

Some Problems on Group Theory
(Groups, homomorphisms, normal subgroups, cyclic subgroups)
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Problem 1 (Hungerford Exercises I.1.14). If G is a finite group of even order, then G contains an element $a \neq e$ such that $a^2 = e$.

Extra. Show that the converse is true as well.

Problem 2 (Hungerford Exercises I.1.15). Let G be a nonempty finite set with an associative binary operation such that for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$ (in other words, G satisfies left and right cancellation laws). Then G is a group. Show that this conclusion may be false if G is infinite.

Problem 3 (Hungerford Exercises I.3.9). A group that has only a finite number of subgroups must be finite.

Problem 4 (UCLA Fall 2000 A3). Let G be a finite group of order n . Suppose that G has a unique subgroup of order d for each positive divisor d of n . Prove that G is a cyclic group.

Problem 5 (Cambridge Part IB 2007 1/II/10G(iii)). Show that any finite subgroup of the multiplicative group of non-zero elements of a field is cyclic. Is this true if the subgroup is allowed to be infinite?

Problem 6 (UCLA Spring 2007 Groups Problem 3). Let G be a group with cyclic automorphism group $\text{Aut}(G)$. Prove that G is abelian.

Problem 7 (UCLA Fall 2001 G3). Let G be a group of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

where $a \in (\mathbb{Z}/p)^*$ and $b \in \mathbb{Z}/p$. Describe all normal subgroups of G .

Problem 8 (Rotman Exercises 2.29).

(i) Let G be a finite group, and let S and T be (not necessarily distinct) nonempty subsets. Prove that either $G = ST$ or $|G| \geq |S| + |T|$.

(ii) Prove that every element in a finite field \mathbb{F} is a sum of two squares.

Problem 9 (Dummit and Foote Exercises 2.3.25). Let G be a cyclic group of order n and let k be an integer relatively prime to n . Prove that the map $x \mapsto x^k$ is surjective. Use Lagrange's Theorem to prove the same is true for any finite group of order n .

Problem 10 (Dummit and Foote Exercises 2.4.14-15, 3.2.21). Let \mathbb{Z} and \mathbb{Q} be the set of integers and the set of rational numbers, respectively.

(i) Prove that every finitely generated subgroup of the additive group \mathbb{Q} is cyclic.

(ii) Prove that \mathbb{Q} is not finitely generated.

(iii) Prove that \mathbb{Q} has no proper subgroups of finite index. Deduce that \mathbb{Q}/\mathbb{Z} has no proper subgroups of finite index.

Further Problem

I also don't know why I put this.

Neumann Lemma [1]. Let the group G be the union of finitely many, let us say n , cosets of (not necessarily distinct) subgroups C_1, C_2, \dots, C_n :

$$G = \bigcup_{i=1}^n C_i g_i.$$

Then at least one subgroup C_i has finite index in G .

Remark. This covering problem is further studied in [2].

References

- [1] B. H. Neumann. “Groups covered by permutable subsets”. In: *J. Lond. Math. Soc.* 29 (1954), pp. 236–248.
- [2] Elia Gorokhovsky, Nicolás Matte Bon, and Omer Tamuz. “A quantitative Neumann lemma for finitely generated groups”. In: *Isr. J. Math.* 262.1 (2024), pp. 487–500.