1. Translate the following expression from KM notation into our original notation, using daggers to denote creation operators and lines to denote contractions.

$$\vdots\, a^{p^\circ q^\bullet\ \ r^{\bullet\bullet}}_{s^\bullet t^{\bullet\bullet} u^\circ} \,\vdots = ?$$

Your final expression should be a Φ-normal-ordered string of six operators with three contraction lines.

Answer:

$$\vdots\, a^{p^\circ q^\bullet\ \ r^{\bullet\bullet}}_{s^\bullet t^{\bullet\bullet} u^\circ} \,\vdots = \vdots\, a^{p^\circ q^\bullet r^{\bullet\bullet}}_{u^\circ s^\bullet t^{\bullet\bullet}} \,\vdots = \vdots\, a^{p^\circ}_{u^\circ} a^{q^\bullet}_{s^\bullet} a^{r^{\bullet\bullet}}_{t^{\bullet\bullet}} \,\vdots = \vdots\, \overline{a_p^\dagger a_u}(-\overline{a_s a_q^\dagger})(-\overline{a_t a_r^\dagger}) \,\vdots = \vdots\, \overline{a_p^\dagger a_u}\,\overline{a_s a_q^\dagger}\,\overline{a_t a_r^\dagger} \,\vdots$$

2. Expand the following as a linear combination of Φ-normal-ordered operators.

$$\tilde{a}^p_q \tilde{a}^{rs}_{tu} = ?$$

Answer:

$$\begin{aligned}
\tilde{a}^p_q \tilde{a}^{rs}_{tu} &= \tilde{a}^{prs}_{qtu} + \hat{P}^{(r/s)} \tilde{a}^{p\ r^\bullet s}_{q^\bullet t\ u} + \hat{P}_{(t/u)} \tilde{a}^{p^\circ r\ s}_{q\ t^\circ u} + \hat{P}^{(r/s)}_{(t/u)} \tilde{a}^{p^\circ r^\bullet s}_{q^\bullet t^\circ u} \\
&= \tilde{a}^{prs}_{qtu} - \hat{P}^{(r/s)} \tilde{a}^{r^\bullet ps}_{q^\bullet tu} - \hat{P}_{(t/u)} \tilde{a}^{p^\circ rs}_{t^\circ qu} - \hat{P}^{(r/s)}_{(t/u)} \tilde{a}^{p^\circ r^\bullet s}_{t^\circ q^\bullet u} \\
&= \tilde{a}^{prs}_{qtu} - \hat{P}^{(r/s)} (-\eta^r_q) \tilde{a}^{ps}_{tu} - \hat{P}_{(t/u)} \gamma^p_t \tilde{a}^{rs}_{qu} - \hat{P}^{(r/s)}_{(t/u)} \gamma^p_t (-\eta^r_q) \tilde{a}^s_u \\
&= \tilde{a}^{prs}_{qtu} + \hat{P}^{(r/s)} \eta^r_q \tilde{a}^{ps}_{tu} - \hat{P}_{(t/u)} \gamma^p_t \tilde{a}^{rs}_{qu} + \hat{P}^{(r/s)}_{(t/u)} \gamma^p_t \eta^r_q \tilde{a}^s_u
\end{aligned}$$

3. Evaluate the following matrix element.[1]

$$\langle \Phi_i^a | H_e - E_0 | \Phi_j^b \rangle = ?$$

Answer:

$$\begin{aligned}
\langle \Phi_i^a | H_e - E_0 | \Phi_j^b \rangle &= f_p^q \langle \Phi | \tilde{a}_a^i \tilde{a}_q^p \tilde{a}_j^b | \Phi \rangle + \tfrac{1}{4} \overline{g}_{pq}^{rs} \langle \Phi | \tilde{a}_a^i \tilde{a}_{rs}^{pq} \tilde{a}_j^b | \Phi \rangle \\
\langle \Phi | \tilde{a}_a^i \tilde{a}_q^p \tilde{a}_j^b | \Phi \rangle &= \tilde{a}_{a^\bullet q^{\bullet\bullet} j^\circ}^{i^\circ p^\bullet \; b^{\bullet\bullet}} + \tilde{a}_{a^\bullet q^\circ \; j^{\circ\circ}}^{i^\circ p^{\circ\circ} b^\bullet} = \gamma_j^i(-\eta_a^p)(-\eta_q^b) + \gamma_q^i \gamma_j^p (-\eta_a^b) = \gamma_j^i \eta_a^p \eta_q^b - \gamma_q^i \gamma_j^p \eta_a^b \\
\langle \Phi | \tilde{a}_a^i \tilde{a}_{rs}^{pq} \tilde{a}_j^b | \Phi \rangle &= \hat{P}_{(r/s)}^{(p/q)} \tilde{a}_{a^\bullet r^\circ s^{\bullet\bullet} j^{\circ\circ}}^{i^\circ p^\bullet q^{\circ\circ} b^{\bullet\bullet}} = \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i (-\eta_a^p) \gamma_j^q (-\eta_s^b) = \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i \eta_a^p \gamma_j^q \eta_s^b
\end{aligned}$$

Substituting the last two equations into the first gives the final result.

$$\begin{aligned}
\langle \Phi_i^a | H_e - E_0 | \Phi_j^b \rangle &= f_p^q (\gamma_j^i \eta_a^p \eta_q^b - \gamma_q^i \gamma_j^p \eta_a^b) + \tfrac{1}{4} \overline{g}_{pq}^{rs} \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i \eta_a^p \gamma_j^q \eta_s^b \\
&= \gamma_j^i f_a^b - \eta_a^b f_j^i + \overline{g}_{aj}^{ib} \\
&= \delta_j^i f_a^b - \delta_a^b f_j^i + \overline{g}_{aj}^{ib}
\end{aligned}$$

[1] You may use either of the following equivalent expressions for the Hamiltonian. (I recommend the one on the right!)

$$H_e = h_p^q a_q^p + \tfrac{1}{4} \overline{g}_{pq}^{rs} a_{rs}^{pq} \qquad\qquad H_e = E_0 + f_p^q \tilde{a}_q^p + \tfrac{1}{4} \overline{g}_{pq}^{rs} \tilde{a}_{rs}^{pq}$$