

1. Translate the following expression from KM notation into our original notation, using daggers to denote creation operators and lines to denote contractions.

$$:a_{s\bullet t\bullet\bullet}^{p^\circ q^\bullet r^\bullet\bullet}{}_{u^\circ}:=?$$

Your final expression should be a Φ -normal-ordered string of six operators with three contraction lines.

Answer:

$$:a_{s\bullet t\bullet\bullet}^{p^\circ q^\bullet r^\bullet\bullet}{}_{u^\circ}:= :a_{u^\circ s\bullet t\bullet\bullet}^{p^\circ q^\bullet r^\bullet\bullet}:= :a_{u^\circ}^{p^\circ}a_s^{q^\bullet}a_t^{r^\bullet\bullet}:= :\overline{a_p^\dagger}a_u(-\overline{a_s}a_q^\dagger)(-\overline{a_t}a_r^\dagger):= :\overline{a_p^\dagger}a_u\overline{a_s}a_q^\dagger\overline{a_t}a_r^\dagger:$$

2. Expand the following as a linear combination of Φ -normal-ordered operators.

$$\tilde{a}_q^p \tilde{a}_{tu}^{rs} = ?$$

Answer:

$$\begin{aligned}
 \tilde{a}_q^p \tilde{a}_{tu}^{rs} &= \tilde{a}_{qtu}^{prs} + \hat{P}^{(r/s)} \tilde{a}_{q \bullet t}^p \tilde{a}_u^{r \bullet s} + \hat{P}_{(t/u)} \tilde{a}_q^{p^\circ r} \tilde{a}_{t^\circ u}^s + \hat{P}_{(t/u)}^{(r/s)} \tilde{a}_{q \bullet t^\circ u}^{p^\circ r \bullet s} \\
 &= \tilde{a}_{qtu}^{prs} - \hat{P}^{(r/s)} \tilde{a}_{q \bullet tu}^{r \bullet ps} - \hat{P}_{(t/u)} \tilde{a}_{t^\circ qu}^{p^\circ rs} - \hat{P}_{(t/u)}^{(r/s)} \tilde{a}_{t^\circ q \bullet u}^{p^\circ r \bullet s} \\
 &= \tilde{a}_{qtu}^{prs} - \hat{P}^{(r/s)} (-\eta_q^r) \tilde{a}_{tu}^{ps} - \hat{P}_{(t/u)} \gamma_t^p \tilde{a}_{qu}^{rs} - \hat{P}_{(t/u)}^{(r/s)} \gamma_t^p (-\eta_q^r) \tilde{a}_u^s \\
 &= \tilde{a}_{qtu}^{prs} + \hat{P}^{(r/s)} \eta_q^r \tilde{a}_{tu}^{ps} - \hat{P}_{(t/u)} \gamma_t^p \tilde{a}_{qu}^{rs} + \hat{P}_{(t/u)}^{(r/s)} \gamma_t^p \eta_q^r \tilde{a}_u^s
 \end{aligned}$$

3. Evaluate the following matrix element.¹

$$\langle \Phi_i^a | H - E_{\text{ref}} | \Phi_j^b \rangle = ?$$

Answer:

$$\begin{aligned} \langle \Phi_i^a | H - E_{\text{ref}} | \Phi_j^b \rangle &= f_p^q \langle \Phi | \tilde{a}_a^i \tilde{a}_q^p \tilde{a}_j^b | \Phi \rangle + \frac{1}{4} \bar{g}_{pq}^{rs} \langle \Phi | \tilde{a}_a^i \tilde{a}_{rs}^{pq} \tilde{a}_j^b | \Phi \rangle \\ \langle \Phi | \tilde{a}_a^i \tilde{a}_q^p \tilde{a}_j^b | \Phi \rangle &= \tilde{a}_{a \bullet q \bullet \bullet}^{i \circ p \bullet} \tilde{a}_{j \circ}^{b \bullet \bullet} + \tilde{a}_{a \bullet q \circ}^{i \circ p \circ \circ} \tilde{a}_{j \circ \circ}^{b \bullet \bullet} = \gamma_j^i (-\eta_a^p) (-\eta_q^b) + \gamma_q^i \gamma_j^p (-\eta_a^b) = \gamma_j^i \eta_a^p \eta_q^b - \gamma_q^i \gamma_j^p \eta_a^b \\ \langle \Phi | \tilde{a}_a^i \tilde{a}_{rs}^{pq} \tilde{a}_j^b | \Phi \rangle &= \hat{P}_{(r/s)}^{(p/q)} \tilde{a}_{a \bullet r \circ s \bullet \bullet}^{i \circ p \bullet q \circ \circ} \tilde{a}_{j \circ \circ}^{b \bullet \bullet} = \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i (-\eta_a^p) \gamma_j^q (-\eta_s^b) = \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i \eta_a^p \gamma_j^q \eta_s^b \end{aligned}$$

Substituting the last two equations into the first gives the final result.

$$\begin{aligned} \langle \Phi_i^a | H - E_{\text{ref}} | \Phi_j^b \rangle &= f_p^q (\gamma_j^i \eta_a^p \eta_q^b - \gamma_q^i \gamma_j^p \eta_a^b) + \frac{1}{4} \bar{g}_{pq}^{rs} \hat{P}_{(r/s)}^{(p/q)} \gamma_r^i \eta_a^p \gamma_j^q \eta_s^b \\ &= \gamma_j^i f_a^b - \eta_a^b f_j^i + \bar{g}_{aj}^{ib} \\ &= \delta_j^i f_a^b - \delta_a^b f_j^i + \bar{g}_{aj}^{ib} \end{aligned}$$

¹You may use either of the following equivalent expressions for the Hamiltonian. (I recommend the one on the right!)

$$H = h_p^q a_q^p + \frac{1}{4} \bar{g}_{pq}^{rs} a_{rs}^{pq} \qquad H = E_{\text{ref}} + f_p^q \tilde{a}_q^p + \frac{1}{4} \bar{g}_{pq}^{rs} \tilde{a}_{rs}^{pq}$$