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Abstract

Pollution is one of the world's primary causes of premature death, but macroeconomic analysis largely neglects the existence of such negative externality. We build a tractable multi-sector growth model where innovations raise productivity, a polluting primary sector exploits natural resources, emissions increase mortality, and fertility is endogenous. The response of the mortality rate to changes in population size is generally ambiguous and often non-monotonic, and reflects a precise equilibrium relationship that combines emission intensity, dilution effects and labor reallocation effects caused by technology. Deadly spillovers affect welfare through multiple channels – including market-size effects – and create additional steady states, including mortality traps that undermine development in less populated resource-rich countries even for low emission elasticities. Emission taxes yield double dividends in terms of income and population capacity, whereas subsidies to primary production reduce potential population and may trigger population implosion especially if combined with new discoveries of polluting primary resources.

JEL codes O12, O44, Q56

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1 Introduction

Pollution kills. According to the Lancet Commission on pollution and health,

> "Diseases caused by pollution were responsible for an estimated 9 million premature deaths in 2015 – 16% of all deaths worldwide – three times more deaths than from AIDS, tuberculosis, and malaria combined and 15 times more than from all wars and other forms of violence." (The Lancet, 2017: p.5).

These figures revise upwards previous estimates by the Global Burden of Disease Study (IHME, 2016), that had prompted the World Health Organization to consider pollution as one of the world's most significant causes of premature death (WHO, 2016). In fact, among all the risk factors held to explain world deaths in 2017, air pollution alone ranks fourth.[1]

The economics literature on the subject is mostly empirical and confirms the scale and pervasiveness of the problem.[2] Despite this evidence, however, macroeconomic analysis generally neglects the role of the *deadly spillovers*: there are no macroeconomic models that account for the simultaneous endogeneity of economic growth, environmental degradation, mortality and fertility. This type of models are however necessary to address fundamental questions, first and foremost, how does pollution-induced mortality affect macroeconomic performance through the demographic channel? Unlike the conventional pollution externalities studied in environmental economics – i.e., emissions that reduce the utility of individuals and/or the efficiency of firms – deadly spillovers affect consumption and production possibilities through multiple channels. Increased total deaths reduce total labor supply and aggregate expenditures, activate reallocation effects between sectors – including the polluting primary sector and R&D activities that drive productivity growth – and prompt households to revise their saving and fertility decisions. From this perspective, the interplay between mortality rates, fertility rates and income dynamics becomes a key determinant of welfare. Understanding how these interactions affect macroeconomic performance is a necessary first step to tackle a number of questions of direct interest to empirical research and policymaking: what are the consequence of pollution-induced mortality in less populated resource-rich countries that

[1] Ritchie and Roser (2020a; 2020b) show that the four main causes of death – heart disease, cancer, respiratory diseases and infections – all exhibit a strong relationship with air pollution. In our analysis, we will not distinguish between air pollution and other forms of pollution caused by industrial activity. Thus, the evidence provides a sort of lower bound on the importance on pollution for mortality. The empirical relationship between the four main causes of death and air pollution is well established: see Schlenker and Walker (2016) and the literature cited therein.

[2] We discuss below the few theoretical models where pollution reduces life expectancy (Mariani et al. 2010; Goenka et al. 2012; Varvarigos, 2014). Existing empirical work in economics includes Ebenstein et al. (2015), Arceo et al. (2016), Bombardini and Li (2016).

typically display high emissions per capita as a result of large, polluting primary sectors? What are the overall macroeconomic effects of pollution taxes and subsidies to primary sectors once we account for demographic change? If deadly spillovers generate mortality traps, is population implosion a possibility only for underdeveloped countries – like poverty traps – or is it a threat for rich economies too?

All the answers to the above questions crucially hinge on how the mortality rate responds to changes in population size. Clearly, the mortality response is an equilibrium concept: changes in population modify resource scarcity, expenditures, fertility, employment in polluting sectors, the resulting emissions and thereby deaths per capita. We tackle this issue in amodel where a polluting primary sector exploits a natural resource, horizontal and vertical innovations raise the productivity of intermediate producers, emissions increase mortality, and private household choices determine fertility. The fact that economic growth, pollution, mortality and fertility are all endogenous allows us to define the path of the *equilibrium mortality rate* and thereby the consequences of deadly spillovers for macroeconomic performance and welfare.

A distinctive property of our framework is that, even in the absence of deadly spillovers, it can produce equilibrium paths where population converges to a finite endogenous level in the long run while per capita incomes keep on growing via endogenous productivity growth. This property extends the results derived in Peretto and Valente (2015), which builds a theory of the population level where demographic and market forces can stabilize population. In contrast to traditional balanced-growth models predicting exponential population growth, this framework can replicate the fertility decline experienced in most industrialized countries while remaining consistent with the demographers' view that in the long run population must converge to a finite size.[3] In the analysis of Peretto and Valente (2015), there is no pollution, mortality is exogenous, and population is stabilized by the *fertility response to income per capita*. We extend this framework to introduce pollution externalities and endogenous mortality, obtaining a model where the *mortality response to emissions* becomes an independent source of equilibrium paths and creates additional steady states that would not exist in the absence of deadly spillovers.

Our main results may be summarized as follows. First, the response of the equilibrium mortality rate to changes in population is generally ambiguous and often non-monotonic, and reflects a precise relationship among emission intensity, dilution effects, and labor reallocation effects to or from the primary sector. In particular, the mortality rate may well be declining (locally, if not globally) in

[3] Demographers forecast a levelling off of world population within the next century, century and a half, and offer arguments based on first principles for why this must happen due to the feedback mechanisms that operate in a closed system, e.g., on a 'finite planet'.

the population-resource ratio, so that resource-rich countries with low population may exhibit very high mortality rates. Second, deadly spillovers have both quantitative and qualitative effects: they always modify the 'regular' growth path – that is, the dynamics of income and demography that we would observe in a pollution-free world – and they can create new steady states that fundamentally affect long-run outcomes. Under all parametrizations, endogenous mortality matters for welfare – with direct effects on family size and per capita incomes – and for productivity – via changes in market size, which directly affects firms' profitability and incentives to innovate. In a large subset of cases, deadly spillovers can create additional steady states, including regular steady states that would not exist otherwise – i.e., endogenous mortality stabilizes population even in the absence of alternative mechanisms – and mortality traps – i.e., threshold levels of the population below which population implodes due to increasing *net* mortality despite *growing* gross fertility. Third, we show that demography matters for environmental policy. Taxing polluting primary sectors yields double dividends: besides decreasing emissions, reduced mortality increases the carrying capacity of people in the economy, which means a higher long-run population level in the regular steady state as well as a smaller regions of potential mortality traps. Subsidies to the primary sector yield opposite consequences – i.e., reduce population capacity and increase the threat of population implosion – and may be a recipe for disaster if they are implemented after new discoveries of natural resource endowments.

With respect to the existing literature, our analysis delivers many novel insights. In general, the view that demography drives macroeconomic performance is well accepted in the profession, but it is rarely rationalized into empirically consistent models (see Brunnschweiler et al. 2020). The few growth models that successfully incorporate a constant endogenous population level in the long run typically focus on Malthusian structures (Eckstein et al. 1988; Galor and Weil, 2000; Brander and Taylor, 1998) or similar market-based mechanisms where resource scarcity triggers price reactions that eventually bring population growth to a halt in the long run (Strulik and Weisdorf, 2008; Peretto and Valente, 2015).[4] These works do not study deadly spillovers nor more general interactions among pollution, mortality and fertility in combination with fully endogenous productivity growth.

The few existing theories linking emissions to mortality – Mariani et al. (2010), Varvarigos (2014), Goenka et al. (2020) – assume that pollution reduces life expectancy and analyze equilibrium paths in one-sector models where income growth is driven by capital accumulation. In these frameworks, the average mortality rate grows with emissions as the economy develops, but the

[4] An exception is Brunnschweiler et al. (2020), where population is stabilized by the dilution of intangible assets representing financial wealth in an economy populated by disconnected generations of finitely-lived agents.

process can be alleviated by the counter-acting effects of defensive expenditures that ultimately create multiple steady states at different income levels. These conclusions recalls Nelson's (1956) notion of under-development traps: non-linearities in the returns to investment generate, besides the regular high-*income* steady state, low-*income* steady states that may act as poverty traps.[5] The same concept has indeed re-emerged in two related strands of literature. In models of pollution control, managing the trade-off between economic growth and environmental quality requires the existence of a highly efficient pollution-abatement technology (Brock and Taylor, 2005), and if abatement efficiency is positively related to output, the economy may exhibit two steady states, a high-income stable equilibrium and a low-income trap induced by environmental degradation (Xepapadeas, 2005).[6] In models with endogenous lifetime (Blackburn and Cipriani, 2002; Chakraborty, 2004), households optimize over finite horizons and longevity rises with income, e.g., via better nutrition and health care. The interaction produces a stable steady state for high income levels but also a poverty trap in which low income and short lifetime become persistent.[7] Our analysis differs starkly from these contributions in both aims and means. We study the impact of deadly spillovers in a multi-sector economy where demography and productivity dynamics decouple: with or without pollution, the regular steady state features a finite endogenous population level while per capita incomes grow due to R&D investment. In particular, we rule out defensive expenditures and similar accumulation-pollution trade-offs that generate non-linearities in one-sector models. Our key results hinge, instead, on the generally ambiguous impact that population growth has on emissions per capita. Even though total emissions always increase with primary production, the behavior of the mortality *rate* crucially depends on how primary output *per worker* changes in response to changes in overall net labor supply – the *primary-employment effect* – and on the rate at which emissions per capita fall as total population grows – the *dilution effect*. The primary-employment and dilution effects affect deaths per adult in opposite directions as population grows. This makes

[5] Goenka et al. (2020) obtain a pollution-induced poverty trap that depends on the pollution-capital trade-off arising in the one-sector neoclassical model. Mariani et al. (2010) and Varvarigos (2014) consider state-dependent abatement activities and obtain the benchmark combination of high-income (stable) and low-income (unstable) steady states making different hypotheses about the source of non-linearity. Another study assuming a pollution-lifetime tradeoff is Ponthiere (2016), which characterizes utilitarian and egalitarian social optima.

[6] In models with fixed saving and investment rates, a high rate of pollution-reducing technical change is a general pre-condition for sustainable long-term growth. Models of optimal pollution control study whether the sustainability condition is satisfied ex-post once savings and investment in clean technologies are endogenous. The rise of poverty traps induced by pollution with state-dependent abatement efficiency is formally demonstrated in Smulders and Gradus (1996) and Xepapadeas (1997).

[7] Another strand of literature that endogeneizes mortality studies the economic impact of infectious diseases – see Goenka and Liu (2012; 2020) and Aksan and Chakraborty (2014) – a topic that will likely deliver new contributions in the wake of the recent pandemic crisis.

the response of the mortality rate to population generally ambiguous, and opens the door to a number of empirically relevant conclusions – e.g., countries with low population and/or abundant primary resources may actually exhibit very high mortality rates relative to highly populated, resource-poor economies. This outcome is more likely to arise when labor and natural resources are substitutes in primary production: as population grows, primary sector's employment increases less than proportionally, so that total emissions increase but per capita damages and deaths per adult decline. Symmetrically, the fact that labor is increasingly substituted with resource use when population falls implies that the per capita emission damage increases with smaller population and this can give rise to mortality traps in resource-rich, labor-poor economies. In this sense, the rise of mortality traps in our model is disconnected from the *poverty* traps that are typically discussed in the literature on growth and development.[8]

Our results on emission taxes appear to be novel with respect to the literature on environmental macroeconomics and policy, where the role played by demographic change is typically neglected. The possibility that emissions taxes could generate double dividends, or more generally, positive side effects on economic performance, has traditionally been linked to the fact that emission taxes may reduce aggregate efficiency losses by shifting distorsions away from other production factors (Bovenberg and Goulder, 2002) or may encourage productivity-enhancing innovations (Porter and van der Linde, 1995). The positive side effects that we obtain in our model – reduced mortality, higher long-run population along regular paths, and increased distance from mortality traps – work through the demographic response to reduced emissions, which is typically ignored in these debates. The specular result on the negative effects of subsidies to the primary sector is relevant from a policymaking perspective because such subsidies are often observed in resource-rich developing countries (Gupta et al., 2002; Metschies, 2005) and are typically justified by invoking the need to boost income via resource rents (Bretschger and Valente, 2018). In our analysis, these subsidies tend to reduce expenditure per capita as well as population capacity in the economy along 'regular paths', and push the economy closer to mortality traps. Similarly, our results on the consequences of new discoveries of primary resources add to the literature on the Resource Curse hypothesis, which explored several possible mechanisms through which natural abundance may undermine economic performance (e.g., Mehlum et al. 2006) but typically neglects demography-economy interactions.

[8] The nature of our *mortality* traps differs from *poverty* traps á la Nelson (1956), which are triggered by low capital per worker. In our model, what triggers ever-declining population is the abundance of the natural resource used in primary production relative to the labor force: there exists a critical level of the *population-resource ratio* below which even the highest fertility rate that households can choose – which is finite and bounded above by their budget constraint – does not compensate for the high mortality rate caused by pollution. An economy that falls into the mortality trap thus experiences population implosion even though the gross fertility rate increases over time.

2 The model

We study a decentralized economy where the primary sector produces a commodity using labor and a raw natural resource (henceforth, resource). The intermediate sector uses the commodity to produce differentiated goods that the final sector uses to produce a homogeneous consumption good. Endogenous economic growth results from horizontal and vertical innovations in the intermediate sector. Commodity production generates harmful pollution that increases mortality. The decisions of households facing child-rearing costs drive endogenous fertility. We begin our analysis with an overview of the interactions among production, pollution and demography.

2.1 Demography, pollution damage and primary production

Time is continuous and indexed by $t \in [0, \infty)$. The dynamics of population, L, are

$$\dot{L}(t) = B(t) - M(t) = [b(t) - m(t)] \cdot L(t), \tag{1}$$

where B is births and M is deaths. Strictly speaking, L is adult population while total population is $L + B$. However, we follow the convention in the literature and refer to L as population and to variables divided by L as per capita. For future use, we also specify the dynamics in terms of rates: birth rate, $b = B/L$, and the death rate, $m = M/L$, the fraction of adult population that passes away at any point in time.[9]

In general, total deaths depend on poulation size and pollution externalities according to the *emission-damage function* of the type $M = f(L, E)$, where E represents total emissions and the partial derivative $\partial f/\partial E$ is strictly positive to capture the positive impact of pollution on mortality. Empirical evidence from the medical literature suggests that $f(L, E)$ is highly non-linear, but its exact shape and degree of homogeneity in E and L are subject to extensive debate. In our analysis, we posit

$$M(t) = \bar{m}L(t) + \mu E(t)^{\chi} L(t)^{\varphi}, \quad \bar{m}, \mu, \chi \geqslant 0, \quad 0 \leqslant \varphi \leqslant 1, \tag{2}$$

where $\bar{m}, \mu, \chi, \varphi$ are constant, exogenous parameters: $\bar{m}$ is the baseline death rate that prevails in the absence of pollution, μ and χ govern the existence and marginal impact of deadly spillovers, and φ is a convenient parameter that allows us to obtain different specifications of the damage function. A first polar case is $\varphi = 0$, which yields a *level-damage function* whereby total emissions E increase total deaths M. A second polar case is $\varphi = 1$, which yields a *rate-damage function*

[9] Strictly speaking, the fertility rate is $B/(L + B)$. The literature, however, typically refers to $b = B/L$ as the fertility rate and we follow the convention unless necessary to avoid confusion. The empirical referent of b is the crude birth rate, or births per adult.

postulating that total emissions E increase the mortality rate $m = M/L$ in the economy. We will focus on the general specification $0 < \varphi < 1$, which is most empirically plausible case.[10] A possible interpretation is that different values of φ capture the effects of different types of pollutants whereby environmental damages can be rival or non-rival: the average mortality rate may thus be affected more heavily by total emissions or by emissions per capita depending on which pollutant is the prominent source of deadly spillovers.

We do not make restrictive hypotheses about the marginal impact of total emissions: given total population, the damage function may be concave, linear or convex in E depending on whether χ is assumed to be less, equal or above unity. On the one hand, this is motivated by evidence: estimated mortality functions are consistent with linear or even strictly concave damage functions at high emission levels (e.g., Cakmak et al., 1999; Izzotti et al., 2000). On the other hand, the unrestricted damage elasticity will allow us to draw novel insights with respect to conventional models of pollution externalities where emissions cause utility or productivity losses. These conventional models predict that environmental externalities limit economic growth if the emission-damage function is strictly convex. We will show that deadly spillovers, instead, can have serious consequences for economic performance and welfare under all parametrizations, including – and in some cases, especially – when χ is strictly less than unity.

Emissions are the harmful by-product of the exploitation of the resource in commodity production. Our primary sector can thus be interpreted as the resource-processing unit of a mining industry or as an energy sector producing electricity from fossil fuels. For simplicity we abstract from characteristics of natural resources such as renewability and model the resource as a fixed endowment, Ω, providing a constant flow of productive services. The production function of the primary sector is

$$Q(t) = \mathcal{F}(\Omega, L_Q(t)), \tag{3}$$

where Q is output, L_Q is employment in resource processing and $\mathcal{F}$ is a linearly homogeneous function. Resource processing generates a flow of pollution

$$E(t) = \xi \cdot Q(t), \quad \xi > 0 \tag{4}$$

where ξ is the exogenous and constant marginal emission intensity. This linear specification is not

[10] Empirically, the relationship between mortality and pollution is highly non-linear in both aggregate and per capita terms (Cakmak et al., 1999; Izzotti et al., 2000). In our model, setting $\varphi = 1$ would mean that total emissions raises the probability of death regardless of the scale of the economy: the extent to which E increases the mortality rate M/L does not depend on population size L, which is questionable. At the other extreme, assuming $\varphi = 0$ would imply that total emissions raise total deaths regardless of population density, which is an equally questionable hypothesis.

restrictive because the actual damage caused by pollution follows from combining (4) with (2), which encompasses all the relevant cases – i.e., level damage, rate damage, convex or concave – depending on the values assumed by the elasticity parameters χ and φ.

To make the paper's analysis directly policy-relevant with minimal addition of structure, we assume that the government taxes commodity sales at rate τ to discourage commodity production. Given the relation between resource processing and pollution discussed above, we can interpret τ as an environmental (i.e., emission) tax. For simplicity, we assume that τ is constant over time and that the government balances the budget in each instant, rebating to the household the revenues from the commodity tax via a lump-sum transfer. Formally, the government satisfies the budget constraint $S(t) = \tau p_q(t) Q(t)$, where S is the transfer and p_q is the price of the commodity.

2.2 Consumption and reproduction choices

We use the Peretto-Valente (2015) extension of the textbook formulation of fertility theory (see, e.g., Barro and Sala-i-Martin, 2004, Ch. 9). The extension gives full control over expenditure per child to the household and allows for a "quantity-quality" trade-off with no additional complexity. Specifically, a representative household maximizes the dynastic utility function

$$U_0 = \int_0^\infty e^{-\rho t} \ln u\left(c_L(t), c_B(t), L(t), B(t)\right) dt, \quad \rho > 0 \tag{5}$$

where ρ is the individual discount rate, c_L is consumption of each adult, c_B is consumption of each child, L is the mass of adults and B is the mass of children. Instantaneous utility is

$$u(c_L, c_B, L, B) = c_L^\alpha c_B^{1-\alpha}\left(L^\alpha B^{1-\alpha}\right)^\psi, \quad 0 < \alpha < 1, \quad 0 < \psi < 1. \tag{6}$$

In this structure, agents obtain utility from the consumption and presence of adults and from the consumption and presence of children with weights, respectively, α and $1-\alpha$. The parameter ψ regulates the trade-off between the individual consumption of the members of each group (adults and children) and the size of each group. Instantaneous utility can be written $u(c_L, c_B, b, L) = c_L^\alpha c_B^{1-\alpha} b^{\psi(1-\alpha)} L^\psi$, in which case the birth rate, b, is the relevant choice variable and ψ is the gross elasticity of dynastic utility with respect to the mass of adults.[11]

Household expenditure is $p_c C = p_c(c_L L + c_B B)$, where p_c is the price of the final good. The fertility choice is thus characterized by a trade-off between the utility benefit from reproduction and the expenditure on the children's consumption. The household supplies labor to all firms and

[11] The restriction $0 < \psi < 1$ implies that for each group the elasticity of utility with respect to individual consumption exceeds the elasticity of utility with respect to the size of the group. Moreover, as we show in the Appendix, the maximization probem of the household is well defined only if the condition $\psi(1-\alpha) < 1-\alpha$ holds.

earns royalties for resource use from the commodity producers. The household's budget is

$$\dot{A}(t) = r(t)\,A(t) + w(t)\,L(t) + p_\omega(t)\,\Omega + S(t) - p_c(t)\,C(t)\,, \tag{7}$$

where r is the rate of return on financial assets, A is asset holdings, w is the wage, and p_ω is the per-unit resource royalty. The household chooses the time paths of c_L, c_B and B to maximize (5) subject to (7) and (1). The household takes the path of the mortality rate as given because private agents are unable to internalize the effects of emissions on mortality. Nonetheless, the household internalizes the intertemporal trade-off caused by population growth: a larger mass of adults expands the dynasty's consumption possibilities via additional labor income but, at the same time, reduces individual consumption possibilities via dilution effects.

The solution to the household problem is described in the Appendix. The conditions for utility maximization are the familiar Euler equation for consumption growth

$$\frac{\dot{p}_c(t)}{p_c(t)} + \frac{\dot{C}(t)}{C(t)} = r(t) - \rho \tag{8}$$

and the associated equation for the birth rate

$$\frac{\dot{b}(t)}{b(t)} = \frac{b(t)}{(1-\alpha)(1-\psi)}\left[\psi + \frac{w(t)\,L(t) - p_c(t)\,C(t)}{p_c(t)\,C(t)}\right] - \rho. \tag{9}$$

Equation (8) determines the growth rate of household consumption expenditure according to the traditional trade-off: the marginal benefit of asset accumulation versus the marginal cost of sacrificing current consumption. Equation (9) says that the birth rate increases over time when the anticipated rate of return from generating future adults exceeds the utility discount rate, ρ. The term in square brackets shows the components of this rate of return: the gross elasticity of utility to the mass of adults, ψ, plus their contribution to asset accumulation, given by the difference between labor income and consumption expenditure.

2.3 Producers: Final and Intermediate Sectors

Final sector. The final sector is competitive and produces with the technology

$$C(t) = \left(\int_0^{N(t)} x_i(t)^{\frac{\epsilon-1}{\epsilon}}\,di\right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 1 \tag{10}$$

where C is output, N is the mass of intermediate goods, x_i is the quantity of good i and ϵ is the elasticity of substitution between pairs of intermediate goods. Final producers maximize profits taking as given the mass of intermediate goods and the price, p_{x_i}, of each intermediate good. The solution to this problem yields the demand schedule

$$p_{x_i}(t) = \frac{p_c(t)\,C(t)}{\int_0^{N(t)} x_i(t)^{\frac{\epsilon-1}{\epsilon}}\,di} \cdot x_i(t)^{-\frac{1}{\epsilon}} \tag{11}$$

for each intermediate good.

Intermediate sector: incumbents. Each intermediate good is supplied by a monopolist that operates the production technology

$$x_i(t) = z_i(t)^{\theta} \cdot Q_i(t)^{\gamma} \left(L_{x_i}(t) - \phi\right)^{1-\gamma}, \quad 0 < \theta < 1, \quad 0 < \gamma < 1, \tag{12}$$

where x_i is output, Q_i is the commodity input, L_{x_i} is production labor and $\phi > 0$ is overhead labor. The productivity term z_i^{θ} is Hicks-neutral with respect to the rival inputs, labor and the commodity, and depends on the stock of firm-specific knowledge z_i. The firm accumulates firm-specific knowledge according to the technology

$$\dot{z}_i(t) = \kappa \cdot \left[\int_0^{N(t)} \frac{1}{N(t)} z_j(t)\, dj\right] \cdot L_{z_i}(t), \quad \kappa > 0 \tag{13}$$

where L_{z_i} is R&D labor, κ is an exogenous parameter and the term in bracket is the stock of public knowledge that accumulates as a result of spillovers across firms: when one firm develops a new idea, it also generates non-excludable knowledge that benefits the R&D of other firms. The firm's instantaneous profits is

$$\pi_i(t) = p_{x_i}(t)\, x_i(t) - p_q(t)\, Q_i(t) - w(t)\, L_{x_i}(t) - w(t)\, L_{z_i}(t), \tag{14}$$

where p_q is the commodity price. The value of the firm is

$$V_i(t) = \int_t^{\infty} \pi_i(v) \exp\left(-\int_t^{v} (r(s) + \delta)\, ds\right) dv, \quad \delta > 0 \tag{15}$$

where δ is the instantaneous probability of realization of an exit shock. (To avoid confusion with the death rate of people, m, we refer to δ as the obsolescence rate.) At time t the firm chooses the paths of $\{p_{xi}, x_i, Q_i, L_{xi}, L_{zi}\}$ that maximize (15) subject to the demand schedule (11), the production technology (12) and the R&D technology (13). The solution to this problem (see the Appendix) yields the maximized value of the firm given the time path of the mass of firms, $N(t)$.

Intermediate sector: entrants. Entrepreneurs hire labor to develop new intermediate goods and set up firms to serve the market. Denoting the typical entrant i without loss of generality and denoting L_{N_i} the amount of labor required to start the new firm, the cost of entry is $wL_{N_i} = \beta p_{xi} x_i$, where $\beta > 0$ is a parameter representing technological opportunity. This assumption captures the notion that entry requires more effort the larger the anticipated volume of production. The entrant anticipates that once in the market the new firm solves an intertemporal problem identical to that of the generic incumbent and therefore that the value of the new firm is the maximized value $V_i(t)$ defined in (15). Free entry then requires

$$V_i(t) = \beta p_{xi}(t)\, x_i(t) = w(t)\, L_{N_i}(t) \tag{16}$$

for each entrant.

2.4 Primary sector

A representative competitive firm combines the resource with labor under constant returns to scale. The firm maximizes profit

$$\Pi_q = p_q(t)\, Q(t)\, (1-\tau) - p_\omega(t)\, \Omega - w(t)\, L_Q(t) \tag{17}$$

subject to the technology (3) taking all prices and the tax rate as given. To simplify the exposition, we work with the CES specification of (3)

$$Q(t) = \mathcal{F}(\Omega, L_Q(t)) = \left[\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}} + (1-\eta) \cdot L_Q(t)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \quad \sigma \geq 0,\ \eta \in (0,1), \tag{18}$$

where σ is the elasticity of input substitution and η governs the input shares. The resource and labor are complements if $\sigma < 1$ and substitutes if $\sigma > 1$. Letting $\sigma \to 1$ we obtain the Cobb-Douglas case $Q = \Omega^\eta L_Q^{1-\eta}$. Let $\Theta(w, p_\omega) \equiv \eta^\sigma p_\omega^{1-\sigma} + (1-\eta)^\sigma w^{1-\sigma}$ denote the unit cost function associated to the technology (18). The profit-maximizing decisions of the commodity producer yield

$$p_q = \frac{\Theta(w, p_\omega)}{1-\tau} \tag{19}$$

and the resource *cost-share* function (see the Appendix)

$$\Upsilon(t) \equiv \frac{\mathrm{d} \ln \Theta(w(t), p_\omega(t))}{\mathrm{d} \ln p_\omega(t)} = \frac{p_\omega(t)\, \Omega}{p_\omega(t)\, \Omega + w(t)\, L_Q(t)} = \frac{\eta^\sigma p_\omega(t)^{1-\sigma}}{\eta^\sigma p_\omega(t)^{1-\sigma} + (1-\eta)^\sigma w(t)^{1-\sigma}}. \tag{20}$$

The resource cost share Υ is the ratio between royalties paid by firms to resource owners and the firm's total expenditures on inputs. In the Cobb-Douglas case, $\sigma \to 1$, the cost-share is constant, $\Upsilon \to \eta$. In the other cases, a higher resource price reduces (increases) the resource cost-share when primary inputs are substitutes (complements) because strict substitutability (complementarity) makes the primary sector's demand for the resource elastic (inelastic). These *cost-share effects* determine the equilibrium response of household income and consumption expenditure to changes in the relative scarcity of the resource, as we show below.

3 Equilibrium and mortality rates

This section summarizes the key interactions taking place in equilibrium between demographic and economic variables. Expenditures per capita reflect the response of income to changes in resource scarcity, while mortality responds to changes in the population-resource ratio according to precise relationship between the equilibrium mortality rate, emission damages in per capit terms, and labor reallocation effects caused by the primary sector's technology.

3.1 Output and input markets

The equilibrium of the intermediate sector is *symmetric*: as shown in the Appendix, at each instant t each monopolist charges the same price $p_{x_i} = p_x$ and produces the same quantity $x_i = x$. Combining this result with the final producer's behavior, we obtain:

$$p_x(t)\, x(t) = p_c(t)\, C(t) \frac{1}{N(t)}; \tag{21}$$

$$C(t) = N(t)^{\frac{\epsilon}{\epsilon-1}} x(t). \tag{22}$$

Equation (21) says that intermediate sales, $p_x x$, equal consumption expenditure, $p_c C$, and that each monopolist captures a share, $1/N$, of the market. Equation (22) says that final output, C, is equal to the quantity used of each intermediate good, x, times the love-of-variety effect, $N^{\frac{\epsilon}{\epsilon-1}}$.

Next, we have several market-clearing conditions. For clarity, we use the subscript i to denote firm-level variables even though the equilibrium is symmetric. The commodity market clears when supply equals demand by intermediate firms, $Q = NQ_i$. The labor market clears when $L = L_X + L_Z + L_N + L_Q$, where L is labor supply, $L_X + L_Z = N(L_{x_i} + L_{z_i})$ is labor demand by intermediate producers (for production and in-house R&D), L_N is labor demand by entrants and L_Q is labor demand by the primary sector. Finally, the financial market clears when the value of the household's portfolio equals the value of the securities issued by firms, $A = NV_i$. The free-entry condition (16) then yields

$$A(t) = \beta p_c(t)\, C(t). \tag{23}$$

Two further conditions linking expenditure on inputs across sectors are especially relevant for our analysis. Combining the profit-maximizing conditions of commodity and intermediate producers, respectively, we obtain (see the Appendix):

$$p_q(t)\, Q(t) = \gamma \frac{\epsilon - 1}{\epsilon} \cdot p_c(t)\, C(t); \tag{24}$$

$$p_\omega(t)\, \Omega = \Upsilon(t) \cdot p_q(t)\, Q(t)(1-\tau). \tag{25}$$

Equation (24) says that expenditure on the commodity is a constant fraction $\gamma(\epsilon-1)/\epsilon$ of expenditure on final output. Equation (25) says that commodity producers spend on the resource a fraction Υ of the after-tax value of their sales, where Υ is the cost-share function defined in (20).

In the remainder of the analysis we normalize the wage $w(t) \equiv 1$. This choice of numeraire implies that expenditure on final output, $p_c C$, is an index of the value added of labor services.[12]

[12] With the wage set at $w = 1$, p_c is the price of the final good in units of labor. Therefore, the real wage, w/p_c, grows when $\dot{p}_c/p_c < 0$ and a long-run equilibrium featuring constant expenditure $p_c C$ and growth of the physical variable C is characterized by $\dot{C}/C = -\dot{p}_c/p_c$, that is, real growth comes from the rate of decline of the relative price of the final good.

Also, we let $y \equiv p_c C/L$ denote *consumption expenditure per capita* and $\ell \equiv L/\Omega$ denote the ratio of labor supply (population) to resource supply, henceforth *input ratio* for short. High ℓ represents relative abundance of labor or, equivalently, relative scarcity of the resource.

3.2 Expenditure and resource use

Two fundamental relationships between consumption expenditure and resource income characterize the intratemporal equilibrium of the economy. The first follows from combining the household's budget constraint (7) and Euler equation for consumption growth (8) with the equilibrium condition of the assets market (23). It reads

$$y(t) = \frac{1 + \frac{p_\omega(t)}{\ell(t)}}{1 - \beta\rho - \tau\gamma\frac{\epsilon-1}{\epsilon}} \tag{26}$$

and says that consumption expenditure per capita, y, is a constant fraction of income per capita, the sum of the wage and resource income per capita, $p_\omega/\ell = p_\omega\Omega/L$. The presence of the commodity tax at the denominator is due to the balanced-budget assumption and captures the positive effect of public transfers on household expenditure. The second relationship follows from (24) and (25). It reads

$$\frac{p_\omega(t)}{\ell(t)} = \left[(1-\tau)\cdot\Upsilon(p_\omega(t))\cdot\gamma\frac{\epsilon-1}{\epsilon}\right]\cdot y(t) \tag{27}$$

and says that resource income per capita is a fraction of consumption expenditure per capita. We call this fraction, the term in square brackets, the *royalty share.*

The royalty share depends on the technological parameters of all production sectors and on the commodity tax. The tax reduces the royalty share despite the lump-sum rebate because it distorts the use of the commodity in primary production and thus generates a traditional deadweight loss. Note that with $w = 1$ the resource cost-share defined in (20), $\Upsilon \equiv \Upsilon(p_\omega)$, is a function of the resource price only. Therefore, equations (26) and (27) form a system of two equations in three variables (y, p_ω, ℓ). To characterize the interaction of the resource market equilibrium with household consumption-saving decisions, we solve for the resource price p_ω and expenditure per capita y as functions of the input ratio ℓ.

Proposition 1 *Given the input ratio $\ell(t) > 0$, at each instant $t \in [0, \infty)$ the solution of equations (26)-(27) yields a unique equilibrium pair*

$$\{p_\omega^*(\ell(t)), y^*(\ell(t))\}$$

with the following properties. The resource price is monotonically increasing in the input ratio, i.e., $dp_\omega^*(\ell)/d\ell > 0$ for all $\ell > 0$. The effect of the input ratio on expenditure per capita, instead, depends *on the elasticity of substitution between inputs in commodity production. In terms of elasticity:*

$$\frac{d\ln y^*(\ell)}{d\ln \ell} = (1-\tau)\gamma\frac{\epsilon-1}{\epsilon}\ell y^*(\ell)\cdot\frac{d\Upsilon(p_\omega(\ell))}{d\ell},$$

where

$$\frac{d\Upsilon(p_\omega(\ell))}{d\ell} = \begin{cases} <0 & \textit{if } \sigma > 1 \\ =0 & \textit{if } \sigma = 1 \\ >0 & \textit{if } \sigma < 1 \end{cases}.$$

Using equation (19) the equilibrium commodity price is

$$p_q^*(\ell) \equiv \frac{1}{1-\tau}\Theta(1, p_\omega^*(\ell)) \quad \textit{with} \quad \frac{dp_q^*(\ell)}{d\ell} = \begin{cases} <0 & \textit{if } \sigma > 1 \\ =0 & \textit{if } \sigma = 1 \\ >0 & \textit{if } \sigma < 1 \end{cases}.$$

Proof: see Appendix.

The effects of the input ratio, ℓ, on expenditure per capita, y, are a direct consequence of the *cost-share effects* discussed earlier. When ℓ rises, the resource becomes relatively more scarce and its price, p_ω, rises. When labor and the resource are substitutes (complements), an increase in the resource price reduces (increases) the resource cost share in primary production and thereby reduces (increases) resource royalties per capita.[13] The important insight of Proposition 1 is thus that the cost-share effects originating in the primary sector push expenditure per capita in the same direction as resource income per capita. Under substitutability, $\sigma > 1$, we have $\partial y^*(\ell)/\partial\ell < 0$ because more abundant labor results in *lower* p_ω/ℓ via the dominant quantity channel (ℓ at the denominator). With $\sigma < 1$, instead, we have $\partial y^*(\ell)/\partial\ell > 0$ because the price channel at the numerator of p_ω/ℓ dominates as the resource price falls more than one-for-one with ℓ. In the Cobb-Douglas case, changes in the input ratio leave resource income per capita and expenditure per capita unchanged. We will exploit these results in section 4 to characterize the interactions between demography and resource scarcity.

3.3 The equilibrium mortality rate

Substituting the emission function (4) in the mortality function (2) and dividing by population we obtain

$$m(t) = \bar{m} + \mu E(t)^\chi L(t)^{\varphi-1} = \bar{m} + \mu\xi^\chi\frac{Q(t)^\chi}{L(t)^{1-\varphi}}, \tag{28}$$

which shows that the mortality rate, $m \equiv M/L$, responds to changes in L depending on the combined effects of demography on the production possibilities of the primary sector – via changes

[13] Expression (20) yields $\partial\Upsilon(p_\omega)/\partial p_\omega < 0$ if $\sigma > 1$, $\partial\Upsilon(p_\omega)/\partial p_\omega = 0$ if $\sigma = 1$, and $\partial\Upsilon(p_\omega)/\partial p_\omega > 0$ if $\sigma < 1$.

in labor supply and resource scarcity – and on how population size affects damage per adult. We label the first channel as the *primary-employment effect*: an increase in population increases total labor supply and thereby primary employment, L_Q, which raises commodity output Q and the associated emissions. The second channel is the *damage-dilution effect* represented by the denominator in the last term of (28): given primary production, Q, an increase in population, L, reduces the emission damage in per capita terms, $\mu\,(\xi Q)^{\chi}\,/L^{1-\varphi}$. Dilution effects are maximal when $\varphi = 0$ and vanish when $\varphi = 1$.

The relative strength of primary-employment and dilution effects depends on the primary sector's technology and on the parameters of the emission function. Given a one-percent increase in population, the primary-employment effect raises total emissions by

$$\varepsilon_E \equiv \frac{\mathrm{d}\xi Q}{\mathrm{d}L} \cdot \frac{L}{\xi Q} = \underbrace{\left(\frac{\partial \mathcal{F}}{\partial L_Q} \cdot \frac{L_Q}{\mathcal{F}}\right) \cdot \left(\frac{\mathrm{d}L_Q}{\mathrm{d}L} \cdot \frac{L}{L_Q}\right)}_{\text{primary-employment effect}} \tag{29}$$

percentage points, where ε_E is the static elasticity of primary production to variations in total labor supply. At the same time, a one-percent increase in population reduces the per capita emission damage by $1-\varphi$ percentage points. Hence, totally differentiating (28) with respect to population yields

$$\frac{\mathrm{d}\,(m-\bar{m})}{\mathrm{d}L} \cdot \frac{L}{(m-\bar{m})} = \chi \cdot \varepsilon_E - (1-\varphi)\,. \tag{30}$$

Expression (30) summarizes the effects of increased population on the excess deaths per adult caused by deadly spillovers. The term $\chi \cdot \varepsilon_E$ measures the impact of the primary-employment effect on the mortality rate, while $(1-\varphi)$ represents the impact of damage dilution. Whether the equilibrium mortality rate increases or decreases with population thus depends on which of the two effects dominates.

Result (30) implies a clear-cut relationship between mortality and the labor-resource ratio. Since $\ell = L/\Omega$ grows at the same rate as population, the same gap of critical elasticities determines the response of the mortality rate to a rise in the labor-resource ratio. Time-differentiation of (28) yields

$$\frac{\dot{m}\,(t)}{m\,(t)-\bar{m}} = [\chi\varepsilon_E - (1-\varphi)] \cdot \frac{\dot{L}\,(t)}{L\,(t)} = [\chi\varepsilon_E - (1-\varphi)] \cdot \frac{\dot{\ell}\,(t)}{\ell\,(t)}. \tag{31}$$

The next Proposition provides a full characterization of the response of the equilibrium mortality rate to population growth and emphasizes the crucial role played by the primary sector's technology.[14]

[14] Proposition 2 characterizes the equilibrium relations among endogenous variables: to avoid confusion, we drop the time argument unless necessary.

Proposition 2 *The equilibrium mortality rate is a function of the input ratio, i.e.,* $m = m^* (\ell)$*. In the Cobb-Douglas case* $\sigma \to 1$ *we have*

$$m = m^* (\ell) \equiv \bar{m} + \tilde{\mu} \cdot \ell^{\chi(1-\eta)-(1-\varphi)}, \tag{32}$$

where

$$\tilde{\mu} \equiv \mu \left[\xi \left(\frac{(1-\tau)(1-\eta)\gamma^{\frac{\epsilon-1}{\epsilon}}}{1-\beta\rho-[\tau\gamma+(1-\tau)\eta\gamma]^{\frac{\epsilon-1}{\epsilon}}} \right)^{1-\eta} \right]^{\chi} \Omega^{\chi-(1-\varphi)} > 0$$

is constant over time. Under substitutability or complementarity, $\sigma \lesseqgtr 1$*, we have*

$$m = m^* (\ell) \equiv \bar{m} + \bar{\mu} \cdot \Upsilon (\ell)^{\frac{\sigma}{1-\sigma}\chi} \cdot \ell^{-(1-\varphi)}, \tag{33}$$

where $\bar{\mu} \equiv \mu \xi^{\chi} \eta^{\chi \frac{\sigma}{\sigma-1}} \Omega^{\chi-(1-\varphi)} > 0$ *is constant over time and* $\Upsilon (\ell) \equiv \Upsilon (p^*_{\omega} (\ell))$ *is the equilibrium cost share of resource use with the property:*

$$\begin{array}{lll} \sigma > 1 \;\; \rightarrow \;\; \frac{d\Upsilon(\ell)}{d\ell} < 0, & \lim_{\ell \to 0^+} \Upsilon (\ell) = 1, & \lim_{\ell \to \infty} \Upsilon (\ell) = 0; \\ \sigma < 1 \;\; \rightarrow \;\; \frac{d\Upsilon(\ell)}{d\ell} > 0, & \lim_{\ell \to 0^+} \Upsilon (\ell) = 0, & \lim_{\ell \to \infty} \Upsilon (\ell) = 1. \end{array} \tag{34}$$

Proof: see Appendix.

The equilibrium mortality rates defined in Proposition 2 are graphically illustrated in Figure 1, which includes all the subcases generated by the technological characteristics of the primary sector. In general, the mortality response to population size is ambiguos and often non-monotonic. In the Cobb-Douglas case, the equilibrium mortality rate responds to ℓ monotonically, but in different directions depending on the underlying parameters. Under substitutability and complementarity, $m^* (\ell)$ can be non-monotonous because it depends on the resource cost-share, $\Upsilon (\ell) \equiv \Upsilon (p^*_{\omega} (\ell))$, which affects the relative strength of primary-employment and dilution effects. We prove in Appendix all the subcases appearing in Figure 1. In this subsection, we emphasize the intuition behind the results for the cases of Cobb-Douglas and strict substitutability, which are particularly relevant for our results.

Cobb-Douglas. Letting $\sigma \to 1$, the employment share of the primary sector becomes time-invariant, and the critical elasticity (29) reduces to the constant $\varepsilon_E = 1 - \eta$. Therefore, from (31), the response of the mortality rate to population obeys a simple knife-edge condition. When $\chi (1 - \eta) < 1 - \varphi$, the damage-diluting effect dominates: population growth raises the mass of deaths via higher emissions, but the mortality rate declines because the mass of adults grows faster than the mass of deaths. When $\chi (1 - \eta) > 1 - \varphi$, instead, the primary-employment effect dominates: population growth raises the mortality rate because higher emissions cause a death toll that more than offsets per-adult dilution.

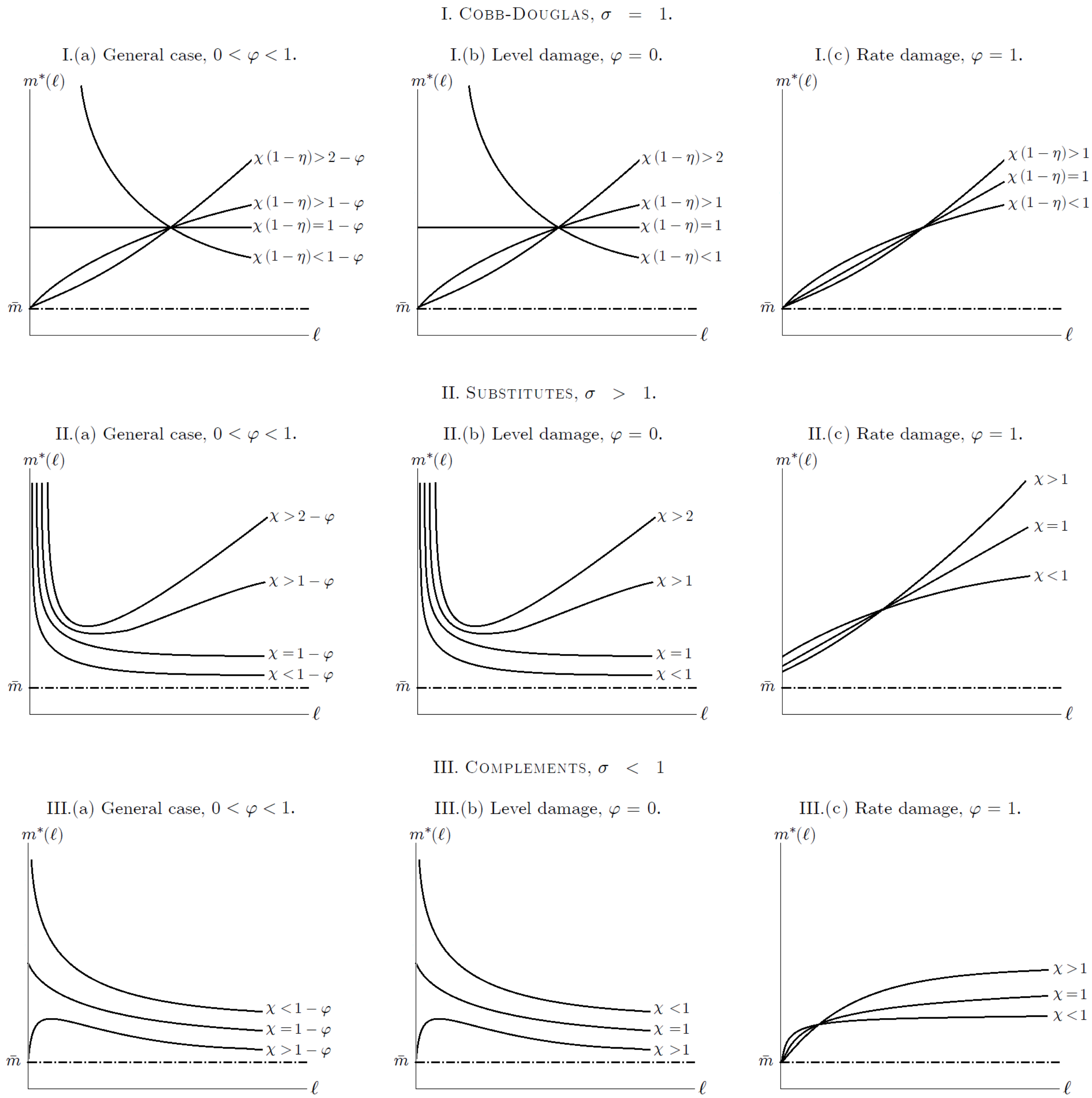

Figure 1: Equilibrium mortality rates as functions of the population-resource ratio, $m = m^*(\ell)$. Without deadly spillovers – i.e., setting $\mu = 0$ – the mortality rate would coincide with the baseline rate $m = \bar{m}$ in all cases.

Substitutability. When $\sigma > 1$, the primary-employment effect is particularly weak at low population levels, and particularly strong at high population levels (see Appendix):

$$\sigma > 1 \;\rightarrow\; \lim_{\ell \to 0^+} \varepsilon_E = 0 \text{ and } \lim_{\ell \to \infty} \varepsilon_E = 1. \tag{35}$$

To grasp the intuition behind (35), assume a persistent decline in population, which reduces total labor supply as well as employment in the primary sector. Given $\sigma > 1$, the primary sector substitutes labor with the primary resource at an increasing rate, so that the magnitude of the primary-employment effect on primary output per worker and on the associated emission damage becomes smaller and smaller: letting $\ell \to 0$, the elasticity of commodity output to employment ε_E approaches zero. The same mechanism in reverse explains why the primary-employment effect becomes stronger when the population-resource ratio increases. Since the intensity of the primary-employment effect depends on ℓ, the mortality response to population growth is generally ambiguous and possibly non-monotonic. In particular, as Figure 1 shows, the equilibrium mortality rate explodes at low population levels for any $0 \leqslant \varphi < 1$, as we establish in the following

Lemma 3 *Assume strict substitutability, $\sigma > 1$, with $0 \leqslant \varphi < 1$. For any value of the damage elasticity, χ, the mortality rate approaches infinity as the input ratio approaches zero:*

$$\sigma > 1 \ \to \lim_{\ell \to 0^+} m^*(\ell) = \bar{m} + \bar{\mu} \cdot \ell^{-(1-\varphi)} = +\infty. \tag{36}$$

Proof: see Appendix.

The intuition for result (36) follows from the fact that the elasticity of commodity output to employment, ε_E, approaches zero as $\ell \to 0$. When population declines, primary producers substitute labor with resource use at increasing rates. This implies that while primary production declines, emissions per worker increase, and the resulting per capita damage – the excess deaths per adult caused by deadly spillovers – eventually explodes at very low levels of population. In other words, under substitutability, population decline weakens the primary-employment effect so much that, below some critical level of population, the dilution effect necessarily dominates.

Lemma 3 implies that, under the assumed conditions, countries with low population and/or abundant primary resources may exhibit very high mortality rates. What happens at high levels of population, instead, depends on parameter values. Since ε_E approaches unity as $\ell \to \infty$, we can observe all the different cases illustrated in Figure 1. If $\chi \leqslant 1 - \varphi$, the mortality rate is *L-shaped* – that is, $m^*(\ell)$ is strictly declining for any level of ℓ – because the primary-employment effects remains waker than dilution at all population levels. If $\chi > 1 - \varphi$, the mortality rate is *U-shaped* – that is, $m^*(\ell)$ reaches a minimum and then increases with ℓ – because relatively high levels of population, combined with a high elasticity of emission damage, make the 'weigthed' primary-employment effect, $\chi\varepsilon_E$, strong enough to dominate dilution.[15]

The generally ambiguous, possibly non-monotonic response of the mortality rate to population deserves attention. The results reported in Figure 1 – especially, those arising under weak and

[15] Figure 1 shows that for any value of $\sigma \gtreqless 1$, there are subcases in which $m^*(\ell)$ is a declining function, at least locally. Decreasing mortality rates become less likely under complementarity, $\sigma < 1$, because the substitution effects underlying result (35) are reversed: with $\sigma < 1$, the primary-employment effect tends to be stronger at low levels of population, so that dilution effects are dominated in a wider range of subcases.

strong substitutability, $\sigma \geqslant 1$ – play a fundamental role in determining the joint dynamics of net fertility, total population and per capita expenditures, as we show below.

4 Population dynamics

The centerpiece of the model is the endogenous mortality rate. This section characterizes the equilibrium dynamics of fertility, mortality and population in a system that implicitly determines the path of per capita expenditures in the short as well as in the long run.

4.1 Demography-scarcity interactions

Since the resource endowment Ω is fixed, the input ratio, $\ell = L/\Omega$, grows over time at the same rate as population, i.e.,

$$\frac{\dot{\ell}(t)}{\ell(t)} = b(t) - m^*(\ell(t)), \tag{37}$$

where $m^*(\ell)$ is the equilibrium mortality rate characterized as a function of the input ratio in Proposition 2. The Euler equation for the birth rate (9) yields

$$\frac{\dot{b}(t)}{b(t)} = \frac{b(t)}{(1-\alpha)(1-\psi)}\left[\frac{1-(1-\psi)\cdot y^*(\ell(t))}{y^*(\ell(t))}\right] - \rho, \tag{38}$$

where $y^*(\ell)$ is expenditure per capita characterized as a function of the input ratio in Proposition 1. Equations (37) and (38) form a two-by-two dynamic system that fully determines the interactions between fertility, resource scarcity and mortality along the equilibrium path. Since the system is capable of generating multiple steady states, we will distinguish among stable and unstable cases by exploiting the following definition.

Definition 4 *A regular steady state is a point (ℓ_{ss}, b_{ss}) in (ℓ, b) space such that the values (ℓ_{ss}, b_{ss}) are positive and finite and satisfy $\dot{b} = \dot{\ell} = 0$. Moreover, the point exhibits (at least local) stability, i.e., there is a thick set of initial conditions $\ell(0) > 0$ starting from which the equilibrium trajectory $(\ell(t), b(t))$ converges to (ℓ_{ss}, b_{ss}) and population converges to the finite value $L_{ss} = \ell_{ss}\Omega > 0$.*

Our notion of regular steady state is conventional in the sense that, being a dynamically stable end-point, it represents the long-run equilibrium of the economy (which will be achieved provided that certain initial conditions are met). The distinctive property is that (ℓ_{ss}, b_{ss}) implies a constant endogenous population level in the lonf run, L_{ss}, but per capita incomes may still grow forever via innovations: if a regular steady state exists and the economy converges to it, population does not grow exponentially in the long run as it would, instead, in traditional models of balanced growth.

The existence of a regular steady state is not guaranteed under all parametrizations. In this respect, our key finding is that pollution-induced mortality bears not only quantitative but also qualitative consequences. On the one hand, if a regular steady state exists independently of pollution, deadly spillovers modify its position and thereby the long-run population level (quantitative effects). On the other hand, deadly spillovers can *create* steady states that would not oterhwise exist, and such additional steady states may be regular or not (qualitative effects). To highlight these findings, we first summarize the model predictions in the special case without pollution externalities (subsection 4.2) and then analyze the complete model by extension (subsection 4.3).

4.2 Special case with exogenous mortality

Setting $\mu = 0$ in (2), we obtain the benchmark model with exogenous mortality and no pollution spillovers (Peretto and Valente, 2015). In this case, the steady-state loci of (37)-(38) read

$$\begin{aligned} \dot{\ell} &= 0 \rightarrow b = \bar{m}, \\ \dot{b} &= 0 \rightarrow b = \frac{(1-\alpha)(1-\psi)\rho}{y^{*}(\ell)^{-1} - (1-\psi)}, \end{aligned}$$

and the model delivers the following results (see Appendix for details and proofs). First, when the primary sector's technology is Cobb-Douglas, $\sigma = 1$, population grows (or declines) at a constant rate forever since there is no regular steady state. More precisely, there is no steady state at all because the steady-state loci are horizontal straight lines that, in general, do not coincide: in Figure 2, phase diagram (a) shows the case in which the equilibrium birth rate exceeds $\bar{m}$, implying a constant and positive net fertility rate.

Second, under strict substitutability, $\sigma > 1$, there exists a regular steady state (ℓ_{ss}, b_{ss}). Phase diagram (e) in Figure 2 shows that the steady state is a saddle point. If the economy starts with $\ell(0) < \ell_{ss}$, the equilibrium path features *positive population growth* with a *declining fertility rate* until b reaches $\bar{m}$, which stabilizes the population level. The reason for these dynamics is that, with $\sigma > 1$, expenditure per capita declines with population because the rising resource scarcity yields lower resource income per capita. This mechanism produces the negative slope of the $\dot{b} = 0$ locus, which is the key to the stability of the process. In fact, in the opposite case of strict complementarity, the income response to population is reversed and the steady state (ℓ_{ss}, b_{ss}) becomes unstable: with $\sigma < 1$, the economy would follow diverging paths leading to either population explosion or human extinction, depending on the initial level of the population-resource ratio (see Appendix for details).

The main takeaway of this subsection is that the cases with $\sigma \geqslant 1$ deserve special emphasis. The Cobb-Douglas case is interesting because the prediction of exponential population growth rests on a knife-edge hypothesis about technology: with $\sigma = 1$, steady states do not seem to exist, but introducing deadly spillovers can create them. The case of strict substitutability, $\sigma > 1$, is even more relevant since it generates a plausible path of economic development without deadly spillovers: assuming $\ell(0) < \ell_{ss}$, population converges to a finite level in the long run because resources per worker and births per adult shrink over time. This is consistent with the well-known fertility decline observed throughout the industrialized world, and with the widely shared idea that population growth outstrips the natural resource base. Introducing deadly spillovers in this context is thus highly significant. We will thus focus on the parametrizations $\sigma \geqslant 1$ in the remainder of the analysis.[16]

4.3 Dynamics with endogenous mortality

The analysis of the previous subsection allows us to study the dynamic system (37)-(38) with endogenous mortality in a straightforward manner. The $\dot{b} = 0$ locus is the same, the $\dot{\ell} = 0$ locus, instead, reads

$$\dot{\ell} = 0 \qquad \rightarrow \qquad b = m^*(\ell) \equiv \begin{cases} \bar{m} + \tilde{\mu} \cdot \ell^{\chi(1-\eta)-(1-\varphi)} & \text{if } \sigma = 1, \\ \bar{m} + \bar{\mu} \cdot \Upsilon(\ell)^{\frac{\sigma}{1-\sigma}\chi} \cdot \ell^{-(1-\varphi)} & \text{if } \sigma \gtrless 1. \end{cases} \tag{39}$$

Expression (39) shows that the shape of the $\dot{\ell} = 0$ locus matches the shape of the equilibrium mortality rate defined in Proposition 2. Figure 2 depicts the resulting phase diagrams for the Cobb-Douglas case and for strict substitutability, and allows for an immediate comparison with the baseline model without deadly spillovers. Both cases deliver novel results.

Cobb-Douglas. With $\sigma = 1$, the gross fertility rate is constant, $b(t) = b_{ss}$ in each t, but deadly spillovers affect net fertility via the mortality rate and create a steady state that would not exist otherwise. The steady state (ℓ_{ss}, b_{ss}) can be stable or unstable depending on the relative strength of emission intensity, labor share in primary production, and damage dilution:

Proposition 5 *(Cobb-Douglas case: stability or mortality traps) For $\sigma = 1$ pollution spillovers create an interior steady state (ℓ_{ss}, b_{ss}), which may be stable or unstable. Assuming $b_{ss} > \bar{m}$ (i.e., population would grow forever in the absence of deadly spillovers) and $0 \leqslant \varphi < 1$, the steady state: creates a mortality trap for $\chi(1-\eta) < 1-\varphi$; is a regular steady state for $\chi(1-\eta) > 1-\varphi$. Proof: see Appendix.*

Figure 2, graph (b), shows the case in which deadly spillovers create mortality traps. Since $\chi(1-\eta) < 1-\varphi$, the $\dot{\ell} = 0$ locus is decreasing in the phase plane, and the steady state denoted by

[16] The analysis of the dynamic system with deadly spillovers and strict complementarity, $\sigma < 1$, is reported in the Appendix for completeness but omitted from the main text to save space.

(ℓ''_{ss}, b_{ss}) is unstable. The input ratio ℓ''_{ss} thus represents an *extinction threshold*. If labor is initially abundant relative to the resource, $\ell_0 > \ell''_{ss}$, the economy experiences sustained population growth whereas in the opposite situation, $\ell_0 < \ell''_{ss}$, the economy falls into a mortality trap characterized by ever-growing mortality rates. The intuition follows our previous discussion on primary-employment and dilution effects (subsection 3.3): a low damage elasticity and/or a low labor share in primary production imply that the mortality rate declines with population size because damage dilution effects dominate. If population is initially large enough to induce positive net fertility at time zero, population will keep on growing everafter because enhanced damage dilution will drive the mortality rate further down. If, instead, population is initially low (or equivalently, the resource is extremely abundant relative to labor supply), net fertility is initially negative and the society falls in a mortality trap with ever-declining population and ever-increasing mortality rates. Figure 2, graphs (c) and (d) consider other parametrizations where $\chi(1-\eta) > 1-\varphi$: high emission intensity and/or a high labor share in primary production make the mortality rate an increasing function of ℓ. In these cases, deadly spillovers create a stable steady state (ℓ'_{ss}, b_{ss}) representing the long-run equilibrium of the economy. Starting from $\ell_0 > \ell'_{ss}$, population increases but its growth rate declines because, as ℓ grows, the primary-employment effect of pollution emissions raise the mortality rate until net fertility becomes zero. Hence, under Cobb-Douglas technology, deadly spillovers either create mortality traps or stabilize the population level in the long run.

Substitutability. The case $\sigma > 1$ delivers even more interesting results. Recalling Lemma 3, substitutability makes mortality rates explode at low population levels in most cases. Phase diagrams (f) and (g) in Figure 2 capture this mechanism: deadly spillovers shift the $\dot{\ell} = 0$ locus up and bend it upwards as ℓ approaches zero. If spillovers are extremely strong, the mortality effect of pollution may even eliminate the regular steady state.[17] But in the more general case where the regular steady state exists, the mortality effect of pollution at low levels of ℓ creates an additional, unstable steady state that yields a mortality trap.

Proposition 6 *(Substitutability: stability and mortality traps) Assume* $\sigma > 1$ *and* $0 \leqslant \varphi < 1$. *Provided there exists a regular steady state* (ℓ'_{ss}, b'_{ss}), *pollution spillovers create a second, unstable steady state* (ℓ''_{ss}, b''_{ss}) *with* $b''_{ss} > b'_{ss}$ *and* $\ell''_{ss} < \ell'_{ss}$. *The interval* $(0, \ell''_{ss})$ *is a mortality trap induced by deadly pollution spillovers. If* $\ell(0) > \ell''_{ss}$, *the economy converges to the regular steady state. If* $\ell(0) < \ell''_{ss}$, *the equilibrium path exhibits* $\lim_{t\to\infty} \ell(t) = 0$ *due to* $\lim_{t\to\infty} L(t) = 0$. *Proof: see Appendix.*

[17] The case with no steady states looks like Figure 2, graphs (f)-(g), but with the $\dot{\ell} = 0$ locus so high that there is no intersection with the $\dot{b} = 0$ locus.

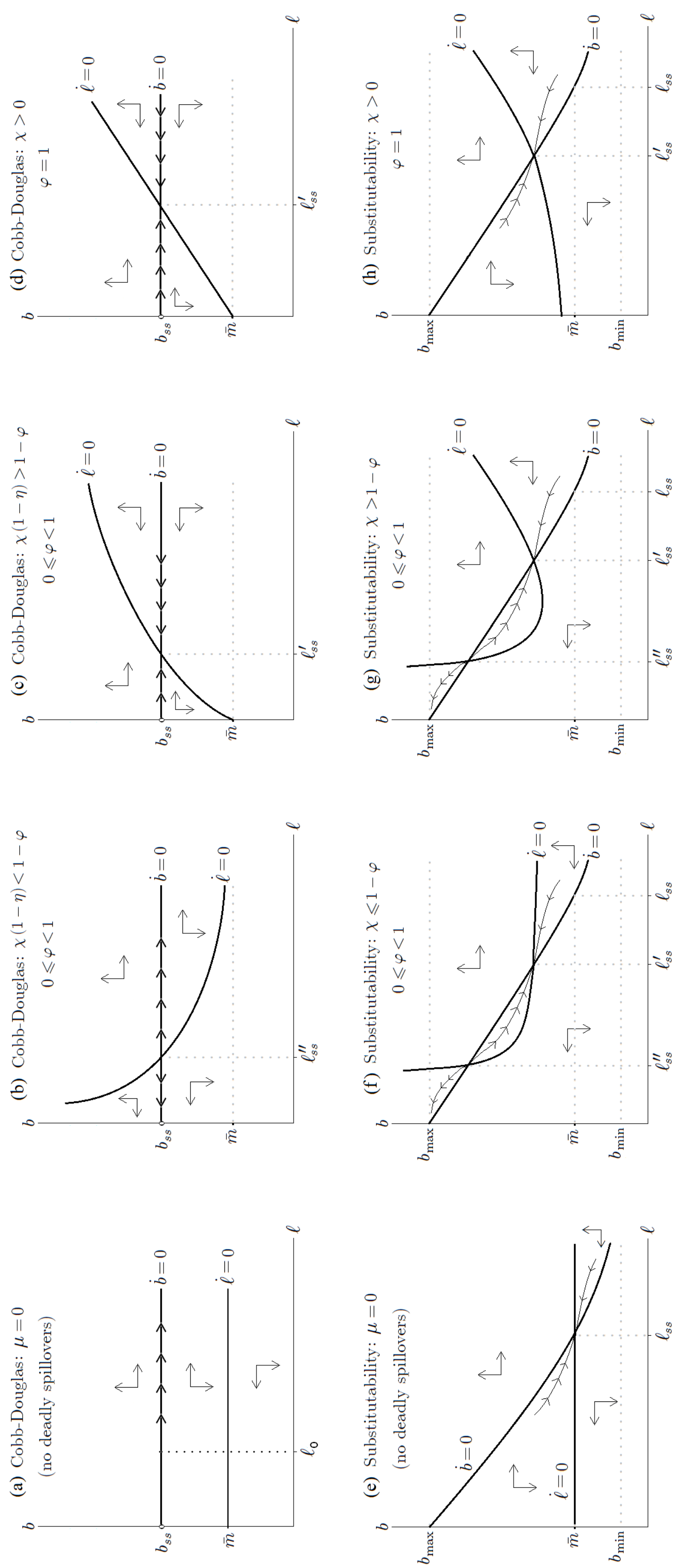

Figure 2 . Phase diagrams of system (37)-(38) under exogenous versus endogenous mortality under $\sigma \geqslant 1$. Diagrams (a) and (e) rule out deadly spillovers by setting $\mu = 0$ in the model. Diagrams (b)-(d) and (f)-(g) assume deadly spillovers under different combinations of parameters. See Appendix (Figure A2) for the case of complementarity.

Proposition 6 establishes that strict substitutability typically produces two steady states because deadly spillovers create a mortality trap, a portion $(0, \ell''_{ss})$ of state space where the implosive dynamics of the population prevail. The unstable steady state (ℓ''_{ss}, b''_{ss}) acts as an extinction threshold: if population is initially too low relative to the resource endowment, $\ell(0) < \ell''_{ss}$, the economy does not converge to the regular steady state (ℓ'_{ss}, b'_{ss}) and follows, instead, an equilibrium path leading to zero population in the long run. Notably, such asymptotic population implosion *does not* result from falling fertility. Rather, starting from $\ell(0) < \ell''_{ss}$, the transition exhibits *increasing fertility as well as increasing mortality*, and the mortality effect prevails. The reason is that the gross fertility rate is constrained by households private wealth, whereas the mortality rate produced by emissions is unbounded: as population shrinks, growing emission damages per capita lead to exploding mortality rates and households may only raise the reproduction rate up to $b_{\max}$, the highest birth rate consistent with their budget constraint. The economy escapes the mortality trap and converges to the regular steady state only if the initial population-resource ratio is sufficiently high, $\ell(0) > \ell''_{ss}$.

Besides their qualitative consequences, we should not overlook the quantitative effects of deadly spillovers. Even when pollution-induced mortality does not prevent the economy from reaching the regular steady state, deadly spillovers *reduce the population capacity* of the society by modifying the position of the regular steady state (ℓ'_{ss}, b'_{ss}). This conclusion is self-evident in Figure 2: with respect to the case with no spillovers, diagram (e), deadly spillovers reduce ℓ'_{ss} and therefore restrict potential population in the long run, in all subcases, even when no mortality traps arise like in diagram (h). The impact of endogenous mortality on population capacity affects the whole equilibrium path of the economy and bears substantial welfare consequences through multiple channels, including firms' incentives to innovate.

Our analysis in this section suggests two further remarks. First, deadly spillovers create mortality traps for any damage elasticity: even with $\chi < 1$, the mortality rate can explode at low levels of the population-resource ratio. The existence of the mortality trap thus depends on the technological properties of the primary sector, not from pessimistic hypotheses about the curvature of the damage function. This result has relevant consequences for applied analysis, in particular the empirical assessment of the harmful effects of pollution externalities, which we discuss in the concluding section of the paper.

Second, Proposition 6 delivers specific insights for less populated, resource-rich economies. Diagrams (f)-(g) in Figure 2 show that economies that are closer to the mortality trap feature a *low* input ratio and a *high* birth rate. Given resource abundance, economies with a small population tend to be ceteris paribus closer to the mortality trap even though they may exhibit higher birth rates. By the same token, exogenous shocks that reduce population push the economy towards the trap via reductions in the input ratio. A similar, though not identical mechanism applies to resource abundance and exogenous shocks expanding the endowment (e.g., discoveries of new stocks of natural resources): given population, a larger resource base can push the economy towards the trap not only by reducing the current population-resource ratio, but also by expanding the mortality trap region $(0, \ell''_{ss})$. We discuss these and related points in the next section.

5 Growth, emission taxes and resource booms

In this section we derive the equilibrium paths of consumption, welfare, output growth and innovation rates. We then study the effects of emission taxes, subsidies to the primary sector, resource booms, and discuss the framework's implications for empirical analysis and policymaking.

5.1 Consumption, growth and welfare

The model's measure of gross domestic product is real consumption, C. Evaluating equations (22) and (12) at the symmetric equilibrium yields

$$C(t) = \frac{L(t)\, y(t)}{p_c(t)} = L(t)\, y(t) \cdot \frac{z(t)^{\theta} N(t)^{\frac{1}{\epsilon-1}}}{(1-\gamma)^{-(1-\gamma)} \gamma^{-\gamma} \frac{\epsilon}{\epsilon-1} p_q(t)^{\gamma}}. \tag{40}$$

This expression says that real consumption equals consumption expenditure divided by the price index of intermediate goods. The price index, in turn, depends on the endogenous components of technology, product variety and firm-specific knowledge, and on the relative price of the commodity. For clarity, we separate the role of endogenous technology from that of the vertical production structure. In the last term of (40), the numerator is a reduced-form representation of *total factor productivity* (TFP), which we henceforth denote as $T \equiv z^{\theta} N^{\frac{1}{\epsilon-1}}$. The denominator is an index of how markup-pricing and the cost of inputs drives the price of intermediates.

Differentiating (40) with respect to time, we obtain

$$g(t) \equiv \frac{\dot{C}(t)}{C(t)} = \frac{\dot{T}(t)}{T(t)} + \left[\frac{\dot{L}(t)}{L(t)} + \frac{\dot{y}(t)}{y(t)}\right] - \gamma \frac{\dot{p}_q(t)}{p_q(t)}. \tag{41}$$

The first term is the growth rate of TFP, which in turn equals a weighted sum of the rates of vertical innovation, $\dot{z}/z$, and horizontal innovation, $\dot{N}/N$. The second term is expenditure growth. The third term uses the dynamics of the commodity price as a summary statistic for the reallocation of inputs across activities in the vertical chain of production that drives the changes in the economy's structure of relative prices. Recalling Proposition 1, the equilibrium commodity price is $p_q^*(\ell) = \frac{1}{1-\tau}\Theta(1, p_\omega^*(\ell))$ and its growth rate over time thus reads

$$\frac{\dot{p}_q(t)}{p_q(t)} = \frac{\mathrm{d}\ln\Theta(w, p_\omega^*(\ell(t)))}{\mathrm{d}\ln p_\omega^*(\ell(t))} \frac{\dot{\ell}(t)}{\ell(t)} = \Upsilon(t) \frac{\dot{\ell}(t)}{\ell(t)}, \tag{42}$$

where Υ is the resource-cost share defined in (20). Therefore, using (42) and the results in Proposition 1, we can write the growth rate of real consumption in compact form as

$$g(t) = \frac{\dot{T}(t)}{T(t)} + \left[1 + \frac{\mathrm{d}\ln y^*(\ell(t))}{\mathrm{d}\ln \ell(t)} - \gamma\Upsilon(t)\right]\frac{\dot{\ell}(t)}{\ell(t)}. \tag{43}$$

The term in square brackets represents transitional effects that operate only when ℓ changes over time: when the input ratio becomes stationary, $\dot{\ell} = 0$, the only source of economic growth is innovation. More precisely, if the economy converges to a regular steady state (ℓ_{ss}, b_{ss}), the only source of economic growth is *vertical* innovation: firm-specific knowledge grows at a constant rate while the mass of firms is constant, $N(t) = N_{ss}$, and proportional to the constant population, $L(t) = L_{ss} = \ell_{ss}\Omega$. The mechanism is that vertical and horizontal innovation exhibit a negative co-movement during the transition: entry of new firms reduces the profitability of firm-specific knowledge investment through market fragmentation while investment in firm-specific knowledge slows down entry by diverting labor away from horizontal R&D. As we show in the Appendix, these co-movements eventually bring the economy to a steady state where the mass of firms is constant and the engine of growth is firm-specific knowledge accumulation.

Proposition 7 *Assume*

$$\frac{\theta(\epsilon-1)(\kappa\phi-\rho-\delta)}{1-\theta(\epsilon-1)-\epsilon\beta(\rho+\delta)} > \rho+\delta$$

and let the economy converge to the steady state (ℓ_{ss}, b_{ss}). Then, the mass of firms is

$$N_{ss} = \frac{1-\theta(\epsilon-1)-\epsilon\beta(\rho+\delta)}{\kappa\phi-\rho-\delta}\cdot\frac{\kappa}{\epsilon}\cdot y^*(\ell_{ss})\cdot L_{ss} > 0, \tag{44}$$

firm-specific knowledge grows at rate

$$\left(\frac{\dot{z}}{z}\right)_{ss} = \frac{\theta(\epsilon-1)(\kappa\phi-\rho-\delta)}{1-\theta(\epsilon-1)-\epsilon\beta(\rho+\delta)} - \rho - \delta > 0, \tag{45}$$

and final output grows at rate

$$g_{ss} = \theta\left(\frac{\dot{z}}{z}\right)_{ss}.$$

Proof: see Appendix.

It is worth noting that the long-run rate of knowledge accumulation, and thus long-run economic growth, is independent of the steady-state values of population, L_{ss}, and expenditure per capita, $y^*(\ell_{ss})$. The reason is that in this class of models the coexistence of vertical and horizontal innovations sterilizes the strong scale effect (Peretto, 1998). An important implication of this mechanism is that the commodity tax as well does not affect long-run growth: changes in τ have transitional effects because they induce net entry or net exit of intermediate firms but leave g_{ss}

unaffected. This does not mean that the tax has negligible effects. To the contrary, a change in τ initiates a transition with first-order welfare effects that we discuss in the next subsection.

The model's key measure of living standards is individual utility in equation (6). Using the utility-maximizing conditions for the allocation of consumption across family members and the equilibrium level of consumption (40), instantaneous utility reads (see the Appendix)

$$\ln u = \underbrace{\bar{\alpha} + \ln T + \ln y^* (\ell) - \gamma \ln p_q^* (\ell)}_{\text{Economic channel } = \ \ln C} + \underbrace{\ln L^{\psi} b^{-(1-\psi)(1-\alpha)}}_{\text{Demographic channel}}, \tag{46}$$

where $\bar{\alpha} \equiv \ln \alpha^{\alpha} \gamma^{\gamma} (1-\alpha)^{1-\alpha} (1-\gamma)^{1-\gamma} \frac{\epsilon-1}{\epsilon}$. Equation (46) allows us to distinguish the different components of instantaneous utility. The term labelled as 'economic channel' equals the logarithm of real consumption and shows how the level of economic activity, and its underlying components, affects welfare at each point in time. The 'demographic channel' summarizes the overall impact of population levels and birth rates on utility: it originates in preferences and combines direct effects, i.e., the household's taste for the mass of adults and children, and the indirect effects of family composition on the allocation of consumption among adults and children. Differentiating (46) with respect to time yields

$$\frac{\dot{u}(t)}{u(t)} = g(t) + \psi \frac{\dot{L}(t)}{L(t)} - (1-\psi)(1-\alpha) \frac{\dot{b}(t)}{b(t)}, \tag{47}$$

where g is the consumption growth rate computed in (43). Equation (47) shows the distinct contribution of economic and demographic channels to the dynamics of utility. Notably, and intuitively, it says that family consumption, C, is a sufficient statistic for the economic channels. The model's dynamics, worked out in detail in the Appendix and briefly discussed above, show that in response to a permanent expansion of the market for intermediate goods – due to, e.g., growing population and/or consumption expenditure per capita – both firm-specific knowledge growth and net entry accelerate until they mean-revert to $(\dot{z}/z)_{ss}$ and N_{ss}. Changes in fundamentals will therefore modify the dynamics of (ℓ, b) and affect welfare through the underlying components of utility, namely, the consumption expenditure channel, $\ln y^* (\ell)$, the commodity price channel, $-\gamma \ln \left(p_q^* (\ell)\right)$, and the two demographic channels, $\ln L^{\psi} b^{-(1-\psi)(1-\alpha)}$. We provide a concrete example in the next subsection by showing the impact of the commodity tax on the economic and demographic channels affecting utility.

5.2 Commodity tax

The most interesting scenarios to study the effects of tax changes are the parametrizations in which strict substitutability and deadly spillovers create a regular steady state a mortality trap

(Proposition 6). The following Proposition provides the comparative statics effects of increasing τ on both the regular steady state, ℓ'_{ss}, and the size of the mortality trap, $(0, \ell''_{ss})$.

Proposition 8 *(Commodity Tax) Assume $\sigma > 1$ and $0 \leqslant \varphi < 1$. Holding constant the input ratio, $\bar{\ell}$, an increase in the commodity tax rate, τ, increases expenditure per capita, reduces the resource price and reduces the mortality rate:*

$$\frac{dy^*\left(\bar{\ell}\right)}{d\tau} > 0, \quad \frac{dp^*_{\omega}\left(\bar{\ell}\right)}{d\tau} < 0, \quad \frac{dp^*_{q}\left(\bar{\ell}\right)}{d\tau} > 0, \quad \frac{dm^*\left(\bar{\ell}\right)}{d\tau} < 0. \tag{48}$$

Hence, the increase in τ shifts the $\dot{\ell} = 0$ locus down and the $\dot{b} = 0$ locus up, yielding a higher regular-steady-state input ratio, $d\ell'_{ss}/d\tau > 0$, as well as a lower mortality-trap threshold, $d\ell''_{ss}/d\tau < 0$. Therefore, the higher commodity tax rate expands the economy's carrying capacity of people, L'_{ss}, and shrinks the mortality trap, $(0, \ell''_{ss})$. Proof: see Appendix.

The commodity tax affects the equilibrium path via two channels. First, an increase in τ reduces the demand for the resource, triggering a reduction in the resource price, $\mathrm{d}p^*_{\omega}(\ell)/\mathrm{d}\tau < 0$, and an increase in expenditure per capita, $\mathrm{d}y^*(\ell)/\mathrm{d}\tau > 0$. Second, given substitutability, the lower resource price raises the resource cost-share and thus drives down the mortality rate via the primary-employment effect, $\mathrm{d}m^*(\ell)/\mathrm{d}\tau > 0$. Figure 3, diagram (a), illustrates the steady-state effects assuming an L-shaped mortality rate for simplicity and without loss of generality. The increase in the expenditure schedule $y^*(\ell)$ yields an upward shift of the $\dot{b} = 0$ locus while the lower mortality rate schedule, $m^*(\ell)$, shifts the $\dot{\ell} = 0$ locus down. These shifts yield a widening gap between the two steady states, with a higher regular-steady-state input ratio, ℓ'_{ss}, and a lower mortality-trap threshold, ℓ''_{ss}.

Considering the welfare effects of tax changes, suppose that the economy is initially in the regular steady state so that the unexpected increase in τ shifts the long-run equilibrium to a higher input-ratio level ℓ'_{ss}. The resulting dynamics are: (i) population L and the input ratio ℓ increase over time, converging to higher steady-state levels from below; (ii) expenditure per capita and the commodity price, $y^*(\ell)$ and $p^*_q(\ell)$, jump up at the time of the shock and then converge from above to the new steady-state levels as ℓ grows;[18] (iii) there is a temporary acceleration in TFP growth with net entry of firms; (iv) the birth rate b jumps on the new saddle path and declines during the transition, converging from above to the new steady state. Therefore, an increase in τ yields: short-run welfare gains through the economic channels, via effects (ii)-(iii), unless the adverse effect of the commodity price is extreme; long-run welfare gains through the demographic channels via

[18] The fact that the increase in τ makes $y^*(\ell)$ and $p^*_q(\ell)$ jump up for given ℓ follows from Proposition 8. The fact that both $y^*(\ell)$ and $p^*_q(\ell)$ decline over time as ℓ grows follows from Proposition 1.

effects (i) and (iv); and ambiguous long-run effects on real consumption in view of the contrasting effects of higher tax and higher steady-state population.

Concerning long-run development prospects, the impact of the commodity tax on the two steady states is a double dividend: it expands the economy's carrying capacity of people *and* reduces the threat of the mortality trap. For example, consider an initial input ratio ℓ_0 such that $\ell''_{ss} < \ell_0 < \ell'_{ss}$ before the change in tax rate, as in Figure 3, diagram (a). The higher tax increases income per capita and reduces the mortality rate, resulting in a larger population in the long run, while pushing the mortality threshold, ℓ''_{ss}, to the left.

The same mechanism in reverse, i.e., reducing the commodity tax, yields a double loss, namely, a lower carrying capacity of people and a higher mortality-trap threshold. A large enough cut of the commodity tax can actually put the economy in the mortality trap, as shown in Figure 3, diagram (b). Given initial input ratio ℓ_2, the economy converges to the regular steady state with the initial tax rate but is in the mortality trap after the tax cut and thus experiences an explosive path of the mortality rate leading to extinction. Equations (46) and (47) then translate these dynamics in lower welfare. This scenario offers a sobering lesson for less populated resource-rich countries that implement low commodity and/or emission taxes and/or subsidize their primary sectors. In an economy with input ratio close to the mortality trap, subsidizing the primary sector amounts to introducing a negative emission tax and can put the country in the mortality trap. Empirical evidence suggests that many real-world economies face such a situation, in particular oil-exporting countries where subsidies to the extractive industry are pervasive (Gupta et al., 2002; Metschies, 2005). Below, we pursue this argument further by showing that the combination of subsidies to the primary sector and new discoveries of the resource can be a recipe for disaster.

5.3 Resource booms

By definition, a *resource boom* – an exogenous increase at time t of the resource endowment, Ω – reduces the input ratio, $\ell(t) = L(t)/\Omega$, immediately. All else equal, this direct effect brings the economy closer to the mortality trap. But the shock may further increase the threat of population implosion by expanding the mortality trap – like subsidies to the primary sector – depending on the value of the emission-damage elasticity.

Proposition 9 *(Resource boom) Assume $\sigma > 1$ and $0 \leqslant \varphi < 1$. An increase in the resource* *endowment, Ω, affects the equilibrium mortality function $m^*(\ell)$ as follows*

$$\frac{dm^*(\ell)}{d\Omega}\left\{\begin{array}{ll} >0 & \text{if } \chi>1-\varphi \\ =0 & \text{if } \chi=1-\varphi \\ <0 & \text{if } \chi<1-\varphi \end{array}\right\} \text{ for any } \ell>0. \tag{49}$$

When $\chi > 1-\varphi$, a resource boom enlarges the mortality trap, $(0, \ell''_{ss})$. Proof: see Appendix.

The mechanism driving this result is that the emission damage incorporated in the mortality function (33) depends on the resource stock with elasticity $\chi - (1-\varphi)$. Therefore, if the damage elasticity exceeds $1-\varphi$, the increase in Ω shifts the equilibrium mortality rate upwards. This phenomenon may be considered a novel type of *resource curse* that has seldom been recognized in the dedicated literature. Diagram (c) in Figure 3 describes the effect of the resource boom assuming $\chi > 1-\varphi$. As the endowment increases from Ω_0 to Ω_1 the $\dot{\ell}=0$ locus shifts up and yields a lower input ratio in the regular steady state, ℓ'_{ss}, and a higher mortality-trap threshold, ℓ''_{ss}. At the same time, the input ratio at time zero moves from the pre-shock level $\ell_0 = L_0/\Omega_0$ to the lower after-shock level $\ell_1 = L_0/\Omega_1$. The welfare effects of these shocks are generally ambiguous. If the economy converges to the new regular steady state ℓ'_{ss} after the shock, utility is likely to be higher via the economic channel since expenditure per adult, population and the mass of firms would be higher in the long run.[19] However, the shock itself may drive the economy into the mortality trap, yielding drastically opposite results: if $\ell_1 < \ell''_{ss}$, the population decline deletes and eventually overturns the consumption gains, while both the demographic components of welfare – population and gross fertility rates – yield net welfare losses both in the transition and in the long run as the mortality rate grows.

In the cases $\chi \leqslant 1-\varphi$, the resource boom does not expand the size of the mortality trap but this does not mean that the trap less threatening: even when the state space of the mortality trap $(0, \ell''_{ss})$ shrinks or does not change, the increase in Ω reduces the population-resource ratio. With $\chi < 1-\varphi$, the $\dot{\ell}=0$ locus would shift down but the current input ratio ℓ may still fall more than the mortality threshold ℓ''_{ss}. With $\chi = 1-\varphi$, the resource boom surely makes the economy closer to population implosion since the steady states do not move. Moreover, the claim that the position of the steady states remains the same only refers to the input ratio, not to population size, since the resource boom affects the mortality-trap threshold in absolute terms, L''_{ss}. Below we further explore this point in a numerical assessment of the effects of resource booms and emission taxes.

[19] The reduction in ℓ'_{ss} is accompanied by a rise in total steady-state population L'_{ss} which falls short of the increase in Ω. Therefore, expenditure per adult $y^*(\ell'_{ss})$, population L'_{ss} and the associated long-run mass of firms N'_{ss} are higher in the post-shock regular steady state. Unless the associated increase in the commodity price (cf. Proposition 1) is extreme, the consumption term in (46) is higher in the new, post-shock regular steady state.

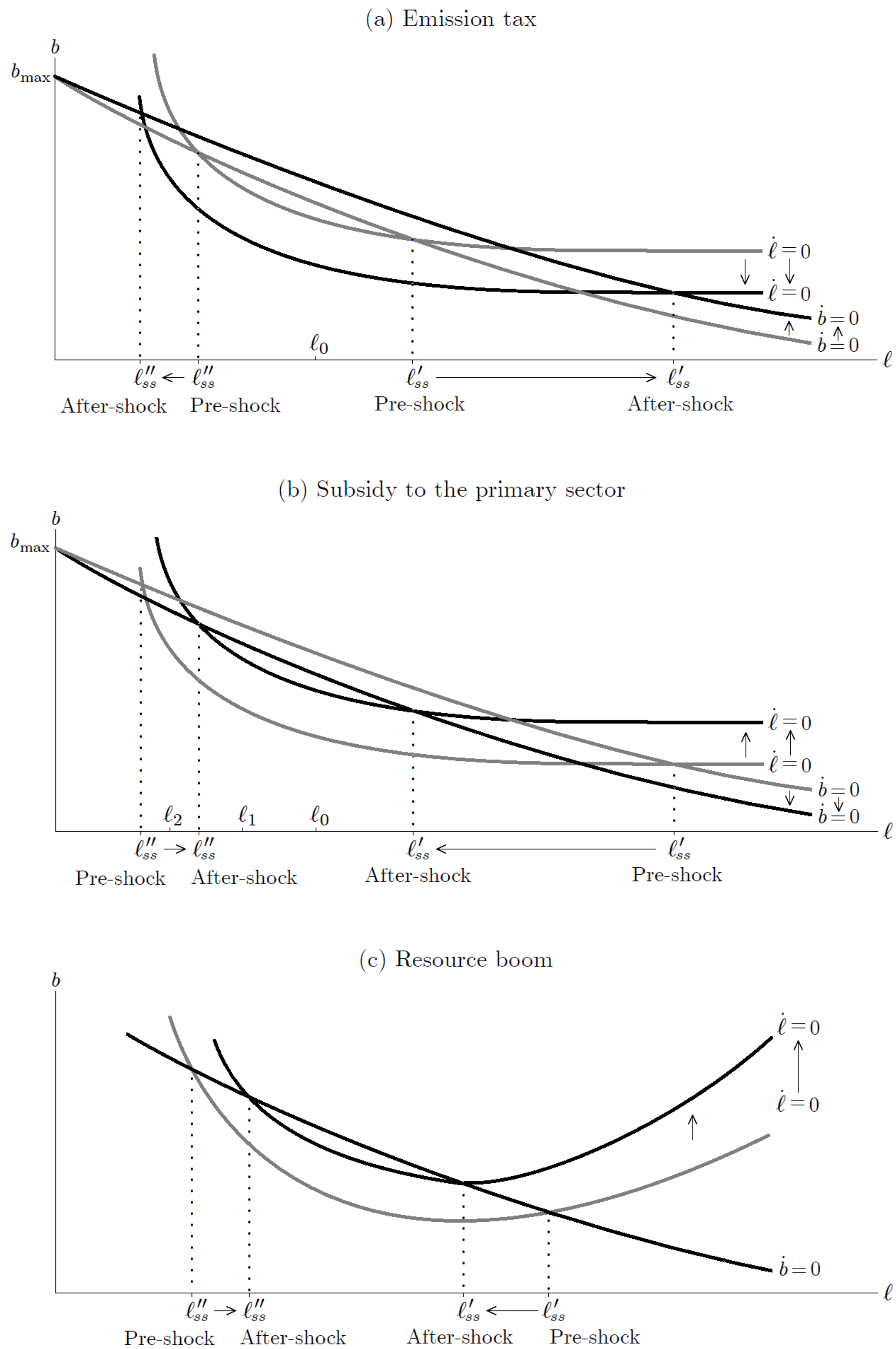

Figure 3. Exogenous shocks under substitutability, $\sigma > 1$. Diagram (a) describes the effects of introducing of an emission tax (increase in τ). Diagram (b) describes the effects of subsidizing purchases of the primary sector's output (decrease in τ). Diagram (c) describes the effect a resource boom (an increase in the resource endowment Ω) when $\chi > 1 - \varphi$.

5.4 Traps, resource booms and subsidies

Consider the polar case of a linear damage function, $\chi = 1$, with $\varphi = 0$ and strict substitutability, $\sigma > 1$. Under these parameters, a resource boom has a straightforward graphical representation, namely, a displacement to the left of the input ratio, with no change in the stationary loci and the associated steady states, (ℓ'_{ss}, b'_{ss}) and (ℓ''_{ss}, b''_{ss}). We then construct the following scenario: the economy experiences a resource boom and the government decides to subsidize the primary sector by reducing the commodity tax rate, τ, below zero. This kind of policies are frequently observed in resource-rich countries, with various justifications. In our model the combination of the two changes moves the economy closer to the mortality trap for two independent reasons: the resource boom shifts the input ratio to the left, from ℓ'_{ss} to $\ell(0) < \ell'_{ss}$, and the lower tax rate shifts the mortality-trap threshold, ℓ''_{ss}, to the right. Figure 3, graph (b), illustrates this mechanism assuming that the pre-shock input ratio is $L_0/\Omega_0 = \ell_0$ and $\ell''_{ss} < \ell_0 < \ell'_{ss}$. The reduction in τ shifts the stationary loci as shown in the phase diagram while the magnitude of the resource boom determines whether the economy falls in the mortality trap. If the increase in Ω is sufficiently small, the initial input ratio is $\ell_1 > \ell''_{ss}$ and the economy experiences an instantaneous jump up in fertility followed by a gradual decline until convergence to the new regular steady state, which is lower than the pre-shock steady state. If instead the increase in Ω is sufficiently large, the initial input ratio is $\ell_2 < \ell''_{ss}$ and the economy falls in the trap: the fertility rate jumps up and keeps growing but never reaches the exploding mortality rate. Consequently, the shock triggers population implosion.

A simple numerical example yields further insights on the conditions under which the economy falls in the mortality trap. Table 1 summarizes the outcomes of a numerical simulation in which the key parameters are set so as to generate an empirically plausible regular steady state.[20] The baseline scenario (i) assumes a zero commodity tax and delivers a long-run birth rate $b'_{ss} = 1.84\%$ in the regular steady state. The regular steady state input ratio and the mortality-trap threshold are $\ell'_{ss} = 2.12$ and $\ell''_{ss} = 0.11$, respectively. The resource stock is $\Omega = 10$mln, which yields a regular steady state population of 21.2mln and a mortality-trap threshold of 1.15mln people. Given the same parameters, scenarios (ii) and (iii) respectively assume a positive tax, $\tau = 5\%$, and a positive subsidy, $\tau = -5\%$, without any change in the resource stock. The effect of such policy shocks exceeds 4mln people in either direction: given the pre-shock steady-state population $L'_{ss} = 21.2$mln, the 5% tax rate generates steady-state population $L'_{ss} = 25.9$mln whereas the 5% subsidy generates steady-state population $L'_{ss} = 17.4$mln. The effects on the mortality-trap threshold is around $40,000$ people in either direction: given the pre-shock threshold $L''_{ss} = 1.15$mln,

[20] The parameter values assumed in calculating the numbers reported in Table 1 are: $\chi = 1$, $\beta = 1.587$, $\rho = 0.015$, $\epsilon = 4.3$, $\gamma = 0.3$, $\eta = 0.5$, $\sigma = 2$, $\alpha = 0.8$, $\psi = 0.3$, $\bar{m} = 0.016$, $\mu = 0.2$, $\xi = 0.05$.

Scenario	τ	Ω	*Regular steady state* ℓ'_{ss}	L'_{ss}	b'_{ss}	*Mortality trap* ℓ''_{ss}	L''_{ss}	b''_{ss}
(*i*) Baseline	0.00	10 mln	2.119	21,192,440	1.84%	0.115	1,150,516	3.9%
(*ii*) Positive tax	0.05	10 mln	2.592	25,921,612	1.81%	0.111	1,111,475	4.0%
(*iii*) Subsidy	-0.05	10 mln	1.747	17,471,691	1.87%	0.120	1,195,633	3.8%
(*iv*) Resource boom	0.00	10.5 mln	2.119	22,252,062	1.84%	0.115	1,208,042	3.9%
(*v*) Boom & Subsidy	-0.05	10.5 mln	1.747	18,345,276	1.87%	0.120	1,255,415	3.8%

Table 1. Taxes, subsidies, resource booms and traps: numerical examples.

the 5% tax rate generates $L''_{ss} = 1.11$mln whereas the 5% subsidy generates $L''_{ss} = 1.19$mln.

Scenarios (iv) and (v) consider a resource boom whereby the endowment Ω increases by 5 percentage points. In scenario (iv), the resource boom occurs in isolation from policy changes: with respect to the baseline scenario, the steady-state input ratios remain the same but the associated *population levels* change. In particular, the population-trap threshold increases by nearly $57,000$ people, from the pre-shock $L''_{ss} = 1.15$mln to the after-shock $L''_{ss} = 1.21$mln. In scenario (v), we address the aforementioned policy exercise, i.e., a combined shock in which the endowment, Ω, rises by 5 percentage points accompanied by the introduction of a five percent subsidy to commodity production, $\tau = -5\%$. The resulting increase in the population-trap threshold is nearly $105,000$ units, from $L''_{ss} = 1.15$mln to $L''_{ss} = 1.25$mln. Comparing scenarios (iv) and (iv), shows that the rise of the population-trap threshold L''_{ss} caused by the resource boom is almost doubled by the concurrent decision of the government to introduce a 5% subsidy to the primary sector.[21]

We stress that this exercise assumes parameter values yielding no effects of the resource boom on the steady state loci ($\chi = 1 - \varphi$). With a stronger elasticity, $\chi > 1 - \varphi$, we would obtain an even larger increase in the mortality-trap threshold for scenarios (iv)-(v) because, as established in Proposition 9, the increase in Ω would further reduce the gap between the two steady states.[22] This and the previous considerations make our general conclusion evident. Labor-poor countries with abundant polluting resources face larger mortality traps in view of their natural endowments. If the governments of these countries respond to new resource discoveries with higher subsidies to the

[21] More precisely, the rise in the population threshold caused by the resource boom is $57,526$ people with no subsidy and is $104,899$ people with a 5% subsidy. The marginal effect of the subsidy is thus a 82% larger increase of the threshold.

[22] In graphical terms, scenario (v) with $\chi > 1 - \varphi$ would feature one downward shift of the $\dot{b} = 0$ locus and *two* upward shifts of the $\dot{\ell} = 0$ locus, one due to the subsidy and one due to the resource boom.

primary sector – a policy that is sometimes justified by the need to overcome under-development traps – the possibility of falling into a different trap characterized by ever-growing mortality should be taken seriously.

6 Conclusion

In stark contrast to the magnitude of pollution-induced mortality reported in the empirical literature, there is little to no recognition of such an important phenomenon in macroeconomic models of growth and development. Filling this gap requires tractable models in which economic growth, fertility and mortality are simultaneously endogenous and interconnected via equilibrium relationships. We have shown that unlike conventional pollution externalities, deadly spillovers affect welfare through multiple channels – labor-supply effects, consumption-saving decisions, reproduction choices, changes in market size that affect incentives to innovate and thereby productivity growth – so that the response of the equilibrium mortality rate to population size is generally ambiguous and often non-monotonic. This relationship between mortality and population reflects not only the emission intensity of primary production but also dilution effects and labor reallocation effects caused by technology. Under parametrizations that yield empirically plausible paths – prominently, a transitional fertility decline leading to a finite endogenous population level – deadly spillovers modify potential population in the long run, and may even create mortality traps that, unlike the typical poverty traps studied in development economics, threaten less populated economies with abundant natural resources.

Our framework delivers novel insights for both applied research and policy making. The conventional view in environmental economics is that pollution externalities – in the form of negative spillovers affecting private utility or firms efficiency – limit growth when emission damages are strictly convex (see, e.g., Xepapadeas, 2005). In our paper, instead, pollution threatens the economy's development and survival through endogenous mortality rates that may reach very high levels even with linear or strictly concave damage functions, like those suggested by empirical evidence (e.g., Cakmak et al., 1999; Izzotti et al., 2000). Our analysis thus suggests that low estimated damage intensities can be deceptively reassuring because they only provide a partial assessment of the actual threat faced by the society. The actual impact of deadly spillovers on economic performance and long-term population capacity stems from the interactions between demography and the technological and market conditions of production sectors. Therefore, empirical research should go beyond estimating damage elasticities to investigate how mortality interacts with labor supply and sectoral output of highly-polluting and less-polluting industries.

With respect to public policy, our model suggests that emission taxes may yield double dividends in terms of both income and population capacity. To the contrary, subsidies to primary production reduce long-run population capacity by increasing the death toll. New discoveries of natural endowments reduce the labor-resource ratio and may increase the risk of population implosion. Consequently, subsidizing commodity production during a resource boom can have disastrous consequences if the primary sector's technology and the mortality function do not change. These considerations suggest that some novel thinking is called for in the debate on the prospects of many developing countries where discoveries of natural resources are accompanied by (implicit or explicit) subsidies designed to foster their exploitation.

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Growth with Deadly Spillovers

(Online Appendix)

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A Appendix: The model

A.1 Consumption and Reproduction Choices

Utility maximization and derivation of equations (8)-(9). The maximization problem can be specified as (omitting time arguments when no ambiguity arises)

$$\max_{\{c_L(t),c_B(t),B(t)\}} \int_0^\infty \ln\left[\left(c_L L^\psi\right)^\alpha \left(c_B B^\psi\right)^{1-\alpha}\right] \cdot \exp\left[-\rho t\right] dt$$

subject to

$$\dot{A} = rA + wL + p_\omega \Omega + S - p_c\left(c_L L + c_B B\right), \tag{A.1}$$

$$\dot{L} = B - mL, \tag{A.2}$$

where (A.1) is the asset accumulation law (7) incorporating the constraint $C = c_L L + c_B B$, and (A.2) is the demographic law (1) where the paths of the mortality rate m is taken as given. The current value Hamiltonian for this problem reads

$$\begin{aligned} \mathcal{L} \equiv\ & \ln\left[\left(c_L L^\psi\right)^\alpha \left(c_B B^\psi\right)^{1-\alpha}\right] + \\ & + \vartheta_A\left[rA + wL + p_\omega \Omega + S - p_c\left(c_L L + c_B B\right)\right] + \\ & + \vartheta_L\left(B - mL\right), \end{aligned} \tag{A.3}$$

where ϑ_A and ϑ_L are the dynamic multipliers associated with asset accumulation and with population growth, respectively. The necessary conditions for utility maximization read

$$\begin{array}{lll l} \mathcal{L}_{C_L} = 0 & \rightarrow & \frac{\alpha}{c_L} = \vartheta_A p_c L & \text{(i)} \\ \mathcal{L}_{C_B} = 0 & \rightarrow & \frac{1-\alpha}{c_B} = \vartheta_A p_c B & \text{(ii)} \\ \mathcal{L}_B = 0 & \rightarrow & \frac{\psi(1-\alpha)}{B} + \vartheta_L = \vartheta_A p_c c_B & \text{(iii)} \\ \mathcal{L}_A = \rho\vartheta_A - \dot{\vartheta}_A & \rightarrow & \vartheta_A r = \rho\vartheta_A - \dot{\vartheta}_A & \text{(iv)} \\ \mathcal{L}_L = \rho\vartheta_L - \dot{\vartheta}_L & \rightarrow & \frac{\psi\alpha}{L} + \vartheta_A\left[w - p_c c_L\right] - \vartheta_L m = \rho\vartheta_L - \dot{\vartheta}_L & \text{(v)} \\ \text{TVC assets} & \rightarrow & \lim_{t\to\infty}\vartheta_A(t) A(t) \exp\left[-\rho t\right] = 0 & \text{(vi)} \\ \text{TVC population} & \rightarrow & \lim_{t\to\infty}\vartheta_L(t) L(t) \exp\left[-\rho t\right] = 0 & \text{(vii)} \end{array} \tag{A.4}$$

Conditions (A.4.i)-(A.4.ii) yield constant consumption shares for adults and children,

$$\frac{c_L L}{c_L B} = \frac{\alpha}{1-\alpha}. \tag{A.5}$$

From (A.5), total households consumption expenditure equals

$$p_c C = p_c \left(c_L L + c_B B\right) = \frac{1}{\vartheta_A}, \tag{A.6}$$

so that the co-state equation for assets (A.4.iv) yields the familiar Euler condition for expenditure growth

$$\frac{\dot{p}_c\left(t\right)}{p_c\left(t\right)} + \frac{\dot{C}\left(t\right)}{C\left(t\right)} = -\frac{\dot{\vartheta}_A\left(t\right)}{\vartheta_A\left(t\right)} = r\left(t\right) - \rho, \tag{A.7}$$

which is equation (8) in the main text. Now consider fertility and population iteractions. From the fertility condition (A.4.iii) we have

$$\psi\left(1-\alpha\right) + \vartheta_L\left(t\right) B\left(t\right) = \vartheta_A p_c\left(t\right) c_B\left(t\right) B\left(t\right) \tag{A.8}$$

where we can substitute $\vartheta_A p_c c_B B = 1-\alpha$ from (A.4.ii), to obtain

$$\vartheta_L\left(t\right) B\left(t\right) = \left(1-\alpha\right)\left(1-\psi\right). \tag{A.9}$$

Expression (A.9) shows that positive fertility $B > 0$ is consistent with a positive marginal shadow value of the population $\vartheta_L > 0$ if and only if $\psi < 1$. Also, equation (A.9) implies a constant shadow value of children, $\vartheta_L\left(t\right) B\left(t\right)$. Time-differentiating (A.9) yields

$$\frac{\dot{\vartheta}_L\left(t\right)}{\vartheta_L\left(t\right)} = -\frac{\dot{B}\left(t\right)}{B\left(t\right)}. \tag{A.10}$$

From (A.6) and (A.9), the ratio between multipliers ϑ_A/ϑ_L equals

$$\frac{\vartheta_A\left(t\right)}{\vartheta_L\left(t\right)} = \frac{1}{\left(1-\alpha\right)\left(1-\psi\right)} \cdot \frac{B\left(t\right)}{p_c\left(t\right) C\left(t\right)}. \tag{A.11}$$

Next consider the co-state equation for population (v), which may be written as

$$-\frac{\dot{\vartheta}_L}{\vartheta_L} = \frac{\psi\alpha}{\vartheta_L L} + \frac{\vartheta_A}{\vartheta_L}\left[w - p_c c_L\right] - m - \rho \tag{A.12}$$

Substituting (A.9), (A.11), and (A.10) in (A.12), we have

$$\frac{\dot{B}}{B} = \frac{b}{\left(1-\alpha\right)\left(1-\psi\right)}\left[\frac{\psi\alpha p_c C + wL - \alpha p_c C}{p_c C}\right] - m - \rho. \tag{A.13}$$

Recalling that the left hand side of (A.13) equals $\frac{\dot{B}}{B} = \frac{\dot{b}}{b} + \frac{\dot{L}}{L}$, the demographic law (1) implies

$$\frac{\dot{b}}{b} = \frac{b}{\left(1-\alpha\right)\left(1-\psi\right)}\left[\frac{\psi\alpha p_c C + wL - \alpha p_c C}{p_c C}\right] - b - \rho$$

and therefore

$$\frac{\dot{b}\left(t\right)}{b\left(t\right)} = \frac{b\left(t\right)}{\left(1-\alpha\right)\left(1-\psi\right)}\left[\psi + \frac{w\left(t\right) L\left(t\right) - p_c\left(t\right) C\left(t\right)}{p_c\left(t\right) C\left(t\right)}\right] - \rho, \tag{A.14}$$

which is equation (A.14) in the main text.

A.2 Producers: Final and Intermediate sectors

Final producers. The final sector behaves like a single competitive firm maximizing aggregate profits from final output sales, $p_c C - \int_0^N p_{x_i} x_i di$, subject to technology (10) taking all prices as given. The first order condition for the quantity x_i of each intermediate variety i yields the demand schedule

$$p_{x_i}(t) = \frac{p_c(t)\, C(t)}{\int_0^{N(t)} x_i(t)^{\frac{\epsilon-1}{\epsilon}} di} \cdot x_i(t)^{-\frac{1}{\epsilon}}, \tag{A.15}$$

which is taken as given by the producer of the intermediate-good variety i.

Incumbents: profit maximization. The maximization problem is

$$\begin{aligned} \max_{\{Q_i, L_{x_i}, L_{z_i}\}} V_i(t) &= \int_t^\infty \pi_i(t) \exp\left(-\int_t^v (r(s)+\delta)\, ds\right) dv \\ & \quad \text{subject to} \\ \pi_i &= p_{x_i} x_i - p_q Q_i - w L_{x_i} - w L_{z_i} & \text{(A.16)} \\ x_i &= z_i^\theta \cdot Q_i^\gamma (L_{x_i} - \phi)^{1-\gamma} & \text{(A.17)} \\ \dot{z}_i &= \kappa \cdot \bar{z} \cdot L_{z_i} & \text{(A.18)} \\ p_{x_i} &= \Psi \cdot x_i^{-\frac{1}{\epsilon}} & \text{(A.19)} \end{aligned}$$

where (A.19) is the final producers' demand for the intermediate (11) after defining the term $\Psi = p_c C / \int_0^N x_i^{\frac{\epsilon-1}{\epsilon}} di$, which is taken as given by the monopolist. Firm-specific knowledge z_i acts as the co-state variable optimized by the firm whereas the path of public knowledge $\bar{z}$ is taken as given by the monopolist. Substituting the constraints (A.16), (A.17) and (A.19) in the objective function, the Hamiltonian for this problem can be written as

$$\bar{\mathcal{L}} = \Psi \left[z_i^\theta Q_i^\gamma (L_{x_i} - \phi)^{1-\gamma} \right]^{\frac{\epsilon-1}{\epsilon}} - p_q Q_i - w L_{x_i} - w L_{z_i} + \vartheta_z \cdot \kappa \cdot \bar{z} \cdot L_{z_i}, \tag{A.20}$$

where ϑ_z is the dynamic multiplier associated to z_i. The necessary conditions for maximization read

$$\begin{array}{llll} \bar{\mathcal{L}}_{Q_i} = 0 & \rightarrow & \gamma \frac{\epsilon-1}{\epsilon} p_{x_i} x_i = p_q Q_i & \text{(i)} \\ \bar{\mathcal{L}}_{L_{x_i}} = 0 & \rightarrow & (1-\gamma) \frac{\epsilon-1}{\epsilon} p_{x_i} x_i = w (L_{x_i} - \phi) & \text{(ii)} \\ \bar{\mathcal{L}}_{L_{z_i}} = 0 & \rightarrow & \vartheta_z \kappa \bar{z} = w & \text{(iii)} \\ \bar{\mathcal{L}}_{z_i} = (r+\delta)\vartheta_z - \dot{\vartheta}_z & \rightarrow & \frac{\epsilon-1}{\epsilon} \cdot \theta \frac{p_{x_i} x_i}{z_i} = (r+\delta)\vartheta_z - \dot{\vartheta}_z & \text{(iv)} \end{array} \tag{A.21}$$

For future reference, note that focs (i)-(ii) in (A.21) imply

$$p_q Q_i + w L_{x_i} = \frac{\epsilon-1}{\epsilon} p_{x_i} x_i + w\phi \tag{A.22}$$

where (iii)-(iv) yield

$$\frac{\dot{\vartheta}_z}{\vartheta_z} = r + \delta - \frac{\epsilon-1}{\epsilon} \cdot \theta \frac{p_{x_i} x_i}{\vartheta_z z_i} = r + \delta - \frac{\epsilon-1}{\epsilon} \cdot \theta p_{x_i} x_i \cdot \frac{\kappa \bar{z}}{w z_i}, \tag{A.23}$$

where the last term follows from using (iii) to substitute ϑ_z.

A.3 Primary sector

Derivation of the cost share of resource use (20). The profit maximizing conditions with respect to resource use and labor respectively yield

$$p_q (1-\tau) \cdot \left[\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}} + (1-\eta) \cdot L_Q^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} \eta \cdot \Omega^{\frac{\sigma-1}{\sigma}} = p_\omega \Omega, \tag{A.24}$$

$$p_q (1-\tau) \cdot \left[\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}} + (1-\eta) \cdot L_Q^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}-1} (1-\eta) \cdot L_Q^{\frac{\sigma-1}{\sigma}} = w L_Q. \tag{A.25}$$

From (A.24) and (A.25), the cost share of resource use is

$$\Upsilon \equiv \frac{p_\omega \Omega}{p_\omega \Omega + w L_Q} = \frac{\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}}}{\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}} + (1-\eta) \cdot L_Q^{\frac{\sigma-1}{\sigma}}}. \tag{A.26}$$

In order to rewrite (A.26) in terms of factor prices, note that (A.24) and (A.25) imply

$$\frac{\eta \cdot \Omega^{\frac{\sigma-1}{\sigma}}}{(1-\eta) \cdot L_Q^{\frac{\sigma-1}{\sigma}}} = \frac{\eta^\sigma p_\omega^{1-\sigma}}{(1-\eta)^\sigma w^{1-\sigma}}. \tag{A.27}$$

Substituting (A.27) in the right hand side of (A.26) yields

$$\Upsilon \equiv \frac{p_\omega \Omega}{p_\omega \Omega + w L_Q} = \frac{\eta^\sigma p_\omega^{1-\sigma}}{\eta^\sigma p_\omega^{1-\sigma} + (1-\eta)^\sigma w^{1-\sigma}}, \tag{A.28}$$

which is expression (20) in the main text.

B Appendix: Equilibrium and Mortality Rates

B.1 Output and Input Markets

Derivation of (21)-(22). Since each intermediate firm has access to the same stock of public knowledge, every monopolist solves the maximization problem given the same level of firm-specific knowledge, which implies a symmetric equilibrium in the intermediate sector. Given $x_i = x$ for each $i \in [0, N]$, integrating across intermediate input quantities inside the final-producers technology (10) yields (21). Similarly, integrating across intermediate input quantities inside the demand schedule for intermediates (A.15) yields (22).

Derivation of (24). From the intermediate producers' problem, the profit maximization condition (A.21.i) can be aggregated as

$$\gamma \frac{\epsilon - 1}{\epsilon} N(t)\, p_{x_i}(t)\, x_i(t) = p_q(t)\, Q(t). \tag{B.1}$$

Substituting the equilibrium condition (21) in (B.1) we obtain

$$\gamma \frac{\epsilon - 1}{\epsilon} p_c(t) C(t) = p_q(t) Q(t), \tag{B.2}$$

which is equation (24) in the main text.

Derivation of (25). The zero-profit condition in the primary sector implies

$$p_q(t) Q(t)(1-\tau) = p_\omega(t)\Omega + w(t) L_Q(t). \tag{B.3}$$

Combining (B.3) with (20) yields (25) in the main text.

B.2 Expenditure and Resource Use

Derivation of (26). Using the government budget constraint (7) to substitute $S = \tau p_q Q$, rewrite the wealth constraint (7) as

$$\frac{\dot{A}(t)}{A(t)} = r(t) + \frac{w(t) L(t)}{A(t)} + \frac{p_\omega(t)\Omega + \tau p_q(t) Q(t)}{A(t)} - \frac{p_c(t) C(t)}{A(t)}. \tag{B.4}$$

Substituting $A = \beta p_c C$ from (23) we have

$$\frac{\dot{p}_c(t)}{p_c(t)} + \frac{\dot{C}(t)}{C(t)} = r(t) + \frac{w(t) L(t)}{\beta p_c(t) C(t)} + \frac{p_\omega(t)\Omega + \tau p_q(t) Q(t)}{\beta p_c(t) C(t)} - \frac{1}{\beta}. \tag{B.5}$$

The Euler condition for consumption (8) can be rewritten in terms of aggregate consumption as

$$\frac{\dot{p}_c(t)}{p_c(t)} + \frac{\dot{C}(t)}{C(t)} = r(t) - \rho \tag{B.6}$$

and substituted into (B.5) to obtain

$$(1-\beta\rho) p_c(t) C(t) = w(t) L(t) + p_\omega(t)\Omega + \tau p_q(t) Q(t). \tag{B.7}$$

Substituting $p_q(t) Q(t)$ by (24), we obtain

$$(1-\beta\rho) p_c(t) C(t) = w(t) L(t) + p_\omega(t)\Omega + \tau\gamma \frac{\epsilon - 1}{\epsilon} p_c(t) C(t). \tag{B.8}$$

Dividing both sides of (B.8) by $L(t)$ and substituting the definitions $y = p_c C/L$ and $\ell = L/\Omega$, and rearranging terms, we obtain equation (26) in the main text.

Proof of Proposition 1. From (20), the cost share of resource use with normalized wage $w = 1$ reads

$$\Upsilon(p_\omega) \equiv \frac{p_\omega \Omega}{p_\omega \Omega + L_Q} = \frac{\eta^\sigma p_\omega^{1-\sigma}}{\eta^\sigma p_\omega^{1-\sigma} + (1-\eta)^\sigma} = \frac{1}{1 + \frac{(1-\eta)^\sigma}{\eta^\sigma p_\omega^{1-\sigma}}} \tag{B.9}$$

and thus exhibits the following properties:

$$\left\{\begin{array}{lcc}(\sigma<1) & \rightarrow & \lim_{p_\omega\to 0}\Upsilon\left(p_\omega\right)=0, \quad \lim_{p_\omega\to\infty}\Upsilon\left(p_\omega\right)=1, \\ (\sigma>1) & \rightarrow & \lim_{p_\omega\to 0}\Upsilon\left(p_\omega\right)=1, \quad \lim_{p_\omega\to\infty}\Upsilon\left(p_\omega\right)=0, \\ (\sigma=1) & \rightarrow & \Upsilon\left(p_\omega\right)=\eta.\end{array}\right\} \tag{B.10}$$

Recalling the definition $\ell = L/\Omega$, rewrite (26) and (27) as

$$y_1\left(p_\omega;\ell\right) = \frac{1+\left(p_\omega/\ell\right)}{1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}} \tag{B.11}$$

$$y_2\left(p_\omega\right) = \frac{1}{1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}\left[\tau+(1-\tau)\cdot\Upsilon\left(p_\omega\right)\right]} \tag{B.12}$$

where (B.12) is obtained by plugging (27) in (26) to eliminate p/ℓ and solving for y. In (B.11) and (B.12), we have defined $y_1\left(p_\omega;\ell\right)$ and $y_2\left(p_\omega\right)$ as functions that treat p_ω as the explicit argument and ℓ as a parameter. The fixed point

$$\left(y^*\left(\ell\right), p_\omega^*\left(\ell\right)\right) = \arg\operatorname{solve}\left\{y_1\left(p_\omega;\ell\right)=y_2\left(p_\omega\right)\right\} \tag{B.13}$$

characterizes the intratemporal equilibrium of the economy. The proof of Proposition 1 involves two steps. First, we prove existence and uniqueness of the equilibrium. Second, we assess the marginal effects of variations in ℓ.

Step #1. System (B.11)-(B.12) can be represented graphically in the (p_ω, y) plane: given ℓ, function $y_1\left(p_\omega;\ell\right)$ is a linear increasing function of p_ω, whereas the behavior of $y_2\left(p_\omega\right)$ depends on the value of σ. From (B.9), $y_2\left(p_\omega\right)$ is decreasing and convex for $\sigma > 1$; a flat horizontal line for $\sigma = 1$; increasing and concave for $\sigma < 1$. The three cases are described in Figure A1. The vertical intercepts and horizontal asymptotes of $y_2\left(p_\omega\right)$ are defined in (B.10) and (B.12). In all cases, the intersection $y_1\left(p_\omega;\ell\right) = y_2\left(p_\omega\right)$ is unique and determines the conditional values $y^*\left(\ell\right)$ and $p_\omega^*\left(\ell\right)$. In particular, $y^*\left(\ell\right)$ exhibits the property

$$y_{\min} \equiv \frac{1}{1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}} < y^*\left(\ell\right) < \frac{1}{1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}} \equiv y_{\max}. \tag{B.14}$$

Step #2. The marginal effects of ℓ can be studied by means of Figure A1. In all cases, an increase in ℓ reduces the slope of $y_1\left(p_\omega;\ell\right)$ leaving $y_2\left(p_\omega\right)$ unchanged, so that the results

$$\frac{\mathrm{d}p_\omega^*\left(\ell\right)}{\mathrm{d}\ell} > 0, \quad \lim_{\ell\to 0^+} p_\omega^*\left(\ell\right) = 0, \quad \lim_{\ell\to\infty} p_\omega^*\left(\ell\right) = \infty, \tag{B.15}$$

hold independently of the elasticity of substitution. With respect to $y^*\left(\ell\right)$, we have

$$(\sigma < 1) \rightarrow \frac{\mathrm{d}y^*\left(\ell\right)}{\mathrm{d}\ell} > 0, \quad \lim_{\ell\to 0^+} y^*\left(\ell\right) = y_{\min}, \quad \lim_{\ell\to\infty} y^*\left(\ell\right) = y_{\max}, \tag{B.16}$$

$$(\sigma > 1) \rightarrow \frac{\mathrm{d}y^*\left(\ell\right)}{\mathrm{d}\ell} < 0, \quad \lim_{\ell\to 0^+} y^*\left(\ell\right) = y_{\max}, \quad \lim_{\ell\to\infty} y^*\left(\ell\right) = y_{\min}, \tag{B.17}$$

$$(\sigma = 1) \rightarrow \frac{\mathrm{d}y^*\left(\ell\right)}{\mathrm{d}\ell} = 0, \quad y = \frac{1}{1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}\left[\tau+(1-\tau)\eta\right]}. \tag{B.18}$$

Finally, the equilibrium commodity price is a function of ℓ via the zero-profit condition in the primary sector: from $\Theta(w, p_\omega) \equiv \eta^\sigma p_\omega^{1-\sigma} + (1-\eta)^\sigma w^{1-\sigma}$ and $p_q = \frac{1}{1-\tau}\Theta(w, p_\omega)$ with $w = 1$, we have

$$p_q^*(\ell) \equiv \frac{1}{1-\tau} \cdot \left[\frac{\eta^\sigma}{p_\omega^*(\ell)^{\sigma-1}} + (1-\eta)^\sigma\right]. \tag{B.19}$$

Since $p_\omega^*(\ell)/\mathrm{d}\ell > 0$, substitutability $\sigma > 1$ implies $p_q^*(\ell)/\mathrm{d}\ell < 0$; complementarity $\sigma < 1$ implies $p_q^*(\ell)/\mathrm{d}\ell > 0$; the Cobb-Douglas case $\sigma = 1$ implies $p_q^*(\ell)/\mathrm{d}\ell = 0$. ■

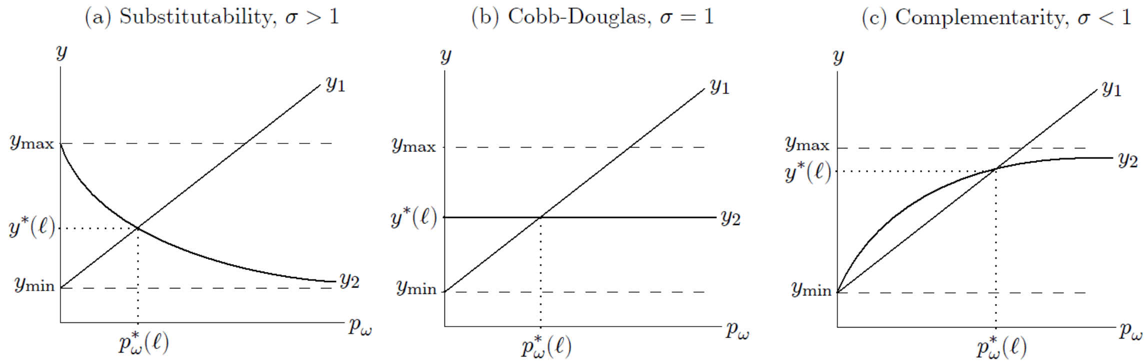

Figure A1. Determination of the equilibrium couple $(y^*(\ell), p_\omega^*(\ell))$ in the proof of Proposition 1.

B.3 The Equilibrium Mortality Rate

Proof of Proposition 2. First, consider the general case $\sigma \gtrless 1$. Denoting commodity output per adult by $q \equiv Q/L$, we can rewrite (28) as

$$m(t) = \bar{m} + \mu\xi^\chi \cdot L(t)^{\chi-(1-\varphi)} q(t)^\chi = \bar{m} + \mu\xi^\chi \cdot L(t)^{\chi-(1-\varphi)} \cdot \ell(t)^{-\chi} (q(t)\ell(t))^\chi. \tag{B.20}$$

Considering the primary sector, the equilibrium zero-profit condition (A.24) and the resource demand schedule of commodity producers (25) can be respectively rewritten as

$$\frac{p_\omega(t)}{p_q(t)} = \Upsilon(t) \cdot q(t)\ell(t)(1-\tau), \tag{B.21}$$

$$\frac{p_\omega(t)}{p_q(t)} = \eta(1-\tau) \cdot (q(t)\ell(t))^{\frac{1}{\sigma}}. \tag{B.22}$$

Combining the above equations to eliminate p_ω/p_q and solving for $q\ell$, we obtain

$$q(t)\ell(t) = \left(\frac{\eta}{\Upsilon(t)}\right)^{\frac{\sigma}{\sigma-1}}. \tag{B.23}$$

Substituting (B.23) into (B.20), the mortality rate becomes

$$m(t) = \bar{m} + \mu\xi^\chi\eta^{\chi\frac{\sigma}{\sigma-1}} \cdot L(t)^{\chi-(1-\varphi)} \cdot \ell(t)^{-\chi} \cdot \Upsilon(\ell(t))^{\frac{\sigma}{1-\sigma}\chi}, \tag{B.24}$$

where the we have substituted $\Upsilon(\ell)$ with the cost share of resource use evaluated in equilibrium, obtained from substituting the equilibrium resource price $p_\omega = p_\omega^*(\ell)$ defined in Proposition 1 inside the expression for the cost share $\Upsilon(p_\omega)$ defined by the last term in equation (20) with normalized wage $w = 1$. Substituting $L(t) = \ell(t)\Omega$ to eliminate population in (B.24), we get

$$m(t) = \bar{m} + \mu\xi^\chi \eta^{\chi\frac{\sigma}{\sigma-1}} \Omega^{\chi-(1-\varphi)} \cdot \ell(t)^{-(1-\varphi)} \cdot \Upsilon(\ell(t))^{\frac{\sigma}{1-\sigma}\chi}, \tag{B.25}$$

which, after defining the convenient constant $\bar{\mu} \equiv \mu\xi^\chi \eta^{\chi\frac{\sigma}{\sigma-1}} \Omega^{\chi-(1-\varphi)} > 0$, reduces to equation (33). Considering the asymptotic behavior of $\Upsilon(\ell)$, we combine results (B.10) with results (B.15) to obtain

$$\begin{cases} \sigma > 1 \ \rightarrow \ \frac{\mathrm{d}\Upsilon(\ell)}{\mathrm{d}\ell} < 0, \quad \lim_{\ell\to\infty} \Upsilon(\ell) = 0, \quad \lim_{\ell\to 0^+} \Upsilon(\ell) = 1, \\ \sigma < 1 \ \rightarrow \ \frac{\mathrm{d}\Upsilon(\ell)}{\mathrm{d}\ell} > 0, \quad \lim_{\ell\to\infty} \Upsilon(\ell) = 1, \quad \lim_{\ell\to 0^+} \Upsilon(\ell) = 0, \end{cases} \tag{B.26}$$

which proves expression (34). Next, consider the Cobb-Douglas case. Profit maximization in the primary sector implies the factor income shares $p_\omega \Omega = \eta \cdot p_q(1-\tau)Q$ and $wL_Q = (1-\eta) \cdot p_q(1-\tau)Q$. Normalizing the wage rate $w = 1$, we can combine these conditions to write

$$p_\omega(t)/\ell(t) = \frac{\eta}{1-\eta} \cdot \frac{L_Q(t)}{L(t)}. \tag{B.27}$$

From Proposition 1, rents per adult p_ω/ℓ are independent of ℓ in the Cobb-Douglas case. Therefore, the employment share of the primary sector $\frac{L_Q(t)}{L(t)}$ is independent of ℓ and, given the constant tax rate τ, we can define the convenient constant

$$\left(\frac{L_Q(t)}{L(t)}\right)^{1-\eta} = \left(\frac{1-\eta}{\eta} p_\omega/\ell\right)^{1-\eta} \equiv \vartheta. \tag{B.28}$$

From the primary sector's technology $Q = \Omega^\eta L_Q^{1-\eta}$, we can use (B.28) to rewrite equilibrium commodity output per adult as

$$q(t) = \ell(t)^{-\eta} \bar{\vartheta}. \tag{B.29}$$

Substituting this result into (28), the equilibrium mortality rate can be written as

$$m(t) = \bar{m} + \mu\xi^\chi \bar{\vartheta}^\chi \Omega^{\chi-(1-\varphi)} \cdot \ell(t)^{\chi(1-\eta)-(1-\varphi)}, \tag{B.30}$$

which, after defining the convenient constant $\tilde{\mu} \equiv \mu\xi^\chi \bar{\vartheta}^\chi \Omega^{\chi-(1-\varphi)} > 0$, reduces to equation (32). ■

Proof of Lemma 3. Under substitutability, result $\lim_{\ell\to 0^+} \Upsilon(\ell) = 1$ in (34) implies that the equilibrium mortality rate (33) exhibits

$$\sigma > 1 \ \rightarrow \lim_{\ell\to 0^+} m^*(\ell) = \bar{m} + \bar{\mu} \cdot \lim_{\ell\to 0^+} \Upsilon(\ell)^{\frac{\sigma}{1-\sigma}\chi} \cdot \ell^{-(1-\varphi)} = \bar{m} + \bar{\mu} \cdot \lim_{\ell\to 0^+} \frac{1}{\ell^{1-\varphi}}, \tag{B.31}$$

so that, for any $0 < \varphi \leqslant 1$, substitutability implies $\lim_{\ell\to 0^+} m(\ell) = +\infty$. ■

Derivation of result (35) and extension to the case of complementarity. For future reference, rewrite equation (29) as

$$\varepsilon_E \equiv \left(\frac{\partial \mathcal{F}}{\partial L_Q} \cdot \frac{L_Q}{\mathcal{F}}\right) \cdot \left(\frac{\mathrm{d}L_Q}{\mathrm{d}L} \cdot \frac{L}{L_Q}\right) = \varepsilon_{Q,L_Q} \cdot \varepsilon_{L_Q,L} \tag{B.32}$$

where both the sub-elasticities ε_{Q,L_Q} and $\varepsilon_{L_Q,L}$ are evaluated in equilibrium. Note that, from (A.27), the ratio between primary inputs can be written as

$$\left(\frac{\Omega}{L_Q}\right)^{\frac{\sigma-1}{\sigma}} = \frac{1-\eta}{\eta} \cdot \frac{\eta^\sigma p_\omega^{1-\sigma}}{(1-\eta)^\sigma w^{1-\sigma}},$$

which can be solved for L_Q with $w = 1$ as

$$L_Q = \Omega \cdot \left(\frac{1-\eta}{\eta}\right)^\sigma \cdot p_\omega^\sigma. \tag{B.33}$$

From the technology (18), the elasticity of commodity output to primary employment reads

$$\varepsilon_{Q,L_Q} \equiv \frac{\partial \mathcal{F}}{\partial L_Q} \cdot \frac{L_Q}{\mathcal{F}} = \frac{(1-\eta) L_Q^{\frac{\sigma-1}{\sigma}}}{\eta \Omega^{\frac{\sigma-1}{\sigma}} + (1-\eta) L_Q^{\frac{\sigma-1}{\sigma}}},$$

where we can substitute L_Q by (B.33) and the equilibrium price $p_\omega = p_\omega^* (\ell)$, obtaining

$$\varepsilon_{Q,L_Q} = \frac{(1-\eta)^\sigma \eta^{1-\sigma} \cdot p_\omega^{\sigma-1}}{\eta + (1-\eta)^\sigma \eta^{1-\sigma} \cdot p_\omega^{\sigma-1}} = \frac{1}{\frac{\eta}{(1-\eta)^\sigma \eta^{1-\sigma} \cdot (p_\omega^*)^{\sigma-1}} + 1}. \tag{B.34}$$

From (B.34), and recalling the asymptotic properties of $p_\omega^* (\ell)$ established in (B.15), the equilibrium elasticity ε_{Q,L_Q} exhibits

$$\begin{array}{lllll} \sigma > 1 & \rightarrow & \frac{\mathrm{d}\varepsilon_{Q,L_Q}}{\mathrm{d}\ell} > 0, & \lim_{\ell \to 0^+} \varepsilon_{Q,L_Q} = 0, & \lim_{\ell \to \infty} \varepsilon_{Q,L_Q} = 1, \\ \sigma < 1 & \rightarrow & \frac{\mathrm{d}\varepsilon_{Q,L_Q}}{\mathrm{d}\ell} < 0, & \lim_{\ell \to 0^+} \varepsilon_{Q,L_Q} = 1, & \lim_{\ell \to \infty} \varepsilon_{Q,L_Q} = 0. \end{array} \tag{B.35}$$

Next, consider the following definitions

$$\varepsilon_{\Upsilon,p_\omega^*} \equiv \frac{\partial \Upsilon (p_\omega^*)}{\partial p_\omega^*} \cdot \frac{p_\omega^*}{\Upsilon}, \quad \varepsilon_{p_\omega^*,\ell} \equiv \frac{\partial p_\omega^* (\ell)}{\partial \ell} \cdot \frac{\ell}{p_\omega^*}, \tag{B.36}$$

where $p_\omega^* (\ell)$ is the equilibrium resource price defined in Proposition 1 and $\Upsilon (p_\omega^*)$ is the resource cost share (20) evaluated in the equilibrium with $w = 1$. Focusing on $\Upsilon (p_\omega^*)$, from (20) we can calculate

$$\varepsilon_{\Upsilon,p_\omega^*} = \frac{(1-\sigma)(1-\eta)^\sigma}{\eta^\sigma (p_\omega^*)^{1-\sigma} + (1-\eta)^\sigma}. \tag{B.37}$$

Next, log-differentiating the static equilibrium conditions (26) and (27) evaluated in equilibrium, we have

$$\varepsilon_{y^*,\ell} = \left(\varepsilon_{p_\omega^*,\ell} - 1\right) \frac{p_\omega^*/\ell}{1 + p_\omega^*/\ell} \quad \text{and} \quad \varepsilon_{p_\omega^*,\ell} - 1 = \varepsilon_{\Upsilon,p_\omega^*} \cdot \varepsilon_{p_\omega^*,\ell} + \varepsilon_{y^*,\ell}, \tag{B.38}$$

where $\varepsilon_{y^*,\ell} \equiv \frac{\partial y^*(\ell)}{\partial \ell} \cdot \frac{\ell}{y^*}$. Combining the two conditions in (B.38) to eliminate $\varepsilon_{y^*,\ell}$ and solving for $\varepsilon_{p_\omega^*,\ell}$, we obtain

$$\varepsilon_{p_\omega^*,\ell} = \frac{1}{1 - \varepsilon_{\Upsilon,p_\omega^*} \left(1 + p_\omega^*/\ell\right)}. \tag{B.39}$$

Using (B.37) to substitute $\varepsilon_{\Upsilon,p_\omega^*}$ in (B.39), we have

$$\varepsilon_{p_\omega^*,\ell} = \frac{1}{1 - \frac{(1-\sigma)(1-\eta)^\sigma (1+p_\omega^*/\ell)}{\eta^\sigma (p_\omega^*)^{1-\sigma} + (1-\eta)^\sigma}} > 0. \tag{B.40}$$

We now have all the elements to characterize the response of primary employment to variations in total labor supply. Time-differentiation of (B.33) yields

$$\frac{\dot{L}_Q(t)}{L_Q(t)} = \sigma \cdot \frac{\dot{p}_\omega(t)}{p_\omega(t)} = \sigma \cdot \varepsilon_{p_\omega^*,\ell} \cdot \frac{\dot{\ell}(t)}{\ell(t)}, \tag{B.41}$$

where the last term follows from substituting the equilibrium price $p_\omega = p_\omega^*(\ell)$. Since $\ell = L/\Omega$, we can rewrite (B.41) as

$$\frac{\dot{L}_Q(t)}{L_Q(t)} = \underbrace{\sigma \varepsilon_{p_\omega^*,\ell}}_{\varepsilon_{L_Q,L}} \cdot \frac{\dot{L}(t)}{L(t)} = \varepsilon_{L_Q,L} \cdot \frac{\dot{L}(t)}{L(t)} \tag{B.42}$$

where $\varepsilon_{L_Q,L}$ is the elasticity of primary employment to total labor supply in equilibrium. Using result (B.40) to substitute $\varepsilon_{p_\omega^*,\ell}$, we obtain

$$\varepsilon_{L_Q,L} = \frac{\sigma}{1 - \frac{(1-\sigma)(1-\eta)^\sigma (1+p_\omega^*/\ell)}{\eta^\sigma (p_\omega^*)^{1-\sigma} + (1-\eta)^\sigma}} > 0. \tag{B.43}$$

The asymptotic behavior of $\varepsilon_{L_Q,L}$ when ℓ ranges from 0 to $+\infty$ is determined by the asymptotic behavior of the resource price p_ω^* and rents per adult p_ω^*/ℓ. The behavior of p_ω^* is already described in expression (B.15). The behavior of p_ω^*/ℓ can be tracked as follows. Evaluating condition (27) in equilibrium, rents per adult equal

$$p_\omega^*(\ell)/\ell = (1-\tau)\gamma \frac{\epsilon - 1}{\epsilon} \cdot \Upsilon(\ell) \cdot y^*(\ell). \tag{B.44}$$

The asymptotic properties of equilibrium expenditure $y^*(\ell)$ are already described in (B.16)-(B.17), and the asymptotic properties of the equilibrium resource cost share $\Upsilon(\ell)$ are already described in (B.26). Therefore, (B.44) implies

$$\begin{array}{llll} \sigma > 1 \;\rightarrow & \frac{\mathrm{d}p_\omega^*/\ell}{\mathrm{d}\ell} < 0, & \lim_{\ell \to 0^+} p_\omega^*/\ell = (1-\tau)\gamma\frac{\epsilon-1}{\epsilon} \cdot y_{\max}, & \lim_{\ell \to \infty} p_\omega^*/\ell = 0. \\ \sigma < 1 \;\rightarrow & \frac{\mathrm{d}p_\omega^*/\ell}{\mathrm{d}\ell} > 0, & \lim_{\ell \to 0^+} p_\omega^*/\ell = 0, & \lim_{\ell \to \infty} p_\omega^*/\ell = (1-\tau)\gamma\frac{\epsilon-1}{\epsilon} \cdot y_{\max}. \end{array} \tag{B.45}$$

Using (B.45), we can establish the following results about the elasticity $\varepsilon_{L_Q,L}$ derived in (B.43).

$$\begin{array}{lll} \sigma > 1 \;\rightarrow & \lim_{\ell \to 0^+} \varepsilon_{L_Q,L} = \sigma, & \lim_{\ell \to \infty} \varepsilon_{L_Q,L} = 1, \\ \sigma < 1 \;\rightarrow & \lim_{\ell \to 0^+} \varepsilon_{L_Q,L} = 1 & \lim_{\ell \to \infty} \varepsilon_{L_Q,L} = \sigma, \end{array} \tag{B.46}$$

Going back to definition of ε_E in (B.32), we can thus substitute results (B.34) and (B.43) to obtain

$$\varepsilon_E = \varepsilon_{Q,L_Q} \cdot \varepsilon_{L_Q,L} = \frac{(1-\eta)^{\sigma} \eta^{1-\sigma} \cdot p_{\omega}^{\sigma-1}}{\eta + (1-\eta)^{\sigma} \eta^{1-\sigma} \cdot p_{\omega}^{\sigma-1}} \cdot \frac{\sigma}{1 - \frac{(1-\sigma)(1-\eta)^{\sigma}(1+p_{\omega}^{*}/\ell)}{\eta^{\sigma}(p_{\omega}^{*})^{1-\sigma}+(1-\eta)^{\sigma}}}, \tag{B.47}$$

which, from (B.35) and (B.46), exhibits the properties

$$\begin{array}{lll} \sigma > 1 \ \rightarrow & \lim_{\ell \to 0^+} \varepsilon_E = 0, & \lim_{\ell \to \infty} \varepsilon_E = 1, \\ \sigma < 1 \ \rightarrow & \lim_{\ell \to 0^+} \varepsilon_E = 1, & \lim_{\ell \to \infty} \varepsilon_E = 0. \end{array} \tag{B.48}$$

Expression (B.48) proves result (35) in the main text for the case of substitutability.

Behavior of the equilibrium mortality rate (comprehensive proof of the diagrams in Figure 1). From (30), we can define the elasticity of the excess mortality rate to population size as

$$\varepsilon_{m,L} \equiv \frac{\mathrm{d}(m-\bar{m})}{\mathrm{d}L} \cdot \frac{L}{(m-\bar{m})} = \chi \varepsilon_E - (1-\varphi), \tag{B.49}$$

Combining results (B.48) with expression (B.49), we have

$$\begin{array}{lll} \sigma > 1 \ \rightarrow & \lim_{\ell \to 0^+} \varepsilon_{m,L} = -(1-\varphi), & \lim_{\ell \to \infty} \varepsilon_{m,L} = \chi - (1-\varphi), \\ \sigma < 1 \ \rightarrow & \lim_{\ell \to 0^+} \varepsilon_{m,L} = \chi - (1-\varphi), & \lim_{\ell \to \infty} \varepsilon_{m,L} = -(1-\varphi). \end{array} \tag{B.50}$$

Result (B.50) provide a comprehensive proof of the behavior of the equilibrium mortality rate in all the subcases depicted in Figure 1 for $\sigma \neq 1$ (the Cobb-Douglas case is already discussed in the main text). First, consider all the sub-cases with $\sigma > 1$. Under substitutability, the limit for $\ell \to 0^+$ is strictly negative for any $0 < \varphi \leqslant 1$, so that $m^*(\ell)$ is surely decreasing in ℓ for low values of ℓ. The limit for $\ell \to \infty$ shows that $m^*(\ell)$ remains declining for $\chi \leqslant (1-\varphi)$ – that is, a monotonically declining 'L-shaped' function – and bends upward, instead, for $\chi > (1-\varphi)$ – that is, a non-monotonic 'U-shaped' function. In the special case $\varphi = 1$, we have an increasing convex function since $\lim_{\ell \to 0^+} \varepsilon_{m,L} = 0$ and $\lim_{\ell \to \infty} \varepsilon_{m,L} = \chi > 0$. Second, consider all the sub-cases with $\sigma < 1$. Under complementarity, the limit for $\ell \to \infty$ is strictly negative for any $0 < \varphi \leqslant 1$, so that $m^*(\ell)$ is surely decreasing in ℓ for high values of ℓ. The limit for $\ell \to 0^+$ shows that $m^*(\ell)$ is declining for $\chi \leqslant (1-\varphi)$ – that is, a monotonically declining 'L-shaped' function – and becomes increasing, instead, for $\chi > (1-\varphi)$ – that is, a non-monotonic 'hump-shaped' function. In the special case $\varphi = 1$, we have an increasing concave function since $\lim_{\ell \to 0^+} \varepsilon_{m,L} = \chi > 0$ and $\lim_{\ell \to \infty} \varepsilon_{m,L} = 0$.

C Appendix: Population Dynamics

C.1 Special Case with Exogenous Mortality

Dynamics with exogenous mortality (including strict complementarity). Setting $m(t) = \bar{m}$ in each $t \in [0, \infty)$, the dynamic system (37)-(38) becomes

$$\frac{\dot{\ell}(t)}{\ell(t)} = b(t) - \bar{m} \tag{C.1}$$

$$\frac{\dot{b}(t)}{b(t)} = \frac{b(t)}{(1-\alpha)(1-\psi)}\left[\frac{1-(1-\psi)\,y^*(\ell(t))}{y^*(\ell(t))}\right] - \rho \tag{C.2}$$

and the stationary loci read

$$\dot{\ell} = 0 \rightarrow \Lambda^{\ell} \equiv b = \bar{m} \tag{C.3}$$

$$\dot{b} = 0 \rightarrow \Lambda^{b}(\ell) \equiv b = \frac{\rho(1-\alpha)(1-\psi)\,y^*(\ell)}{1-(1-\psi)\,y^*(\ell)}. \tag{C.4}$$

From the definition of $\Lambda^b(\ell)$ in (C.4) we can rewrite (C.2) as $\dot{b} = \rho\frac{b^2}{\Lambda^b(\ell)} - \rho b$. Therefore, system (C.1)-(C.2) exhibits the coefficient matrix

$$\Xi \equiv \begin{pmatrix} \frac{\partial \dot{\ell}}{\partial \ell} = b - \bar{m} & \frac{\partial \dot{\ell}}{\partial b} = \ell \\ & \\ \frac{\partial \dot{b}}{\partial \ell} = -\frac{\rho b^2}{\Lambda^b(\ell)^2}\frac{\partial \Lambda^b(\ell)}{\partial \ell} & \frac{\partial \dot{b}}{\partial b} = 2\rho\frac{b}{\Lambda^b(\ell)} - \rho \end{pmatrix} \tag{C.5}$$

which can be evaluated in any generic simultaneous steady state $\dot{\ell} = \dot{b} = 0$ as

$$\Xi_{ss} \equiv \begin{pmatrix} \frac{\partial \dot{\ell}}{\partial \ell} = 0 & \frac{\partial \dot{\ell}}{\partial b} = \ell_{ss} \\ & \\ \frac{\partial \dot{b}}{\partial \ell} = -\rho\frac{\partial \Lambda^b(\ell)}{\partial \ell} & \frac{\partial \dot{b}}{\partial b} = \rho \end{pmatrix} \tag{C.6}$$

The determinant of (C.6) is given by

$$|\Xi_{ss}| = \rho \ell_{ss}\frac{\partial \Lambda^b(\ell)}{\partial \ell} \tag{C.7}$$

and the eigenvalues $(\varkappa_1, \varkappa_2)$ of (C.6) are determined by the second-order equation

$$\varkappa^2 - \rho\varkappa + \rho\ell_{ss}\frac{\partial \Lambda^b(\ell)}{\partial \ell} = 0 \tag{C.8}$$

The three possible cases (Cobb-Douglas, substitutability, complementarity) are discussed below.

Cobb-Douglas case. From (B.18), setting $\sigma = 1$ implies a constant expenditure level

$$y(t) = \frac{1}{1 - \beta\rho - \gamma\frac{\epsilon-1}{\epsilon}\left[\tau + (1-\tau)\eta\right]} \equiv \bar{y}. \tag{C.9}$$

From (C.9), the stationary locus $\dot{b}=0$ in (C.4) becomes

$$\dot{b}=0 \rightarrow \Lambda^{b}(\ell) \equiv b=\frac{\rho(1-\alpha)(1-\psi) \bar{y}}{1-(1-\psi) \bar{y}} \tag{C.10}$$

which is independent of the population-resource ratio ℓ. Therefore, as shown in phase diagram (a) of Figure 2, the two loci are horizontal straight lines. In order to satisfy all the utility-maximizing conditions, the fertility rate must jump onto the $\dot{b}=0$ at time zero, which implies a constant gross fertility rate forever. When the parameters satisfy

$$\frac{\rho(1-\alpha)(1-\psi) \bar{y}}{1-(1-\psi) \bar{y}}>\bar{m} \tag{C.11}$$

the gross fertility rate exceeds the mortality rate, $\Lambda^{b}(\ell) \equiv b>\bar{m}$, in which case the economy displays positive population growth. This is the case depicted in Figure 2, graph (a). In the long run, the economy converges asymptotically to zero resources per capita and an infinite population.

Substitutability. By Proposition 1, setting $\sigma>1$ implies that $y^{*}(\ell)$ is strictly decreasing in ℓ. Therefore, the stationary locus $\dot{b}=0$ in (C.4) is also decreasing in ℓ. In particular, combining (C.4) with results (B.17), we have

$$\frac{\partial \Lambda^{b}(\ell)}{\partial \ell}<0, \quad \lim_{\ell \rightarrow 0} \Lambda^{b}(\ell)=\frac{\rho(1-\alpha)(1-\psi) y_{\max}}{1-(1-\psi) y_{\max}}, \quad \lim_{\ell \rightarrow \infty} \Lambda^{b}(\ell)=\frac{\rho(1-\alpha)(1-\psi) y_{\min}}{1-(1-\psi) y_{\min}}, \tag{C.12}$$

where $y_{\min} \equiv 1/\left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right)$ and $y_{\max} \equiv 1/\left(1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}\right)$ from (B.14). Since the $\dot{\ell}=0$ locus is a horizontal straight line, $\Lambda^{\ell} \equiv b=\bar{m}$, result (C.12) allows us to define suitable parameter restrictions such that there exists a simultaneous steady state (b_{ss}, ℓ_{ss}) in which $b=\bar{m}$ and $\ell_{ss}>0$. The fact that such steady state (b_{ss}, ℓ_{ss}) is saddle-point stable is proved as follows. From $\partial \Lambda^{b}(\ell)/\partial \ell<0$ in (C.12), we have $\rho^{2}-4\rho\ell_{ss}\frac{\partial \Lambda^{b}(\ell)}{\partial \ell}>0$ and this implies that both the eigenvalues $(\varkappa_{1}, \varkappa_{2})$ solving (C.8) are real. Moreover, the fact that $\sqrt{\rho^{2}-4\rho\ell_{ss}\frac{\partial \Lambda^{b}(\ell)}{\partial \ell}}>\rho$ guarantees that $(\varkappa_{1}, \varkappa_{2})$ have opposite sign. The direction of the arrows shown in phase diagram (e) of Figure 2 is determined by the signs of the coefficients in matrix (C.6). Therefore, under substitutability, the steady state (b_{ss}, ℓ_{ss}) is a global attractor of the dynamics. If the economy has initial endowments such that $\ell(0)>\ell_{ss}$, the economy jumps on the branch of the stable arm featuring negative population growth. Instead, if the economy has initial endowments such that $\ell(0)<\ell_{ss}$, the economy jumps on the opposite branch of the stable arm featuring positive population growth. In either case, the economy approaches asymptotically a finite endogenous level of population $L_{ss}=\ell_{ss}\Omega$ and constant resources per capita in the long run.

Complementarity, $\sigma<1$. By Proposition 1, setting $\sigma<1$ implies that $y^{*}(\ell)$ is strictly increasing in ℓ. Therefore, the stationary locus $\dot{b}=0$ in (C.4) is also increasing in ℓ. In particular, combining (C.4) with results (B.16), we have

$$\frac{\partial \Lambda^{b}(\ell)}{\partial \ell}>0, \quad \lim_{\ell \rightarrow 0} \Lambda^{b}(\ell)=\frac{\rho(1-\alpha)(1-\psi) y_{\min}}{1-(1-\psi) y_{\min}}, \quad \lim_{\ell \rightarrow \infty} \Lambda^{b}(\ell)=\frac{\rho(1-\alpha)(1-\psi) y_{\max}}{1-(1-\psi) y_{\max}}, \tag{C.13}$$

where $y_{\min} \equiv 1/\left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right)$ and $y_{\max} \equiv 1/\left(1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}\right)$ from (B.14). Since the $\dot{\ell}=0$ locus is a horizontal straight line, $\Lambda^{\ell} \equiv b=\bar{m}$, result (C.13) allows us to define suitable parameter restrictions such that there exists a unique simultaneous steady state (b_{ss},ℓ_{ss}) in which $b=\bar{m}$ and $\ell_{ss}>0$. The fact that such steady state (b_{ss},ℓ_{ss}) is globally unstable is proved as follows. From $\partial\Lambda^{b}(\ell)/\partial\ell>0$ in (C.13), the determinant (C.7) is strictly positive, and $|\Xi_{ss}|>0$ implies that the eigenvalues $(\varkappa_1,\varkappa_2)$ are both real and have the same sign. From (C.8), the polynomial exhibits the signs $(+,-,+)$, which implies by Descartes rule that no root can be negative. Hence, both $(\varkappa_1,\varkappa_2)$ must be strictly positive. Therefore the simultaneous steady state (b_{ss},ℓ_{ss}) under complementarity is globally unstable. Since this case is not discussed in the main text, we report the associated phase diagram in Figure A2, graph (a). The direction of the arrows is determined by the signs of the coefficients in matrix (C.6). If the economy has initial endowments such that $\ell(0)>\ell_{ss}$, the economy jumps on the diverging path featuring positive population growth and increasing population-resource ratio, and approaches asymptotically an infinite population and zero resources per capita. Instead, if the economy has initial endowments such that $\ell(0)<\ell_{ss}$, the economy jumps on the diverging path featuring declining population, and ultimately implosion in the long run. The cause of the instability is that expenditure per capita increases with population because the rising resource scarcity yields higher resource income per capita. When labor is initially relatively scarce, the rising resource *abundance* drives down resource income per capita inducing further reductions in fertility and, eventually, population implosion. When labor is initially abundant relative to the resource base, a positive initial net fertility rate triggers a self-reinforcing circle of rising incomes and rising population via sustained fertility rates.

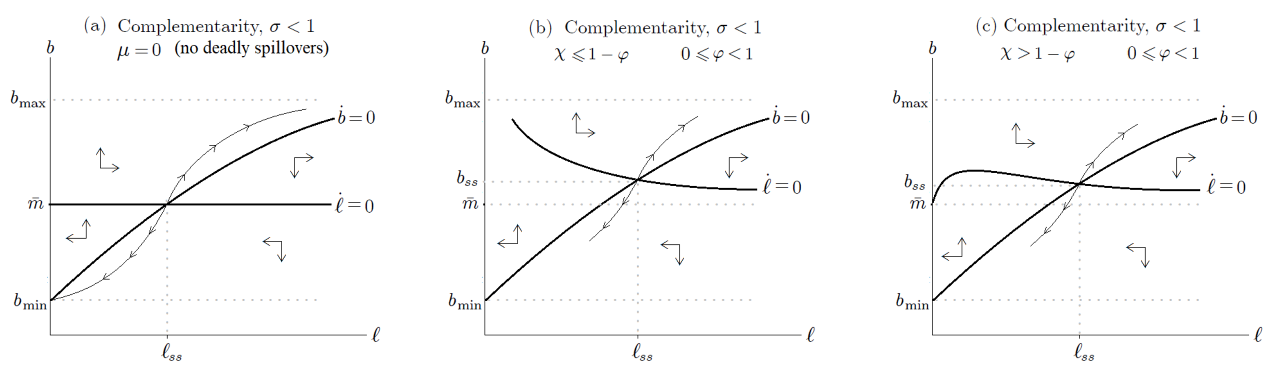

Figure A2. Demographic dynamics with and without deadly spillovers under complementarity.

C.2 Dynamics with Endogenous Mortality

Dynamics with endogenous mortality (including strict complementarity). The dynamic system (37)-(38) with endogenous mortality reads

$$\begin{aligned}\frac{\dot{\ell}(t)}{\ell(t)} &= b(t) - m^*(\ell(t)) \\ \frac{\dot{b}(t)}{b(t)} &= \frac{b(t)}{(1-\alpha)(1-\psi)}\left[\frac{1-(1-\psi)\,y^*(\ell(t))}{y^*(\ell(t))}\right] - \rho\end{aligned}$$

and the stationary loci read

$$\dot{\ell} = 0 \to \Lambda^{\ell}(\ell) \equiv b = m^*(\ell) \tag{C.14}$$

$$\dot{b} = 0 \to \Lambda^{b}(\ell) \equiv b = \frac{\rho(1-\alpha)(1-\psi)\,y^*(\ell)}{1-(1-\psi)\,y^*(\ell)}. \tag{C.15}$$

For future reference, note that the elasticity of the stationary locus (C.4) with respect to ℓ is

$$\frac{\partial\Lambda^{b}(\ell)}{\partial\ell}\frac{\ell}{\Lambda^{b}(\ell)} = \frac{1}{1-(1-\psi)\,y^*(\ell)}\cdot\frac{\partial y^*(\ell)}{\partial\ell}\frac{\ell}{y^*(\ell)} \tag{C.16}$$

From the definition of $\Lambda^{b}(\ell)$ in (C.15) we can rewrite (38) as $\dot{b} = \rho\frac{b^2}{\Lambda^{b}(\ell)} - \rho b$. Therefore, system (37)-(38) exhibits the coefficient matrix

$$\Xi \equiv \begin{pmatrix} \frac{\partial\dot{\ell}}{\partial\ell} = b - m^*(\ell) - \ell\cdot\frac{\partial m^*(\ell)}{\partial\ell} & \frac{\partial\dot{\ell}}{\partial b} = \ell \\ & \\ \frac{\partial\dot{b}}{\partial\ell} = -\frac{\rho b^2}{\Lambda^{b}(\ell)^2}\frac{\partial\Lambda^{b}(\ell)}{\partial\ell} & \frac{\partial\dot{b}}{\partial b} = 2\rho\frac{b}{\Lambda^{b}(\ell)} - \rho \end{pmatrix} \tag{C.17}$$

which can be evaluated in any generic simultaneous steady state $\dot{\ell} = \dot{b} = 0$ as

$$\Xi_{ss} \equiv \begin{pmatrix} \frac{\partial\dot{\ell}}{\partial\ell} = -\ell_{ss}\cdot\frac{\partial m^*(\ell)}{\partial\ell} & \frac{\partial\dot{\ell}}{\partial b} = \ell_{ss} \\ & \\ \frac{\partial\dot{b}}{\partial\ell} = -\rho\frac{\partial\Lambda^{b}(\ell)}{\partial\ell} & \frac{\partial\dot{b}}{\partial b} = \rho \end{pmatrix} \tag{C.18}$$

The determinant of (C.6) is given by

$$|\Xi_{ss}| = \rho\ell_{ss}\left[\frac{\partial\Lambda^{b}(\ell)}{\partial\ell} - \frac{\partial m^*(\ell)}{\partial\ell}\right] \tag{C.19}$$

and the eigenvalues $(\varkappa_1, \varkappa_2)$ of (C.6) are determined by the second-order equation

$$\varkappa^2 - \varkappa\left(\rho - \ell_{ss}\cdot\frac{\partial m^*(\ell)}{\partial\ell}\right) + \rho\ell_{ss}\left[\frac{\partial\Lambda^{b}(\ell)}{\partial\ell} - \frac{\partial m^*(\ell)}{\partial\ell}\right] = 0 \tag{C.20}$$

Note that expression (C.14) implies that in the (ℓ, b) plane, the $\dot{\ell} = 0$ locus exhibits the same shape as that of the equilbrium mortality rate $m^*(\ell)$ characterized in Proposition 2. Expression (C.15) is the same as that for the case of exogenous mortality (see (C.4) above) so that it satisfies all the properties previously derived. The cases with strict substitutability $\sigma > 1$ and Cobb-Douglas $\sigma = 1$ are discussed in the proofs of Propositions 6 and 5 below. The cases with strict substitutability $\sigma < 1$ are discussed further below.

Substitutability: dynamics with $\sigma > 1$ and proof of Proposition 6. Throughout this proof we assume $0 < \varphi \leqslant 1$. By Proposition 1, setting $\sigma > 1$ implies that $y^*(\ell)$ is strictly decreasing in ℓ. Therefore, the stationary locus $\dot{b} = 0$ in (C.15) is also decreasing in ℓ. In particular, combining (C.15) with results (B.17), we have

$$\frac{\partial \Lambda^b(\ell)}{\partial \ell} < 0, \quad \lim_{\ell \to 0} \Lambda^b(\ell) = \frac{\rho(1-\alpha)(1-\psi)\, y_{\max}}{1-(1-\psi)\, y_{\max}}, \quad \lim_{\ell \to \infty} \Lambda^b(\ell) = \frac{\rho(1-\alpha)(1-\psi)\, y_{\min}}{1-(1-\psi)\, y_{\min}}, \tag{C.21}$$

where $y_{\min} \equiv 1/\left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right)$ and $y_{\max} \equiv 1/\left(1-\beta\rho-\gamma\frac{\epsilon-1}{\epsilon}\right)$ from (B.14). Consider now the stationary locus $\dot{\ell} = 0$ in (C.14). Recalling Proposition 2 and results (36) and (35), substitutability implies

$$\lim_{\ell \to 0^+} \Lambda^\ell(\ell) = \bar{m} + \frac{\bar{\mu}}{\ell^{1-\varphi}} = +\infty \quad \text{with} \ \lim_{\ell \to 0^+} \varepsilon_{m,L} = -(1-\varphi) \ \text{and} \ \lim_{\ell \to \infty} \tilde{\varepsilon} = \varepsilon_{m,L} = \chi - (1-\varphi) \tag{C.22}$$

and the exact shape of the $\dot{\ell} = 0$ locus for high values of ℓ depends on the sign of $\chi - (1-\varphi)$. For $\chi \leqslant 1-\varphi$, the $\dot{\ell} = 0$ locus is monotonously decreasing and asymptotically horizontal,

$$\chi \leqslant 1 \to \frac{\mathrm{d}\Lambda^\ell(\ell)}{\mathrm{d}\ell} < 0 \text{ with } \lim_{\ell \to +\infty} \Lambda^\ell(\ell) = \bar{m} \tag{C.23}$$

whereas for $\chi > 1-\varphi$, the $\dot{\ell} = 0$ locus is *U-shaped*,

$$\chi > 1 \to \exists\, \hat{\ell} > 0 : \frac{\mathrm{d}\Lambda^\ell(\ell)}{\mathrm{d}\ell} \begin{cases} < 0 & \text{for} \quad 0 < \ell < \hat{\ell} \\ > 0 & \text{for} \quad \hat{\ell} < \ell < \infty \end{cases} < 0 \text{ and } \lim_{\ell \to +\infty} \Lambda^\ell(\ell) = \infty \tag{C.24}$$

Recalling the properties of the $\dot{b} = 0$ locus derived in (C.21), it follows that the existence of simultaneous steady states satisfying $\dot{b} = \dot{\ell} = 0$ falls into the following cases and subcases:

$\chi \leqslant 1-\varphi$ In this case, the combination of (C.21) and (C.23) implies that, provided the general existence condition $b_{\max} < \bar{m} < b_{\min}$ is satisfied, there certainly exist *two* simultaneous steady states $\dot{b} = \dot{\ell} = 0$ respectively charcterized by the labor-resource ratios ℓ'_{ss} and ℓ''_{ss} with $\ell'_{ss} > \ell''_{ss}$, as shown in Figure 2, phase diagram (f).

$\chi > 1-\varphi$ In this case, the combination of (C.21) and (C.24) implies that, provided the general existence condition $b_{\max} < \bar{m} < b_{\min}$ is satisfied, we can either have no steady state (that is, the $\dot{\ell} = 0$ locus is always strictly above the $\dot{b} = 0$ locus) or *two* simultaneous steady states charcterized by the labor-resource ratios ℓ'_{ss} and ℓ''_{ss} with $\ell'_{ss} > \ell''_{ss}$, as shown in Figure 2, phase diagram (g).[23] The case with no steady state arises when the spillover is extremely strong: in graphical terms, the $\dot{\ell} = 0$ locus shifts upwards so much that no intersection with the $\dot{b} = 0$ locus exists. But when spillovers are not that strong, the intersections between the two loci are two, as shown in Figure 2, diagram (g).

Assuming that two steady states (b'_{ss}, ℓ'_{ss}) and (b''_{ss}, ℓ''_{ss}) exist, their stability properties can be derived as follows. First, consider the steady state (b''_{ss}, ℓ''_{ss}) characterized by low labor-resource ratio. As shown in Figure 2, this is an intersection in which the $\dot{\ell} = 0$ locus cuts the $\dot{b} = 0$ locus from above while both loci are strictly declinling, that is,

$$\left(\frac{\partial \Lambda^b(\ell)}{\partial \ell} - \frac{\partial m^*(\ell)}{\partial \ell}\right)\bigg|_{\ell=\ell''_{ss}} > 0. \tag{C.25}$$

Result (C.25) implies that the determinant (C.19) evaluated in the steady state (b''_{ss}, ℓ''_{ss}) is strictly positive, $|\Xi''_{ss}| > 0$, and this implies that the eigenvalues $(\varkappa_1, \varkappa_2)$ are both real and have the same sign. Since $\frac{\partial m^*(\ell)}{\partial \ell}\Big|_{\ell=\ell''_{ss}} < 0$, the polynomial in (C.20) exhibits the signs $(+, -, +)$, which implies by Descartes rule that no root can be negative. Hence, both $(\varkappa_1, \varkappa_2)$ must be strictly positive. Therefore the steady state (b''_{ss}, ℓ''_{ss}) is an *unstable node* and acts as a "mortality trap": if the initial labor-resource ratio $\ell(0)$ is strictly below ℓ''_{ss}, the equilibrium path diverges to $\lim_{t\to\infty} \ell(t) = 0$ and thereby population implosion, $\lim_{t\to\infty} L(t) = 0$.

Next, consider the steady state (b'_{ss}, ℓ'_{ss}) characterized by low-fertility and high labor-resource ratio. As shown in Figure 2, this is an intersection in which the $\dot{\ell} = 0$ locus cuts the $\dot{b} = 0$ locus from below, that is,

$$\left(\frac{\partial \Lambda^b(\ell)}{\partial \ell} - \frac{\partial m^*(\ell)}{\partial \ell}\right)\bigg|_{\ell=\ell'_{ss}} < 0. \tag{C.26}$$

Result (C.26) implies real roots because the solution to (C.20) includes the positive term

$$\left(\rho - \ell_{ss} \cdot \frac{\partial m^*(\ell)}{\partial \ell}\right)^2 - 4\rho\ell_{ss}\left[\frac{\partial \Lambda^b(\ell)}{\partial \ell} - \frac{\partial m^*(\ell)}{\partial \ell}\right] > 0.$$

Also, result (C.26) implies that the determinant (C.19) evaluated in the steady state (b'_{ss}, ℓ'_{ss}) is strictly negative, $|\Xi'_{ss}| > 0$, so that the real roots have opposite sign. Therefore the steady state (b'_{ss}, ℓ'_{ss}) is *saddle-point stable* and acts as an attractor: if the initial labor-resource ratio $\ell(0)$ is strictly above ℓ''_{ss}, the equilibrium path converges to $\lim_{t\to\infty} \ell(t) = \ell'_{ss}$ along the stable arm of the saddle.

[23] The only potential exception would be the rather implausible case in which the $\dot{\ell} = 0$ locus is tangent to the $\dot{b} = 0$ locus from above, which we do not discuss for simplicity.

Cobb-Douglas: dynamics with $\sigma = 1$ and proof of Proposition 6. The properties of the stationary locus (C.14) under $\sigma = 1$ directly follow from result (32) in Proposition 2. On the one hand, From (C.10), the stationary locus $\dot{b} = 0$ is the horizontal straight line

$$\dot{b} = 0 \rightarrow \Lambda^b \equiv b = \frac{\rho (1-\alpha)(1-\psi)\bar{y}}{1-(1-\psi)\bar{y}}.$$

On the other hand, from (C.14) and (32), the stationary locus $\dot{\ell} = 0$ becomes

$$\dot{\ell} = 0 \rightarrow \Lambda^\ell (\ell) \equiv b = m^* (\ell) = \bar{m} + \tilde{\mu} \cdot \ell (t)^{\chi(1-\eta)-(1-\varphi)}.$$

Assuming that the general existence condition $\Lambda^b > \bar{m}$ is satisfied, we obtain the general cases described in diagrams (b)-(c)-(d) of Figure 2. If $\chi (1-\eta) < 1-\varphi$, the $\dot{\ell} = 0$ locus is decreasing: since

$$\frac{\partial \Lambda^b}{\partial \ell} - \frac{\partial m^* (\ell)}{\partial \ell} = -\frac{\partial m^* (\ell)}{\partial \ell} > 0, \tag{C.27}$$

the eigenvalues are both real and positive, the steady state is an *unstable node* and thus acts as a mortality trap generated by deadly spillovers. Instead, if $\chi (1-\eta) > 1-\varphi$, the $\dot{\ell} = 0$ locus is increasing: since

$$\frac{\partial \Lambda^b}{\partial \ell} - \frac{\partial m^* (\ell)}{\partial \ell} = -\frac{\partial m^* (\ell)}{\partial \ell} < 0 \tag{C.28}$$

the eigenvalues are both real and have opposite signs, the steady state is *saddle-point stable* and thus acts as a regular steady state generated by deadly spillovers. The knife-edge case $\chi (1-\eta) = 1-\varphi$ predicts exponential population growth or decline (including the possibility that population implodes exponentially with deadly spillovers though it would explode exponentially without spillovers).

Complementarity: dynamics with $\sigma < 1$. Under complementarity, deadly spillovers bear quantitative effects by modifying the position of the unstable steady state but do not yield qualitative effects: contrary to the case with substitutability, pollution externalities do not create additional steady states. The intuition for this result can be easily verified in Figure A2, where phase diagram (a) refers to the model without deadly spillovers (see the Appendix section on "Dynamics with exogenous mortality" above) and phase diagrams (b) and (c) refer to the model with deadly spillovers under different parametrizations. Phase diagrams (b) and (c) can be straightforwardly obtained by superimposing the equilibrium mortality rates derived in Figure 1 in the phase diagram without spillovers 2A.(a). Without deadly spillovers, there only exists one steady state, which is unstable. With deadly spillovers, the unstable steady state still exists and is pushed north-east, but there no additional steady states created by endogenous mortality.

D Appendix: Growth, Emission Taxes and Resource Booms

D.1 Consumption, growth and welfare

Derivation of (40). Using (24) to substitute $p_c C$ in (21), and rearranging terms, we have the static equilibrium condition

$$p_x(t)\,x(t) = \frac{1}{\gamma\frac{\epsilon-1}{\epsilon}} \cdot \frac{p_q(t)\,Q(t)}{N(t)} = \frac{1}{\gamma\frac{\epsilon-1}{\epsilon}} \cdot p_q(t) \cdot Q_i(t)\,, \tag{D.1}$$

where $Q_i = Q/N$ is commodity use by each intermediate producer. From the first-order condition (ii) in expression (A.21), net employment in intermediate production equals $L_{x_i} - \phi = (1-\gamma)\frac{\epsilon-1}{\epsilon}p_x x \cdot \frac{1}{w}$. Substituting this expression with $w = 1$ in the technology of intermediate producers (12), we have

$$x(t) = z_i(t)^\theta\,Q_i(t)^\gamma\,(L_{x_i}(t) - \phi)^{1-\gamma} = z_i(t)^\theta\,Q_i(t)^\gamma\left[(1-\gamma)\,\frac{\epsilon-1}{\epsilon}p_x(t)\,x(t)\right]^{1-\gamma},$$

where we can use (D.1) to substitute $p_x x$ in the last term, obtaining

$$x(t) = z_i(t)^\theta \cdot \left(\frac{1-\gamma}{\gamma} \cdot p_q(t)\right)^{1-\gamma} \cdot \frac{Q(t)}{N(t)}. \tag{D.2}$$

Substituting $x(t) = C(t)\,N(t)^{-\frac{\epsilon}{\epsilon-1}}$ from (22), we obtain

$$C(t) = N(t)^{\frac{1}{\epsilon-1}}\,z_i(t)^\theta \cdot \left(\frac{1-\gamma}{\gamma}\right)^{1-\gamma} p_q(t)^{-\gamma} \cdot p_q(t)\,Q(t)\,. \tag{D.3}$$

Substituting $p_q(t)\,Q(t) = \gamma\frac{\epsilon-1}{\epsilon}p_c(t)\,C(t)$ from (24) in (D.3) yields

$$C(t) = N(t)^{\frac{1}{\epsilon-1}}\,z_i(t)^\theta \cdot \left(\frac{1-\gamma}{\gamma}\right)^{1-\gamma} p_q(t)^{-\gamma} \cdot \gamma\frac{\epsilon-1}{\epsilon} \cdot p_c(t)\,C(t)$$

and therefore

$$\frac{1}{p_c(t)} = N(t)^{\frac{1}{\epsilon-1}}\,z_i(t)^\theta \cdot \left(\frac{1-\gamma}{\gamma}\right)^{1-\gamma} \gamma\frac{\epsilon-1}{\epsilon}p_q(t)^{-\gamma}\,. \tag{D.4}$$

Substituting $\frac{1}{p_c(t)}$ from (D.4) in the definition $C(t) = \frac{L(t)y(t)}{p_c(t)}$ gives equation (40) in the text.

Equilibrium utility: derivation of (46). Combining the utility-maximizing condition (A.5) with the definition of total consumption $C = c_L L + c_B B$ yields

$$c_L^\alpha c_B^{1-\alpha} = c_L \cdot \left(\frac{1-\alpha}{\alpha} \cdot \frac{L}{B}\right)^{1-\alpha} \tag{D.5}$$

as well as

$$c_L = \alpha\frac{C}{L} \tag{D.6}$$

Substituting (D.5) in the instantaneous utility (6), and then using (D.6) to eliminate c_L from the resulting expression yields

$$\begin{aligned} u &= c_L^{\alpha} c_B^{1-\alpha} b^{\psi(1-\alpha)} L^{\psi} = c_L \cdot \left(\frac{1-\alpha}{\alpha} \cdot \frac{L}{B} \right)^{1-\alpha} \cdot b^{\psi(1-\alpha)} L^{\psi}, \\ u &= \alpha^{\alpha} (1-\alpha)^{1-\alpha} \cdot C \cdot \left(\frac{L}{B} \right)^{1-\alpha} \cdot b^{\psi(1-\alpha)} L^{\psi-1}, \\ u &= \alpha^{\alpha} (1-\alpha)^{1-\alpha} \cdot C \cdot b^{(\psi-1)(1-\alpha)} L^{\psi-1}. \end{aligned} \tag{D.7}$$

From (40), consumption equals

$$C = \frac{\epsilon - 1}{\epsilon} \gamma^{\gamma} (1-\gamma)^{1-\gamma} \cdot L \cdot y \cdot T \cdot p_q^{-\gamma} \tag{D.8}$$

Substituting (D.8) in (D.7), we obtain

$$u = \alpha^{\alpha} \gamma^{\gamma} (1-\alpha)^{1-\alpha} (1-\gamma)^{1-\gamma} \frac{\epsilon - 1}{\epsilon} \cdot y \cdot T \cdot p_q^{-\gamma} \cdot b^{(\psi-1)(1-\alpha)} L^{\psi}.$$

and therefore

$$\ln u = \bar{\alpha} + \ln y + \ln T - \gamma \ln p_q - (1-\psi)(1-\alpha) \ln b + \psi \ln L$$

where we have defined $\bar{\alpha} \equiv \ln \alpha^{\alpha} \gamma^{\gamma} (1-\alpha)^{1-\alpha} (1-\gamma)^{1-\gamma} \frac{\epsilon-1}{\epsilon}$.

Proof of Proposition 7. As a first step, we derive the equilibrium growth rate of firms knowledge. From (A.23) with $w = 1$ and $\bar{z} = z_i$ by symmetry, we have

$$\frac{\dot{\vartheta}_z}{\vartheta_z} = r + \delta - \frac{\epsilon - 1}{\epsilon} \cdot \kappa \theta p_{x_i} x_i. \tag{D.9}$$

From foc (iii) in expression (A.21), symmetry implies $\vartheta_z \kappa z_i = 1$ where κ is constant, so that $\dot{\vartheta}_z / \vartheta_z = -\dot{z}_i / z_i$. Equation (D.9) thus yields

$$\frac{\dot{z}_i}{z_i} = \frac{\epsilon - 1}{\epsilon} \cdot \kappa \theta p_{x_i} x_i - r - \delta. \tag{D.10}$$

Exploiting the definitions $y = p_c C / L$ and $\ell = L / \Omega$, we can rewrite the equilibrium output condition (21) and the Keynes-Ramsey rule (8), respectively, as

$$p_{x_i} x_i = \frac{yL}{N}, \tag{D.11}$$

$$r = \frac{\dot{y}}{y} + \frac{\dot{\ell}}{\ell} + \rho. \tag{D.12}$$

Substituting both (D.11) and (D.12) in (D.10), the equilibrium growth rate of firms knowledge reads

$$\frac{\dot{z}_i(t)}{z_i(t)} = \frac{\epsilon - 1}{\epsilon} \cdot \kappa \theta \frac{y(t) L(t)}{N(t)} - \frac{\dot{y}(t)}{y(t)} - \frac{\dot{\ell}(t)}{\ell(t)} - \rho - \delta. \tag{D.13}$$

Next consider horizontal innovations. Time-differentiating the free-entry condition (16) we have

$$\frac{\dot{V}_i}{V_i} = \frac{\dot{(p_{x_i}x_i)}}{p_{x_i}x_i} = \frac{\dot{y}}{y} + \frac{\dot{\ell}}{\ell} - \frac{\dot{N}}{N}, \tag{D.14}$$

where the last term follows from time-differentiation of (D.11). From the definition of present-value profits (15), the growth rate of V_i must obey the dynamic no-arbitrage condition $\dot{V}_i/V_i = r+\delta-(\pi_i/V_i)$. Substituting this condition in (D.14) and solving the resulting expression for $\dot{N}/N$, we obtain

$$\frac{\dot{N}}{N} = \frac{\pi_i}{V_i} + \frac{\dot{y}}{y} + \frac{\dot{\ell}}{\ell} - (r+\delta) = \frac{\pi_i}{V_i} - \rho - \delta, \tag{D.15}$$

where the last term follows from substituing r with (D.12). From (14) and (A.22), the profit rate with $w=1$ can be written as

$$\frac{\pi_i}{V_i} = \frac{\frac{1}{\epsilon}p_{x_i}x_i - \phi - L_{z_i}}{V_i} = \frac{\frac{1}{\epsilon}p_{x_i}x_i - \phi - L_{z_i}}{\beta p_{x_i}x_i} \tag{D.16}$$

where the last term follows from the free-entry condition (16). Substituting (D.16) in (D.15), and using (D.11) to substitute $p_{x_i}x_i$, yields

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\beta}\left(\frac{1}{\epsilon} - (\phi + L_{z_i}(t))\frac{N(t)}{y(t)L(t)}\right) - \rho - \delta. \tag{D.17}$$

From the knowledge accumulation equation (13) under symmetry, we can substitute $L_{z_i} = \frac{1}{\kappa}\frac{\dot{z}_i}{z_i}$ in (D.17) to obtain the equilibrium growth rate of the mass of firms

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\beta}\left[\frac{1}{\epsilon} - \left(\phi + \frac{1}{\kappa}\frac{\dot{z}_i(t)}{z_i(t)}\right)\frac{N(t)}{y(t)L(t)}\right] - \rho - \delta. \tag{D.18}$$

Equations (D.13) and (D.18) determine the joint dynamics of vertical and horizontal innovation rates. The growth rate of knowledge may be either strictly positive – i.e., the case in which vertical R&D activities are operative – or zero – i.e., the case in which parameters are such that no labor is invested in knowledge accumulation. In either case, it is already apparent from (D.18) that the mass of firms $N(t)$ follows a logistic process with time-varying coefficients. For the sake of generality, we hereby focus on the case in which vertical R&D is operative. Using (D.13) to substitute the growth rate of knowledge in (D.18), we obtain

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\beta}\left[\frac{1-\theta(\epsilon-1)}{\epsilon} - \phi\frac{N(t)}{y(t)L(t)} - \frac{1}{\kappa}\left(\frac{\dot{y}(t)}{y(t)} + \frac{\dot{\ell}(t)}{\ell(t)} + \rho + \delta\right)\frac{N(t)}{y(t)L(t)}\right] - \rho - \delta. \tag{D.19}$$

Clearly, if the economy converges to a regular steady state (b_{ss}, ℓ_{ss}), population $L = L_{ss}$ and expenditure per adult $y^*(\ell_{ss}) = y_{ss}$ are both constant and the growth rate of the mass of firms reduces to

$$\frac{\dot{N}(t)}{N(t)} = \frac{1}{\beta}\left[\frac{1-\theta(\epsilon-1)}{\epsilon} - \left(\phi - \frac{\rho+\delta}{\kappa}\right)\frac{N(t)}{y_{ss}L_{ss}}\right] - \rho - \delta, \tag{D.20}$$

which converges to zero with a constant mass of firms given by

$$N_{ss} \equiv \lim_{t\to\infty} N\left(t\right) = \frac{1-\theta\left(\epsilon-1\right)-\epsilon\beta\left(\rho+\delta\right)}{\kappa\phi-\rho-\delta} \cdot \frac{\kappa}{\epsilon} \cdot y_{ss} L_{ss}, \tag{D.21}$$

which proves expression (44) in Proposition 7. Note that, from the equilibrium condition (26) evaluated in the steady state, the product $y_{ss} \cdot L_{ss}$ equals

$$y_{ss} \cdot L_{ss} = \frac{L_{ss} + p_{\omega,ss}\Omega}{1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}}. \tag{D.22}$$

Based on result (D.21), we can calculate the long-run growth rate of firms knowledge from (D.13) as

$$\lim_{t\to\infty} \frac{\dot{z}_i\left(t\right)}{z_i\left(t\right)} = \frac{\epsilon-1}{\epsilon} \cdot \kappa\theta \frac{y_{ss} L_{ss}}{N_{ss}} - \rho - \delta = \frac{\theta\left(\epsilon-1\right)\left(\kappa\phi-\rho-\delta\right)}{1-\theta\left(\epsilon-1\right)-\epsilon\beta\left(\rho+\delta\right)} - \rho - \delta, \tag{D.23}$$

which proves expression (45) in Proposition 7.

D.2 Commodity Tax

Proof of Proposition 8. The proof comprises five steps, namely, (i) proving $\mathrm{d}y^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$, (ii) proving $\mathrm{d}p_\omega^*\left(\bar{\ell}\right)/\mathrm{d}\tau < 0$, (iii) proving $\mathrm{d}m^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$, (iv) proving $\mathrm{d}\ell'_{ss}/\mathrm{d}\tau > 0$ and $\mathrm{d}\ell''_{ss}/\mathrm{d}\tau < 0$, and (v) proving that $\mathrm{d}p_\omega^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$.

Step 1: proof of $\mathrm{d}y^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$. This result can be easily proved graphically by means of Figure A1. From the static equilibrium conditions (B.11)-(B.12), recalling that $0 < \Upsilon\left(p_\omega\right) < 1$, it is easily established that changes in τ for given ℓ yield the following marginal effects

$$\frac{\partial y_1\left(p_\omega;\ell\right)}{\partial\tau} > 0, \quad \frac{\partial y_2\left(p_\omega\right)}{\partial\tau} > 0 \quad \text{and} \quad \frac{\partial y_{\min}}{\partial\tau} > 0. \tag{D.24}$$

Since both $y_1\left(p_\omega;\ell\right)$ and $y_2\left(p_\omega\right)$ shift upwards following an increase in τ, it follows that the equilibrium level of expenditure per adult is also increasing in the tax rate, $\mathrm{d}y^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$ for given $\bar{\ell}$.

Step 2: proof of $\mathrm{d}p_\omega^*\left(\bar{\ell}\right)/\mathrm{d}\tau < 0$. To simplify notation, in the remainder of this proof we will denote $y^*\left(\bar{\ell}\right)$, $p_\omega^*\left(\bar{\ell}\right)$ and $\Upsilon\left(p_\omega^*\left(\bar{\ell}\right)\right)$ by y^*, p_ω^*, and $\Upsilon\left(p_\omega^*\right)$, respectively. Total differentiation of (26) and (27) with respect to τ in equilibrium gives, respectively,

$$\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau} \cdot \frac{1}{\ell} = \frac{\mathrm{d}y^*}{\mathrm{d}\tau} \cdot \left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right) - y^* \cdot \gamma\frac{\epsilon-1}{\epsilon}, \tag{D.25}$$

$$\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau} \cdot \frac{1}{p_\omega^*} = -\frac{1}{1-\tau} + \frac{\mathrm{d}\Upsilon\left(p_\omega^*\right)}{\mathrm{d}\tau} \cdot \frac{1}{\Upsilon\left(p_\omega^*\right)} + \frac{\mathrm{d}y^*}{\mathrm{d}\tau} \cdot \frac{1}{y^*}. \tag{D.26}$$

Focusing on (D.26), note that by construction, $\mathrm{d}\Upsilon\left(p_\omega^*\right)/\mathrm{d}\tau = \left(\partial\Upsilon/\partial p_\omega^*\right) \cdot \left(\mathrm{d}p_\omega^*/\mathrm{d}\tau\right)$ so that, from the definition of $\varepsilon_{\Upsilon,p_\omega^*}$ in (B.37), we have

$$\frac{\mathrm{d}\Upsilon\left(p_\omega^*\right)}{\mathrm{d}\tau} \cdot \frac{1}{\Upsilon\left(p_\omega^*\right)} = \underbrace{\frac{\partial\Upsilon\left(p_\omega^*\right)}{\partial p_\omega^*} \cdot \frac{p_\omega^*}{\Upsilon\left(p_\omega^*\right)}}_{\varepsilon_{\Upsilon,p_\omega^*}} \cdot \frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau} \cdot \frac{1}{p_\omega^*} = \varepsilon_{\Upsilon,p_\omega^*} \cdot \frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau} \cdot \frac{1}{p_\omega^*}. \tag{D.27}$$

Substituting result (D.27) into (D.26), and solving for $(\mathrm{d}p_\omega^*/\mathrm{d}\tau)$, we obtain

$$\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau}\cdot\frac{1}{p_\omega^*}\cdot\left(1-\varepsilon_{\Upsilon,p_\omega^*}\right)=\frac{\mathrm{d}y^*}{\mathrm{d}\tau}\cdot\frac{1}{y^*}-\frac{1}{1-\tau}. \tag{D.28}$$

Now consider equation (D.25): rearranging terms to solve for $(\mathrm{d}y^*/\mathrm{d}\tau)$, we have

$$\frac{\mathrm{d}y^*}{\mathrm{d}\tau}\cdot\frac{1}{y^*}\cdot\left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right)=\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau}\cdot\frac{1}{\ell}\cdot\frac{1}{y^*}+\gamma\frac{\epsilon-1}{\epsilon},$$

where we can substitute $\left(1-\beta\rho-\tau\gamma\frac{\epsilon-1}{\epsilon}\right)=\frac{1+p_\omega^*/\ell}{y^*}$ from (26) as well as $\gamma\frac{\epsilon-1}{\epsilon}=\frac{p_\omega^*/\ell}{(1-\tau)\cdot\Upsilon\cdot y^*}$ from (27), to obtain

$$\frac{\mathrm{d}y^*}{\mathrm{d}\tau}\cdot\frac{1}{y^*}=\frac{p_\omega^*/\ell}{1+p_\omega^*/\ell}\cdot\left[\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau}\cdot\frac{1}{p_\omega^*}+\frac{1}{(1-\tau)\cdot\Upsilon\left(p_\omega^*\right)}\right]. \tag{D.29}$$

Using (D.29) to substitute $(\mathrm{d}y^*/\mathrm{d}\tau)$ into the right hand side of (D.28), we have

$$\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau}\cdot\frac{1}{p_\omega^*}\cdot\underbrace{\left(1-\varepsilon_{\Upsilon,p_\omega^*}-\frac{p_\omega^*/\ell}{1+p_\omega^*/\ell}\right)}_{\text{strictly positive}}=\frac{1}{1-\tau}\left[\frac{p_\omega^*/\ell}{1+p_\omega^*/\ell}\cdot\frac{1}{\Upsilon\left(p_\omega^*\right)}-1\right], \tag{D.30}$$

where the term in round brackets in the left hand side is strictly positive because, under strict substitutability, $\varepsilon_{\Upsilon,p_\omega^*}<0$ holds (see expression (B.37) above). Therefore, the sign of $(\mathrm{d}p_\omega^*/\mathrm{d}\tau)$ is determined by the last term in square brackets in the right hand side of (D.30). By definition (20), the cost share of resource use can be written as $\Upsilon\equiv\frac{p_\omega/\ell}{(p_\omega/\ell)+(L_Q/L)}$. Therefore, we have

$$\frac{p_\omega^*/\ell}{1+p_\omega^*/\ell}\cdot\frac{1}{\Upsilon\left(p_\omega^*\right)}=\frac{(L_Q/L)+(p_\omega^*/\ell)}{1+(p_\omega^*/\ell)}<1, \tag{D.31}$$

where the strict inequality must hold because $L_Q/L<1$ is necessary to have positive production in the intermediate sector. It follows from (D.31) that the last term in square brackets in the right hand side of (D.30) is strictly negative. Hence, the equilibrium resource price is strictly decreasing in the tax rate, $\mathrm{d}p_\omega^*/\mathrm{d}\tau<0$ for given ℓ.

Step 3: proof of $\mathrm{d}m^*\left(\bar{\ell}\right)/\mathrm{d}\tau<0$. To simplify notation, denote $m^*\left(\bar{\ell}\right)$ by m^*. Since $\sigma>1$ implies $\partial\Upsilon/\partial p_\omega^*<0$, it follows from the previous result $\mathrm{d}p_\omega^*/\mathrm{d}\tau<0$ that, for given ℓ,

$$\frac{\mathrm{d}\Upsilon\left(p_\omega^*\right)}{\mathrm{d}\tau}=\frac{\partial\Upsilon\left(p_\omega^*\right)}{\partial p_\omega^*}\cdot\frac{\mathrm{d}p_\omega^*}{\mathrm{d}\tau}>0. \tag{D.32}$$

From the equilibrium mortality rate (33) in Proposition 2 we thus have

$$\frac{\mathrm{d}m^*}{\mathrm{d}\tau}=\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\bar{m}+\bar{\mu}\cdot\Upsilon^{\frac{\sigma}{1-\sigma}\chi}\cdot\ell^{-(1-\varphi)}\right)=\bar{\mu}\ell^{-(1-\varphi)}\cdot\frac{\sigma}{1-\sigma}\chi\cdot\frac{\mathrm{d}\Upsilon\left(p_\omega^*\right)}{\mathrm{d}\tau}<0, \tag{D.33}$$

where the negative sign comes from $\sigma>1$ combined with (D.32) above.

Step 4: proof of $\mathrm{d}\ell'_{ss}/\mathrm{d}\tau>0$ *and* $\mathrm{d}\ell''_{ss}/\mathrm{d}\tau<0$. This result hinges on two effects that correspond to two shifts in the steady-state loci of the dynamic system (37)-(38), as graphically shown in Figure

3, diagram (a). First, the $\dot{\ell} = 0$ locus reads $b = m^*(\ell)$ and shifts downwards in the phase plane in view of the result $\mathrm{d}m^*\left(\bar{\ell}\right)/\mathrm{d}\tau < 0$. Second, from (C.15), the $\dot{b} = 0$ locus is strictly increasing in expenditure per adult y^* and therefore shifts upwards in the phase plane in view of the result $\mathrm{d}y^*\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$. Both these shifts imply that an increase in τ widens the distance between the two steady states, pushing the regular input ratio ℓ'_{ss} to the right and the extinction threshold ℓ''_{ss} to the left, as shown in Figure 3, diagram (a).

Step 5: proof of $\mathrm{d}p^*_q\left(\bar{\ell}\right)/\mathrm{d}\tau > 0$. From (B.19), we have

$$p^*_q\left(\bar{\ell}\right) = \frac{1}{1-\tau} \cdot \left[\frac{\eta^\sigma}{p^*_\omega\left(\bar{\ell}\right)^{\sigma-1}} + (1-\eta)^\sigma\right]$$

so that an increase in τ for given $\bar{\ell}$ has two effects: a direct one, which is strictly positive, and an indirect on working through $p^*_\omega\left(\bar{\ell}\right)$. Having proved that $\mathrm{d}p^*_\omega\left(\bar{\ell}\right)/\mathrm{d}\tau < 0$, the fact that $\sigma > 1$ implies a negative relationship between $p^*_\omega\left(\bar{\ell}\right)$ on $p^*_q\left(\bar{\ell}\right)$ yields a strictly positive indirect effect of τ on $p^*_q\left(\bar{\ell}\right)$ and thereby a strictly positive overall effect of τ on $p^*_q\left(\bar{\ell}\right)$.

D.3 Resource Booms

Proof of Proposition 9. From Proposition 2, the $\dot{\ell} = 0$ locus with $\sigma > 1$ reads

$$\dot{\ell} = 0 \rightarrow b = m^*(\ell) \equiv \bar{m} + \mu\xi^\chi\eta^{\chi\frac{\sigma}{\sigma-1}}\Omega^{\chi-(1-\varphi)} \cdot \Upsilon(\ell)^{\frac{\sigma}{1-\sigma}\chi} \cdot \ell^{-(1-\varphi)}$$

where the term $\Omega^{\chi-(1-\varphi)}$ implies that

$$\frac{\mathrm{d}m^*(\ell)}{\mathrm{d}\Omega}\left\{\begin{array}{ll} >0 & \text{if } \chi > 1-\varphi \\ =0 & \text{if } \chi = 1-\varphi \\ <0 & \text{if } \chi < 1-\varphi \end{array}\right\} \text{ for any } \ell > 0 \tag{D.34}$$

From (D.34), following an increase in the resource base Ω, the $\dot{\ell} = 0$ locus in the phase diagram shifts upwards when $\chi > 1-\varphi$, shifts downwards when $\chi < 1-\varphi$, and does not shift when $\chi = 1$. Since the position $\dot{b} = 0$ locus is not affected by the resource base Ω, the input ratio levels associated to the mortality threshold and the regular steady state respectively react to the resource boom as follows

$$\frac{\mathrm{d}\ell''_{ss}}{\mathrm{d}\Omega} = \left\{\begin{array}{ll} >0 & \text{if } \chi > 1-\varphi \\ =0 & \text{if } \chi = 1-\varphi \\ <0 & \text{if } \chi < 1-\varphi \end{array}\right\} \quad \text{and} \quad \frac{\mathrm{d}\ell'_{ss}}{\mathrm{d}\Omega} = \left\{\begin{array}{ll} <0 & \text{if } \chi > 1-\varphi \\ =0 & \text{if } \chi = 1-\varphi \\ >0 & \text{if } \chi < 1-\varphi \end{array}\right\}$$

which completes the proof. ■

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Growth with Deadly Spillovers

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