3. Derive by induction:

$$\frac{\partial^{m+1}}{\partial \lambda^{m+1}} \langle \Phi | \lambda V_c | \Psi(\lambda) \rangle = (m+1) \langle \Phi | V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle + \langle \Phi | \lambda V_c | \frac{\partial^{m+1}}{\partial \lambda^{m+1}} \Psi(\lambda) \rangle$$

then use result to prove

$$E^{(m+1)} = \langle \Phi | V_c | \Psi^{(m)} \rangle .$$

$m=0$ case:

using product rule

$$\frac{\partial}{\partial \lambda} \langle \Phi | \lambda V_c | \Psi(\lambda) \rangle = \langle \Phi | V_c | \Psi(\lambda) \rangle + \langle \Phi | \lambda V_c | \frac{\partial}{\partial \lambda} \Psi(\lambda) \rangle \quad \checkmark$$

Assume for mth case:

... Assuming $\frac{\partial^m}{\partial \lambda^m} \langle \Phi | \lambda V_c | \Psi(\lambda) \rangle = m \langle \Phi | V_c | \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \Psi(\lambda) \rangle + \langle \Phi | \lambda V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle$

Show $(m+1)$th case.

$$\frac{\partial^{m+1}}{\partial \lambda^{m+1}} \langle \Phi | \lambda V_c | \Psi(\lambda) \rangle = \frac{\partial}{\partial \lambda} \left[\frac{\partial^m}{\partial \lambda^m} \langle \Phi | \lambda V_c | \Psi(\lambda) \rangle \right]$$

$$= \frac{\partial}{\partial \lambda} \left[m \langle \Phi | V_c | \frac{\partial^{m-1}}{\partial \lambda^{m-1}} \Psi(\lambda) \rangle \right] + \frac{\partial}{\partial \lambda} \left[\langle \Phi | \lambda V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle \right]$$

$$= m \langle \Phi | V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle + \langle \Phi | V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle + \langle \Phi | \lambda V_c | \frac{\partial^{m+1}}{\partial \lambda^{m+1}} \Psi(\lambda) \rangle$$

$$= (m+1) \langle \Phi | V_c | \frac{\partial^m}{\partial \lambda^m} \Psi(\lambda) \rangle + \langle \Phi | \lambda V_c | \frac{\partial^{m+1}}{\partial \lambda^{m+1}} \Psi(\lambda) \rangle$$

QED

(By induction)

3. (cont'd)

We have

$$\frac{\partial^{m+1}}{\partial\lambda^{m+1}}\langle\Phi|\lambda V_c|\Psi(\lambda)\rangle = (m+1)\langle\Phi|V_c|\frac{\partial^m}{\partial\lambda^m}\Psi(\lambda)\rangle + \langle\Phi|\lambda V_c|\frac{\partial^{m+1}}{\partial\lambda^{m+1}}\Psi(\lambda)\rangle$$

and

$$\cancel{E_c^{(m+1)} = \langle\Phi|V_c|\Psi^{(m)}\rangle, \text{ where } \Psi^{(m)} \equiv \frac{1}{m!}\frac{\partial^m\Psi(\lambda)}{\partial\lambda^m}\Big|_{\lambda=0}}$$

$$E_c^{(m)} \equiv \frac{1}{m!}\frac{\partial^m E(\lambda)}{\partial\lambda^m}\Big|_{\lambda=0}, \quad E(\lambda) = \langle\Phi|\lambda V_c|\Psi(\lambda)\rangle$$

$$\Rightarrow E_c^{(m+1)} = \frac{1}{(m+1)!}\frac{\partial^{m+1}}{\partial\lambda^{m+1}}\Big[\langle\Phi|\lambda V_c|\Psi(\lambda)\rangle\Big]\Big|_{\lambda=0}$$

0 for $\lambda=0$

$$= \frac{1}{(m+1)!}\Big[(m+1)\langle\Phi|V_c|\frac{\partial^m}{\partial\lambda^m}\Psi(\lambda)\rangle + \cancel{\langle\Phi|\lambda V_c|\frac{\partial^{m+1}}{\partial\lambda^{m+1}}\Psi(\lambda)\rangle}\Big]\Big|_{\lambda=0}$$

$$= \frac{1}{\cancel{(m+1)}m!}\cdot\cancel{(m+1)}\langle\Phi|V_c|\frac{\partial^m}{\partial\lambda^m}\Psi(\lambda)\rangle\Big|_{\lambda=0}$$

$$= \frac{1}{m!}\langle\Phi|V_c|\frac{\partial^m}{\partial\lambda^m}\Psi(\lambda)\rangle\Big|_{\lambda=0}$$

$$= \langle\Phi|V_c|\frac{1}{m!}\frac{\partial^m}{\partial\lambda^m}\Psi(\lambda)\rangle\Big|_{\lambda=0}$$

$$\Psi^{(m)} \equiv \frac{1}{m!}\frac{\partial^m\Psi(\lambda)}{\partial\lambda^m}\Big|_{\lambda=0}$$

$$= \langle\Phi|V_c|\Psi^{(m)}\rangle$$