

The Group All-Pay Auction with Heterogeneous Impact Functions^{*}

Subhasish M. Chowdhury ^a and Iryna Topolyan ^b

^a School of Economics, Centre for Behavioural and Experimental Social Science, and Centre for Competition Policy, University of East Anglia, Norwich NR4 7TJ, UK

^b Department of Economics, University of Cincinnati, Cincinnati, OH 45221, USA

This version: 12 March, 2015

Abstract

We analyze a group all-pay auction with a group specific public good prize in which one group follows a weakest-link and the other group follows a best-shot impact function. This type of game depicts situations in which the best-shot group is an attacker and the other group is a defender. We show that when the per-capita valuations are equal across groups, there exists a continuum of mixed strategy equilibria in which both groups randomize continuously without a gap over the same interval whose lower bound is zero. There are two further types of equilibria with discontinuous strategies. For the first type, each player in the best-shot group puts mass at the upper bound of the support whereas each player in the other group puts mass at the lower bound of the support. For the second type, players in the best-shot group put masses at both the lower and the upper bounds, while the other group randomizes without an atom. If players in the best-shot group could coordinate on the mass they put at the upper bound of the support, they would want to make it as large as possible (within the relevant range).

JEL Classification: C72; D70; D72; D74; H41

Keywords: group contest; all-pay auction; best-shot; weakest-link; group-specific public goods

Corresponding author: Iryna Topolyan, email: topolyia@gmail.com

^{*} We appreciate the comments of Walter Enders, Matt Van Essen, Paan Jindapon, David Malueg, Paul Pecorino, Seth Streitmatter, Ted Turocy, and the seminar participants at the University of Alabama and the University of East Anglia. Any remaining errors are our own.

1. Introduction

In various situations economic agents expend costly resources (in the form of money, time, investment etc.) in order to win a prize. These activities, known as contests, include rent-seeking, patent race, promotional tournament, warfare, litigation, sports – among others. A crucial part of modeling contests is to define the probability of winning the contest as a function of the resources expended. This function is often called a Contest Success Function (CSF). One of the most popular forms of CSF is an *all-pay auction*. In an all-pay auction players expend resources, but only the player that expends the highest amount wins the contest whereas all the players forgo the resources expended.¹ It is called an ‘all-pay auction’ due to its likeness with a standard first-price auction, if one considers the resources expended as sunk-bids. Equilibria in the all-pay auction are characterized by Baye et al. (1996). They find no pure strategy equilibrium. Players randomize on a continuous support and may have a mass point. The upper bound of the support, amount of the mass, and the equilibrium payoffs depend on the prize valuation.

Group sports events, alliances in a war, rent-seeking groups lobbying for a favorable policy decision, conflict between defense authorities and terrorist groups, etc. are instances in which all-pay auction occurs between groups. A bid-aggregation function that maps individual group member bids into a ‘group bid’ is called an impact function (Wärneryd, 1998). Baik et al. (2001) are the first to analyze a group all-pay auction. Here the sum of all group members’ bids constitutes the group-bid, and the group with the highest group-bid wins. They find free-riding equilibria in which the highest valuation player in a group makes a positive bid with a positive probability, whereas all the other group-members free-ride by placing zero bids. This is later generalized by Topolyan (2014) who characterizes further equilibria. Chowdhury et al. (2013b) and Barbieri et al. (2014) extend this stream of research by considering a weakest-link (the minimum bid within a group represents the group-bid) and a best-shot (the maximum bid within a group represents the group-bid) impact function, respectively. Both the studies find multiple equilibria that can be ranked according to payoffs.²

¹ This type of contest is introduced by Hillman and Samet (1986), Hillman and Riley (1989). Two other canonical forms of CSFs, each of which incorporates randomness in the outcome, are the logit type (Tullock, 1980) CSF, and the Tournament (Lazear and Rosen, 1981). See the survey of Konrad (2009) for a detailed discussion.

² There also is a stream of research that considers group contests with the logit-type CSF. The earliest study in this area is by Katz et al. (1990) who employ an additive impact function. This is extended in various ways by Baik (1993, 2008), Lee (2012), Kolmar and Rommeswinkel (2013), Chowdhury et al. (2013a) and Chowdhury and Topolyan (2013). The last study considers heterogeneous impact functions and is the closest to the current analysis.

Note that in each of the existing studies the impact functions of the contesting groups are homogeneous, i.e., the bid aggregation technology remains the same for the contesting groups. There are real life situations, however, where different groups follow different impact functions. In specific, there are several examples from Industrial Organization, Political Economy, and Defense and Peace economics in which one group follows a best-shot while the other follows a weakest-link impact function. In this study we construct such a group all-pay auction, characterize the equilibria, and provide implications of the results.

Field observations of such situations consist of group competitions and conflicts in which one group defends from another group's attack. Examples include conflict between police groups and drug cartels, internal security groups versus terrorist organizations, computer viruses versus computer security system, etc. The current framework, in which the highest bid among the attacking group members specifies the intensity of the attack and the lowest bid among defending group members specifies the strength of the defense, perfectly captures these situations. A military security issue (Conybeare et al., 1994), in which terrorist groups attack and the best-shot of the attack constitutes the power of the attack, but for the internal security groups the defense depends on the weakest-link, is a prominent example. The same logic holds for the attack and defense mechanisms in computer security and in system reliability (Varian, 2004; Hausken 2008), or the siege game between a police group and defending drug cartels. Even in a market, a firm trying to enter a market with new product features advertises about its best features whereas an incumbent spreads negative campaigns about the weakest features of the entrant product (Thomas, 1999).

There also are applications of this structure beyond the attack and defense cases. In an innovation race in which one firm employs a parallel innovation process among its R&D teams but another one employs a sequential process (Nelson, 1961; Abernathy and Rosenbloom, 1968) can be modeled with this structure. The best output among the parallel processes constitutes the strength of one innovation, whereas the weakest part of the sequential innovation process constitutes its strength in the innovation race.

Note that if a group (e.g., either the police or the drug cartel) wins the contest, then it implies a win for every group member. Hence, the prize has the features of a group-specific public good. Rightfully, in this study we consider a group all-pay auction with a group specific public good prize in which one group has a best-shot and another group has a weakest-link

impact function. We find that no pure strategy equilibrium exists. There exists a continuum of mixed strategy equilibria in which both groups randomize, without a gap, over the same interval whose lower bound is zero. If the per capita valuation in the attacker group, v_1/m , is greater than or equal to the per capita valuation in the defender group, v_2/n , then each player in the attacker group puts mass at the upper bound of the support ranging from (and including) zero to (but not including) 1; each player in the defender group puts mass at zero ranging from $(1 - mv_2/nv_1)$ to (but not including) 1. If the per capita valuation in the attacker group is less than the per capita valuation in the defender group, then two types of equilibria are possible: (I) each player in the attacker group puts mass at the upper bound of the support ranging from $(1 - nv_1/mv_2)$ to (but not including) 1; each player in the defender group puts mass at zero ranging from (and including) zero to (but not including) 1; (II) each player in the attacker group puts mass at the lower bound of the support (zero) ranging from $(1 - nv_1/mv_2)$ to (but not including) zero, and at the upper bound of the support ranging from (and including) zero to $(1 - nv_1/mv_2)$.

Our analysis contributes to various distinct areas of literature. It extends the literature on all-pay auction by achieving equilibria in which players place mass at the upper bound of the support, or place two atoms. These types of strategies are not observed in the existing literature. It also extends the research on group contest itself as it considers heterogeneous impact functions among groups. Chowdhury and Topolyan (2013) analyze this with a Tullock CSF, but fail to capture the situations in which the competition is cut-throat, and only find pure strategy equilibria that are very different from the current study. Since the prize is a group-specific public good and the group members face externalities through the network structure of the impact functions, this study also contributes to the long and established literature on public goods with network externalities (Hirshleifer, 1983, 1985; Bliss and Nalebuff, 1984; Bergstrom et al., 1986; Cornes, 1993; Barbieri and Malueg, 2008a, 2008b, 2014). Here also, the equilibria characterized turn out to be very different from the existing studies due to the conflict component included in the current analysis. Finally, the current study adds to the understanding of the issues of attack and defense such as information security and system reliability, as described in Varian (2004), or terrorism and inland security as discussed in Conybeare et al. (1994) and Arce et al. (2012).

The rest of the paper proceeds as follows. In the next section we formally set up the model. In Section 3 we solve for equilibrium and interpret the results. We provide two specific examples of the model and the corresponding equilibria in Section 4. Section 5 concludes.

2. Model

Consider a situation in which two groups are engaged in a group all-pay auction. Group 1 (2) consists of m (n) risk-neutral players, respectively, who make irreversible costly bids to win a group-specific public-good prize. This bid can be in the form of money, time, effort or any other resources – depending on the specific contest, but is measured in the same units as the prize values. The individual group members' valuation for the prize may differ across groups; however it is the same within a group. Let $v_g > 0$ represent the group-specific common valuation for the prize in group g . Furthermore, let $x_{gi} \geq 0$ represent the bid by player i in group g .

Now we specify the *group impact function* as $f_g: \mathbb{R}_+^{m_g} \rightarrow \mathbb{R}_+$. The group-bid of group 1 is $X_1 = f_1(x_{11}, x_{12}, \dots, x_{1m})$. Similar notation is employed for group 2. The following assumptions specify a best-shot impact function for group 1 and a weakest-link impact function for group 2.

Assumption 1. The group-bid of group 1 is represented by the maximum bid placed among the players in group 1, i.e., $X_1 = \max\{x_{11}, x_{12}, \dots, x_{1m}\}$.

Assumption 2. The group-bid of group 2 is represented by the minimum bid placed among the players in group 2, i.e., $X_2 = \min\{x_{21}, x_{22}, \dots, x_{2n}\}$.

Next, denote $p_g(X_1, X_2): \mathbb{R}_+^2 \rightarrow [0, 1]$ as a *contest success function* (CSF) to specify the winning probability of group g . We employ a group all-pay auction as follows.

Assumption 3. The probability of winning the prize for group 1 is

$$p_1(X_1, X_2) = \begin{cases} 1 & \text{if } X_1 > X_2 \\ 1/2 & \text{if } X_1 = X_2 \\ 0 & \text{if } X_1 < X_2 \end{cases}$$

The probability that group 2 wins is: $1 - p_1(X_1, X_2)$.

We assume that only the members of the winning group receive the group-specific public good prize but all players forgo their bids. For all the players, the cost of bid is the bid itself.

Hence, the payoff for player i in group g is:

$$u_{gi} = p_g v_g - x_{gi}. \tag{1}$$

To close the structure we assume that every player in the contest simultaneously and independently chooses her bid, and that all of the above, including the parameter values and the rules of the game, is common knowledge. We employ Nash equilibrium as our solution concept and use the following definitions throughout the paper. To simplify notations, Definition 1 is given for group 1, though it is analogous for group 2.

Definition 1. A *strategy* of player i in group 1 is a probability distribution over a subset of \mathbb{R}_+ , and is denoted by s_{1i} . Group 1's *strategy* is an m -tuple of its members' strategies (s_{11}, \dots, s_{1m}) , and is denoted by s_1 .

Definition 2. Given a strategy s_{gi} of player i in group g , the *individual support* S_{gi} is the closure of the set of all points of increase of the cumulative distribution function F_{gi} that corresponds to s_{gi} , i.e., $S_{gi} = Cl\{x \in \mathbb{R}_+ : F_{gi}(x - \varepsilon) < F_{gi}(x) < F_{gi}(x + \varepsilon) \text{ for all } \varepsilon > 0\}$.

Definition 3. Given group g 's strategy s_g , the *group support* S_g is the closure of the set of all points of increase of the cumulative distribution function F_g that corresponds to the group strategy s_g , i.e., $S_g = Cl\{x \in \mathbb{R}_+ : F_g(x - \varepsilon) < F_g(x) < F_g(x + \varepsilon) \text{ for all } \varepsilon > 0\}$.

Definition 4. If player i in group g distributes positive mass over some nonempty subset of \mathbb{R}_{++} , then the player is called *active*. Otherwise the player is called *inactive*.

Definition 5. If group g distributes positive mass over some nonempty subset of \mathbb{R}_{++} , then the group is called *active*. Otherwise the group is called *inactive*.

3. Equilibria

In this section we focus on and characterize (adopting the terminology by Barbieri et al., 2014) all semi-symmetric equilibrium strategies, in which all players within the same group employ the same strategy. Asymmetric equilibria may exist, where otherwise identical players are employing different strategies. However, since all players within a group are symmetric, it is natural to look at the possible semi-symmetric equilibria, which is the scope of this study.

We begin by stating *Lemma 1* and *Lemma 2*. The first lemma points out that in equilibrium, if one exists, both groups actively participate in the contest. The second lemma

reinstates a standard result from the all-pay auction literature. The proofs of both the lemmas are trivial and are not included here.

Lemma 1. *There exists no equilibrium in which both groups are inactive.*

Lemma 2. *There exists no other pure strategy Nash equilibrium.*

Next we characterize equilibrium supports. To do so, we first define the bounds. Following the notation of Chowdhury et al. (2013b), for each group g let S_{g+}^* denote $S_g^* \cap \mathbb{R}_{++} = \{x > 0: x \in S_g^*\}$, where S_g^* represents group g 's equilibrium support. Also, let $\underline{s}_g^* = \inf\{x: x \in S_g^*\}$, $\underline{s}_{g+}^* = \inf\{x: x \in S_{g+}^*\}$, and $\bar{s}_g^* = \sup\{x: x \in S_g^*\}$.

Lemma 3. *Let $s^* = \{s_1^*, s_2^*\}$ be a mixed strategy Nash equilibrium. Then the following statements are true.*

- i. $S_{1+}^* = S_{2+}^*$.
- ii. $\min\{\underline{s}_1^*, \underline{s}_2^*\} = 0$.
- iii. “No gaps in the support”: i.e., there exists no interval $(a, b) \subseteq S_g^*$, where $a < b$, such that $F_g(a) = F_g(b)$, $g = 1, 2$.
- iv. No player in group 2 puts mass at \bar{s}_2^* .
- v. No group g puts mass at x for any $x \in (\underline{s}_g^*, \bar{s}_g^*)$.
- vi. At most one group puts mass at zero.

Proof: See Appendix.

Combining Lemmas 1-3, we state the main result of this analysis.

Theorem 1. *All semi-symmetric equilibria of this all-pay auction are as follows.*

- 1) If $\frac{nv_1}{mv_2} \geq 1$, then every player in group 1 puts a mass of a_1 at \bar{s}^* , where $0 \leq a_1 < 1$, and every player in group 2 puts a mass of $\left[1 - \frac{mv_2}{nv_1}(1 - a_1)\right]$ at zero.
- 2) If $\frac{nv_1}{mv_2} < 1$, then there are two possibilities:

- I. Every player in group 1 puts a mass of a_1 at \bar{s}^* , where $1 - \frac{nv_1}{mv_2} \leq a_1 < 1$, and every player in group 2 puts a mass of $\left[1 - \frac{mv_2}{nv_1}(1 - a_1)\right]$ at zero.
- II. Every player in group 1 puts a mass of $\left[1 - a_1 - \frac{nv_1}{mv_2}\right]$ at zero, where $0 \leq a_1 \leq 1 - \frac{nv_1}{mv_2}$, and a mass of a_1 at \bar{s}^* .

Proof: See Appendix.

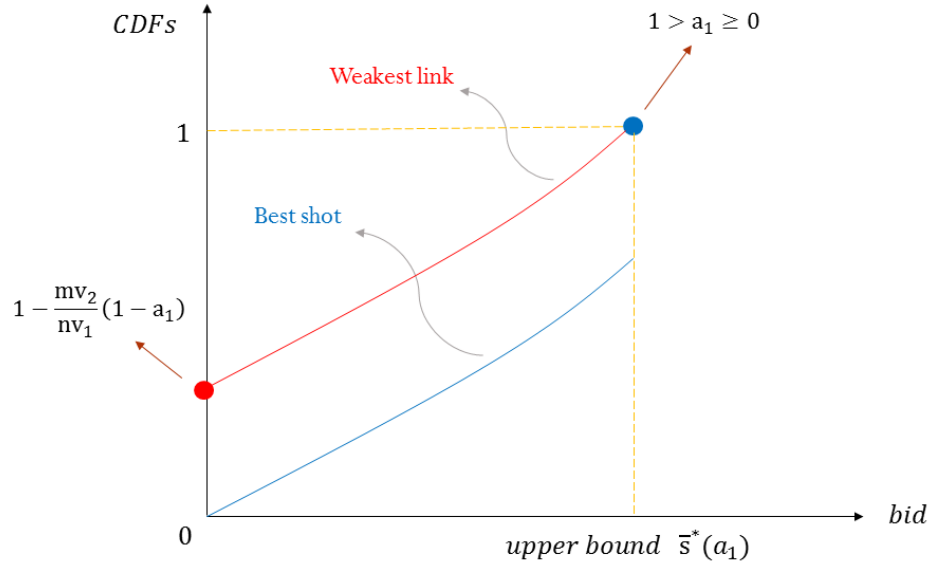


Figure 1. Equilibrium CDFs when $(nv_1/mv_2) \geq 1$.

Equilibrium strategies (CDFs) of players for the case $nv_1 \geq mv_2$ are depicted in Figure 1. As the size of the weakest-link group, n , increases (or the size of the best-shot group, m , decreases), each player in group 2 puts greater mass at zero, which improves the payoffs in group 1.³ If players in the best-shot group could coordinate on the mass they put at the upper bound of the support, they would want to make it as close to one as possible, however the optimum at one is never achieved. If the unit mass is put at the upper bound of the support, the group employs a pure strategy, which is not possible in equilibrium. Note that players in group 2 always receive zero payoff as long as they put positive mass at zero. Thus, their payoff is unaffected by the

³ Note that the mass that each player in group 2 puts at zero is inversely related to a_1 . The upper bound of the support, \bar{s}^* , is determined from equation $A_1(1 - a_1) = \bar{s}^*$, where A_1 is strictly increasing in the relevant range. Thus when every player in group 2 puts greater mass at zero, \bar{s}^* decreases, making players in group 1 better-off.

changes in the group sizes, as long as $nv_1 \geq mv_2$ is satisfied, although the players bid less aggressively.

If $nv_1 < mv_2$, there are two types of equilibria. Type I equilibria are depicted in Figure 2. Group 1 puts mass at the upper bound of the common support, and group 2 puts mass at zero. Similar to the previous case, as n increases (or m decreases), the welfare of each player in group 1 improves, while group 2 receives zero payoff.

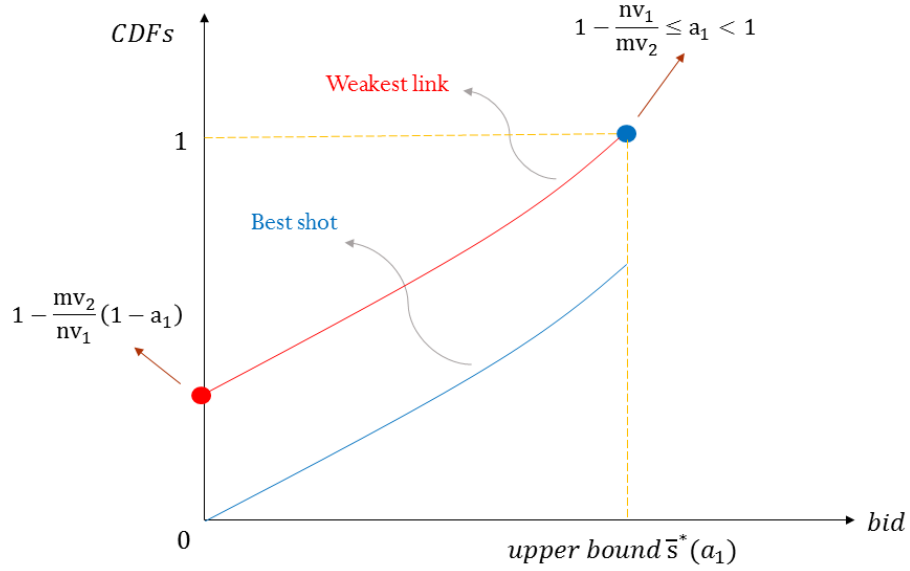


Figure 2. Equilibrium CDFs when $(nv_1/mv_2) < 1$, type I equilibria.

Figure 3 shows type II equilibria. Group 1 puts atoms at both ends of the support, while group 2 randomizes continuously. Every player in the weakest-link group receives a positive expected payoff of $(1 - a_1 - \frac{nv_1}{mv_2})$. Once again, an increase in n (or a decrease in m) results in a lower payoff for the weakest-link group, as long as $\frac{nv_1}{mv_2} < 1$.⁴ If players in the best-shot group could coordinate on the masses they put at the extreme ends of the support, they would choose to put the maximum possible mass, $1 - \frac{nv_1}{mv_2}$, at \bar{s}^* and no mass at zero.

⁴ Note that in general it is difficult to say how the upper bound of the group support varies with the group sizes, as the expression in (7) is complicated. Therefore it is difficult to say how players in group 1 fare as the group sizes change.

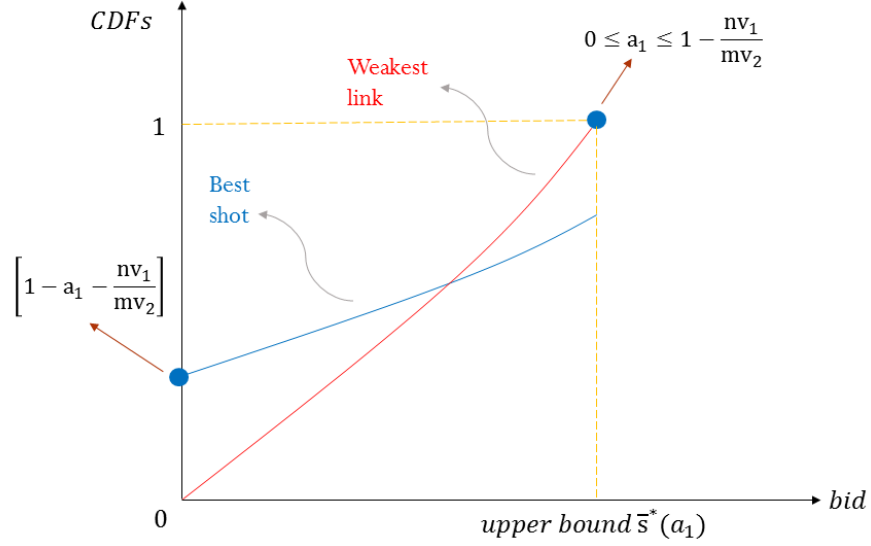


Figure 3. Equilibrium CDFs when $(nv_1/mv_2) < 1$, type II equilibria.

This finding is similar to the group-size paradox, which states that free-riding makes smaller groups more effective (Olson, 1965; Bergstrom et al., 1986; Esteban and Ray, 2001; Barbieri and Malueg, 2008a and 2008b; Pecorino, 2015). Our result is also reminiscent to the volunteer's dilemma, which arises in a single-group setting and reflects the lack of incentives for each player to contribute when she can rely on the contributions of other group members (Diekmann, 1984; Harrington, 2001; Barbieri and Malueg, 2014). However, in contrast to the free-rider paradox and the volunteer's dilemma, the payoffs of players in the weakest-link group in type I equilibria stay the same as the group sizes vary.

4. Examples

In the previous section we characterized the equilibria of the game. We observed that two cases may arise and the natures of the equilibria are very different for those two cases. Here we provide with examples that delineates the two most basic types of the equilibria, but covers both the technicalities and intuitions of the symmetry vs. asymmetry in group size and in prize valuations. The first one considers symmetric groups with common prize value whereas the second one incorporates different group sizes with different prize value per group. These also correspond to the two types of the equilibria characterized.

4.1. Example 1: the symmetric group size, symmetric value case

Suppose the groups are symmetric, with $m = n = 2$ and $v_1 = v_2 = 1$. Theorem 1 implies there exists a continuum of equilibria, in which each player in group 1 puts a mass of a_1 at the upper bound of the common support, where $0 \leq a_1 < 1$, and every player in group 2 puts the same mass at zero. Equation (5) implies:

$$2F(x)(1 - F(x) - a_1)F'(x) = 1. \quad (8)$$

Let $y = F(x)$, then the following equation implicitly defines y as a function of x under the condition $0 \leq y \leq a_1$:

$$y^2(1 - a_1) - \frac{2}{3}y^3 = x. \quad (9)$$

The implicit plot of (9) for the case $a_1 = 0$ is depicted in Figure 4. Indeed, when we restrict our attention to the range $\{y: 0 \leq y \leq 1\}$, y is implicitly defined as an increasing function of x .

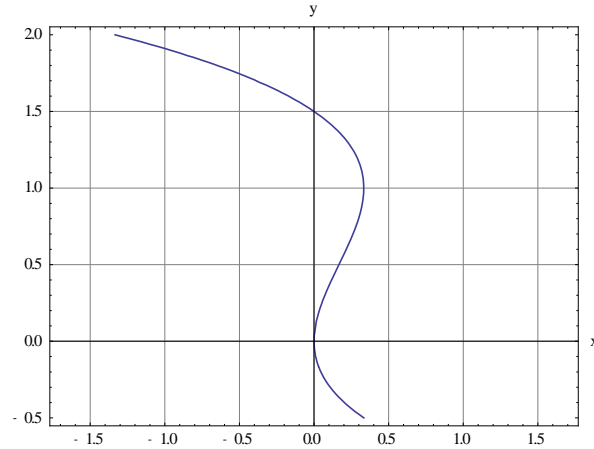


Figure 4. Implicit plot of $y^2 - \frac{2}{3}y^3 = x$.

The family of CDFs of a player in group 1, when a_1 takes values of 0, 0.1, 0.2, and 0.5, is depicted in Figure 5. Here effort levels are depicted on the horizontal axis, and values that CDF takes are depicted on the vertical axis. Theorem 1 implies that every player in group 2 puts the same mass at zero. It is easy to see that the upper bound of the groups' support is decreasing in a_1 and ranges from 0.33 when $a_1 = 0$ to 0.04 when $a_1 = 0.5$. The upper bound of the support

goes to zero as a_1 goes to 1. Notice that in equilibrium $a_1 = 1$ is never achieved (since no group could sustain a pure strategy), suggesting that the Nash equilibrium correspondence is not upper hemicontinuous. This happens because the players' payoffs are discontinuous due to the all-pay auction assumption.

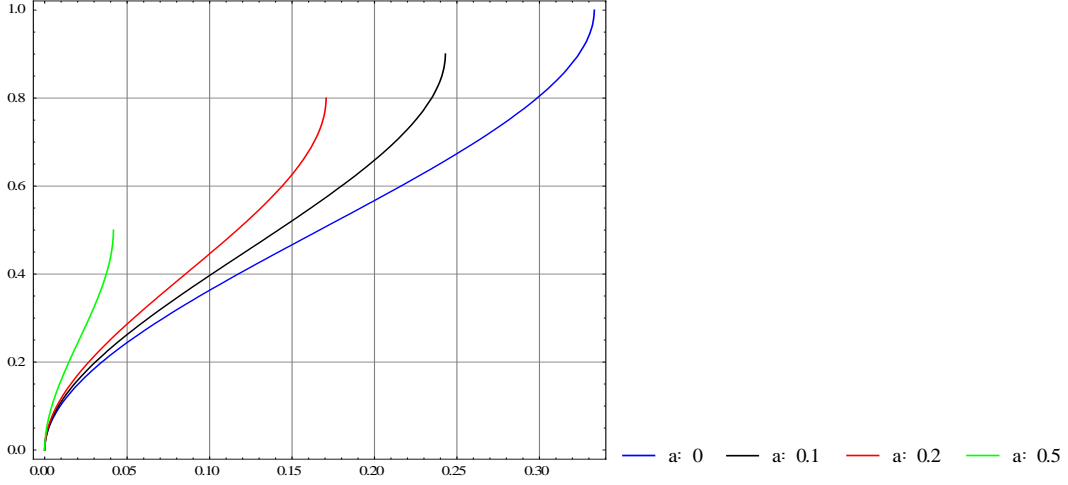


Figure 5. Strategy (CDF) of a player in group 1 when $a_1 = 0, 0.1, 0.2$, and 0.5 .

4.2. Example 2: the asymmetric group size, asymmetric value case

Now suppose group 1 has three players with valuation $v_1 = 1$, and group 2 has two players with valuation $v_2 = 3$. There are two types of equilibria.

Type I equilibria: each player in group 1 puts a mass of a_1 at the upper bound of the common support, where $1/2 \leq a_1 < 1$, and every player in group 2 puts a mass of $a_G = \frac{1}{2}(1 + a_1)$ at zero. Equation (5) implies

$$24F(x)(1 - F(x) - a_1)^2 F'(x) = 1. \quad (10)$$

Letting $y = F(x)$, we obtain the following equation relating y and x , where $0 \leq y \leq a_1$:

$$6y^4 + 16(a_1 - 1)y^3 + 12(a_1 - 1)^2 y^2 = x. \quad (11)$$

CDFs of a player in group 1, when a_1 takes values of $0.5, 0.6$, and 0.7 , are depicted in Figure 6. Again, the upper bound of the groups' support diminishes from 0.125 to zero as a_1

increases. As a_1 increases, players in group 2 put larger mass at zero, ranging from 0.75 to 1. As before, the mass gets infinitely close to one, however the maximum of one is never achieved.

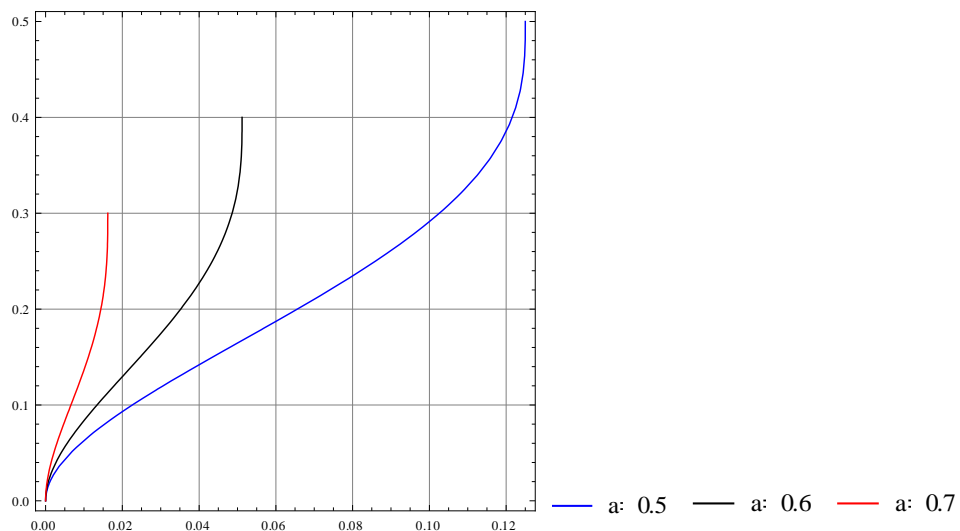


Figure 6. Strategy (CDF) of a player in group 1 when $a_1 = 0.5, 0.6$, and 0.7 .

Type II equilibria: each player in group 1 puts a mass of a_1 at the upper bound of the common support, where $0 \leq a_1 \leq \frac{1}{2}$, and a mass of $a_F = \frac{1}{2} - a_1$ at zero. Equilibrium CDFs of a player in group 2 when a_1 takes values of 0, 0.25, and 0.5 are depicted in Figure 7.

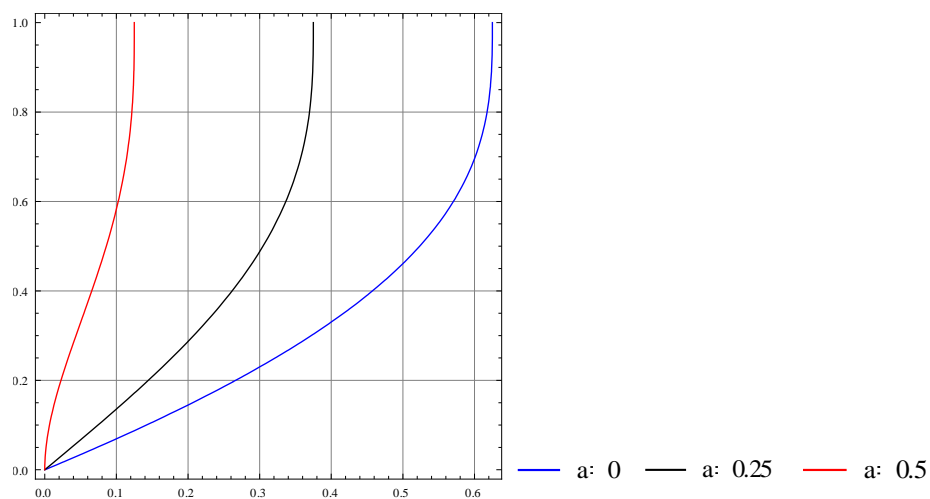


Figure 7. Strategy (CDF) of a player in group 2 when $a_1 = 0, 0.25$, and 0.5 .

As can be observed from the figure above, the upper bound of the support diminishes, similar to the last case, as a_1 increases.

5. Discussion

We introduce a group all-pay auction with heterogeneous impact functions. In specific, we consider the case in which one group has a best-shot whereas the other group has a weakest-link impact function. This structure reflects various real life situations in which one group (with the weakest-link impact function) is defending from the other (with the best-shot impact function) group's attack and the competition is cut-throat.

We characterize all semi-symmetric equilibria of the game in which symmetric players play the same strategy. We show that when the groups share the same per-capita valuation, i.e., prize valuation scaled with the number of group members, there exists a mixed strategy equilibrium where both groups randomize continuously without a gap over the same interval whose lower bound is zero. If the per-capita valuation in the group with the best-shot impact function is higher, then the best-shot group places mass at the upper bound of the support while the other group places mass at zero. Otherwise two types of equilibria are possible. For both types of equilibria the best-shot group puts mass at the upper bound of the support. However, for the first type of equilibria the weakest-link group also puts mass at the lower bound of the support (zero), whereas for the second type the best-shot group puts mass at both the lower and the upper bound of the support and the weakest-link group randomizes without an atom.

At a technical level, we introduce two new types of equilibria into the literature on all-pay auction in which players may place an atom at the upper bound of the support, or may have two atoms. These equilibria are achieved because of the weakest-link technology which restricts players to shift mass down. Compared to the existing studies of group all-pay auctions with homogenous impact functions, we find equilibria that are different. Whereas Barbieri et al. (2014), imposing best-shot impact functions for both groups, find a wide variety of equilibria in which a player puts mass at either the upper or the lower bound of the support (but not both), we find that atoms at both extremes are possible in the attacker group, because the defender group follows the weakest-link effort technology. While Chowdhury et al. (2013b), imposing weakest-link impact functions for both groups, find equilibria in which pure and mixed strategies may co-exist, only mixed strategies could be sustained in equilibrium in our model. Also, unlike Chowdhury et al. (2013b), it is not possible for both groups to be inactive in equilibrium. These results stem from the fact that one group adopts the best-shot effort technology. As expected, our

results are also very different from that of Baik et al. (2001) who employ perfect substitute impact functions for both groups and find free-riding equilibria that are similar to the standard all-pay auction equilibria of Baye et al. (1996).

Chowdhury and Topolyan (2013) analyze group contest with heterogeneous impact functions but with logit-type (Tullock, 1980) CSF; they characterize pure strategy equilibria and show that in any equilibrium only one player in the attacker group is active, while all players in the defender group exert the same positive effort. However, note that when the competition is neck to neck, as in these examples, the CSF is likely to be represented by an all-pay auction. We show, compared to Chowdhury and Topolyan (2013), that much wider participation is possible in mixed strategy equilibria. Particularly, we show that it is possible that all members in the attacker group are active and randomize over the same support, and that members of the defender group may fail to coordinate with respect to the effort level and exert different efforts in a mixed strategy equilibrium.

Supporting the literature on ‘group size paradox’ (Olson, 1965) we show that an increase in the relative group size of the weakest-link group results in a lower payoff for that group, as long as the per-capita valuation in the weakest-link group is lower. Our result also contributes to the issue of volunteer's dilemma (Diekmann, 1984) in a multi-group contest setting in the sense of group contribution. However, adding to the group-size paradox and the volunteer's dilemma literatures, we specify conditions for which the payoffs of players in the weakest-link group can stay the same as the group sizes vary. Furthermore, Clark and Konrad (2007), Hausken (2008) and Arce et al. (2012) consider attack-and-defense in multiple battle conflict with network. We extend the same to group conflicts.

There are various ways to broaden this analysis. The obvious, but technically difficult, extensions would be to allow more than two groups and to introduce heterogeneous valuations. Endogenizing the choice of impact function, endogenizing group formation, considering private good prizes are further possible topics. Finally, we consider a complete information setting. Introducing incomplete information may provide interesting results. We leave these as areas for future research.

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APPENDIX

Proof of Lemma 3.

(i) Suppose $S_{1+}^* \neq S_{2+}^*$. Let $x \in S_{1+}^*$ be such that $x \notin S_{2+}^*$ (observe that $x > 0$). Consider the following possibilities:

- (a) If $x > \bar{s}_{2+}^*$, then any player in group 1 would deviate from x to $x - \varepsilon$ for some small $\varepsilon > 0$, leading to a contradiction.
- (b) If $x < \underline{s}_{2+}^*$, then any player in group 1 would deviate from x to zero.
- (c) If $\underline{s}_{2+}^* < x < \bar{s}_{2+}^*$, then any player in group 1 would deviate from x to $x - \varepsilon$ for some sufficiently small $\varepsilon > 0$, leading to a contradiction.

The case where $x \in S_{2+}^*$ and $x \notin S_{1+}^*$ is similar.

(ii) Suppose, by contradiction, that $\min\{\underline{s}_1, \underline{s}_2\} > 0$, then by part (i) we have $S_1^* = S_2^*$. Denote the common lower bound of the support by \underline{s} . We claim that every group must put mass at \underline{s} . Suppose not, and let group g be such that its players do not put mass at \underline{s} . Then if any player in group $-g$ (i.e., not in group g) deviates from \underline{s} to zero, the winning probability of group $-g$ will remain the same. Therefore every group must put mass at \underline{s} , but then every player could improve by deviating from \underline{s} to $\underline{s} + \varepsilon$.

(iii) Suppose for some group g and some a, b with $0 \leq a < b$ we have $F_g(a) = F_g(b)$. Without loss of generality assume there exists a $\Delta > 0$ such that $F_g(b) + t > F_g(b)$ and $F_g(a) - t < F_g(a)$ for all $0 < t < \Delta$. In other words, there is a gap in the support of group g 's bid equal to (a, b) . Next, we claim that at most one group puts mass at b . Indeed, if both groups put mass at b , then any player would be better-off by slightly increasing her bid from b to $b + \varepsilon$. Assume group g does not put mass at b , then any player in the other group would shift mass from $[b, b + \delta)$ to $a + \delta$ for some $\delta > 0$ sufficiently small. But then it is not rational for a player in group g to randomize over $[b, b + \delta)$, which leads to a contradiction.

(iv) Follows immediately from the fact that group 1 utilizes the best-shot bid technology.

(v) Suppose group g puts mass at some $x \in (\underline{s}_+^*, \bar{s}^*)$, which implies that every player in group g puts mass at x since players within each group employ a symmetric strategy. This implies any

player in the other group, $-g$, would shift mass from $(x - \varepsilon, x]$ to $(x + \varepsilon)$ for some sufficiently small $\varepsilon > 0$, but then it is not optimal for a player in group g to put mass at x .

(vi) Suppose both groups put mass at zero, then any player would be better-off shifting mass from zero to some very small $\varepsilon > 0$, but then it is not optimal for players in the rival group to put mass at zero. ■

Proof of Theorem 1.

Let each player in group 1 randomize according to some CDF $F(\cdot)$, and each player of group 2 randomize according to a CDF $G(\cdot)$. Following Barbieri et al. (2014), the expected payoff of player i in group 1 from exerting an effort of x is given by

$$u_{1i}(x) = -x + v_1 \left[1 - \int_x^\infty (F(z))^{m-1} n(1 - G(z))^{n-1} dG(z) \right]$$

Then, $u'_{1i}(x) = nv_1(F(x))^{m-1}(1 - G(x))^{n-1}G'(x) - 1 = 0$, which implies

$$nv_1(F(x))^{m-1}(1 - G(x))^{n-1}G'(x) = 1. \quad (2)$$

Similarly,

$$mv_2(F(x))^{m-1}(1 - G(x))^{n-1}F'(x) = 1. \quad (3)$$

Let a_1 be the (possibly zero) mass that every player in group 1 puts at \bar{s}^* , then

$$1 - G(x) = \frac{mv_2}{nv_1}(1 - F(x) - a_1) \text{ for all } 0 < x < \bar{s}^*. \quad (4)$$

Consider two possibilities.

Case I. Each player in group 2 puts a mass of a_G at zero, where $0 \leq a_G < 1$. Lemma 3(vi) implies that players in group 1 do not put mass at zero. Equation (4) could be written as $G(x) = 1 - \frac{mv_2}{nv_1}(1 - a_1) + \frac{mv_2}{nv_1}F(x)$, therefore in order to satisfy $0 \leq a_G < 1$, we must have $a_1 \geq 1 - \frac{nv_1}{mv_2}$. Notice that if $\frac{nv_1}{mv_2} \geq 1$, this condition is satisfied for all $0 < a_1 < 1$.

Plugging (4) into (3) yields

$$\frac{(mv_2)^n}{(nv_1)^{n-1}} (F(x))^{m-1} (1 - F(x) - a_1)^{n-1} F'(x) = 1. \quad (5)$$

Denote $\frac{(mv_2)^n}{(nv_1)^{n-1}} y^{m-1} (1 - y - a_1)^{n-1}$ by $D_1(y)$. Note that since $D_1(y)$ is a polynomial, its anti-derivative exists and is also a polynomial. Denote by $A_1(y)$ the antiderivative of $D_1(y)$. Then the solution to (5) has the form $A_1[F(x)] = x + C$, where C is some constant. Since by assumption $F(0) = 0$, the constant C is equal to zero. Thus a solution to (5) satisfies

$$A_1[F(x)] = x. \quad (6)$$

Furthermore, $D_1[F(x)] \geq 0$ if and only if $0 \leq F(x) \leq 1 - a_1$, with strict inequality whenever $F(x)$ is in the interior of the interval. Notice that since $D_1(y) = \frac{\partial}{\partial y} [A_1(y)]$, $A_1(\cdot)$ is strictly increasing on $(0, 1 - a_1)$. Therefore the inverse function A_1^{-1} is defined for all $0 < F(x) < 1 - a_1$, and thus there exists a unique solution to (5) satisfying $0 < F(x) < 1 - a_1$. The upper bound of the support \bar{s}^* is determined uniquely (by continuity of F) from equation $A_1(1 - a_1) = \bar{s}^*$. Notice that by assumption $F(0) = 0$ and $F(\bar{s}^*) = 1 - a_1$.

Case II. Each player in group 1 puts a mass of a_F at zero, where $0 \leq a_F < 1$, while players in group 2 do not put mass at zero. Rewrite Equation (4) as $F(x) = 1 - a_1 - \frac{nv_1}{mv_1} + \frac{nv_1}{mv_2} G(x)$, then in order to satisfy $0 \leq a_F < 1$, we must have $a_1 \leq 1 - \frac{nv_1}{mv_2}$, which holds only if $\frac{nv_1}{mv_2} < 1$. The latter condition is satisfied for all $0 < a_1 < 1$ when $\frac{nv_1}{mv_2} \geq 1$. Similarly to the previous case,

$$nv_1 \left(1 - a_1 - \frac{nv_1}{mv_2} + \frac{nv_1}{mv_2} G(x) \right)^{m-1} (1 - G(x))^{n-1} G'(x) = 1. \quad (7)$$

Letting $D_2(y) = nv_1 \left(1 - a_1 - \frac{nv_1}{mv_2} + \frac{nv_1}{mv_2} y \right)^{m-1} (1 - y)^{n-1}$ and denoting by $A_2(y)$ the antiderivative of $D_2(y)$ with zero constant term, and then following the lines of Case I and noting that A_2 is strictly increasing on $(0,1)$ we conclude that there exists a unique solution to (7). Finally, from $A_2(1) = \bar{s}^*$ we solve for \bar{s}^* . ■