Lecture 3.4: Deriving Wick's Theorem

1 Induction Proofs

We will use proofs by induction in order to prove Wick's theorem. An induction proof has two steps:

1. Prove statement is true for n=1
2. Assuming that the statement is true for n=k, prove it is true for k+1

The idea is that if you can show a statement is true for the first case n=1, and show that it is true if you add one to n, then it is generally true.

2 Proof of Wick's Theorem

To prove Wick's Theorem, we will first prove two lemmas:

Lemma 1:

$$n[x_1 \cdots x_m]x_{m+1} = n[x_1 \cdots x_m x_{m+1}] + \sum_{i}^{m} n[x_1 \cdots \underbracket{x_i \cdots x_m x_{m+1}}]$$

Lemma 2:

$$n[x_1 \cdots \underbracket{x_i \cdots} x_m]x_{m+1} = n[x_1 \cdots \underbracket{x_i \cdots} x_m x_{m+1}] + \sum_{\substack{j=1 \\ j \notin C}}^{m} n[x_1 \cdots \underbracket{x_i \cdots} \underbracket{x_j \cdots x_m x_{m+1}}]$$

Then we will use these lemmas to prove Wick's Theorem by induction.

2.1 Lemma 1

$$n[x_1 \cdots x_m]x_{m+1} = n[x_1 \cdots x_m x_{m+1}] + \sum_{i}^{m} n[x_1 \cdots \underbracket{x_i \cdots x_m x_{m+1}}]$$

In words, this lemma says that if you multiply a creation or annihilation operator x_{m+1} with a normal ordered string of creation and annihilation operators, it is equal to the normal ordered string of creation and annihilation operators, including x_{m+1}, plus all possible contractions involving x_{m+1} and the rest of the operators.

There are two cases to look at for this proof: A, x_{m+1} is an annihilation operator a_r and B, x_{m+1} is a creation operator $a_r^\dagger$.

Case A:

The case of $x_{m+1} = a_r$ is easy to prove. If $x_{m+1} = a_r$, then a_r can simply be absorbed into the normal product because it is on the right side of the expression:

$$n[x_1 \cdots x_m]a_r = n[x_1 \cdots x_m x_{m+1} a_r]$$

Furthermore, all contractions involving a_r has a_r on the right side, and will go to 0. We have thus proved Lemma 1 for $x_{m+1} = a_r$.

Case B:

There are two subcases for case B, $x_{m+1} = a_r^\dagger$.

Case B1: all $x_1 \cdots x_m$ are annihilation operators, $a_{p_1} \cdots a_{p_m}$.
Case B2: some $x_1 \cdots x_m$ are creation operators.

We will first prove case B1 by induction.

For m = 1:

$$
\begin{aligned}
n[a_{p_1}]a_r^\dagger &= a_{p_1} a_r^\dagger \\
&= \delta_{p_1 r} - a_r^\dagger a_{p_1} \\
&= n[\underbracket{a_{p_1} a_r^\dagger}] + n[a_{p_1} a_r^\dagger] \\
&= n[a_{p_1} a_r^\dagger + n[\underbracket{a_{p_1} a_r^\dagger}]
\end{aligned}
$$

Lemma 1 holds for m = 1. Now, we will take the induction step. Assume that Lemma 1 is true for $m = l$,

$$
n[a_{p_1} \cdots a_{p_l}]a_r^\dagger = n[a_{p_1} \cdots a_{p_l} a_r^\dagger] + \sum_i^l n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}]
$$

show that it is true for $m = l + 1$:

We will first start by multiplying $a_{p_{l+1}}$ from the left on both sides:

$$
a_{p_{l+1}} n[a_{p_1} \cdots a_{p_l}]a_r^\dagger = a_{p_{l+1}} n[a_{p_1} \cdots a_{p_l} a_r^\dagger] + \sum_i^l a_{p_{l+1}} n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}]
$$

We can now rewrite the right side of the equation by realizing that $a_{p_1} \cdots a_{p_l}$ is already in normal order and can be taken out of the normal product. After some rearrangement, we get:

$$
\begin{aligned}
a_{p_{l+1}} n[a_{p_1} \cdots a_{p_l}]a_r^\dagger &= a_{p_{l+1}} n[a_{p_1} \cdots a_{p_l} a_r^\dagger] + \sum_i^l a_{p_{l+1}} n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}] \\
a_{p_{l+1}} a_{p_1} \cdots a_{p_l} a_r^\dagger &= \text{“ ”} \\
(-1)^l a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger &= \text{“ ”} \\
(-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}]a_r^\dagger &= \text{“ ”}
\end{aligned}
$$

We now rearrange the first term on the righthand side of the equation. We first write the normal product of $n[a_{p_1} \cdots a_{p_l} a_r^\dagger]$, and then rewrite $a_{p_{l+1}} a_r^\dagger$:

$$
\begin{aligned}
(-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}]a_r^\dagger &= a_{p_{l+1}} n[a_{p_1} \cdots a_{p_l} a_r^\dagger] + \sum_i^l a_{p_{l+1}} n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}] \\
&= a_{p_{l+1}} (-1)^l a_r^\dagger a_{p_1} \cdots a_{p_l} + \text{“ ”} \\
&= (-1)^l a_{p_{l+1}} a_r^\dagger a_{p_1} \cdots a_{p_l} + \text{“ ”} \\
&= (-1)^l (n[a_{p_{l+1}} a_r^\dagger] + \underbracket{a_{p_{l+1}} a_r^\dagger}) a_{p_1} \cdots a_{p_l} + \text{“ ”} \\
&= (-1)^l n[a_{p_{l+1}} a_r^\dagger] a_{p_1} \cdots a_{p_l} + (-1)^l \underbracket{a_{p_{l+1}} a_r^\dagger} a_{p_1} \cdots a_{p_l} + \text{“ ”}
\end{aligned}
$$

We can now reorder the terms and manipulate them to be in normal ordering:

$$
\begin{aligned}
&= (-1)^{l+1} a_r^\dagger a_{p_{l+1}} a_{p_1} \cdots a_{p_l} + (-1)^l n[\underbracket{a_{p_{l+1}} a_r^\dagger} a_{p_1} \cdots a_{p_l}] + \text{“ ”} \\
&= (-1)^{l+1} (-1)^l a_r^\dagger a_{p_1} \cdots a_{p_l} a_{p_{l+1}} + (-1)^l n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + \text{“ ”} \\
&= (-1)^{l+1} (-1)^l n[a_r^\dagger a_{p_1} \cdots a_{p_l} a_{p_{l+1}}] + (-1)^l n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + \text{“ ”} \\
&= (-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger] + (-1)^l n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + \text{“ ”}
\end{aligned}
$$

Finally, we will rearrange the last term to put $a_{p_{l+1}}$ in the normal product:

$$
\begin{aligned}
(-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}] a_r^\dagger &= (-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger] + (-1)^l n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + \sum_i^l a_{p_{l+1}} n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}] \\
&= \text{“ ”} + \sum_i^l n[a_{p_{l+1}} a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_r^\dagger}] \\
&= \text{“ ”} + \sum_i^l (-1)^l n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger}]
\end{aligned}
$$

Our final expression is thus:

$$
\begin{aligned}
(-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}] a_r^\dagger &= (-1)^l n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger] + (-1)^l n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + (-1)^l \sum_i^l n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger}] \\
n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}] a_r^\dagger &= n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger] + n[a_{p_1} \cdots a_{p_l} \underbracket{a_{p_{l+1}} a_r^\dagger}] + \sum_i^l n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger}] \\
n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}}] a_r^\dagger &= n[a_{p_1} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger] + \sum_i^{l+1} n[a_{p_1} \cdots \underbracket{a_{p_i} \cdots a_{p_l} a_{p_{l+1}} a_r^\dagger}]
\end{aligned}
$$

This proves Lemma 1 for case B1.

Case B2: Prove

$$
n[x_1 \cdots x_m] a_r^\dagger = n[x_1 \cdots x_m a_r^\dagger] + \sum_i^m n[x_1 \cdots \underbracket{x_i \cdots x_m a_r^\dagger}]
$$

where some $x_1 \cdots x_m$ are creation operators.

First, we can rewrite $n[x_1 \cdots x_m]$ such that all the creation operators are to the left in the normal product:

$$
\begin{aligned}
n[x_1 \cdots x_m] a_r^\dagger &= n[x_1 \cdots x_m a_r^\dagger] + \sum_i^m n[x_1 \cdots \underbracket{x_i \cdots x_m a_r^\dagger}] \\
(-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}] a_r^\dagger &= (-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
&\quad + (-1)^R \sum_i^m n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots \underbracket{a_{R_i} \cdots a_{p_{R_m}} a_r^\dagger}]
\end{aligned}
$$

Next, we can split the final term between contractions of $a_r^\dagger$ with creations operators $a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger$, and $a_r^\dagger$ with annihilation operators $a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}$

$$
\begin{aligned}
(-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}] a_r^\dagger = {} & (-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
& + (-1)^R \sum_{i=1}^{k} n[a_{p_{R_1}}^\dagger \cdots a_{R_i}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
& + (-1)^R \sum_{i=k+1}^{m} n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{R_i} \cdots a_{p_{R_m}} a_r^\dagger]
\end{aligned}
$$

All contractions between 2 creation operators is zero, so we just get:

$$
\begin{aligned}
(-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}] a_r^\dagger = {} & (-1)^R n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
& + (-1)^R \sum_{i=k+1}^{m} n[a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger a_{p_{R_{k+1}}} \cdots a_{R_i} \cdots a_{p_{R_m}} a_r^\dagger]
\end{aligned}
$$

We can take $a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger$ out of the normal product because this substring is already in normal product form:

$$
\begin{aligned}
(-1)^R (a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) n[a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}] a_r^\dagger = {} & (-1)^R (a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) n[a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
& + (-1)^R (a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) \sum_{i=k+1}^{m} n[a_{p_{R_{k+1}}} \cdots a_{R_i} \cdots a_{p_{R_m}} a_r^\dagger] \\
(a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) n[a_{p_{R_{k+1}}} \cdots a_{p_{R_m}}] a_r^\dagger = {} & (a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) n[a_{p_{R_{k+1}}} \cdots a_{p_{R_m}} a_r^\dagger] \\
& + (a_{p_{R_1}}^\dagger \cdots a_{p_{R_k}}^\dagger) \sum_{i=k+1}^{m} n[a_{p_{R_{k+1}}} \cdots a_{R_i} \cdots a_{p_{R_m}} a_r^\dagger]
\end{aligned}
$$

We see that what we have in orange is a statement of Lemma 1 in the B1 case. Thus, in the B2 case, we just have a factor multiplied by a statement of Lemma 1 in the B1 case, which we have already proved to be true.

We have therefore proved Lemma 1 to be true.

2.2 Lemma 2

Now, let's prove Lemma 2:

$$
n[x_1 \cdots x_i \cdots x_m] x_{m+1} = n[x_1 \cdots x_i \cdots x_m x_{m+1}] + \sum_{\substack{j=1 \\ j \notin C}}^{m} n[x_1 \cdots x_i \cdots x_j \cdots x_m x_{m+1}]
$$

In words, Lemma 2 says that a normal product with some contractions multiplied by another operator x_{m+1} is equal to a normal product including x_{m+1} with the same contractions, plus a sum over j of the normal product including x_{m+1} with the same contraction as before and an additional contraction between x_{m+1} and a previously uncontracted operator x_j. Operators that are contracted belong to set C. Here, j does not include any operators that are already contracted ($j \notin C$).
We will use the following index definitions in the proof:

$$
2\lambda + \mu = m
$$

$$(i, j, \cdots, i_\lambda, j\lambda) \in C$$
$$k_1 \cdots k_\mu \notin C$$

First, a normal product with some contractions multiplied by an operator x_{m+1} can be written with the contractions taken out:

$$n[x_1 \cdots \underbrace{x_{i_1} \cdots x_{j_1}} \cdots \underbrace{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_m]x_{m+1} = (-1)^R \underbrace{x_{i_1} x_{j_1}} \cdots \underbrace{x_{i_\lambda} x_{j_\lambda}} n[x_{k_1} \cdots x_{k_\mu}]x_{m+1}$$

We can then use Lemma 1 to expand the expression:

$$\begin{aligned}(-1)^R \underbrace{x_{i_1} x_{j_1}} \cdots \underbrace{x_{i_\lambda} x_{j_\lambda}} n[x_{k_1} \cdots x_{k_\mu}]x_{m+1} &= (-1)^R \underbrace{x_{i_1} x_{j_1}} \cdots \underbrace{x_{i_\lambda} x_{j_\lambda}} n[x_{k_1} \cdots x_{k_\mu} x_{m+1}] \\ &+ (-1)^R \underbrace{x_{i_1} x_{j_1}} \cdots \underbrace{x_{i_\lambda} x_{j_\lambda}} \sum_{j=1}^{k} n[x_{k_1} \cdots \underbrace{x_{k_j} \cdots x_{k_\mu} x_{m+1}}]\end{aligned}$$

We can now put the contractions back in their original places and see that this is the statement of Lemma 2:

$$\begin{aligned}n[x_1 \cdots \underbrace{x_{i_1} \cdots x_{j_1}} \cdots \underbrace{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_m]x_{m+1} &= n[x_1 \cdots \underbrace{x_{i_1} \cdots x_{j_1}} \cdots \underbrace{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_m x_{m+1}] \\ &+ \sum_{j=1}^{m} n[x_1 \cdots \underbrace{x_{i_1} \cdots x_{j_1}} \cdots \underbrace{x_{k_j} \cdots \underbrace{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_m x_{m+1}}]\end{aligned}$$

We have thus proved Lemma 2 to be true.

2.3 Wick's Theorem

We are now ready to prove Wick's theorem by induction.

Formally defined, Wick's theorem states:

$$\begin{aligned}x_1 \cdots x_m = n[x_1 \cdots x_m] & \\ &+ \sum_{i<j} n[x_1 \cdots \underbrace{x_i \cdots x_j} x_2] \\ &+ \sum_{\substack{i_1<j_1, i_2<j_2 \\ i_1<j_2, j_1 \neq j_2}} n[x_1 \cdots x_{i_1} \cdots x_{i_2} \cdots x_{j_1} \cdots x_{j_2} \cdots x_m] \\ &+ \cdots \\ &+ \sum_{f.c.} n\overline{\overline{[x_1 \cdots x_m]}}\end{aligned}$$

where "f.c." stands for fully contracted. Note that in cases where m is odd, the last terms will not be fully contracted but have one uncontracted operator. In words, Wick's theorem states that a string of operators can be written as its normal product plus all possible contractions of the normal product. We will adopt a shorthand notation for expressing all possible contractions of the normal product and write Wick's theorem as:

$$x_1 \cdots x_m = n[x_1 \cdots x_m] + \sum_{a.c.} n\overline{[x_1 \cdots x_m]}$$

where 'a.c.' stands for 'all possible contractions' and $n\overline{[x_1 \cdots x_m]}$ is just representative of some general normal order product with contractions.

We will first prove Wick's Theorem for the case m=1:
One operator x_1 can just be written as a normal product, and cannot form contractions:

$$x_1 = n[x_1]$$

Wick's theorem is trivially proved. We can prove the more interesting case of m=2 as well: We can begin by rewriting $x_1 x_2$ using the definition of a contraction:

$$x_1 x_2 = n[x_1 x_2] + \underbracket{x_1 x_2}$$

$\underbracket{x_1 x_2}$ is equivalently:

$$x_1 x_2 = n[x_1 x_2] + \underbracket{x_1 x_2} n[\varnothing]$$

$$x_1 x_2 = n[x_1 x_2] + n[\underbracket{x_1 x_2}]$$

And Wick's theorem is proved by simply applying the definition of a contraction.

We will now take the induction step. Assuming Wick's theorem is true for $m = l$, we will prove that it is true for $m = l + 1$: We can multiply the statement of Wick's theorem on the right by x_{l+1}:

$$\begin{aligned}
x_1 \cdots x_l x_{l+1} = {} & n[x_1 \cdots x_l] x_{l+1} \\
& + \sum_{1 \leq i_1 < j_1}^{l} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots x_l] x_{l+1} \\
& + \sum_{\substack{i_1 < j_1, i_2 < j_2 \\ i_1 < i_2, j_1 \neq j_2}}^{l} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_2} \cdots x_{j_2}} \cdots x_l] x_{l+1} \\
& + \cdots \\
& + \sum_{\substack{i_1 < j_1 \cdots i_{\lambda-1} < j_{\lambda-1} \\ i_1 < \cdots < i_{\lambda-1}, j_1 \neq \cdots \neq j_{\lambda-1}}}^{l} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_{\lambda-1}} \cdots x_{j_{\lambda-1}}} \cdots x_l] x_{l+1} \\
& + \sum_{\substack{i_1 < j_1 \cdots i_\lambda < j_\lambda \\ i_1 < \cdots < i_\lambda, j_1 \neq \cdots \neq j_\lambda}}^{l} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_l] x_{l+1}
\end{aligned}$$

If l is even, we have for the last term:

$$\cdots + \sum_{f.c.} n\overline{\overline{[x_1 \cdots x_l]}} x_{l+1}$$

If l is odd, we have for the last term:

$$\sum_k n[\underbracket{x_1 \cdots} x_k \underbracket{x_{k+1} \cdots x_l}] x_{l+1}$$

where in the normal product only x_k is uncontracted.

We can expand each of the terms using Lemma 1 and 2:

$$n[x_1 \cdots x_l] x_{l+1} = n[x_1 \cdots x_l x_{l+1}] + n[x_1 \cdots \underbracket{x_k \cdots x_l x_{l+1}}]$$

$$\sum_{1\le i_1<j_1}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots x_l]x_{l+1} \quad = \sum_{1\le i_1<j_1}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots x_l x_{l+1}]$$
$$+ \sum_{1\le i_1<j_1}^{l}\ \sum_{k\ne i_1,j_1}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_k\cdots x_l x_{l+1}}]$$

$$\sum_{\substack{i_1<j_1,i_2<j_2\\ i_1<i_2,j_1\ne j_2}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_2}\cdots x_{j_2}}\cdots x_l]x_{l+1} \quad = \sum_{\substack{i_1<j_1,i_2<j_2\\ i_1<i_2,j_1\ne j_2}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_2}\cdots x_{j_2}}\cdots x_l x_{l+1}]$$
$$+ \sum_{\substack{i_1<j_1,i_2<j_2\\ i_1<i_2,j_1\ne j_2}}^{l}\ \sum_{\substack{k\ne i_1,i_2\\ k\ne j_1,j_2}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_2}\cdots x_{j_2}}\cdots \underbracket{x_k\cdots x_l x_{l+1}}]$$

$$\sum_{\substack{i_1<j_1\cdots i_{\lambda-1}<j_{\lambda-1}\\ i_1<\cdots<i_{\lambda-1},j_1\ne\cdots\ne j_{\lambda-1}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda-1}}\cdots x_{j_{\lambda-1}}}\cdots x_l]x_{l+1}$$
$$= \sum_{\substack{i_1<j_1\cdots i_{\lambda-1}<j_{\lambda-1}\\ i_1<\cdots<i_{\lambda-1},j_1\ne\cdots\ne j_{\lambda-1}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda-1}}\cdots x_{j_{\lambda-1}}}\cdots x_l x_{l+1}]$$
$$+ \sum_{\substack{i_1<j_1\cdots i_{\lambda-1}<j_{\lambda-1}\\ i_1<\cdots<i_{\lambda-1},j_1\ne\cdots\ne j_{\lambda-1}}}^{l}\ \sum_{\substack{k\ne i_1\cdots i_{\lambda-1}\\ k\ne j_1\cdots j_{\lambda-1}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda-1}}\cdots x_{j_{\lambda-1}}}\cdots \underbracket{x_k\cdots x_l x_{l+1}}]$$

$$\sum_{\substack{i_1<j_1\cdots i_{\lambda}<j_{\lambda}\\ i_1<\cdots<i_{\lambda},j_1\ne\cdots\ne j_{\lambda}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda}}\cdots x_{j_{\lambda}}}\cdots x_l]x_{l+1}$$
$$= \sum_{\substack{i_1<j_1\cdots i_{\lambda}<j_{\lambda}\\ i_1<\cdots<i_{\lambda},j_1\ne\cdots\ne j_{\lambda}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda}}\cdots x_{j_{\lambda}}}\cdots x_l x_{l+1}]$$
$$+ \sum_{\substack{i_1<j_1\cdots i_{\lambda}<j_{\lambda}\\ i_1<\cdots<i_{\lambda},j_1\ne\cdots\ne j_{\lambda}}}^{l}\ \sum_{\substack{k\ne i_1\cdots i_{\lambda}\\ k\ne j_1\cdots j_{\lambda}}}^{l} n[x_1\cdots \underbracket{x_{i_1}\cdots x_{j_1}}\cdots \underbracket{x_{i_{\lambda}}\cdots x_{j_{\lambda}}}\cdots \underbracket{x_k\cdots x_l x_{l+1}}]$$

If l is even, the last term is just:

$$\sum_{f.c.} n[\overline{\overline{x_1\cdots x_l}}]x_{l+1} = n[\overline{\overline{x_1\cdots x_l}}x_{l+1}]$$

where only x_{l+1} is uncontracted.

If l is odd, the last term becomes:

$$\sum_{k}^{l} n[x_1 \cdots x_k x_{k+1} \cdots x_l]x_{l+1} = \sum_{k}^{l} n[x_1 \cdots x_k x_{k+1} \cdots x_l x_{l+1}] + \sum_{f.c.} n\overline{\overline{[x_1 \cdots x_k x_{k+1} \cdots x_l x_{l+1}]}}$$

Putting all the terms together, we get:

$$\begin{aligned}
x_1 \cdots x_l x_{l+1} = {} & n[x_1 \cdots x_l x_{l+1}] + n[x_1 \cdots x_k \cdots x_l x_{l+1}] \\
& + \sum_{1 \le i_1 < j_1}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_l x_{l+1}] \\
& + \sum_{1 \le i_1 < j_1}^{l} \sum_{k \ne i_1, j_1}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_k \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1, i_2 < j_2 \\ i_1 < i_2, j_1 \ne j_2}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_2} \cdots x_{j_2} \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1, i_2 < j_2 \\ i_1 < i_2, j_1 \ne j_2}}^{l} \sum_{\substack{k \ne i_1, i_2 \\ k \ne j_1, j_2}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_2} \cdots x_{j_2} \cdots x_k \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1 \cdots i_{\lambda-1} < j_{\lambda-1} \\ i_1 < \cdots < i_{\lambda-1}, j_1 \ne \cdots \ne j_{\lambda-1}}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_{\lambda-1}} \cdots x_{j_{\lambda-1}} \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1 \cdots i_{\lambda-1} < j_{\lambda-1} \\ i_1 < \cdots < i_{\lambda-1}, j_1 \ne \cdots \ne j_{\lambda-1}}}^{l} \sum_{\substack{k \ne i_1 \cdots i_{\lambda-1} \\ k \ne j_1 \cdots j_{\lambda-1}}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_{\lambda-1}} \cdots x_{j_{\lambda-1}} \cdots x_k \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1 \cdots i_\lambda < j_\lambda \\ i_1 < \cdots < i_\lambda, j_1 \ne \cdots \ne j_\lambda}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_\lambda} \cdots x_{j_\lambda} \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1 \cdots i_\lambda < j_\lambda \\ i_1 < \cdots < i_\lambda, j_1 \ne \cdots \ne j_\lambda}}^{l} \sum_{\substack{k \ne i_1 \cdots i_\lambda \\ k \ne j_1 \cdots j_\lambda}}^{l} n[x_1 \cdots x_{i_1} \cdots x_{j_1} \cdots x_{i_\lambda} \cdots x_{j_\lambda} \cdots x_k \cdots x_l x_{l+1}] \\
& + \cdots
\end{aligned}$$

We notice that on the righthand side we can combine terms 2 and 3 (all single contractions), terms 4 and 5 (all double contractions), and so on, to recover the statement of Wick's theorem for $m = l + 1$:

$$
\begin{aligned}
x_1 \cdots x_l x_{l+1} = {} & n[x_1 \cdots x_l x_{l+1}] \\
& + \sum_{1 \le i_1 < j_1}^{l+1} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1, i_2 < j_2 \\ i_1 < i_2, j_1 \neq j_2}}^{l+1} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_2} \cdots x_{j_2}} \cdots x_l x_{l+1}] \\
& + \cdots \\
& + \sum_{\substack{i_1 < j_1 \cdots i_{\lambda-1} < j_{\lambda-1} \\ i_1 < \cdots < i_{\lambda-1}, j_1 \neq \cdots \neq j_{\lambda-1}}}^{l+1} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_{\lambda-1}} \cdots x_{j_{\lambda-1}}} \cdots x_l x_{l+1}] \\
& + \sum_{\substack{i_1 < j_1 \cdots i_\lambda < j_\lambda \\ i_1 < \cdots < i_\lambda, j_1 \neq \cdots \neq j_\lambda}}^{l+1} n[x_1 \cdots \underbracket{x_{i_1} \cdots x_{j_1}} \cdots \underbracket{x_{i_\lambda} \cdots x_{j_\lambda}} \cdots x_l x_{l+1}] \\
& + \cdots
\end{aligned}
$$

Finally, we note that for the case $l + 1$, the last terms have one uncontracted operator if l is even and are fully contracted if l is odd.

3 Proof of Generalized Wick's Theorem

In some cases, substring of operator strings are already in the normal product form, and we can use the generalized Wick's theorem, which states:

$$x_1 \cdots x_m = n[x_1 \cdots x_m] + \sum_{a.c.}{}' n\overline{[x_1 \cdots x_m]}$$

where $\sum_{a.c.}{}'$ denotes skipping contractions of operators that originated from the same normal ordered group.

Proof:

We write a string of operators where a substring is already in normal order:

$$x_1 \cdots x_{k_{\mu-1}} n[x_{k_{\mu-1}+1} \cdots x_{k_\mu}] x_{k_\mu+1} \cdots x_m$$

We can permute indices inside the normal-ordered string and take it out of the normal product:

$$(-1)^R x_1 \cdots x_{k_{\mu-1}} a^\dagger_{p_1} \cdots a^\dagger_{p_\alpha} a_{p_{\alpha+1}} \cdots a_{p_\beta} x_{k_\mu+1} \cdots x_m$$

Using Wick's theorem, we can expand the string as:

$$
\begin{aligned}
& (-1)^R x_1 \cdots x_{k_{\mu-1}} a^\dagger_{p_1} \cdots a^\dagger_{p_\alpha} a_{p_{\alpha+1}} \cdots a_{p_\beta} x_{k_\mu+1} \cdots x_m = \\
& \qquad (-1)^R n[x_1 \cdots x_{k_{\mu-1}} a^\dagger_{p_1} \cdots a^\dagger_{p_\alpha} a_{p_{\alpha+1}} \cdots a_{p_\beta} x_{k_\mu+1} \cdots x_m] \\
& \qquad + (-1)^R \sum_{a.c.} n\overline{[x_1 \cdots x_{k_{\mu-1}} a^\dagger_{p_1} \cdots a^\dagger_{p_\alpha} a_{p_{\alpha+1}} \cdots a_{p_\beta} x_{k_\mu+1} \cdots x_m]}
\end{aligned}
$$

Let's consider all contractions involving only $a^\dagger_{p_1} \cdots a^\dagger_{p_\alpha} a_{p_{\alpha+1}} \cdots a_{p_\beta}$:

- If $\alpha = 0$ and there are no creation operators in the string, there are only annihilation operators in the normal-ordered group, and any $\underbracket{a_r a_s} = 0$.

- If $\beta = 0$ and there are no annihilation operators in the string, there are only creation operators in the normal-ordered group, and any $\underbracket{a^\dagger_r a^\dagger_s} = 0$.

- If $1 \leq \alpha < \beta$, we have both annihilation and creation operators, and within the normal ordered group will encounter contractions of type: $\underbracket{a_r a_s} = 0$, $\underbracket{a^\dagger_r a^\dagger_s} = 0$, and $\underbracket{a^\dagger_r a_s} = 0$

Thus, we see that any contractions within a normal ordered group will always be zero, and we can leave them out in the final Wick expansion. This proves the generalized Wick's theorem.