9 Introduction to Permutation Group Theory

9.1 Notations

A **permutation group** of a set Ω is a subgroup of $\mathrm{Sym}(\Omega)$. If G is a permutation group on Ω, then G acts on Ω via the inclusion map, i.e. $\omega^g = g(\omega)$, and this is a faithful action. The **degree** of such an action is $|\Omega|$. Conversely, if G is a faithful action on Ω, then G can be identified as a permutation group of Ω. For simplicity, we reintroduce notations for notions in group actions. Let G act on Ω.

$$\begin{aligned}
\omega^G &:= O_G(\omega) = \{\omega^g \mid g \in G\}, && \text{(\textbf{Orbit} of } \omega \in \Omega\text{)}\\
G_\omega &:= S_G(\omega) = \{g \in G \mid \omega^g = \omega\}, && \text{(\textbf{Point stabilizer} of } \omega \in \Omega\text{)}\\
G_X &:= \{g \in G \mid X^g = X\}, && \text{(\textbf{Setwise stabilizer} of } X \subseteq \Omega\text{)}\\
G_{(X)} &:= \{g \in G \mid x^g = x \text{ for all } x \in X\}. && \text{(\textbf{Elementwise stabilizer} of } X \subseteq \Omega\text{)}
\end{aligned}$$

9.2 Isomorphic Actions

Definition 9.1. Let G and H be groups acting on the sets Ω and Δ, respectively. The two actions (or the pairs (G, Ω) and (H, Δ)) are said to be **permutationally isomorphic** if there exists a bijection $\vartheta : \Omega \to \Delta$ and an isomorphism $\chi : G \to H$ such that

$$\vartheta(\omega^g) = \vartheta(\omega)^{\chi(g)}$$

for all $\omega \in \Omega, g \in G$. In other words, for every $g \in G$ the following diagram commutes.

$$\begin{tikzcd}
\Omega \arrow[r, "g"] \arrow[d, "\vartheta"'] & \Omega \arrow[d, "\vartheta"] \\
\Delta \arrow[r, "\chi(g)"'] & \Delta
\end{tikzcd}$$

If such conditions hold, the pair (ϑ, χ) is said to be a **permutational isomorphism**. Similarly, the pair (ϑ, χ) is a **permutational embedding** of the permutation group G on Ω into the permutation group H on Δ, if $\chi : G \to H$ is a monomorphism and $(\vartheta, \hat{\chi})$ is a permutational isomorphism, where $\hat{\chi} : G \to \operatorname{Im} \chi$ is obtained from χ by simply restricting the range of χ.

Proposition 9.2. *Let G act on a set Ω. Let Δ be a set and let $\vartheta : \Omega \to \Delta$ be a bijection. Define a G-action on Δ by $\delta^g = \vartheta((\vartheta^{-1}(\delta))^g)$. Then $(\vartheta, \mathrm{Id}_G)$ is a permutational isomorphism from the G-action on Ω to the G-action on Δ.*

Proposition 9.3. *Let G and H be groups acting transitively on Ω and Δ, respectively. Then the following are equivalent.*

(1) *The actions of G and H on Ω and Δ, respectively, are permutationally isomorphic.*

(2) *There exist $\omega \in \Omega$ and $\delta \in \Delta$ and an isomorphism $\varphi : G \to H$ such that $\varphi(G_\omega) = H_\delta$.*

(3) *For all $\omega \in \Omega$ and $\delta \in \Delta$, there exists an isomorphism $\varphi : G \to H$ such that $\varphi(G_\omega) = H_\delta$.*

Proposition 9.4. *Let Ω be a set and let G_1, G_2 be permutation groups on* $\mathrm{Sym}(\Omega)$. *Then G_1 and G_2 are permutationally isomorphic if and only if they are conjugate in* $\mathrm{Sym}(\Omega)$. *Moreover, if (ϑ, φ) is a permutational isomorphism, then $\vartheta \in \mathrm{Sym}(\Omega)$ and $\varphi(g) = \vartheta^{-1} g \vartheta$, for all $g \in G_1$.*

Recall that if H is a subgroup of a group G, then the right coset action of G on the set Γ_H of right cosets of H is defined by $(Hx)^g = Hxg$ for $x, g \in G$. In view of Theorem **??**, this action is transitive. In fact, every transitive action is permutationally isomorphic to a coset action.

Proposition 9.5. *Let G act transitively on Ω and let $\omega \in \Omega$. Then the G-action on Ω is permutationally isomorphic to the G-action on Γ_{G_ω}.*

Remark. In case of a permutation group, or more generally, a faithful action of G on Ω, we can verify that this action is faithful on Γ_{G_ω}. Let ρ be the associated homomorphism. Then $\rho(G) \cong G$ and so we can establish a permutational isomorphism between (G, Ω) and $(\rho(G), \Gamma_{G_\omega})$.

Proposition 9.6 (Frattini's Argument)**.** *Let G act transitively on Ω and let $\omega \in \Omega$. Then a subgroup H of G is transitive if and only if $G = G_\omega H$.*

9.3 Blocks

Definition 9.7. Let G act transitively on Ω. The nonempty subset Δ of Ω is called a **block** if for every $g \in G$, either $\Delta^g = \Delta$ or $\Delta^g \cap \Delta = \emptyset$. All the singletons of Ω and the set Ω itself are blocks, and so they are said to be **trivial**.

Proposition 9.8. *Let G act transitively on Ω. Then the following propositions hold.*

(i) *If Δ is a block of Ω, then G_Δ acts transitively on Δ.*

(ii) *If Δ is a block of Ω, then $|\Delta^g| = |\Delta|$ and Δ^g is a block for each $g \in G$.*

(iii) *If Δ is a subset of Ω, then Δ is a block if and only if $\{\Delta^g \mid g \in G\}$ forms a partition of Ω.*

Definition 9.9. Let G act on Ω. An equivalence relation $\sim$ on Ω is called a G-**congruence** if

$$\omega_1 \sim \omega_2 \iff \omega_1^g \sim \omega_2^g$$

for all $\omega_1, \omega_2 \in \Omega$ and $g \in G$. We also say that G preserves the relation.

Proposition 9.10. *Let G act transitively on Ω.*

(i) *If $\sim$ is a G-congruence on Ω, then each equivalence class is a block of Ω.*

(ii) *Let Σ be the set of equivalence classes of a G-congruence. Then G acts transitively on Σ.*

(iii) *If Δ is a block, then $\Sigma = \{\Delta^g \mid g \in G\}$ is the set of equivalence classes of a G-congruence on Ω.*

Definition 9.11. Let G act transitively on Ω. The set Σ of equivalence classes associated to a G-congruence on Ω is called a **system of blocks** (or a **system of imprimitivity**). Such a system is said to be **trivial** if it only contains trivial blocks.

Remark. Let Σ be a system of blocks. By Proposition 9.10 (i) and (ii), we can choose an equivalence class Δ from Σ so that $\Sigma = \{\Delta^g \mid g \in G\}$.

9.4 Primitive Actions

Definition 9.12. Let G act transitively on Ω. The action (or G-set) is said to be **primitive** (or G is **primitive** on Ω) if G has no nontrivial blocks; otherwise, it is **imprimitive**.

Proposition 9.13. *Let G acts transitively on Ω. Let $\omega \in \Omega$ be fixed. Then there is a one-to-one correspondence between the set of blocks of Ω containing ω and the set of subgroups which contains the stabilizer G_ω of ω.*

Remark. This correspondence is order-preserving, i.e., if Δ_1, Δ_2 are blocks of Ω containing ω, then $\Delta_1 \subseteq \Delta_2$ if and only if $\theta(\Delta_1) \subseteq \theta(\Delta_2)$. Indeed $\Delta_1 \subseteq \Delta_2$ implies that $\omega^g \in \Delta_1 \subseteq \Delta_2$ for all $g \in H_{\Delta_1}$. Conversely, suppose that $H_{\Delta_1} \leq H_{\Delta_2}$. Let $\delta \in \Delta_1$. So there exists $g \in G$ such that $\delta = \omega^g$. This implies $g \in H_{\Delta_1} \subseteq H_{\Delta_2}$. So $\delta = \omega^g \in \Delta_2$. This shows that θ is order-preserving.

Corollary 9.14. *Let G act transitively on Ω. Then G is primitive if and only if the point stabilizers are maximal subgroups.*

9.5 Centralizers and Normalizers of Transitive Permutation Groups

Definition 9.15. Let G act on a set Ω. We say that G acts **semiregularly** on Ω (G or the G-set Ω is **semiregular**) if nonidentity elements fix no point, i.e., $G_\omega = 1$ for all $\omega \in \Omega$. We say that G acts **regularly** on Ω if G is transitive and semiregular.

Let H be a subgroup of a group G and let Γ_H be the set of right cosets of H in G. We have the right coset action of G on Γ_H, i.e., $(Hx)^g = Hxg$. In fact, we can also define another right coset action of $N_G(H)$ on Γ_H by left multiplication: $(Hx)^g = Hg^{-1}x$. Note that g^{-1} is required and G is restricted to $N_G(H)$ to ensure that this action is a well-defined right action.

Lemma 9.16. *Let G be a group with a subgroup H, and put $K := N_G(H)$. Let Γ_H denote the set of right cosets of H in G, and let ρ and λ denote the right and left multiplication of G and K, respectively, on Γ_H as defined above. Then the following hold.*

(i) $\operatorname{Ker}\lambda = H$ *and* $\lambda(K)$ *is semiregular.*

(ii) The centralizer C of $\rho(G)$ in $\operatorname{Sym}\Gamma_H$ *equals $\lambda(K)$.*

(iii) $H \in \Gamma_H$ has the same orbit under $\lambda(K)$ as under $\rho(K)$.

(iv) If $\lambda(K)$ is transitive, then $K = G$, and $\lambda(G)$ and $\rho(G)$ are conjugate in $\operatorname{Sym}(\Gamma_H)$.

Theorem 9.17. *Let G be a transitive subgroup of $\mathrm{Sym}(\Omega)$ and let $\omega \in \Omega$. Let C be the centralizer of G in $\mathrm{Sym}(\Omega)$. Then the following hold.*

(i) *C is semiregular, and $C \cong N_G(G_\omega)/G_\omega$.*

(ii) *C is transitive if and only if G is regular.*

(iii) *If C is transitive, then it is conjugate to G in $\mathrm{Sym}(\Omega)$ and hence C is regular.*

(iv) *$C = 1$ if and only if G_ω is self-normalizing in G, i.e., $N_G(G_\omega) = G_\omega$.*

(v) *If G is abelian, then $C = G$.*

(vi) *If G is primitive and nonabelian, then $C = 1$.*

Theorem 9.18. *Let G be a transitive subgroup of* $\mathrm{Sym}(\Omega)$*, let N be the normalizer of G in* $\mathrm{Sym}(\Omega)$ *and let $\alpha \in \Omega$. If $\Psi : N \to \mathrm{Aut}\, G$ is the homomorphism defined by conjugation, and $\sigma \in \mathrm{Aut}(G)$, then $\sigma \in \mathrm{Im}\, \Psi$ if and only if $(G_\alpha)^\sigma$ is a point stabilizer for G, i.e., $(G_\alpha)^\sigma = G_\beta$ for some $\beta \in \Omega$.*

Definition 9.19. Let G be a group. The **holomorph** of G, denoted by $\mathrm{Hol}(G)$, is the semidirect product $G \rtimes \mathrm{Aut}(G)$ with respect to the natural action of $\mathrm{Aut}(G)$ on G.

In the case where G is regular, the normalizer of G in the symmetric group is the holomorph of G.

Corollary 9.20. *Let G be a transitive subgroup of $\mathrm{Sym}(\Omega)$ and let N be the normalizer of G in $\mathrm{Sym}(\Omega)$. If G is regular, then $\mathrm{Im}\,\Psi = \mathrm{Aut}(G)$, where Ψ is the homomorphism defined in Theorem 9.18. Let $\alpha \in \Omega$. In this case $N_\alpha \cong \mathrm{Aut}(G)$ and $N \cong \mathrm{Hol}(G)$.*

Remark. It seems to be true that permutational isomorphisms preserve transitivity, primitivity, regularity and faithfulness. More specifically, let (G, Ω) and (H, Δ) be permutational isomorphic group actions, then the following holds.

(i) (G, Ω) is transitive if and only if (H, Δ) is transitive.

(ii) (G, Ω) is primitive if and only if (H, Δ) is primitive.

(iii) (G, Ω) is regular if and only if (H, Δ) is regular.

(iv) (G, Ω) is faithful if and only if (H, Δ) is faithful.

These might be too trivial until they are not mentioned anywhere, but it is important to know. Siapa boleh tolong provekan?

Main References. [PS18; DM96; Cam99; Jac85]