Local and global estimates of solutions of Hamilton-Jacobi parabolic equation with absorption

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Abstract

We obtain new a priori estimates for the nonnegative solutions of the equation

$$u_t - \Delta u + |\nabla u|^q = 0$$

in $Q_{\Omega,T} = \Omega \times (0,T)$, $T \leqq \infty$, where $q > 0$, and $\Omega = \mathbb{R}^N$, or Ω is a smooth bounded domain of $\mathbb{R}^N$ and $u = 0$ on $\partial\Omega \times (0,T)$.

In case $\Omega = \mathbb{R}^N$, we show that any solution $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$ of equation (1.1) in $Q_{\mathbb{R}^N,T}$ (in particular any weak solution if $q \leqq 2$), without condition as $|x| \to \infty$, satisfies the universal estimate

$$|\nabla u(.,t)|^q \leqq \frac{1}{q-1}\frac{u(.,t)}{t}, \qquad \text{in } Q_{\mathbb{R}^N,T}.$$

Moreover we prove that the growth of u is limited by $C(t+t^{-1/(q-1)})(1+|x|^{q'})$, where C depends on u.

We also give existence properties of solutions in $Q_{\Omega,T}$, for initial data locally integrable or even unbounded Radon measures. We give a nonuniqueness result in case $q > 2$. Finally we show that besides the local regularizing effect of the heat equation, u satisfies a second effect of type $L^R_{loc} - L^\infty_{loc}$, due to the gradient term.

Keywords Hamilton-Jacobi equation; Radon measures; initial trace; universal bounds., regularizing effects.

A.M.S. Subject Classification 35K15, 35K55, 35B33, 35B65, 35D30

Contents

1 Introduction

Here we consider the *nonnegative* solutions of the parabolic Hamilton-Jacobi equation

$$u_t - \nu\Delta u + |\nabla u|^q = 0, \tag{1.1}$$

where $q > 1$, in $Q_{\Omega,T} = \Omega \times (0,T)$, where Ω is any domain of $\mathbb{R}^N$, $\nu \in (0,1]$. We study the problem of a priori estimates of the *nonnegative* solutions, with possibly rough *unbounded* initial data

$$u(x,0) = u_0 \in \mathcal{M}^+(\Omega), \tag{1.2}$$

where we denote by $\mathcal{M}^+(\Omega)$ the set of nonnegative Radon measures in Ω, and $\mathcal{M}_b^+(\Omega)$ the subset of bounded ones. We say that u is a solution of (1.1) if it satisfies (1.1) in $Q_{\Omega,T}$ in the weak sense of distributions, see Section 2. We say that u *has a trace* u_0 *in* $\mathcal{M}^+(\Omega)$ if $u(.,t)$ converges to u_0 in the weak* topology of measures:

$$\lim_{t\to 0}\int_\Omega u(.,t)\psi dx = \int_\Omega \psi du_0, \qquad \forall\psi \in C_c(\Omega). \tag{1.3}$$

Our purpose is to obtain a priori estimates valid for any solution in $Q_{\Omega,T} = \Omega \times (0,T)$, without assumption on the boundary of Ω, or for large $|x|$ if $\Omega = \mathbb{R}^N$.

Fisrt recall some known results. The Cauchy problem in $Q_{\mathbb{R}^N,T}$

$$(P_{\mathbb{R}^N,T})\left\{\begin{array}{l} u_t - \nu\Delta u + |\nabla u|^q = 0, \quad \text{in } Q_{\mathbb{R}^N,T}, \\ u(x,0) = u_0 \qquad \text{in } \mathbb{R}^N, \end{array}\right. \tag{1.4}$$

is the object of a rich literature, see among them [2],[9], [5], [11], [26],[12], [13], and references therein. The first studies concern *classical* solutions, that means $u \in C^{2,1}(Q_{\mathbb{R}^N,T})$, with *smooth bounded initial data* $u_0 \in C_b^2\left(\mathbb{R}^N\right)$: there a unique global solution such that

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|u_0\|_{L^\infty(\mathbb{R}^N)}, \text{ and } \|\nabla u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq \|\nabla u_0\|_{L^\infty(\mathbb{R}^N)}, \qquad \text{in } Q_{\mathbb{R}^N,T},$$

see [2]. Then universal a priori estimates of the gradient are obtained *for this solution*, by using the Bersnstein technique, which consists in computing the equation satisfied by $|\nabla u|^2$: first from [23],

$$\|\nabla u(.,t)\|^q_{L^\infty(\mathbb{R}^N)} \leqq \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}}{t},$$

in $Q_{\mathbb{R}^N,T}$, , then from [9],

$$|\nabla u(.,t)|^q \leqq \frac{1}{q-1}\frac{u(.,t)}{t}, \tag{1.5}$$

$$\|\nabla(u^{\frac{q-1}{q}})(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-1/2}\|u_0\|^{\frac{q-1}{q}}_{L^\infty(\mathbb{R}^N)}, \qquad C = C(N,q,\nu). \tag{1.6}$$

Existence and uniqueness was extended to any $u_0 \in C_b\left(\mathbb{R}^N\right)$ in [20]; then the estimates (1.6) and (1.5) are still valid, see [5]. In case of nonnegative rough initial data $u_0 \in L^R\left(\mathbb{R}^N\right)$, $R \geqq 1$, or $u_0 \in \mathcal{M}_b^+(\mathbb{R}^N)$, the problem was studied in a semi-group formulation [9], [11], [26], then in the larger class of weak solutions in [12], [13]. Recall that two critical values appear: $q = 2$, where the equation can be reduced to the heat equation, and

$$q_* = \frac{N+2}{N+1}.$$

Indeed the Cauchy problem with initial value $u_0 = \kappa\delta_0$, where δ_0 is the Dirac mass at 0 and $\kappa > 0$, has a weak solution u^κ if and only if $q < q_*$, see [9], [12]. Moreover as $\kappa \to \infty$, (u^κ) converges to a unique very singular solution Y, see [25], [10], [8], [12]. And $Y(x,t) = t^{-a/2}F(|x|/\sqrt{t})$, where

$$a = \frac{2-q}{q-1}, \tag{1.7}$$

and F is bounded and has an exponential decay at infinity.

In [13, Theorem 2.2] it is shown that for any $R \geqq 1$ global regularizing L^R-L^∞ properties of two types hold for the Cauchy problem in $Q_{\mathbb{R}^N,T}$: one due to the heat operator:

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{2R}}\|u_0\|_{L^R(\mathbb{R}^N)}, \qquad C = C(N,R,\nu), \tag{1.8}$$

and the other due to the gradient term, independent of ν ($\nu > 0$):

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \leqq Ct^{-\frac{N}{qR+N(q-1)}}\|u_0\|_{L^R(\mathbb{R}^N)}^{\frac{qR}{qR+N(q-1)}}, \qquad C = C(N,q,R). \tag{1.9}$$

A great part of the results has been extended to the Dirichlet problem in a bounded domain Ω :

$$(P_{\Omega,T})\begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0, \end{cases} \tag{1.10}$$

where $u_0 \in \mathcal{M}_b^+(\Omega)$, and $u(.,t)$ converges to u_0 weakly in $\mathcal{M}_b^+(\Omega)$, see [6], [26], [12], [13]. Universal estimates are given in [16], see also [12]. Note that (1.5) cannot hold, since it contradicts the Höpf Lemma.

Finally local estimates in any domain Ω were proved in [26]: for any classical solution u in $Q_{\Omega,T}$ and any ball $B(x_0, 2\eta) \subset \Omega$, there holds in $Q_{B(x_0,\eta),T}$

$$|\nabla u|(.,t) \leqq C(t^{-\frac{1}{q}} + \eta^{-1} + \eta^{-\frac{1}{q-1}})(1 + u(.,t)), \qquad C = C(N,q,\nu). \tag{1.11}$$

1.1 Main results

In Section 3 we give *local integral estimates* of the solutions *in terms of the initial data, and a first regularizing effect,* local version of (1.8), see Theorem 3.3.

Theorem 1.1 *Let $q > 1$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega,T}$, and let $B(x_0, 2\eta) \subset\subset \Omega$ such that u has a trace $u_0 \in L^R_{loc}(\Omega)$, $R \geqq 1$ and $u \in C([0,T); L^R_{loc}(\Omega))$. Then for any $0 < t \leqq \tau < T$,*

$$\sup_{x\in B(x_0,\eta/2)} u(x,t) \leqq Ct^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0,\eta))}), \qquad C = C(N,q,\nu,R,\eta,\tau).$$

If $R = 1$, the estimate remains true when $u_0 \in \mathcal{M}^+(\Omega)$ (with $\|u_0\|_{L^1(B(x_0,\eta))}$ replaced by $\int_{B(x_0,\eta)} du_0$).

In Section 4, we give *global estimates* of the solutions of (1.1) in $Q_{\mathbb{R}^N,T}$, and this is our main result. We show that *the universal estimate (1.5) in* $\mathbb{R}^N$ *holds without assuming that the solutions are initially bounded*:

Theorem 1.2 *Let* $q > 1$. *Let* u *be any classical solution, in particular* ***any weak solution*** *if* $q \leqq 2$, *of equation (1.1) in* $Q_{\mathbb{R}^N,T}$. *Then*

$$|\nabla u(.,t)|^q \leqq \frac{1}{q-1}\frac{u(.,t)}{t}, \qquad \text{in } Q_{\mathbb{R}^N,T}. \tag{1.12}$$

And we prove that *the growth of the solutions is limited, in* $|x|^{q'}$ *as* $|x| \to \infty$ *and in* $t^{-1/(q-1)}$ *as* $t \to 0$:

Theorem 1.3 *Let* $q > 1$. *Let* u *be any classical solution, in particular* ***any weak solution*** *if* $q \leqq 2$, *of equation (1.1) in* $Q_{\mathbb{R}^N,T}$, *such that there exists a ball* $B(x_0, 2\eta)$ *such that* u *has a trace* $u_0 \in \mathcal{M}^+((B(x_0, 2\eta))$. *Then*

$$u(x,t) \leqq C(q)t^{-\frac{1}{q-1}}\,|x-x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0,\eta)} du_0), \qquad C = C(N,q,\eta). \tag{1.13}$$

In [14], we show that there exist solutions with precisely this type of behaviour of order $t^{-1/(q-1)}\,|x|^{q'}$ as $|x| \to \infty$ or $t \to 0$. Moreover we prove that the condition on the trace is always satisfied for $q < q_*$.

In Section 5 we complete the study by giving *existence results* with only *local assumptions on* u_0, extending some results of [5] where u_0 is continuous:

Theorem 1.4 *Let* $\Omega = \mathbb{R}^N$ *(resp.* Ω *bounded).*

(i) If $1 < q < q_*$, *then for any* $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$ *(resp.* $\mathcal{M}^+(\Omega)$*), there exists a weak solution* u *of equation (1.1) (resp. of* $(D_{\Omega,T})$*) with trace* u_0.

(ii) If $q_* \leqq q \leqq 2$, *then existence still holds for any nonnegative* $u_0 \in L^1_{loc}(\mathbb{R}^N)$ *(resp.* $L^1_{loc}(\Omega)$*).*

(iii) If $q > 2$, *existence holds for any nonnegative* $u_0 \in L^1_{loc}(\mathbb{R}^N)$ *(resp.* $L^1_{loc}(\Omega)$*) which is limit of a nondecreasing sequence of continuous functions.*

Our proof of (ii) (iii) is based on approximations by nonincreasing sequences, following the methods of [11], [13]. Another proof can be obtained when $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and $q \leqq 2$, by techniques of equiintegrability, see [22] for a connected problem.

Moreover we give a result of *nonuniqueness* of weak solutions in case $q > 2$:

Theorem 1.5 *Assume that* $q > 2$, $N \geq 2$.*Then the Cauchy problem* $(P_{\mathbb{R}^N,\infty})$ *with initial data*

$$\tilde{U}(x) = \tilde{C}\,|x|^{|a|} \in C(\mathbb{R}^N), \qquad \tilde{C} = \frac{q-1}{q-2}(\frac{(N-1)q-N}{q-1})^{\frac{1}{q-1}},$$

admits at least two weak solutions: the stationary solution $\tilde{U}$, *and a radial self-similar solution of the form*

$$U_{\tilde{C}}(x,t) = t^{|a|/2} f(|x|/\sqrt{t}), \tag{1.14}$$

where f *is increasing on* $[0,\infty)$, $f(0) > 0$, *and* $\lim_{\eta\to\infty}\eta^{-|a|/2}f(\eta) = \tilde{C}$.

Finally we give in Section 6a second type of regularizing effects giving a local version of (1.9).

Theorem 1.6 *Let $q > 1$, and let u be any nonnegative classical solution (resp. any weak solution if $q \leqq 2$) of equation (1.1) in $Q_{\Omega,T}$, and let $B(x_0, 2\eta) \subset \Omega$. Assume that $u_0 \in L^R_{loc}(\Omega)$ for some $R \geqq 1$, $R > q-1$, and $u \in C([0,T)\,; L^R_{loc}(\Omega))$. Then for any $\varepsilon > 0$, and for any $\tau \in (0,T)$, then there exists $C = C(N,q,R,\eta,\varepsilon,\tau)$ such that*

$$sup_{B_{\eta/2}} u(.,t) \leqq Ct^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} + Ct^{\frac{1-\varepsilon}{R+1-q}} \|u_0\|^{\frac{R}{R+1-q}}_{L^R(B_\eta)}\,. \tag{1.15}$$

If $q < 2$, the estimates for $R = 1$ are also valid when u has a trace $u_0 \in \mathcal{M}^+(\Omega)$, with $\|u_0\|_{L^1(B_\eta)}$ replaced by $\int_{B_\eta} du_0$.

In conclusion, note that a part of our results could be extended to more general quasilinear operators, for example to the case of equation involving the p-Laplace operator

$$u_t - \nu\Delta_p u + |\nabla u|^q = 0$$

with $p > 1$, following the results of [13], [4], [21], [19].

2 Classical and weak solutions

We set $Q_{\Omega,s,\tau} = \Omega \times (s,\tau)$, for any $0 \leqq s < \tau \leqq \infty$, thus $Q_{\Omega,T} = Q_{\Omega,0,T}$.

Definition 2.1 *Let $q > 1$ and Ω be any domain of $\mathbb{R}^N$. We say that a nonnegative function u is a **classical** solution of (1.1) in $Q_{\Omega,T}$ if $u \in C^{2,1}(Q_{\Omega,T})$. We say that u is a **weak solution** (resp. weak subsolution) of (1.1) in $Q_{\Omega,T}$, if $u \in C((0,T); L^1_{loc}(Q_{\Omega,T})) \cap L^1_{loc}((0,T); W^{1,1}_{loc}(\Omega))$, $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$ and u satisfies (1.1) in the distribution sense:*

$$\int_0^T \int_\Omega (-u\varphi_t - \nu u\Delta\varphi + |\nabla u|^q \varphi) = 0, \quad \forall\varphi \in \mathcal{D}(Q_{\Omega,T}), \tag{2.1}$$

(resp.

$$\int_0^T \int_\Omega (-u\varphi_t - \nu u\Delta\varphi + |\nabla u|^q \varphi) \leqq 0, \quad \forall\varphi \in \mathcal{D}^+(Q_{\Omega,T}).) \tag{2.2}$$

And then for any $0 < s < t < T$, and any $\varphi \in C^1((0,T), C^1_c(\Omega))$,

$$\int_\Omega (u\varphi)(.,t) - \int_\Omega (u\varphi)(.,s) + \int_s^t \int_\Omega (-u\varphi_t + \nu\nabla u.\nabla\varphi + |\nabla u|^q \varphi) = 0 \textit{ (resp. } \leqq 0). \tag{2.3}$$

Remark 2.2 *Any weak subsolution u is locally bounded in $Q_{\Omega,T}$. Indeed, since u is ν-subcaloric, there holds for any ball $B(x_0,\rho) \subset\subset \Omega$ and any $\rho^2 \leqq t < T$,*

$$\sup_{B(x_0,\frac{\rho}{2}) \times \left[t-\frac{\rho^2}{4},t\right]} u \leqq C(N,\nu)\rho^{-(N+2)} \int_{t-\frac{\rho^2}{2}}^t \int_{B(x_0,\rho)} u. \tag{2.4}$$

Any nonnegative function $u \in L^1_{loc}(Q_{\Omega,T})$, such that $|\nabla u|^q \in L^1_{loc}(Q_{\Omega,T})$, and u satisfies (2.1), is a weak solution and $|\nabla u| \in L^2_{loc}(Q_{\Omega,T})), u \in C((0,T); L^s_{loc}(Q_{\Omega,T})), \forall s \geqq 1$, see [12, Lemma 2.4].

Next we recall the regularity of the weak solutions of (1.1) for $q \leqq 2$, see [12, Theorem 2.9], [13, Corollary 5.14]:

Theorem 2.3 *Let $1 < q \leqq 2$. Let Ω be any domain in $\mathbb{R}^N$. Let u be any weak nonnegative solution of (1.1) in $Q_{\Omega,T}$. Then $u \in C^{2+\gamma,1+\gamma/2}_{loc}(Q_{\Omega,T})$ for some $\gamma \in (0,1)$, and for any smooth domains $\omega \subset\subset \omega' \subset\subset \Omega$, and $0 < s < \tau < T$, $\|u\|_{C^{2+\gamma,1+\gamma/2}(Q_{\omega,s,\tau})}$ is bounded in terms of $\|u\|_{L^\infty(Q_{\omega',s/2,\tau})}$. Thus for any sequence (u_n) of nonnegative weak solutions of equation (1.1) in $Q_{\Omega,T}$, uniformly locally bounded, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$.*

Remark 2.4 *Let $q > 1$. From the estimates (1.11), for any sequence of classical nonnegative solutions (u_n) of (1.1) in $Q_{\Omega,T}$, uniformly bounded in $L^\infty_{loc}(Q_{\Omega,T})$, one can extract a subsequence converging in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ to a classical solution u of (1.1).*

Remark 2.5 *Let us mention some results of concerning the trace, valid for any $q > 1$, see [12, Lemma 2.14]. Let u be any nonnegative weak solution u of (1.1) in $Q_{\Omega,T}$. Then u has a trace u_0 in $\mathcal{M}^+(\Omega)$ if and only if $u \in L^\infty_{loc}([0,T);L^1_{loc}(\Omega))$, and if and only if $|\nabla u|^q \in L^1_{loc}(\Omega \times [0,T))$. And then for any $t \in (0,T)$, and any $\varphi \in C^1_c(\Omega \times [0,T))$, and any $\zeta \in C^1_c(\Omega)$,*

$$\int_\Omega u(.,t)\varphi dx + \int_0^t \int_\Omega (-u\varphi_t + \nu\nabla u.\nabla\varphi + |\nabla u|^q \varphi) = \int_\Omega \varphi(.,0)du_0, \tag{2.5}$$

$$\int_\Omega u(.,t)\zeta + \int_0^t \int_\Omega (\nu\nabla u.\nabla\zeta + |\nabla u|^q \zeta) = \int_\Omega \zeta du_0. \tag{2.6}$$

If $u_0 \in L^1_{loc}(\Omega)$, then $u \in C([0,T);L^1_{loc}(\Omega))$.

Finally we consider the Dirichlet problem in a smooth bounded domain Ω:

$$(D_{\Omega,T})\begin{cases} u_t - \Delta u + |\nabla u|^q = 0, & \text{in } Q_{\Omega,T}, \\ u = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{2.7}$$

Definition 2.6 *We say that a function u is a **weak solution of** $(D_{\Omega,T})$ if it is a weak solution of equation (1.1) such that $u \in C((0,T);L^1(\Omega)) \cap L^1_{loc}((0,T);W^{1,1}_0(\Omega))$, and $|\nabla u|^q \in L^1_{loc}((0,T);L^1(\Omega))$. We say that u is a **classical** solution of $(D_{\Omega,T})$ if $u \in C^{2,1}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega} \times (0,T)\right)$.*

3 Local integral properties and first regularizing effect

3.1 Local integral properties

Lemma 3.1 *Let Ω be any domain in $\mathbb{R}^N$, $q > 1, R \geqq 1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega,T}$, such that $u \in C((0,T);L^R_{loc}(\Omega))$. Let $\xi \in C^1((0,T);C^1_c(\Omega))$, with values in $[0,1]$. Let $\lambda > 1$. Then there exists $C = C(q,R,\lambda)$, such that, for any $0 < s < t \leqq \tau < T$,*

$$\begin{aligned} &\textstyle\int_\Omega u^R(.,t)\xi^\lambda + \frac{1}{2}\int_s^\tau \int_\Omega u^{R-1}|\nabla u|^q \xi^\lambda + \nu\frac{R-1}{2}\int_s^\tau \int_\Omega u^{R-2}|\nabla u|^2\xi^\lambda \\ &\leqq \textstyle\int_\Omega u^R(.,s)\xi^\lambda + \lambda R\int_s^t \int_\Omega u^R \xi^{\lambda-1}|\xi_t| + C\int_s^t \int_\Omega u^{R-1}\xi^{\lambda-q'}|\nabla\xi|^{q'}. \end{aligned} \tag{3.1}$$

Proof. (i) Let $R=1$. Taking $\varphi=\xi^{\lambda}$ in (2.3), we obtain, since $\nu \leqq 1$,

$$\begin{aligned}
&\int_{\Omega} u(.,t)\xi^{\lambda} + \int_s^t \int_{\Omega} |\nabla u|^q \xi^{\lambda} \leqq \int_{\Omega} u(s,.)\xi^{\lambda} + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u \xi_t + \lambda\nu \int_s^t \int_{\Omega} \xi^{\lambda-1} \nabla u . \nabla \xi \\
&\leqq \int_{\Omega} u(.,s)\xi^{\lambda} + \lambda \int_s^t \int_{\Omega} \xi^{\lambda-1} u \, |\xi_t| + \frac{1}{2} \int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} + C(q,\lambda) \int_s^t \int_{\Omega} \xi^{\lambda-q'} |\nabla \xi|^{q'},
\end{aligned}$$

hence (3.1) follows.

(ii) Next assume $R>1$. Consider $u_{\delta,n}=((u+\delta)*\varphi_n)$, where (φ_n) is a sequence of mollifiers, and $\delta>0$. Then by convexity, $u_{\delta,n}$ is also a subsolution of (1.1):

$$(u_{\delta,n})_t - \nu \Delta u_{\delta,n} + |\nabla u_{\delta,n}|^q \leqq 0.$$

Multiplying by $u_{\delta,n}^{R-1}\xi^{\lambda}$ and integrating between s and t, and going to the limit as $\delta \to 0$ and $n \to \infty$, see [13], we get with different constants $C=(N,q,R,\lambda)$, independent of ν,

$$\begin{aligned}
&\frac{1}{R}\int_{\Omega} u^R(.,t)\xi^{\lambda} + \nu(R-1)\int_s^t \int_{\Omega} u^{R-2}|\nabla u|^2 \xi^{\lambda} + \int_s^t \int_{\Omega} u^{R-1}|\nabla u|^q \xi^{\lambda} \\
&\leqq \frac{1}{R}\int_{\Omega} u^R(.,s)\xi^{\lambda} + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R \, |\xi_t| + \lambda\nu \int_\theta^t \int_{\Omega} u^{R-1} |\nabla u| \, |\nabla \xi| \, \xi^{\lambda-1} \\
&\leqq \frac{1}{R}\int_{\Omega} u^R(.,s)\xi^{\lambda} + \lambda \int_s^t \int_{B_\rho} \xi^{\lambda-1} u^R \, |\xi_t| \\
&+ \frac{1}{2}\int_s^{\tau} \int_{\Omega} u^{R-1}|\nabla u|^q \xi^{\lambda} + C(\lambda,R)\int_s^t \int_{\Omega} u^{R-1} \xi^{\lambda-q'} |\nabla \xi|^{q'},
\end{aligned}$$

and (3.1) follows again. ■

Then we give local integral estimates of $u(.,t)$ in terms of the initial data:

Lemma 3.2 *Let $q>1$. Let $\eta>0$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega,T}$, with trace $u_0 \in \mathcal{M}^+(\Omega)$, and let $B(x_0,2\eta) \subset\subset \Omega$. Then for any $t \in (0,T)$,*

$$\int_{B(x_0,\eta)} u(x,t) \leqq C(N,q)\eta^{N-q'} t + \int_{B(x_0,2\eta)} du_0. \tag{3.2}$$

Moreover if $u_0 \in L^R_{loc}(\Omega)$ $(R>1)$, and $u \in C([0,T)\,; L^R_{loc}(\Omega))$, then

$$\|u(.,t)\|_{L^R(B(x_0,\eta))} \leqq C(N,q,R)\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B(x_0,2\eta))} . \tag{3.3}$$

If $u \in C(\overline{B(x_0,2\eta)} \times [0,T))$, then

$$\|u(.,t)\|_{L^\infty(B(x_0,\eta))} \leqq C(N,q)\eta^{-q'} t + \|u_0\|_{L^\infty(B(x_0,2\eta))} . \tag{3.4}$$

Proof. We can assume that $0 \in \Omega$ and $x_0=0$. We take $\xi \in C^1_c(\Omega)$, independent of t, with values in $[0,1]$, and $R=1$ in (3.1), $\lambda=q'$. Then for any $0<s<t<T$,

$$\int_{\Omega} u(.,t)\xi^{q'} + \frac{1}{2}\int_s^t \int_{\Omega} |\nabla u|^q \xi^{q'} \leqq \int_{\Omega} u(.,s)\xi^{q'} + C(q)\int_s^t \int_{\Omega} |\nabla \xi|^{q'} \leqq \int_{\Omega} u(.,s)\xi^{q'} + C(q)t\int_{\Omega} |\nabla \xi|^{q'}.$$

Hence as $s \to 0$, we get

$$\int_\Omega u(.,t)\xi^{q'} + \frac{1}{2}\int_0^t \int_\Omega |\nabla u|^q \xi^{q'} \leqq C(q)t\int_\Omega |\nabla\xi|^{q'} + \int_\Omega \xi^{q'} du_0. \tag{3.5}$$

Then taking $\xi = 1$ in B_η with support in $B_{2\eta}$ and $|\nabla\xi| \leqq C_0(N)/\eta$,

$$\int_{B_\eta} u(x,t) \leqq C(N,q)\eta^{N-q'}t + \int_{B_{2\eta}} \xi^{q'} du_0, \tag{3.6}$$

hence we get (3.2). Next assume $u_0 \in L^R_{loc}(\Omega)$ $(R > 1)$, and $u \in C([0,T) ; L^R_{loc}(\Omega))$. Then from (3.1), for any $0 < s < t \leqq \tau < T$, we find,

$$\begin{aligned} \textstyle\int_\Omega u^R(.,t)\xi^\lambda + \frac{1}{2}\int_s^\tau \int_\Omega u^{R-1}|\nabla u|^q \xi^\lambda &\leqq \textstyle\int_\Omega u^R(.,s)\xi^\lambda + \int_s^t \int_\Omega u^{R-1}\xi^{\lambda-q'}|\nabla\xi|^{q'} \\ &\leqq \textstyle\int_\Omega u^R(.,s)\xi^\lambda + \varepsilon\int_s^t \int_{B_{2\eta}} u^R\xi^\lambda + \varepsilon^{1-R}\int_s^t \int_{B_{2\eta}} \xi^{\lambda-Rq'}|\nabla\xi|^{Rq'}. \end{aligned}$$

Taking $\lambda = Rq'$, and ξ as above, we find

$$\int_{B_{2\eta}} u^R(.,t)\xi^{Rq'} \leqq \int_{B_{2\eta}} u^R(.,s)\xi^{Rq'} + \varepsilon\int_s^t \int_{B_{2\eta}} u^R\xi^{Rq'} + \varepsilon^{1-R}C(N)C_0^{Rq'}(N)\eta^{N-Rq'}t.$$

Next we set $\varpi(t) = \sup_{\sigma\in[s,t]} \int_{B_{2\eta}} u^R(.,\sigma)\xi^{Rq'}$. Then

$$\varpi(t) \leqq \int_{B_{2\eta}} u^R(.,s)\xi^{Rq'} + \varepsilon(t-s)\varpi(t) + \varepsilon^{1-R}C(N)C_0^{Rq'}(N)\eta^{N-Rq'}t.$$

Taking $\varepsilon = 1/2t$, we get

$$\frac{1}{2}\int_{B_{2\eta}} u^R(.,t)\xi^{Rq'} \leqq \int_{B_{2\eta}} u^R(.,s)\xi^{rq'} + C(N)C_0^{Rq'}(N)\eta^{N-Rq'}t^R.$$

Then going to the limit as $s \to 0$,

$$\int_{B_\eta} u^R(x,t) \leqq C(N)C_0^{Rq'}(N)\eta^{N-Rq'}t^R + \int_{B_{2\eta}} u_0^R\xi^{Rq'}, \tag{3.7}$$

thus (3.3) follows.

If $u \in C(\overline{B_{2\rho}} \times [0,T))$, then (3.7) holds for any $R \geqq 1$, implying

$$\|u(.,t)\|_{L^R(B_\eta)} \leqq C^{\frac{1}{R}}(N)C_0^{q'}(N)\eta^{\frac{N}{R}-q'}t + \|u_0\|_{L^R(B_{2\eta})},$$

and (3.3) follows as $R \to \infty$. ∎

3.2 Regularizing effect of the heat operator

We first give a first regularizing effect due to the Laplace operator in $Q_{\Omega,T}$, for any domain Ω, for classical or weak solutions in terms of the initial data.

Theorem 3.3 *Let $q>1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega,T}$, and let $B(x_0,2\eta)\subset\Omega$ such that u has a trace $u_0\in\mathcal{M}^+(B(x_0,2\eta))$. Then for any $\tau<T$, and any $t\in(0,\tau]$,*

$$\sup_{x\in B(x_0,\eta/2)} u(x,t)\leqq Ct^{-\frac{N}{2}}(t+\int_{B(x_0,\eta)} du_0),\qquad C=C(N,q,\nu,\eta,\tau). \tag{3.8}$$

Moreover if $u_0\in L^R_{loc}(\Omega)$ $(R>1)$, and $u\in C([0,T)\,;L^R_{loc}(\Omega))$, then

$$\sup_{x\in B(x_0,\eta/2)} u(x,t)\leqq Ct^{-\frac{N}{2R}}(t+\|u_0\|_{L^R(B(x_0,\eta))}),\qquad C=C(N,q,\nu,R,\eta,\tau). \tag{3.9}$$

Proof. We still assume that $x_0=0\in\Omega$. Let $\xi\in C^1_c(B_{2\eta})$ be nonnegative, radial, with values in $[0,1]$, with $\xi=1$ on B_η and $|\nabla\xi|\leqq C_0(N)/\eta$. Since u is ν-subcaloric, from (2.4), for any $\rho\in(0,\eta)$ such that $\rho^2\leqq t<\tau$,

$$\sup_{B_{\eta/2}} u(.,t)\leqq C(N,\nu)\rho^{-(N+2)}\int_{t-\rho^2/4}^{t}\int_{B_\eta} u, \tag{3.10}$$

hence from Lemma 3.2,

$$\sup_{B_{\eta/2}} u(.,t)\leqq C(N,q,\nu)\rho^{-N}(\eta^{N-q'}t+\int_{B_{2\eta}} du_0).$$

Let $k_0\in\mathbb{N}$ such that $k_0\eta^2/2\geqq\tau$. For any $t\in(0,\tau]$, there exists $k\in\mathbb{N}$ with $k\leqq k_0$ such that $t\in\left(k\eta^2/2,(k+1)\eta^2/2\right]$. Taking $\rho^2=t/(k+1)$, we find

$$\begin{aligned}\sup_{B_{\eta/2}} u(.,t) &\leqq C(N,q,\nu)(k_0+1)^{\frac{N}{2}}t^{-\frac{N}{2}}(\eta^{N-q'}t+\int_{B_{2\eta}} du_0)\\ &\leqq C(N,q,\nu)(\eta^{-N}\tau^{\frac{N}{2}}+1)t^{-\frac{N}{2}}(\eta^{N-q'}t+\int_{B_{2\eta}} du_0).\end{aligned} \tag{3.11}$$

Thus we obtain (3.8). Next assume that $u\in C([0,T)\,;L^R_{loc}(B_{2\eta}))$, with $R>1$. We still approximate u by $u_{\delta,n}=(u+\delta)*\varphi_n$, where (φ_n) is a sequence of mollifiers, and $\delta>0$. Since u is ν-subcaloric, then $u^R_{\delta,n}$ is also ν-subcaloric. Then for any $\rho\in(0,\eta)$ such that $\rho^2\leqq t<\tau$, we have

$$\sup_{B_{\eta/2}} u^R_{\delta,n}(.,t)\leqq C(N,\nu)\rho^{-(N+2)}\int_{t-\rho^2/4}^{t}\int_{B\rho/2} u^R_{\delta,n},$$

hence as $\delta\to0$ and $n\to\infty$, from Lemma (3.2),

$$\sup_{B_{\eta/2}} u^R(.,t)\leqq C(N,\nu)\rho^{-(N+2)}\int_{t-\rho^2/4}^{t}\int_{B\rho/2} u^R\leqq C(N,q,\nu,R)(\eta^{-N}\tau^{\frac{N}{2}}+1)(\eta^{N-Rq'}t^R+\int_{B_{2\eta}} u_0^R). \tag{3.12}$$

We deduce (3.9) as above. ■

4 Global estimates in $\mathbb{R}^N$

We first show that the universal estimate of the gradient (1.12) implies the estimate (1.13) of the function:

Theorem 4.1 *Let $q > 1$. Let u be a classical solution of equation (1.1) in $Q_{\mathbb{R}^N,T}$. Assume that there exists a ball $B(x_0, 2\eta)$ such that u has a trace $u_0 \in \mathcal{M}^+((B(x_0, 2\eta))$. If u satisfies (1.12), then for any $t \in (0, T)$,*

$$u(x,t) \leqq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \int_{B(x_0,\eta)} du_0), \qquad C = C(N, q, \eta), \tag{4.1}$$

If $u_0 \in L^R_{loc}(\Omega)$, $R \geqq 1$ and $u \in C([0, T) ; L^R_{loc}(\Omega))$, then

$$u(x,t) \leqq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + Ct^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0,\eta))}), \qquad C = C(N, q, R, \nu, \eta). \tag{4.2}$$

$$u(x,t) \leqq C(q)t^{-\frac{1}{q-1}} |x - x_0|^{q'} + C(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0,\eta))}), \qquad C = C(N, q, R, \eta). \tag{4.3}$$

Proof. Estimate (1.12) is equivalent to

$$\left|\nabla(u^{\frac{1}{q'}})\right|(.,t) \leqq \frac{(q-1)^{\frac{1}{q'}}}{q} t^{-\frac{1}{q}}, \qquad \text{in } Q_{\mathbb{R}^N,T}. \tag{4.4}$$

Then with constants $C(q)$ only depending of q,

$$u^{\frac{1}{q'}}(x,t) \leqq u^{\frac{1}{q'}}(x_0,t) + C(q)t^{-\frac{1}{q}} |x - x_0|, \tag{4.5}$$

then

$$u(x,t) \leqq C(q)(u(x_0,t) + t^{-\frac{1}{q-1}} |x - x_0|^{q'}), \tag{4.6}$$

and, from Theorem 3.3,

$$u(x_0,t) \leqq C(N, q, R, \nu, \eta)t^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0,\eta))}).$$

Therefore (4.2) follows. Also, interverting x and x_0, for any $R \geqq 1$,

$$u^R(x_0,t) \leqq C(q,R)(u^R(x,t) + t^{-\frac{R}{q-1}} |x - x_0|^{Rq'}).$$

Integrating on $B(x_0, \eta/2)$, we get

$$\eta^N u^R(x_0,t) \leqq C(q,R)(\int_{B(x_0,\eta/2)} u^R(.,t) + t^{-\frac{R}{q-1}} \eta^{N-Rq'});$$

using Lemma 3.2, we deduce

$$u(x_0,t) \leqq C(N, q, R, \eta)(t^{-\frac{1}{q-1}} + t + \int_{B(x_0,\eta)} du_0),$$

and if $u_0 \in L^R_{loc}(\Omega)$,

$$u(x_0,t) \leqq C(N, q, R, \eta)(t^{-\frac{1}{q-1}} + t + \|u_0\|_{L^R(B(x_0,\eta))}),$$

and the conclusions follow from (4.6). ∎

Remark 4.2 *In particular, the estimates (4.1)-(4.3) hold for solutions with $u_0 \in C_b(\mathbb{R}^N)$, and more generally for limits a.e. of such solutions, that we can call* ***reachable*** *solutions. Inegality (4.5) was used in [5, Theorem 3.3] for obtaining local estimates of classical of bounded solutions.in $Q_{\mathbb{R}^N,T}$.*

In order to prove Theorem 1.2, we first give an estimate of the type of (1.13) on a time interval $(0,\tau]$, *with constants depending on τ and ν*, which is *not obtained from any estimate of the gradient.* Our result is based on the construction of suitable supersolutions in annulus of type $Q_{B_{3\rho}\backslash\overline{B_\rho},\infty}$, $\rho>0$. For the construction we consider the function $t\in(0,\infty)\longmapsto\psi_h(t)\in(1,\infty)$, where $h>0$ is a parameter, solution of the problem

$$(\psi_h)_t+h(\psi_h^q-\psi_h)=0 \quad \text{in } (0,\infty)\,, \qquad \psi_h(0)=\infty, \quad \psi_h(\infty)=1, \tag{4.7}$$

given explicitely by $\psi_h(t)=(1-e^{-h(q-1)t})^{-\frac{1}{q-1}}$; hence $\psi_h^q-\psi_h\geqq 0$, and for any $t>0$,

$$((q-1)ht)^{-\frac{1}{q-1}}\leqq\psi_h(t)\leqq 2^{\frac{1}{q-1}}(1+((q-1)ht)^{-\frac{1}{q-1}}). \tag{4.8}$$

since, for $x>0$, $x(1-x/2)\leqq 1-e^{-x}\leqq x$, hence $x/2\leqq 1-e^{-x}\leqq x$, for $x\leqq 1$.

Proposition 4.3 *Let $q>1$. Then there exists a nonnegative function V defined in $Q_{B_3\times(0,\infty)}$, such that V is a supersolution of equation (1.1) on $Q_{B_3\backslash\overline{B_1},\infty}$, and V converges to ∞ as $t\to 0$, uniformly on B_3 and converges to ∞ as $x\to\partial B_3$, uniformly on $(0,\tau)$ for any $\tau<\infty$. And V has the form*

$$V(x,t)=e^t\Phi(|x|)\psi_h(t) \qquad \text{in } Q_{B_3,\infty} \tag{4.9}$$

for some $h=h(N,q,\nu)>0$, where ψ_h is given by (4.7), and Φ is a suitable radial function depending on N,q,ν, such that

$$-\nu\Delta\Phi+\Phi+|\nabla\Phi|^q\geqq 0 \qquad \text{in } B_3. \tag{4.10}$$

Proof. We first construct Φ. Let $\sigma>0$, such that $\sigma\geqq a=(2-q)/(q-1)$. Let φ_1 be the first eigenfunction of the Laplacian in B_3 such that $\varphi_1(0)=1$, associated to the first eigenvalue λ_1, hence φ_1 is radial ; let $m_1=\min_{\overline{B_1}}\varphi_1>0$ and $M_1=\min_{\overline{B_3}\backslash B_1}|\nabla\varphi_1|$. Let us take $\Phi=\Phi_K=\Phi_0+K$, where $\Phi_0=\gamma\varphi_1^{-\sigma}$, $K>0$ and $\gamma>0$ are parameters Then

$$-\nu\Delta\Phi+\Phi+|\nabla\Phi|^q=F(\Phi_0)+K, \qquad \text{with}$$

$$F(\Phi_0)=\gamma\varphi_1^{-(\sigma+2)}(\gamma^{q-1}\sigma^q\varphi_1^{(q-1)(a-\sigma)}\left|\varphi_1'\right|^q+(1-\nu\sigma\lambda_1)\varphi_1^2-\nu\sigma(\sigma+1)\varphi_1'^2).$$

There holds $\lim_{r\to 3}|\varphi_1'|=c_1>0$ from the Höpf Lemma. Taking $\sigma>a$ we fix $\gamma=1$, and then $\lim_{r\to 3}F(\Phi_0)=\infty$. If $q<2$ we can also take $\sigma=a$, we get

$$F(\Phi_0)=\gamma\varphi_1^{-q'}(\gamma^{q-1}a^q\left|\varphi_1'\right|^q+(1-\nu a\lambda_1)\varphi_1^2-aq'\varphi_1'^2),$$

hence fixing $\gamma>\gamma(N,q,\nu)$ large enough, we still get $\lim_{r\to 3}F(\Phi_0)=\infty$. Thus F has a minimum μ in B_3. Taking $K=K(N,q,\nu)>|\mu|$ we deduce that Φ satisfies (4.10), and $\lim_{r\to 3}\Phi=\infty$.

Observe that $\Phi'^q/\Phi = \gamma^q\sigma^q/(\gamma\varphi_1^{q+\sigma(q-1)} + K\varphi_1^{q(\sigma+1)})$ is increasing, then $m_K = m_K(N,q,\nu) = \min_{[1,3]} |\Phi'|^q/\Phi = |\Phi'(1)|^q/\Phi(1) > 0$. We define V by (4.9) and compute

$$\begin{aligned} V_t - \nu\Delta V + |\nabla V|^q &= e^t(\Phi\psi_h + \Phi(\psi_h)_t - \nu\Delta\Phi) + e^{qt}|\nabla\Phi|^q\psi_h^q \\ &\geqq e^t(\Phi\psi_h + \Phi\psi_t - \nu\Delta\Phi + |\nabla\Phi|^q\psi^q) = e^t(\psi^q - \psi_h)(|\nabla\Phi|^q - h\Phi). \end{aligned}$$

We take $h = h(N,q,\nu) < m_K$. Then on $B_3\backslash B_1$ we have $|\nabla\Phi|^q - h\Phi > 0$, and $\psi^q \geqq \psi_h$, then V is a supersolution on $B_3\backslash B_1$. Moreover V is radial and increasing with respect to $|x|$, then

$$\begin{aligned} \sup_{\overline{B_2}} V(x,t) = \sup_{\overline{\partial B_2}} V(x,t) = e^t\Phi(2)\psi_h(t) &\leqq 2^{\frac{1}{q-1}}e^t\Phi(2)(1 + ((q-1)ht)^{-\frac{1}{q-1}}) \\ &\leqq C(N,q,\nu)e^t\Phi(2)(1 + t^{-\frac{1}{q-1}}). \end{aligned} \tag{4.11}$$

■

Theorem 4.4 *Let u be a classical solution, (in particular any weak solution if $q \leqq 2$) of equation (1.1) in $Q_{\mathbb{R}^N,T}$. Assume that there exists a ball $B(x_0, 2\eta)$ such that u admits a trace $u_0 \in \mathcal{M}^+(B(x_0, 2\eta))$.*

(i) Then for any $\tau \in (0,T)$, and $t \leqq \tau$,

$$u(x,t) \leqq C(t^{-\frac{1}{q-1}}|x - x_0|^{q'} + t^{-\frac{N}{2}}(t + \int_{B(x_0,\eta)} du_0)), \qquad C = C(N,q,\nu,\eta,\tau), \tag{4.12}$$

(ii) Also if $u \in C([0,T)\,; L^R_{loc}(B(x_0,2\eta)))$,

$$u(x,t) \leqq C(t^{-\frac{1}{q-1}}|x - x_0|^{q'} + t^{-\frac{N}{2R}}(t + \|u_0\|_{L^R(B(x_0,\eta))})), \qquad C = C(N,q,\nu,R,\eta,\tau), \tag{4.13}$$

if $u \in C([0,T) \times B(x_0,2\eta))$, then

$$u(x,t) \leqq C(t^{-\frac{1}{q-1}}|x - x_0|^{q'} + t + \sup_{B(x_0,\eta)} u_0), \qquad C = C(N,q,\nu,\eta,\tau). \tag{4.14}$$

Proof. We use the function V constructed above. We can assume $x_0 = 0$. For any $\rho > 0$, we consider the function V_ρ defined in $B_{3\rho} \times (0,\infty)$ by

$$V_\rho(x,t) = \rho^{-a}V(\rho^{-1}x, \rho^{-2}t).$$

It is a supersolution of the equation (1.1) on $B_{3\rho}\backslash\overline{B_\rho} \times (0,\infty)$, infinite on $\partial B_{3\rho} \times (0,\infty)$ and on $B_{3\rho} \times \{0\}$, and from (4.11)

$$\begin{aligned} \sup_{\overline{B_{2\rho}}} V_\rho(x,t) = \sup_{\partial B_{2\rho}} V_\rho(x,t) &\leqq C_1(N,q,\nu)\rho^{-a}e^{\frac{t}{\rho^2}}\Phi(2)(1 + \rho^{\frac{2}{q-1}}t^{-\frac{1}{q-1}}) \\ &\leqq C_2(N,q,\nu)\rho^{q'}e^{\frac{t}{\rho^2}}(\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \end{aligned} \tag{4.15}$$

(i) First suppose that $u \in C([0,T) \times \mathbb{R}^N))$. Let $\tau \in (0,T)$, and $C(\tau) = \sup_{Q_{B_\rho,\tau}} u$. Then $w = C(\tau) + V_\rho$ is a supersolution in $Q = (B_{3\rho}\backslash\overline{B_\rho}) \times (0,\tau]$, and from the comparison principle we obtain $u \leqq C(\tau) + V_\rho$ in that set. Indeed let $\epsilon > 0$ small enough. Then there exists $\tau_\epsilon < \epsilon$ and

$r_\epsilon \in (3\rho - \epsilon, 3\rho)$, such that $w(.,s) \geqq \max_{\overline{B_{3\rho}}} u(.,\epsilon)$ for any $s \in (0, \tau_\epsilon]$, and $w(x,t) \geqq \max_{\overline{B_{3\rho}} \times [0,\tau]} u$ for any $t \in (0,\tau]$ and $r_\epsilon \leqq |x| < 3\rho$. We compare $u(x,t+\epsilon)$ with $w(x,t+s)$ on $[0,\tau-\epsilon] \times \overline{B_{r_\epsilon}} \backslash \overline{B_\rho}$. And for $|x| = \rho$, we have $u(x,t+\epsilon) \leqq C(\tau) \leqq w(x,t+s)$. Then $u(.,t+\epsilon) \leqq w(.,t+s)$ in $\overline{B_{r_\epsilon}} \backslash \overline{B_\rho} \times (0,\tau-\epsilon]$. As $s,\epsilon \to 0$, we deduce that $u \leqq w$ in Q.

Hence in $\overline{B_{2\rho}} \times (0,\tau)$, we find from (4.15)

$$u \leqq C(\tau) + \sup_{\overline{B_{2\rho}}} V_\rho(x,t) \leqq C(\tau) + C_2 \rho^{q'} e^{\frac{t}{\rho^2}} (\rho^{-\frac{2}{q-1}} + t^{-\frac{1}{q-1}}). \tag{4.16}$$

Making t tend to τ, this proves that

$$\sup_{Q_{B_{2\rho},\tau}} u \leqq \sup_{Q_{B_\rho,\tau}} u + C_2 \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}})$$

By induction, we get

$$\begin{aligned}\sup_{Q_{B_{2^{n+1}\rho},\tau}} u &\leqq \sup_{Q_{B_{2^n\rho},\tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{4^n\rho^2}} ((2^n\rho)^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leqq \sup_{Q_{B_{2^n\rho},\tau}} u + C_2 2^{nq'} \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}});\end{aligned}$$

$$\begin{aligned}\sup_{Q_{B_{2^{n+1}\rho},\tau}} u &\leqq \sup_{Q_{B_\rho}} u + C_2 (1 + 2^{q'} + .. + 2^{nq'}) \rho^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \\ &\leqq \sup_{Q_{B_\rho,\tau}} u + C_2 2^{q'} (2^n\rho)^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}).\end{aligned}$$

For any $x \in \mathbb{R}^N$ such that $|x| \geqq \rho$, there exists $n \in \mathbb{N}^*$ such that $x \in B_{2^{n+1}\rho} \backslash \overline{B_{2^n\rho}}$, then

$$u(x,\tau) \leqq \sup_{Q_{B_\rho,\tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}) \tag{4.17}$$

thus

$$\sup_{Q_{\mathbb{R}^N,\tau}} u \leqq \sup_{Q_{B_\rho,\tau}} u + C_2 2^{q'} |x|^{q'} e^{\frac{\tau}{\rho^2}} (\rho^{-\frac{2}{q-1}} + \tau^{-\frac{1}{q-1}}). \tag{4.18}$$

(ii) Next we consider any classical solution u in $Q_{\mathbb{R}^N,T}$ with trace u_0 in $B(x_0, 2\eta)$. We still assume $x_0 = 0$. Then for $0 < \epsilon \leqq t \leqq \tau$, from (3.4) in Lemma 3.2, there holds

$$\sup_{B_{\eta}/2} u(x,t) \leqq C(N,q) \eta^{-q'} t + \sup_{B_\eta} u(x,\epsilon).$$

Then from (4.18) with $\rho = \eta/2$, we deduce that for any $(x,t) \in Q_{\mathbb{R}^N,\epsilon,\tau}$,

$$u(x,t) \leqq C(N,q) \eta^{-q'} t + \sup_{B_{\eta/2}} u(.,\epsilon) + C(1 + (t-\epsilon)^{-\frac{1}{q-1}}) |x|^{q'},$$

with $C = C(N, q, \nu, \eta, \tau)$. Next we take $\epsilon = t/2$. Then for any $t \in (0, \tau]$, from (3.8) in Theorem 3.3,

$$u(x,t) \leqq C(N, q, \eta)t + Ct^{-1(q-1)} |x|^{q'} + Ct^{-\frac{N}{2}}(t + \int_{B_\eta} du_0).$$

with $C = C(N, q, \nu, \eta, \tau)$ and we obtain (4.12). And (4.13), (4.14) follow from (3.9) and (3.4). ∎

Next we show our main Theorem 1.2. We use a *local* Bernstein technique, as in [26]. The idea is to compute the equation satisfied by the function $v = u^{(q-1)/q}$, introduced in [9], and the equation satisfied by $w = |\nabla v|^2$, to obtain estimates of w in a cylinder $Q_{B_M,T}$, $M > 0$. The difficulty is that this equation involves an elliptic operator $w \mapsto w_t - \Delta w + b.\nabla w$, where b depends on v, and may be unbounded. However it can be controlled by the estimates of v obtained at Theorem 4.4. Then as $M \to \infty$, we can prove nonuniversal L^∞ estimates of w. Finally we obtain universal estimates of w by application of the maximum principle in $Q_{\mathbb{R}^N,T}$, valid because w is bounded. First we give a slight improvement of a comparison principle shown in [26, Proposition 2.2].

Lemma 4.5 *Let Ω be any domain of $\mathbb{R}^N$, and $\tau, \kappa \in (0, \infty)$, $A, B \in \mathbb{R}$. Let $U \in C([0, \tau); L^2_{loc}(\overline{\Omega}))$ such that $U_t, \nabla u, D^2u \in L^2_{loc}(\overline{\Omega} \times (0, \tau))$, $\operatorname{ess\,sup}_{Q_{\Omega,\tau}} U < \infty$, $U \leqq B$ on the parabolic boundary of $Q_{\Omega,\tau}$, and*

$$U_t - \Delta U \leqq \kappa(1 + |x|) |\nabla U| + f \qquad \text{in } Q_{\Omega,\tau}$$

where $f = f(x,t)$ such that $f(.,t) \in L^2_{loc}(\overline{\Omega})$ for a.e. $t \in (0, \tau)$ and $f \leqq 0$ on $\{(x,t) \in Q_{\Omega,\tau} : U(x,t) \geqq A\}$. Then $\operatorname{ess sup}_{Q_{\Omega,\tau}} U \leqq \max(A, B)$.

Proof. We set $\varphi(x,t) = \Lambda t + \ln(1 + |x|^2)$, $\Lambda > 0$. Then $\nabla\varphi = 2x/(1 + |x|^2)$, $0 \leqq \Delta\varphi \leqq 2N/(1 + |x|^2) \leqq 2N$. Let $\varepsilon > 0$ and $Y = U - \max(A, B) - \varepsilon\varphi$. Taking $\Lambda = 2\sqrt{2}\kappa + 2N$, we obtain

$$Y_t - \Delta Y - f - \kappa(1 + |x|) |\nabla Y| \leqq \varepsilon(K(1 + |x|) |\nabla\varphi| - \varphi_t + \Delta\varphi) \leqq \varepsilon(2\sqrt{2}\kappa + 2N - \Lambda) = 0.$$

Since $\operatorname{esssup}_{Q_{\Omega,\tau}} U < \infty$, for R large enough, and any $t \in (0, \tau)$, we have $Y(.,t) \leqq 0$ *a.e.* in $\Omega \cap \{|x| > R\}$. And $Y^+ \in C([0, \tau); L^2(\Omega)) \cap W^{1,2}((0, \tau); L^2(\Omega))$, $Y^+(0) = 0$ and $Y^+(.,t) \in W^{1,2}(\Omega \cap B_R)$ for *a.e.* $t \in (0, \tau)$, and $fY^+(.,t) \leqq 0$. Then

$$\frac{1}{2}\frac{d}{dt}(\int_\Omega Y^{+2}(.,t) \leqq -\int_\Omega \left|\nabla Y^+(.,t)\right|^2 + \kappa(1 + R)\int_\Omega |\nabla Y(.,t)| Y^+(.,t) \leqq \frac{\kappa^2(1 + R)^2}{4}\int_\Omega Y^{+2}(.,t),$$

hence by integration $Y \leqq 0$ *a.e.* in $Q_{\Omega,\tau}$. We conclude as $\varepsilon \to 0$. ∎

Proof of Theorem 1.2. We can assume $x_0 = 0$. By setting $u(x,t) = \nu^{q'/2}U(x/\sqrt{\nu}, t)$, for proving (4.4) we can suppose that u is a classical solution of (1.1) with $\nu = 1$. We set

$$\delta + u = v^{\frac{q}{q-1}}, \qquad \delta \in (0, 1).$$

(i) Local problem relative to $|\nabla v|^2$. Here u is any classical solution u of equation (1.1) in a cylinder $Q_{B_M,T}$ with $M > 0$. Then v satisfies the equation

$$v_t - \Delta v = \frac{1}{q-1}\frac{|\nabla v|^2}{v} - cv |\nabla v|^q, \qquad c = (q')^{q-1}. \tag{4.19}$$

Setting $w = |\nabla v|^2$, we define

$$\mathcal{L}w = w_t - \Delta w + b.\nabla w, \qquad b = (qcvw^{\frac{q-2}{2}} - \frac{2}{q-1}\frac{1}{v})\nabla v.$$

Differentiating (4.19) and using the identity $\Delta w = 2\nabla(\Delta w).\nabla w + 2\left|D^2 v\right|^2$, we obtain the equation

$$\mathcal{L}w + 2cw^{\frac{q+2}{2}} + 2\left|D^2 v\right|^2 + \frac{2}{q-1}\frac{w^2}{v^2} = 0. \tag{4.20}$$

As in [26], for $s \in (0,1)$, we consider a test function $\zeta \in C^2(\overline{B}_{3M/4})$ with values in $[0,1]$, $\zeta = 0$ for $|x| \geq 3M/4$ and $|\nabla\zeta| \leqq C(N,s)\zeta^s/M$ and $|\Delta\zeta| + |\nabla\zeta|^2/\zeta \leqq C(N,s)\zeta^s/M^2$ in $B_{3M/4}$. We set $z = w\zeta$. We have

$$\mathcal{L}z = \zeta\mathcal{L}w + w\mathcal{L}\zeta - 2\nabla w.\nabla\zeta \leqq \zeta\mathcal{L}w + w\mathcal{L}\zeta + \left|D^2 v\right|^2\zeta + 4w\frac{|\nabla\zeta|^2}{\zeta}.$$

It follows that in $Q_{B_M,T}$,

$$\mathcal{L}z + 2cw^{\frac{q+2}{2}}\zeta + \frac{2}{q-1}\frac{w^2}{v^2}\zeta \leqq \frac{C\zeta^s w}{M^2} + \frac{C\zeta^s w^{\frac{3}{2}}}{M}\left|cqvw^{\frac{q-2}{2}} - \frac{2}{q-1}\frac{1}{v}\right| \leqq C\zeta^s(\frac{w}{M^2} + \frac{vw^{\frac{q+1}{2}}}{M} + \frac{w^{\frac{3}{2}}}{Mv}),$$

with constants $C = C(N,q,s)$. Since $\zeta \leqq 1$, from the Young inequality, taking $s \geqq \max(q+1),3)/(q+2)$, for any $\varepsilon > 0$,

$$\frac{C}{M}\zeta^s vw^{\frac{q+1}{2}} = \frac{C}{M}\zeta^{\frac{q+1}{q+2}}\zeta^{s-\frac{q+1}{q+2}}vw^{\frac{q+1}{2}} \leqq \varepsilon\zeta w^{\frac{q+2}{2}} + C(N,q,\varepsilon)\frac{v^{q+2}}{M^{q+2}},$$

and

$$\frac{C}{M^2}\zeta^s w \leqq \varepsilon\zeta w^{\frac{q+2}{2}} + C(N,q,\varepsilon)\frac{1}{M^{\frac{2(q+2)}{q}}},$$

$$\frac{C}{M}\zeta^s\frac{w^{\frac{3}{2}}}{v} \leqq \frac{1}{\delta M}\zeta^s w^{\frac{3}{2}} = \frac{1}{\delta M}\zeta^{s-\frac{3}{q+2}}\zeta^{\frac{3}{q+2}}w^{\frac{3}{2}} \leqq \varepsilon\zeta w^{\frac{q+2}{2}} + C(N,q,\varepsilon)\frac{1}{(\delta M)^{\frac{q+2}{q-1}}}.$$

Then with a new $C = C(N,q,\delta)$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leqq C(\frac{v^{q+2}}{M^{q+2}} + \frac{1}{M^{\frac{2(q+2)}{q}}} + \frac{1}{M^{\frac{q+2}{q-1}}}). \tag{4.21}$$

(ii) Nonuniversal estimates of w. Here we assume that u is a classical solution of (1.1) in whole $Q_{\mathbb{R}^N,T}$, such that $u \in C(\mathbb{R}^N \times [0,T))$. From Theorem 4.4, for any $\tau \in (0,T)$, there holds in $Q_{\mathbb{R}^N,\tau}$

$$v(x,t) = (\delta + u(x,t))^{\frac{q-1}{q}} \leqq C(t^{-\frac{1}{q}}|x| + (t + \sup_{B_{2\eta}} u_0)^{\frac{q-1}{q}}), \qquad C = C(N,q,\eta,\tau). \tag{4.22}$$

hence for $M \geqq M(q, \sup_{B_{2\eta}} u_0, \tau) \geqq 1$, we deduce

$$v(x,t) \leqq 2Ct^{-\frac{1}{q}}M, \qquad \text{in } Q_{B_M,\tau}.$$

Then with a new constant $C = C(N, q, \eta, \tau, \delta)$, there holds in $Q_{B_{3M/4},\tau}$

$$\mathcal{L}z + cz^{\frac{q+2}{2}} \leqq Ct^{-\frac{q+2}{q}}. \tag{4.23}$$

Next we consider $\Psi(t) = Kt^{-2/q}$. It satisfies

$$\Psi_t + c\Psi^{\frac{q+2}{2}} = (cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geqq Ct^{-\frac{q+2}{q}}$$

if $K \geqq \overline{K} = \overline{K}(N, q, \eta, \tau, \delta)$. Fixing $\epsilon \in (0, T)$ such that $\tau + \epsilon < T$, there exists $\tau_\epsilon \in (0, \epsilon)$ such that $\Psi(\theta) \geqq \sup_{B_M} z(., \epsilon)$ for any $\theta \in (0, \tau_\epsilon)$. We have

$$\begin{aligned} &z_t(., t+\epsilon) - \Delta z(., t+\epsilon) + b(., t+\epsilon).\nabla(z, t+\epsilon) + cz^{\frac{q+2}{2}}(t+\epsilon) \\ &\leqq C(t+\epsilon)^{-\frac{q+2}{q}} \leqq C(t+\theta)^{-\frac{q+2}{q}} \leqq \Psi_t(t+\theta) + c\Psi^{\frac{q+2}{2}}(t+\theta). \end{aligned}$$

Therefore, setting $\tilde{z}(., t) = z(., t+\epsilon) - \Psi(t+\theta)$,on the set $\mathcal{V} = \left\{(x, t) \in Q_{B_{3M/4},\tau+\epsilon} : \tilde{z}(x, t) \geqq 0\right\}$,

$$\tilde{z}(., t) - \Delta\tilde{z}(., t) + b(., t+\epsilon).\nabla\tilde{z}(., t) \leqq 0;$$

and $\tilde{z}(., t) \leqq 0$ for sufficiently small $t > 0$, and $\tilde{z} \leqq 0$ on $\partial B_{3M/4} \times [0, \tau]$. Then from Lemma 4.5, we get $z(., t+\epsilon) \leqq \Psi(t+\theta)$ in $Q_{B_{3M/4},\tau}$, since $|b| \leqq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1}\frac{1}{\delta}w^{1/2})$, hence bounded on $Q_{B_{3M/4},\tau+\epsilon}$. Going to the limit as $\theta, \epsilon \to 0$, we deduce that $z(., t) \leqq \overline{K}t^{-\frac{2}{q}}$ in $Q_{B_{3M/4},\tau}$, thus $w(., t) \leqq \overline{K}t^{-\frac{2}{q}}$ in $Q_{B_{M/2},\tau}$. Next we go to the limit as $M \to \infty$ and deduce that $w(., t) \leqq \overline{K}t^{-\frac{2}{q}}$ in $Q_{\mathbb{R}^N,\tau}$, namely

$$(q')^q\,|\nabla v(., t)|^q = \frac{|\nabla u|^q}{\delta + u}(., t) \leqq Ct^{-1}, \qquad C = C(N, q, \eta, \delta, \tau).$$

In turn for any ϵ as above, *there holds* $w \in L^\infty(Q_{\mathbb{R}^N,\epsilon,T})$, that means $|\nabla v| \in L^\infty(Q_{\mathbb{R}^N,\epsilon,\tau})$.

(iii) Universal estimate (4.4) for $u \in C(\mathbb{R}^N \times [0, T))$: we prove the universal estimate (4.4). Taking again $\Psi(t) = Kt^{-2/q}$, with now $K = K(N, q) = q^{-2}(q-1)^{2/q'}$, we have

$$\Psi_t + 2c\Psi^{\frac{q+2}{2}} \geqq (2cK^{\frac{q+2}{2}} - 2q^{-1}K)t^{-\frac{q+2}{q}} \geqq 0.$$

And $\mathcal{L}w + 2cw^{\frac{q+2}{2}} \leqq 0$ from (4.20). Moreover there exists $\tau_\epsilon \in (0, \tau)$ such that $\Psi(\theta) \geqq \sup_{\mathbb{R}^N} w(., \epsilon)$ for any $\theta \in (0, \tau_\epsilon)$. Setting $y(., t) = w(., t+\epsilon) - \Psi(., t+\theta)$, hence on the set $\mathcal{U} = \left\{(x, t) \in Q_{\mathbb{R}^N,\tau} : y(x, t) \geqq 0\right\}$, there holds in the same way

$$y(., t) - \Delta y(., t) + b(., t+\epsilon).\nabla y(., t) \leqq 0.$$

Here we only have from (4.22)

$$|b| \leqq (qcvw^{\frac{q-1}{2}} + \frac{2}{q-1}\frac{1}{\delta}w^{1/2}) \leqq \kappa_\epsilon(1 + |x|)$$

on $Q_{\mathbb{R}^N,\epsilon,\tau}$, for some $\kappa_\epsilon = \kappa_\epsilon(N, q, \eta, \sup_{B_{2\eta}} u_0, \tau, \epsilon)$. It is sufficient to apply Lemma 4.5. We deduce that $w(., t+\epsilon) \leqq \Psi(t+\theta)$ on $(0, \tau)$. As $\theta, \epsilon \to 0$ we obtain that $w(., t) \leqq \Psi(t) = q^{-2}(q-1)^{2/q'}t^{-2/q}$, which shows now that in $(0, T)$

$$|\nabla v(., t)|^q = (q')^{-q}\frac{|\nabla u|^q}{\delta + u}(., t) \leqq q^{-q}(q-1)^{(q-1)}t^{-1}.$$

As $\delta \to 0$, we obtain (4.4).

(iv) General universal estimate. Here we relax the assumption $u \in C(\mathbb{R}^N \times [0, T))$: For any $\epsilon \in (0, T)$ such that $\tau + \epsilon < T$, we have $u \in C(\mathbb{R}^N \times [\epsilon, \tau + \epsilon))$, then from above,

$$|\nabla v(., t+\epsilon)|^q \leqq \frac{1}{q-1}\frac{1}{t},$$

and we obtain (4.4) as $\epsilon \to 0$, on $(0, \tau)$ for any $\tau < T$, hence on $(0, T)$. ■

Proof of Theorem 1.3. It is a direct consequence of Theorems 1.2 and 4.1. ■

5 Existence and nonuniqueness results

Next we mention some known uniqueness and comparison results, for the Cauchy problem, see [11, Theorems 2.1,4.1,4.2 and Remark 2.1],[13, Theorem 2.3, 4.2, 4.25, Proposition 4.26], and for the Dirichlet problem, see [1, Theorems 3.1, 4.2], [6], [13, Proposition 5.17], [24].

Theorem 5.1 *Let $\Omega = \mathbb{R}^N$ (resp. Ω bounded). (i) Let $1 < q < q_*$, and $u_0 \in \mathcal{M}_b(\mathbb{R}^N)$(resp. $u_0 \in \mathcal{M}_b(\Omega)$). Then there exists a unique weak solution u of (1.1) with trace u_0 (resp. a weak solution of $(D_{\Omega,T})$, such that $\lim_{t\to 0} u(.t) = u_0$ weakly in $\mathcal{M}_b(\Omega)$)). If $v_0 \in \mathcal{M}_b(\Omega)$ and $u_0 \leqq v_0$, and v is the solution associated to v_0, then $u \leqq v$.*

(ii) Let $u_0 \in L^R(\Omega)$, $1 \leqq R \leqq \infty$. If $1 < q < (N+2R)/(N+R)$, or if $q = 2$, $R < \infty$, there exists a unique weak solution u of (1.1) (resp. $(D_{\Omega,T})$) such that $u \in C([0, T); L^R(\Omega)$ and $u(0) = u_0$. If $v_0 \in L^R(\mathbb{R}^N)$ and $u_0 \leqq v_0$, then $u \leqq v$. If u_0 is nonnegative, then for any $1 < q \leqq 2$, there still exists at least a weak nonnegative solution u satisfying the same conditions.

Next we prove Theorem 1.4. We begin by the subcritical case:

Remark 5.2 *From [3, Lemma 3.3], for any reals $s < \tau$, the Dirichlet problem*

$$\begin{cases} u_t - \Delta u = g, & \text{in } \mathcal{D}'(Q_{\Omega,\tau}), \\ u = 0, & \text{on } \partial\Omega \times (0, \tau), \\ u(., s) = u_s, & \text{in } \Omega, \end{cases}$$

in a bounded domain Ω, with data $g \in L^1(Q_{\Omega,\tau})$ and $u_s \in L^1(\Omega)$, has a unique solution $u \in C([0, \tau], L^1(\Omega)) \cap L^1((0, \tau); W_0^{1,1}(\Omega))$. And $u \in L^k((s, \tau); W_0^{1,k}(\Omega))$, for any $k \in [1, q_*)$, and

$$\|u\|_{L^k((s,\tau);W_0^{1,k}(\Omega))} \leqq C(k, \Omega)(\|u(., s)\|_{L^1(\Omega)} + \|g\|_{L^1(Q_{\Omega,s,\tau})}). \tag{5.1}$$

This implies local estimates in any domain Ω of $\mathbb{R}^N$: for any $g \in L^1_{loc}((0,T), L^1(\Omega))$, and $u \in C((0,T), L^1(\Omega)) \cap L^1_{loc}((0,T); W^{1,1}_{loc}(\Omega))$, such that

$$u_t - \Delta u = g, \quad in\ \mathcal{D}'(Q_{\Omega,T}),$$

there holds $u \in L^1_{loc}((0,T); W^{1,k}_{loc}(\Omega)$. And for any domain $\omega \subset\subset \Omega$, and any $0 < s < \tau < T$

$$\|u\|_{L^k((s,\tau);W^{1,k}(\omega))} \leqq C(k,\omega)(\|u(s,.)\|_{L^1(\omega)} + \|g + |\nabla u| + |u|\|_{L^1(Q_{\omega,s,\tau})}). \tag{5.2}$$

Proposition 5.3 *Let $1 < q < q_*$, and $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Then for any $u_0 \in \mathcal{M}^+(\mathbb{R}^N)$ (resp. $\mathcal{M}^+(\Omega)$), there exists a weak solution u of equation (1.1) (resp. of $(D_{\Omega,T})$) with trace u_0.*

Proof. Assume $\Omega = \mathbb{R}^N$ (resp. Ω bounded). Let $u_{0,n} = u_0 \llcorner B_n$ (resp. $u_{0,n} = u_0 \llcorner \overline{\Omega'_{1/n}}$, where $\Omega_n = \{x \in \Omega : d(x, \partial\Omega) > 1/n\}$, for n large enough). From Theorem 5.1, there exists a unique weak solution u_n of (1.1) (resp. of $(D_{\Omega,T})$) with trace $u_{0,n}$, and (u_n) is nondecreasing; and $u_n \in C^{2,1}(Q_{\mathbb{R}^N,T})$ since $q \leqq 2$. From (3.1), (3.5), for any $\xi \in C^{1+}_c(\Omega)$,

$$\int_\Omega u_n(.,t)\xi^{q'} + \frac{1}{2}\int_0^t \int_\Omega |\nabla u_n|^q \xi^{q'} \leqq Ct \int_\Omega |\nabla \xi|^{q'} + \int_\Omega \xi^{q'} du_0. \tag{5.3}$$

Hence (u_n) is bounded in $L^\infty_{loc}\left([0,T)\,; L^1_{loc}(\Omega)\right)$, and $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}\left([0,T)\,; L^1_{loc}(\Omega)\right)$. In turn (u_n) is bounded in $L^\infty_{loc}((0,T); L^\infty_{loc}(\Omega))$, from Theorem 3.3. From Remark 2.4, (u_n) converges in $C^{2,1}_{loc}(Q_{\mathbb{R}^N,T})$ (resp. $C^{2,1}_{loc}(Q_{\Omega,T}) \cap C^{1,0}\left(\overline{\Omega} \times (0,T)\right))$ to a weak solution u of (1.1) in $Q_{\mathbb{R}^N,T}$ (resp. of $(D_{\Omega,T})$). Also (u_n) is bounded in $L^k_{loc}([0,T)\,; W^{1,k}_{loc}(\mathbb{R}^N))$ (resp. $L^k_{loc}([0,T)\,; W^{1,k}_0(\Omega))$) for any $k \in [1, q^*)$ from Remark 5.2. Since $q < q_*$, $(|\nabla u_n|^q)$ is equiintegrable in $Q_{B_R,\tau}$ for any $R > 0$ (resp. in $Q_{\Omega,\tau}$) and $\tau \in (0,T)$, then $(|\nabla u|^q) \in L^1_{loc}\left([0,T)\,; L^1_{loc}(\Omega)\right)$. From (2.6),

$$\int_\Omega u_n(t,.)\xi + \int_0^t \int_\Omega |\nabla u_n|^q \xi = -\int_0^t \int_\Omega \nabla u_n.\nabla \xi + \int_\Omega \xi du_0. \tag{5.4}$$

As $n \to \infty$ we obtain

$$\int_\Omega u(t,.)\xi + \int_0^t \int_\Omega |\nabla u|^q \xi = -\int_0^t \int_\Omega \nabla u.\nabla \xi + \int_\Omega \xi du_0.$$

Thus $\lim_{t\to 0} \int_\Omega u(.,t)\xi = \int_\Omega \xi du_0$, for any $\xi \in C^{1+}_c(\Omega)$, hence for any $\xi \in C^+_c(\Omega)$; hence u admits the trace u_0. ∎

Next we consider the supercritical case $q \geqq q_*$. From [1], if the Dirichlet problem $(P_{\Omega,T})$ has a solution with $u_0 \in \mathcal{M}_b(\Omega)$, then u_0 does not charge the sets of $W^{\frac{2-q}{q},q'}(\Omega)$ capacity 0 if $q < 2$. And $u_0 \in L^1(\Omega)$ if $q \geqq 2$, see another proof in [12]. In the same way, if the Cauchy problem $(P_{\mathbb{R}^N,T})$ has a solution with trace $u_0 \in \mathcal{M}(\mathbb{R}^N)$, then u_0 does not charge the sets of $W^{\frac{2-q}{q},q'}(\mathbb{R}^N)$-capacity 0 if $q < 2$, and $u_0 \in L^1_{loc}(\mathbb{R}^N)$ if $q \geqq 2$. The converse question is to know what are the "admissible" measures for which the problem has a solution. It is widely open, and we give here a few results in that direction, extending some results of [11].

Theorem 5.4 *(i) Let $1 < q \leqq 2$, and $\Omega = \mathbb{R}^N$ (resp. Ω bounded). For any nonnegative $u_0 \in L^1_{loc}\left(\mathbb{R}^N\right)$ (resp. $u_0 \in L^1_{loc}(\Omega)$), there exists a weak nonnegative solution of equation (1.1) in $Q_{\Omega,T}$ with trace u_0. And then $u \in C([0,T); L^1_{loc}\left(\mathbb{R}^N\right)$ (resp. $u \in C([0,T); L^1_{loc}(\Omega))$.*

(ii) Let $q > 2$. The existence is valid for any nonnegative $u_0 \in L^1_{loc}\left(\mathbb{R}^N\right)$ (resp. $L^1_{loc}(\Omega)$) which is a limit of an nondecreasing sequence of nonnegative functions in $C_b\left(\mathbb{R}^N\right)$ (resp. in $C_0(\Omega)$).

Proof. (i) Let $\Omega = \mathbb{R}^N$(resp. Ω bounded). As in Proposition 5.3, we set $u_{0,n} = \min(u_0, n)\chi_{B_n}$ (resp. $u_{0,n} = \min(u_0, n)\chi_{\overline{\Omega'_{1/n}}}$ for n large enough). Then $u_{0,n} \in L^r(\Omega)$ for any $r \geqq 1$. From Theorem 5.1, the problem admits a solution u_n , and it is unique in $C([0,T); L^r(\Omega))$ for any $r > (2-q)/N(q-1)$ and then (u_n) is nondecreasing. As in Proposition 5.3, (u_n) is bounded in $L^\infty_{loc}\left([0,T); L^1_{loc}(\Omega)\right)$, and $(|\nabla u_n|^q)$ is bounded in $L^1_{loc}\left([0,T); L^1_{loc}(\Omega)\right)$. Moreover, (u_n) is bounded in $L^\infty_{loc}((0,T); L^\infty_{loc}(\Omega))$ from Lemma 3.3. From Theorem 2.3, (u_n) converges in $C^{2,1}_{loc}(Q_{\Omega,T})$ to a weak solution u of (1.1) in $Q_{\Omega,T}$, such that $u \in L^\infty_{loc}\left([0,T); L^1_{loc}(\Omega)\right)$ and $|\nabla u|^q \in L^1_{loc}\left([0,T); L^1_{loc}(\Omega)\right)$.

Then from Remark 2.5, u admits a trace $\mu_0 \in \mathcal{M}^+(\Omega)$ as $t \to 0$. Applying (5.4) to u_n, since $u_n \leqq u$, we get

$$\lim_{t\to 0}\int_\Omega u(.,t)\xi = \int_\Omega \xi d\mu_0 \geqq \lim_{t\to 0}\int_\Omega u_n(.,t)\xi = \int_\Omega \xi du_0,$$

for any $\xi \in C_c^+(\Omega)$; thus $u_0 \leqq \mu_0$. Moreover

$$\int_\Omega u_n(t,.)\xi + \int_0^t\int_\Omega |\nabla u_n|^q\xi = \int_0^t\int_\Omega u_n\Delta\xi dx + \int_\Omega \xi du_0.$$

And (u_n) is bounded in $L^k(Q_{\omega,\tau})$ for any $k \in (1, q_*)$; then for any domain $\omega \subset\subset \Omega$, (u_n) converges strongly in $L^1(Q_{\omega,\tau})$; then from the convergence *a.e.* of the gradients, and the Fatou Lemma,

$$\int_{\mathbb{R}^N} u(t,.)\xi + \int_0^t\int_{\mathbb{R}^N} |\nabla u|^q\xi \leqq \int_0^t\int_{\mathbb{R}^N} u\Delta\xi dx + \int_{\mathbb{R}^N} \xi du_0.$$

But from Remark 2.5,

$$\int_{\mathbb{R}^N} u(t,.)\xi + \int_0^t\int_{\mathbb{R}^N} |\nabla u|^q\xi = \int_0^t\int_{\mathbb{R}^N} u\Delta\xi dx + \int_{\mathbb{R}^N} \xi d\mu_0,$$

then $\mu_0 \leqq u_0$, hence $\mu_0 = u_0$. Finally we prove the continuity: Let $\xi \in \mathcal{D}^+(\Omega)$ and $\omega \subset\subset \Omega$ containing the support of ξ. Then $z = u\xi$ is solution of the Dirichlet problem

$$\begin{cases} z_t - \Delta z = g, & \text{in } Q_{\omega,T}, \\ z = 0, & \text{on } \partial\omega \times (0,T), \\ \lim_{t\to 0} z(.,t) = \xi u_0, & \text{weakly in } \mathcal{M}_b(\omega), \end{cases}$$

with $g = -|\nabla u|^q\xi + v(-\Delta\psi) - 2\nabla v.\nabla\psi \in L^1(Q_{\omega,T})$. The solution is unique, see [6, Proposition 2.2]. Since $u_0 \in L^1_{loc}(\Omega)$, there also exists a unique solution such that $z \in C([0,T), L^1(\omega))$ from [3, Lemma 3.3], hence $u \in C([0,T), L^1_{loc}(\Omega))$.

(ii) As above, by taking for $(u_{0,n})$ a nondecreasing sequence in $C_b\left(\mathbb{R}^N\right)$ (resp. in $C_0(\Omega)$), converging to u_0, and using Remark 2.4 for classical solutions. ■

In particular this ends the proof of Theorem 1.4.

Next we show the nonuniqueness of the weak solutions when $q > 2$: here the coefficient a defined at (1.7) is negative, and $|a| = (q-2)/(q-1) < 1$.

Proof of Theorem 1.5. Since $q > 2$ and $N \geq 2$, the function $\tilde{U}$ is a solution in $\mathcal{D}'\left(\mathbb{R}^N\right)$ of the stationary equation

$$-\Delta u + |\nabla u|^q = 0$$

Indeed $\tilde{U} \in W^{1,q}_{loc}\left(\mathbb{R}^N\right) \cap W^{2,1}_{loc}\left(\mathbb{R}^N\right)$ because $N > q'$, and $\tilde{U}$ is a classical solution in $\mathbb{R}^N \backslash \{0\}$. Then it is a weak solution of $(P_{\mathbb{R}^N,\infty})$, and $\tilde{U} \notin C^1(Q_{\mathbb{R}^N,\infty})$. Since $\tilde{U} \in C\left(\mathbb{R}^N\right)$, from Theorem 5.4, or from [5], there exists also a classical solution $U_{\tilde{C}} \in C^{2,1}(Q_{\mathbb{R}^N,\infty})$ of the problem, thus $U_{\tilde{C}} \neq U_0$.

More generally, for any $C > 0$, there exists a classical solution U_C with trace $C\,|x|^{|a|}$. And U_C is obtained as the limit of the nondecreasing sequence of the unique solutions $U_{n,C}$ with trace $\min(C\,|x|^{|a|}, n)$, then it is radial. Moreover for any $\lambda > 0$, the function $U_{n,C,\lambda}(x,t) = \lambda^{-a}U_{n,C}(\lambda x, \lambda^2 t)$ admits the trace $\min(C\,|x|^{|a|}, n\lambda^{-a})$. Therefore, denoting by $k_{\lambda,n}$ the integer part of $n\lambda^{-a}$, there holds $U_{k_{\lambda,n},C} \leq U_{n,C,\lambda} \leq U_{k_{\lambda,n}+1}$ from the comparison principle. And $U_{n,C,\lambda}(x,t)$ converges everywhere to $\lambda^{-a}U_C(\lambda x, \lambda^2 t)$, thus $U_C(x,t) = \lambda^{-a}U_C(\lambda x, \lambda^2 t)$, that means U_C is self-similar. Then U_C has the form (1.14), where $f \in C^2([0,\infty))$, $f(0) \geqq 0$, $f'(0) = 0$, and $\lim_{\eta\to\infty} \eta^{-|a|/2} f(\eta) = C$, and for any $\eta > 0$,

$$f''(\eta) + (\frac{N-1}{\eta} + \frac{\eta}{2})f'(\eta) - \frac{|a|}{2}f(\eta) - \left|f'(\eta)\right|^q = 0. \tag{5.5}$$

From the Cauchy-Lipschitz Theorem, we find $f(0) > 0$, since $f \not\equiv 0$, hence $f''(0) > 0$. The function f is increasing: indeed if there exists a first point $\eta_0 > 0$ such that $f'(\eta_0) = 0$, then $f''(\eta_0) > 0$, which is contradictory. ■

6 Second local regularizing effect

Here we show the second regularizing effect. We prove an estimate, playing the role of the subcaloricity estimate (2.4). Our proof follows the general scheme of Stampacchia's method, developped by many authors, see [17] and references there in, and [19].

First we write estimate (3.1) in another form, and from Gagliardo estimate, we obtain the following:

Lemma 6.1 *Let $q > 1$. Let $\eta > 0, r \geqq 1$. Let u be any nonnegative weak subsolution of equation (1.1) in $Q_{\Omega,T}$. Let $B_{2\eta} \subset\subset \Omega$, $0 < \theta < \tau < T$, and $\xi \in C^1((0,T), C^1_c(\Omega))$, with values in $[0,1]$, such that $\xi(.,t) = 0$ for $t \leqq \theta$. Let $\lambda \geqq \max(2, q')$.*

Then for any $\nu \in (0,1]$,

$$\sup_{[\theta,\tau]} \int_\Omega u^r(.,t)\xi^\lambda + \frac{\int_\theta^\tau \int_\Omega u^{(q+r-1)(1+\frac{\mu}{N})} \xi^{\lambda(1+\frac{\mu}{N})}}{(\sup_{t\in[\theta,\tau]} \int_\Omega u^r \xi^{\frac{\lambda r}{q+r-1}})^{\frac{q}{N}}} \leqq C \int_\theta^\tau \int_\Omega (u^r\,|\xi_t| + u^{r-1}\,|\nabla\xi|^{q'} + u^{q+r-1}\,|\nabla\xi|^q), \tag{6.1}$$

where $\mu = rq/(q+r-1)$, $C = C(N,q,r,\lambda)$.

Proof. From Remark 2.2, $u \in L^\infty_{loc}(Q_{\Omega,T}))$, and hence $u^{\frac{q+r-1}{q}}\xi^{\frac{\lambda}{q}} \in W^{1,q}(Q_{\Omega,\theta,t})$ and

$$\int_\theta^t \int_\Omega |\nabla(u^{\frac{q+r-1}{q}}\xi^{\frac{\lambda}{q}})|^q = \int_\theta^t \int_\Omega \left| \frac{q+r-1}{q} u^{\frac{r-1}{q}}\xi^{\frac{\lambda}{q}}\nabla u + \frac{\lambda}{q} u^{\frac{q+r-1}{q}}\xi^{\frac{\lambda-q}{q}}\nabla\xi \right|^q$$
$$\leqq C(\int_\theta^t \int_\Omega u^{r-1}|\nabla u|^q\xi^\lambda + \int_\theta^t \int_\Omega u^{q+r-1}|\nabla\xi|^q\xi^{\lambda-q}),$$

with $C = C(q,r,\lambda)$. From (3.1), since $\nu \leqq 1$, we get

$$\sup_{[\theta,\tau]} \int_\Omega u^r(.,t)\xi^\lambda + \int_\theta^\tau \int_\Omega |\nabla(u^{\frac{q+r-1}{q}}\xi^{\frac{\lambda}{q}})|^q \leqq C\int_\theta^\tau \int_\Omega (u^r\,|\xi_t| + u^{r-1}\,|\nabla\xi|^{q'} + u^{q+r-1}\,|\nabla\xi|^q), \quad (6.2)$$

where $C = C(q,r,\lambda)$. Next we use a Galliardo type estimate, see [17, Proposition 3.1]: for any $\mu \geqq 1$, and any $w \in L^\infty_{loc}((0,T),L^\mu(\Omega)) \cap L^q_{loc}((0,T),W^{1,q}(\Omega))$,

$$\int_\theta^\tau \int_\Omega w^{q(1+\frac{\mu}{N})}) \leqq C(\int_\theta^\tau \int_\Omega |\nabla w|^q)(\sup_{t\in[\theta,\tau]} \int_\Omega |w|^\mu)^{\frac{q}{N}}, \qquad C = C(N,q,\mu).$$

Taking $w = u^{\frac{q+r-1}{q}}\xi^{\frac{\lambda}{q}}$ and $\mu = qr/(q+r-1) \geqq r \geqq 1$, setting $s = 1+\mu/N$, it comes

$$\int_\theta^\tau \int_\Omega u^{(q+r-1)s}\xi^{\lambda s} \leqq C(\int_\theta^\tau \int_\Omega |\nabla w|^q)(\sup_{t\in[\theta,\tau]} \int_\Omega u^r\xi^{\frac{\lambda r}{q+r-1}})^{\frac{q}{N}},$$

hence (6.1) follows. ■

Theorem 6.2 *Let $q > 1$. Let u be any nonnegative weak solution of equation (1.1) in $Q_{\Omega,T}$. Let $B(x_0,\rho) \subset\subset \Omega$. Let $R > q-1$ (in particular any $R \geqq 1$ if $q < 2$). Then there exists $C = C(N,q,R)$ such that, for any t,θ such that $0 < t-2\theta < t < T$,*

$$\sup_{B(x_0,\frac{\rho}{2})\times[t-\theta,t]} u \leqq C\theta^{-\frac{N+q}{qR+N(q-1)}}(\int_{t-2\theta}^t \int_{B(x_0,\rho)} u^R)^{\frac{q}{qR+N(q-1)}}$$
$$+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}}(\int_{t-2\theta}^t \int_{B(x_0,\rho)} u^R)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}}(\int_{t-2\theta}^t \int_{B(x_0,\rho)} u^R)^{\frac{1}{R+1-q}}. \quad (6.3)$$

Proof. Since $u \in C((0,T);L^R_{loc}(Q_{\Omega,T}))$, by regularization we can assume that u is a classical solution in $Q_{\Omega,T}$. Let t,θ such that $0 < t-2\theta < t < T$. We can assume $x_0 = 0 \in \Omega$. By translation of $t-\theta$, we are lead to prove that for any solution in $Q_{\Omega,-\tau/2,\tau/2}$ $(\tau < T)$,

$$\sup_{Q_{B_{\rho/2},0,\theta}} u \leqq C\theta^{-\frac{N+q}{qR+N(q-1)}}(\int_{-\theta}^\theta \int_{B_\rho} u^R)^{\frac{q}{qR+N(q-1)}}$$
$$+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}}(\int_{-\theta}^\theta \int_{B_\rho} u^R)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}}(\int_{-\theta}^\theta \int_{B_\rho} u^R)^{\frac{1}{R+1-q}}. \quad (6.4)$$

For given $k>0$ we set $u_k=(u-k)^+$. Then $u_k\in C(0,T);L^R_{loc}(Q_{\Omega,T}))$, and u_k is a weak subsolution of equation (1.1), from the Kato inequality. We set

$$\begin{aligned}
&\rho_n=(1+2^{-n})\rho/2, \qquad t_n=-(1+2^{-n})\theta/2,\\
&Q_n=B_{\rho_n}\times(t_n,\theta), \qquad Q_0=B_\rho\times(-\theta,\theta), \qquad Q_\infty=B_{\rho/2}\times(-\theta/2,\theta),\\
&k_n=(1-2^{-(n+1)})k, \qquad \tilde{k}=(k_n+k_{n+1})/2.
\end{aligned}$$

and set $M_\sigma=\sup_{Q_\infty}u$, $M=\sup_{Q_0}u$. Let $\xi(x,t)=\xi_1(x)\xi_2(t)$ where $\xi_1\in C^1_c(\Omega)$, $\xi_2\in C^1(\mathbb{R})$, with values in $[0,1]$, such that

$$\begin{aligned}
&\xi_1=1 \quad \text{on } B_{\rho_{n+1}}, \qquad \xi_1=0 \quad \text{on } \mathbb{R}^N\backslash B_{\rho_n}, \qquad |\nabla\xi_1|\leqq C(N)2^{n+1}/\rho;\\
&\xi_2=1 \quad \text{on } \left[\theta_{n+1},\infty\right), \qquad \xi_2=0 \quad \text{on } \left(-\infty,\theta_n\right], \qquad |\xi_{2,t}|\leqq C(N)2^{n+1}/\theta.
\end{aligned}$$

From Lemma 6.1 we get, with $\mu=qr/(q+r-1)$,

$$\sup_{t\in[t_{n+1},\theta]}\int_{B_{\rho_{n+1}}}u^r_{k_{n+1}}(.,t)+\frac{\int_{t_{n+1}}^\theta\int_{B_{\rho_{n+1}}}u_{k_{n+1}}^{(q+r-1)(1+\frac{\mu}{N})}}{(\sup_{t\in[t_n,\theta]}\int_{B_{\rho_n}}u^r_{k_n})^{\frac{q}{N}}}\leqq CX_n, \text{ where}$$

$$X_n=\int_{t_n}^\theta\int_{B_{\rho_n}}(u^r_{k_{n+1}}|\zeta_t|+u^{r-1}_{k_{n+1}}|\nabla\xi|^{q'}+u^{q+r-1}_{k_{n+1}}|\nabla\xi|^q)).$$

Let us define

$$Y_n=\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_n}^{q+r-1}, \quad Z_n=\sup_{t\in[t_n,\theta]}\int_{B_{\rho_n}}u^r_{k_n}, \quad W_n=\int_{t_n}^\theta\int_{B_{\rho_n}}\chi_{\{u\geqq k_n\}}.$$

Thus, from the Hölder inequality,

$$Z_{n+1}+Z_n^{-\frac{q}{N}}W_{n+1}^{-\frac{\mu}{N}}Y_{n+1}^{1+\frac{\mu}{N}}\leqq CX_n. \tag{6.5}$$

Morever, for any $\gamma,\beta>0$,

$$\begin{aligned}
\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_n}^{\gamma+\beta}&\geqq\int_{t_n}^\theta\int_{B_{\rho_n}}(k_n-k_{n+1})^{\gamma+\beta}\chi_{\{u\geqq k_{n+1}\}}\\
&\geqq(k2^{-(n+2)})^{\gamma+\beta}\int_{t_n}^\theta\int_{B_{\rho_n}}\chi_{\{u\geqq k_{n+1}\}}\geqq(k2^{-(n+2)})^{\gamma+\beta}\int_{t_{n+1}}^\theta\int_{B_{\rho_{n+1}}}\chi_{\{u\geqq k_{n+1}\}},
\end{aligned}$$

and from the Hölder inequality,

$$\begin{aligned}
\int_{t_n}^\theta\int_{B_{\rho_n}}u^\gamma_{k_{n+1}}&\leqq(\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_{n+1}}^{\gamma+\beta})^{\frac{\gamma}{\gamma+\beta}}(\int_{t_n}^\theta\int_{B_{\rho_n}}\chi_{\{u\geqq k_{n+1}\}})^{\frac{\beta}{\gamma+\beta}}\\
&\leqq(\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_n}^{\gamma+\beta})(k^{-1}2^{(n+2)})^\beta(\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_n}^{\gamma+\beta})^{\frac{\beta}{\gamma+\beta}}\\
&\leq(k^{-1}2^{(n+2)})^\beta\int_{t_n}^\theta\int_{B_{\rho_n}}u_{k_n}^{\gamma+\beta}.
\end{aligned}$$

Thus in particular

$$W_{n+1} \leqq C(\frac{2^{n+1}}{k})^{q+r-1}Y_n, \qquad \int_{t_n}^{\theta}\int_{B_{\rho_n}} u_{k_{n+1}}^r \leqq C(\frac{2^{n+1}}{k})^{q-1}Y_n, \qquad \int_{t_n}^{\theta}\int_{B_{\rho_n}} u_{k_{n+1}}^{r-1} \leqq C(\frac{2^{n+1}}{k})^{q}Y_n. \tag{6.6}$$

Otherwise

$$X_n \leqq \int_{t_n}^{\theta}\int_{B_{\rho_n}} (2^{n+1}\theta^{-1}u_{k_{n+1}}^r + 2^{q'(n+1)}\rho^{-q'}u_{k_{n+1}}^{r-1} + 2^{q(n+1)}\rho^{-q}u_{k_{n+1}}^{q+r-1}),$$

then from (6.6),

$$X_n \leqq Cb_0^n f(\theta,\rho,k)Y_n, \qquad \text{where } f(\theta,\rho,k) = (\theta^{-1}\frac{1}{k^{q-1}} + \frac{1}{k^q}\rho^{-q'} + \rho^{-q}). \tag{6.7}$$

for some b_0 depending on q, r. Then from (6.5), (6.6) and (6.7),

$$Z_{n+1} \leqq Cb_0^n f(\theta,\rho,k)Y_n, \qquad Y_{n+1}^{1+\frac{\mu}{N}} \leqq CZ_n^{\frac{q}{N}}(\frac{2^{n+1}}{k})^{(q+r-1)\frac{\mu}{N}}b_0^n f(\theta,\rho,k)Y_n^{1+\frac{\mu}{N}}.$$

Since $Y_{n+1} \leqq Y_n$,setting $\alpha = q/(N+\mu)$ and denoting by b_1, b some new constants depending on N, q, r,

$$\begin{aligned} Y_{n+2} &\leqq CZ_{n+1}^{\frac{q}{N+\mu}}b_1^{n+1}k^{-(q+r-1)\frac{\mu}{N+\mu}}f^{\frac{N}{N+\mu}}(\theta,\rho,k)Y_{n+1} \\ &\leqq C(b_0^n f(\theta,\rho,k)Y_n)^{\frac{q}{N+\mu}}b_1^{n+1}k^{-(q+r-1)\frac{\mu}{N+\mu}}f^{\frac{N}{N+\mu}}(\theta,\rho,k)Y_n \\ &\leqq Cb^n f^{\frac{N+q}{N+\mu}}k^{-(q+r-1)\frac{\mu}{N+\mu}}Y_n^{1+\frac{q}{N+\mu}} := Db^nY_n^{1+\alpha}. \end{aligned}$$

From [17, Lemma 4.1], $Y_n \to 0$ if

$$Y_0^{\alpha}\delta^{1/\alpha} \leqq D^{-1} = C^{-1}k^{(q+r-1)\frac{\mu}{N+\mu}}f^{-\frac{N+q}{N+\mu}},$$

that means

$$k^{qr} \geqq cY_0^q((\theta^{-1}\frac{1}{k^{q-1}} + \frac{1}{k^q}\rho^{-q'} + \rho^{-q}))^{N+q}. \tag{6.8}$$

For getting (6.8) it is sufficient that

$$k^{qr+(q-1)(N+q)} \geqq \frac{c}{2}Y_0^q\theta^{-(N+q)}, \quad k^{(r+N+q)} \geqq (\frac{c}{2})^{1/q}Y_0\rho^{-\frac{N+q}{q-1}}, \quad \text{and } k^r \geqq \frac{c}{2}Y_0\rho^{-(N+q)}.$$

Thus we deduce that

$$\begin{aligned} \sup_{Q_\infty} u &\leqq C\theta^{-\frac{N+q}{qr+(N+q)(q-1)}}(\int_{-\theta}^{\theta}\int_{B_\rho} u^{q+r-1})^{\frac{q}{qr+(N+q)(q-1)}} \\ &\quad + C\rho^{-\frac{N+q}{(q-1)(r+N+q)}}(\int_{-\theta}^{\theta}\int_{B_\rho} u^{q+r-1})^{\frac{1}{r+N+q}} + C\rho^{-\frac{N+q}{r}}(\int_{-\theta}^{\theta}\int_{B_\rho} u^{q+r-1})^{\frac{1}{r}}. \end{aligned} \tag{6.9}$$

If we set $q + r - 1 = R$, we obtain (6.4) for any $R \geqq q$.

Next we consider the case $R < q$. From (6.9) we get

$$\begin{aligned}\sup_{B_{\sigma\rho}\times(-\theta/2,\theta)} u &\leqq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}}\Big(\int_0^\theta\int_{B_\rho} u^q\Big)^{\frac{q}{q+(q-1)(N+q)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(1+N+q)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^q\Big)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)}\int_{-\theta}^\theta\int_{B_\rho} u^q \\ &\leqq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}}\Big(\sup_{B_\rho\times 0,\theta)} u\Big)^{\frac{q(q-R)}{q+(q-1)(N+q)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{q}{q+(q-1)(N+q)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(1+N+q)}}\Big(\sup_{B_\rho\times 0,\theta)} u\Big)^{\frac{q(q-R)}{1+N+q}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{1}{1+N+q}} \\ &+ C\rho^{-(N+q)}\Big(\sup_{B_\rho\times 0,\theta)} u\Big)^{(q-R)}\int_{-\theta}^\theta\int_{B_\rho} u^R.\end{aligned}$$

We define

$$\tilde\rho_n = (1+2^{-(n+1)})\rho, \qquad \theta_n = -(1+2^{-(n+1)})\theta, \qquad \tilde Q_n = B_{\tilde\rho_n}\times(\theta_n,\theta), \qquad M_n = \sup_{\tilde Q_n} u,$$

hence $M_0 = \sup_{B_{\rho/2}\times(-\theta/2,\theta)} u$. We find

$$\begin{aligned}M_n &\leqq C\theta^{-\frac{N+q}{q+(q-1)(N+q)}} M_{n+1}^{\frac{q(q-R)}{q+(q-1)(N+q)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{q}{q+(q-1)(N+q)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(1+N+q)}} M_{n+1}^{\frac{q(q-R)}{1+N+q}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{1}{1+N+q}} + C\rho^{-(N+q)}M_{n+1}^{q-R}\int_{-\theta}^\theta\int_{B_\rho} u^R.\end{aligned}$$

We set

$$\begin{aligned}I &= C\theta^{-\frac{N+q}{q+(q-1)(N+q)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{q}{q+(q-1)(N+q)}}, \\ J &= C\rho^{-(N+q)}\int_0^\theta\int_{B_\rho} u^R, \qquad L = C\rho^{-\frac{N+q}{(q-1)(1+N+q)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{1}{1+N+q}}.\end{aligned}$$

Note that $R > q-1$, that means $q-R<1$. Then from Hölder inequality,

$$M_n \leqq \frac{1}{2}M_{n+1} + C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}), \qquad \sigma = \frac{q+(q-1)(N+q)}{N(q-1)+qR}, \quad \delta = \frac{1+N+q}{R+N+1}.$$

Thus $M_0 \leqq 2^{-n}M_n + 2C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}})$, and finally

$$\begin{aligned}M_0 = \sup_{Q_0} u &\leqq C(I^\sigma + L^\delta + J^{\frac{1}{R+1-q}}) = C\theta^{-\frac{N+q}{N(q-1)+qR}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{q}{N(q-1)+qR}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}}\Big(\int_{-\theta}^\theta\int_{B_\rho} u^R\Big)^{\frac{1}{R+1-q}},\end{aligned}$$

which shows again (6.4). Then (6.4) holds for any $R > q-1$, in particular for any $R \geqq 1$ if $q < 2$. ■

Now we prove our second regularing effect due to the effect of the gradient:

Proof of Theorem 1.6. We assume $x_0 = 0$. Let $\kappa > 0$ be a parameter. From (6.3), for any $\rho \in (0, \eta)$ such that $\rho^\kappa \leqq t < \tau$,

$$\sup_{B_{\frac{\rho}{2}} \times [t-\rho^\kappa, t]} u \leqq C\rho^{-\frac{\kappa(N+q)}{qR+N(q-1)}} (\int_{t-\rho^\kappa}^{t} \int_{B_\rho} u^R)^{\frac{q}{qR+N(q-1)}}$$

$$+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)}} (\int_{t-\rho^\kappa}^{t} \int_{B_\rho} u^R)^{\frac{1}{R+N+1}} + C\rho^{-\frac{N+q}{R+1-q}} (\int_{t-\rho^\kappa}^{t} \int_{B_\rho} u^R)^{\frac{1}{R+1-q}},$$

where $C = C(N, q, R)$. Now from estimate (3.3) of Lemma 3.2,

$$\begin{aligned}\sup_{B_{\eta/2}} u(.,t) &\leqq C\rho^{-\frac{\kappa N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C\rho^{-\frac{N+q}{(q-1)(R+N+1)} + \frac{\kappa}{R+N+1}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C\rho^{\frac{-(N+q)+\kappa}{R+1-q}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}}.\end{aligned}$$

Let $\tau < T$, and $k_0 \in \mathbb{N}$ such that $k_0 \eta^\kappa/2 \geqq \tau$. For any $t \in (0, \tau]$, there exists $k \in \mathbb{N}$ with $k \leqq k_0$ such that $t \in (k\eta^\kappa/2, (k+1)\eta^\kappa/2]$. taking $\rho^\kappa = t/(k+1)$, we find for any $0 < t < \tau$, and $C = C(N, q, R)$,

$$\begin{aligned}\sup_{B_{\eta/2}} u(.,t) &\leqq C(\frac{1+\eta^{-\kappa}\tau}{t})^{\frac{N}{qR+N(q-1)}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ C(\frac{1+\eta^{-\kappa}\tau}{t})^{\frac{\frac{N+q}{\kappa(q-1)}-1}{R+N+1}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} \\ &+ C(\frac{1+\eta^{-\kappa}\tau}{t})^{\frac{\frac{N+q}{\kappa}-1}{R+1-q}} (\eta^{\frac{N}{R}-q'} t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}}. \qquad (6.10)\end{aligned}$$

If we choose κ such that $\kappa\varepsilon(N+q)q' \geqq 1$, we obtain, with $C = C(N, q, R, \eta, \varepsilon, \tau)$,

$$\begin{aligned}\sup_{B_{\eta/2}} u(.,t) &\leqq Ct^{-\frac{N}{qR+N(q-1)}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{Rq}{qR+N(q-1)}} \\ &+ Ct^{\frac{1-\varepsilon}{R+N+1}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+N+1}} + Ct^{\frac{1-\varepsilon}{R+1-q}} (t + \|u_0\|_{L^R(B_\eta)})^{\frac{R}{R+1-q}} \qquad (6.11)\end{aligned}$$

And in fact the second term can be absorbed by the first one, with a new constant depending on τ, and we finally obtain (1.15). ■

Remark 6.3 *These estimate in $t^{-N/(qR+N(q-1))}$ improves the estimate in $t^{-N/2R}$ of the first regularizing effect when $q > q_*$. And it appears to be sharp. Indeed consider for example the particular solutions given in [25] of the form $u_C(x,t) = Ct^{-a/2} f(|x|/\sqrt{t})$, where $\eta \mapsto f(\eta)$ is bounded, $f'(0) = 0$ and $\lim_{\eta\to\infty} \eta^a f(\eta) = C$. Then u_C is solution of (1.1) in $Q_{\mathbb{R}^N \setminus \{0\},\infty}$, with initial data $C|x|^{-a}$. When $a < N$, that means $q > q_*$, then $|x|^{-a} \in L^R_{loc}(\mathbb{R}^N)$ for any $R \in [1, N/a)$, and u_C is solution in $Q_{\mathbb{R}^N,\infty}$. We have $\sup_{B_1} u(.,t) = Cf(0)t^{-a/2}$. Taking $N/R = a(1+\delta)$, for small $\delta > 0$ our estimate near $t = 0$ gives $\sup_{B_1} u(.,t) \leqq C_\delta t^{-\frac{a}{2}(1+\delta)}$.*

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