On the behavior of $n \,/\, \sqrt[n]{n!}$ as n approaches infinity

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Let $\mathbb{N}$ denote set of all *positive* integers and let Euler's number be defined as:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}\,.$$

The aim of this paper is to present a simple proof for the following theorem:

Theorem 1. *If*

$$u_n = \frac{n}{\sqrt[n]{n!}}$$

for every $n \in \mathbb{N}$ then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and converges to e.

The convergence to e can easily be deduced from *Stirling's formula* [1, 3] but using such a tool is clearly an overkill. Our proof of Theorem 1 is self-contained.

As an introduction for our first lemma, recall that for every $(u_n)_{n\in\mathbb{N}} \in \left]0,\infty\right[^{\mathbb{N}}$, we have

$$\liminf_{n\to\infty} \frac{u_{n+1}^{n+1}}{u_n^n} \leq \liminf_{n\to\infty} u_n \leq \limsup_{n\to\infty} u_n \leq \limsup_{n\to\infty} \frac{u_{n+1}^{n+1}}{u_n^n}$$

[3], so if $\left(u_{n+1}^{n+1} u_n^{-n}\right)_{n\in\mathbb{N}}$ converges then $(u_n)_{n\in\mathbb{N}}$ converges to the same limit. In particular, every series that passes the *ratio test* also passes the *root test*. The latter results are closely related to:

Lemma 1. *Let $(u_n)_{n\in\mathbb{N}} \in \left]0,\infty\right[^{\mathbb{N}}$ be such that $u_1 < u_2$ and $\left(u_{n+1}^{n+1} u_n^{-n}\right)_{n\in\mathbb{N}}$ is monotonically increasing. Then, $(u_n)_{n\in\mathbb{N}}$ is increasing and converges to the same limit as $\left(u_{n+1}^{n+1} u_n^{-n}\right)_{n\in\mathbb{N}}$.*

Proof. Put $v_n = u_{n+1}u_n^{-1}$ and $w_n = u_{n+1}^{n+1}u_n^{-n}$ for every $n \in \mathbb{N}$.

Let us first prove that $(u_n)_{n\in\mathbb{N}}$ is increasing. More precisely, let us prove by induction on n that v_n is greater than 1 for every $n \in \mathbb{N}$. Since v_1 is greater than 1 by assumption, the basis of our induction holds true. Now, let $n \in \mathbb{N}$. Straightforward computations yield

$$w_n = u_{n+1}v_n^n\,, \tag{1}$$

and subsequently, $w_{n+1} = u_{n+1}v_{n+1}^{n+2}$. It follows $v_{n+1}^{n+2} = w_{n+1}w_n^{-1}v_n^n \geq v_n^n$, and thus $v_n > 1$ implies $v_{n+1} > 1$, as desired.

It remains to prove $\bar{u} = \bar{w}$, where $\bar{u} = \lim_{n\to\infty} u_n$ and $\bar{w} = \lim_{n\to\infty} w_n$. Let $n \in \mathbb{N}$. Since $v_n > 1$, Equation (1) yields $w_n > u_{n+1}$, and thus we obtain $\bar{w} \geq \bar{u}$ by letting n approach ∞. In particular, $\bar{u} = \infty$ implies $\bar{w} = \infty$. Hence, the final step is to prove $\bar{u} \geq \bar{w}$ under the assumption $\bar{u} \neq \infty$. Equality

$$\frac{u_{2n}^{2n}}{u_n^n} = \prod_{k=n}^{2n-1} w_k$$

holds true because the latter product is telescoping. Moreover, each factor w_k of that product is larger than or equal to w_n. It follows $u_{2n}^{2n}u_n^{-n} \geq w_n^n$, or equivalently, $u_{2n}^2u_n^{-1} \geq w_n$. Since $\bar{u} \neq \infty$, $u_{2n}^2u_n^{-1}$ approaches $\bar{u}$ as n approaches ∞. It follows $\bar{u} \geq \bar{w}$, as desired. □

As an aside, let us check that the converse of Lemma 1 is false. Set $u_n = \exp\left(-n^{-2}\right)$ for each $n \in \mathbb{N}$. Clearly, $(u_n)_{n\in\mathbb{N}}$ is increasing whereas $\left(u_{n+1}^{n+1}u_n^{-n}\right)_{n\in\mathbb{N}}$ is decreasing because

$$\frac{u_{n+1}^{n+1}}{u_n^n} = \exp\left(\frac{1}{n(n+1)}\right)$$

for every $n \in \mathbb{N}$.

Lemma 2. *If*

$$u_n = \left(1 + \frac{1}{n}\right)^n$$

for every $n \in \mathbb{N}$ then the sequence $(u_n)_{n\in\mathbb{N}}$ is increasing and converges to e.

Proof. Let us first prove that $(u_n)_{n\in\mathbb{N}}$ is increasing. Let $n \in \mathbb{N}$ and let $x \in [-1, 1[$. Since x^k is not greater than 1 for every $k \in \mathbb{N}$, we have

$$\frac{1 - x^{n+1}}{1 - x} = \sum_{k=0}^{n} x^k \leq n + x < n + 1\,,$$

and consequently,

$$x^{n+1} > 1 + (n+1)(x-1)\,. \tag{2}$$

Now, set

$$x = \frac{1 + (n+1)^{-1}}{1 + n^{-1}}\,,$$

and then multiply Equation (2) by $1 + n^{-1}$ on both sides. We obtain $u_{n+1}u_n^{-1} > 1$, as desired.

It remains to prove $\bar{u} = e$, where $\bar{u} = \lim_{n\to\infty} u_n$. Put

$$v_n = \sum_{k=0}^{n} \frac{1}{k!}$$

for each $n \in \mathbb{N}$. Let m, $n \in \mathbb{N}$ be such that $m \le n$. The binomial theorem yields

$$u_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \cdot \frac{n!}{n^k(n-k)!}\,.$$

Besides, $n!$ is less than or equal to $n^k(n-k)!$ for each $k \in \{0, 1, 2, \ldots, n\}$. It follows $u_n \le v_n$, and consequently, $\bar{u} \le e$. The final step is to prove the other inequality. For each fixed $k \in \{0, 1, 2, \ldots, n\}$,

$$\frac{n!}{(n-k)!n^k} = \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right)$$

approaches 1 as n approaches ∞. Therefore, by letting m be fixed and n approach ∞ in

$$\sum_{k=0}^{m} \frac{1}{k!} \cdot \frac{n!}{n^k(n-k)!} \le u_n\,,$$

we obtain $v_m \le \bar{u}$, and consequently, $e \le \bar{u}$. $\square$

Let us briefly comment the proof of Lemma 2. Our proof of the fact that $\big((1+n^{-1})^n\big)_{n\in\mathbb{N}}$ is increasing is taken from [4]. Another approach [1] is to closely examine the right-hand side of equality

$$\left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!} \prod_{j=0}^{k-1} \left(1 - \frac{j}{n}\right)$$

which holds true for every $n \in \mathbb{N}$. Equation (2) is usually called *Bernoulli's inequality*; it holds true for every $n \in \mathbb{N}$ and every $x \in [-1, 1[\cup]1, \infty[$ [2]. Our proof of the fact that $\left((1 + n^{-1})^n\right)_{n \in \mathbb{N}}$ converges to e is taken from [3].

Proof of Theorem 1. Straightforward computations yield

$$\frac{u_{n+1}^{n+1}}{u_n^n} = \left(1 + \frac{1}{n}\right)^n$$

for every $n \in \mathbb{N}$, so Lemma 2 ensures that $\left(u_{n+1}^{n+1} u_n^{-n}\right)_{n \in \mathbb{N}}$ is (monotonically) increasing and converges to e. Besides, we have $u_1 = 1 < \sqrt{2} = u_2$. Therefore, the desired result follows from Lemma 1. □

References

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