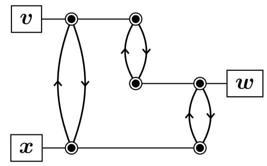
1. Give an example of each of the following.

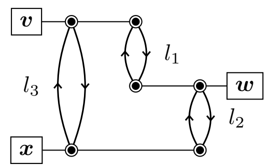
   (a) A closed, connected graph of at least two operators.

   Answer:



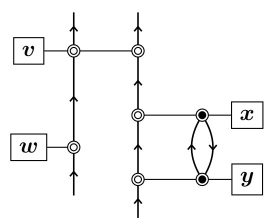
   (b) A Hugenholtz path of at least three lines that doesn't qualify as a Goldstone path.

   Answer: The sequence of lines (l_1, l_2, l_3) in the following graph.



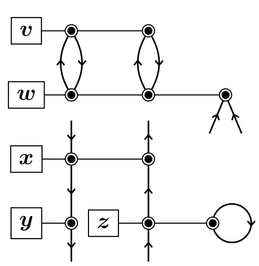
   (c) Non-equivalent, interchangeable subgraphs, where at least one subgraph contains multiple operators.

   Answer: The subgraphs $G[\{\boldsymbol{w}\}]$ and $G[\{\boldsymbol{x}, \boldsymbol{y}\}]$ in the following.

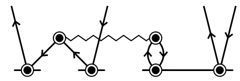


   (d) A graph that is disconnected and linked.

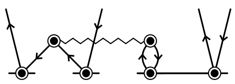
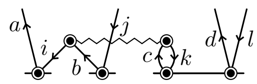
   Answer:



2. Interpret the following graph algebraically, and then simplify your expression as much as possible.[1]



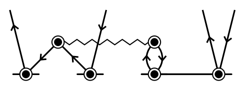
Answer:

 $= \sum\limits_{\substack{abcd\\ ijkl}}$ 

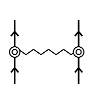
$= \sum\limits_{\substack{abcd\\ ijkl}} \overline{g}^{bc}_{ik} c^{i}_{a} c^{j}_{b} c^{kl}_{cd} \,\vdots\, a^{i^{\circ} k^{\circ\circ}}_{b^{\bullet} c^{\bullet\bullet}} a^{a}_{i^{\circ}} a^{b^{\bullet}}_{j} a^{c^{\bullet\bullet} d}_{k^{\circ\circ} l} \,\vdots$

$$\vdots\, a^{i^{\circ} k^{\circ\circ}}_{b^{\bullet} c^{\bullet\bullet}} a^{a}_{i^{\circ}} a^{b^{\bullet}}_{j} a^{c^{\bullet\bullet} d}_{k^{\circ\circ} l} \,\vdots = \tilde{a}^{i^{\circ} k^{\circ\circ} a b^{\bullet} c^{\bullet\bullet} d}_{b^{\bullet} c^{\bullet\bullet} i^{\circ} j k^{\circ\circ} l} = -\tilde{a}^{i^{\circ} k^{\circ\circ} a b^{\bullet} c^{\bullet\bullet} d}_{b^{\bullet} c^{\bullet\bullet} j i^{\circ} k^{\circ\circ} l} = -\gamma^{i}_{i} \gamma^{k}_{k} (-\eta^{b}_{b})(-\eta^{c}_{c}) \tilde{a}^{ad}_{jl} = -\tilde{a}^{ad}_{jl}$$

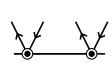
Substituting the second equation into the first gives the final answer.

 $= -\sum\limits_{\substack{abcd\\ ijkl}} \overline{g}^{bc}_{ik} c^{i}_{a} c^{j}_{b} c^{kl}_{cd}\, \tilde{a}^{ad}_{jl}$

[1] The operators in this graph are defined as follows.

 $\equiv \left(\frac{1}{2!}\right)^2 \sum\limits_{pqrs} \overline{g}^{rs}_{pq} \tilde{a}^{pq}_{rs}$

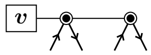
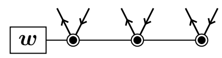
 $\equiv \sum\limits_{ia} c^{i}_{a} \tilde{a}^{a}_{i}$

 $\equiv \left(\frac{1}{2!}\right)^2 \sum\limits_{ijab} c^{ij}_{ab} \tilde{a}^{ab}_{ij}$

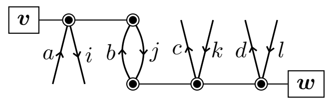
3. Write the following algebraic expression as a graph.[2]

$$\sum_{\substack{abcd\\ijkl}} \overline{v}_{ij}^{ab}\overline{w}_{bcd}^{jkl}\vdots a_{ab^\bullet}^{ij^\circ} a_{j^\circ kl}^{b^\bullet cd}\vdots$$

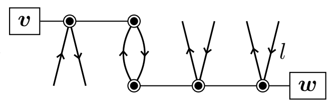
Answer: This is a contraction of the following two diagrams


$= \left(\frac{1}{2!}\right)^2 \sum_{ijab} \overline{v}_{ij}^{ab}\tilde{a}_{ij}^{ab}$

$= \left(\frac{1}{3!}\right)^2 \sum_{ijkabc} \overline{w}_{abc}^{ijk}\tilde{a}_{abc}^{ijk}$

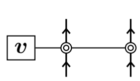
joining them by a hole line and a particle line. The labeled graph looks as follows.

$$\overline{v}_{ij}^{ab}\overline{w}_{bcd}^{jkl}\vdots a_{ab^\bullet}^{ij^\circ} a_{j^\circ kl}^{b^\bullet cd}\vdots =$$


Noting that $\boldsymbol{w}$ has two pairs of equivalent lines leads to the final result.

$$\sum_{\substack{abcd\\ijkl}} \overline{v}_{ij}^{ab}\overline{w}_{bcd}^{jkl}\vdots a_{ab^\bullet}^{ij^\circ} a_{j^\circ kl}^{b^\bullet cd}\vdots = 4 \cdot$$


[2]Use the following to denote the operators in your graph.

$\left(\frac{1}{2!}\right)^2 \sum_{pqrs} \overline{v}_{pq}^{rs}\tilde{a}_{rs}^{pq} \equiv$

$\left(\frac{1}{3!}\right)^2 \sum_{\substack{pqr\\stu}} \overline{w}_{pqr}^{stu}\tilde{a}_{stu}^{pqr} \equiv$