Some Problems in Group Theory
(Groups, homomorphisms, normal subgroups, cyclic subgroups)
July 2, 2025

**Problem 1** (Hungerford Exercises I.1.14)**.** If $G$ is a finite group of even order, then $G$ contains an element $a \neq e$ such that $a^2 = e$.

*Extra.* Show that the converse is true as well.

**Problem 2** (Hungerford Exercises I.1.15)**.** Let $G$ be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$ (in other words, $G$ satisfies left and right cancellation laws). Then $G$ is a group. Show that this conclusion may be false if $G$ is infinite.

**Problem 3** (Hungerford Exercises I.3.9)**.** A group that has only a finite number of subgroups must be finite.

**Problem 4** (UCLA Fall 2000 A3)**.** Let $G$ be a finite group of order $n$. Suppose that $G$ has a unique subgroup of order $d$ for each positive divisor $d$ of $n$. Prove that $G$ is a cyclic group.

**Problem 5** (Cambridge Part IB 2007 1/II/10G(iii))**.** Show that any finite subgroup of the multiplicative group of non-zero elements of a field is cyclic. Is this true if the subgroup is allowed to be infinite?

**Problem 6** (UCLA Spring 2007 Groups Problem 3)**.** Let $G$ be a group with cyclic automorphism group $\mathrm{Aut}(G)$. Prove that $G$ is abelian.

**Problem 7** (UCLA Fall 2001 G3)**.** Let $G$ be a group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in (\mathbb{Z}/p)^*$ and $a \in \mathbb{Z}/p$. Describe all normal subgroups of $G$.

**Problem 8** (Rotman Exercises 2.29)**.**

(i) Let $G$ be a finite group, and let $S$ and $T$ be (not necessarily distinct) nonempty subsets. Prove that either $G = ST$ or $|G| \geq |S| + |T|$.

(ii) Prove that every element in a finite field $\mathbb{F}$ is a sum of two squares.

**Problem 9** (Dummit and Foote Exercises 2.3.25)**.** Let $G$ be a cyclic group of order $n$ and let $k$ be an integer relatively prime to $n$. Prove that the map $x \mapsto x^k$ is surjective. Use Lagrange's Theorem to prove the same is true for any finite group of order $n$.

**Problem 10** (Dummit and Foote Exercises 2.4.14-15, 3.2.21)**.** Let $\mathbb{Z}$ and $\mathbb{Q}$ be the set of integers and the set of rational numbers, respectively.

(i) Prove that every finitely generated subgroup of the additive group $\mathbb{Q}$ is cyclic.

(ii) Prove that $\mathbb{Q}$ is not finitely generated.

(iii) Prove that $\mathbb{Q}$ has no proper subgroups of finite index. Deduce that $\mathbb{Q}/\mathbb{Z}$ has no proper subgroups of finite index.

Further Problem

I also don't know why I put this.

**Neumann Lemma [1].** Let the group $G$ be the union of finitely many, let us say $n$, cosets of (not necessarily distinct) subgroups $C_1, C_2, \ldots, C_n$:

$$G = \bigcup_{i=1}^{n} C_i g_i.$$

Then at least one subgroup $C_i$ has finite index in $G$.

**Remark.** This covering problem is further studied in [2].

# References


[1] B. H. Neumann. "Groups covered by permutable subsets". In: *J. Lond. Math. Soc.* 29 (1954), pp. 236–248.

[2] Elia Gorokhovsky, Nicolás Matte Bon, and Omer Tamuz. "A quantitative Neumann lemma for finitely generated groups". In: *Isr. J. Math.* 262.1 (2024), pp. 487–500.