1. Translate the following expression from KM notation into our original notation, using daggers to denote creation operators and lines to denote contractions.

$$\vdots\, a^{p^{\circ}q^{\bullet}\;\;r^{\bullet\bullet}}_{s^{\bullet}t^{\bullet\bullet}u^{\circ}} \,\vdots = ?$$

Your final expression should be a Φ-normal-ordered string of six operators with three contraction lines.

Answer:

$$\vdots\, a^{p^{\circ}q^{\bullet}\;\;r^{\bullet\bullet}}_{s^{\bullet}t^{\bullet\bullet}u^{\circ}} \,\vdots = \vdots\, a^{p^{\circ}q^{\bullet}r^{\bullet\bullet}}_{u^{\circ}s^{\bullet}t^{\bullet\bullet}} \,\vdots = \vdots\, a^{p^{\circ}}_{u^{\circ}} a^{q^{\bullet}}_{s^{\bullet}} a^{r^{\bullet\bullet}}_{t^{\bullet\bullet}} \,\vdots = \vdots\, \overline{a_p^{\dagger} a_u}(-\overline{a_s a_q^{\dagger}})(-\overline{a_t a_r^{\dagger}}) \,\vdots = \vdots\, \overline{a_p^{\dagger} a_u}\,\overline{a_s a_q^{\dagger}}\,\overline{a_t a_r^{\dagger}} \,\vdots$$

2. Expand the following as a linear combination of Φ-normal-ordered operators.

$$\tilde{a}^{p}_{q}\tilde{a}^{rs}_{tu} = ?$$

Answer:

$$\begin{aligned}
\tilde{a}^{p}_{q}\tilde{a}^{rs}_{tu} &= \tilde{a}^{prs}_{qtu} + \hat{P}^{(r/s)}\tilde{a}^{p\phantom{{}^{\bullet}t}r^{\bullet}s}_{q^{\bullet}t\phantom{r^{\bullet}}u} + \hat{P}_{(t/u)}\tilde{a}^{p^{\circ}r\phantom{t^{\circ}}s}_{qt^{\circ}u} + \hat{P}^{(r/s)}_{(t/u)}\tilde{a}^{p^{\circ}r^{\bullet}s}_{q^{\bullet}t^{\circ}u} \\
&= \tilde{a}^{prs}_{qtu} - \hat{P}^{(r/s)}\tilde{a}^{r^{\bullet}ps}_{q^{\bullet}tu} - \hat{P}_{(t/u)}\tilde{a}^{p^{\circ}rs}_{t^{\circ}qu} - \hat{P}^{(r/s)}_{(t/u)}\tilde{a}^{p^{\circ}r^{\bullet}s}_{t^{\circ}q^{\bullet}u} \\
&= \tilde{a}^{prs}_{qtu} - \hat{P}^{(r/s)}(-\eta^{r}_{q})\tilde{a}^{ps}_{tu} - \hat{P}_{(t/u)}\gamma^{p}_{t}\tilde{a}^{rs}_{qu} - \hat{P}^{(r/s)}_{(t/u)}\gamma^{p}_{t}(-\eta^{r}_{q})\tilde{a}^{s}_{u} \\
&= \tilde{a}^{prs}_{qtu} + \hat{P}^{(r/s)}\eta^{r}_{q}\tilde{a}^{ps}_{tu} - \hat{P}_{(t/u)}\gamma^{p}_{t}\tilde{a}^{rs}_{qu} + \hat{P}^{(r/s)}_{(t/u)}\gamma^{p}_{t}\eta^{r}_{q}\tilde{a}^{s}_{u}
\end{aligned}$$

3. Evaluate the following matrix element.[1]

$$\langle\Phi_i^a|H_e - E_{\mathrm{ref}}|\Phi_j^b\rangle = \,?$$

Answer:

$$\begin{aligned}
\langle\Phi_i^a|H_e - E_{\mathrm{ref}}|\Phi_j^b\rangle &= f_p^q\langle\Phi|\tilde{a}_a^i\tilde{a}_q^p\tilde{a}_j^b|\Phi\rangle + \tfrac{1}{4}\overline{g}_{pq}^{rs}\langle\Phi|\tilde{a}_a^i\tilde{a}_{rs}^{pq}\tilde{a}_j^b|\Phi\rangle \\
\langle\Phi|\tilde{a}_a^i\tilde{a}_q^p\tilde{a}_j^b|\Phi\rangle &= \tilde{a}^{i^{\circ}p^{\bullet}b^{\bullet\bullet}}_{a^{\bullet}q^{\bullet\bullet}j^{\circ}} + \tilde{a}^{i^{\circ}p^{\circ\circ}b^{\bullet}}_{a^{\bullet}q^{\circ}j^{\circ\circ}} = \gamma_j^i(-\eta_a^p)(-\eta_q^b) + \gamma_q^i\gamma_j^p(-\eta_a^b) = \gamma_j^i\eta_a^p\eta_q^b - \gamma_q^i\gamma_j^p\eta_a^b \\
\langle\Phi|\tilde{a}_a^i\tilde{a}_{rs}^{pq}\tilde{a}_j^b|\Phi\rangle &= \hat{P}_{(r/s)}^{(p/q)}\tilde{a}^{i^{\circ}p^{\bullet}q^{\circ\circ}b^{\bullet\bullet}}_{a^{\bullet}r^{\circ}s^{\bullet\bullet}j^{\circ\circ}} = \hat{P}_{(r/s)}^{(p/q)}\gamma_r^i(-\eta_a^p)\gamma_j^q(-\eta_s^b) = \hat{P}_{(r/s)}^{(p/q)}\gamma_r^i\eta_a^p\gamma_j^q\eta_s^b
\end{aligned}$$

Substituting the last two equations into the first gives the final result.

$$\begin{aligned}
\langle\Phi_i^a|H_e - E_{\mathrm{ref}}|\Phi_j^b\rangle &= f_p^q(\gamma_j^i\eta_a^p\eta_q^b - \gamma_q^i\gamma_j^p\eta_a^b) + \tfrac{1}{4}\overline{g}_{pq}^{rs}\hat{P}_{(r/s)}^{(p/q)}\gamma_r^i\eta_a^p\gamma_j^q\eta_s^b \\
&= \gamma_j^i f_a^b - \eta_a^b f_j^i + \overline{g}_{aj}^{ib} \\
&= \delta_j^i f_a^b - \delta_a^b f_j^i + \overline{g}_{aj}^{ib}
\end{aligned}$$

[1] You may use either of the following equivalent expressions for the Hamiltonian. (I recommend the one on the right!)

$$H_e = h_p^q a_q^p + \tfrac{1}{4}\overline{g}_{pq}^{rs}a_{rs}^{pq} \qquad\qquad H_e = E_{\mathrm{ref}} + f_p^q\tilde{a}_q^p + \tfrac{1}{4}\overline{g}_{pq}^{rs}\tilde{a}_{rs}^{pq}$$