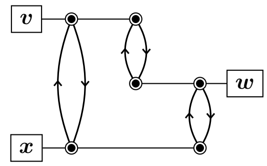
1. Give an example of each of the following.

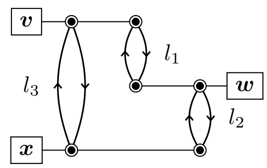
   (a) A closed, connected graph of at least two operators.

   Answer:



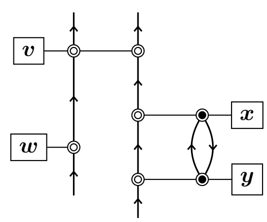
   (b) A Hugenholtz path of at least three lines that doesn't qualify as a Goldstone path.

   Answer: The sequence of lines (l_1, l_2, l_3) in the following graph.



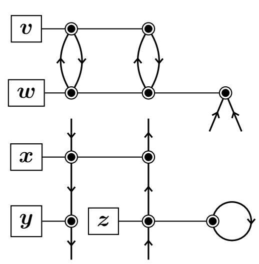
   (c) Non-equivalent, interchangeable subgraphs, where at least one subgraph contains multiple operators.

   Answer: The subgraphs $G[\{\boldsymbol{w}\}]$ and $G[\{\boldsymbol{x}, \boldsymbol{y}\}]$ in the following.

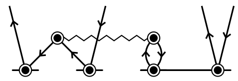


   (d) A graph that is disconnected and linked.

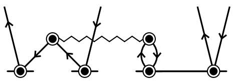
   Answer:



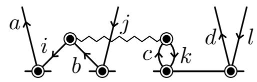
2. Interpret the following graph algebraically, and then simplify your expression as much as possible.[1]



Answer:



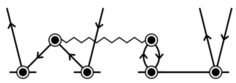
$$= \sum_{\substack{abcd\\ijkl}}$$



$$= \sum_{\substack{abcd\\ijkl}} \overline{g}^{bc}_{ik} c^i_a c^j_b c^{kl}_{cd} \vdots a^{i^\circ k^{\circ\circ}}_{b^\bullet c^{\bullet\bullet}} a^a_{i^\circ} a^{b^\bullet}_j a^{c^{\bullet\bullet} d}_{k^{\circ\circ} l} \vdots$$

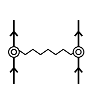
$$\vdots a^{i^\circ k^{\circ\circ}}_{b^\bullet c^{\bullet\bullet}} a^a_{i^\circ} a^{b^\bullet}_j a^{c^{\bullet\bullet} d}_{k^{\circ\circ} l} \vdots = \tilde{a}^{i^\circ k^{\circ\circ} a b^\bullet c^{\bullet\bullet} d}_{b^\bullet c^{\bullet\bullet} i^\circ j k^{\circ\circ} l} = -\tilde{a}^{i^\circ k^{\circ\circ} a b^\bullet c^{\bullet\bullet} d}_{b^\bullet c^{\bullet\bullet} j i^\circ k^{\circ\circ} l} = -\gamma^i_i \gamma^k_k (-\eta^b_b)(-\eta^c_c)\tilde{a}^{ad}_{jl} = -\tilde{a}^{ad}_{jl}$$

Substituting the second equation into the first gives the final answer.



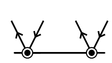
$$= -\sum_{\substack{abcd\\ijkl}} \overline{g}^{bc}_{ik} c^i_a c^j_b c^{kl}_{cd}\, \tilde{a}^{ad}_{jl}$$

[1] The operators in this graph are defined as follows.



$$\equiv \left(\tfrac{1}{2!}\right)^2 \sum_{pqrs} \overline{g}^{rs}_{pq} \tilde{a}^{pq}_{rs}$$

$$\equiv \sum_{ia} c^i_a \tilde{a}^a_i$$

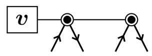
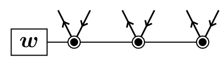


$$\equiv \left(\tfrac{1}{2!}\right)^2 \sum_{ijab} c^{ij}_{ab} \tilde{a}^{ab}_{ij}$$

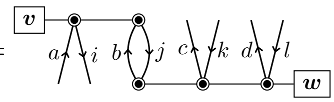
3. Write the following algebraic expression as a graph.[2]

$$\sum_{\substack{abcd\\ijkl}} \overline{v}^{ab}_{ij}\overline{w}^{jkl}_{bcd}\,\vdots\, a^{ij^\circ}_{ab^\bullet}\, a^{b^\bullet cd}_{j^\circ kl}\,\vdots$$

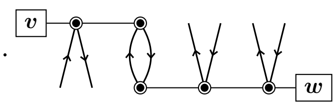
Answer: This is a contraction of the following two diagrams


$= \left(\frac{1}{2!}\right)^2 \sum_{ijab} \overline{v}^{ab}_{ij}\tilde{a}^{ab}_{ij}$

$= \left(\frac{1}{3!}\right)^2 \sum_{ijkabc} \overline{w}^{ijk}_{abc}\tilde{a}^{ijk}_{abc}$

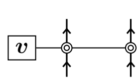
joining them by a hole line and a particle line. The labeled graph looks as follows.

$$\overline{v}^{ab}_{ij}\overline{w}^{jkl}_{bcd}\,\vdots\, a^{ij^\circ}_{ab^\bullet}\, a^{b^\bullet cd}_{j^\circ kl}\,\vdots =$$


Noting that $\boldsymbol{w}$ has two pairs of equivalent lines leads to the final result.

$$\sum_{\substack{abcd\\ijkl}} \overline{v}^{ab}_{ij}\overline{w}^{jkl}_{bcd}\,\vdots\, a^{ij^\circ}_{ab^\bullet}\, a^{b^\bullet cd}_{j^\circ kl}\,\vdots = 4\,\cdot$$


[2]Use the following to denote the operators in your graph.

$\left(\frac{1}{2!}\right)^2 \sum_{pqrs} \overline{v}^{rs}_{pq}\tilde{a}^{pq}_{rs} \equiv$

$\left(\frac{1}{3!}\right)^2 \sum_{\substack{pqr\\stu}} \overline{w}^{stu}_{pqr}\tilde{a}^{pqr}_{stu} \equiv$