1. Prove the Hausdorff expansion.[1]

$$e^X Y e^{-X} = Y + [X, Y] + \tfrac{1}{2!}[X, [X, Y]] + \tfrac{1}{3!}[X, [X, [X, Y]]] + \cdots \tag{1}$$

2. Prove the following.

$$[H_\mathrm{c}, T] = \vdots H_\mathrm{c} T \vdots \qquad [[H_\mathrm{c}, T], T] = \vdots H_\mathrm{c} T T \vdots \qquad [[[H_\mathrm{c}, T], T], T] = \vdots H_\mathrm{c} T T T \vdots \qquad \cdots \tag{2}$$

3. Using equation 2, explain the following.

$$[\,\cdot\,, T]^n (H_\mathrm{c}) = \vdots H_\mathrm{c} T T T T \vdots \text{ for } n \geq 4 \tag{3}$$

4. Prove that the determinant basis consists of eigenfunctions of the diagonal Fock operator.

$$H_0 \Phi_{i_1 \cdots i_k}^{a_1 \cdots a_k} = \mathcal{E}_{i_1 \cdots i_k}^{a_1 \cdots a_k} \Phi_{i_1 \cdots i_k}^{a_1 \cdots a_k} \qquad H_0 \equiv f_p^p \tilde{a}_p^p \qquad \mathcal{E}_{i_1 \cdots i_k}^{a_1 \cdots a_k} \equiv \sum_{r=1}^{k} f_{a_r}^{a_r} - \sum_{r=1}^{k} f_{i_r}^{i_r} \tag{4}$$

5. Use equations 2 and 4 to write the coupled-cluster amplitude equation $\langle \Phi_{ij\cdots}^{ab\cdots} | \overline{H}_\mathrm{c} | \Phi \rangle = 0$ as follows.

$$t_{ab\cdots}^{ij\cdots} = (\mathcal{E}_{ab\cdots}^{ij\cdots})^{-1} \langle \Phi_{ab\cdots}^{ij\cdots} | V_\mathrm{c} \exp(T) | \Phi \rangle_\mathrm{C} \qquad V_\mathrm{c} \equiv H_\mathrm{c} - H_0 \tag{5}$$

6. Explain why the following terms vanish.[3]

$$\tfrac{1}{4!} \langle \Phi_i^a | V_\mathrm{c} T_1^4 | \Phi \rangle_\mathrm{C} \qquad \langle \Phi_{ijk}^{abc} | V_\mathrm{c} | \Phi \rangle_\mathrm{C} \qquad \langle \Phi_{ijk}^{abc} | V_\mathrm{c} T_1 | \Phi \rangle_\mathrm{C}$$

[1] For a slick alternative to the proof in the notes, look at the solution to exercise 3.1.6 in Helgaker's big purple book, but note that you should give a proper proof by induction.[2]

[2] See `https://en.wikipedia.org/wiki/Mathematical_induction`.

[3] Hint: Use arguments about the excitation levels of their operators.