1. Derive the Hausdorff expansion.[1]

$$e^X Y e^{-X} = Y + [X, Y] + \tfrac{1}{2!}[X,[X,Y]] + \tfrac{1}{3!}[X,[X,[X,Y]]] + \cdots \tag{1}$$

2. Prove the following.

$$[H_\mathrm{c}, T] = \vdots\contraction{}{H_\mathrm{c}}{}{T}H_\mathrm{c}T\vdots \qquad [[H_\mathrm{c}, T], T] = \vdots\contraction{}{H_\mathrm{c}}{T}{T}\contraction{}{H_\mathrm{c}}{}{T}H_\mathrm{c}TT\vdots \qquad [[[H_\mathrm{c}, T], T], T] = \vdots\contraction{}{H_\mathrm{c}}{TT}{T}\contraction{}{H_\mathrm{c}}{T}{T}\contraction{}{H_\mathrm{c}}{}{T}H_\mathrm{c}TTT\vdots \qquad \cdots \tag{2}$$

3. Using equation 2, explain the following.

$$[\,\cdot\,, T]^n(H_\mathrm{c}) = 0 \ \text{ for } n > 4 \tag{3}$$

4. Prove that the determinant basis consists of eigenfunctions of the diagonal Fock operator.

$$H_0 \Phi_{i_1 \cdots i_k}^{a_1 \cdots a_k} = \mathcal{E}_{i_1 \cdots i_k}^{a_1 \cdots a_k} \Phi_{i_1 \cdots i_k}^{a_1 \cdots a_k} \qquad H_0 \equiv f_p^p \tilde{a}_p^p \qquad \mathcal{E}_{i_1 \cdots i_k}^{a_1 \cdots a_k} \equiv \sum_{r=1}^{k} f_{a_r}^{a_r} - \sum_{r=1}^{k} f_{i_r}^{i_r} \tag{4}$$

5. Use equations 2 and 4 to write the coupled-cluster amplitude equation $\langle \Phi_{ij\cdots}^{ab\cdots} | \overline{H}_\mathrm{c} | \Phi \rangle = 0$ as follows.

$$t_{ab\cdots}^{ij\cdots} = (\mathcal{E}_{ab\cdots}^{ij\cdots})^{-1} \langle \Phi_{ij\cdots}^{ab\cdots} | V_\mathrm{c} \exp(T) | \Phi \rangle_\mathrm{C} \qquad V_\mathrm{c} \equiv H_\mathrm{c} - H_0 \tag{5}$$

6. Explain why the following terms vanish.[3]

$$\tfrac{1}{2} \langle \Phi_i^a | V_\mathrm{c} T_2^2 | \Phi \rangle_\mathrm{C} \qquad \langle \Phi_{ijk}^{abc} | V_\mathrm{c} | \Phi \rangle_\mathrm{C} \qquad \langle \Phi_{ijk}^{abc} | V_\mathrm{c} T_1 | \Phi \rangle_\mathrm{C}$$

[1] For an alternative to the direct, somewhat bullheaded proof shown in the notes look at the solution to Exercise 3.1.6 in Helgaker's big purple book, but note that you should give a proper proof by induction.[2]

[2] See `https://en.wikipedia.org/wiki/Mathematical_induction`.

[3] Hint: Use arguments about the excitation levels of their operators.