1. Explain why each of the following terms vanishes.

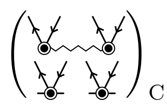
   (a) $\frac{1}{5!}\langle\Phi_{ijklm}^{abcde}|V_\mathrm{c}T_1^5|\Phi\rangle_\mathrm{C}$ (b) $\langle\Phi_{ij}^{ab}|V_\mathrm{c}T_2T_3|\Phi\rangle_\mathrm{C}$ (c) $\frac{1}{2!}\langle\Phi_{ijkl}^{abcd}|V_\mathrm{c}T_1^2|\Phi\rangle_\mathrm{C}$ (d) $\frac{1}{2!}\langle\Phi_{ijk}^{abc}|V_\mathrm{c}T_1^2|\Phi\rangle_\mathrm{C}$

   Answer:

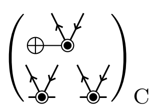
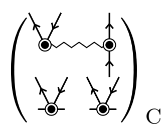
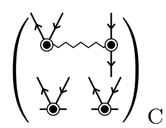
   (a) Because V_c has at most four lines available for contraction and there are five T-operators, there is no way to satisfy the connectedness requirement.

   (b) Because the net excitation level of the T-operators and the bare excitation operator is $2+3-2=+3$, and V_c has no contribution with excitation level -3.

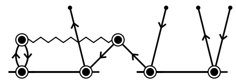
   (c) The net excitaton level of the T-operators and the bare excitation operator is $1+1-4=-2$, so we need the $+2$ component of V_c to balance the product. This diagram has no quasiparticle annihilation lines and therefore cannot contract with the T operators to satisfy the connectedness requirement.

    $= 0$

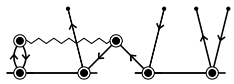
   (d) The net excitation level of the T-operators and the bare excitation operator is $1+1-3=-1$, so we need a $+1$ component from V_c to balance the product. Of the three diagrams in V_c with excitation level $+1$, one of them has no quasiparticle annihilation lines and the other two have only one. Therefore, only one of the T operators can be connected to V_c and there is no way to satisfy the connectedness requirement for the other one.

    $= 0$  $= 0$  $= 0$

2. Interpret the following graph and fully simplify your answer.



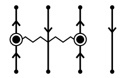
Answer: Denote the bare excitation operator at the top by $\tilde{a}^{ij}_{ab}$.



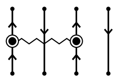
$$= (-)^{4+3} \frac{1}{2!2!} \hat{P}^{(i/j)}_{(a/b)} \overline{g}^{cd}_{kl} t^{kl}_{ca} t^{ij}_{db} = -\frac{1}{2!} \hat{P}_{(a/b)} \overline{g}^{cd}_{kl} t^{kl}_{ca} t^{ij}_{db}$$

(Implicit summation over k, l, c, d.)

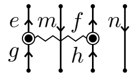
3. Interpret the following graph and fully simplify it the "long way." That is, you may use Rules 1-3 but you must start from Axiom 1 and show each step to get to your final answer.



Answer: Denote the top bare excitation operator by $\tilde{a}^{ij}_{ab}$ and the bottom one by $\tilde{a}^{cd}_{kl}$. Axiom 1 gives



$$= \frac{1}{2!2!2!} \sum_{\substack{efgh\\mn}}$$

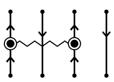


$$= \frac{1}{2!2!2!} \sum_{\substack{efgh\\mn}} \hat{P}^{(i/j)}_{(a/b)} \delta^i_m \delta^j_n \delta^e_a \delta^f_b \, \overline{g}^{gh}_{ef} \, \hat{P}^{(c/d)}_{(k/l)} \delta^m_k \delta^n_l \delta^c_g \delta^d_h \vdots \tilde{a}^{m^{\circ 1} n^{\circ 2}}_{e^{\bullet 1} f^{\bullet 2}} \, \tilde{a}^{e^{\bullet 1} f^{\bullet 2}}_{g^{\bullet 3} h^{\bullet 4}} \tilde{a}^{g^{\bullet 3} h^{\bullet 4}}_{m^{\circ 1} n^{\circ 2}} \vdots$$

where I have used Rule 1 to determine the degeneracy factor: three pairs of equivalent lines contribute 2! each, and there are no equivalent subgraphs. Using Rule 2, the operator string evaluates as follows

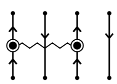
$$\vdots \tilde{a}^{m^{\circ 1} n^{\circ 2}}_{e^{\bullet 1} f^{\bullet 2}} \, \tilde{a}^{e^{\bullet 1} f^{\bullet 2}}_{g^{\bullet 3} h^{\bullet 4}} \tilde{a}^{g^{\bullet 3} h^{\bullet 4}}_{m^{\circ 1} n^{\circ 2}} \vdots = (-1)^{2+2} = +1$$

since there are two hole lines and two loops. Summing over the line labels gives the following.



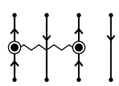
$$= \frac{1}{2!2!2!} \sum_{\substack{efgh\\mn}} \hat{P}^{(i/j)}_{(a/b)} \delta^i_m \delta^j_n \delta^e_a \delta^f_b \, \overline{g}^{gh}_{ef} \, \hat{P}^{(c/d)}_{(k/l)} \delta^m_k \delta^n_l \delta^c_g \delta^d_h = \frac{1}{2!2!2!} \hat{P}^{(i/j)}_{(a/b)} \hat{P}^{(c/d)}_{(k/l)} \overline{g}^{cd}_{ab} \delta^i_k \delta^j_l$$

Using Rule 3, we can cancel the degeneracy factors for equivalent lines connected to the top bare excitation operator against $\hat{P}^{(i/j)}_{(a/b)}$.



$$= \frac{1}{\cancel{2!}\cancel{2!}2!} \cancel{\hat{P}^{(i/j)}_{(a/b)}} \hat{P}^{(c/d)}_{(k/l)} \overline{g}^{cd}_{ab} \delta^i_k \delta^j_l = \frac{1}{2!} \hat{P}^{(c/d)}_{(k/l)} \overline{g}^{cd}_{ab} \delta^i_k \delta^j_l$$

Applying Rule 3 to the lower operator, we can cancel the permutation over c, d but not the one over k, l, since the degeneracy factor for the hole lines was already canceled in the previous step.



$$= \frac{\cancel{1}}{\cancel{2!}} \hat{P}^{\cancel{(c/d)}}_{(k/l)} \overline{g}^{cd}_{ab} \delta^i_k \delta^j_l = \hat{P}_{(k/l)} \overline{g}^{cd}_{ab} \delta^i_k \delta^j_l$$

Extra Credit. Prove Rule 3 for a closed graph with a single bare excitation operator of the following form.

$$\tilde{a}^{i_1\cdots i_m}_{a_1\cdots a_m} = (\tfrac{1}{m!})^2\, \bar{\delta}^{b_1\cdots b_m}_{j_1\cdots j_m}\, \tilde{a}^{j_1\cdots j_m}_{b_1\cdots b_m} \qquad\qquad \bar{\delta}^{b_1\cdots b_m}_{j_1\cdots j_m} \equiv \hat{P}^{(i_1/\cdots/i_m)}_{(a_1/\cdots/a_m)}\delta^{i_1}_{j_1}\cdots\delta^{i_m}_{j_m}\delta^{b_1}_{a_1}\cdots\delta^{b_m}_{a_m}$$

Answer: Using Axiom 1, a closed graph containing this bare excitation operator will have the form

$$\frac{1}{|I_1|!\cdots|I_k|!|A_1|!\cdots|A_h|!}\sum_{\substack{b_1\cdots b_m\\ j_1\cdots j_m}} \bar{\delta}^{b_1\cdots b_m}_{j_1\cdots j_m}\, T^{j_1\cdots j_m}_{b_1\cdots b_m} = \frac{1}{|I_1|!\cdots|I_k|!|A_1|!\cdots|A_h|!}\hat{P}^{(i_1/\cdots/i_m)}_{(a_1/\cdots/a_m)}T^{i_1\cdots i_m}_{a_1\cdots a_m}$$

where $I_1\cup\cdots\cup I_k = \{i_1,\ldots,i_m\}$ and $A_1\cup\cdots\cup A_h = \{a_1,\ldots,a_m\}$ partition the indices of the bare excitation operator into subsets that fall on equivalent coefficient lines. Any remaining degeneracy factors, interaction tensors, or contracted operators are contained in $T^{i_1\cdots i_m}_{a_1\cdots a_m}$. According to the definition of equivalent lines, then, the indices in a given subset I_p or A_q must occur on a single interaction tensor in $T^{i_1\cdots i_m}_{a_1\cdots a_m}$, which is therefore already antisymmetric with respect to these indices. Denoting the indices in I_p by $\{i_{p,1},\ldots,i_{p,|I_p|}\}$, this enables the following cancellation.

$$\hat{P}^{(i_1/\cdots/i_m)}_{(a_1/\cdots/a_m)}T^{i_1\cdots i_m}_{a_1\cdots a_m} = \hat{P}^{(i_1/\cdots/I_p/\cdots/i_m)}_{(a_1/\cdots/a_m)}\,\hat{P}^{(i_{p,1},\ldots,i_{p,|I_p|})}T^{i_1\cdots i_m}_{a_1\cdots a_m} = |I_p|!\hat{P}^{(i_1/\cdots/I_p/\cdots/i_m)}_{(a_1/\cdots/a_m)}T^{i_1\cdots i_m}_{a_1\cdots a_m}$$

Repeating this procedure for the remaining subsets completes the proof.

$$\frac{1}{|I_1|!\cdots|I_k|!|A_1|!\cdots|A_h|!}\hat{P}^{(i_1/\cdots/i_m)}_{(a_1/\cdots/a_m)}T^{i_1\cdots i_m}_{a_1\cdots a_m} = \frac{1}{\cancel{|I_1|!}\cdots\cancel{|I_k|!}\cancel{|A_1|!}\cdots\cancel{|A_h|!}}\hat{P}^{(I_1/\cdots/I_k)}_{(A_1/\cdots/A_h)}T^{i_1\cdots i_m}_{a_1\cdots a_m} = \hat{P}^{(I_1/\cdots/I_k)}_{(A_1/\cdots/A_h)}T^{i_1\cdots i_m}_{a_1\cdots a_m}$$

Appendix.

Axiom 1. The algebraic of a graph G is obtained from a corresponding summand graph $\Sigma(G)$ as follows.

$$G = \frac{1}{\mathrm{dg}(G)}\sum_{\text{labels}}\Sigma(G)$$

Rule 1. Each set of k equivalent lines or equivalent subgraphs contributes a factor of $k!$ to the degeneracy.
Rule 2. The overall sign of a closed graph is $(-)^{h+l}$, where h and l denote the total number of hole lines and loops.
Rule 3. For bare excitation operators, cancel the degeneracy factors from their equivalent coefficient lines by replacing the full antisymmetrizer, $P^{(p_1/\cdots/p_m)}_{(q_1/\cdots/q_m)}$, with a reduced antisymmetrizer over inequivalent coefficient lines, $\hat{P}^{(P_1/\cdots/P_h)}_{(Q_1/\cdots/Q_k)}$.[1] [2]

[1]Here $\{p_1,\ldots,p_m\} = P_1\cup\cdots\cup P_h$ and $\{q_1,\ldots,q_m\} = Q_1\cup\cdots\cup Q_k$ are the upper and lower indices on the bare excitation operator $\tilde{a}^{p_1\cdots p_m}_{q_1\cdots q_m}$, and the P_i's and Q_i's label subsets of equivalent coefficient lines.

[2]For equivalent lines connecting two bare excitation operators, this cancellation can only be performed once.