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# A Generalized Tullock Contest and the Existence of Multiple Equilibria*

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## Abstract

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# 1. Introduction

Contests are economic or social interactions in which two or more players expend costly resources in order to win a prize. The resources expended by players determine their probability of winning a prize. In this article we construct a generalized Tullock contest under complete information. We consider a simple two-player contest where, contingent upon winning or losing, a player receives different prizes. Players' outcome-contingent payoffs are linear functions of prizes, own effort, and the effort of the rival.[1] Under this structure we characterize the equilibria, and show that multiple equilibria might exist even under symmetric prize values, which is a new result in the literature. We also demonstrate how this structure nests most of the Tullock contests in the literature. Finally, we introduce and characterize several new contests in which multiple equilibria may arise under very general conditions.

The contest literature originated with Tullock (1980). In this model, player $i$'s probability of winning is $p_i(x_i, x_j) = x_i^r/(x_i^r + x_j^r)$, where $x_i$ and $x_j$ are the efforts of players $i$ and $j$. So, the player expending the highest effort has a higher probability of winning the contest. The function, $p_i(x_i, x_j)$, that maps efforts into probabilities of winning is called the contest success function (CSF). The most popular versions of the Tullock CSF are the lottery ($r$ = 1) and the all-pay auction ($r = \infty$).[2] There are several reasons why Tullock CSF is widely employed. First, a number of studies have provided axiomatic justification for the Tullock CSF (Skaperdas, 1996; Kooreman and Lambart, 1997; Clark and Riis, 1998). Second, Baye and Hoppe (2003) have identified conditions under which a variety of rent-seeking contests, innovation tournaments, and patent-race games are strategically equivalent to the Tullock contest.

[1] Contests are characterized by three attributes such as prizes, players, and the efforts of the players (Konrad, 2009).
[2] A special limiting case ($r = \infty$) of the Tullock CSF links the theory of contests to the first-price all-pay auction, where the winner is the player that expends the highest effort (Baye et al., 1996).

Economists often use modified payoffs in Tullock contest in order to address specific research questions. For example, Skaperdas and Gan (1995) restrict the losing payoff to study the effect of risk aversion in a "limited liability" contest. Cohen and Sela (2005) restrict the winning payoff to show that in certain contests a weaker contestant can win with higher probability than a stronger contestant. Other studies that use modified payoffs in Tullock contest include Garfinkel and Skaperdas (2000), Grossman and Mendoza (2001), and Öncüler and Croson (2005), Matros and Armanios (2009). In this article we propose a generalized Tullock contest in which payoffs are linear functions of prizes, own effort, and the effort of the rival.[3] Our model nests a number of the existing contests in the literature and it also provides a framework for studying new contests with spillovers.

One of the main motivations of this article comes from the fact that in many real life contests payoffs depend not only on the individual but also on the rival's effort. For example, in innovation contests one firm's R&D effort may provide information spillover that benefits its rival (D'Aspremont and Jacquemin, 1988). Kamien et al. (1992) show how positive input spillovers can affect R&D decisions when firms are engaged in a two-stage game of innovation. Another example where spillovers are important is litigation (Baye et al., 2005). Depending on the litigation system, losers have to compensate winners for a portion of their legal expenditures or up to the amount actually spent by the loser. This creates either negative or positive spillover effect of one party's expenditure on another. We explicitly model such spillovers in the context of a Tullock lottery contest.

The main finding of this article is that multiple equilibria may arise in Tullock contests under very general conditions. This is a very important finding for a number of reasons. First, in

[3] The only studies that use a general payoff structure (though under the all-pay auction CSF) are done by Baye et al. (2005, 2009).

multi-stage or repeated games the existence of multiple non-payoff equivalent equilibria at a stage game means that one can condition equilibrium selection in the subgame based on past behavior. This allows for a wide range of payoffs to be supported as subgame perfect equilibria. Second, in the presence of multiple equilibria, comparative statics have to be conditioned on a particular equilibrium since different equilibria may lead to different comparative statics results. Finally, the existence of multiple equilibria is important for designing both static and dynamic contests. A contest designer needs to account for the full profile of equilibria and corresponding comparative statics in order to achieve a given objective.

## 2. Theoretical Model

We consider a two-player contest with two prizes. The players are denoted by $i$ and $j$. Both players value the winning prize as $W > 0$ and the losing prize as $L \geq 0$. We assume that winning the prize provides higher valuation than losing the prize, i.e. $W > L$. Players simultaneously expend irreversible and costly efforts $x_i \geq 0$ and $x_j \geq 0$. The probability of player $i$ winning the contest is described by a Tullock lottery CSF:

$$p_i(x_i, x_j) = \begin{cases} x_i/(x_i + x_j) & \text{if } x_i + x_j \neq 0 \\ 1/2 & \text{if } x_i = x_j = 0 \end{cases} \tag{1}$$

Contingent upon winning or losing, the payoff for player $i$ is a linear function of prizes, own effort, and the effort of the rival:

$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i + \beta_1 x_j & \text{with probability} \quad p_i(x_i, x_j) \\ L + \alpha_2 x_i + \beta_2 x_j & \text{with probability} \quad 1 - p_i(x_i, x_j) \end{cases} \tag{2}$$

where $\alpha_1$, $\alpha_2$ are cost parameters, and $\beta_1$, $\beta_2$ are spillover parameters. To ensure that a player has no incentive to expend infinite effort, we impose conditions that a player's own effort has a negative direct impact on his winning payoff and non-positive direct impact on his losing payoff, that is $\alpha_1 < 0$ and $\alpha_2 \leq 0$.

We define the contest described by (1) and (2) as $\Gamma(i, j, \Omega)$ , where $\Omega = \{W, L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$ is the parameter space. All parameters in $\Omega$ along with the CSF are common knowledge for both players. The players are assumed to be risk neutral, therefore, for a given effort pair $(x_i, x_j)$, the expected payoff for player $i$ in contest $\Gamma(i, j, \Omega)$ is:

$$E\big(\pi_i(x_i, x_j)\big) = \frac{x_i}{x_i+x_j}\big(W + \alpha_1 x_i + \beta_1 x_j\big) + \frac{x_j}{x_i+x_j}\big(L + \alpha_2 x_i + \beta_2 x_j\big) \tag{3}$$

where $(x_i, x_j) \neq (0,0)$. For $x_i = x_j = 0$, the expected payoff is $E\big(\pi_i(x_i, x_j)\big) = (W + L)/2$. By setting $A = W - L + (\beta_1 - \beta_2)x_j$ , $B = \alpha_1 - \alpha_2$ , and $C = L + \beta_2 x_j$ , expression (3) can be rewritten as:

$$E\big(\pi_i(x_i, x_j)\big) = B\frac{x_i^2}{x_i+x_j} + A\frac{x_i}{x_i+x_j} + \alpha_2 x_i + C \tag{4}$$

Player $i$'s best response is derived by maximizing $E\big(\pi_i(x_i, x_j)\big)$ with respect to $x_i$. Differentiating equation (4) with respect to $x_i$ yields the following first order condition:

$$(B + \alpha_2)x_i^2 + 2x_j(B + \alpha_2)x_i + Ax_j + \alpha_2 x_j^2 = 0 \tag{5}$$

In Appendix we show that the payoff function for player $i$ is concave as long as:

$$x_j \leq \frac{W-L}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)} \tag{6}$$

If (6) holds then first order condition is necessary and sufficient for payoff maximization of player $i$. Consequently by solving (5) for $x_i$ and by substituting back the values of $A$ and $B$, we receive the best response function of $x_i$ in terms of the effort choice of $x_j$:

$$x_i^{BRF} = -x_j + \sqrt{\frac{\{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)\}x_j^2 - \{W-L\}x_j}{\alpha_1}} \tag{7}$$

if $x_j \leq (W - L)/(-\alpha_2 - \beta_1 + \beta_2)$, and otherwise, $x_i^{BRF} = 0$.[4] It is clear that the best response function (7) depends on $\alpha_1$, $\alpha_2$, the difference between $\beta_1$ and $\beta_2$, and the spread between the winning and the losing prize valuations. The slope of the best response function is defined by:

$$\frac{\partial x_i^{BRF}}{\partial x_j} = -1 + \frac{2\{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)\}x_j - \{W-L\}}{2\sqrt{\alpha_1\left[\{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)\}x_j^2 - \{W-L\}x_j\right]}} \tag{8}$$

Depending on the rival's effort and the parameters, this slope can either be positive or negative. It is also clear that the curvature of (7) will be different for different values of parameters. By simultaneously solving best response functions (7), and accounting for symmetric Nash equilibrium we obtain equilibrium in which player $i$ and $j$ expend

$$x_i^* = x_j^* = x = \frac{(W-L)}{-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2)} \tag{9}$$

The restriction $-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2) > 0$ also ensures that the second order condition holds (see Appendix). The expected equilibrium payoff in the symmetric equilibrium is given by:

$$E^*(\pi) = \frac{(\beta_2-\alpha_1)(W-L)}{-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2)} + L \tag{10}$$

The incentive compatibility constraint, $E^*(\pi) \geq L$, implies that the equilibrium payoff is more than the sure payoff of losing. To insure that both players are willing to expend equilibrium efforts, we need further restriction of $\beta_2 - \alpha_1 \geq 0$. This restriction means that the cost of winning is lower than the spillover benefit from losing.

In addition to the symmetric equilibrium, the generalized contest $\Gamma(i, j, \Omega)$ can generate multiple asymmetric equilibria. The restriction $(5\alpha_1 - \alpha_2) - (\beta_1 - \beta_2) > 0$ guarantees the existence of two asymmetric equilibria $\{x_i^* = \overline{x}; x_j^* = \underline{x}\}$ and $\{x_i^* = \underline{x}; x_j^* = \overline{x}\}$, where

[4] Note that the restriction (6) is weaker than the restriction needed for (7) to be well defined. Hence, when the best response is positive then solving the best response functions, if solutions exist, will lead us to equilibria.

$$\overline{x} = \frac{1}{2}\frac{(W-L)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}\left[1+\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right] \quad (11)$$

$$\underline{x} = \frac{1}{2}\frac{(W-L)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}\left[1-\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right] \quad (12)$$

These equilibria result from the fact that the best response functions in (7) are parabolas and two parabolas can intersect in multiple points (see Figure 4.1 and Figure 4.3). Equations (8) and (9) are defined only if we impose the restriction $(5\alpha_1 - \alpha_2) - (\beta_1 - \beta_2) > 0$. Basically, this means that existence of asymmetric equilibria requires sufficient difference in cost and spillover parameters. The second order condition under the asymmetric equilibria holds if $\alpha_1 < 0$. By imposing further incentive compatibility restriction $\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}} \geq \frac{(4\alpha_1-\alpha_2)-(\beta_1-2\beta_2)}{(2\alpha_1-\alpha_2)-\beta_1}$ we ensure that both players are willing to expend asymmetric equilibria efforts, i.e. $E\left(\pi_i(x_i^* = \overline{x}, x_j^* = \underline{x})\right) \geq L$ and $E\left(\pi_j(x_i^* = \overline{x}, x_j^* = \underline{x})\right) \geq L$. These results are summarized in the following Proposition.

**Proposition:** In contest $\Gamma(i, j, \Omega)$, if $-(3\alpha_1 + \alpha_2) - (\beta_1 - \beta_2) > 0$ and $\beta_2 - \alpha_1 \geq 0$ then there exists a symmetric equilibrium defined by (9). Furthermore, if $(5\alpha_1 - \alpha_2) - (\beta_1 - \beta_2) > 0$ and $\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}} \geq \frac{(4\alpha_1-\alpha_2)-(\beta_1-2\beta_2)}{(2\alpha_1-\alpha_2)-\beta_1}$ then in addition to the symmetric equilibrium there exist two asymmetric equilibria defined by (11) and (12).

Szidarovszky and Okuguchi (1997) as well as Cornes and Hartley (2005) proved the existence and uniqueness of the symmetric equilibrium for a simple Tullock contest. In this article we showed that uniqueness of the equilibrium crucially depends on the specification of payoff function and the cost and spillover parameters. From the Proposition one may conclude that under a generalized payoff structure, the equilibrium of the Tullock contest is not necessarily unique.[5]

## 3. Existing Contests in the Literature

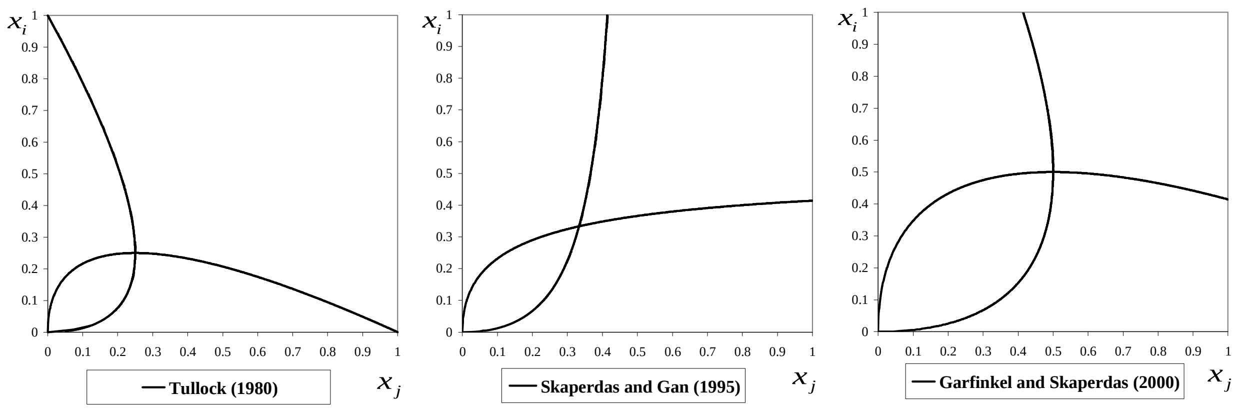
In the standard contest defined by Tullock (1980), both players have the same valuation for the prize and despite the outcome of the contest both players completely forgo their efforts. In such a case, $W > 0$, $\alpha_1 = \alpha_2 = -1$, and the other parameters in $\Omega$ are zero. The best response function for player $i$ is $x_i = -x_j + \sqrt{W x_j}$ (Figure 3.1). The unique equilibrium is the symmetric equilibrium with $x_i^* = x_j^* = W/4$. Skaperdas and Gan (1995) examine a "limited liability" case in which the loser's payoff is independent of the efforts expended. The authors motivate this example by stating that contestants may be entrepreneurs who borrow money to spend on research and development and thus are not legally responsible in case of loss. The loser of such a contest is unable to repay the loan and goes bankrupt. In such a case, $W > 0$, $\alpha_1 = -1$, and the other parameters in $\Omega$ are zero. The best response function for player $i$ is $x_i = -x_j + \sqrt{x_j^2 + W x_j}$ (Figure 3.1). Under the symmetric equilibrium we have $x_i^* = x_j^* = W/3$.

Garfinkel and Skaperdas (2000) consider a case where two players contest to win a war. In this game player $i$ and player $j$ have resource endowments of $V_i$ and $V_j$ which they can use to win the contest. The winner receives the sum of resources minus the sum of efforts expended by both players. It is also assumed that war destroys a fraction $(1-\phi) \in (0,1)$ of the total payoff.

[5] Lee and Kang (1998) and Gurtler (2005) consider contests with externalities. These contests can be captured by setting $\alpha_1 = \alpha_2 = -(1-\beta)$, and $\beta_1 = \beta_2 = \beta$, where $\beta \in (0,1)$ is the externality parameter. It is easy to show that besides the symmetric equilibrium shown by the authors, asymmetric equilibria exist. Similarly, one can demonstrate that the contest described by Cohen and Sela (2005) has multiple equilibria. The later point has been independently shown by Matros (2009).

Thus, the needed restrictions are $W = \phi(V_i + V_j)$, $\alpha_1 = \beta_1 = -\phi$, and the other parameters in $\Omega$ are zero. The best response function is $x_i = -x_j + \sqrt{(V_i + V_j)x_j}$ (Figure 3.1, where $V_i + V_j = 2$). Even though $V_i$ and $V_j$ can be different, the equilibrium efforts for players $i$ and $j$ are the same, $x_i^* = x_j^* = (V_i + V_j)/4$.

**Figure 3.1 – Best Response Functions and Resulting Equilibria (*W* = 1)**




Baye et al. (2005) examine and compare different litigation systems under the all-pay auction CSF ($r = \infty$). We use the Tullock lottery CSF ($r = 1$) in Baye et al. (2005) structure by restricting $L = 0$, $\alpha_1 = -\beta$, $\beta_1 = -(1-\alpha)$, $\alpha_2 = -\alpha$, and $\beta_2 = -(1-\beta)$, where $\alpha \in (0,1)$ and $\beta \in (0,1)$. Interesting enough, when we restrict the parameters to match their model, the best response function $x_i = -x_j + \sqrt{W x_j/\beta}$ is independent of the value of $\alpha$. Note that when $\beta = 1$ (i.e., the case of American, Marshall, and Quayle systems of litigation), the best response function as well as the symmetric equilibrium turns out to be qualitatively equivalent to that in Tullock (1980).

# 4. New Contests

In the previous Sections we discussed several well known contests in the literature. For example, we discussed a standard Tullock contest in which the restriction $\alpha_1 = \alpha_2 = -1$ implies that the cost of losing is same as the cost of winning. However, in many real life situations we observe that the winner of the contest pays less than the loser. For example, in many East-European universities students who receive A as an average grade in recitations do not have to take the final exam and thus automatically receive A as their final grade. Another example is the government procurement auction for defense weapons. Different companies make costly investments to produce prototypes and the government shares the prototype's production cost with the winner.[6] In these cases, the winner of the contest faces lower marginal cost than the loser. Rightfully, this contest can be called a "lazy winner" contest. We can capture this by setting $W > 0$, $\alpha_2 < \alpha_1 < 0$ and other parameters in $\Omega$ to zero. Therefore, the payoff for player $i$ is given by

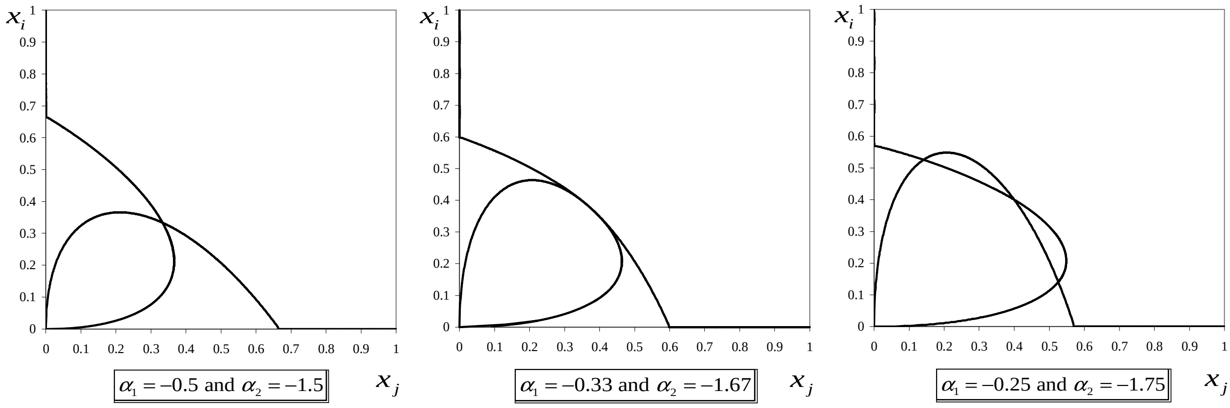
$$\pi_i(x_i, x_j) = \begin{cases} W + \alpha_1 x_i & \text{with probability} \quad p_i(x_i, x_j) \\ \alpha_2 x_i & \text{with probability} \quad 1 - p_i(x_i, x_j) \end{cases} \tag{13}$$

The resulting best response function is $x_i = -x_j + \sqrt{\{(\alpha_1 - \alpha_2)x_j^2 - W x_j\}/\alpha_1}$. Under symmetric equilibrium, both players expend equal efforts $x_i^* = x_j^* = W/(-3\alpha_1 - \alpha_2)$. The scenario described by (13) can generate multiple equilibria if the difference between $\alpha_1$ and $\alpha_2$ is sufficiently high. In Figure 4.1 we plot the best response functions for different parameters of $\alpha_1$ and $\alpha_2$. When $\alpha_1 = -0.25$ and $\alpha_2 = -1.75$, the best response functions intersect three times, indicating one symmetric and two asymmetric equilibria. The two asymmetric equilibria of the contest defined by (13) are given by $\{x_i^* = \overline{x}; x_j^* = \underline{x}\}$ and $\{x_i^* = \underline{x}; x_j^* = \overline{x}\}$, where $\overline{x} =$

[6] See Kaplan et al. (2002) for a detailed example.

$\frac{1}{2}\frac{\sqrt{\alpha_1-\alpha_2}+\sqrt{5\alpha_1-\alpha_2}}{\sqrt{(\alpha_1-\alpha_2)^3}}W$ and $\underline{x} = \frac{1}{2}\frac{\sqrt{\alpha_1-\alpha_2}-\sqrt{5\alpha_1-\alpha_2}}{\sqrt{(\alpha_1-\alpha_2)^3}}W$ . For the existence of asymmetric equilibria we need the condition $5\alpha_1 > \alpha_2$. The intuition behind this result comes from the perceptive behavior of the players. One player may give more weight to the fact that the loser has a higher marginal cost and thus may expend a low effort $\underline{x}$ in equilibrium. On the contrary, the other player envisions a lower marginal cost of winning and thus may expend a higher effort $\overline{x}$. This result is completely new to the literature, since nobody has shown that under such mild restrictions multiple equilibria can arise in the Tullock contest.

**Figure 4.1 – Best Response Functions for "Lazy Winner" Contest ($W$ = 1)**




Next, we consider an "input spillover" contest where the effort expended by player $j$ partially benefits player $i$ and vice versa. This case can be interpreted as the input spillover effect in R&D innovation contest (Kamien et al., 1992).[7] In our model we assume that the winner of the contest receives a benefit proportional to the loser's effort and the loser receives a benefit proportional to the winner's effort. After setting $\alpha_1 = \alpha_2 = -1$, and $L = 0$ the payoff function of "input spillover" contest takes the form:

$$\pi_i(x_i, x_j) = \begin{cases} W - x_i + \beta_1 x_j & \text{with probability } p_i(x_i, x_j) \\ -x_i + \beta_2 x_j & \text{with probability } 1 - p_i(x_i, x_j) \end{cases} \tag{16}$$

[7] Baye et al. (2009) provide a similar analysis using an all-pay auction CSF.

where $\beta_1 \geq 0$, $\beta_2 \geq 0$, and $\beta_1 - \beta_2 < 4$. The interesting part about this type of contest is that the best response function, $x_i = -x_j + \sqrt{(\beta_1 - \beta_2)x_j^2 + Wx_j}$, changes dramatically with changes in parameters. In Figure 4.2 we display best response functions and resulting equilibria for different values of $\beta_1$ and $\beta_2$. Note that when $\beta_1 = \beta_2 = 0.5$ we have standard best response functions as in Tullock (1980). This is the case when both the winner and the loser receive the same benefit from other player's effort. When only the winner receives the benefit, $\beta_1 = 1$ and $\beta_2 = 0$, then the "input spillover" contest is equivalent to Skaperdas and Gan (1995).

**Figure 4.2 – Best Response Functions for "Input Spillover" Contest (*W* = 1)**

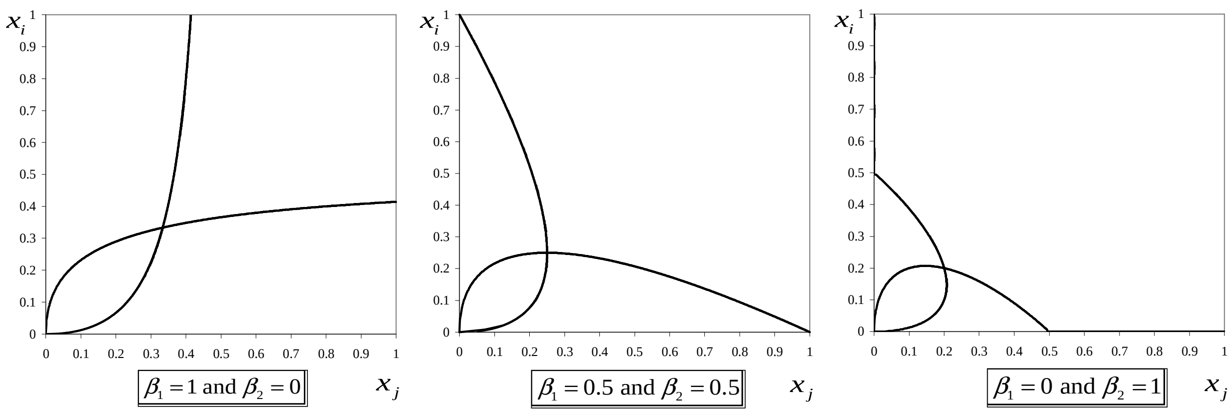
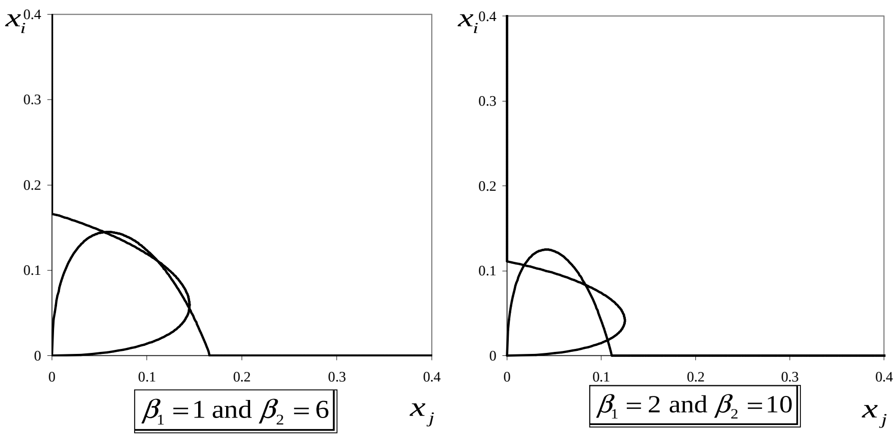



**Figure 4.3 – Asymmetric Equilibria in "Input Spillover" Contest (*W* = 1)**



The symmetric equilibrium effort in the contest defined by (16) is given by $x_i^* = x_j^* = W/(4 - \beta_1 + \beta_2)$. As $(\beta_1 - \beta_2)$ increases, the total effort expended in the contest increases. This has a simple intuition: if the positive externality gathered by winning increases relative to that of losing then the players will spend more effort to win the contest. This can also be seen from Figure 4.2. Under symmetric equilibrium, as we move from left to right, $(\beta_1 - \beta_2)$ decreases and as a result the total effort expended in the contest decreases.

It is also important to emphasize that when spillover gain of the loser $(\beta_2)$ is sufficiently higher than the spillover gain of the winner $(\beta_1)$, we arrive at the case of multiple equilibria. In particular, any combination of $\beta_1$ and $\beta_2$, such that $\beta_1 - \beta_2 < -4$, will generate one symmetric and two asymmetric equilibria (Figure 4.3). When the difference in spillover is substantial, then the total equilibrium effort expended in any of the equilibria is very small relative to the prize value. This case resembles R&D contests in countries where property rights are not protected by the government and the spillover in case of losing is very high. Therefore, there is a strong incentive to free ride on the effort of the others.

## 5. Discussion

In this article we construct a generalized Tullock contest under complete information. We show how different existing contests in the literature can be nested under this generalized structure. We also find conditions for the existence of multiple asymmetric equilibria even under symmetric prize values. Finally, we introduce and characterize several contests new to the literature. Our results can be applied to the fields of labor economics, law and economics, industrial organization, public economics, and political economy. By applying certain parameter restrictions to our model one can also imitate the rent-seeking contest, patent race, military combat, or legal conflict.

There are number of interesting extensions of our analysis. For example, one can use our generalized structure to meet a given objective of a contest designer. This objective varies between contests. In sports or social benefit programs the designer may want to maximize the total effort expenditures, whereas in rent-seeking or election contests the designer may want to minimize the total effort expenditures. For a given objective, one can set the parameters of our model so that the desired outcome is achieved. Other extensions include contests with more than two players, the effects of risk aversion and incomplete information. Finally, this study opens questions of stability analysis in the Tullock type contests with multiple equilibria.

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# Appendix

## Second Order Condition and Concavity of Payoff Function

From (4) the F.O.C. and S.O.C. are as following

$$\frac{dE(\pi_i)}{dx_i} = B\frac{x_i^2+2x_ix_j}{(x_i+x_j)^2} + A\frac{x_j}{(x_i+x_j)^2} + \alpha_2 \tag{A.1}$$

$$\frac{d^2E(\pi_i)}{dx_i^2} = 2B\frac{(x_i+x_j)^3-(x_i^2+2x_ix_j)(x_i+x_j)}{(x_i+x_j)^4} - 2A\frac{x_j}{(x_i+x_j)^3} = (Bx_j - A)\frac{2x_j}{(x_i+x_j)^3} \tag{A.2}$$

Replacing in (A.2) the values of $A$ and $B$ from (4) we get:

$$Sign\left\{\frac{d^2E(\pi_i)}{dx_i^2}\right\} = Sign\{Bx_j - A\} = Sign\{(\alpha_1-\alpha_2-\beta_1+\beta_2)x_j - (W-L)\} \tag{A.3}$$

Assuming symmetric equilibrium as in (9) the necessary condition for S.O.C. is

$$Sign\left\{\frac{\alpha_1-\alpha_2-\beta_1+\beta_2}{-3\alpha_1-\alpha_2-\beta_1+\beta_2}(W-L) - (W-L)\right\} < 0 \tag{A.4}$$

Since $W > L$, $(W-L)$ does not affect the sign of the S.O.C. Hence, we can rewrite (A.4) as:

$$Sign\left\{\frac{4\alpha_1}{-3\alpha_1-\alpha_2-\beta_1+\beta_2}\right\} < 0 \tag{A.5}$$

Therefore, the sufficient condition for S.O.C. to hold is $-(3\alpha_1+\alpha_2)-(\beta_1-\beta_2) > 0$.

When the equilibrium is asymmetric as in (11) or (12) the necessary conditions for S.O.C. are

$$Sign\left\{-1+\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right\} < 0 \text{ and } Sign\left\{-1-\sqrt{\frac{(5\alpha_1-\alpha_2)-(\beta_1-\beta_2)}{(\alpha_1-\alpha_2)-(\beta_1-\beta_2)}}\right\} < 0 \tag{A.6}$$

It straightforward to see that as long as $\alpha_1 < 0$, both conditions in (A.6) hold.