{0,t}s^M_t + D^M_{1,t}m_{t,1} + D^M_{2,t}m_{t,2} + D^M_{3,t}m_{t,3}\right) + u^i_t, \tag{12}$$

with u_t^i denoting a white noise error term, β_i being an unknown slope parameter, and $D_{j,t}^M$, $j = 0, 1, 2, 3$, denoting dummy variables to indicate the length of reporting delays with the sub-index M reflecting that the macroeconomic variables are announced each month. Correspondingly, $D_{0,t}^M = 1$ indicates that there is no time delay in terms of the data announcement with RL being zero (i.e., data are announced before the current month is completed). Likewise, $D_{j,t}^M = 1$, $j > 0$, indicates a RL of less than j months and zero otherwise. In matrix form, equation (12) can be summarized as

$$\begin{pmatrix} \widetilde{z}_t^1 \\ \vdots \\ \widetilde{z}_t^n \end{pmatrix} = \begin{pmatrix} \beta_1 D_{0,t}^M & \beta_1 D_{1,t}^M & \beta_1 D_{2,t}^M & \beta_1 D_{3,t}^M & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \beta_n D_{0,t}^M & \beta_n D_{1,t}^M & \beta_n D_{2,t}^M & \beta_n D_{3,t}^M & 0 \end{pmatrix} \begin{pmatrix} s_t^M \\ m_{t,1} \\ m_{t,2} \\ m_{t,3} \\ s_t^d \end{pmatrix} + \begin{pmatrix} u_t^1 \\ \vdots \\ u_t^n \end{pmatrix}. \quad (13)$$

Extending the set of announcement dummies introduced above, we define the dummy variables $D_{3+i,t}^d$, $i = 1, \ldots, n$, to indicate the announcement day of the macroeconomic release i. Then, mimicking the argument above, the relationship between the *hypothetical* and the *actual* releases is given by

$$\begin{pmatrix} z_t^1 \\ \vdots \\ z_t^i \\ \vdots \\ z_t^n \end{pmatrix} = \begin{pmatrix} D_{3+1,t}^d & \cdots & & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ & & D_{3+i,t}^d & & \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & & \cdots & D_{3+n,t}^d \end{pmatrix} \begin{pmatrix} \widetilde{z}_t^1 \\ \vdots \\ \widetilde{z}_t^i \\ \vdots \\ \widetilde{z}_t^n \end{pmatrix}. \quad (14)$$

Stacking (10) on (13), and (11) on (14), we obtain

$$\underbrace{\begin{pmatrix} \widetilde{y}_t^3 \\ \widetilde{y}_t^2 \\ \widetilde{y}_t^1 \\ \widetilde{z}_t^1 \\ \vdots \\ \widetilde{z}_t^n \end{pmatrix}}_{\widetilde{Y}_t} = \underbrace{\begin{pmatrix} 0 & D_{1,t}^Q & D_{2,t}^Q & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{1,t}^Q & D_{2,t}^Q & 0 & 0 & 0 & 0 & 0 \\ 0 & D_{1,t}^Q & D_{2,t}^Q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_1 D_{0,t}^M & \beta_1 D_{1,t}^M & \beta_1 D_{2,t}^M & \beta_1 D_{3,t}^M & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \beta_n D_{0,t}^M & \beta_n D_{1,t}^M & \beta_n D_{2,t}^M & \beta_n D_{3,t}^M & 0 \end{pmatrix}}_{C_t} \underbrace{\begin{pmatrix} s_t^Q \\ q_{t,1} \\ q_{t,2} \\ s_t^M \\ m_{t,1} \\ m_{t,2} \\ m_{t,3} \\ s_t^d \end{pmatrix}}_{\mathbb{Z}_t} + \underbrace{\begin{pmatrix} e_t^3 \\ e_t^2 \\ e_t^1 \\ u_t^1 \\ \vdots \\ u_t^n \end{pmatrix}}_{U_t},$$

where $U_t \sim MN(\mathbf{0}, \Sigma_U)$, and

$$\underbrace{\begin{pmatrix} y_t^3 \\ y_t^2 \\ y_t^1 \\ z_t^1 \\ \vdots \\ z_t^n \end{pmatrix}}_{Y_t} = \underbrace{\begin{pmatrix} D_{3,t}^d & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & D_{2,t}^d & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & D_{1,t}^d & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & D_{3+n,t}^d \end{pmatrix}}_{B_t} \underbrace{\begin{pmatrix} \widetilde{y}_t^3 \\ \widetilde{y}_t^2 \\ \widetilde{y}_t^1 \\ \widetilde{z}_t^1 \\ \vdots \\ \widetilde{z}_t^n \end{pmatrix}}_{\widetilde{Y}_t}.$$

In short notation, the two matrix equations above can be written as

$$\widetilde{Y}_t = C_t \mathbb{Z}_t + U_t, \tag{15}$$

and

$$Y_t = B_t \widetilde{Y}_t. \tag{16}$$

Consequently, we obtain

$$Y_t = B_t C_t \mathbb{Z}_t + B_t U_t, \tag{17}$$

where

$$B_t U_t = \begin{pmatrix} D_{3,t}^d e_t^3, & D_{2,t}^d e_t^2, & D_{1,t}^d e_t^1, & D_{4,t}^d u_t^1, & \cdots & , D_{3+n,t}^d u_t^n \end{pmatrix}',$$

and

$$B_t C_t \mathbb{Z}_t = \begin{pmatrix} q_{t,1} D_{1,t}^d D_{1,t}^Q + q_{t,2} D_{1,t}^d D_{2,t}^Q \\ q_{t,1} D_{2,t}^d D_{1,t}^Q + q_{t,2} D_{2,t}^d D_{2,t}^Q \\ q_{t,1} D_{3,t}^d D_{1,t}^Q + q_{t,2} D_{3,t}^d D_{2,t}^Q \\ \beta_1 \left(s_t^M D_4^d D_{0,t}^M + m_{t,1} D_4^d D_{1,t}^M + m_{t,2} D_4^d D_{2,t}^M + m_{t,3} D_4^d D_{3,t}^M \right) \\ \beta_2 \left(s_t^M D_5^d D_{0,t}^M + m_{t,1} D_5^d D_{1,t}^M + m_{t,2} D_5^d D_{2,t}^M + m_{t,3} D_5^d D_{3,t}^M \right) \\ \vdots \\ \beta_n \left(s_t^M D_{n+3}^d D_{0,t}^M + m_{t,1} D_{n+3}^d D_{1,t}^M + m_{t,2} D_{n+3}^d D_{2,t}^M + m_{t,3} D_{n+3}^d D_{3,t}^M \right) \end{pmatrix} = \begin{pmatrix} x_{1t} \\ x_{2t} \\ x_{3t} \\ x_{4,t}\beta_1 \\ x_{5,t}\beta_2 \\ \vdots \\ x_{n+3,t}\beta_n \end{pmatrix},$$

where $B_t C_t \mathbb{Z}_t$ is re-parametrized with

$$\begin{aligned}
x_{1,t} &= q_{t,1} D^d_{1,t} D^Q_{1,t} + q_{t,2} D^d_{1,t} D^Q_{2,t}, \\
x_{2,t} &= q_{t,1} D^d_{2,t} D^Q_{1,t} + q_{t,2} D^d_{2,t} D^Q_{2,t}, \\
&\vdots \\
x_{3+i,t} &= s^M_t D^d_{i+3} D^M_{0,t} + m_{t,1} D^d_{i+3} D^M_{1,t} + m_{t,2} D^d_{i+3} D^M_{2,t}.
\end{aligned}$$

Accordingly, the $i+3$-th equation is then given by

$$y_{3+i,t} = x_{3+i,t}\beta_i + u^i_t$$

for $i = 1, \ldots, n$. Finally, define

$$\mathbf{b} = \begin{pmatrix} 1 & 1 & 1 & \beta_1 & \beta_2 & \ldots & \beta_n \end{pmatrix}' \quad \text{and} \quad \mathbf{X}_t = \begin{pmatrix} x_{1,t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & x_{n+3,t} \end{pmatrix},$$

then, (17) can be written as

$$Y_t = \mathbf{X}_t \mathbf{b} + \mathbf{u}_t, \quad \text{where } \mathbf{u}_t = B_t U_t. \tag{18}$$

3 Bayesian Inference

3.1 Gibbs Sampling

Equations (6) and (17) can be fitted into a time-varying state space form given as

$$\begin{pmatrix} \mathbb{Z}_{t+1} \\ Y_t \end{pmatrix} = \Phi_t \mathbb{Z}_t + \epsilon_t, \quad \epsilon_t = \begin{pmatrix} V_t \\ B_t U_t, \end{pmatrix} \text{ where } \Phi_t = \begin{pmatrix} A_t \\ B_t C_t \end{pmatrix}, \tag{19}$$

for $t = 2, \ldots, T$.

V_t and U_t given above with $V_t \overset{i.i.d}{\sim} MN(\mathbf{0}, \Sigma_V)$ and $U_t \overset{i.i.d}{\sim} MN(\mathbf{0}, \Sigma_U)$, where $MN(\mu, \Sigma)$ denoting a multivariate normal distribution with mean μ and covariance Σ. Σ_U gives the covariance matrix of the BEA estimation errors and the errors arising from linking GDP growth to the variables $\tilde{z}^i_t$, in eq. (12).

As the dummy variables $D_{l,t}^{Q/M/d}$ are obviously time-varying, the matrix $B_t C_t$, as well as A_t in Φ_t, are time-varying. The model has to be represented in a time-varying state space form. We denote the precision matrix as Σ_U^{-1}, and $h_e = \sigma_e^{-2}$ as the error precision in eq. (6). The covariance matrix Ω_t is given by

$$\Omega_t = \begin{pmatrix} \Sigma_V & \Sigma_{VU} \\ \Sigma_{UV} & \Sigma_U \end{pmatrix},$$

which is assumed to be constant over time.

State space models can be estimated using the Kalman filter, see, for example, Carter and Kohn (1994), De Jong and Shephard (1995) or Durbin and Koopman (2001). Using Bayesian inference has the advantage that not only is it straightforward to compute (nowcast) densities, but also it is possible to take parameter uncertainties and estimation uncertainties into account.

Denote $\beta' = (\beta_1, \beta_2, \ldots, \beta_n)$, and collecting all parameters to be estimated in the vector $\theta = \left\{\beta', \phi_1, h_e, \Sigma_U^{-1}\right\}$, we elicit priors as follows: For the precision matrix Σ_U^{-1} we choose a wishart prior (Chib and Greenberg, 1995), i.e., $\Sigma_U^{-1} \sim W\left(\underline{A}_U, \underline{\upsilon}\right)$, where $W\left(\underline{A}_U, \underline{\upsilon}\right)$ denotes the wishart distribution with $\underline{A}_U$ being a positive definite symmetric (PDS) matrix, and with $\underline{\upsilon} > 0$ a scalar degree of freedom parameter; A gamma prior is elicited for h_e, $h_e \sim \Gamma\left(\underline{a}_e, \underline{b}_e\right)$, where $\Gamma\left(\underline{a}_e, \underline{b}_e\right)$ denotes a gamma distribution with $\underline{a}_e$ being the shape parameter and $\underline{b}_e$ the scale parameter; As the prior for $\mathbf{b}$, we choose $\mathbf{b} \sim MN\left(\underline{\mu}_{\mathbf{b}}, \underline{V}_{\mathbf{b}}\right)$. For parsimony, we restrict $\phi_2 = \phi_3 = 0$ and choose a normal prior for ϕ_1 with $\phi_1 \sim N\left(\underline{\mu}_{\phi_1}, \underline{V}_{\phi_1}\right) 1\left(|\phi_1| < 1\right)$, where $1(\cdot)$ is an indicating function. This prior ensures ϕ_1 lies inside the unit circle, thereby ensuring weak stationarity of GDP growth. For computational convenience, we assume independence amongst all priors.

Define the $NT \times 1$ vector $\mathbf{Y} = (Y_1', Y_2', \ldots, Y_T')'$. Denote $\mathbb{Z} = (\mathbb{Z}_1, \mathbb{Z}_2, \ldots, \mathbb{Z}_T)$ containing the latent data which has to be estimated. Let $p\left(\mathbf{Y}, \mathbb{Z} \mid \theta\right)$ be the joint density of $(\mathbf{Y}, \mathbb{Z})$ given θ, $p\left(\mathbf{Y} \mid \mathbb{Z}, \theta\right)$ be the conditional density of $\mathbf{Y}$ given $(\mathbb{Z}, \theta)$, and $p\left(\mathbb{Z} \mid \theta\right)$ be the density of $\mathbb{Z}$ given θ, respectively. Based on Bayes Theorem, the joint density of $(\mathbf{Y}, \mathbb{Z})$ given θ is then

$$p\left(\mathbf{Y}, \mathbb{Z} \mid \theta\right) = p\left(\mathbf{Y} \mid \mathbb{Z}, \theta\right) p\left(\mathbb{Z} \mid \theta\right) \tag{20}$$

with the posterior given by

$$p(\theta \mid \mathbf{Y}, \mathbb{Z}) \propto p(\mathbf{Y}, \mathbb{Z} \mid \theta)\, p(\theta)\,.$$

To estimate this time varying state space model with a large proportion of missing values in the data set, we can use a Kalman filter within an MCMC algorithm.

It is not straightforward to draw Σ_U^{-1} from its posterior conditional because the derivation of the posterior Σ_U^{-1} does not allow for missing dimensions in the residuals, where the missing residuals are induced by a large proportion of missing observations on (daily) BEA announcements. To address this issue we impute the missing values by random draws conditional on the observable data. The imputation is implemented directly in the underlying MCMC algorithm. As derived in more detail in Appendix A, a Gibbs sampling algorithm is proposed to estimate the parameters of interest and the latent states.

Gibbs sampling algorithm to estimate θ and $\mathbb{Z}$:

1. Initialize the parameters $\theta = \left\{\beta', \phi_1, h_e, \Sigma_U^{-1}\right\}$.

2. Draw the latent state $\mathbb{Z}$ using the Kalman filter smoother within a time-varying state space form.

3. Draw θ as follows:

   (a) Draw $h_e \mid \beta, \phi_1, \Sigma_U^{-1}; \mathbb{Z}, \mathbf{Y}$ from $h_e \sim \Gamma\left(\overline{a}_e, \overline{b}_e\right)$.

   (b) Draw $\phi_1 \mid \beta, h_e, \Sigma_U^{-1}; \mathbb{Z}, \mathbf{Y}$ with $\phi_1 \sim N\left(\overline{\mu}_{\phi_1}, \overline{V}_{\phi_1}\right) 1\left(|\phi_1| < 1\right)$.

   (c) Draw β and Σ_U^{-1} with imputations

      i. Impute missing values in $\mathbf{Y}$. The imputed data is denoted by $\mathbf{Y}^*$.

      ii. As $\mathbf{b} = \begin{pmatrix} 1, & 1, & 1, & \beta' \end{pmatrix}'$ and $\mathbf{b} \sim MN\left(\overline{\mu}_{\mathbf{b}}, \overline{V}_{\mathbf{b}}\right)$, calculate the posterior mean $\overline{\mu}_{\mathbf{b}}$, and posterior covariance $\overline{V}_{\mathbf{b}}$.

      iii. Partition $\overline{\mu}_{\mathbf{b}}$ and $\overline{V}_{\mathbf{b}}$ to obtain $\overline{\mu}_{\beta}$ and $\overline{V}_{\beta}$. Get a random draw of $\beta \sim MN\left(\overline{\mu}_{\beta}, \overline{V}_{\beta}\right)$.

      iv. Draw $\Sigma_U^{-1} \mid \beta, \phi_1, h_e; \mathbb{Z}, \mathbf{Y}^*$ using $\Sigma_U^{-1} \sim W\left(\overline{A}_U, \overline{v}\right)$ with a dimension of $n+3$ by $n+3$.

4. Run step 2 – step 3 S times and discard the initial S_0 draws.

If we are to impose the restriction $e_t^3 = 0$ on the variance, as in the model step-up of Evans (2005), the dimension of Σ_U^{-1} is $(n+2) \times (n+2)$ instead of $(n+3) \times (n+3)$ and a corresponding $\mathbf{b} = \begin{pmatrix} 1, & 1, & \beta' \end{pmatrix}'$.

3.2 Stochastic Search Variable Selection

Using the SSVS method yields a data-driven way of selecting the most relevant predictors of GDP growth and thus keeps the model parsimonious. It has been introduced by George et al. (2008) to perform model (de-)selection in a Bayesian VAR framework. The underlying idea is to assign a priori distribution for each coefficient, b_j in $\mathbf{b}$ with $j = 1, \dots, N$, being zero (and thus the associated variable *not* belonging to the model). Then, computing the resulting posteriors and performing recursive forecasting or filtering, at each point in time, we can compute $\Pr(b_j = 0|\text{data})$ and thus can determine the time-varying set of underlying GDP growth predictors.

SSVS is performed by assigning a prior for each coefficient b_j consisting of a mixture of two normal distributions:

$$b_j \mid \gamma_j \sim (1-\gamma_j)\, N\left(0, \kappa_{0j}^2\right) + \gamma_j N\left(0, \kappa_{1j}^2\right),$$

where γ_j denotes a selection dummy taking the values one and zero. The prior variance κ_{0j}^2 is chosen as being very small, setting the parameter b_j virtually to zero. Conversely, κ_{1j}^2 is chosen as being comparatively large, making the prior relatively non-informative. Consequently, the variable associated with b_j is selected whenever $\gamma_j = 1$. As proposed by George et al. (2008) we set the hyper-parameters $\kappa_{0j}^2 = c_0\sqrt{\widehat{var\,(b_j)}}$ and $\kappa_{1j}^2 = c_1\sqrt{\widehat{var\,(b_j)}}$, where $c_0 = 0.1$, $c_1 = 10$, and $\widehat{var\,(b_j)}$ is an estimate of the variance of the corresponding coefficient (based on a preliminary MCMC run using a non-informative prior). George et al. (2008) refer to this choice of the hyper-parameters as being 'default semi-automatic', making the approach hierarchical.

By assuming γ_j being independent of each other, the conditional prior of γ_j, given all other elements $\gamma_{\backslash -j}$, is

$$\gamma_j \mid \gamma_{\backslash -j} \sim B\left(1, \underline{q}_j\right),$$

where $B(\cdot)$ denotes a Bernoulli distribution. Hence, the probabilities of $b_j|\gamma_j$, following the two possible priors, are given by

$$\Pr(\gamma_j = 1) = \underline{q}_j \text{ and } \Pr(\gamma_j = 0) = 1 - \underline{q}_j.$$

Following Jochmann et al. (2010), $\underline{q}_j$ is chosen as 0.5 indicating that all coefficients are (a priori) equally likely to be included/excluded.

SSVS can be also applied to shrink the elements of the (potentially high-dimensional) precision matrix Σ_U^{-1} by re-parametrization of Σ_U. Applying a Cholesky decomposition, Σ_U can be decomposed as

$$\Sigma_U = \psi^{-1'}\psi^{-1}$$

or

$$\Sigma_U^{-1} = \psi\psi',$$

where ψ is the upper-triangular matrix. As noted in George et al. (2008), "the elements of the precision matrix Σ_U^{-1} (σ^{ij}) are natural quantities of interest because $-\sigma^{ij}/\sqrt{\sigma^{ii}\sigma^{jj}}$ is a partial correlation coefficient. For example $\sigma^{12} = 0$, the first two components of the errors are independent given the rest of the components."

Denote the diagonal elements as $\psi = (\psi_{11}, \psi_{22}, \ldots, \psi_{jj})'$. To ensure the positive definiteness of Σ_U, we assign gamma priors for the diagonal elements ψ_{jj}^2 and thus restrict them to being positive. SSVS is applied only to the off-diagonal elements of ψ. Collect the off-diagonal upper triangular elements in $\eta_j = (\psi_{1j}, ...\psi_{j-1,j})'$, for $j = 2, ..., N$ with $\eta = (\eta_2', ..., \eta_N')'$. For each ψ_{ij}, we define a corresponding selection (zero/one) dummy ω_{ij} and denote $\omega_j = (\omega_{1j}, ..., \omega_{j-1,j})'$. Similar to the selection of variables above, SSVS is performed by eliciting a hierarchical prior for η_j of the form

$$\eta_j \mid \omega_j \sim MN_{j-1}(\mathbf{0}, D_j R_j D_j),$$

where $D_j = diag(h_{1j}, ..., h_{j-1,j})$ is a $(j-1) \times (j-1)$ diagonal matrix of prior variances and R_j denotes a (given) correlation matrix which, for simplicity, is set to an identity matrix.

Following George et al. (2008), we set

$$h_{ij} = \left\langle \begin{array}{l} \kappa_{0ij} \text{ if } \omega_{ij} = 0 \\ \kappa_{1ij} \text{ if } \omega_{ij} = 1 \end{array} \right. \qquad i = 1, \ldots, j-1 \ ,$$

with the hyper-parameters chosen as $\kappa_{0ij}^2 = 0.1$ and $\kappa_{1ij}^2 = 5$. Consequently, given ω_{ij}, the elements $\psi_{ij} \mid \omega_{ij}$ follow a mixture of normal distributions with

$$\psi_{ij} \mid \omega_{ij} \sim (1-\omega_{ij})\, N\left(0, \kappa_{0ij}^2\right) + \omega_{ij} N\left(0, \kappa_{1ij}^2\right). \tag{21}$$

If ω_{ij} is 0, the parameter ψ_{ij} reflects only very small variations around zero, and thus a small value of ψ_{ij} implies little correlation between component i and component j given the rest of the components. Otherwise, it follows a quite non-informative prior and can clearly deviate from zero. As in George et al. (2008), the choice of κ_{0ij}^2 and κ_{1ij}^2 could also depend on the estimated variances of ψ_{ij} and thus being 'semi-automatic'. However, given the relatively high dimension of the underlying state space mode, we refrain from allowing for this extra flexibility in hyper-parameters. Implementing the final SSVS procedure into the Gibbs sampling algorithm we proceed as follows:

Gibbs sampling algorithm with SSVS

1. Initialize the parameters $\theta = \{\beta', \gamma, \phi_1, h_e, \psi, \omega\}$.
2. Draw $\mathbb{Z}$ using the Kalman filter smoother.
3. Draw $\theta = \{\beta', \gamma, \phi_1, h_e, \psi, \omega\}$.
   (a) Draw $h_e \mid \beta', \gamma, \phi_1, \psi, \omega; \mathbb{Z}, \mathbf{Y}$ from $h_e \sim \Gamma\left(\overline{a}_e, \overline{b}_e\right)$.
   (b) Draw $\phi_1 \mid \beta', \gamma, h_e, \psi, \omega; \mathbb{Z}, \mathbf{Y}$ with $\phi_1 \sim N\left(\overline{\mu}_{\phi_1}, \overline{V}_{\phi_1}\right) 1\left(|\phi_1| < 1\right)$.
   (c) Draw β and $\Sigma_U^{-1}\,(\psi\psi')$ using the SSVS approach:
      i. Impute the missing values in $\mathbf{Y}$. Imputed data is denoted by $\mathbf{Y}^*$.
      ii. Draw ψ as follows:
         A. The diagonal elements ψ_{jj} are drawn from a gamma distribution, i.e., $\psi_{jj}^2 \mid \beta', \gamma, \phi_1, h_e, \omega; \mathbb{Z},\ \mathbf{Y}^* \sim \Gamma\left(\overline{a}_j, \overline{b}_j\right)$.
         B. The off-diagonal elements $\eta_j = (\psi_{1j}, \ldots \psi_{j-1,j})'$ are drawn from a normal distribution, $\eta_j \mid \beta', \gamma, \phi_1, h_e, \omega; \mathbb{Z}\, \mathbf{Y}^* \sim MN_{j-1}\left(0, D_j D_j\right)$.

C. Calculate $\Sigma_U^{-1} = \psi\psi'$, where ψ is re-parametrized with ψ_{jj} and η_j.

iii. Draw ω_{ij} from Bernoulli, $\omega_{ij} \mid \beta', \gamma, \phi_1, h_e, \psi; \mathbb{Z}, \mathbf{Y}^* \sim B\left(u_{ij} / \left(u_{ij1} + u_{ij2}\right)\right)$ indicating the selection of ψ_{ij}.

iv. Calculate $\overline{\mu}_{\mathbf{b}}, \overline{V}_{\mathbf{b}}$ where $\mathbf{b} = \begin{pmatrix} 1, & 1, & 1, & \beta' \end{pmatrix}'$, $\mathbf{b} \sim MN\left(\overline{\mu}_{\mathbf{b}}, \overline{V}_{\mathbf{b}}\right)$.

v. By partitioning $\overline{\mu}_{\mathbf{b}}, \overline{V}_{\mathbf{b}}$, we get $\overline{\mu}_{\beta}$ and $\overline{V}_{\beta}$. Get a random draw of β from $MN\left(\overline{\mu}_{\beta}, \overline{V}_{\beta}\right)$.

vi. Draw γ_j from a Bernoulli distribution, $\gamma_j \mid \mathbf{b}, \phi_1, h_e, \psi, \omega; \mathbb{Z}, \mathbf{Y}^* \sim B\left(u_j / \left(u_{j1} + u_{j2}\right)\right)$.

4. Run step 2 – step 3 S times and discard the initial S_0 draws.

This algorithm can also be combined with a restriction of the variance of e_t^3 to equal 0 with a corresponding vector $\mathbf{b} = \begin{pmatrix} 1, & 1, & \beta' \end{pmatrix}'$. Note from eq.(12) – eq.(14), SSVS visiting on the coefficient $\mathbf{b}$ (β) will not help us to decide which macroeconomic variable is best related to GDP growth. If β_i $(i = 1, \ldots, n)$ is restricted to zero, it only indicates that little variation in the macroeconomic variable can be explained using latent growth. In other words, it does not really tell us how much variation in the growth is contributed by the macroeconomic variables. However, the visit of SSVS on the covariance matrix Σ_U (re-parametrized Σ_U to be precise), on the other hand, can tell us how the variables are associated. Therefore, the inference of the best predictors, is made from the visit of SSVS on ψ ($\Sigma_U^{-1} = \psi\psi\prime$). As pointed out in George et al. (2008), "restriction of the regression coefficient helps in selecting the correct restrictions for ψ", we propose the above sampling algorithm and consider SSVS for restriction for both $\mathbf{b}$ and ψ. The choice of ψ_{ij} (where the SSVS index ω_{ij} is small close to zero) means zero partial correlation between the i-th component and the j-th component of the error term for $i, j > 1$.

3.3 Forecasting, Nowcasting and Backcasting

One of the advantages of using a Bayesian approach is that we are able to provide the back-/now-/fore-cast density, thereby reflecting the uncertainty associated with a point back-/now-/fore-cast. Using the designed Gibbs sampler, we obtain a draw of the latent state vector, $\mathbb{Z}_t^{(i)}$, arising from the Kalman filter after each iteration. The latter is computed as the average of S number MCMC runs after the burn-in period,

$$\widehat{\mathbb{Z}}_t = \frac{1}{S}\sum_{i=1}^{S}\mathbb{Z}_t^{(i)}.$$

Then, having the estimates $\hat{\mathbb{Z}}_t = (\hat{s}_t^Q, \hat{q}_{t,1}, \hat{q}_{t,2}, \hat{s}_t^M, \hat{m}_{t,1}, \hat{m}_{t,2}, \hat{m}_{t,3}, \hat{s}_t^d)'$, it is straightforward to conduct back-/now-/forecastings. For instance, with t as the reference point, *backcasts* of last months' or quarters' growth are directly given by $\hat{m}_{t,i}$, for $i = 1,\ 2,\ 3$ or $\hat{q}_{t,i}$, for $i = 1, 2$, respectively. Likewise back-/nowcasts of monthly or quarterly growth up to t are given by $\hat{s}_t^M$ or $\hat{s}_t^Q$, respectively. *Nowcasts* of today's daily growth are then obtained by $\hat{s}_t^d$.

Recalling the assumption of constant daily growth rates during a month, *forecasts* of future daily growth rates until the end of the current month are given by $\hat{s}_t^d$. Accordingly, expected growth until the end of the current month, $\hat{m}_{t,0}$, is computed as $\hat{m}_{t,0} = \hat{s}_t^M + k\hat{s}_t^d$ with k denoting the remaining days till the end of the month. Finally, expected future monthly growth rates over horizons beyond the current month are computed exploiting the imposed AR(1) structure in eq.(5).

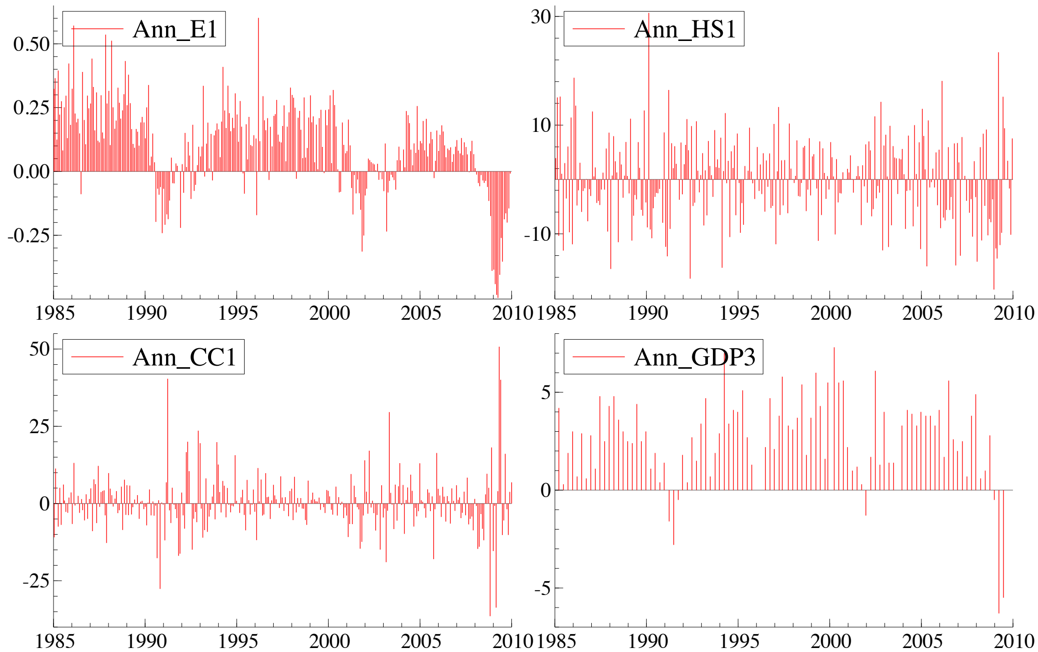
In Clements and Hendry, 1998, p.168, there are various sources of forecast uncertainties. Plotting the simulated back-/now-/forecasting distributions using all post burn-in Gibbs sampling draws, $\mathbb{Z}_t^{(i)}$, we are able to illustrate one sort of uncertainty induced by the parameter estimation uncertainty. Further, because forecast errors depend on Σ_U, the variance-covariance in the innovation process, we can plot the simulated back-/now-/forecast distributions using innovations generated from the estimates of Σ_U ($\hat{\Sigma}_U$) from data sets with both the unreduced dimension (i.e. with all variables) and reduced dimension (i.e. variables selected by SSVS).

4 Empirical Analysis

4.1 Data

The data-set covers January 2nd, 1985 to December 31st, 2009. It contains twenty-three macroeconomic leading variables announced at monthly frequency, and three sets of GDP estimates announced at quarterly frequency. Therefore,

Figure 1: Plots of E1(non-farm payrolls), HS1(housing starts), CC1(consumer confidence), and the quarterly announced 'final' estimate of GDP growth.



we can observe at least one announcement each working day. Because the dates of announcements are not fixed, and the first observations for many macroeconomic variables are not available at the start of some data series, the data set is unbalanced, unsynchronized, and observed at mixed frequency. For instance, the first observation for the manager purchase index (ISM) variable is announced on January 5th, 1985, while the first observation for the new home sales (NHS1) variable is not available until March 30th, 1988. To standardize the announcement figures, we do not employ the actual announcements, but compute the implied growth rate (percentage change) of the released macroeconomic variables. Released GDP growth is computed on a quarterly basis based on the original BEA announcements.

Monitored on a daily frequency, the data contains $9,129$ observations. The missing rates are about 97% for monthly announced macroeconomic variables, and 99% for quarterly announced GDP estimates. Since the macroeconomic leading variables are announced at monthly frequency, the time delay of announcements indicated by the reporting lags, RL, is at most two months. There are a few macroeconomic variables - such as factory orders (FI2), trade deficit (TRD1), and construction spending (CS1) - occasionally announced with a RL of three months. In contrast, consumer confidence (CC1) is typically announced within the current month. Thus, the RL of CC1 is at most one month. Table (1) provides an overview of the employed variables and gives corresponding descriptive statistics. To illustrate, figure (1) plots time series of three macroeconomic leading indicators, i.e., non-farm payrolls (E1), housing starts (HS1), consumer confidence (CC1), and the quarterly announced 'final' estimate of GDP growth.

4.2 Estimation with the MCMC

The modified nowcasting model based on Evans (2005) and the SSVS approach are applied to our data set. First, we analyse the full sample from January 2nd, 1985 to December 31st, 2009. To look for empirical evidence that GDP growth is correlated with a set of leading indicators, and because this set changes over time, we apply the data-driven SSVS approach to five separate 5-year sub-samples. Figure (2) plots the $GDP3$ data announced by the BEA.

Figure 2: This figure plots the GDP3 data announced by the BEA from January 2nd, 1985 to December 31st, 2009.

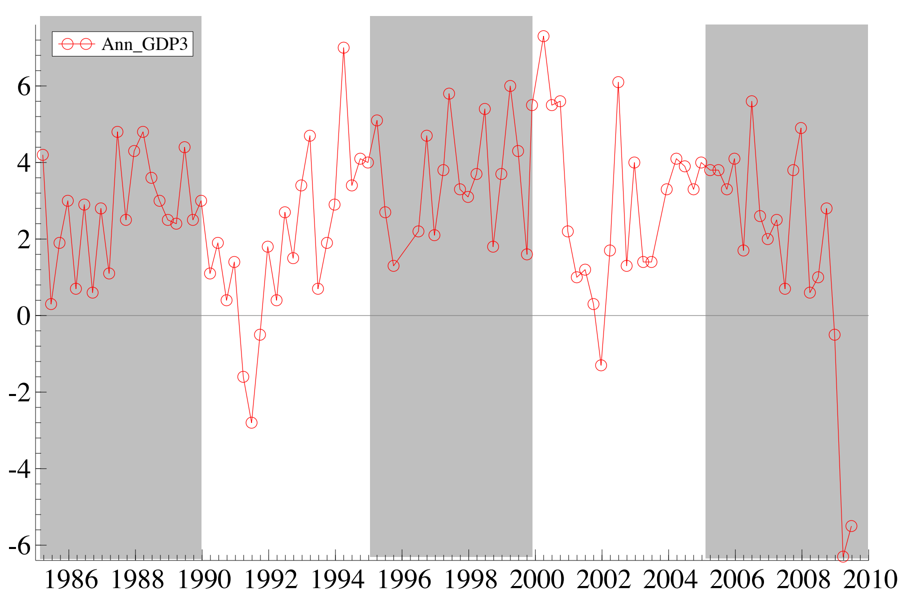


Table 1: Descriptive statistics of the underlying variables. All series announced at monthly frequency are computed in monthly percentage changes. BEA announces GDP growth on a quarterly frequency. ‘RL’, the reporting lag, indicates the length of time delay (in months) between the data collection period and the date of the actual announcement. i.e., if the time delay is greater or equal than one day but smaller than one month, RL equals one. If there is no time delay, the data is announced before the data collection period is complete, RL equals zero.

id	Variable Name	frequency	obs	miss	min	mean	max	minRL	maxRL	avRL
E1	non-farm payrolls	monthly	300	8829	-0.496	0.096	0.602	1	1	1
E2	unemployment rate	monthly	300	8829	-0.4	-0.005	0.5	1	1	1
E3	hourly earnings	monthly	265	8864	-1.3	0.274	1.6	1	1	1
RS1	retail sales	monthly	300	8829	-5.8	0.272	7	1	1	1
RS2	core retail sales	monthly	246	8883	-3.1	0.285	2.2	1	1	1
PPI1	producer price index	monthly	300	8829	-2.8	0.181	3.2	1	2	1.007
PPI2	core PPI	monthly	246	8883	-1	0.137	1.3	1	2	1.007
CPI1	consumer price index	monthly	299	8830	-1.7	0.238	1.2	1	2	1.003
CPI2	core CPI	monthly	245	8884	-0.1	0.224	0.8	1	2	1.003
IP1	industrial production	monthly	300	8829	-2.8	0.127	1.7	1	1	1
IP2	capacity utilization	monthly	261	8868	-2.3	-0.036	1.6	1	1	1
PI1	personal income	monthly	298	8831	-2.3	0.417	3.7	1	2	1.252
PI2	pers. cons. expendit.	monthly	297	8832	-2	0.418	2.9	1	3	1.255
ISM1	ISM / NAPM	monthly	300	8829	-6.4	0.011	5.9	0	1	0.937
DGO1	durable goods orders	monthly	299	8830	-12.4	0.235	12.8	1	2	1.007
HS1	housing starts	monthly	299	8830	-20.253	-0.049	30.645	1	2	1.007
NHS1	new home sales	monthly	261	8868	-23.968	0.228	25.252	1	3	1.241
BI1	business inventories	monthly	262	8867	-1.5	0.222	1.1	2	3	2.004
LI1	leading indicators	monthly	300	8829	-1.7	0.143	2.1	1	2	1.393
CS1	construction spending	monthly	260	8869	-3.3	0.204	3.1	1	3	1.973
FI2	factory orders	monthly	261	8868	-7.5	0.247	7.1	1	3	1.851
TRD1	trade deficit	monthly	299	8830	-69.9	-24.973	-3.4	1	3	1.920
CC1	consumer confidence	monthly	301	8828	-36.455	0.189	50.769	0	1	0.199
GDP1	GDP 1st estimate (‘advance’)	quarterly	98	9031	-6.1	2.526	7.2	1	2	1.031
GDP2	GDP 2nd estimate (‘preliminary’)	quarterly	98	9031	-6.2	2.669	8.2	1	4	2.071
GDP3	GDP 3rd estimate (‘final’)	quarterly	95	9034	-6.3	2.6	7.3	2	6	3.011

The Gibbs sampler is run for $2,500$ iterations with the first $1,000$ iterations discarded. To demonstrate that the sampling algorithm is efficient, we plot two examples of the MCMC draws and histograms for two of the parameters we estimated. From the plots of MCMC draws, figure (3) shows that there is no indication that the chain does not converge.

Figure 3: This figure plots two examples of the MCMC draws and histograms. The top panel plots the MCMC draws and histogram for ϕ_1, which connects daily growth s_{t+1} and monthly growth $m_{t,1}$ as in equation (5), where $s^d_{t+1} = \phi_1 \left(1 - \iota^M_t\right) m_{t,1} + \phi_1 \iota^M_t \left(s^M_t + s^d_t\right) + \phi_2 m_{t,2} + \phi_3 m_{t,3} + v_t$. The bottom panel plots the MCMC draws and histogram for β_{23}, the coefficient of $CC1$, where the monthly macroeconomic variable and the latent monthly growth are connected as in equation (17), $z^i_t = \beta_i \left(s^M_t D^d_{i+3} D^M_{0,t} + m^M_{t,1} D^d_{i+3} D^M_{1,t} + m^M_{t,2} D^d_{i+3} D^M_{2,t}\right) + D^d_{i+3} u^i_t$, $i = 23$.

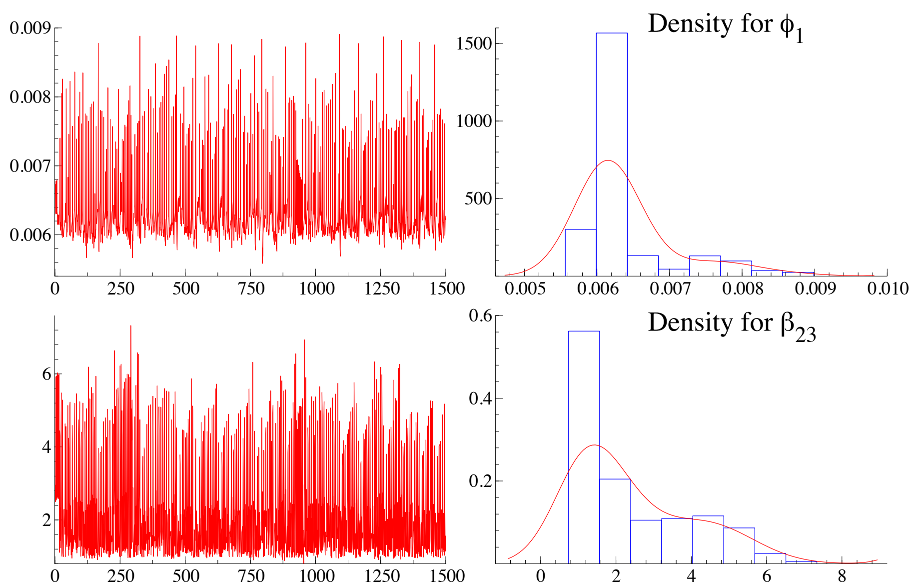
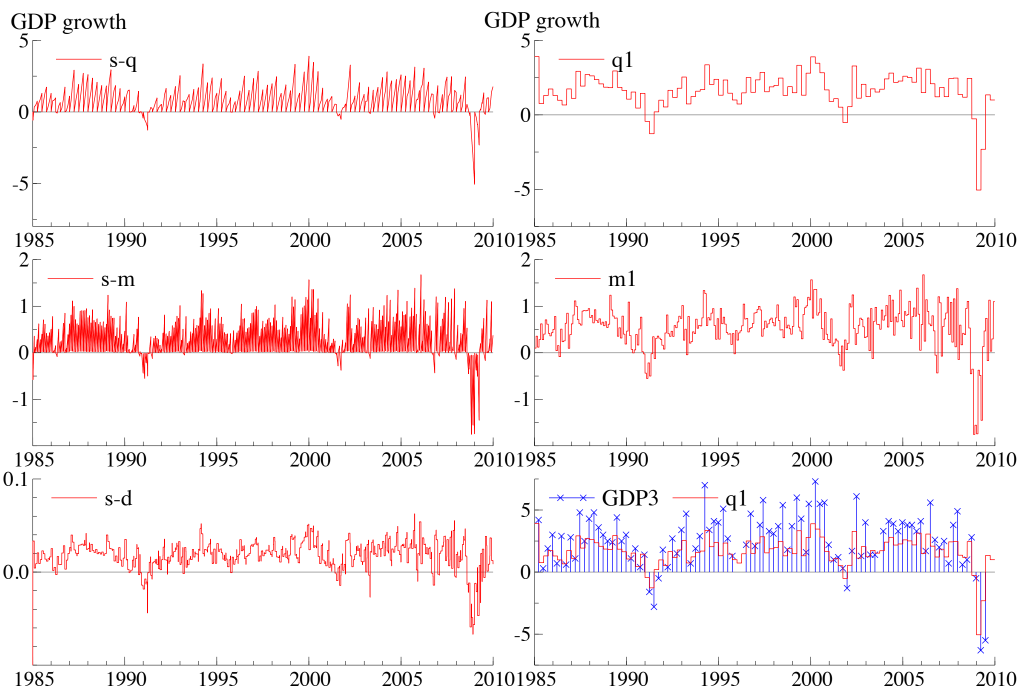


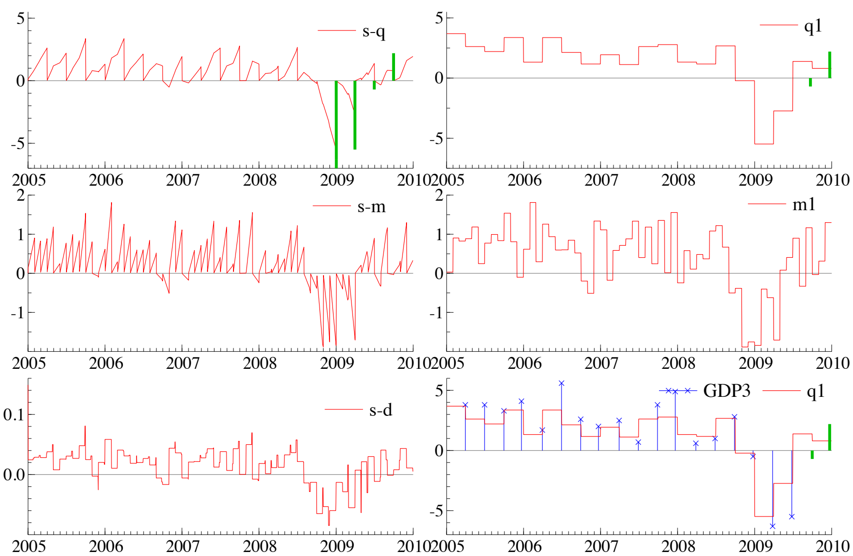
Figure (4) plots the filtered latent state of GDP growth using the full data set. The ‘final’ estimate of GDP growth, $GDP3$, is typically announced with one quarter RL. In other words, the length of time delay between the data collection period and the date of the actual announcement is longer than one day but shorter than a quarter. Because the estimated $\hat{q}_{t,1}$ in this nowcast state space model represents the estimate of $q_{t,1}$, which is the growth of the previous completed quarter with t as the reference point, it is thus intuitive to take $\hat{q}_{t,1}$ as the fitted value of previous quarter’s GDP growth, $q_{t,1}$, and compare it with the BEA’s ‘final’ estimate of GDP growth. In other words, plotting $\hat{q}_{t,1}$ (the estimate of underlying growth $q_{t,1}$ using this nowcast state space model) against $GDP3$ (the'final' estimate of $q_{t,1}$ made by the BEA) can illustrate how close are the estimates of $q_{t,1}$ achieved from this nowcasting model to the BEA's 'final' estimates. In the bottom-right of figure (4), the index lines plot the BEA's 'final' estimate, $GDP3$, which represent the BEA's estimate of the 'true' underlying growth, $q_{t,1}$. In the bottom-right of figure (4), by plotting $\hat{q}_{t,1}$ and $GDP3$ – an approximate measurement of the unobservable $q_{t,1}$ in this model setup – the model fits the data well, where $GDP3$ and $\hat{q}_{t,1}$ follow a similar pattern.

Figure 4: This figure shows the plots of the filtered latent state of GDP growth, where the full sample of data, covering the period from Jan 2nd, 1985 to Dec 31st, 2009, is analysed. $s - q$ represents the cumulation of growth from the beginning of a quarter. $q1$ represents the lagged complete quarterly growth. $s - m$ represents the cumulation of growth from the beginning of a month. $m1$ represents the lagged monthly growth. $s - d$ represents daily growth. In the bottom-right, $GDP3$ is plotted against $\hat{q}_{t,1}$, where $GDP3$ is the announced 'final' estimate of growth by the BEA. The y-axis measures GDP growth and the x-axis indicates the time span.

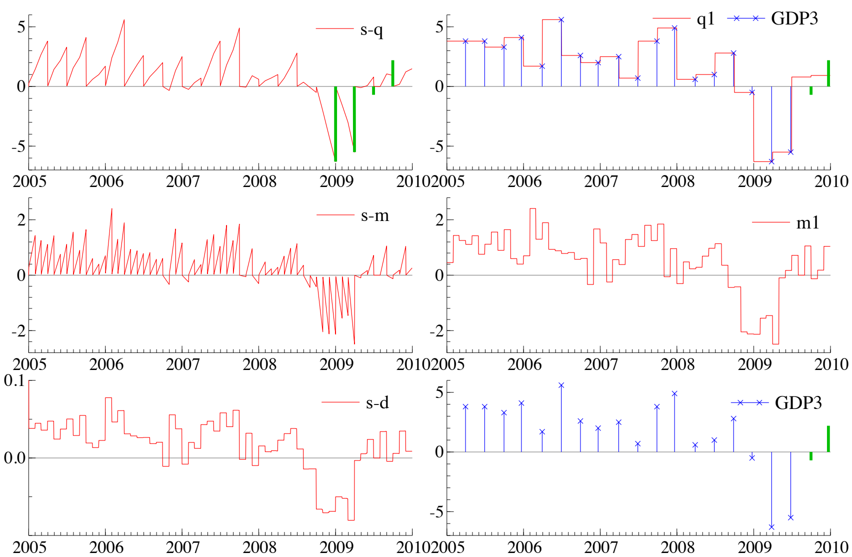


As stated in eq.(11), an estimation error e_t^3 is allowed for in our model setup. Recall our earlier statement that a restriction on the variance of e_t^3 in Evans (2005) assumes the BEA's 'final' estimate $GDP3$ is equivalent to the 'true' underlying growth $q_{t,1}$ and thus error-free. Relaxing this assumption without imposing restrictions on e_t^3, then, naturally allows for future revisions.

Figure 5: These figures plot the filtered latent state of GDP growth, where a sub-sample of data, covering the period from Jan 1st, 2005 to Dec 31st, 2009, is analysed. $s-q$ represents the cumulation of growth from the beginning of a quarter. $q1$ represents the complete quarterly growth with one quarter lag. $s-m$ represents the cumulation of growth from the beginning of a month. $m1$ represents the monthly growth with one month RL. $s-d$ represents daily growth. In the bottom-right of figure (a) and top-right in figure (b), $GDP3$ are plotted against $\hat{q}_{t,1}$, where $GDP3$ is the announced 'final' estimate of growth by the BEA. The y-axis measures GDP growth and the x-axis indicates the time span.



(a) No restriction on Σ_U was imposed.



(b) Restriction on Σ_U was imposed, where e_t^3 is restricted to 0

This nowcast state space model and the Kalman filter smoother allow us to perform backcasts/nowcasts/forecasts without extra computational cost. For instance, figure (5) plots the filtered latent state of GDP growth using a sub-sample covering the period from January 1st, 2005 to December 31st, 2009. Figure 5(a) presents results where no restriction on e_t^3 is imposed, and figure 5(b) presents results where e_t^3 is restricted to equal to 0. The blue index lines, labelled as $GDP3$, represent the 'final' estimates of GDP growth announced by the BEA, which are corresponding to the announcement dates. In our data set, the last blue index line in the bottom-right of figure 5(a) represents the BEA's 'final' estimate of GDP growth of quarter one, 2009, which was announced on June 25th, 2009.

The solid red line (labelled as $q1$) in the bottom-right of figure 5(a) plots $\hat{q}_{t,1}$, which are the estimates of $q_{t,1}$. Therefore, the red solid line in the bottom-right of figure 5(a) between July 1st, 2009 and December 31st, 2009 represents the estimates of GDP growth of quarter two (April, May, and June) and quarter three (July, August, and September) using this nowcasting state space model, whereas the data set does not cover the BEA's estimates of GDP growth in these two quarters. The two green index lines in the top-right of figure 5(a) and figure 5(b) correspond to the BEA's 'final' estimates of GDP growth in quarter two and quarter three, 2009. The BEA's 'final' estimates of GDP growth in quarter two and quarter three, which were subsequently estimated at -0.7 and 2.2.

One advantage of using this nowcasting state space model is that simple linear interpolations cannot capture the nonlinear changes in s_t^Q and s_t^M. For example, as explained in equations (10) and (11), $q_{t,1}$, the 'true' latent growth, is related with y_t^3 (observable 'final' estimate by the BEA), by considering both the RL ($D_{i,t}^Q, i = 1, 2$) and the date that the data is announced ($D_{3,t}^d$). $q_{t,1}$ is also connected with s_t^Q and s_t^M as stated in equation (6). Therefore, the changes in s_t^Q and s_t^M represented in this nowcasting state space model, of course, depend on $D_{i,t}^Q$, $D_{3,t}^d$, and y_t^3. Particularly, when the SSVS approached is applied, the estimates of s_t^Q and s_t^M depend on both the selected macroeconomic variables and the estimate of Σ_U. As a result, the changes in s_t^Q and s_t^M are not linear in time and cannot be achieved by using simple linear interpolations.

In figure 5(a), $s - q$ and $s - m$ represent the cumulative growth from the beginning of each quarter (s_t^Q) and the cumulative growth from the beginning of each month (s_t^M), respectively. From figure 5(a), the values for both s_t^Q and s_t^M

are 0 at the start of the quarter/month and gradually, but nonlinearly, change toward the end of the quarter/month. Moreover, the estimates of $q_{t,1}$, $q_{t,2}$, and $m_{t,i}$ with $i = 1, 2$ can give us a fuller picture of the latent state of the economy, especially when the BEA's estimates of the GDP growth are not yet announced.

To illustrate how the sampling algorithm can be applied to the nowcasting model proposed in Evans (2005), where e_t^3 is restricted to equal to 0, figure 5(b) plots the filtered latent state of GDP growth using a sample covering the period from January 1st, 2005 to December 31st, 2009. In the bottom-right of figure 5(a) and the top-right of figure 5(b), $GDP3$ are plotted against $\hat{q}_{t,1}$, where $GDP3$ is the announced 'final' estimate of growth by the BEA. As illustrated in the top-right of 5(b), if $e_t^3 = 0$, the BEA's 'final' estimates are taken as the 'true' underlying growth under this restriction. In other words, the BEA's 'final' estimates are error-free. As a result, no future revision or backcastings for $q_{t,1}$ is necessary, and the values of $GDP3$ are simply taken as the true underlying GDP growth, $q_{t,1}$, which is indicated in eq.(10) and eq.(11). Therefore, a nowcasting state space model without imposing restriction on e_t^3 is more realistic in the sense that future revisions of the estimates of growth, $\hat{q}_{t,1}$, are allowed for.

4.3 Stochastic Search Variable Selection in the Nowcasting Model

Figure (6) plots $\hat{\omega}_{ij}$, the estimate of the restriction index under SSVS for ψ_{ij} using the whole sample of data, and five seperate 5-year sub-samples. As stated in eq. (21), if ω_{ij} is larger than 0.5, ψ_{ij} is then selected, which indicates variable i is more likely to be correlated with the variable j. Because we are particularly interested in the forecasts/nowcasts/backcasts of y_t^3 ($GDP3$ variable), and the partial correlation between y_t^3 and other variables, we focus on the first row in each of the heat maps. If ω_{1j}, the element in the first row, is larger than 0.5, ψ_{1j} is then selected as illustrated in eq. (21). This indicates variable j is more likely to be correlated with the first variable, y_t^3, the BEA's 'final' estimate of GDP growth. Therefore, variable j has more forecasting/nowcasting/backcasting power with respect to y_t^3 in the chosen sample period. The greater the importance of a variable j in forecasting/nowcasting/backcasting y_t^3, the closer ω_{1j} is to 1.

Results from figure (6) indicate that the possible restrictions on ψ_{1j} may have changed over time, and therefore, the partial correlation between $GDP3$ and other macroeconomic variables may vary over time. This result implies that the set of important macroeconomic variables that can be used to nowcast/forecast/backcast the GDP growth is time-varying. Therefore, we would like to give more weight to the important macroeconomic indicators that were visited by the SSVS, when we nowcast/forecast/backcast the GDP growth at different periods of time.

To illustrate that different sets of macroeconomic variables are selected by the SSVS approach over time, we then estimate the data with a rolling window for 240 months, where the window size is fixed and each step rolls forward with 30 observations. The initial sample is from 1985:01 to 1989:11. To illustrate how often each variable is selected in different time period (T), we calculate the selection frequency as n/T, where n is the number of times that ω_{1j} goes above 0.5, and T is the total duration (in months) of each time period. The time periods are selected according to the US business cycle expansion and contraction dates estimated by the NBER.

Table (2) presents how frequent each variable is visited by the SSVS approach if we wish to select important variables to forecast/nowcast/backcast GDP3. From table (2), quarterly announced GDP variables and monthly non-farm payrolls (E1), unemployment rate (E2), hourly earnings (E3), retails sales (RS1), core retail sales (RS2), and producer price index (PPI1) are selected most of the time over the 240-month period. Factory orders (FI2), trade deficit (TRD1), and consumer confidence (CC1), on the other hand, have less forecast/nowcast/backcast power in y_t^3 (GDP3). Manager purchasing index (ISM1), housing starts (HS1) and new home sales (NHS1) are selected more often in the II (91:03 – 01:02) and III (01:11 – 07:11) expansion periods.

We take six monthly announced macroeconomic variables as an example to illustrate that the set for selected important variables changes over time. Figure (7) plots the estimated selection index $\hat{\omega}_{1j}$ for these six monthly announced macroeconomic variables over 240-month rolling windows. Note that variable j is selected as an important variable in forecasting/nowcasting/backcasting y_t^3, the BEA's final estimate of the GDP growth, if the estimated values of ω_{1j} go above the grey line ($\hat{\omega}_{1j} > 0.5$).

Table 2: This table presents how frequent each variable is selected based on the estimations of selection index ω_{1j} in the corresponding periods (T). The selection frequency is calculated as n/T, where n is the number of times that $\omega_{1j} > 0.5$ in the estimation period and T is the duration of a estimation period. For example, $\omega_{1,10}$ is larger than 0.5 n (62) out of T (240) times, where T is the total number of estimation period. The frequency of variable PPI2 to be selected from 1989:12 to 2009:11 is $62/240 = 0.2583$. In the first contraction period between 1990:07 and 1991:02, PPI was selected 7 out of 8 times, the corresponding selection frequency for PPI2 in this estimation period is 0.875. The first estimation sample is from Jan 2nd, 1985 to Dec 31st, 1989. The nowcasting is carried out by a rolling window for 240 months, with a fixed window size. In each step, the window rolls forward with 30 observations. Estimation periods are selected according to the US business cycle expansion and contraction dates on http://www.nber.org/cycles.html

	Total	Total Exp.	Total Cont.	Exp. I	Cont. I	Exp. II	Cont. II	Exp. III	Cont. III	Exp. IV
dur.	240	206	34	7	8	120	8	73	18	6
	89:12-09:11			89:12-90:06	90:07-91:02	91:03-01:02	01:03-01:10	01:11-07:11	07:12-09:05	09:06-09:11
GDP3	1	1	1	1	1	1	1	1	1	1
GDP2	1	1	1	1	1	1	1	1	1	1
GDP1	1	1	1	1	1	1	1	1	1	1
E1	1	1	1	1	1	1	1	1	1	1
E2	1	1	1	1	1	1	1	1	1	1
E3	0.9958	1	0.9706	1	1	1	0.875	1	1	1
RS1	0.7625	0.7669	0.7352	1	0.875	0.725	0.625	0.8082	0.7222	0.8333
RS2	0.7833	0.8058	0.6470	1	0.875	0.8416	0.5	0.7123	0.6111	1
PPI1	0.6875	0.6941	0.6470	1	1	0.7083	0	0.6438	0.7778	0.6667
PPI2	0.2583	0.2524	0.2941	1	0.875	0.3167	0	0.0958	0.1667	0
CPI1	0.1542	0.1602	0.1176	0.4285	0.25	0.1833	0	0.1095	0.1111	0
CPI2	0.3083	0.3107	0.2941	1	0.625	0.2417	0.125	0.3835	0.2222	0
IP1	0.3583	0.3786	0.2352	1	0.375	0.325	0.5	0.4383	0.0556	0
IP2	0.2333	0.2135	0.3529	0	0.125	0.1917	0.75	0.2739	0.2778	0.1667
PI1	0.1167	0.1262	0.0588	0	0	0.0167	0	0.3287	0.1111	0
PI2	0.4708	0.4854	0.3823	0.1428	0	0.4	0.625	0.6438	0.4444	0.6667
ISM1	0.4167	0.4417	0.2647	0.14285	0.125	0.4667	0.375	0.4657	0.2778	0
DGO1	0.1833	0.2087	0.0294	0	0	0.275	0.125	0.1369	0	0
HS1	0.2791	0.3155	0.0588	0	0.125	0.2833	0	0.3424	0.0556	1
NHS1	0.025	0.0291	0	0	0	0.0333	0	0.0274	0	0
BI1	0.0708	0.0776	0.0294	0	0	0.0833	0	0.0822	0.0556	0
LI1	0.0208	0.0243	0	0	0	0.025	0	0.0274	0	0
CS1	0.1667	0.1699	0.1470	0.7142	0.125	0.0917	0	0.2602	0.2222	0
FI2	0	0	0	0	0	0	0	0	0	0
TRD1	0	0	0	0	0	0	0	0	0	0
CC1	0	0	0	0	0	0	0	0	0	0

4.4 Nowcast Densities

Following the forecasting/nowcasting/backcasting procedure described in section 3.3, figure (8) plots the rolling window nowcasting densities for y_t^3, which is achieved by applying the VAR approach to all data series. It is called as nowcasting densities because y_t^3 are simulated using $\hat{q}_{t,1}$ and $\hat{\Sigma}_U$, which are estimated using data up to day t. Figure (9) plots the rolling window nowcasting densities, where the SSVS approach is applied. The first sample we use for estimation is from January 2nd, 1989 to December 1st, 1989. In each step, the window rolls forward by 30 observations. With both approaches, the nowcast distribution for y_t^3 is simulated using $\hat{\Sigma}_U$, the estimated variance-covariance.

With the SSVS approach, the nowcasting is carried out in two steps. In the first step, the sample is estimated to achieve $\hat{\omega}_{1j}$. Then, the selection is determined by the selection index $\hat{\omega}_{1j}$. Therefore, different sets of variables are selected as the window rolls forward in each step. In the second step, the selected variables are picked out and used for a second-step estimation. In other words, if $\hat{\psi}_{1j}$, the indication of the partial correlation between variable j and y_t^3, is not selected according to $\hat{\omega}_{1j}$, data series j will not be included in the second step for estimation. Finally, estimations of latent variable $\hat{q}_{t,1}$ are used for point nowcasts of y_t^3.

From figure (9), the simulated distributions of y_t^3 achieved using the SSVS change over time, i.e. the distributions are wider in some time periods and narrower in others, which implies that the nowcast uncertainties are time-varying. In figure (8), the distributions using the VAR approach with all 26 data series are included, have a similar shape across time. In other words, the estimated variances ($\hat{\Sigma}_U$) of the simulated distributions do not change much over time. Comparing figure (8) and figure (9), the densities with the SSVS approach (in figure 9) are narrower, relative to those achieved from using all variables (in figure 8). Because nowcast errors of y_t^3 depend on the variance-covariance, Σ_U, it is crucial to allow Σ_U to be estimated depending on a set of selected important indicators. In other words, the SSVS approach allows for a more certain prediction of GDP growth than the traditional approach of using all possible explanatory variables.

In order to investigate how the forecast/nowcast/backcast distributions change as time evolves, we focuse on forecasting/nowcasting/backcasting the growth of quarter one (Q1), 2009. Quarter one starts from January 1st, and ends on March 31st. However, the BEA had not announced their estimate of the Q1's growth until June 25th (at -5.5). The RL of the BEA's 'final' estimate of GDP growth in quarter one is almost three months.

Figure (10) plots the forecasting/nowcasting/backcasting distributions for the GDP growth in Q1, 2009 using both the VAR and the SSVS approaches. The VAR approach uses all available information by including all data series for estimation. The SSVS approach, on the other hand, follows a two-step estimation procedure. The initial sample covers January 1st, 2005 to October 15th, 2008. Then, an expanding rolling window forecasting/nowcasting/backcasting is carried out for 35 steps. In each step, the window rolls forward and expands by 10 observations. The last sample used for estimation covers January 1st, 2005 to September 20th, 2009.

If the sample is up to January 1st, 2009, where data used for estimation does not cover any information in 2009 Q1, GDP growth in 2009 Q1 is achieved by following the forecasting procedure as illustrated in section 3.3. If the sample used for estimation contains data between January 1st, 2009 and March 31st, 2009, the estimations are, then, used to obtain the point/density nowcast of the GDP growth in 2009 Q1 by following the nowcasting procedure. If the sample used for estimation includes the whole time period of quarter one 2009, backcasting procedure is used. In the last step, the estimation uses the whole sample, which covers January 1st, 2005 to September 20th, 2009. Therefore, $\hat{q}_{t,2}$ achieved from estimation is the corresponding point backcast for GDP growth in quarter one.

From figure (10), the forecasting/nowcasting/backcasting distributions achieved using the SSVS approach (top-right) are time-varying. Plotting the point forecasts/nowcasts/backcasts achieved from both the SSVS and VAR approach together in figure (10), there is less fluctuation in the point backcasts samples than that in the forecast and nowcasts samples.

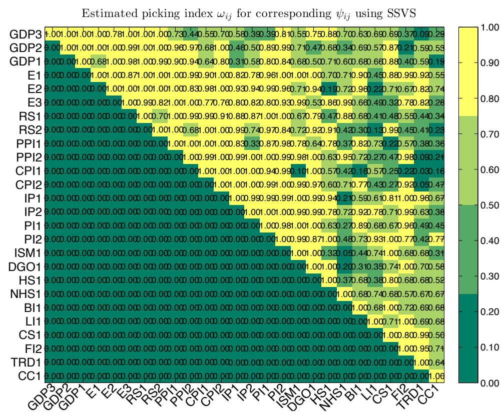
Figure (11) plots the estimated selection index $\hat{\omega}_{1j}$ for $GDP2$, $GDP1$, and 23 monthly macroeconomic variables. Variable j is more likely to be correlated with $GDP3$, if the value of $\hat{\omega}_{1j}$ are above 0.5. For example, the 18th step on the x-axis corresponds to date April 3rd, 2009. In other words, the selection index $\hat{\omega}_{1j}$ at the 18th step are estimated using data from January 1st, 2005 to April 3rd, 2009. Similarly, the 27th step on the x-axis corresponds to date July 2nd, 2009, which implies the selection index $\hat{\omega}_{1j}$ at and after the 27th step contains information well beyond the end of quarter one, 2009.

From figure (11), the weights of the importance of variables in forecasting/nowcasting/backcasting GDP growth are clearly time-varying. For instance, the construction spending (CS1) variable is not selected before April 3rd, 2009 in forecasting and nowcasting the quarter one's GDP growth. However, it is selected several times in the periods when we carry out the backcasting for the GDP growth in quarter one.

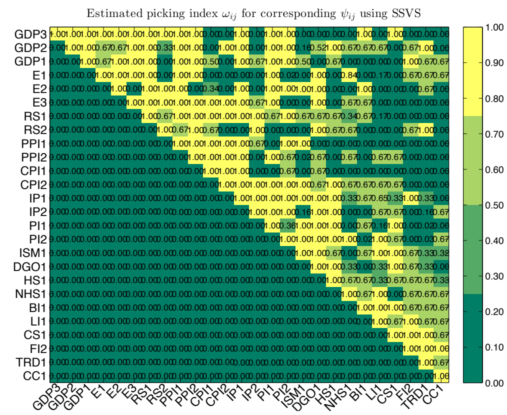
Another similar exercise is carried out to forecast/nowcast/backcast the GDP growth in quarter four (Q4), 2008. Figure (12) plots the forecasting/nowcasting/backcasting densities for GDP growth in quarter four (Q4), 2008 using both the VAR and the SSVS approaches. The VAR approach uses all available information by including all data series for estimation. The SSVS approach follows a two-step estimation procedure. The initial sample covers from July 10th, 2004 to July 22th, 2008. Then, an expanding rolling window is carried out for 35 steps. In each step, the window rolls forward and expands by 10 observations. The last sample used for estimation covers July 10th, 2004 to June 27th, 2009. Forecasting for GDP growth in 2008 Q4 is carried out when the estimation sample is up to Oct 1st, 2008. Nowcasting is carried out when the sample used for estimation contains data between Oct 1st, 2008 and Dec 31st, 2008. If the sample used for estimation includes the whole time period of quarter four 2008, the backcasting procedure is used.

In figure (12), the point forecast/nowcast/backcast for GDP growth in Q4, 2008 using both the SSVS and the VAR approach are plotted together. In figure (12), we illustrate that as time passes - and our nowcast becomes a backcast - our estimates of economic activity become more precise. In other words, the forecasts and nowcasts for GDP growth appear to have more fluctuations than the backcasts.

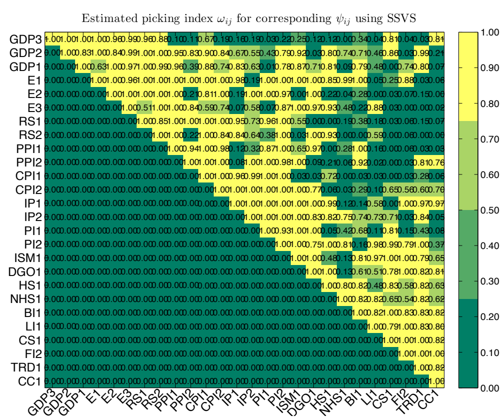
Figure 6: The upper triangular in these heatmaps display the SSVS index ω_{ij} for choices of ψ_{ij}. The first row in each of the heat map is the SSVS index ω_{1j} for choices of ψ_{1j}, where ψ_{1j} reflects the partial correlation between $GDP3$ and other variables. If the element ω_{1j} is smaller than 0.5, variable j is less likely to be correlated with $GDP3$. This indicates that variable j has little forecasting power with respect to GDP growth in the chosen sample period.



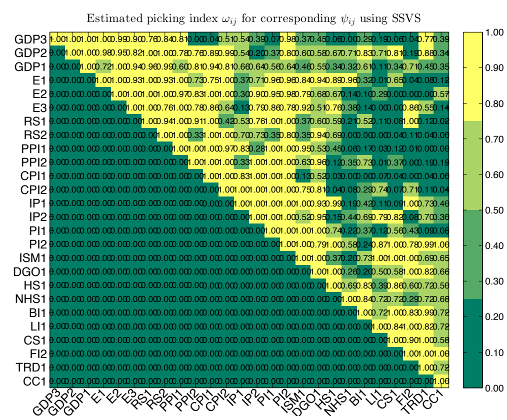
(a) Total sample Jan 1985 – Dec 2009



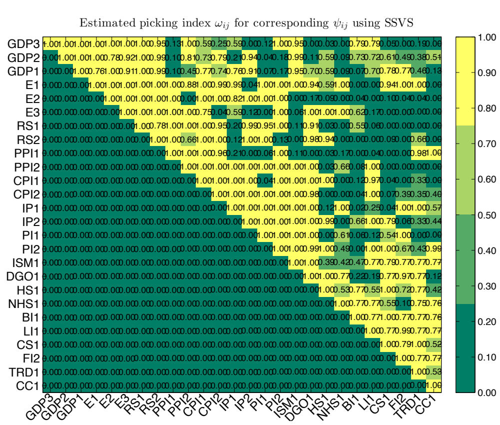
(b) Segment Period I: Jan 1985 – Dec 1989



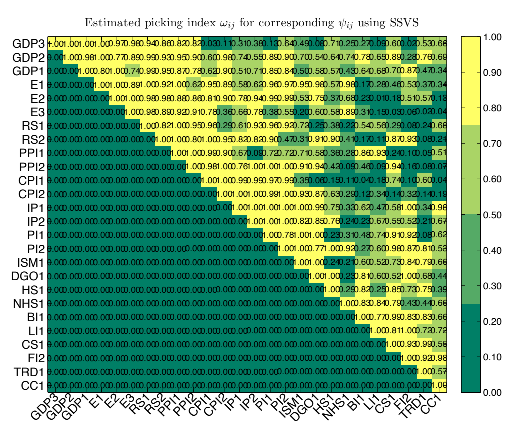
(c) Segment Period II: Jan 1990 – Dec 1994



(d) Segment Period III: Jan 1995 – Dec 1999



(e) Segment Period IV: Jan 2000 – Dec 2004



(f) Segment Period V: Jan 2005 – Dec 2009

Figure 7: This figure plots six examples of the estimated selection index $\hat{\omega}_{1j}$ for six monthly macroeconomic variables. From top to bottom: ISM1 (manager purchasing index), DGO1(durable goods orders), HS1(housing starts), NHS1(new home sales), BI1(business inventories), and LI1(leading indicators). The rolling window estimation period is from 1989:12 to 2009:11 with a time span of 240 months. If the value of $\hat{\omega}_{1j}$ goes above the grey line, variable j is more correlated with y_t^3, the first variable in vector $\mathbf{Y}_t$ in eq.(17). y_t^3 represents the BEA's 'final' estimate of the GDP growth.

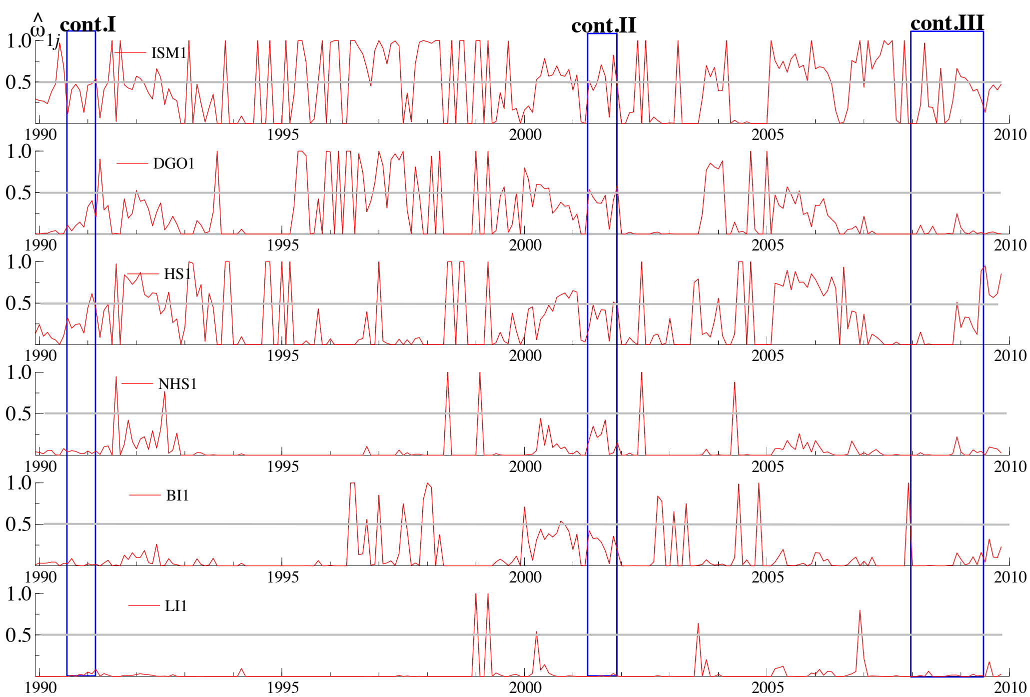
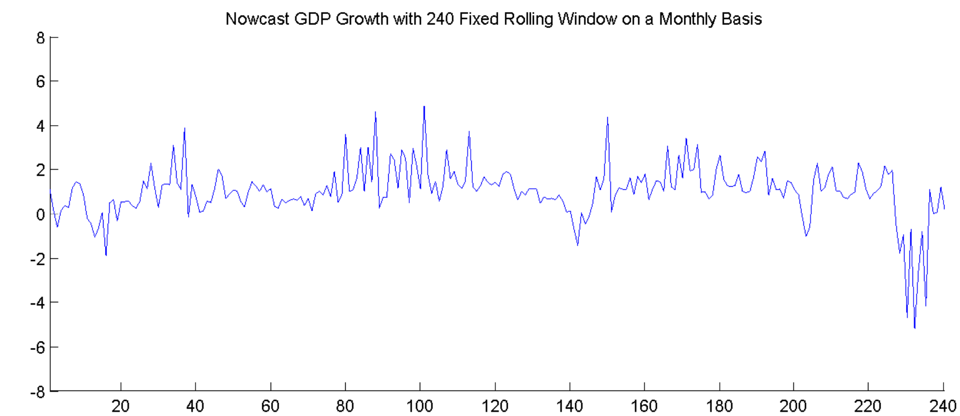


Figure 8: This figure plots the nowcasting densities where Σ_U is estimated using all 26 variables. Top: plots the point nowcast for GDP growth y_t^3. Bottom: plots the nowcast densities for 240 months covering the period 1989:12 – 2009:11. The densities are achieved by simulating innovations from the estimate $\hat{\Sigma}_U$.



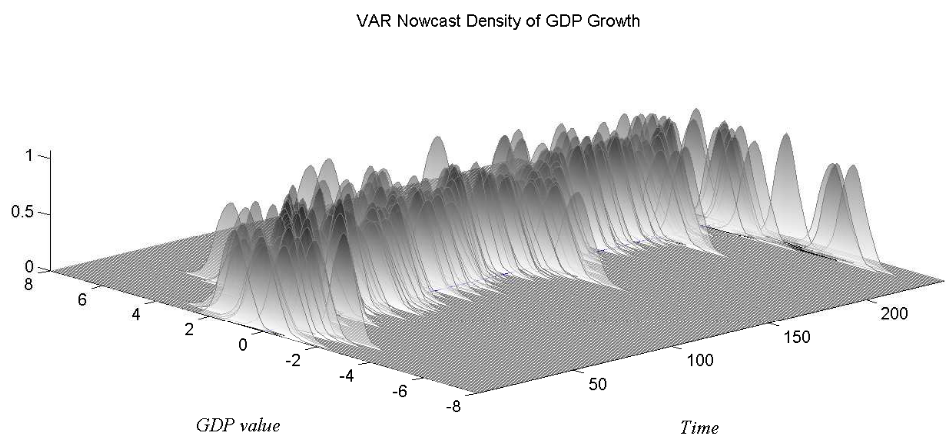
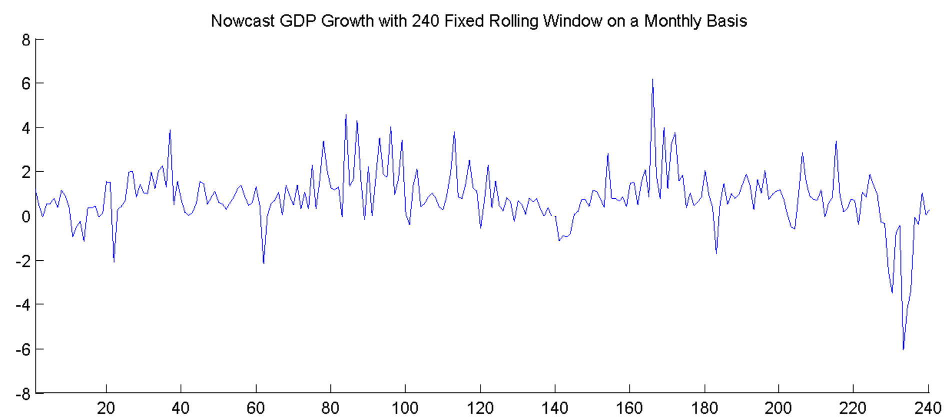


Figure 9: This figure plots the nowcasting densities where Σ_U is estimated using the SSVS method with a two-step variable selection procedure. Top: plots the point nowcast for GDP growth y_t^3. Bottom: plots the nowcast densities with a rolling window for 240-month covering the period 1989:12 – 2009:11. In each step, the data is reduced to a set of important variables, which are selected based on the estimates of ω_{1j} from the first step. The densities are achieved by simulating innovations from $\hat{\Sigma}_U$, which are estimated using data sets with reduced dimensions.



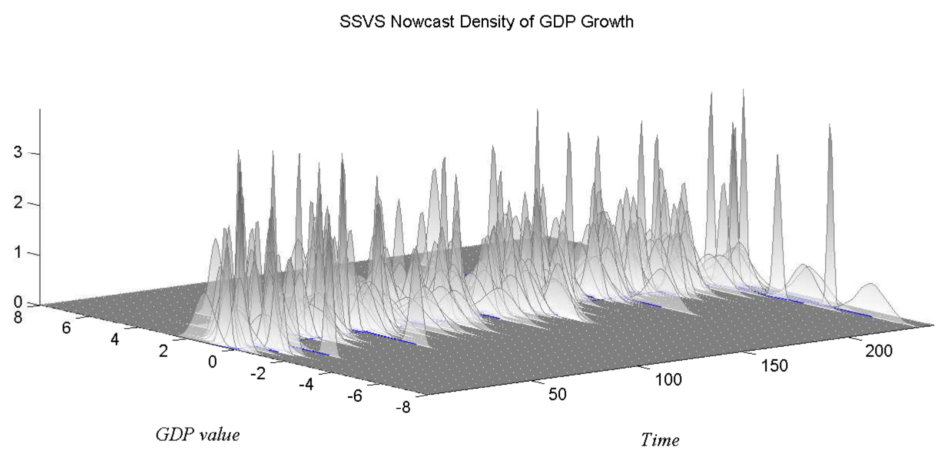


Figure 10: This figure plots the forecasting/nowcasting/backcasting densities for GDP growth in quarter one (Q1), 2009 using both the VAR and the SSVS approaches. The VAR approach uses all available information by including all data series for estimation. The SSVS approach follows a two-step estimation procedure. The initial sample covers from January 1st, 2005 to October 15th, 2008. Then, an expanding rolling window is carried out for 35 steps. In each step, the window rolls forward and expands by 10 observations. The last sample used for estimation covers January 1st, 2005 to September 20th, 2009. Forecasting for GDP growth in 2009 Q1 is carried out when the estimation sample is up to January 1st, 2009. Nowcasting is carried out when the sample used for estimation contains data between January 1st, 2009 and March 31st, 2009. If the sample used for estimation includes the whole time period of quarter one 2009, a backcasting procedure is, then, applied. Top-left: plots the point forecast/nowcast/backcast for GDP growth in Q1, 2009 using the SSVS approach. Top-right: plots the point forecast/nowcast/backcast for GDP growth in Q1, 2009 using the VAR approach including all data series. Middle-left: plots the forecast/nowcast/backcast distributions for GDP growth in Q1, 2009 using the SSVS approach. Middle-right: plots the forecast/nowcast/backcast distributions for GDP growth in Q1, 2009 using the VAR approach including all data series. Bottom: plots the point forecast/nowcast/backcast for GDP growth in Q1, 2009 using both the SSVS and the VAR approach.

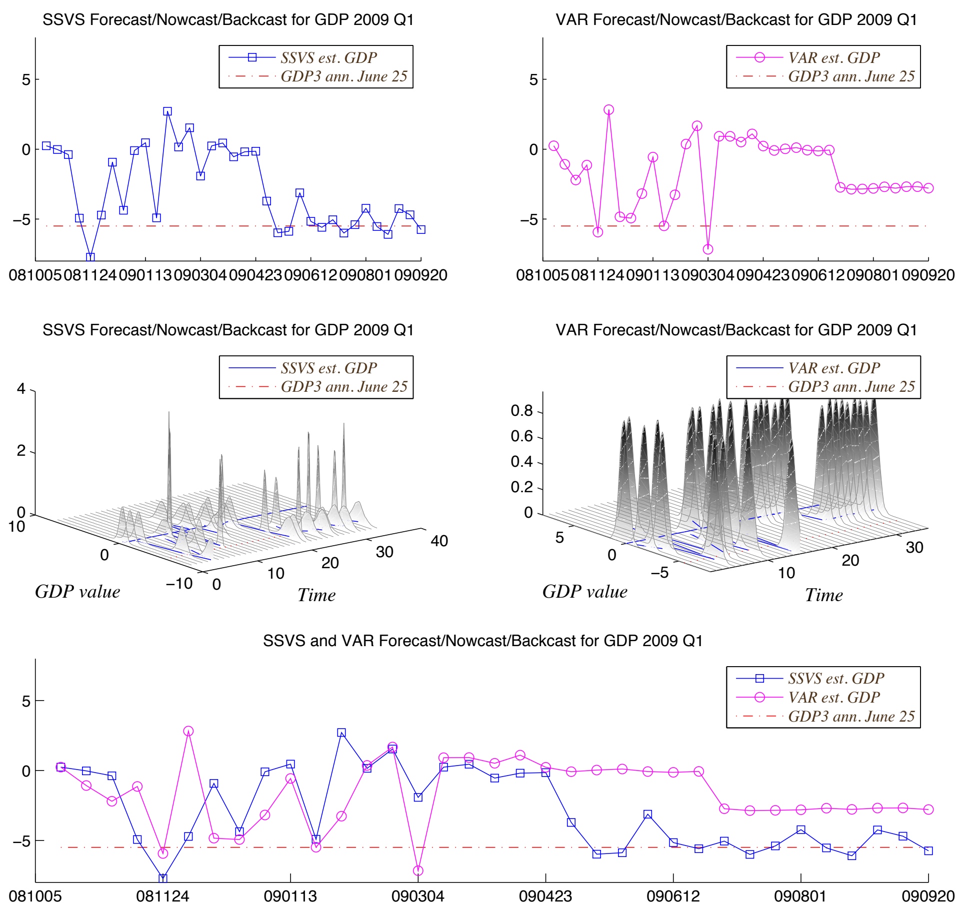


Figure 11: This figure plots the estimated selection index $\hat{\omega}_{1j}$ for $GDP2$, $GDP1$, and 23 monthly macroeconomic variables. Variable j is more likely to be correlated with $GDP3$, if the value of $\hat{\omega}_{1j}$ are above 0.5. For example, the 18th step on the x-axis corresponds to date April 3rd, 2009. In other words, the selection index $\hat{\omega}_{1j}$ at the 18th step are estimated using sample from January 1st, 2005 to April 3rd, 2009. Similary, the 27th step on the x-axis corresponds to date July 2nd, 2009, which implies the selection index $\hat{\omega}_{1j}$ at and after the 27th step contains information well beyond the end of quarter one, 2009.

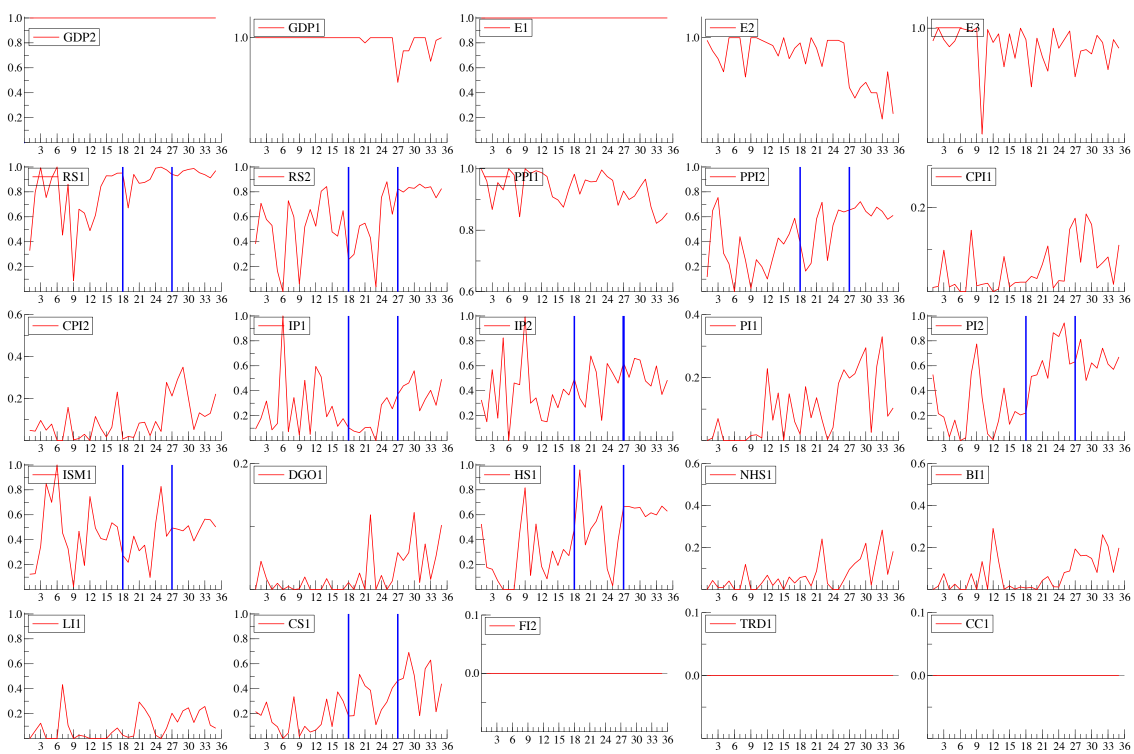
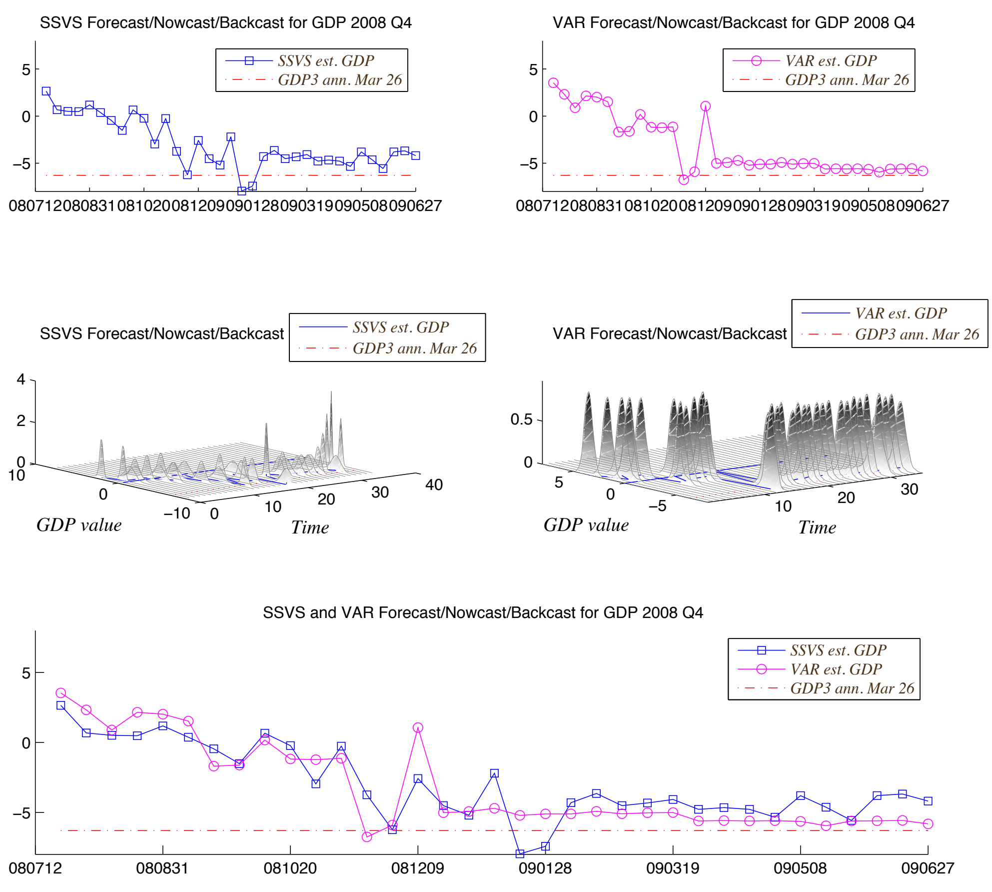


Figure 12: This figure plots the forecasting/nowcasting/backcasting densities for GDP growth in quarter four (Q4), 2008 using both the VAR and the SSVS approaches. The VAR approach uses all available information by including all data series for estimation. The SSVS approach follows a two-step estimation procedure. The initial sample covers from July 10th, 2004 to July 22nd, 2008. Then, an expanding rolling window is carried out for 35 steps. In each step, the window rolls forward and expands by 10 observations. The last sample used for estimation covers July 10th, 2004 to June 27th, 2009. Forecasting for GDP growth in 2008 Q4 is carried out when the estimation sample is up to Oct 1st, 2008. Nowcasting is carried out when the sample used for estimation contains data between Oct 1st, 2008 and Dec 31st, 2008. If the sample used for estimation includes the whole time period of quarter four 2008, the backcasting procedure is used. Top-left: plots the point forecast/nowcast/backcast for GDP growth in Q4, 2008 using the SSVS approach. Top-right: plots the point forecast/nowcast/backcast for GDP growth in Q4, 2008 using the VAR approach including all data series. Middle-left: plots the forecast/nowcast/backcast distributions for GDP growth in Q4, 2008 using the SSVS approach. Middle-right: plots the forecast/nowcast/backcast distributions for GDP growth in Q4, 2008 using the VAR approach including all data series. Bottom: plots the point forecast/nowcast/backcast for GDP growth in Q4, 2008 using both the SSVS and the VAR approach.



5 Conclusion

Using the updated macroeconomic variable announcements at a variety of mixed frequencies, not only could we find out the possible model restrictions visited by the SSVS, but we can also evaluate the uncertainties that are associated with the backcasting/nowcasting/forecasting.

Based on Evans (2005), we modified a nowcast state space model, where unbalanced, unsynchronized, and mixed frequency variables can be fitted to estimate GDP growth at various frequencies. A Kalman filter smoother within a Gibbs sampling algorithm is designed to estimate the high dimensional latent variables.

Using the draws from the iterative MCMC algorithm, backcasting, nowcasting, and forecasting densities can be achieved without extra computational cost. Therefore, the uncertainties that are associated with the point backcast, nowcast, and forecast can be evaluated. A SSVS approach is implemented within the designed MCMC algorithm, which provides a data-driven approach to help us to select relevant macroeconomic variables with respect to forecasting GDP growth.

To illustrate the potential practical use of the designed MCMC algorithm, we analyse a data set covering the period from January 2nd, 1985 to December 31st, 2009. The designed MCMC is able to filter out the latent growth on a daily, monthly, and quarterly basis. We found that the restrictions index ω_{ij} visited by the SSVS were time varying across different sample periods.

Similar to the model set-up in Evans (2005), and yet with more flexible generalizations, this study removed the restriction on e_t^3, allowing for future revisions of the estimates of GDP growth. This modified model fits the data well, where the estimated $\hat{q}_{t,1}$ can provide a more complete picture of the lagged quarterly growth corresponding to the 'final' estimate of GDP growth announced by the BEA. Moreover, the estimates $\hat{q}_{t,1}$, $\hat{q}_{t,2}$, and $\hat{m}_{t,i}$ with $i = 1, 2$ using the nowcasting model are able to give us a fuller picture of the latent state of the economy, especially when part of the data is not available. Also, backcasting/nowcasting/forecasting densities can be simulated by using the draws from the MCMC sampler with no extra cost.

Using the SSVS approach, this study provides empirical evidence that the importance of macroeconomic indicators in forecasting GDP growth has varied across the sample period from 1989:12 to 2009:11. By reducing the data dimensions, the designed SSVS within a Gibbs sampler algorithm offers a data-driven way to estimate Σ_U using a set of important variables. The simulated backcast/nowcast/forecast distribution, using the innovations generated from $\hat{\Sigma}_U$, can then provide crucial information of backcast/nowcast/forecast errors, as well as uncertainties.

We carried out a forecasting/nowcasting/backcasting exercise focusing on the GDP growth in quarter four, 2008 and quarter one, 2009. We illustrate that as time evolves, the backcasts appear to have less fluctuations than the forecasts and nowcasts for GDP growth.

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A Posteriors Derivations

The parameters of interest are collected in $\theta = \left\{\beta', \phi_1, h_e, \Sigma_U^{-1}\right\}$. Let $p\left(\mathbf{Y}, \mathbb{Z} \mid \theta\right)$, $p\left(\mathbf{Y} \mid \mathbb{Z}, \theta\right)$ and $p\left(\mathbb{Z} \mid \theta\right)$ be the joint density of $(\mathbf{Y}, \mathbb{Z})$ given θ, the conditional density of $\mathbf{Y}$ given $(\mathbb{Z}, \theta)$, and the density of $\mathbb{Z}$ given θ, respectively. The joint density of $(\mathbf{Y}, \mathbb{Z})$ given θ then is

$$p\left(\mathbf{Y}, \mathbb{Z} \mid \theta\right) = p\left(\mathbf{Y} \mid \mathbb{Z}, \theta\right) p\left(\mathbb{Z} \mid \theta\right).$$

The posterior can be derived from the likelihood and the elicited priors according to the Bayes Theorem:

$$p\left(\theta \mid \mathbf{Y}, \mathbb{Z}\right) \propto p\left(\mathbf{Y}, \mathbb{Z} \mid \theta\right) p\left(\theta\right) \tag{22}$$

We restrict ϕ_2 and ϕ_3, in eq.(6), to be zero. And therefore, only h_e and ϕ_1 have to be estimated. Denoting, $\mathbb{Z} = (\mathbb{Z}_2, ..., \mathbb{Z}_T)$,

$$p\left(\mathbb{Z} \mid \theta\right) = \left(\frac{h_e}{2\pi}\right)^{\frac{T-1}{2}} \exp\left\{-\frac{h_e}{2}\sum_{t=2}^{T}\left[\begin{array}{c} s_t^d - \phi_1\left(1 - \iota_{t-1}^M\right) m_{t-1,1} \\ -\phi_1 \iota_{t-1}^M\left(s_{t-1}^M + s_{t-1}^d\right)\end{array}\right]^2\right\}. \tag{23}$$

The derivation of likelihood $p(\mathbf{Y} \mid \mathbb{Z},\theta)$ is not straightforward. Recall eq.(18),

$$Y_t = \mathbf{X}_t\mathbf{b}+B_tU_t$$

where $\mathbf{b} = \begin{pmatrix} 1, & 1, & 1, & \beta' \end{pmatrix}'$, and $U_t \overset{i.i.d}{\sim} MN(0, \Sigma_U)$. Σ_U is treated as time invariant, which is unknown and has to be estimated. For further research, we may relax this assumption and allow for a time varying Σ_U. We restrict Σ_U to be a positive definite matrix, which permits a Cholesky decomposition, and the precision matrix Σ_U^{-1} is denoted as

$$\Sigma_U^{-1} = \psi\psi'. \tag{24}$$

Because of the missing dimensions in $\mathbf{Y}$, the posterior conditional of Σ_U^{-1} will no longer follow a wishart distribution. Therefore, we impute the missing dimensions in $\mathbf{Y}$ and $\mathbf{X}_t$ to achieve $\mathbf{Y}^*$ and $\mathbf{X}_t^*$. The data is imputed in a such way that the likelihood function is not changed. Therefore, the likelihood, with imputed Y_t^* and $\mathbf{X}_t^*$ is

$$\begin{aligned} p(\mathbf{Y}^* \mid \mathbb{Z},\theta) &= \left(\frac{1}{2\pi\left|\Sigma_U\right|}\right)^{\frac{T-1}{2}} \exp\left\{-\frac{1}{2}\sum_{t=2}^{T}\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)'\Sigma_U^{-1}\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)\right\} \\ &\propto \left|\Sigma_U\right|^{-\frac{T-1}{2}} \exp\left\{-\frac{1}{2}trace\left(\sum_{t=2}^{T}\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)'\Sigma_U^{-1}\right)\right\}. \end{aligned} \tag{25}$$

According to eq.(22) – (25) the posterior $p(\theta \mid \mathbf{Y}, \mathbb{Z})$ then is

$$\begin{aligned} p(\theta \mid \mathbf{Y}^*, \mathbb{Z}) &= \left|\psi\right|^{T-1} \exp\left\{-\frac{1}{2}trace\left(\sum_{t=2}^{T}\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)\left(Y_t^* - \mathbf{X}_t^*\mathbf{b}\right)'\psi\psi'\right)\right\} \\ &\quad \times \left(\frac{h_e}{2\pi}\right)^{\frac{T-1}{2}} \exp\left\{-\frac{h_e}{2}\sum_{t=2}^{T}\left(s_t^d - \phi_1 m_{t-1,1}\right)^2\right\} p(\theta). \end{aligned}$$

Posterior of ϕ_1

Given the prior for ϕ_1, $\phi_1 \sim N\left(\underline{\mu}_{\phi_1}, \underline{V}_{\phi_1}\right) 1\left(|\phi_1| < 1\right)$, the posterior conditional is given by

$$\overline{\mu}_{\phi_1} = \frac{\underline{V}_{\phi_1} \sum_{t=2}^{T} \left[m_{t-1,1}\left(1-\iota_{t-1}^{M}\right) + \left(s_{t-1}^{M} + s_{t-1}^{d}\right)\iota_{t-1}^{M}\right] s_t^d + \underline{\mu}_{\phi_1}/h_e}{\underline{V}_{\phi_1} \sum_{t=2}^{T} \left[m_{t-1,1}\left(1-\iota_{t-1}^{M}\right) + \left(s_{t-1}^{M} + s_{t-1}^{d}\right)\iota_{t-1}^{M}\right]^2 + 1/h_e},$$

$$\overline{V}_{\phi_1} = \frac{\underline{V}_{\phi_1}/h_e}{\underline{V}_{\phi_1} \sum_{t=2}^{T} \left[m_{t-1,1}\left(1-\iota_{t-1}^{M}\right) + \left(s_{t-1}^{M} + s_{t-1}^{d}\right)\iota_{t-1}^{M}\right]^2 + 1/h_e}.$$

Posterior of h_e

Given the gamma prior for $h_e \sim \Gamma\left(\underline{a}_e, \underline{b}_e\right)$, we have

$$\begin{aligned} p\left(h_e \mid \mathbb{Z}, \mathbf{Y}, \theta\right) &\propto \left(\frac{h_e}{2\pi}\right)^{\frac{T-1}{2}} \exp\left\{-\frac{h_e}{2}\sum_{t=2}^{T}\left[\begin{array}{c} s_t^d - \phi_1\left(1-\iota_{t-1}^{M}\right)m_{t-1,1} \\ -\phi_1\iota_{t-1}^{M}\left(s_{t-1}^{M}+s_{t-1}^{d}\right)\end{array}\right]^2\right\} \\ &\quad \cdot h_e^{\underline{a}_e-1}\exp\left\{-\underline{b}_e h_e\right\} \\ &\propto h_e^{\frac{T}{2}-1+\underline{a}_e}\exp\left\{-h_e\left[\frac{1}{2}\sum_{t=2}^{T}\left[\begin{array}{c} s_t^d - \phi_1\left(1-\iota_{t-1}^{M}\right)m_{t-1,1} \\ -\phi_1\iota_{t-1}^{M}\left(s_{t-1}^{M}+s_{t-1}^{d}\right)\end{array}\right]^2 + \underline{b}_e\right]\right\}. \end{aligned}$$

and thus

$$h_e \sim \Gamma\left(\overline{a}_e, \overline{b}_e\right),$$

with

$$\overline{a}_e = \frac{T-1}{2} + \overline{a}_e,$$

$$\overline{b}_e = \underline{b}_e + \frac{1}{2}\sum_{t=2}^{T}\left[s_t^d - \phi_1\left(1-\iota_{t-1}^{M}\right)m_{t-1,1} - \phi_1\iota_{t-1}^{M}\left(s_{t-1}^{M}+s_{t-1}^{d}\right)\right]^2.$$

Posterior conditionals for Σ_U^{-1} and β

We are interested in estimating the precision matrix Σ_U^{-1} and β, which is a subvector of $\mathbf{b}$. As B_t is a 'pick-out' matrix to indicate the variables that are observed on day t, the diagonal elements $D_{i,t}^{d}$ $(i=1,...,3+n)$ equal 1 when the data is actually announced and missing (N/A) otherwise.

$$
B_t \;=\; \begin{pmatrix} D_{3,t}^{d} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & D_{2,t}^{d} & 0 & \cdots & \cdots & \vdots \\ 0 & 0 & D_{1,t}^{d} & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & D_{3+n,t}^{d} \end{pmatrix},
$$

and

$$
B_t U_t \;=\; \mathbf{u}_t \;=\left(\begin{array}{cccccc} D_{3,t}^{d} e_t^3 & D_{2,t}^{d} e_t^2 & D_{1,t}^{d} e_t^1 & D_{4,t}^{d} u_t^1 & \cdots & D_{3+n,t}^{d} u_t^n \end{array} \right)'.
$$

The residuals can be calculated using $\mathbf{u}_t = Y_t - \mathbf{X}_t\mathbf{b}$, where Y_t and $\mathbf{X}_t$ all contain missing dimensions. Suppose $\mathbf{u}_t$ does not contain any missing dimension, the posterior conditional of Σ_U^{-1} follows a wishart distribution, $W\left(\overline{A}_U, \overline{v}\right)$ where $\overline{A}_U = \left(\sum_{t=2}^{T}\left(Y_t - \mathbf{X}_t\mathbf{b}\right)\left(Y_t - \mathbf{X}_t\mathbf{b}\right)' + \underline{A}_U^{-1}\right)^{-1}$. Because of the missing dimensions in the data structure, the full posterior conditional of Σ_U^{-1}, no longer follows a wishart distribution. Motivated by Rubin (1996), we apply the imputation technique, which is imputing the missing residuals conditional on the non-missing residuals, in order to construct an MCMC algorithm that we can simulate the posterior distribution of Σ_U^{-1}. The imputation method has been applied in various research areas, including finance applications, see Kofman and Sharpe (2003).

The imputation technique is based on the properties of multivariate distributions, refer to Koop et al. (2007). The corresponding missing elements in Y_t are replaced with the imputed residuals, and the corresponding missing elements in $\mathbf{X}_t$ are replaced with 0. The data is imputed in a such way that the likelihood function is not changed. This imputation method is also used in Zeithammer and Lenk (2006) and Schafer (1997) p.227 to solve the problem of missing dimensions in the data. With the imputed residuals, $\mathbf{u}_t$ becomes $\mathbf{u}_t^*$. The posterior conditional of Σ_U^{-1} calculated using $\mathbf{u}_t^*$ then follows a wishart distribution as if no dimension in $\mathbf{u}_t$ was absent. If all dimension in $\mathbf{u}_t$ are missing, we do not have to impute as there is no information contributing to the likelihood.

Denote ε_t^m as the missing residuals and ε_t^o as the available residuals in $\mathbf{u}_t$, then $\{\varepsilon_t^o, \varepsilon_t^m\} \sim MN(\mathbf{0}, \Sigma_U)$. We can partition the mean vector $\mathbf{0}$ and covariance matrix Σ_U as

$$\begin{pmatrix} \mathbf{0}^o \\ \mathbf{0}^m \end{pmatrix}, \text{ and } \begin{pmatrix} \Sigma_{oo} & \Sigma_{om} \\ \Sigma_{mo} & \Sigma_{mm} \end{pmatrix}$$

According to the properties of the multi-normal, the missing residuals ε_t^m can be imputed from $\varepsilon_t^m \sim f_{MN}\left(\Sigma_{mo}(\Sigma_{oo})^{-1}\varepsilon_t^o, \Sigma_{mm} - \Sigma_{mo}(\Sigma_{oo})^{-1}\Sigma_{om}\right)$. Once the missing dimensions in $\mathbf{u}_t$ are replaced by the imputed residuals, whereas missing dimensions in Y_t are imputed with ε_t^m and missing dimensions in $\mathbf{X}_t$ are replaced with 0, Y_t is imputed as Y_t^* and $\mathbf{X}_t$ is imputed as $\mathbf{X}_t^*$ correspondingly.

With an elicit wishart prior of $\Sigma_U^{-1} \sim f_W(\underline{A}_U, \underline{\upsilon})$, the posterior conditional of Σ_U^{-1} follow a wishart distribution, $\Sigma_U^{-1} \mid \theta; \mathbf{Y}^*, \mathbb{Z} \sim W\left(\overline{A}_U, \overline{\upsilon}\right)$ with

$$\overline{\upsilon} = \underline{\upsilon} + \frac{1}{2}(T-1)$$

and

$$\overline{A}_U = \left(\sum_{t=2}^{T} (Y_t^* - \mathbf{X}_t^*\mathbf{b})(Y_t^* - \mathbf{X}_t^*\mathbf{b})' + \underline{A}_U^{-1}\right)^{-1}. \tag{26}$$

The posterior conditional for $\mathbf{b}$ then has to be derived using the imputed Y_t^* and $\mathbf{X}_t^*$ for $t = 2, ..., T$, $\mathbf{b} \sim MN\left(\overline{\mu}_{\mathbf{b}}, \overline{V}_{\mathbf{b}}\right)$, where

$$\overline{V}_{\mathbf{b}}^{-1} = \underline{V}_{\mathbf{b}}^{-1} + \sum_{t=2}^{T} \mathbf{X}_t^{*\prime}\Sigma_U^{-1}\mathbf{X}_t^*,$$

and

$$\overline{\mu}_{\mathbf{b}} = \overline{V}_{\mathbf{b}}\left(\underline{V}_{\mathbf{b}}^{-1}\underline{\mu}_{\mathbf{b}} + \sum_{t=2}^{T} \mathbf{X}_t^{*\prime}\Sigma_U^{-1}\mathbf{Y}_t^*\right).$$

Because $\mathbf{b} = \begin{pmatrix} 1, & 1, & 1, & \beta' \end{pmatrix}'$, we can partition $\mathbf{b}$ into $\widetilde{\beta}_1 = \begin{pmatrix} 1, & 1, & 1 \end{pmatrix}'$ and $\widetilde{\beta}_2 = \beta = \begin{pmatrix} \beta_1, & \beta_2, & \cdots & \beta_n \end{pmatrix}'$. Correspondingly, the posterior conditional of $\mathbf{b}$ can be partitioned as

$$\overline{\mu}_{\mathbf{b}} = \begin{pmatrix} \overline{\mu}_1 \\ \overline{\mu}_2 \end{pmatrix} \text{ and } \overline{V}_{\mathbf{b}} = \begin{pmatrix} \overline{V}_{11} & \overline{V}_{12} \\ \overline{V}_{21} & \overline{V}_{22} \end{pmatrix}.$$

Note that the missing dimensions are imputed such way that the posterior conditional of $\mathbf{b}$ is not affected by the imputed values. Once again, based on the properties of the multi-normal, $\widetilde{\beta}_2$ conditional on a fixed vector $\widetilde{\beta}_1$ follows a multivariate normal, $\widetilde{\beta}_2 \mid \widetilde{\beta}_1 \sim MN\left(\overline{\mu}_{\widetilde{\beta}_{2|1}}, \overline{V}_{\widetilde{\beta}_{2|1}}\right)$, where $\overline{V}_{\widetilde{\beta}_{2|1}} = \overline{V}_{22} - \overline{V}_{21}\left(\overline{V}_{11}\right)^{-1}\overline{V}_{12}$, and $\overline{\mu}_{\widetilde{\beta}_{2|1}} = \overline{\mu}_2 + \overline{V}_{21}\left(\overline{V}_{11}\right)^{-1}\left(\widetilde{\beta}_1 - \overline{\mu}_1\right)$.

Therefore, to simulate posterior distribution of β, we can used the random draws of $\widetilde{\beta}_2$ conditional on $\widetilde{\beta}_1$ according to Theorem 5.3 in (Härdle and Simar, 2007, p.147 – 149). Details regarding the imputation techniques, refer to Schafer (1997) p.183, and Little and Rubin (2002) .

Note that the missingness in the data is independent of both Y_t and $\mathbf{X}_t$, and therefore, we can treat the missing data as missing-at-random. The repeated series of imputation and posterior steps should be iterated long enough for the results to be reliable for a multiply imputed data set, see Schafer (1997).

Sampling $\mathbb{Z}$

The latent state vector $\mathbb{Z}_t$ for $t = 2, ..., T$ is filtered out by using a Kalman filter and smoothing within the MCMC. Denote $P = I_p \times 10^6$, where I_p is a p dimensional identity matrix and p is the dimension of matrix A_t. De Jong and Shephard (1995) propose to use a Kalman filter with a simulation smoother to get draws of $\mathbb{Z} \mid y, \theta$, where θ collects all parameters in the model. Recall

$$\begin{pmatrix} \mathbb{Z}_{t+1} \\ Y_t \end{pmatrix} = \Phi_t \mathbb{Z}_t + \epsilon_t, \qquad \epsilon_t = \begin{pmatrix} V_t \\ \mathbf{u}_t \end{pmatrix},$$

for $t = 2, \ldots, T$, where V_t and U_t given above with $V_t \overset{i.i.d}{\sim} f_{MN}(\mathbf{0}, \Sigma_V)$ and $U_t \overset{i.i.d}{\sim} f_{MN}(\mathbf{0}, \Sigma_U)$. Because B_t is a dummy matrix, in the filtering procedure, we treat β' in C_t as known and set B_t as an identity matrix with ones on the diagonal.

$$\Phi_t = \begin{pmatrix} A_t \\ B_t C_t \end{pmatrix} = \begin{pmatrix} A_t \\ C_t \end{pmatrix}$$

To estimate the latent state $\mathbb{Z}_t$, Durbin and Koopman (2001) propose a new simulation smoother, which requires the generation of simulated observations from the model together with a smoothing algorithm, and a standard Kalman filter procedure:

$$
\begin{aligned}
U_t &= Y_t - C_t \mathbb{Z}_t, \qquad F_t = C_t P_t C_t' + \Sigma_U, \quad K_t = (A_t P_t C_t')\, F_t^{-1} \\
L_t &= A_t - K_t \mathbb{Z}_t, \quad \mathbb{Z}_{t+1} = A_t \mathbb{Z}_t + K_t U_t, \quad P_{t+1} = A_t P_t A_t' + \Sigma_V - K_t F_t K_t'
\end{aligned}
$$

Refer to Durbin and Koopman (2001) and Koopman et al. (1999) for details of the state smoothing algorithm. Note that sampling $\mathbb{Z}$ and unknown parameters θ directly is not possible because of degeneracies (see Koopman et al., 1999). However, using the state smoothing within a Gibbs sampler, we can avoid the problems of choosing initial values and maximizing the likelihood function in high dimensions. See also Durbin and Koopman (2001), p.159. The derivations of the posterior conditionals for ϖ_{ij} and γ_j are well documented in Jochmann et al. (2010). We do not provide any further discussion.

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