On the behavior of $n \,/\, \sqrt[n]{n!}$ as n approaches infinity

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Let $\mathbb{N}$ denote set of all *positive* integers and let Euler's number be defined as:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}\,.$$

The aim of this paper is to present a simple proof for the following theorem:

Theorem 1. *If*

$$u_n = \frac{n}{\sqrt[n]{n!}}$$

for every $n \in \mathbb{N}$ then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and converges to e.

The convergence to e can easily be deduced from *Stirling's formula* [1, 3] but using such a tool is clearly an overkill. Our proof of Theorem 1 is self-contained and does not rely on any differential or integral calculus.

As an introduction for our first lemma, recall that for every $(u_n)_{n\in\mathbb{N}} \in \left]0,\infty\right[^{\mathbb{N}}$, we have

$$\liminf_{n\to\infty} \frac{u_{n+1}^{n+1}}{u_n^n} \le \liminf_{n\to\infty} u_n \le \limsup_{n\to\infty} u_n \le \limsup_{n\to\infty} \frac{u_{n+1}^{n+1}}{u_n^n}$$

[3], so if $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$ converges then $(u_n)_{n\in\mathbb{N}}$ converges to the same limit. In particular, every series that passes the *ratio test* also passes the *root test.* The latter results are closely related to:

Lemma 1. *Let $(u_n)_{n\in\mathbb{N}} \in \left]0,\infty\right[^{\mathbb{N}}$ be such that $u_1 < u_2$ and $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$ is monotonically increasing. Then, $(u_n)_{n\in\mathbb{N}}$ is increasing and converges to the same limit as $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$.*

Proof. Put $v_n = u_{n+1}u_n^{-1}$ and $w_n = u_{n+1}^{n+1}u_n^{-n}$ for every $n \in \mathbb{N}$.

Let us first prove that $(u_n)_{n\in\mathbb{N}}$ is increasing. More precisely, let us prove by induction on n that v_n is greater than 1 for every $n \in \mathbb{N}$. Since v_1 is greater than 1 by assumption, the basis of our induction holds true. Now, let $n \in \mathbb{N}$. Straightforward computations yield

$$w_n = u_{n+1}v_n^n\,, \tag{1}$$

and subsequently, $w_{n+1} = u_{n+1}v_{n+1}^{n+2}$. It follows $v_{n+1}^{n+2} = w_{n+1}w_n^{-1}v_n^n \geq v_n^n$, and thus $v_n > 1$ implies $v_{n+1} > 1$, as desired.

It remains to prove $\bar{u} = \bar{w}$, where $\bar{u} = \lim_{n\to\infty} u_n$ and $\bar{w} = \lim_{n\to\infty} w_n$. Let $n \in \mathbb{N}$. Since $v_n > 1$, Equation (1) yields $w_n > u_{n+1}$, and thus we obtain $\bar{w} \geq \bar{u}$ by letting n approach ∞. In particular, $\bar{u} = \infty$ implies $\bar{w} = \infty$. Hence, the final step is to prove $\bar{u} \geq \bar{w}$ under the assumption $\bar{u} \neq \infty$. Equality

$$\frac{u_{2n}^{2n}}{u_n^n} = \prod_{k=n}^{2n-1} w_k$$

holds true because the latter product is telescoping. Moreover, each factor w_k of that product is larger than or equal to w_n. It follows $u_{2n}^{2n}u_n^{-n} \geq w_n^n$ or, equivalently, $u_{2n}^2u_n^{-1} \geq w_n$. Since $0 < \bar{u} < \infty$, $u_{2n}^2u_n^{-1}$ approaches $\bar{u}$ as n approaches ∞. It follows $\bar{u} \geq \bar{w}$, as desired. □

As an aside, let us check that the converse of Lemma 1 is false. Set $u_n = \exp\left(-n^{-2}\right)$ for each $n \in \mathbb{N}$. Clearly, $(u_n)_{n\in\mathbb{N}}$ is increasing whereas $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$ is decreasing because

$$\frac{u_{n+1}^{n+1}}{u_n^n} = \exp\left(\frac{1}{n(n+1)}\right)$$

for every $n \in \mathbb{N}$.

Lemma 2. *Inequality* $(1-x)^n \geq 1 - nx$ *holds true for every* $x \in [0,1]$ *and every* $n \in \mathbb{N}$.

Proof. Let $x \in [0,1]$ be fixed. Put

$$u_n = (1-x)^n + nx$$

for each $n \in \mathbb{N}$. Straightforward computations yield

$$u_{n+1} - u_n = \left(1 - (1-x)^n\right)x \geq 0$$

for every $n \in \mathbb{N}$, so the sequence $(u_n)_{n\in\mathbb{N}}$ is monotonically increasing. In particular, we have $u_n \geq u_1 = 1$ for every $n \in \mathbb{N}$, and thus the desired inequality holds true. □

Lemma 2 states a very special case of *Bernoulli's inequality* [2].

Lemma 3. *If*

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

for every $n \in \mathbb{N}$ *then the sequence* $(u_n)_{n\in\mathbb{N}}$ *is increasing and converges to* e*.*

Proof. Let us first check that $(u_n)_{n\in\mathbb{N}}$ is increasing. Let $n \in \mathbb{N}$. Our task is to prove $u_{n+1}u_n^{-1} > 1$. Straightforward computations yield

$$\frac{1}{u_n} = \left(\frac{n}{n+1}\right)^n ,$$

$$u_{n+1} = \left(\frac{n+2}{n+1}\right)^n \frac{n+2}{n+1} ,$$

and

$$\frac{n}{n+1} \cdot \frac{n+2}{n+1} = 1 - \frac{1}{(n+1)^2} ,$$

whence

$$\frac{u_{n+1}}{u_n} = \frac{1}{u_n} \cdot u_{n+1} = \left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right)^n \frac{n+2}{n+1} = \left(1 - \frac{1}{(n+1)^2}\right)^n \frac{n+2}{n+1} .$$

Besides, we get

$$\left(1 - \frac{1}{(n+1)^2}\right)^n \geq 1 - \frac{n}{(n+1)^2}$$

by letting $x = (n+1)^{-2}$ in Lemma 2. It follows

$$\frac{u_{n+1}}{u_n} \geq \left(1 - \frac{n}{(n+1)^2}\right) \frac{n+2}{n+1} = 1 + \frac{1}{(n+1)^3} > 1 ,$$

as desired.

It remains to prove $\bar{u} = e$, where $\bar{u} = \lim_{n\to\infty} u_n$. Put

$$v_n = \sum_{k=0}^{n} \frac{1}{k!}$$

for each $n \in \mathbb{N}$. Let m, $n \in \mathbb{N}$ be such that $m \leq n$. The binomial theorem yields

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{n!}{n^k(n-k)!} .$$

Besides, $n!$ is less than or equal to $n^k(n-k)!$ for each $k \in \{0,1,2,\ldots,n\}$. It follows $u_n \leq v_n$, and consequently, $\bar{u} \leq e$. The final step is to prove the other inequality. For each fixed $k \in \{0,1,2,\ldots,n\}$,

$$\frac{n!}{n^k(n-k)!} = \prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)$$

approaches 1 as n approaches ∞. Therefore, by letting m be fixed and n approach ∞ in

$$\sum_{k=0}^{m}\frac{1}{k!}\cdot\frac{n!}{n^k(n-k)!} \leq u_n\,,$$

we obtain $v_m \leq \bar{u}$, and consequently, $e \leq \bar{u}$. $\square$

Let us briefly comment the proof of Lemma 3. Our proof of the fact that $\left((1+n^{-1})^n\right)_{n\in\mathbb{N}}$ is increasing is taken from [4]. Another approach [1] is to closely examine the right-hand side of equality

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\frac{1}{k!}\prod_{j=0}^{k-1}\left(1-\frac{j}{n}\right)$$

which holds true for every $n \in \mathbb{N}$. Our proof of the fact that $\left((1+n^{-1})^n\right)_{n\in\mathbb{N}}$ converges to e is taken from [3].

Proof of Theorem 1. Straightforward computations yield

$$\frac{u_{n+1}^{n+1}}{u_n^n} = \left(1+\frac{1}{n}\right)^n$$

for every $n \in \mathbb{N}$, so Lemma 3 ensures that $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$ is (monotonically) increasing and converges to e. Besides, we have $u_1 = 1 < \sqrt{2} = u_2$. Therefore, the desired result follows from Lemma 1. $\square$

References

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