

1 Hartree-Fock Theory

The goal of electronic structure theory is to solve the “clamped-nuclei” Schrödinger equation

$$\hat{H}\Psi_k = E_k\Psi_k \quad \hat{H} = V_{\text{nuc}} + \hat{H}_e = \sum_{a < b}^{\text{nuc.}} \frac{Z_a Z_b}{|\mathbf{R}_a - \mathbf{R}_b|} - \frac{1}{2} \sum_i^{\text{elec.}} \nabla_i^2 - \sum_a^{\text{nuc.}} \sum_i^{\text{elec.}} \frac{Z_a}{|\mathbf{R}_a - \mathbf{r}_i|} + \sum_{i < j}^{\text{elec.}} \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (1.1)$$

with an optimal balance of accuracy and efficiency for the problem of interest. The most accurate solution possible for a given atomic orbital (AO) basis set¹ results from expanding the wavefunction

$$\Psi_k = \sum_{\mu} \Phi_{\mu} c_{\mu k} \quad (1.2)$$

in terms of all possible Slater determinants Φ_{μ} that can be formed from an orthonormal one-electron basis of spin-orbitals, $\{\psi_p\}$. The expansion coefficients $(\mathbf{c})_k = c_{\mu k}$ are eigenvectors of the matrix $(\mathbf{H})_{\mu\nu} = \langle \Phi_{\mu} | \hat{H} | \Phi_{\nu} \rangle$, which is the matrix representation of the Hamiltonian in the determinant basis. This is called the *full configuration-interaction* (FCI) solution.

Any one-electron basis spans the same “function space” as the AO basis set itself, and the full n -electron basis $\{\Phi_{\mu}\}$ spans the same space of n -electron functions regardless of how one forms spin orbitals from the AO basis set. As a result, one obtains the same FCI solution for any choice of spin-orbitals. In general, however, FCI solutions are completely unfeasible for basis sets of sufficient size to approach the complete basis set limit. One can think of this as a simple counting problem: if there are m functions in the AO basis, then there are $2m$ spin-orbitals in the one-electron basis,² and there are “ $2m$ choose n ”³

$$\binom{2m}{n} \equiv \frac{(2m)!}{n!(2m-n)!}$$

unique Slater determinants in the n -electron basis that can be formed from the spin MOs. The upshot is that we usually have to omit some Slater determinants in order to get an answer in a reasonable amount of time.

As soon as we truncate our determinant expansion (1.2), our choice of spin MOs makes a significant difference in the quality of our results. In particular, we need to choose our set of one-electron functions to minimize the number of Slater determinants it takes to “get close to” the exact wavefunction.

1.1 The Hartree-Fock optimization problem

It can be shown that optimizing $\langle \Psi | \hat{H}_e | \Psi \rangle$ by varying Ψ subject to the normalization constraint $\langle \Psi | \Psi \rangle = 1$ is equivalent to solving the Schrödinger equation. When we further constrain the form of Ψ this is no longer true, but it *does* generally allow us to get the best approximation to Ψ for a given approach (or “Ansatz”).

In order to make the wavefunction expansion converge with a relatively small number of Φ_{μ} s, we wish to find the best single-determinant approximation to Ψ . That is, we wish to optimize

$$\langle \Phi | \hat{H}_e | \Phi \rangle \quad \Phi(1, \dots, n) = \frac{1}{\sqrt{n!}} \begin{vmatrix} \psi_1(1) & \psi_2(1) & \cdots & \psi_n(1) \\ \psi_1(2) & \psi_2(2) & \cdots & \psi_n(2) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_1(n) & \psi_2(n) & \cdots & \psi_n(n) \end{vmatrix} \quad (1.3)$$

with respect to variation of the orbitals $\{\psi_p\}$, enforcing the normalization constraint by keeping the spin orbitals orthonormal. Note that here the function argument (i) is shorthand for (\mathbf{r}_i, s_i) where \mathbf{r}_i denotes the position of the i^{th} electron and s_i denotes its spin. This optimization problem is the idea behind *Hartree-Fock theory*.

¹cc-pVXZ, 6-31G, ANO1, etc.

² m α -orbitals and m β -orbitals.

³The number of unique sets of n marbles that can be drawn from a bag of $2m$ marbles. See <http://en.wikipedia.org/wiki/Combination>

Once we have solved for the Hartree-Fock optimization problem, the expectation value $\langle \Phi | \hat{H}_e | \Phi \rangle$ is itself a good first approximation to the electronic energy. More importantly, however, when we use this new set of Hartree-Fock spin-orbitals, $\{\psi_p\}$, the FCI expansion tends to converge much more quickly to the true wavefunction. Specifically, when we rewrite equation 1.2 in terms of single $\{\Phi_i^a\}$, double $\{\Phi_{ij}^{ab}\}$, triple $\{\Phi_{ijk}^{abc}\}$, etc. replacements⁴ of the orbitals in the Hartree-Fock determinant Φ with the remaining orbitals in the basis

$$\Psi = \Phi + \sum_i \Phi_i^a c_a^i + \sum_{\substack{a < b \\ i < j}} \Phi_{ij}^{ab} c_{ab}^{ij} + \sum_{\substack{a < b < c \\ i < j < k}} \Phi_{ijk}^{abc} c_{abc}^{ijk} + \dots \quad (1.4)$$

the coefficients tend to be very small, and are often virtually negligible for higher than quadruple replacements.

1.2 The Hartree-Fock equations

The electronic Hamiltonian \hat{H}_e contains one- and two-electron operators.

$$\hat{H}_e = \sum_i \hat{h}(i) + \sum_{i < j} \hat{g}(i, j) \quad \hat{h}(i) \equiv -\frac{1}{2} \nabla_i^2 + \sum_a \frac{Z_a}{|\mathbf{r}_i - \mathbf{R}_a|} \quad \hat{g}(i, j) \equiv \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \quad (1.5)$$

Its expectation value with respect to a single determinant Φ is given by the first Slater rule

$$\langle \Phi | \hat{H}_e | \Phi \rangle = \sum_i^n \langle \psi_i | \hat{h} | \psi_i \rangle + \frac{1}{2} \sum_{ij}^n \langle \psi_i \psi_j | | \psi_i \psi_j \rangle \quad \langle \psi_p \psi_q | | \psi_r \psi_s \rangle \equiv \langle \psi_p \psi_q | \psi_r \psi_s \rangle - \langle \psi_p \psi_q | \psi_s \psi_r \rangle \quad (1.6)$$

where the one- and two-electron integrals are defined as follows.

$$\langle \psi_p | \hat{h} | \psi_q \rangle \equiv \int d(1) \psi_p^*(1) \hat{h}(1) \psi_q(1) \quad \langle \psi_p \psi_q | \psi_r \psi_s \rangle \equiv \int d(1) d(2) \psi_p^*(1) \psi_q^*(2) \hat{g}(1, 2) \psi_r(1) \psi_s(2) \quad (1.7)$$

We wish to optimize equation 1.6 while constraining the orbitals to be normalized and orthogonal.⁵

$$\langle \psi_i | \psi_j \rangle \stackrel{!}{=} \delta_{ij} \quad (1.8)$$

The corresponding Lagrangian functional (see appendix A) is

$$\mathcal{L}[\{\psi_i\}, \{\psi_i^*\}, \{\epsilon_{ij}\}] = \sum_{i=1}^n \langle \psi_i | \hat{h} | \psi_i \rangle + \frac{1}{2} \sum_{i,j=1}^n \langle \psi_i \psi_j | | \psi_i \psi_j \rangle - \sum_{i,j=1}^n \epsilon_{ij} (\langle \psi_i | \psi_j \rangle - \delta_{ij}) \quad (1.9)$$

where $\{\epsilon_{ij}\}$ are our Lagrangian multipliers for the orthonormality constraint. Note that the complex conjugates of the orbitals $\{\psi_i^*\}$ are included as separate arguments of \mathcal{L} , since the real and imaginary components of ψ_i can be varied independently.⁶

The stationarity conditions for the Hartree-Fock Lagrangian are (see appendix B)

$$\left. \frac{d\mathcal{L}[\psi_k^* + \varepsilon \eta^*]}{d\varepsilon} \right|_{\varepsilon=0} \stackrel{!}{=} 0 \quad \text{and} \quad \left. \frac{d\mathcal{L}[\psi_k + \varepsilon \eta]}{d\varepsilon} \right|_{\varepsilon=0} \stackrel{!}{=} 0 \quad \text{for all } \eta \quad k = 1, \dots, n \quad (1.10)$$

which can be stated in words as follows: For each orbital ψ_1, \dots, ψ_n in the determinant Φ , mixing in a little bit of an arbitrary function $\eta = \eta(\mathbf{r}, s)$ doesn't change the Lagrangian.

⁴It is typical to use dummy indices i, j, k, l to count over the orbitals in the reference determinant Φ – the “occupied orbitals” – and to use a, b, c, d , to count over the orbitals not contained in Φ – the “unoccupied” or “virtual orbitals.” Dummy indices p, q, r, s are generally used to count over the full set of spin-orbitals, whether occupied or not.

⁵The $\stackrel{!}{=}$ sign means “must equal” – these are conditions to be satisfied.

⁶One can explicitly show that for complex variables $\frac{\partial z}{\partial z^*} = 0$. See https://en.wikipedia.org/wiki/Wirtinger_derivatives#Functions_of_one_complex_variable.

Separating out the terms in eq (1.9) involving a particular orbital ψ_k , we can write

$$\begin{aligned}\mathcal{L} = & \langle \psi_k | \hat{h} | \psi_k \rangle + \sum_i \langle \psi_k \psi_i | | \psi_k \psi_i \rangle - \sum_i \epsilon_{ki} (\langle \psi_k | \psi_i \rangle - \delta_{ki}) - \sum_i \epsilon_{ik} (\langle \psi_i | \psi_k \rangle - \delta_{ik}) \\ & + \sum_{i \neq k} \langle \psi_i | \hat{h} | \psi_i \rangle + \frac{1}{2} \sum_{i \neq k, j \neq k} \langle \psi_i \psi_j | | \psi_i \psi_j \rangle - \sum_{i \neq k, j \neq k} \epsilon_{ij} (\langle \psi_i | \psi_j \rangle - \delta_{ij})\end{aligned}$$

using $\langle \psi_k \psi_i | | \psi_k \psi_i \rangle = \langle \psi_i \psi_k | | \psi_i \psi_k \rangle$, which follows from exchanging integration variables in eq (1.7). The functional directional derivative for varying ψ_k^* along η^* is then

$$\left. \frac{d\mathcal{L}[\psi_k^* + \varepsilon \eta^*]}{d\varepsilon} \right|_{\varepsilon=0} = \frac{d}{d\varepsilon} \left(\langle \psi_k + \varepsilon \eta | \hat{h} | \psi_k \rangle + \sum_i \langle (\psi_k + \varepsilon \eta) \psi_i | | \psi_k \psi_i \rangle - \sum_i \epsilon_{ki} \langle \psi_k + \varepsilon \eta | \psi_i \rangle \right) \Big|_{\varepsilon=0}$$

where we have dropped ε -independent terms since their derivatives vanish. Evaluating the right-hand side gives

$$\left. \frac{d\mathcal{L}[\psi_k^* + \varepsilon \eta^*]}{d\varepsilon} \right|_{\varepsilon=0} = \langle \eta | \hat{h} | \psi_k \rangle + \sum_i \langle \eta \psi_i | | \psi_k \psi_i \rangle - \sum_i \epsilon_{ki} \langle \eta | \psi_i \rangle$$

where the first two terms can be written as follows.⁷

$$\langle \eta | \hat{h} | \psi_k \rangle + \sum_i \langle \eta \psi_i | | \psi_k \psi_i \rangle = \int d(1) \eta^*(1) \left(\hat{h}(1) + \sum_i \langle \psi_i(2) | \hat{g}(1, 2) (1 - \hat{P}(1, 2)) | \psi_i(2) \rangle \right) \psi_k(1)$$

Here, the *coordinate exchange operator* $\hat{P}(1, 2)$ allows us to write $(1 - \hat{P}(1, 2))\psi_i(2)\psi_k(1)$ as a shorthand for $\psi_i(2)\psi_k(1) - \psi_i(1)\psi_k(2)$. The expression in parentheses constitutes the *Fock operator*.⁸

$$\hat{f}(1) \equiv \hat{h}(1) + \sum_i \langle \psi_i(2) | \hat{g}(1, 2) (1 - \hat{P}(1, 2)) | \psi_i(2) \rangle \quad (1.11)$$

Note that it implicitly depends on orbital set, $\hat{f} = \hat{f}[\{\psi_i\}]$.

Using a similar procedure for the variation of ψ_k along η , eq (1.10) evaluates to

$$\int d(1) \eta^*(1) \left(\hat{f}(1) \psi_k(1) - \sum_i \epsilon_{ki} \psi_i(1) \right) \stackrel{!}{=} 0 \quad \text{and} \quad \int d(1) \left(\psi_k^*(1) \hat{f}(1) - \sum_i \psi_i^*(1) \epsilon_{ik} \right) \eta(1) \stackrel{!}{=} 0 \quad \text{for all } \eta$$

which, by the Fundamental Lemma of Calculus of Variations (appendix C), is equivalent to the following

$$\hat{f}(1) \psi_k(1) \stackrel{!}{=} \sum_i \epsilon_{ki} \psi_i(1) \quad \text{and} \quad \hat{f}(1) \psi_k^*(1) \stackrel{!}{=} \sum_i \epsilon_{ik} \psi_i^*(1) \quad (1.12)$$

using the Hermitian-ness of the Fock operator, $\langle \psi_k | \hat{f} \eta \rangle = \langle \hat{f}^\dagger \psi_k | \eta \rangle = \langle \hat{f} \psi_k | \eta \rangle$. Subtracting the complex conjugate of the right equation from the left gives

$$\sum_i (\epsilon_{ki} - \epsilon_{ik}^*) \psi_i(1) \stackrel{!}{=} 0$$

which, since the orbitals are linearly independent,⁹ implies that $\epsilon = [\epsilon_{ij}]$ forms a Hermitian matrix.

$$\epsilon_{k1} - \epsilon_{1k}^* = \dots = \epsilon_{kn} - \epsilon_{nk}^* = 0 \quad (1.13)$$

⁷Defining $\langle \psi_p(2) | \hat{g}(1, 2) | \psi_q(2) \rangle \equiv \int d(2) \psi_p^*(2) \hat{g}(1, 2) \psi_q(2)$

⁸You may see this written as $\hat{f}(1) = \hat{h}(1) + \sum_i (\hat{J}_i(1) - \hat{K}_i(1))$ where $\hat{J}_i(1) \equiv \langle \psi_i(2) | \hat{g}(1, 2) | \psi_i(2) \rangle$ and $\hat{K}_i \equiv \langle \psi_i(2) | \hat{g}(1, 2) \hat{P}(1, 2) | \psi_i(2) \rangle$ are the *Coloumb* and *exchange operators*.

⁹http://en.wikipedia.org/wiki/Linear_independence\#Definition

Requiring the multiplier matrix to be Hermitian makes the second condition in eq (1.12) redundant, so that the final *Hartree-Fock equations* can be expressed as follows.

$$\hat{f}\psi_i \stackrel{!}{=} \sum_j \epsilon_{ij}\psi_j \quad \text{and} \quad \epsilon \stackrel{!}{=} \epsilon^\dagger \quad (1.14)$$

To review, these conditions define orbitals which optimize $\langle \Phi | \hat{H}_e | \Phi \rangle$ subject to the constraint $\langle \psi_i | \psi_j \rangle = \delta_{ij}$.

1.3 The canonical Hartree-Fock equations

Appendix D shows that the Hartree-Fock energy and the orthogonality relations are invariant to unitary mixing of the orbitals in Φ . This implies that the solution to the Hartree-Fock optimization problem is not unique, because any unitary transformation of the orbitals in Φ is also a solution. In this section we show how to use this freedom to our advantage, by choosing orbitals which diagonalize the Lagrange multiplier matrix, partially decoupling eq (1.14). These orbitals are known as *canonical Hartree-Fock orbitals*.

In matrix notation, the Hartree-Fock equations can be written as follows.

$$\hat{f}\psi \stackrel{!}{=} \epsilon\psi \quad \text{and} \quad \epsilon = \epsilon^\dagger \quad \epsilon = \begin{pmatrix} \epsilon_{11} & \cdots & \epsilon_{1n} \\ \vdots & \ddots & \vdots \\ \epsilon_{n1} & \cdots & \epsilon_{nn} \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \quad (1.15)$$

Since the matrix ϵ is Hermitian, it can be diagonalized by a unitary transformation \mathbf{U} .

$$\epsilon = \mathbf{U}\tilde{\epsilon}\mathbf{U}^\dagger \quad \tilde{\epsilon} = \begin{pmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_n \end{pmatrix} \quad (1.16)$$

Inserting this decomposition into eq (1.15) and multiplying both sides from the left by \mathbf{U}^\dagger , we get

$$\hat{f}(\mathbf{U}^\dagger\psi) = \tilde{\epsilon}(\mathbf{U}^\dagger\psi)$$

which shows that the problem can be decoupled by using a new set of orbitals $\tilde{\psi}_1, \dots, \tilde{\psi}_n$, defined as follows.¹⁰

$$\tilde{\psi}_i = \sum_{j=1}^n U_{ji}^* \psi_j \quad (1.17)$$

It can be shown that the Fock operator \hat{f} is invariant to this type of transformation (see appendix D).

Substituting the new orbitals into eq (1.15) and dropping tildes yields the *canonical Hartree-Fock equations*.

$$\hat{f}\psi_i = \epsilon_i\psi_i \quad i = 1, \dots, n$$

Since ϵ is Hermitian, the Lagrangian eigenvalues are real. Note that these equations are not fully decoupled, since \hat{f} still depends on the full orbital set $\{\psi_i\}$. Solving them amounts to solving for the *self-consistent field*

$$\hat{v}(1) \equiv \sum_i \langle \psi_i(2) | \hat{g}(1,2)(1 - \hat{P}(1,2)) | \psi_i(2) \rangle = \sum_i (\hat{J}_i(1) - \hat{K}_i(1)) \quad (1.18)$$

in $\hat{f} = \hat{h} + \hat{v}$ that allows all n equations to hold true simultaneously.

¹⁰In matrix notation this reads $\tilde{\psi} = \mathbf{U}^\dagger\psi$.

A Constrained Optimization

The standard method of optimizing a function subject to a constraint is called Lagrangian optimization. Taking a function of two variables $f(x, y)$ as an example, suppose we want to optimize it subject to a constraint of the form $g(x, y) = c$. In this approach, we define the “Lagrangian function” \mathcal{L} as

$$\mathcal{L}(x, y, \lambda) \equiv f(x, y) - \lambda(g(x, y) - c) \quad (\text{A.1})$$

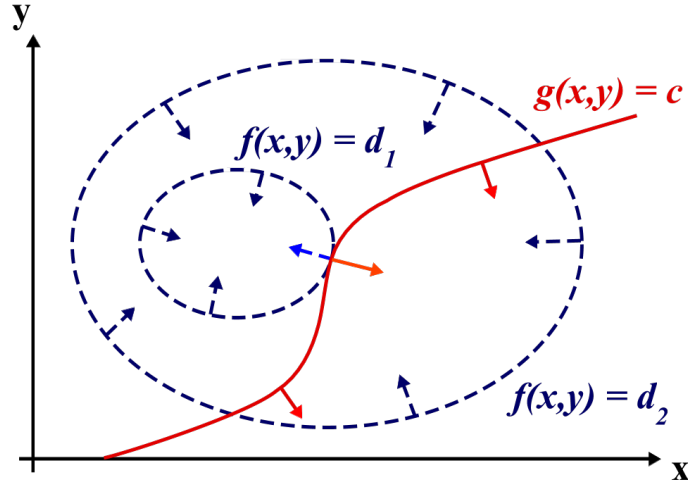
where the parameter λ is called the Lagrange multiplier. The constrained optimization problem can be solved by optimizing \mathcal{L} with respect to x , y , and λ . To see why, consider the stationarity conditions for \mathcal{L} .

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} \stackrel{!}{=} 0 \quad \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} \stackrel{!}{=} 0 \quad \frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x, y) \stackrel{!}{=} 0 \quad (\text{A.2})$$

The last equation is simply the requirement that the constraint $g(x, y) = c$ be satisfied – i.e. that the point (x, y) lies along the contour of $g(x, y)$ specified by $g(x, y) = c$. The first two equations correspond to the requirement that the gradients of the function $f(x, y)$ and the constraint surface $g(x, y)$ be parallel

$$\nabla f = \lambda \nabla g \quad (\text{A.3})$$

which is always true at the point (x, y) of closest approach along the line $g(x, y) = c$ to a minimum or maximum of the function $f(x, y)$. This is best understood visually.



If the gradients were not parallel, we could move along $g(x, y) = c$ to a higher contour of $f(x, y)$ by following the component of ∇f parallel to $g(x, y) = c$.

B Functional Derivatives

A functional is just a function of a function – i.e. some rule F that maps a function f into a number $F[f]$. Definite integrals are a common example. In order to optimize a functional F with respect to its argument f , one needs to take a *functional derivative*.¹¹ To motivate the definition of a functional derivative, first consider the definition of an ordinary derivative

$$\frac{df(x)}{dx} \equiv \lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon} \quad (\text{B.1})$$

and note the following identity, which you can verify using $f(x + \varepsilon) = f(x) + \frac{df(x)}{d\varepsilon}\varepsilon + \mathcal{O}(\varepsilon^2)$.

$$\lim_{\varepsilon \rightarrow 0} \frac{f(x + \varepsilon) - f(x)}{\varepsilon} = \left. \frac{df(x + \varepsilon)}{d\varepsilon} \right|_{\varepsilon=0} \quad (\text{B.2})$$

For multivariate functions, we have the concept of a *directional derivative*

$$\mathbf{y} \cdot \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \lim_{\varepsilon \rightarrow 0} \frac{f(\mathbf{x} + \varepsilon \mathbf{y}) - f(\mathbf{x})}{\varepsilon} \quad (\text{B.3})$$

which measures the change in $f(\mathbf{x})$ in the direction \mathbf{y} . Using equation B.2, the directional derivative can be evaluated as an ordinary scalar derivative with respect to ε .

$$\mathbf{y} \cdot \frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} = \left. \frac{df(\mathbf{x} + \varepsilon \mathbf{y})}{d\varepsilon} \right|_{\varepsilon=0} \quad (\text{B.4})$$

The functional derivative $\frac{\delta F}{\delta f}$ is defined to satisfy an equation analogous to B.3, playing the role of the gradient.

$$\int_{-\infty}^{\infty} dx' g(x') \frac{\delta F[f]}{\delta f(x')} \equiv \lim_{\varepsilon \rightarrow 0} \frac{F[f + \varepsilon g] - F[f]}{\varepsilon} \quad (\text{B.5})$$

This left-hand side could be called a *functional directional derivative*, giving the change in F upon displacing its argument along the function g . Here, the integral takes the role of the dot product in B.3. Using the same trick as in equation B.4, the functional derivative can be expressed as an ordinary scalar derivative.

$$\int_{-\infty}^{\infty} dx' g(x') \frac{\delta F[f]}{\delta f(x')} = \left. \frac{dF[f + \varepsilon g]}{d\varepsilon} \right|_{\varepsilon=0} \quad (\text{B.6})$$

The standard procedure for evaluating the functional derivative is to first evaluate the right-hand side of equation B.6 for an arbitrary g and then infer what $\frac{\delta F[f]}{\delta f(x)}$ must be by comparing to the left-hand side. Equivalently, $g(x')$ can be replaced with a Dirac delta $\delta(x - x')$ in order to arrive at $\frac{\delta F[f]}{\delta f(x)}$ directly.

Using eq. B.6 and the lemma in appendix C, we find that the stationarity condition for a functional

$$\frac{\delta F[f]}{\delta f} \stackrel{!}{=} 0 \quad (\text{B.7})$$

is equivalent to the following condition.

$$\left. \frac{dF[f + \varepsilon g]}{d\varepsilon} \right|_{\varepsilon=0} \stackrel{!}{=} 0 \quad \text{for all } g(x) \quad (\text{B.8})$$

¹¹http://en.wikipedia.org/wiki/Functional_derivative

C Fundamental Lemma of Calculus of Variations

The *Fundamental Lemma of Calculus of Variations*¹² says that, for continuous functions, the condition

$$\int_{-\infty}^{\infty} dx f(x) \eta(x) = 0 \quad \text{for all } \eta(x) \quad (\text{C.1})$$

holds only when $f(x) = 0$ for all x . We can see this by considering the case $\eta(x) = f(x)$. Since $f(x)^2$ is nonnegative everywhere, the integral yields a positive number whenever $f(x) \neq 0$ on a finite range of x values.

¹²http://en.wikipedia.org/wiki/Fundamental_lemma_of_calculus_of_variations

D Unitary Invariances for Hartree-Fock Orbitals

Orthonormality. By definition, unitary transformations preserve overlaps. This can be verified as follows

$$\langle \tilde{\psi}_i | \tilde{\psi}_j \rangle = \sum_{kl} U_{ki} U_{lj}^* \langle \psi_k | \psi_l \rangle = \sum_{kl} U_{ki} U_{lj}^* \delta_{kl} = \sum_k U_{ki} U_{kj}^* = \delta_{ij}$$

using $\sum_k U_{ki} U_{kj}^* = (\mathbf{U} \mathbf{U}^\dagger)_{ji} = (\mathbf{1})_{ji} = \delta_{ji}$.

Fock operator. Only the Coulomb and exchange parts of the Fock operator depend on the orbital set. For the Coulomb part, we have

$$\sum_i \langle \tilde{\psi}_i(2) | \hat{g}(1,2) | \tilde{\psi}_i(2) \rangle = \sum_{ijk} U_{ji} U_{ki}^* \langle \psi_j(2) | \hat{g}(1,2) | \psi_k(2) \rangle = \sum_{jk} \delta_{jk} \langle \psi_j(2) | \hat{g}(1,2) | \psi_k(2) \rangle = \sum_j \langle \psi_j(2) | \hat{g}(1,2) | \psi_j(2) \rangle$$

using the fact that $\sum_i U_{ji} U_{ki}^* = \delta_{jk}$. For the exchange part, we have the same thing with a $\hat{P}(1,2)$ sandwiched in there.

Hamiltonian expectation value. The vector notation $\boldsymbol{\psi}$ for our orbitals allows us to express Φ and $\tilde{\Phi}$ as

$$\Phi(1, \dots, n) = \frac{1}{\sqrt{n!}} |\boldsymbol{\psi}(1) \cdots \boldsymbol{\psi}(n)| \quad \tilde{\Phi}(1, \dots, n) = \frac{1}{\sqrt{n!}} |\tilde{\boldsymbol{\psi}}(1) \cdots \tilde{\boldsymbol{\psi}}(n)|$$

which, noting that the matrix $(\tilde{\boldsymbol{\psi}}(1) \cdots \tilde{\boldsymbol{\psi}}(n))$ is simply

$$(\tilde{\boldsymbol{\psi}}(1) \cdots \tilde{\boldsymbol{\psi}}(n)) = (\mathbf{U}^\dagger \boldsymbol{\psi}(1) \cdots \mathbf{U}^\dagger \boldsymbol{\psi}(n)) = \mathbf{U}^\dagger (\boldsymbol{\psi}(1) \cdots \boldsymbol{\psi}(n))$$

implies $\tilde{\Phi} = \det(\mathbf{U}^\dagger) \Phi = \det(\mathbf{U})^* \Phi$. Therefore, $\tilde{\Phi}$ and Φ have the same energy expectation values.

$$\langle \tilde{\Phi} | \hat{H}_e | \tilde{\Phi} \rangle = \det(\mathbf{U} \mathbf{U}^\dagger) \langle \Phi | \hat{H}_e | \Phi \rangle = \langle \Phi | \hat{H}_e | \Phi \rangle \quad (\text{D.1})$$