1. Prove the Thouless theorem.

2. Assuming the spectral theorem for normal matrices, prove that every unitary matrix can be written as $\mathbf{U} = \exp(\mathbf{X} - \mathbf{X}^\dagger)$ for some square matrix $\mathbf{X}$. If the dimension of these matrices is m, explain why $m(m+1)/2$ of the elements in $\mathbf{X}$ are redundant for this parametrization.

3. Prove the following.

$$\exp(G)\, a_p^\dagger \exp(-G) = \sum_q a_q^\dagger \,(\exp(\mathbf{G}))_{qp} \qquad\qquad G = \sum_{pq} (\mathbf{G})_{pq}\, a_p^\dagger a_q \tag{1}$$

4. Show that the creation and annihilation operators associated with a set of spin-orbitals $\{\psi_p'\}$ transformed from the original basis by $\mathbf{U} = \exp(\mathbf{X} - \mathbf{X}^\dagger)$ can be written as follows.

$$\begin{aligned} a_p'^\dagger &= \exp(X - X^\dagger) a_p^\dagger \exp(X^\dagger - X) \\ a_p' &= \exp(X - X^\dagger) a_p \exp(X^\dagger - X) \end{aligned} \qquad\qquad X \equiv \sum_{pq} (\mathbf{X})_{pq}\, a_p^\dagger a_q \tag{2}$$

5. Prove the following.

$$|\Phi'_{(p_1 \cdots p_n)}\rangle = \exp(X - X^\dagger) |\Phi_{(p_1 \cdots p_n)}\rangle \tag{3}$$