

1. Explain why each of the following terms vanishes.

$$(a) \frac{1}{5!} \langle \Phi_{ijklm}^{abcde} | V_c T_1^5 | \Phi \rangle_C \quad (b) \langle \Phi_{ij}^{ab} | V_c T_2 T_3 | \Phi \rangle_C \quad (c) \frac{1}{2!} \langle \Phi_{ijkl}^{abcd} | V_c T_1^2 | \Phi \rangle_C \quad (d) \frac{1}{2!} \langle \Phi_{ijk}^{abc} | V_c T_1^2 | \Phi \rangle_C$$

Answer:

- (a) Because V_c has at most four lines available for contraction and there are five T -operators, there is no way to satisfy the connectedness requirement.
- (b) Because the net excitation level of the T -operators and the bare excitation operator is $2 + 3 - 2 = +3$, and V_c has no contribution with excitation level -3 .
- (c) The net excitaton level of the T -operators and the bare excitation operator is $1 + 1 - 4 = -2$, so we need the $+2$ component of V_c to balance the product. This diagram has no quasiparticle annihilation lines and therefore cannot contract with the T operators to satisfy the connectedness requirement.

$$\left(\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \right)_C = 0$$

- (d) The net excitation level of the T -operators and the bare excitation operator is $1 + 1 - 3 = -1$, so we need a $+1$ component from V_c to balance the product. Of the three diagrams in V_c with excitation level $+1$, one of them has no quasiparticle annihilation lines and the other two have only one. Therefore, only one of the T operators can be connected to V_c and there is no way to satisfy the connectedness requirement for the other one.

$$\left(\begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \right)_C = 0$$

$$\left(\begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \right)_C = 0$$

$$\left(\begin{array}{c} \text{Diagram 7} \\ \text{Diagram 8} \end{array} \right)_C = 0$$

2. Interpret the following graph and fully simplify your answer.

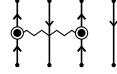


Answer: Denote the bare excitation operator at the top by \tilde{a}_{ab}^{ij} .

$$\text{Diagram} = (-)^{4+3} \frac{1}{2!2!} \hat{P}_{(a/b)}^{(i/j)} \bar{g}_{kl}^{cd} t_{ca}^{kl} t_{db}^{ij} = -\frac{1}{2!} \hat{P}_{(a/b)} \bar{g}_{kl}^{cd} t_{ca}^{kl} t_{db}^{ij}$$

(Implicit summation over k, l, c, d .)

3. Interpret the following graph and fully simplify it the “long way.” That is, you may use Rules 1-3 but you must start from Axiom 1 and show each step to get to your final answer.



Answer: Denote the top bare excitation operator by \tilde{a}_{ab}^{ij} and the bottom one by \tilde{a}_{kl}^{cd} . Axiom 1 gives

$$\left| \begin{array}{c} \text{Diagram} \end{array} \right| = \frac{1}{2!2!2!} \sum_{\substack{efgh \\ mn}} \left| \begin{array}{c} e \quad m \quad f \quad n \\ g \quad h \end{array} \right| = \frac{1}{2!2!2!} \sum_{\substack{efgh \\ mn}} \hat{P}_{(a/b)}^{(i/j)} \delta_m^i \delta_n^j \delta_a^e \delta_b^f \bar{g}_{ef}^{gh} \hat{P}_{(k/l)}^{(c/d)} \delta_k^m \delta_l^n \delta_g^c \delta_h^d \left| \begin{array}{c} \tilde{a}_{e \bullet 1 f \bullet 2}^{m \circ 1 n \circ 2} \tilde{a}_{g \bullet 3 h \bullet 4}^{c \bullet 1 f \bullet 2} \tilde{a}_{m \circ 1 n \circ 2}^{g \bullet 3 h \bullet 4} \end{array} \right|$$

where I have used Rule 1 to determine the degeneracy factor: three pairs of equivalent lines contribute $2!$ each, and there are no equivalent subgraphs. Using Rule 2, the operator string evaluates as follows

$$\left| \begin{array}{c} \tilde{a}_{e \bullet 1 f \bullet 2}^{m \circ 1 n \circ 2} \tilde{a}_{g \bullet 3 h \bullet 4}^{c \bullet 1 f \bullet 2} \tilde{a}_{m \circ 1 n \circ 2}^{g \bullet 3 h \bullet 4} \end{array} \right| = (-1)^{2+2} = +1$$

since there are two hole lines and two loops. Summing over the line labels gives the following.

$$\left| \begin{array}{c} \text{Diagram} \end{array} \right| = \frac{1}{2!2!2!} \sum_{\substack{efgh \\ mn}} \hat{P}_{(a/b)}^{(i/j)} \delta_m^i \delta_n^j \delta_a^e \delta_b^f \bar{g}_{ef}^{gh} \hat{P}_{(k/l)}^{(c/d)} \delta_k^m \delta_l^n \delta_g^c \delta_h^d = \frac{1}{2!2!2!} \hat{P}_{(a/b)}^{(i/j)} \hat{P}_{(k/l)}^{(c/d)} \bar{g}_{ab}^{cd} \delta_k^i \delta_l^j$$

Using Rule 3, we can cancel the degeneracy factors for equivalent lines connected to the top bare excitation operator against $\hat{P}_{(a/b)}^{(i/j)}$.

$$\left| \begin{array}{c} \text{Diagram} \end{array} \right| = \frac{1}{2!2!2!} \cancel{\hat{P}_{(a/b)}^{(i/j)}} \hat{P}_{(k/l)}^{(c/d)} \bar{g}_{ab}^{cd} \delta_k^i \delta_l^j = \frac{1}{2!} \hat{P}_{(k/l)}^{(c/d)} \bar{g}_{ab}^{cd} \delta_k^i \delta_l^j$$

Applying Rule 3 to the lower operator, we can cancel the permutation over c, d but not the one over k, l , since the degeneracy factor for the hole lines was already canceled in the previous step.

$$\left| \begin{array}{c} \text{Diagram} \end{array} \right| = \frac{1}{2!} \cancel{\hat{P}_{(k/l)}^{(c/d)}} \bar{g}_{ab}^{cd} \delta_k^i \delta_l^j = \hat{P}_{(k/l)} \bar{g}_{ab}^{cd} \delta_k^i \delta_l^j$$

Extra Credit. Prove Rule 3 for a closed graph with a single bare excitation operator of the following form.

$$\tilde{a}_{a_1 \dots a_m}^{i_1 \dots i_m} = \left(\frac{1}{m!}\right)^2 \bar{\delta}_{j_1 \dots j_m}^{b_1 \dots b_m} \tilde{a}_{b_1 \dots b_m}^{j_1 \dots j_m} \quad \bar{\delta}_{j_1 \dots j_m}^{b_1 \dots b_m} \equiv \hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/i_m)} \delta_{j_1}^{i_1} \dots \delta_{j_m}^{i_m} \delta_{a_1}^{b_1} \dots \delta_{a_m}^{b_m}$$

Answer: Using Axiom 1, a closed graph containing this bare excitation operator will have the form

$$\frac{1}{|I_1|! \dots |I_k|! |A_1|! \dots |A_h|!} \sum_{\substack{b_1 \dots b_m \\ j_1 \dots j_m}} \bar{\delta}_{j_1 \dots j_m}^{b_1 \dots b_m} T_{b_1 \dots b_m}^{j_1 \dots j_m} = \frac{1}{|I_1|! \dots |I_k|! |A_1|! \dots |A_h|!} \hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/i_m)} T_{a_1 \dots a_m}^{i_1 \dots i_m}$$

where $I_1 \cup \dots \cup I_k = \{i_1, \dots, i_m\}$ and $A_1 \cup \dots \cup A_h = \{a_1, \dots, a_m\}$ partition the indices of the bare excitation operator into subsets that fall on equivalent coefficient lines. Any remaining degeneracy factors, interaction tensors, or contracted operators are contained in $T_{a_1 \dots a_m}^{i_1 \dots i_m}$. According to the definition of equivalent lines, then, the indices in a given subset I_p or A_q must occur on a single interaction tensor in $T_{a_1 \dots a_m}^{i_1 \dots i_m}$, which is therefore already antisymmetric with respect to these indices. Denoting the indices in I_p by $\{i_{p,1}, \dots, i_{p,|I_p|}\}$, this enables the following cancellation.

$$\hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/i_m)} T_{a_1 \dots a_m}^{i_1 \dots i_m} = \hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/I_p/\dots/i_m)} \hat{P}^{(i_{p,1}, \dots, i_{p,|I_p|})}_{(a_1/\dots/a_m)} T_{a_1 \dots a_m}^{i_1 \dots i_m} = |I_p|! \hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/I_p/\dots/i_m)} T_{a_1 \dots a_m}^{i_1 \dots i_m}$$

Repeating this procedure for the remaining subsets completes the proof.

$$\frac{1}{|I_1|! \dots |I_k|! |A_1|! \dots |A_h|!} \hat{P}_{(a_1/\dots/a_m)}^{(i_1/\dots/i_m)} T_{a_1 \dots a_m}^{i_1 \dots i_m} = \frac{1}{|I_1|! \dots |I_k|! |A_1|! \dots |A_h|!} \hat{P}_{(A_1/\dots/A_h)}^{(I_1/\dots/I_k)} T_{a_1 \dots a_m}^{i_1 \dots i_m} = \hat{P}_{(A_1/\dots/A_h)}^{(I_1/\dots/I_k)} T_{a_1 \dots a_m}^{i_1 \dots i_m}$$

Appendix.

Axiom 1. The algebraic of a graph G is obtained from a corresponding summand graph $\Sigma(G)$ as follows.

$$G = \frac{1}{\text{dg}(G)} \sum_{\text{labels}} \Sigma(G)$$

Rule 1. Each set of k equivalent lines or equivalent subgraphs contributes a factor of $k!$ to the degeneracy.

Rule 2. The overall sign of a closed graph is $(-)^{h+l}$, where h and l denote the total number of hole lines and loops.

Rule 3. For bare excitation operators, cancel the degeneracy factors from their equivalent coefficient lines by replacing the full antisymmetrizer, $P_{(q_1/\dots/q_m)}^{(p_1/\dots/p_m)}$, with a reduced antisymmetrizer over inequivalent coefficient lines, $\hat{P}_{(Q_1/\dots/Q_k)}^{(P_1/\dots/P_h)}$.^{1 2}

¹Here $\{p_1, \dots, p_m\} = P_1 \cup \dots \cup P_h$ and $\{q_1, \dots, q_m\} = Q_1 \cup \dots \cup Q_k$ are the upper and lower indices on the bare excitation operator $\tilde{a}_{q_1 \dots q_m}^{p_1 \dots p_m}$, and the P_i 's and Q_i 's label subsets of equivalent coefficient lines.

²For equivalent lines connecting two bare excitation operators, this cancellation can only be performed once.