

THE MAX-MIN GROUP CONTEST

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ABSTRACT. We investigate a group all-pay auction with weakest-link impact function and group-specific public good prize. Since only the minimum effort exerted among all group members represents the group effort and the group with the maximum group effort wins the contest, this is termed as the ‘Max-Min group contest’. Examples of such structure include various sporting events, territorial conflicts, negative product or political campaigns etc. We fully characterize pure strategy equilibria for the case of two groups and show that a continuum of pure strategy equilibria exists, in which all (active) players exert the same effort. We also fully characterize symmetric mixed strategy equilibria for the case of two groups. There are two types of non-degenerate mixed strategy equilibria - with and without continuous supports. A semi-pure strategy equilibrium may exist in which all the members of one group play the same pure strategy whereas all the members of the other group play the same mixed strategy. We also fully characterize pure strategy equilibria for a general case of n groups and specify candidates for mixed strategy equilibria.

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JEL classifications: C72; D70; D72; H41

1. Introduction

Contests are a family of games in which players exert costly resources such as physical effort, time, money etc. in expectation of winning a valuable prize (Konrad, 2009). Contests are ubiquitous and examples include sports, rent-seeking, litigation, war among others. Modeling a contest involves specifying the probability of winning as a function of efforts put forth by the players. This function is called a Contest Success Function (CSF). The two most popular CSFs used in the literature are the all-pay auction (Baye et al., 1996) in which the player with the highest effort wins with certainty, and the lottery (Tullock, 1980) in which a higher effort is associated with a higher probability of winning. In the case of group contests, groups engage in costly confrontations and each group member has the opportunity to make a costly investment for its group. A function that translates the efforts of individual group members into the group effort is called an Impact Function (Wärneryd, 2001).

In the current study we investigate a group contest with the weakest-link impact function and the all-pay auction CSF. In other words, the minimum among the efforts exerted by the individual members within a group represents the effort of that group, and the group whose group effort is the maximum among all the groups wins the prize with certainty. As the Max effort among the Min efforts turns out to be the winner, we call the contest described above as the Max-Min group contest. We make two further assumptions standard in the literature (and in line with the examples provided later). First, the prize has the nature of a group specific public good in the sense that the prize is achieved by all the members of the winning group. Even if a member does not expend effort, but its group wins, that member is also entitled to the prize. Second, we assume complete information, i.e., the game structure and the parameter values are common knowledge.

The literature on group contest starts with the work by Katz et al. (1990) who use a perfectly substitute (linear) impact function and a lottery CSF under symmetric valuation. This is later extended by Baik (2008) with asymmetric valuation.¹ It is found that low valuation players free ride on the highest valuation group member(s). The analyses of non-linear impact functions are rather recent. Under lottery CSF, Lee (2012) uses a weakest-link impact function and finds multiple equilibria with no free-riding, among which it is

¹Ursprung (1990), Baik (1993), Riaz et al. (1995), Dijkstra (1998), and Münster (2009) also study the group contests with a lottery CSF and group-specific public good prize. Whereas Nitzan (1991a,b), Lee (1995), Katz and Tokatlidu (1996) and Wärneryd (1998) consider the prize as a private good.

possible to identify the Pareto dominant equilibrium. Kolmar and Rommeswinkel (2013) use a CES impact function ranging from perfect substitute to weakest link and pin down the convergence conditions for equilibria. Chowdhury et al. (2013) instead use a best-shot impact function in which the best effort among the group members represents the group effort. They find equilibria in which it is possible for the highest valuation player to free ride on a lower value group member's effort. All these studies consider every group to follow the same impact function, but Chowdhury and Topolyan (2013) analyze a group contest in which it is possible for different groups to follow different impact functions and fully characterize related equilibria.

Turning to the all-pay auction CSF, Baik et al. (2001) and Topolyan (2013) consider a perfectly substitute impact function and find free-riding equilibria analogous to the lottery CSF results. Barbieri et al. (2013) consider a best-shot impact function and found conditions for which multiple agents per group may be active. Both of these studies found equilibria different from individual all-pay auctions. We contribute to the group contest literature by employing the weakest link impact function and all-pay auction CSF.²

There are several field examples that involve a Max-Min group contest. Consider, for instance, team pursuit sporting events, such as team races in speed skating and female cycling. In these events each teams record is measured as the time when the last member of the team finishes the race and the winning prize goes to the team whose record is the fastest. Also, both summer and winter Olympic team events based on synchronization, such as synchronized swimming, rhythmic gymnastics, or pairs (figure) skating, the team performance literally depends on the lowest performer of the team. All these cases are copybook examples of a Max-Min group contest.

In industrial organization, the race between two research joint ventures to set the standard for a new O-ring (Kremer, 1993) type product, or the system reliability war between two system provider groups also have similar features with this contest. In both cases the weakest part of the product/system portrays its strength while competing with another

²Since the prize in this group contest has the nature of a public good within the winning group, the current study is also related to the literature on the provision of public goods under network externalities; and contributes to a well established area of research including studies by Hirshleifer (1983), Bergstrom et al. (1986), Harrison and Hirshleifer (1989), Vicary (1990), Conybeare et al. (1994), Sandler and Vicary (2001), Arce (2001), Vicary and Sandler (2002), Varian (2004), Cornes and Hartley (2007), and Lei et al. (2007).

product or system. Similarly, when rival organizations are involved in negative advertisement about several dimensions of a product, then the product with the weakest feature in a particular dimension eventually loses market. Similarly in the case of negative electoral campaigning on a set of political issues (Skaperdas and Grofman, 1995), the weakest issue determines the strength of a candidate. Hence, situations in which multi-dimensional negative campaigning is involved can also be modeled as a Max-Min group contest.

Another distinct example of the Max-Min group contest is territorial conflict. National Geographic (2006) observes Turf battle among Borneo Sea Eagles in which a pair of Sea Eagles fight off intruder Eagles who try to catch fish in the territory covered by the pair.³ It is also observed that often different computer viruses and bots get involved into conflict to capture the same computer or program. The ‘Computer World’ magazine reported in 2010 that *“An upstart Trojan horse program has decided to take on its much-larger rival by stealing data and then removing the malicious program from infected computers. Security researchers say that the relatively unknown Spy Eye toolkit added this functionality just a few days ago in a bid to displace its larger rival, known as Zeus. The feature, called “Kill Zeus,” apparently removes the Zeus software from the victim’s PC, giving Spy Eye exclusive access to usernames and passwords.”*⁴

All these examples are composed of diverse background and nature. Nevertheless they all reflect situations in which groups engage in conflict and the strength of a group depends on the weakest effort exerted by a member of the group. Consider the computer virus for an example. Each virus is a collection of codes. While in conflict with another virus, whether one virus will ultimately be the victim of its rival and be abolished depends crucially on the weakest part of its codes. Similarly, when two Sea Eagle pairs get involved into a territorial conflict, which pair will be evicted depends on the strength of its weaker partner - who leaves the conflict first. The same logic follows for other examples as well. This, however, is not surprising. When groups get involved in attack games such as the examples given above, then the group members are essentially connected with a weakest-link technology, and survival of a group depends on its weakest group member (Clark and Konrad, 2007; Kovenock and Roberson, 2010).

³<http://rockyforkfarms.com/eagle-vs-eagle-in-mid-air/>

⁴http://www.computerworld.com/s/article/9154618/New_Russian_botnet_tries_to_kill_rival. This, however, is not an exception. Similar incidences noted by, among others, Layden (2008) in Channel Register Magazine. Details are available here: http://www.channelregister.co.uk/2008/02/28/rootkit_wars/.

In the continuation we aim to characterize the possible equilibrium strategies for the Max-Min group contest and discuss the related implications. Standard all-pay auction literature, be it regarding individual players (Baye et al., 1996) or groups (Baik et al., 2001), show that pure strategy equilibrium does not exist. However, in a two group case, we find that a continuum of pure strategy equilibria might co-exist with four characteristically different mixed strategy equilibria. In these pure strategy equilibria, all (active) players expend the same effort. A special type of mixed strategy equilibrium exists in which every member of a group exerts the same effort whereas every member of the other group plays the same mixed strategy. We call this a semi-pure strategy equilibrium. There are two types of non-degenerate mixed strategy equilibria - with and without continuous supports. Section 2 constructs the model with two groups. This helps to keep brevity in the analysis as many of the quantitative results follow through to Section 3 which analyzes a general model with n groups with any number of group members. We characterize the pure strategy equilibria and introduce candidates for mixed strategy equilibria for the general case. We discuss the results and corresponding implications in Section 4.

2. A Two-group Group-Specific Common Value Model

Assume there are two groups, 1 and 2, having k_1 and k_2 risk-neutral players, respectively, where $k_1, k_2 \geq 2$. Denote by I_g the index set of all players of group g . The prize is a group-specific public good, and the valuation of the prize is the same for every player of group g (denoted by v_g). Assume without loss of generality that $v_1 \geq v_2$.

Denote a bid of player i of group g by $x_{g,i}$, and the vector of bids $(x_{g,1}, \dots, x_{g,k_g})$ by x_g . Given player i of group g and a tuple of bids (x_1, x_2) , denote by x_{-i}^g the bids of all players other than i , i.e.,

$$x_{-i}^g = \{(x_{g,1}, \dots, x_{g,i-1}, x_{g,i+1}, \dots, x_{g,k_g}), x_{-g}\},$$

where x_{-g} is the vector of bids from the group not g .

Definition 2.1. *Given a vector of bids of group g 's members x_g , **group g 's bid** is a non-negative real number $B(x_g)$, where B is a (weakest link) function of k_g variables of the following form:*

$$B(x_g) = \min_{i \in I_g} x_{g,i}. \quad (2.1)$$

The cost function of bid is the same for all players and $c(x) = x$ for any effort level $x \geq 0$.

We assume both groups compete in an all-pay auction (see Hillman and Riley, 1989; Baye et al., 1996). Let P_g denote the winning probability of group g , then the contest success function has the following form:

$$P_g(x_g, x_{-g}) = \begin{cases} 1 & \text{if } B(x_g) > B(x_{-g}) \\ \frac{1}{2} & \text{if } B(x_g) = B(x_{-g}) \\ 0 & \text{if } B(x_g) < B(x_{-g}). \end{cases} \quad (2.2)$$

Consequently, the payoff function of player i of group g has the following form:

$$u_{g,i}(x_g, x_{-g}) = \begin{cases} v_g - x_{g,i} & \text{if } B(x_g) > B(x_{-g}) \\ \frac{v_g}{2} - x_{g,i} & \text{if } B(x_g) = B(x_{-g}) \\ -x_{g,i} & \text{if } B(x_g) < B(x_{-g}). \end{cases} \quad (2.3)$$

Definition 2.2. A **strategy** of player i of group g is a probability distribution over a subset of \mathbb{R}_+ (the set of bids of player i). A strategy profile of player i of group g is denoted by $s_{g,i}$. Denote the set of all strategies of player i by $\mathbb{S}_{g,i}$.

Definition 2.3. Group g 's **strategy** is a k_g -tuple of its members' strategies $(s_{g,1}, \dots, s_{g,k_g})$. Group g 's strategy is denoted by s_g . The set of all strategies of group g is denoted by \mathbb{S}_g , and the set of all strategies for the game by \mathbb{S} .

We extend the payoff function u from the set of pure strategies $\mathbb{R}_+^{k_1+k_2}$ to the set of mixed strategies \mathbb{S} as follows. Fix a strategy $s \in \mathbb{S}$; let F_g , $F_{g,i}$, and F_{-g} be the probability measures on \mathbb{R}_+ induced by s_g , $s_{g,i}$, and s_{-g} , respectively. Intuitively, F_g for instance represents the probability distribution induced on \mathbb{R}_+ by s_g . Then the payoff of player i of group g is computed as follows:

$$u_{g,i}(s) = \int_{\mathbb{R}_+} v_g F_{-g}(x) dF_g(x) - \int_{\mathbb{R}_+} x dF_{g,i}(x). \quad (2.4)$$

The first integral in Equation 2.4 represents the expected value of winning the prize, and the second integral represents the expected cost of effort for player i .

Definition 2.4. Given a strategy $s_{g,i}$ of player i of group g , the **individual support** $S_{g,i}$ is the closure of set of all points of increase of the cumulative distribution function $F_{g,i}$ corresponding to $s_{g,i}$:

$$S_{g,i} = Cl(\{x \in \mathbb{R}_+ : F_{g,i}(x - \epsilon) < F_{g,i}(x) < F_{g,i}(x + \epsilon) \text{ for all } \epsilon > 0\}).$$

Definition 2.5. Similarly, given group g 's strategy s_g , the **group support** of s_g is the closure of set of all points of increase of the cumulative distribution function F_g corresponding to s_g , and will denoted by S_g :

$$S_g = Cl(\{x \in \mathbb{R}_+ : F_g(x - \epsilon) < F_g(x) < F_g(x + \epsilon) \text{ for all } \epsilon > 0\}).$$

Note that a support may fail to be connected. This happens when a subset of the convex hull of the support is omitted by a player, i.e., chosen with probability zero.

Definition 2.6. A tuple $x^* = (x_1^*, x_2^*)$, where x_g^* is group g 's bid (see Definition 2.1), is called a **pure strategy equilibrium** if

$$u_{g,i}(x_{g,i}^*, x_{-i}^{g*}) \geq u_{g,i}(x_{g,i}, x_{-i}^{g*}) \text{ for every player } i \text{ of group } g \text{ and every } x_{g,i} \in \mathbb{R}_+.$$

Definition 2.7. A tuple $s^* = (s_1^*, s_2^*)$, where s_g^* is group g 's strategy (see Definition 2.3), is called a **mixed strategy equilibrium** if

$$u_{g,i}(s_{g,i}^*, s_{-i}^{g*}) \geq u_{g,i}(s_{g,i}, s_{-i}^{g*}) \text{ for every player } i \text{ of group } g \text{ and every } s_{g,i} \in \mathbb{S}_{g,i}.$$
⁵

Note that a pure strategy equilibrium is a special case of a mixed strategy equilibrium, but we specifically define it here because generally we do not get a pure strategy equilibrium in all-pay auctions. The following result is obvious and we state it without a formal proof.

Theorem 2.8. All equilibria in pure strategies are as follows.

$$x_{g,i} = \lambda \quad \text{for all } i \in I_g, g = 1, 2 \tag{2.5}$$

where $\lambda \in [0, \frac{v_2}{2}]$. Furthermore the total effort expended in the pure strategy equilibria is $(k_1 + k_2)\lambda$, which varies between 0 and $\frac{(k_1 + k_2)v_2}{2}$.

⁵The term s_{-i}^{g*} is defined analogously to x_{-i}^{g*} .

Definition 2.9. Given an individual support $S_{g,i}$ of player i , the **upper bound** of the support is

$$\bar{s}_{g,i} = \sup \{x : x \in S_{g,i}\}.$$

Similarly, the **lower bound** of the support is

$$\underline{s}_{g,i} = \inf \{x : x \in S_{g,i}\}.$$

The upper and lower bounds of a group support S_g are defined analogously, with S_g in place of $S_{g,i}$ in the above definition.

For each group g , let $S_{g+}^* = S_g^* \cap \mathbb{R}_{++} = \{x > 0 : x \in S_g^*\}$. Let $\underline{s}_{g+}^* = \inf \{x : x \in S_{g+}^*\}$ and $\bar{s}_{g+}^* = \sup \{x : x \in S_{g+}^*\}$.

Now we will discuss a non-degenerate mixed strategy equilibrium, and for the purpose of brevity in the continuation we use the term mixed strategy equilibria when we discuss non-degenerate mixed strategy equilibria.

Lemma 2.10. Let s^* be a mixed strategy equilibrium, then the following statements are true.

- (1) $\bar{s}_{g,i}^* = \bar{s}_g^*$ for all $i \in I_g$.
- (2) $S_{1+}^* = S_{2+}^*$.
- (3) If $S_1 = S_2$, then $\underline{s}^* = 0$.

Proof. (1) Suppose without loss of generality that $\bar{s}_{g,i}^* > \bar{s}_{g,j}^*$, then player j is better-off shifting mass from $(\bar{s}_{g,i}^*, \bar{s}_{g,j}^*]$ to $\bar{s}_{g,i}^*$, leading to a contradiction.

- (2) Suppose $S_{1+}^* \neq S_{2+}^*$. Without loss of generality let x be such that $x \in S_{1+}^*$ but $x \notin S_{2+}^*$. Consider the following cases.

- (a) If $x > \bar{s}_{2+}^*$, then any player of group 1 is better-off switching from x to $x - \epsilon$ for some small $\epsilon > 0$, which is a contradiction.
- (b) If $x < \underline{s}_{2+}^*$, then any player of group 1 is better-off deviating from x to 0 (observe that $x > 0$ since $x \in S_{1+}^*$).
- (c) If $\underline{s}_2^* < x < \bar{s}_2^*$ (in which case the support of group 2 is not connected), then any member of group 1 would deviate from x to $x - \epsilon$ for some small $\epsilon > 0$.

- (3) Suppose that $S_1 = S_2$ and the common lower bound is $\underline{s}^* > 0$. We claim that every group must put an atom at \underline{s}^* . Suppose not, say group 1 does not put an atom at \underline{s}^* , then the payoff of any player i of group 2 from playing \underline{s}^* is

$$u_{2,i}(\underline{s}^*, s_{-i}^*) = -\underline{s}^* < 0.$$

Hence player i is better-off deviating to 0, which establishes the claim. But if every group puts some mass at \underline{s}^* , then any player could increase her payoff by deviating from \underline{s}^* to $\underline{s}^* + \epsilon$ for some small $\epsilon > 0$, which is a contradiction.

■

Lemma 2.11. *Let s^* be a mixed strategy equilibrium. Then no player puts mass at x for any $x \in (\underline{s}_+^*, \bar{s}^*)$.*

Proof. Note that by Lemma 2.10(2), $S_{1+}^* = S_{2+}^*$, therefore $\underline{s}_{1+}^* = \underline{s}_{2+}^* = \underline{s}_+^*$. Suppose without loss of generality some player i of group 1 puts a positive mass at some $a \in (\underline{s}_+^*, \bar{s}^*)$. It is easy to see that F_1 has a jump discontinuity at a . Let player j of group 2 be such that $a \in S_{2,j+}^*$, then $u_{2,j}(a) < u_{2,j}(a + \epsilon)$ for some $\epsilon > 0$. Then any player j of group 2 such that $\underline{s}_{2,j+}^* \geq \underline{s}_+^*$ would want to shift mass from $(a - \epsilon, a]$ to $a + \delta$ for some small $\epsilon, \delta > 0$. If no member of group 2 puts mass in $(a - \epsilon, a]$, then it is not optimal for player i to put mass at a (she is better off deviating to $a - \epsilon$). ■

Let us investigate the so-called semi-pure strategy equilibria in which some players employ a pure strategy, while others use a mixed strategy. We focus on the symmetric equilibria, in which players within a group employ the same strategy.

Lemma 2.12. *All symmetric semi-pure strategy equilibria are as follows. For every $0 < a < 1$, every player of group 1 always contributes $\frac{v_2}{2} (1 - a)^{k_2 - 1}$, and every player of group 2 contributes zero with probability $1 - a$ and $\frac{v_2}{2} (1 - a)^{k_2 - 1}$ with probability a .⁶ If $v_1 = v_2 = v$, then in addition there exists a continuum of equilibria where for every*

⁶Note that when $a = 0$ or $a = 1$, the continuum of equilibria degenerates to the pure strategy equilibria where each player contributes zero or $\frac{v}{2}$, respectively.

$0 < a < 1$, every player of group 2 always contributes $\frac{v}{2}(1-a)^{k_1-1}$, and every player of group 1 contributes zero with probability $1-a$ and $\frac{v}{2}(1-a)^{k_1-1}$ with probability a .

Proof. Let s^* be an equilibrium which is symmetric and semi-pure. By Lemma 2.10(2) we have $S_{1+}^* = S_{2+}^*$. By Lemma 2.11 no group puts mass at more than one point above zero. Notice that it is not possible for both groups to put mass at zero, for otherwise any player of the other group could increase her payoff by deviating from zero to some small $\epsilon > 0$. Therefore the only possibility is for every player in one group to always contribute some $x > 0$ and for every player in the other group (group g) to randomize between zero and x . Let every player in group g put mass a at x . Since every player in group g earns zero payoff and is indifferent between his pure strategies of zero and x , it must be that

$$x = \frac{v_g}{2}a^{k_g-1}.$$

Clearly, no player j in group $(-g)$ wants to deviate from x provided that

$$u_{-g,j}(s^*) = v_{-g}\left(1 - \frac{1}{2}a^{k_g}\right) - x \geq 0,$$

which is satisfied if and only if

$$\frac{2 - a^{k_g}}{a^{k_g-1}} \geq \frac{v_g}{v_{-g}}. \quad (2.6)$$

Condition 2.6 holds if and only $v_g \leq v_{-g}$, i.e., $v_g = v_2$ and $v_{-g} = v_1$, in which case it holds for all $0 \leq a \leq 1$. This completes the proof. ■

Recall that for any group strategy s_g the corresponding support S_g is unique. This follows directly from the definition of a support.

Corollary 2.13. *Let s^* be a mixed strategy equilibrium which is not semi-pure, then the equilibrium group support S_g^* has uncountably many elements for each $g = 1, 2$.*

Proof. Immediate consequence of Lemmata 2.10(2), 2.10(3), and 2.11. ■

Lemma 2.14. *Let s^* be a symmetric mixed strategy equilibrium for which there exists $0 < z < \bar{s}^*$ such that $x \in S_1 \cap S_2$ for all $z \leq x \leq \bar{s}^*$. Then no single group puts an atom at \bar{s}^* (i.e., it is not the case that all players of some group put an atom at \bar{s}^*).*

Proof. Suppose not, and without loss of generality group 1 puts an atom at \bar{s}^* . Consider two possible cases.

- (1) Group 2 also puts a positive mass at \bar{s}^* . Then any player i of group 1 is better-off increasing her effort from $\bar{s}^* - \epsilon$ to \bar{s}^* for a sufficiently small $\epsilon > 0$ for the following reason. If player i expends effort level \bar{s}^* , then group 1 ties with group 2 with a positive probability (in the event that the effort level of both groups is \bar{s}^*). However if player i expends effort $\bar{s}^* - \epsilon$, then group 1's effort is always less than \bar{s}^* due to the weakest-link effort technology, and group 1 does not get a chance to tie with group 2 at \bar{s}^* , resulting in a loss of payoffs for all players of group 1. For a sufficiently small ϵ such loss exceeds the economy on the effort cost for player i , which is equal to ϵ . This implies that the support $S_{1,i}^*$ is disconnected, contradicting our initial assumption.
- (2) Group 2 does not put a positive mass at \bar{s}^* . Again, we claim that any player i of group 1 is better-off increasing her effort from $\bar{s}^* - \epsilon$ to \bar{s}^* for a sufficiently small $\epsilon > 0$. If player i chooses \bar{s}^* , then with a positive probability group 1's effort exceeds that of group 2 (which happens when group 1's effort is \bar{s}^*). If player i expends effort level $\bar{s}^* - \epsilon$, then group 1's effort is always less than \bar{s}^* , which results in a payoff loss for player i . Consequently, the support $S_{1,i}^*$ is disconnected, contradicting our initial assumption.

■

Note that it is not an equilibrium for each player in one group to randomize over two *strictly positive* effort levels, while all players of the other group expend some positive effort level with probability one. This is because any player of the former group would deviate from the lower bound of their group support to zero. Notice that supports of the type $[s^1, s^2] \cup [s^3, s^4]$, that is, piecewise- continuous supports, are not possible for the

following reason. By Lemma 2.11, no player puts mass at either s^2 or s^3 , thus for any player i of group g , $u_{g,i}(s^2, s_{g,-i}^*) > u_{g,i}(s^3, s_{g,-i}^*)$ because of the gap (s^2, s^3) , leading to a contradiction.

Therefore (using Lemmata 2.11, 2.14 and Corollary 2.13) if the equilibrium supports are nontrivial (i.e., non-singletons), then the following are the only remaining candidates for symmetric equilibria. The support of each player i of group g is either:

- (1) the interval $[\underline{s}^*, \bar{s}^*]$;
- (2) the union $[\underline{s}^*, \tilde{s}^*] \cup \bar{s}^*$;
- (3) the union $0 \cup [\hat{s}^*, \bar{s}^*]$;
- (4) the union $0 \cup [q^*, r^*] \cup \bar{s}^*$.

In cases (1) and (2), $\underline{s}^* = 0$ by Lemma 2.10(3). Theorems 2.15, 2.18, 2.20, and 2.21 take care of these four cases.

Theorem 2.15. *A mixed strategy equilibrium where all players randomize over some interval without a gap is unique.*

1. If $\frac{k_1 v_2}{k_2 v_1} \leq 1$, then the support is $S_{g,i}^* = [0, \bar{s}^*]$ for all players, where $\bar{s}^* = \left(\frac{k_1 v_2}{k_2 v_1}\right)^{k_2}$.
 $\frac{k_2 v_1}{k_1 + k_2 - 1}$. Each player of group 1 randomizes according to the cdf

$$F(x) = 1 - \left(1 - \frac{x}{\bar{s}^*}\right)^{\frac{1}{k_1 + k_2 - 1}},$$

and each player of group 2 employs the strategy

$$G(x) = \left(1 - \frac{k_1 v_2}{k_2 v_1}\right) + \frac{k_1 v_2}{k_2 v_1} F(x) = 1 - \frac{k_1 v_2}{k_2 v_1} \left(1 - \frac{x}{\bar{s}^*}\right)^{\frac{1}{k_1 + k_2 - 1}}.$$

2. If $\frac{k_1 v_2}{k_2 v_1} > 1$, then the support is $S_{g,i}^* = [0, \bar{s}^*]$ for all players, where $\bar{s}^* = \left(\frac{k_2 v_1}{k_1 v_2}\right)^{k_1}$.
 $\frac{k_1 v_2}{k_1 + k_2 - 1}$. Each player of group 2 randomizes according to the cdf

$$G(x) = 1 - \left(1 - \frac{x}{\bar{s}^*}\right)^{\frac{1}{k_1 + k_2 - 1}},$$

and each player of group 1 employs the strategy

$$F(x) = \left(1 - \frac{k_2 v_1}{k_1 v_2}\right) + \frac{k_2 v_1}{k_1 v_2} G(x) = 1 - \frac{k_2 v_1}{k_1 v_2} \left(1 - \frac{x}{\bar{s}^*}\right)^{\frac{1}{k_1 + k_2 - 1}}.$$

Proof. Suppose s^* is an equilibrium such that the support of each player's strategy is the interval $[0, \bar{s}^*]$. Note that, because we are working with the weakest link effort technology, the effort distribution of group g , given the individual distribution functions $F_{g,i}$'s of its players, is given by

$$F_g(x) = 1 - \prod_{i \in I_g} (1 - F_{g,i}(x)) \text{ for each } 0 \leq x \leq \bar{s}^*. \quad (2.7)$$

Equation 2.4 implies that the payoff of player i of group 1 from expending an effort $x \in (0, \bar{s}^*]$ is

$$u_{1,i}(x) = v_1 \left[\int_0^x F_2(z) dF_{1,-i}(z) + F_2(x) (1 - F_{1,-i}(x)) \right] - x \quad (2.8)$$

Since F_2 and $F_{1,-i}$ are continuous on $(0, \bar{s}^*]$ (due to the continuity of each player's strategy by Lemmata 2.11 and 2.14), $u_{1,i}(x)$ is continuous on the same interval. Because player i is randomizing over $(0, \bar{s}^*]$ without a gap, $u_{1,i}$ is constant on a dense subset of $(0, \bar{s}^*]$. Combined with the fact that $u_{1,i}$ is continuous, we conclude that $u_{1,i}$ is constant on that interval. Therefore, $u_{1,i}$ is continuously differentiable on $(0, \bar{s}^*)$, and the derivative is equal to zero.⁷ Thus,

$$\begin{aligned} u'_{1,i}(x) &= v_1 [F_2(x)F'_{1,-i}(x) + F'_2(x) - (F'_2(x)F_{1,-i}(x) + F_2F'_{1,-i}(x))] - 1 \\ &= v_1 F'_2(x) [1 - F_{1,-i}(x)] - 1 \\ &= 0 \end{aligned}$$

Denote the cdf of a generic player of group 1 by F , and the cdf of a generic player of group 2 by G , then 2.7 implies that $F_2(x) = 1 - (1 - G(x))^{k_2}$ and $F_{1,-i}(x) = 1 - (1 - F(x))^{k_1-1}$. Therefore

$$v_1 k_2 (1 - G(x))^{k_2-1} G'(x) (1 - F(x))^{k_1-1} = 1 \quad (2.9)$$

Now fix player j of group 2, and follow the same lines to conclude that

$$v_2 k_1 (1 - G(x))^{k_2-1} F'(x) (1 - F(x))^{k_1-1} = 1 \quad (2.10)$$

⁷We are indebted to David A. Malueg and Stefano Barbieri for introducing this technique to us.

Equations 2.9 and 2.10 imply that $\frac{v_2 k_1}{v_1 k_2} F'(x) = G'(x)$ for all $x \in (0, \bar{s}^*)$.

1. Assume $\frac{v_2 k_1}{v_1 k_2} \leq 1$. Notice that

$$1 - G(x) = \int_x^{\bar{s}^*} G'(z) dz = \frac{k_1 v_2}{k_2 v_1} \int_x^{\bar{s}^*} F'(z) dz = \frac{k_1 v_2}{k_2 v_1} (1 - F(x)) \quad (2.11)$$

Substituting this into 2.9 yields

$$k_2 v_1 \left(\frac{k_1 v_2}{k_2 v_1} \right)^{k_2} (1 - F(x))^{k_1 + k_2 - 2} F'(x) = 1 \quad (2.12)$$

The above differential equation determines F . To calculate F , integrate both sides of 2.25.

$$k_2 v_1 \left(\frac{k_1 v_2}{k_2 v_1} \right)^{k_2} \frac{1}{k_1 + k_2 - 1} (1 - F(x))^{k_1 + k_2 - 1} = C - x, \quad (2.13)$$

where C is a constant of integration. Therefore, F is of the form:

$$F(x) = 1 - \left(\frac{c_1 - x}{c_2} \right)^{\frac{1}{k_1 + k_2 - 1}},$$

where c_1 and c_2 are some constants. Observing that $F(0) = 0$ and $F(\bar{s}^*) = 1$, we conclude that $c_1 = c_2 = \bar{s}^*$, therefore

$$F(x) = 1 - \left(1 - \frac{x}{\bar{s}^*} \right)^{\frac{1}{k_1 + k_2 - 1}}, \text{ and} \quad (2.14)$$

$$G(x) = \left(1 - \frac{k_1 v_2}{k_2 v_1} \right) + \frac{k_1 v_2}{k_2 v_1} F(x) = 1 - \frac{k_1 v_2}{k_2 v_1} \left(1 - \frac{x}{\bar{s}^*} \right)^{\frac{1}{k_1 + k_2 - 1}} \quad (2.15)$$

Notice that each player of group 2 places mass $\left[1 - \frac{k_1 v_2}{k_2 v_1} \right]$ at zero (hence group 2 places mass $\left[1 - \left(\frac{k_1 v_2}{k_2 v_1} \right)^{k_2} \right]$ at zero), while group 1 randomizes continuously over $[0, \bar{s}^*]$. Therefore the expected payoff of any player of group 1 is $v_1 \left[1 - \left(\frac{k_1 v_2}{k_2 v_1} \right)^{k_2} \right]$, while any player of group 2 earns zero expected payoff. To pin down \bar{s}^* , observe that by differentiating 2.14 we obtain

$$F'(x) = \frac{1}{\bar{s}^* (k_1 + k_2 - 1)} \left(1 - \frac{x}{\bar{s}^*} \right)^{2 - k_1 - k_2} = \frac{1}{\bar{s}^* (k_1 + k_2 - 1)} (1 - F(x))^{2 - k_1 - k_2} \quad (2.16)$$

Express $F'(x)$ from 2.25 and equate it to the right hand side of 2.16, which yields

$$\bar{s}^* = \left(\frac{k_1 v_2}{k_2 v_1} \right)^{k_2} \cdot \frac{k_2 v_1}{k_1 + k_2 - 1} \quad (2.17)$$

Equations 2.14, 2.15, and 2.17 characterize a mixed strategy equilibrium under the condition $\frac{v_2 k_1}{v_1 k_2} \leq 1$, which is unique in the subclass of strategy profiles where players randomize over an interval without a gap.

2. Assume $\frac{v_2 k_1}{v_1 k_2} > 1$. Follow the lines of the previous case to conclude that $\bar{s}^* = \left(\frac{k_2 v_1}{k_1 v_2} \right)^{k_1} \cdot \frac{k_1 v_2}{k_1 + k_2 - 1}$, $G(x) = 1 - \left(1 - \frac{x}{\bar{s}^*} \right)^{\frac{1}{k_1 + k_2 - 1}}$, and $F(x) = \left(1 - \frac{k_2 v_1}{k_1 v_2} \right) + \frac{k_2 v_1}{k_1 v_2} G(x)$. ■

Corollary 2.16. (*Symmetric valuations*) Assume $v_1 = v_2 = v$ and without loss of generality $k_1 \leq k_2$, then the unique mixed strategy equilibrium where all players randomize over some interval without a gap is as follows. The support is $S_{g,i}^* = [0, \bar{s}^*]$ for all players, where $\bar{s}^* = \left(\frac{k_1}{k_2} \right)^{k_2} \cdot \frac{k_2 v}{k_1 + k_2 - 1}$. Each player of group 1 randomizes according to the cdf

$$F(x) = 1 - \left(1 - \frac{x}{\bar{s}^*} \right)^{\frac{1}{k_1 + k_2 - 1}},$$

and each player of group 2 employs the strategy

$$G(x) = \left(1 - \frac{k_1}{k_2} \right) + \frac{k_1}{k_2} F(x) = 1 - \frac{k_1}{k_2} \left(1 - \frac{x}{\bar{s}^*} \right)^{\frac{1}{k_1 + k_2 - 1}}.$$

Corollary 2.17. Let s^* be the mixed strategy equilibrium of Theorem 2.15. If $\frac{k_1 v_2}{k_2 v_1} \leq 1$, then every player of group 1 earns equilibrium payoff $u_1(s^*) = 1 - \frac{k_1 v_2}{k_2 v_1}$, and every player of group 2 gets $u_2(s^*) = 0$. If $\frac{k_1 v_2}{k_2 v_1} > 1$, then $u_1(s^*) = 0$ and $u_2(s^*) = 1 - \frac{k_2 v_1}{k_1 v_2}$.

Given any function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, let $J_f(x)$ denote the jump of f at $x \in \mathbb{R}$. In other words, $J_f(x)$ is the mass that f puts at x (recall that if f is continuous at x , then $J_f(x) = 0$).

Theorem 2.18. Equilibria where the support of each player is of the form $[0, \tilde{s}^*] \cup \bar{s}^*$ are as follows (\tilde{s} is the upper bound of the continuous part of the support).

- (1) If $k_1 = k_2$, then $G(x) = 1 - \frac{v_2}{v_1} + \frac{k_1 v_2}{k_2 v_1} F(x)$, $F(0) = 0$, and F is continuous for all $0 \leq x \leq \tilde{s}_+^*$; $J_F(\tilde{s}_+^*) = a$, and $J_G(\tilde{s}_+^*) = a \frac{v_2}{v_1}$, where $0 < a < 1$.

- (2) (a) If $k_1 > k_2$, then $G(x) = 1 - \frac{v_2}{v_1} + \frac{k_1 v_2}{k_2 v_1} F(x)$, $F(0) = 0$, and F is continuous for all $0 \leq x \leq \tilde{s}_\dagger$; $J_F(\bar{s}_\dagger) = a$, and $J_G(\bar{s}_\dagger) = a \frac{v_2}{v_1}$, where $\max\{0, \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}\} \leq a < 1$.
- (b) If $\frac{k_1 v_2}{k_2 v_1} > 1$, then in addition the following equilibria exist.
 $F(x) = \left[1 - a_1 - \frac{k_2 v_1}{k_1 v_2} + \frac{k_2}{k_1} a_1\right] + \frac{k_2 v_1}{k_1 v_2} G(x)$, $G(0) = 0$, and G is continuous for all $0 \leq x \leq \tilde{s}_\diamond$; $J_F(\bar{s}_\diamond) = a$, and $J_G(\bar{s}_\diamond) = a \frac{v_2}{v_1}$, where $0 < a \leq \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}$.
- (3) If $k_1 < k_2$, then $G(x) = 1 - \frac{v_2}{v_1} + \frac{k_1 v_2}{k_2 v_1} F(x)$, $F(0) = 0$, and F is continuous for all $0 \leq x \leq \tilde{s}_\nabla$; $J_F(\bar{s}_\nabla) = a$, and $J_G(\bar{s}_\nabla) = a \frac{v_2}{v_1}$, where $0 < a < 1$.

Proof. Suppose s^* is a symmetric equilibrium such that the support of each player is the union $[0, \tilde{s}^*] \cup \bar{s}^*$. Let a_g be the mass that each player of group g puts at \bar{s}^* . As any player i is indifferent between \tilde{s}^* and \bar{s}^* , for $g = 1, 2$ we have

$$u_{g,i}(\bar{s}_{g,i}^*, s_{-i}^{g*}, s_{-g}^*) - u_{g,i}(\hat{s}_{g,i}^*, s_{-i}^{g*}, s_{-g}^*) = \frac{v_g}{2} (a_{-g})^{k_{-g}} (a_g)^{k_g - 1} - (\bar{s}^* - \hat{s}^*) = 0, \quad (2.18)$$

Consequently,

$$\frac{a_1}{a_2} = \frac{v_1}{v_2}, \quad (2.19)$$

which yields

$$\bar{s}^* - \hat{s}^* = \frac{v_1}{2} \left(\frac{v_2}{v_1} \right)^{k_2} a_1^{k_1 + k_2 - 1}. \quad (2.20)$$

Clearly, $0 < a_1 < 1$ and $0 < a_2 < 1$ must be satisfied. Observe that player i is indifferent between expending effort levels \hat{s}^* and z for any $0 < z < \hat{s}^*$. For each $x \in [0, \hat{s}^*]$ the expected payoff of player i of group 1 is

$$u_{1,i}(x) = v_1 \left[\int_0^x F_2(z) dF_{1,-i}(z) + F_2(x) (1 - F_{1,-i}(x)) \right] - x. \quad (2.21)$$

Follow the lines of Theorem 2.15 to conclude that Equations 2.9 and 2.10 hold for all $0 \leq x \leq \hat{s}^*$, thus $\frac{k_1 v_2}{k_2 v_1} F' = G'$. Note that for $0 \leq x \leq \hat{s}^*$ we have

$$1 - a_1 \cdot \frac{v_2}{v_1} - G(x) = \int_x^{\hat{s}^*} G'(z) dz = \frac{k_1 v_2}{k_2 v_1} \int_x^{\hat{s}^*} F'(z) dz = \frac{k_1 v_2}{k_2 v_1} (1 - a_1 - F(x)). \quad (2.22)$$

We thus have the following candidates for equilibrium cdfs on $[0, \hat{s}^*]$.

(1) Type I equilibrium:

$$G(x) = \left[1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1) \right] + \frac{k_1 v_2}{k_2 v_1} F(x), \quad (2.23)$$

where F is strictly increasing and $F(0) = 0$.⁸

(2) Type II equilibrium:

$$F(x) = \left[1 - a_1 - \frac{k_2 v_1}{k_1 v_2} + \frac{k_2}{k_1} a_1 \right] + \frac{k_2 v_1}{k_1 v_2} G(x), \quad (2.24)$$

such that G is strictly increasing and $G(0) = 0$.

Consider three possible cases:

- (1) $k_1 = k_2$; it should be evident that type II equilibrium does not exist because $1 - a_1 - \frac{k_2 v_1}{k_1 v_2} + \frac{k_2}{k_1} a_1 = 1 - \frac{v_1}{v_2} < 0$. Let us investigate type I equilibria. When $k_1 = k_2$, Equation 2.23 implies that G puts mass $1 - \frac{v_2}{v_1}$ at zero for any $0 < a_1 < 1$. The case $a_1 = 0$ is a degenerate case where all players randomize without a gap; it is taken care of in Theorem 2.15. When $a_1 = 1$ the equilibrium degenerates to the semi-pure equilibrium of Lemma 2.12.
- (2) $k_1 > k_2$, then type I equilibrium exists only if $\max\{0, \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}\} \leq a_1 < 1$,⁹ and type II equilibrium exists only if $\frac{k_1 v_2}{k_2 v_1} > 1$ and $0 < a_1 \leq \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}$ (these conditions are implied by the fact that the atoms at zero should be non-negative). Note that when $a_1 = \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}$, neither F nor G put mass at zero, so type I and type II equilibria blend into a single equilibrium with $F(0) = G(0) = 0$ and $G(x) = \frac{k_1 v_2}{k_2 v_1} F(x)$ for all $0 \leq x \leq \hat{s}^*$.
- (3) $k_1 < k_2$, then type I equilibrium exists only if $0 < a_1 \leq \frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)}$, while type II equilibrium fails to exist because under the condition $v_1 > v_2$ the mass placed by F at zero cannot be at most one. Note that since $\frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)} > 1$, the necessary condition for type I equilibrium becomes $0 < a_1 < 1$.

⁸This is because F and G cannot both have atoms at zero, for otherwise any player would deviate from zero to a very small positive bid. Note that this is not always the case in an individual all-pay auction. We proved earlier that every player within a group plays the same strategy, thus if F had an atom at zero, then the effort distribution of the entire group (group 1) would have an atom at zero.

⁹It is easy to verify that $\frac{k_1 v_2 - k_2 v_1}{v_2(k_1 - k_2)} < 1$ since $v_1 > v_2$.

To verify that the above candidates are equilibria and solve for F and G one needs to plug Equations 2.23 and 2.24 into 2.9. This yields the initial-value problems.

$$k_1 v_2 \left[\frac{k_1 v_2}{k_2 v_1} - a_1 \frac{v_2}{v_1} \left(\frac{k_1}{k_2} - 1 \right) - \frac{k_1 v_2}{k_2 v_1} F(x) \right]^{k_2-1} [1 - F(x)]^{k_1-1} F'(x) = 1, \quad F(0) = 0 \quad (2.25)$$

$$k_2 v_1 \left[a_1 + \frac{k_2 v_1}{k_1 v_2} - \frac{k_2}{k_1} a_1 - \frac{k_2 v_1}{k_1 v_2} G(x) \right]^{k_2-1} [1 - G(x)]^{k_2-1} G'(x) = 1, \quad G(0) = 0 \quad (2.26)$$

To solve it, we need to employ the basic techniques of the theory of the first-order ordinary differential equations. There is no closed-form solution to the initial-value problems 2.25 and 2.26, however each possesses a unique solution by Picard-Lindelöf theorem (Coddington and Levinson, 1955, Theorem 3.1, p.12). Indeed, F' is differentiable in F and continuous in x , when one expresses F' as

$$F' = \frac{1}{k_1 v_2} \left[\frac{k_1 v_2}{k_2 v_1} - a_1 \frac{v_2}{v_1} \left(\frac{k_1}{k_2} - 1 \right) - \frac{k_1 v_2}{k_2 v_1} F(x) \right]^{1-k_2} [1 - F(x)]^{1-k_1}.$$

Similarly, G' is differentiable in G and continuous in x . Thus, the strategy profiles listed in the above three cases are indeed equilibria. To solve for a type I equilibrium, one needs to solve the initial-value problem 2.25, then find \tilde{s} as the solution to $F(\tilde{s}) = 1 - a_1$, and finally pin down \bar{s} from Equation 2.20. Type II equilibrium is solved similarly using 2.26.

■

Corollary 2.19. *Equilibrium payoffs in mixed strategy equilibria of Theorem 2.18 are as follows.*

- (1) If $k_1 = k_2$, then each player of group 1 earns $u_1(s^*) = 1 - \frac{v_2}{v_1}$, and every player of group 2 earns $u_2(s^*) = 0$.
- (2) If $k_1 > k_2$, then in the type I equilibria $u_1(s^*) = 1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1)$ and $u_2(s^*) = 0$.
If $\frac{mv_2}{nv_1} > 1$, then in the type II equilibria $u_1(s^*) = 0$ and $u_2(s^*) = 1 - a_1 - \frac{k_2 v_1}{k_1 v_2} + \frac{k_2}{k_1} a_1$.
- (3) If $k_1 < k_2$, then $u_1(s^*) = 1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1)$ and $u_2(s^*) = 0$.

Theorem 2.20. *There exists a continuum of equilibria where each player in one group randomizes over $[\hat{s}^*, \bar{s}^*]$ without a gap and puts mass at \hat{s}^* , and the support of each player in the other group is $0 \cup [\hat{s}^*, \bar{s}^*]$. Particularly,*

- (1) *If $\frac{v_2 k_1}{v_1 k_2} \leq 1$, then:*
 - (a) *Every player in group 1 puts mass a_1 at zero and every player in group 2 puts mass a_2 at \hat{s}^* , where $a_1 \in [0, 1]$, and $a_2 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1\right]$.*
 - (b) *Every player in group 1 puts mass a_1 at \hat{s}^* and every player in group 2 puts mass a_2 at zero, where $a_1 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1\right]$ and $a_2 \in [0, 1]$.*
- (2) *If $\frac{v_2 k_1}{v_1 k_2} > 1$, then:*
 - (a) *Every player in group 1 puts mass a_1 at zero and every player in group 2 puts mass a_2 at \hat{s}^* , where then $a_1 \in \left[1 - \frac{v_1 k_2}{v_2 k_1}, 1\right]$, and $a_2 \in [0, 1]$.*
 - (b) *Every player in group 1 puts mass a_1 at \hat{s}^* and every player in group 2 puts mass a_2 at zero, where $a_1 \in [0, 1]$ and $a_2 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1\right]$.*

Proof. Let s^* be a symmetric equilibrium such that the support of each player in some group g is the union $0 \cup [\hat{s}^*, \bar{s}^*]$. By Lemma 2.10(2), $S_{1+}^* = S_{2+}^* = \hat{s}^*$. The other group, $(-g)$, does not put mass at zero, for otherwise any player would do better by increasing her effort slightly from zero to some $\epsilon > 0$. Therefore $S_2^* = [\hat{s}^*, \bar{s}^*]$. We claim that group $(-g)$ puts mass at \hat{s}^* . Suppose not, then $u_{g,i}(\hat{s}^*, s_{-i}^*) = -\hat{s}^*$ and player i would do better by deviating to zero.

Denote by a_g and a_{-g} the mass that each player of group g and group $(-g)$ puts at zero and \hat{s}^* , respectively. Since group g puts mass at zero, equilibrium payoff of a generic player of group g is $u_{g,i}(s^*) = 0$. As before, we conclude that the payoff of player i of group $(-g)$ from expending effort $\hat{s}^* \leq x \leq \bar{s}^*$ is

$$u_{-g,i}(x) = v_{-g} \left[\int_{\hat{s}^*}^x F_g(z) dF_{-g,-i}(z) + F_g(x) (1 - F_{-g,-i}(x)) + 1 - (1 - a_g)^k \right] - x. \quad (2.27)$$

Similarly, the payoff of player j of group g from expending effort $\hat{s}^* < x \leq \bar{s}^*$ is

$$u_{g,j}(x) = v_g \left[\int_{\hat{s}^*}^x F_{-g}(z) dF_{g,-j}(z) + F_{-g}(x) (1 - F_{g,-j}(x)) + \right] - x. \quad (2.28)$$

Consider the following cases.

(1) $g = 1$, i.e., group 1 puts mass at zero. Recall that the cdf of a generic player of group 1 and 2 is denoted by F and G , respectively. Notice that since $u_{2,j}(\hat{s}^*, s_{-j}^*) = 0$, we have

$$\hat{s}^* = \frac{v_2}{2} (1 - a_2)^{k_2-1} \left[1 - (1 - a_1)^{k_1} \right]. \quad (2.29)$$

Following the lines of Theorem 2.15, we conclude that $\frac{v_2 k_1}{v_1 k_2} F'(x) = G'(x)$ for all $x \in [\hat{s}^*, \bar{s}^*]$, therefore

$$1 - F(x) = 1 - a_1 = \frac{v_1 k_2}{v_2 k_1} (1 - G(x)) \text{ for all } x \in [\hat{s}^*, \bar{s}^*], \quad (2.30)$$

$$1 - F(\hat{s}^*) = 1 - a_1 = \frac{v_1 k_2}{v_2 k_1} (1 - G(\hat{s}^*)) = \frac{v_1 k_2}{v_2 k_1} (1 - a_2), \text{ and} \quad (2.31)$$

$$v_1 k_2 (1 - G(x))^{k_2-1} G'(x) (1 - F(x))^{k_1-1} = 1. \quad (2.32)$$

Performing routine algebraic transformations and using the fact that $G(\hat{s}^*) = a_2$, we obtain that for all $\hat{s}^* \leq x \leq \bar{s}^*$,

$$G(x) = 1 - \left[(1 - a_2)^{k_1+k_2-1} + \frac{c_1}{c_2} - \frac{x}{c_2} \right]^{\frac{1}{k_1+k_2-1}} \text{ and} \quad (2.33)$$

$$F(x) = \frac{v_2 k_1 - v_1 k_2}{v_2 k_1} + \frac{v_1 k_2}{v_2 k_1} G(x), \quad (2.34)$$

where $c_1 = \frac{v_2}{2} (1 - a_2)^{k_2} [1 - (1 - a_1)^{k_1}]$ and $c_2 = \frac{v_1 k_2}{k_1+k_2-1} \left(\frac{v_1 k_2}{v_2 k_1} \right)^{k_1-1}$.

Finally, to pin down \bar{s}^* , use Equation 2.33 together with the condition $G(\bar{s}^*) = 1$. Observe from Equation 2.31 that when $a_2 = 1$, we have $a_1 = 1$, in which case $\hat{s}^* = \bar{s}^* = 0$, and the equilibrium degenerates to the pure-strategy equilibrium where no player exerts any effort. Since a_1 and a_2 cannot be negative, Equations 2.33, 2.34, 2.29, together with the expression \bar{s}^* characterize a continuum of equilibria such that:

- (a) if $\frac{v_2 k_1}{v_1 k_2} \leq 1$, then $a_1 \in [0, 1]$, and $a_2 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1 \right]$;
- (b) if $\frac{v_2 k_1}{v_1 k_2} > 1$, then $a_1 \in \left[1 - \frac{v_1 k_2}{v_2 k_1}, 1 \right]$, and $a_2 \in [0, 1]$.

(2) $g = 2$, i.e., group 2 puts mass at zero. Follow similar lines to derive the following equilibria:

$$\hat{s}^* = \frac{v_1}{2} (1 - a_1)^{k_1-1} \left[1 - (1 - a_2)^{k_2} \right], \quad (2.35)$$

$$F(x) = 1 - \left[(1 - a_1)^{k_1+k_2-1} + \frac{c'_1}{c'_2} - \frac{x}{c'_2} \right]^{\frac{1}{k_1+k_2-1}} \quad \text{and} \quad (2.36)$$

$$G(x) = \frac{v_1 k_2 - v_2 k_1}{v_1 k_2} + \frac{v_2 k_1}{v_1 k_2} F(x), \quad (2.37)$$

where $c'_1 = \frac{v_1}{2} (1 - a_1)^{k_1} [1 - (1 - a_2)^{k_2}]$ and $c'_2 = \frac{v_2 k_1}{k_1+k_2-1} \left(\frac{v_2 k_1}{v_1 k_2} \right)^{k_2-1}$. As before, use Equation 2.36 with the condition $F(\bar{s}^*) = 1$ to pin down \bar{s}^* .

Use the following analog of Equation 2.31:

$$1 - a_2 = \frac{v_2 k_1}{v_1 k_2} (1 - a_1) \quad (2.38)$$

to conclude that Equations 2.35, 2.36, 2.37, together with \bar{s}^* characterize a continuum of equilibria where:

- (a) if $\frac{v_1 k_2}{v_2 k_1} \leq 1$, then $a_2 \in [0, 1]$, and $a_1 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1 \right]$;
- (b) if $\frac{v_1 k_2}{v_2 k_1} > 1$, then $a_2 \in \left[1 - \frac{v_2 k_1}{v_1 k_2}, 1 \right]$, and $a_1 \in [0, 1]$.

■

Theorem 2.21. *There exists a continuum of symmetric equilibria where each player in one group randomizes over $0 \cup [q^*, r^*] \cup \bar{s}^*$. Such equilibria are of two kinds.*

(1) *Each player of group 1 earns zero payoff and puts mass a'_1 at zero and a_1 at \bar{s}^* . Each player of group 2 earns a positive payoff and puts mass a'_2 at q^* and a_2 at \bar{s}^* , where $0 \leq 1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1) + \frac{k_1 v_2}{k_2 v_1} a'_1 \leq 1$ and $0 \leq a'_1 \leq 1$.*

(2) *Each player of group 2 zero payoff and puts mass a'_2 at zero and a_2 at \bar{s}^* . Each player of group 1 earns a positive payoff and puts mass a'_1 at q^* and a_1 at \bar{s}^* , where $0 \leq 1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1) + \frac{k_1 v_2}{k_2 v_1} a'_1 \leq 1$ and $0 \leq a'_1 \leq 1$.*

Proof. Let s^* be an equilibrium such that each player in one group randomizes over $0 \cup [q^*, r^*] \cup \bar{s}^*$. The arguments of Theorem 2.20 imply that this group puts no mass at q^*

and the other group randomizes over $[q^*, r^*] \cup \bar{s}^*$ and puts mass at q^* . Denote by a_g the mass that group g puts at \bar{s}^* . Denote by a'_1 (a'_2) the mass that each player in group 1 (group 2), respectively, puts at q^* or zero. Recall that the cdf of a generic player of group 1 and group 2 is denoted by F and G , respectively. Follow the lines of Theorem 2.18 to conclude that $\frac{k_1 v_2}{k_2 v_1} F' = G'$ and

$$1 - a_1 \cdot \frac{v_2}{v_1} - G(x) = \frac{k_1 v_2}{k_2 v_1} (1 - a_1 - F(x)). \quad (2.39)$$

for all $q^* \leq x \leq r^*$. Therefore,

$$1 - a_1 \frac{v_2}{v_1} - a'_2 = \frac{k_1 v_2}{k_2 v_1} (1 - a_1 - a'_1) \quad (2.40)$$

Follow the reasoning of Theorem 2.20 to establish the following candidates for equilibrium cdfs on $[q^*, r^*]$. Note that because both F and G are positive at q^* , type I and type II equilibria which are analogous to those of Theorem 2.18 blend into a single equilibrium candidate:

$$G(x) = \left[1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1) \right] + \frac{k_1 v_2}{k_2 v_1} F(x), \quad (2.41)$$

where F is strictly increasing and $F(q^*) = a'_1$. Note that, because the mass placed at zero or q^* cannot be negative or greater than 1, type I equilibrium exists only if

$$0 \leq 1 - a_1 \frac{v_2}{v_1} - \frac{k_1 v_2}{k_2 v_1} (1 - a_1) + \frac{k_1 v_2}{k_2 v_1} a'_1 \leq 1 \quad \text{and} \quad 0 \leq a'_1 \leq 1 \quad (2.42)$$

The following initial-value problems, which is analogous to Theorem 2.18, characterizes equilibrium.

$$k_1 v_2 \left[\frac{k_1 v_2}{k_2 v_1} - a_1 \frac{v_2}{v_1} \left(\frac{k_1}{k_2} - 1 \right) - \frac{k_1 v_2}{k_2 v_1} F(x) \right]^{k_2-1} [1 - F(x)]^{k_1-1} F'(x) = 1, \quad F(q^*) = a'_1 \quad (2.43)$$

It possesses a unique solution by Picard-Lindelöf theorem (Coddington and Levinson, 1955, Theorem 3.1, p.12), thus equilibrium exists if and only if condition 2.42 is satisfied.

To finish the characterization of equilibria, it remains to solve for q^* and r^* . Notice that to any solution of 2.43 there correspond two equilibria:

- (1) Each player of group 1 puts mass a'_1 at zero, while each player of group 2 – mass a'_2 at q^* . Thus equilibrium payoff of a group 1 player is zero, therefore $u_{1,i}(q_i^*, s_{-i}^*) = \frac{v_1}{2}(1 - a'_2)^{k_2} \left[1 - (1 - a'_1)^{k_1-1} \right] - q^* = 0$. Hence

$$q^* = \frac{v_1}{2}(1 - a'_2)^{k_2} \left[1 - (1 - a'_1)^{k_1-1} \right] \quad (2.44)$$

Then, find r^* as the solution to $F(r^*) = 1 - a_1$, and finally solve for \bar{s}^* from

$$\bar{s}^* - r^* = \frac{v_1}{2} \left(\frac{v_2}{v_1} \right)^{k_2} a_1^{k_1+k_2-1} \quad (2.45)$$

- (2) Each player of group 2 puts mass a'_2 at zero, while each player of group 1 – mass a'_1 at q^* . Similarly, it can be shown that

$$q^* = \frac{v_2}{2}(1 - a'_1)^{k_1} \left[1 - (1 - a'_2)^{k_2-1} \right] \quad (2.46)$$

Find r^* as the solution to $G(r^*) = 1 - a_2$, and solve for \bar{s}^* from Equation 2.45.

■

Lemma 2.12 and Theorems 2.8, 2.15, 2.18, 2.20, and 2.21 characterize all equilibria. Note that the expected payoff of each player in the equilibria of Theorem 2.15 is zero, while in equilibria of Theorems 2.18 and 2.20 all players of one group earn a positive payoff while players of the other group earn zero. In contrast, there is a continuum of pure strategy equilibria where all players earn a positive payoff. Figure 1 helps to visualize the distinction between equilibria of Theorems 2.15, 2.18, and 2.20.

[Figure 1 about here]

Note that though we cannot obtain a closed-form solution for \tilde{s} and \bar{s} in Theorem 2.18 in the general case, in some simple cases we are able to do so, as the following suggests.

Example 2.22. Suppose two groups compete in an all-pay auction with the weakest-link effort technology. Group 1 has two players, and group 2 has three players. The group valuations are $v_1 = 2$ and $v_2 = 1$. Group 1 is a stylized representation of a small

and strong group (in the sense of group valuation). There exists a continuum of pure strategy equilibria where each player expends an effort λ , where $0 \leq \lambda \leq \frac{1}{2}$. In the unique semi-pure equilibrium each player in group 1 always contributes $\frac{1}{8}$, while each player in group 2 contributes zero and $\frac{1}{8}$ with equal probability. Since $\frac{k_1 v_2}{k_2 v_1} = \frac{1}{3} < 1$, Theorem 2.15.1 implies that there exists a mixed strategy equilibrium of the following form. Each group randomizes over the interval $[0, \frac{1}{18}]$, each player of group 1 randomizes according to the cdf $F(x) = 1 - (1 - 18x)^{\frac{1}{4}}$, and each player of group 2 employs the strategy $G(x) = 1 - \frac{1}{3}(1 - 18x)^{\frac{1}{4}}$. Each player earns an equilibrium payoff of $\frac{2}{3}$, while each player of group 2 earns zero. As expected, the larger group is disadvantaged and earns zero, while the “small and strong” group enjoys a positive payoff.

Moreover, Theorem 2.18.3 implies that there exists a continuum of equilibria with piecewise-continuous supports possessing atoms, which are as follows. Each player of group 1 randomizes according to the cdf F such that $F(0) = 0$, F is continuous for all $0 \leq x \leq \tilde{s}$, and F puts mass a at some \bar{s} , where $0 < a < 1$. Each player of group 2 randomizes according to the cdf $G(x) = \frac{1}{2} + \frac{1}{3}F(x)$ over $[0, \tilde{s}]$ and puts mass $\frac{a}{2}$ at \bar{s} . We compute the parameters of the distributions of the players’ strategies as follows. Equation 2.25, which determines F , becomes

$$\frac{1}{18} (2 + a - 2F(x))^2 (1 - F(x)) dF = dx \quad (2.47)$$

Integrating both sides of 2.47 and using the condition $F(0) = 0$, we obtain the following implicit function which defines F as a function of x .

$$(2 + a)^2 F(x) - \frac{1}{2}(2 + a)(6 + a)F(x)^2 + \frac{4}{3}(3 + a)F(x)^3 - F(x)^4 = 18x \quad (2.48)$$

Observe that since $0 < a < 1$ and $0 < F(x) < 1 - a$ for all $0 < x < \tilde{s}$, the expression $(2 + a - 2F(x))^2 (1 - F(x))$ is always nonzero. Therefore by the Implicit Function Theorem there exists a unique continuously differentiable function $F(x)$ satisfying Equation 2.48. Using the condition $F(\tilde{s}) = 1 - a$ we can calculate \tilde{x} from Equation 2.48.

$$\tilde{s} = \frac{1}{108} (-17a^4 + 3a^2 + 8a + 6) \quad (2.49)$$

Finally, we can calculate the point where each player puts mass, \bar{s} , from Equation 2.20.

$$\bar{s} = \frac{1}{216} (-7a^4 + 6a^2 + 16a + 12) \quad (2.50)$$

Notice that when $a = 0$, such equilibrium degenerates to the mixed strategy equilibrium of Theorem 2.15.1 with the upper bound of the support equal to $\frac{1}{18}$, and when $a = 1$, the equilibrium degenerates to the semi-pure equilibrium. The plot of \tilde{s} and \bar{s} for each value of $a \in [0, 1]$ is presented in Figure 2. The equilibrium cdfs of the example are presented in Figure 3.

[Figure 2 about here]

[Figure 3 about here]

Each player of group 1 receives the payoff $u_1 = \frac{2}{3} - \frac{1}{6}a$, which ranges from $\frac{2}{3}$ to $\frac{1}{2}$, and each member of group 2 earns $u_2(s^*) = 0$.

3. Generalization to n -group asymmetric value model

In the previous section we only addressed the case of two groups where the valuation of the prize was the same for all players within a group. We now extend that model and assume that there are $n > 2$ groups, and that individual valuations may differ within as well as across groups. As in Section 2, denote a generic group by g . As before, suppose group g has $k_g \geq 2$ players. Without loss of generality, within each group $v_{g,1} \geq v_{g,2} \geq \dots \geq v_{g,k_g}$.

Definition 3.1. *Given group g 's pure strategy x_g , group g **participates** if $B(x_g) > 0$.*

Without loss of generality groups are ordered according to the lowest valuation in the descending order, i.e., $v_{1,k_1} \geq v_{2,k_2} \geq \dots \geq v_{g,k_g}$. The following result suggests that there exist pure strategy equilibria in which not all groups participate. The number of participating groups depends upon the distribution of valuations.

Let \mathcal{P}_n denote the set of all subsets of $\{1, 2, \dots, n\}$ that contain at least two elements. The following result is a generalization of Theorem 2.8.

Theorem 3.2. *All equilibria in pure strategy are as follows. For each subset J of \mathcal{P}_n ,*

$$x_{g,i} = \lambda \quad \text{for all } i \in I_g, g \in J \quad \text{and,} \quad (3.1)$$

$$x_{g,i} = 0 \quad \text{for all } i \in I_g, g \notin J, \quad (3.2)$$

where $0 \leq \lambda \leq \frac{1}{m} \min_{g \in J} v_{g,k_g}$, and m is the cardinality of J .

Proof. Fix a subset J of $\{1, 2, \dots, n\}$ having $m \geq 2$ elements, and $\lambda \in [0, \frac{1}{m} \min_{g \in J} v_{g,k_g}]$. Intuitively, J is the list of participating groups. Consider a strategy profile s such that each player of any group in J contributes λ , and the rest of the players do not participate. It is clear that no player of group $\ell \notin J$ wants to deviate by contributing a positive amount, because group k 's effort would still be zero, and the deviator would get a negative payoff. It remains to show that no player of any group in J wants to deviate. Suppose player i of some group $k \in J$ deviates to some $s' > \lambda$. It is easy to see that the winning probability of group k is the same, hence $u_{k,i}(s', s_{-i}) < u_{k,i}(s)$, which is a contradiction. Suppose player i deviates to some $\hat{s} < \lambda$, then group k loses with certainty, hence $u_{k,i}(\hat{s}, s_{-i}) < u_{k,i}(s)$, leading to a contradiction. This completes the proof. ■

To better understand this result, consider the following example.

Example 3.3. Suppose there are 3 groups with valuations $\{1, 2\}$, $\{2, 3, 4\}$, and $\{2, 2\}$. Then the following pure strategy equilibria could arise.

- (1) No group participates.
- (2) All groups participate, all players put the same effort level λ , where $0 \leq \lambda \leq \frac{1}{3}$.
- (3) Only groups 1 and 2 (or groups 1 and 3) participate, and every player of the participating groups expends the same effort λ such that $0 \leq \lambda \leq \frac{1}{2}$.
- (4) Only groups 2 and 3 participate, and every player of the participating groups expends the same effort λ such that $0 \leq \lambda \leq 1$.

One can verify that Lemmata 2.10, 2.12, 2.11, 2.14 and Corollary 2.13 generalize to the case of asymmetric valuation because the proofs of these results do not rely on the assumption that the valuation is the same within a group. Note that it is hard to solve for equilibria in a more general setting for the following reason. First, for the common value case where $n > 2$, one would need to solve a system of n differential equations, containing terms of the form $F'_1 F'_2 \cdots F'_{n-1}$. Second, even if $n = 2$ but the valuations are asymmetric within a group, different groups may put different atoms at zero, which would significantly

complicate the derivation. Now think what happens when we combine these two problems together. We conjecture, however, that all symmetric mixed strategy equilibria (if any) must be of the following form.

- (1) All players in any of the participating groups randomize continuously and without a gap over some interval $[0, \bar{s}]$, possibly putting mass at zero.
- (2) All players in any of the participating groups randomize continuously and without a gap over some interval $[0, \tilde{s}]$ and put mass at some $\bar{s} > \tilde{s}$.
- (3) The support of one group is of the form $0 \cup [\hat{s}^*, \bar{s}^*]$, and all players in other active groups are randomizing over $[\hat{s}^*, \bar{s}^*]$, possibly putting mass at \hat{s}^* .

Though the derivation of these equilibria (as well as the question of existence or nonexistence of mixed strategy equilibria) is beyond the scope of our paper and may be an avenue for future research, we conjecture that high-valuation players put larger mass at \bar{s} .

4. Discussion

We construct and analyze the ‘Max-Min group contest’, in which the minimum effort exerted by a group member represents the group effort and the group with the maximum group effort becomes the winner. There are several field examples of this type of contest. We fully characterize the pure strategy and symmetric mixed strategy equilibria for the case of two groups. There exists a continuum of pure strategy equilibria. In addition, a semi-pure strategy equilibrium exists, in which all the members of a group play the same pure strategy, whereas all the members of the other group play the same mixed strategy. There are two further types of non-degenerate mixed strategy equilibria, with and without continuous supports. We find that different combinations of the prize valuations and the number of players in each group may result in different types of equilibria and corresponding rent dissipation and payoffs. For the case of more than two groups, we fully characterize the pure strategy equilibria and point out candidates for mixed strategy equilibria.

Our results are very different from that of weakest link public goods games, or weakest link group contest with Tullock CSF. While players within each group employ the same strategy due to perfect complementarity, both the variety of equilibria and the corresponding payoffs remain different. The most important part of the difference is the ‘group size effect’, i.e., in the current model the relative group size plays an important role, unlike

the other studies. The results are also distinguishable from other all-pay auction group contests with perfectly substitute or best-shot impact functions. Those contests do not possess pure strategy equilibria, while in our model there is a continuum of pure strategy equilibria. In such equilibria a very wide range of rent dissipation is possible. The rent dissipation ranges from full to negligible. In any equilibrium all players are active provided that the entire group is active due to the weakest-link effort technology. This is in contrast with the free-riding equilibria with the best-shot or substitute effort technologies (Baik et. al., 2001; Barbieri et. al., 2013) where only a few (possibly one, in the extreme case) players are active.

Given the existing literature on all-pay auctions, some of the results such as the existence of a pure strategy or a semi-pure strategy equilibrium, discontinuity in equilibrium support, or the effects of the group size are new. First, it is known that pure strategy equilibrium may exist in the case of incomplete information (Krishna and Morgan, 1997) and for a non-monotonic payoff function (Chowdhury, 2009). Although those conditions are not satisfied in the current structure, due to the specific network externalities, a continuum of pure strategy equilibria is obtained. It is also surprising to observe the semi-pure strategy equilibrium, in which some players play pure strategy whereas some other players play mixed strategy. This type of semi-pure strategy equilibrium is noted by Krishna (1989) in trade theory and by Clark and Konrad (2007) in the context of a multiple battlefield contest with network effects. We show that such equilibria may exist in group contests due to the network effects. The network externalities and the effects of relative group size result in a jump in equilibrium support which is also not seen in the existing all-pay auction studies. Finally, as mentioned earlier, unless value distribution is changed, the group size does not affect equilibrium strategies and payoffs in standard group contests. We find that in case of weakest link network and public good prize, the ‘relative value’ scaled by the group size determines equilibrium effort, and this makes parts of the current results comparable to the group contests with private good prizes.

Similar to the minimum-effort coordination game of Van Huyck et al. (1990) and Anderson and Goeree (2001) in which players’ effort levels are perfect complements in providing public goods, there exist multiple pure and mixed strategy Nash equilibria in our model, because the group impact function for each group depends on the minimum effort level within that group. Hence, the existence of multiple equilibria brings in the relevant and important issues of coordination problems and equilibrium selection principles. Especially,

in the common value case where all the players in a group have the same valuation on the prize, those issues may be highlighted. Furthermore, there are standard assumptions used in our model that can be relaxed as extensions of the current study. It is possible to extend the structure to situations in which the prize is a private good. An incomplete information structure may portray some real life situations better. Finally, throughout the paper we have assumed risk neutrality that can be relaxed to incorporate risk aversion. We leave these issues for future research.

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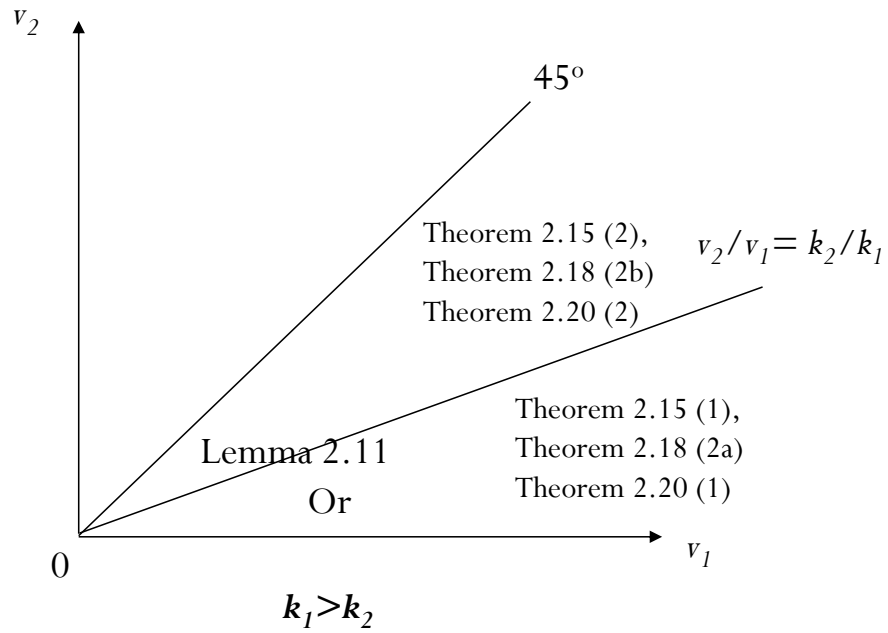
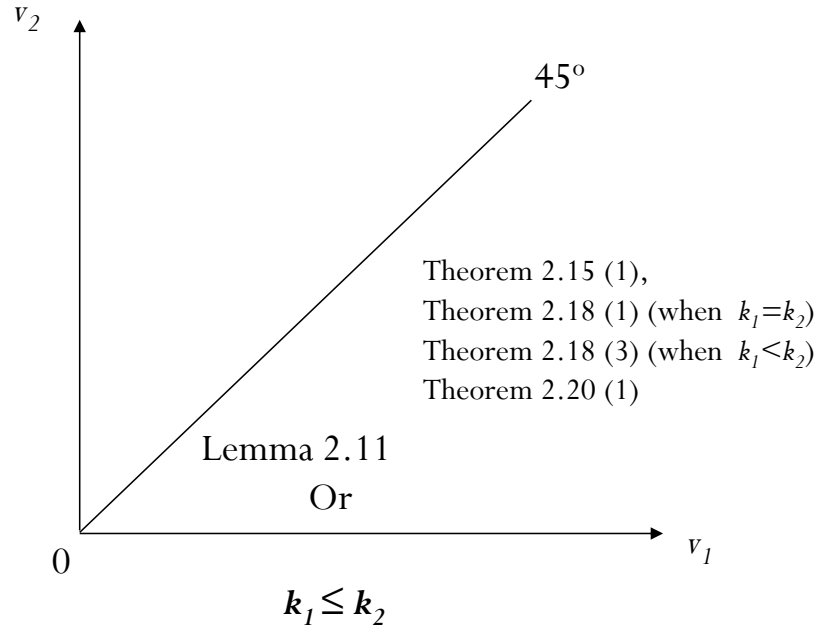


FIGURE 1. The equilibria in the model

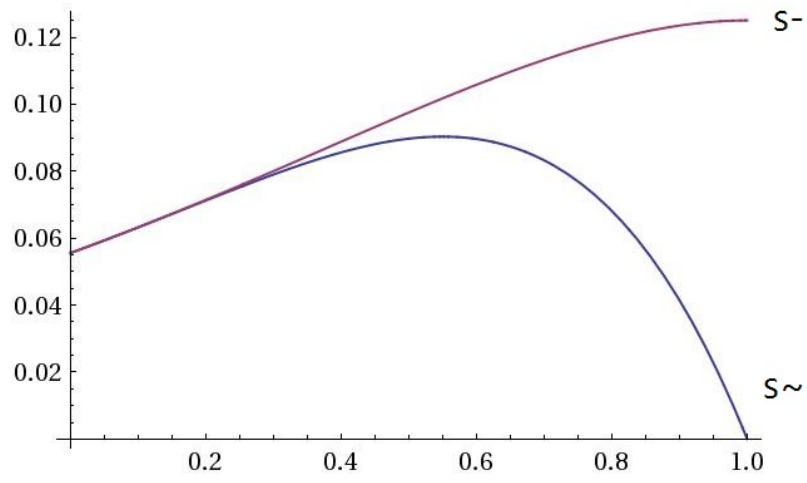
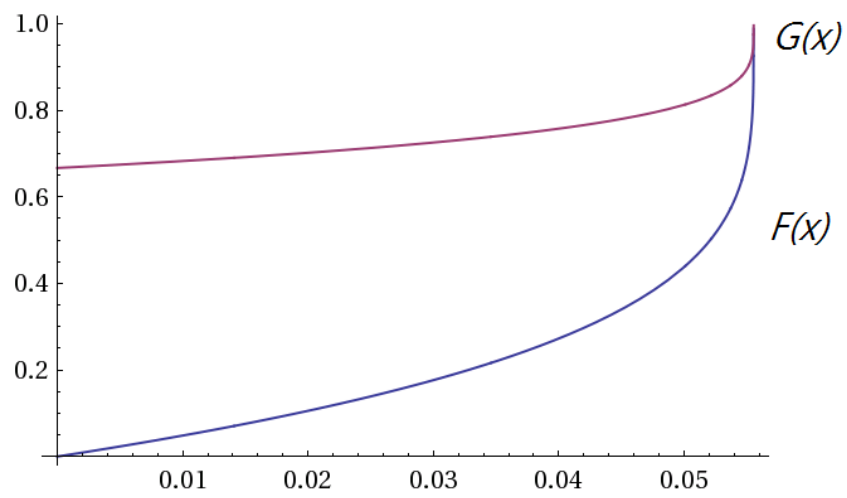
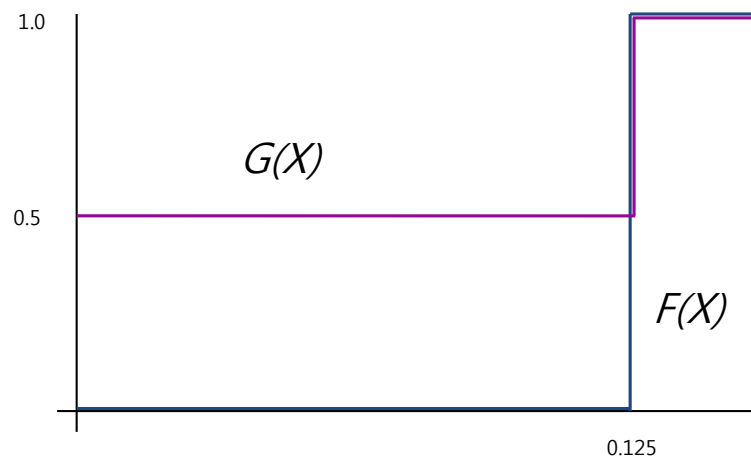


FIGURE 2. The plot of \tilde{s} and \bar{s}



$a = 0$



$a = 1$

FIGURE 3. The equilibrium CDF when $a = 0$ and $a = 1$