Wick's Theorem

Wick's Theorem

normal order

normal order $\quad a_{p_1}^\dagger \cdots a_{p_m}^\dagger a_{q_1} \cdots a_{q_n}$

normal order $a^+_{p_1} \cdots a^+_{p_m} a_{q_1} \cdots a_{q_n}$

vanishing vac expectation value

normal order $a^\dagger_{p_1} \cdots a^\dagger_{p_m} a_{q_1} \cdots a_{q_n}$

vanishing vac expectation value

normal ordered

normal order $a^\dagger_{p_1} \cdots a^\dagger_{p_m} a_{q_1} \cdots a_{q_n}$

vanishing vac expectation value

normal ordered

$$:q_1 \cdots q_n: = \varepsilon_\pi \, q_{\pi(1)} \cdots q_{\pi(n)}$$

normal order $a^{\dagger}_{p_1} \cdots a^{\dagger}_{p_m} a_{q_1} \cdots a_{q_n}$

vanishing vac expectation value

normal ordered

$$:q_1 \cdots q_n: = \varepsilon_\pi\, q_{\pi(1)} \cdots q_{\pi(n)}$$

where π puts string in normal order

$$:a_p a_q: = a_p a_q$$

$$:a_p a_q: = a_p a_q$$

$$:a_p^\dagger a_q^\dagger: = a_p^\dagger a_q^\dagger$$

$$:a_p a_q: = a_p a_q$$

$$:a_p^\dagger a_q^\dagger: = a_p^\dagger a_q^\dagger$$

$$:a_p^\dagger a_q: = a_p^\dagger a_q$$

$$:a_p a_q: = a_p a_q$$

$$:a_p^\dagger a_q^\dagger: = a_p^\dagger a_q^\dagger$$

$$:a_p^\dagger a_q: = a_p^\dagger a_q$$

$$:a_p a_q^\dagger: = -a_q^\dagger a_p$$

contraction:

contraction:

$$\overbracket{q_1 \, q_2}$$

contraction:

$$\overline{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

contraction:

$$\overbrace{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbrace{a_p a_q}$$

contraction:

$$\overbrace{q_1 \, q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbrace{a_p \, a_q} = a_p a_q - a_p a_q = 0$$

contraction:

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{a_p a_q} = a_p a_q - a_p a_q = 0$$

$$\overbracket{a_p^{\dagger} a_q^{\dagger}} = 0$$

contraction:

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{a_p a_q} = a_p a_q - a_p a_q = 0$$

$$\overbracket{a_p^{\dagger} a_q^{\dagger}} = 0$$

$$\overbracket{a_p^{\dagger} a_q} = 0$$

contraction:

$$\overline{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overline{a_p a_q} = a_p a_q - a_p a_q = 0$$

$$\overline{a_p^\dagger a_q^\dagger} = 0$$

$$\overline{a_p^\dagger a_q} = 0$$

$$\overline{a_p a_q^\dagger}$$

contraction:

$$q_1 q_2 \equiv q_1 q_2 - :q_1 q_2:$$

$$a_p a_q = a_p a_q - a_p a_q = 0$$

$$a_p^\dagger a_q^\dagger = 0$$

$$a_p^\dagger a_q = 0$$

$$a_p a_q^\dagger = a_p a_q^\dagger - (-a_q^\dagger a_p)$$

contraction:

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{a_p a_q} = a_p a_q - a_p a_q = 0$$

$$\overbracket{a_p^\dagger a_q^\dagger} = 0$$

$$\overbracket{a_p^\dagger a_q} = 0$$

$$\overbracket{a_p a_q^\dagger} = a_p a_q^\dagger - (-a_q^\dagger a_p) = [a_p, a_q^\dagger]_+$$

contraction:

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{a_p a_q} = a_p a_q - a_p a_q = 0$$

$$\overbracket{a_p^\dagger a_q^\dagger} = 0$$

$$\overbracket{a_p^\dagger a_q} = 0$$

$$\overbracket{a_p a_q^\dagger} = a_p a_q^\dagger - (-a_q^\dagger a_p) = [a_p, a_q^\dagger]_+$$

$$= \delta_{pq}$$

note:

note:

$$\wick{\c q_1 \c q_2} = q_1 q_2 - :q_1 q_2:$$

note:

$$\overbracket{q_1 q_2} = q_1 q_2 - :q_1 q_2:$$

$$\Longrightarrow \quad q_1 q_2 = :q_1 q_2: + \overbracket{q_1 q_2}$$

note:

$$\wick{\c{q}_1 \c{q}_2} = q_1 q_2 - :q_1 q_2:$$

$$\Longrightarrow \quad q_1 q_2 = \underbrace{:q_1 q_2:}_{\text{normal-ordered}} + \underbrace{\wick{\c{q}_1 \c{q}_2}}_{\text{contraction}}$$

normal-ordered with contraction

normal-ordered with contraction

$$: q_1 \cdots q_i \cdots q_j \cdots q_n :$$

normal-ordered with contraction

$$:q_1 \cdots q_i \cdots q_j \cdots q_n:$$

$$\equiv (-)^{j-i+1}\, q_i q_j \; :q_1 \cdots \cancel{q_i} \cdots \cancel{q_j} \cdots q_n:$$

Wick's theorem:

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \overbracket{q_1 q_2}$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \contraction{}{q}{_1}{q}q_1 q_2$$

normal-ordered contraction

Wick's theorem:

$$q_1 q_2 = \underbrace{:q_1 q_2:}_{\text{normal-ordered}} + \underbrace{\overbracket{q_1 q_2}}_{\text{contraction}}$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \contraction{}{q}{{}_1}{q}q_1 q_2$$

generalizes to

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \contraction{}{q}{{}_1}{q} q_1 q_2$$

generalizes to

$$q_1 \cdots q_n =$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \overbracket{q_1 q_2}$$

generalizes to

$$q_1 \cdots q_n = :q_1 \cdots q_n:$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \overbracket{q_1 q_2}$$

generalizes to

$$q_1 \cdots q_n = :q_1 \cdots q_n:$$
$$+ \sum :q_1 \cdots \overbracket{q_i \cdots q_j} \cdots q_n:$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \overbracket{q_1 q_2}$$

generalizes to

$$q_1 \cdots q_n = :q_1 \cdots q_n:$$

$$+ \sum :q_1 \cdots \overbracket{q_i \cdots q_j} \cdots q_n:$$

$$+ \sum :q_1 \cdots q_i \cdots q_k \cdots q_j \cdots q_l \cdots q_n:$$

Wick's theorem:

$$q_1 q_2 = :q_1 q_2: + \wick{\c q_1 \c q_2}$$

generalizes to

$$
\begin{aligned}
q_1 \cdots q_n = {} & :q_1 \cdots q_n: \\
& + \sum :q_1 \cdots \wick{\c q_i \cdots \c q_j} \cdots q_n: \\
& + \sum :q_1 \cdots \wick{\c1 q_i \cdots \c2 q_k \cdots \c1 q_j \cdots \c2 q_l} \cdots q_n: \\
& + \cdots
\end{aligned}
$$

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

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$$Q = :Q: + :\overline{Q}:$$

$\downarrow$

sum of unique single, double, triple, etc. contractions

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

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$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

$$:Q::Q': = :QQ': + :\overline{Q Q'}:$$

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

cross-contractions

$$:Q::Q': = :QQ': + :\overline{QQ'}:$$

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

$$:Q::Q': = :QQ': + :\overline{Q\,Q'}:$$

Corollary 2.

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

$$:Q::Q': = :QQ': + :\overline{QQ'}:$$

Corollary 2.

$$\langle vac|Q|vac\rangle = :\overline{\overline{Q}}:$$

Wick's theorem:

$$Q = :Q: + :\overline{Q}:$$

Corollary 1.

$$:Q::Q': = :QQ': + :\overbracket{QQ'}:$$

Corollary 2. complete contractions

$$\langle \mathrm{vac} | Q | \mathrm{vac} \rangle = :\overline{\overline{Q}}:$$

Φ-normal order

Φ-normal order $\quad b^\dagger_{p_1} \cdots b^\dagger_{p_m} b_{q_1} \cdots b_{q_n}$

Φ-normal order $\quad b^{+}_{p_1} \cdots b^{+}_{p_m} b_{q_1} \cdots b_{q_n}$

vanishing Φ expectation value

Φ-normal order $\quad b^{+}_{p_1} \cdots b^{+}_{p_m} b_{q_1} \cdots b_{q_n}$

vanishing Φ expectation value

Φ-normal ordered

Φ-normal order $\quad b^{+}_{p_1} \cdots b^{+}_{p_m} b_{q_1} \cdots b_{q_n}$

vanishing value $\quad \Phi$ expectation

Φ-normal ordered

$$:q_1 \cdots q_n: \; = \; \varepsilon_\pi \, q_{\pi(1)} \cdots q_{\pi(n)}$$

where π puts string in Φ-normal order

Φ-normal contraction

$$\overbracket{q_1\, q_2} \equiv q_1 q_2 - :q_1 q_2:$$

Φ-normal contraction

$$\wick{\c{q}_1 \c{q}_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\wick{\c{b}_p \c{b}_q} = 0$$

Φ-normal contraction

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - \vdots q_1 q_2 \vdots$$

$$\overbracket{b_p b_q} = 0$$

$$\overbracket{b_p^\dagger b_q^\dagger} = 0$$

Φ-normal contraction

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{b_p b_q} = 0$$

$$\overbracket{b_p^\dagger b_q^\dagger} = 0$$

$$\overbracket{b_p^\dagger b_q} = 0$$

Φ-normal contraction

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - {:}q_1 q_2{:}$$

$$\overbracket{b_p b_q} = 0$$

$$\overbracket{b_p^\dagger b_q^\dagger} = 0$$

$$\overbracket{b_p^\dagger b_q} = 0$$

$$\overbracket{b_p b_q^\dagger} = \delta_{pq}$$

Φ-normal contraction

$$\overline{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overline{b_p b_q} = 0$$

$$\overline{b_p^\dagger b_q^\dagger} = 0$$

$$\overline{b_p^\dagger b_q} = 0 \longrightarrow \overline{a_a^\dagger a_b} = 0$$

$$\overline{b_p b_q^\dagger} = \delta_{pq}$$

Φ-normal contraction

$$\wick{\c q_1 \c q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\wick{\c b_p \c b_q} = 0$$

$$\wick{\c b_p^\dagger \c b_q^\dagger} = 0$$

$$\wick{\c b_p^\dagger \c b_q} = 0 \longrightarrow \wick{\c a_a^\dagger \c a_b} = 0 \qquad \wick{\c a_i \c a_j^\dagger} = 0$$

$$\wick{\c b_p \c b_q^\dagger} = \delta_{pq}$$

Φ-normal contraction

$$\contraction{}{q_1}{}{q_2} q_1 q_2 \equiv q_1 q_2 - :q_1 q_2:$$

$$\contraction{}{b_p}{}{b_q} b_p b_q = 0$$

$$\contraction{}{b_p^\dagger}{}{b_q^\dagger} b_p^\dagger b_q^\dagger = 0$$

$$\contraction{}{b_p^\dagger}{}{b_q} b_p^\dagger b_q = 0 \longrightarrow \contraction{}{a_a^\dagger}{}{a_b} a_a^\dagger a_b = 0 \qquad \contraction{}{a_i}{}{a_j^\dagger} a_i a_j^\dagger = 0$$

$$\contraction{}{b_p}{}{b_q^\dagger} b_p b_q^\dagger = \delta_{pq} \longrightarrow \contraction{}{a_a}{}{a_b^\dagger} a_a a_b^\dagger = \delta_{ab}$$

Φ-normal contraction

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\overbracket{b_p b_q} = 0$$

$$\overbracket{b_p^\dagger b_q^\dagger} = 0$$

$$\overbracket{b_p^\dagger b_q} = 0 \longrightarrow \overbracket{a_a^\dagger a_b} = 0 \qquad \overbracket{a_i a_j^\dagger} = 0$$

$$\overbracket{b_p b_q^\dagger} = \delta_{pq} \longrightarrow \overbracket{a_a a_b^\dagger} = \delta_{ab} \qquad \overbracket{a_i^\dagger a_j} = \delta_{ij}$$

Φ-normal contraction

$$\overbracket{a_a^\dagger a_b} = 0 \qquad \overbracket{a_i a_j^\dagger} = 0$$

$$\overbracket{a_a a_b^\dagger} = \delta_{ab} \qquad \overbracket{a_i^\dagger a_j} = \delta_{ij}$$

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

Φ-normal contraction

$$\overline{a_a^\dagger a_b} = 0 \qquad \overline{a_i a_j^\dagger} = 0$$

$$\overline{a_a a_b^\dagger} = \delta_{ab} \qquad \overline{a_i^\dagger a_j} = \delta_{ij}$$

$$\overline{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\Longrightarrow \overline{q_1 q_2} =$$

Φ-normal contraction

$$\overbracket{a_a^\dagger a_b} = 0 \qquad \overbracket{a_i a_j^\dagger} = 0$$

$$\overbracket{a_a a_b^\dagger} = \delta_{ab} \qquad \overbracket{a_i^\dagger a_j} = \delta_{ij}$$

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - {:}q_1 q_2{:}$$

$$\Longrightarrow \overbracket{q_1 q_2} = \langle \Phi | q_1 q_2 - {:}q_1 q_2{:} | \Phi \rangle$$

Φ-normal contraction

$$\overbracket{a_a^\dagger\, a_b} = 0 \qquad \overbracket{a_i\, a_j^\dagger} = 0$$

$$\overbracket{a_a\, a_b^\dagger} = \delta_{ab} \qquad \overbracket{a_i^\dagger\, a_j} = \delta_{ij}$$

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\Longrightarrow \overbracket{q_1 q_2} = \langle \Phi | q_1 q_2 - \overset{\text{kills } \Phi}{\cancel{:q_1 q_2:}} | \Phi \rangle$$

Φ-normal contraction

$$\overbracket{a_a^\dagger\, a_b} = 0 \qquad \overbracket{a_i\, a_j^\dagger} = 0$$

$$\overbracket{a_a\, a_b^\dagger} = \delta_{ab} \qquad \overbracket{a_i^\dagger\, a_j} = \delta_{ij}$$

$$\overbracket{q_1 q_2} \equiv q_1 q_2 - :q_1 q_2:$$

$$\Longrightarrow \overbracket{q_1 q_2} = \langle \Phi | q_1 q_2 - \overbrace{\cancel{:q_1 q_2:}}^{\text{kills } \Phi} | \Phi \rangle$$

$$= \langle \Phi | q_1 q_2 | \Phi \rangle$$

one-hole density matrix

$$\overline{a_p \, a_q^\dagger} = \langle \Phi | a_p \, a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

one-hole density matrix

$$\overbracket{a_p \, a_q^\dagger} = \langle \Phi | a_p \, a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \end{array}\right]$$

one-hole density matrix

$$\overbracket{a_p a_q^\dagger} = \langle \Phi | a_p a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \end{array}\right]$$

$$\overbracket{a_i a_j^\dagger} = 0 \qquad \overbracket{a_a a_b^\dagger} = \delta_{ab}$$

one-hole density matrix

$$\overline{a_p a_q^\dagger} = \langle \Phi | a_p a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

$$\overline{a_i a_j^\dagger} = 0 \qquad \overline{a_a a_b^\dagger} = \delta_{ab}$$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \end{array}\right]$$

one-particle density matrix

$$\overline{a_p^\dagger a_q} = \langle \Phi | a_p^\dagger a_q | \Phi \rangle \equiv \gamma_{pq}$$

one-hole density matrix

$$\overline{a_p a_q^\dagger} = \langle \Phi | a_p a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

$$\overline{a_i a_j^\dagger} = 0 \qquad \overline{a_a a_b^\dagger} = \delta_{ab}$$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} \end{array}\right]$$

one-particle density matrix

$$\overline{a_p^\dagger a_q} = \langle \Phi | a_p^\dagger a_q | \Phi \rangle \equiv \gamma_{pq}$$

$$\left[\begin{array}{c|c} \begin{matrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

one-hole density matrix

$$\overbracket{a_p a_q^\dagger} = \langle \Phi | a_p a_q^\dagger | \Phi \rangle \equiv \eta_{pq}$$

$$\overbracket{a_i a_j^\dagger} = 0 \qquad \overbracket{a_a a_b^\dagger} = \delta_{ab}$$

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1\end{smallmatrix} \end{array}\right]$$

one-particle density matrix

$$\overbracket{a_p^\dagger a_q} = \langle \Phi | a_p^\dagger a_q | \Phi \rangle \equiv \gamma_{pq}$$

$$\overbracket{a_i^\dagger a_j} = \delta_{ij} \qquad \overbracket{a_a^\dagger a_b} = 0$$

$$\left[\begin{array}{c|c} \begin{smallmatrix}1 & & \\ & \ddots & \\ & & 1\end{smallmatrix} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Wick's thm. $Q = :Q: + :\overline{Q}:$

Corollary 1. $:Q::Q': = :QQ': + :\overbracket{Q\,Q'}:$

Corollary 2. $\langle \mathrm{vac} | Q | \mathrm{vac} \rangle = :\overline{\overline{Q}}:$

Wick's thm. $\quad Q = :Q: + :\overline{Q}:$

Corollary 1. $\quad :Q::Q': = :QQ': + :\overline{QQ'}:$

Corollary 2. $\quad \langle \mathrm{vac}|Q|\mathrm{vac}\rangle = :\overline{\overline{Q}}:$

now, contractions are $\overline{a_p^\dagger a_q} = \gamma_{pq}$ and $\overline{a_p a_q^\dagger} = \eta_{pq}$

Examples:

Examples:

$$a_p^\dagger a_q$$

Examples:

$$a_p^\dagger a_q = :a_p^\dagger a_q: + :\overbracket{a_p^\dagger a_q}:$$

Examples:

$$a_p^\dagger a_q = :a_p^\dagger a_q: + :\overbracket{a_p^\dagger a_q}:$$

$$\langle \Phi | a_p^\dagger a_q | \Phi \rangle$$

Examples:

$$a_p^\dagger a_q = :a_p^\dagger a_q: + :\wick{\c{a}_p^\dagger \c{a}_q}:$$

$$\langle \Phi | a_p^\dagger a_q | \Phi \rangle = :\wick{\c{a}_p^\dagger \c{a}_q}:$$

Examples:

$$a_p^\dagger a_q = :a_p^\dagger a_q: + :\overbracket{a_p^\dagger a_q}:$$

$$\langle \Phi | a_p^\dagger a_q | \Phi \rangle = :\overbracket{a_p^\dagger a_q}:$$

$$= \gamma_{pq}$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r:$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :\contraction{}{a}{_p^\dagger a_q^\dagger}{a} a_p^\dagger a_q^\dagger a_s a_r: + :\contraction{}{a}{_p^\dagger a_q^\dagger a_s}{a} a_p^\dagger a_q^\dagger a_s a_r:$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :\wick{\c a_p^\dagger a_q^\dagger \c a_s a_r}: + :\wick{\c a_p^\dagger a_q^\dagger a_s \c a_r}:$$

$$+ :\wick{a_p^\dagger \c a_q^\dagger \c a_s a_r}:$$

Examples:

$$
\begin{aligned}
a_p^\dagger a_q^\dagger a_s a_r = {} & :a_p^\dagger a_q^\dagger a_s a_r: \\
& + :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r: \\
& + :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:
\end{aligned}
$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r:$$

Examples:

$$
\begin{aligned}
a_p^\dagger a_q^\dagger a_s a_r = {} & :a_p^\dagger a_q^\dagger a_s a_r: \\
& + \wick{:\c a_p^\dagger a_q^\dagger \c a_s a_r:} + \wick{:\c a_p^\dagger a_q^\dagger a_s \c a_r:} \\
& + \wick{:a_p^\dagger \c a_q^\dagger \c a_s a_r:} + \wick{:a_p^\dagger \c a_q^\dagger a_s \c a_r:} \\
& + \wick{:\c1 a_p^\dagger \c2 a_q^\dagger \c2 a_s \c1 a_r:} + \wick{:\c1 a_p^\dagger \c2 a_q^\dagger \c1 a_s \c2 a_r:}
\end{aligned}
$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$\langle \Phi | a_p^\dagger a_q^\dagger a_s a_r | \Phi \rangle = :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

Examples:

$$a_p^\dagger a_q^\dagger a_s a_r = :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$\langle \Phi | a_p^\dagger a_q^\dagger a_s a_r | \Phi \rangle = :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r:$$

$$= \gamma_{pr} \gamma_{qs} - \gamma_{ps} \gamma_{qr}$$

$$\hat{H}_e = \sum_{pq} h_{pq}\, a_p^\dagger a_q + \frac{1}{2} \sum_{pqrs} \langle pq | rs \rangle\, a_p^\dagger a_q^\dagger a_s a_r$$

$$\hat{H}_e = \sum_{pq} h_{pq}\, a_p^\dagger a_q + \frac{1}{2}\sum_{pqrs} \langle pq|rs\rangle\, a_p^\dagger a_q^\dagger a_s a_r$$

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$$
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&+ :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r: + :a_p^\dagger a_q^\dagger a_s a_r: \\
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$$

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$$\sum_i h_{ii} + \frac{1}{2}\sum_{ij}\langle ij||ij\rangle \qquad h_{pq} + \sum_i \langle pi||qi\rangle$$

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1. $\langle \Phi | H_e | \Phi_i^a \rangle$
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3. $\langle \Phi_i^a | H_e | \Phi_j^b \rangle$

The end.