

天线与电波传播 ANTENNAS AND WAVE PROPAGATION

LECTURE 4

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Lecture 4

Dipole/Monopole Antenna

Use vector potential to find far-field characteristics

Infinitesimal dipole ($L \leq \lambda/50$)

Short dipole ($\lambda/50 < L \leq \lambda/10$)

Linear dipole

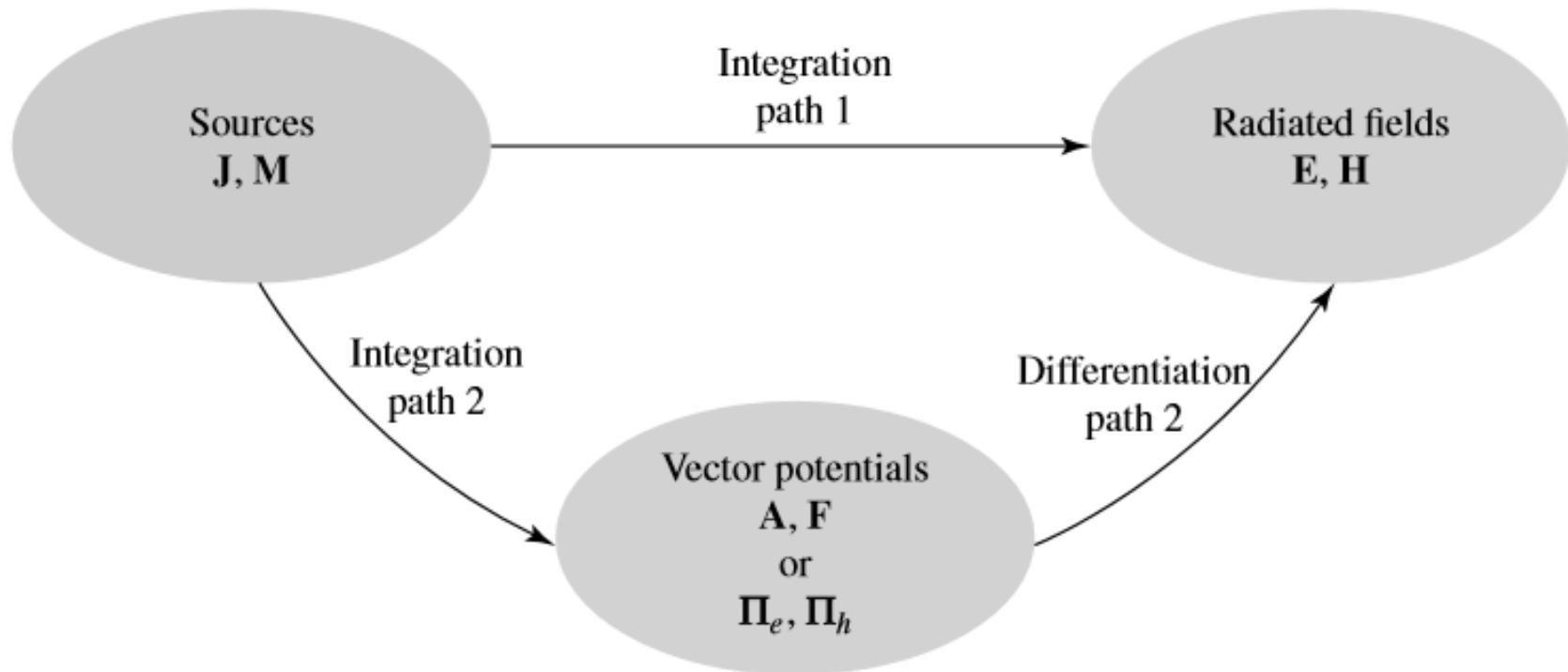
- current distribution and side lobes
- wavelength vs radiation pattern/directivity etc.)

Balun

Monopole (Method of images)

- Wavelength vs radiation pattern/directivity
- Effects of ground plane size

Vector Potential



- (Chapter 3, Fig. 3.1, Balanis Book)

Far-Field Radiation Characteristics

- Specify electric and/or magnetic current densities J, M (Chapter 3, Fig. 3.1, Balanis Book)
- Determine **vector potential** components A_θ, A_ϕ and/or F_θ, F_ϕ (3-46)–(3-54)
- Find far-zone E and H radiated fields ($E_\theta, E_\phi; H_\theta, H_\phi$) using (3-58a)–(3-58b)

Far-Field Radiation Characteristics

- Find Radiation Power Density (W_{rad} , W/m²) or Radiation Power Intensity (U_{rad} , W/Sr) at all directions through integration
- Find Directivity

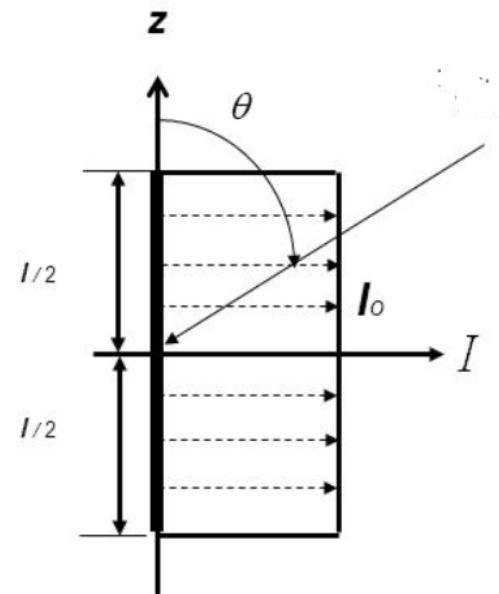
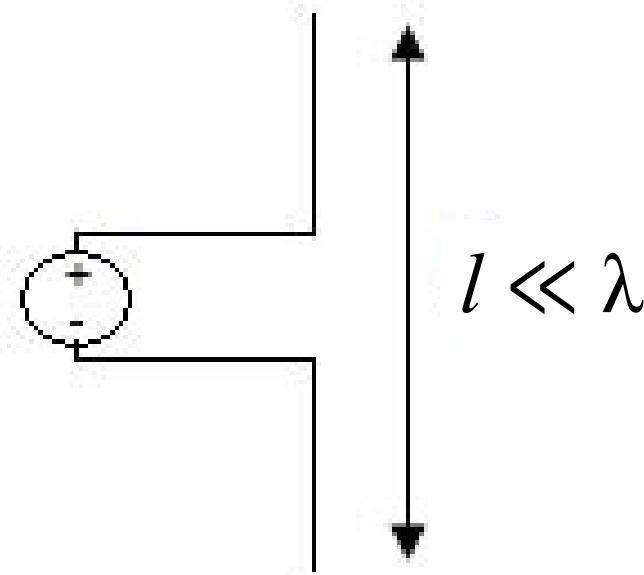
$$D_0 = D_{\max} = D(\theta, \phi)|_{\max} = \frac{U(\theta, \phi)|_{\max}}{U_0} = \frac{4\pi U(\theta, \phi)|_{\max}}{P_{\text{rad}}}$$

- Form normalized power amplitude pattern:

$$P_n(\theta, \phi) = \frac{U(\theta, \phi)}{U_{\max}}$$

Infinitesimal Dipole Antenna

- very small $l \ll \lambda$
- very thin $a \ll \lambda$



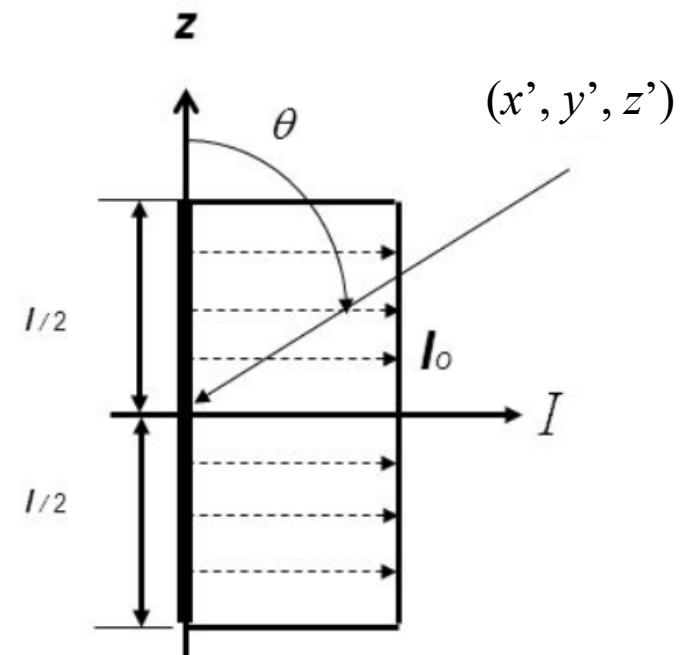
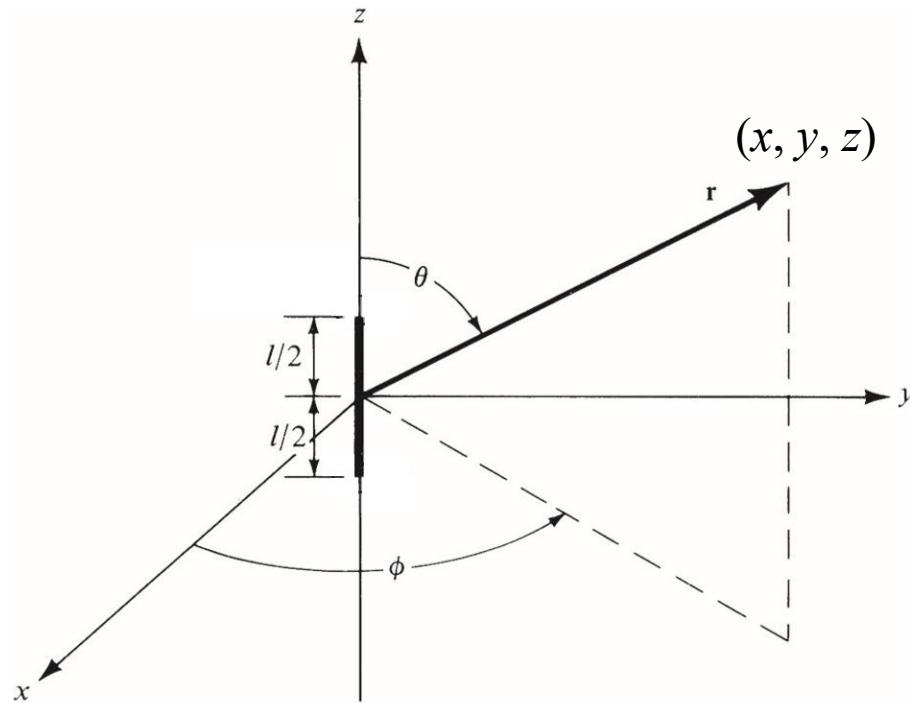
$$\mathbf{I}(z') = \hat{\mathbf{a}}_z I_o$$

Constant current

Infinitesimal Dipole Antenna

Find vector potential at an observation point (x, y, z)

A point on the source (x', y', z')

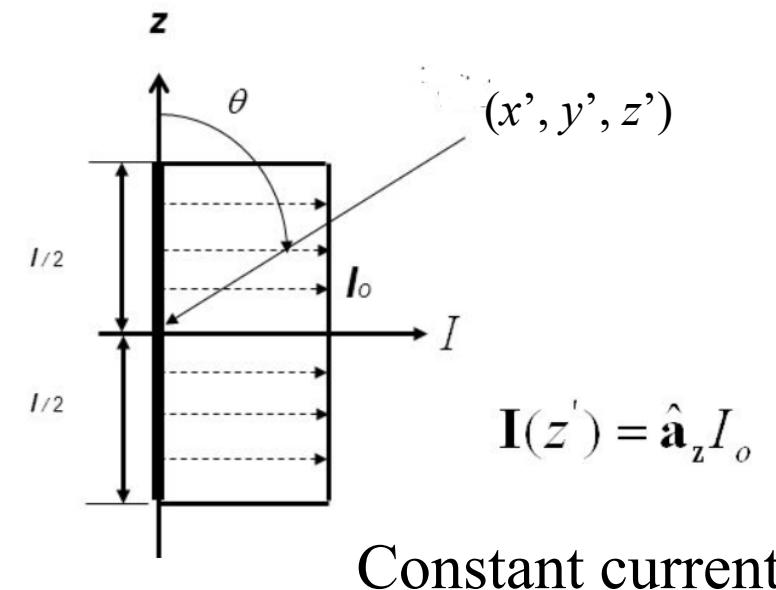
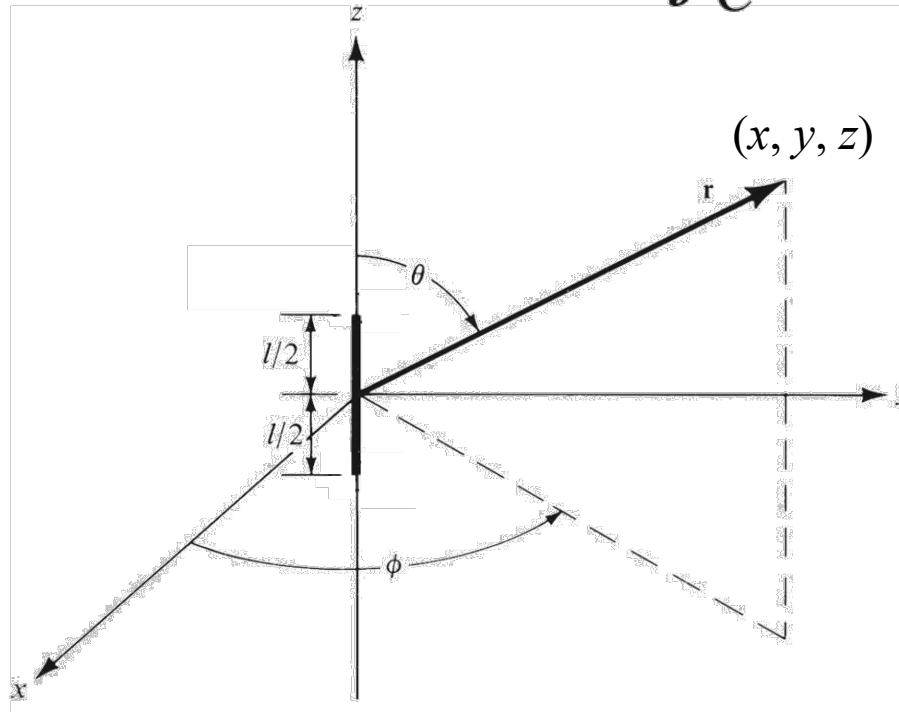


$$\mathbf{I}(z') = \hat{\mathbf{a}}_z I_o$$

Constant current

Infinitesimal Dipole Antenna Vector Potential

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$



$$\mathbf{I}(z') = \hat{\mathbf{a}}_z I_o$$

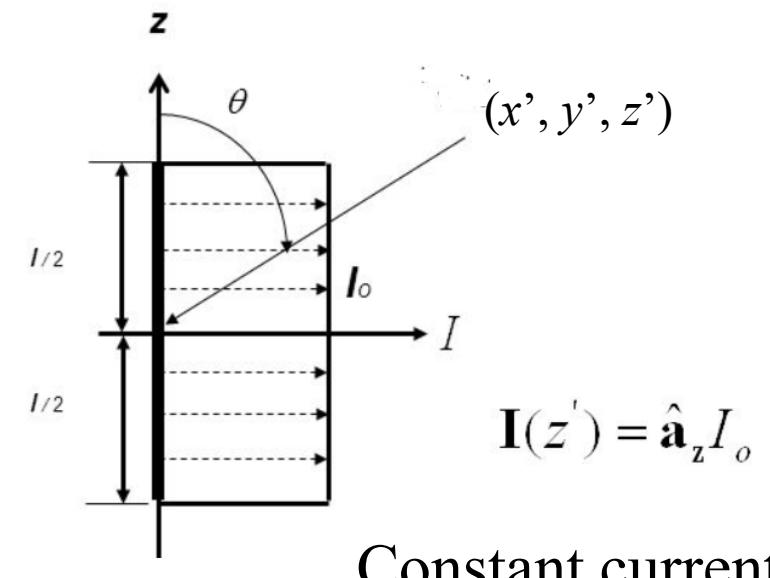
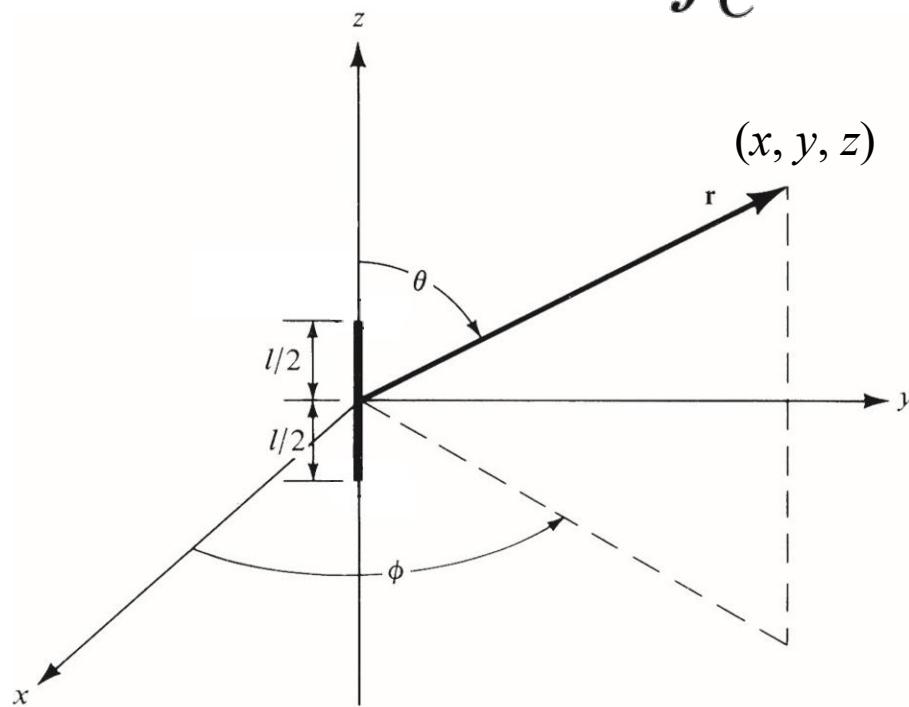
Constant current

Infinitesimal Dipole Antenna Vector Potential

$$\mathbf{I}_e(x', y', z') = \hat{\mathbf{a}}_z I_0$$

$x' = y' = z' = 0$ (infinitesimal dipole)

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$



Constant current

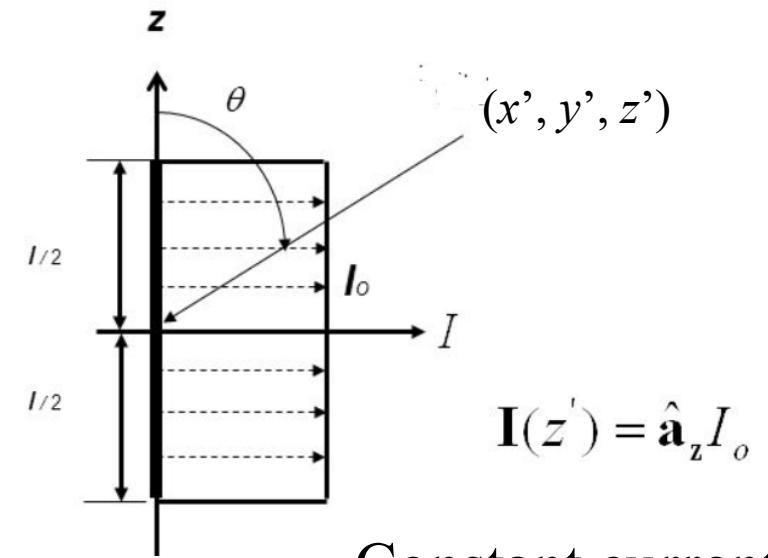
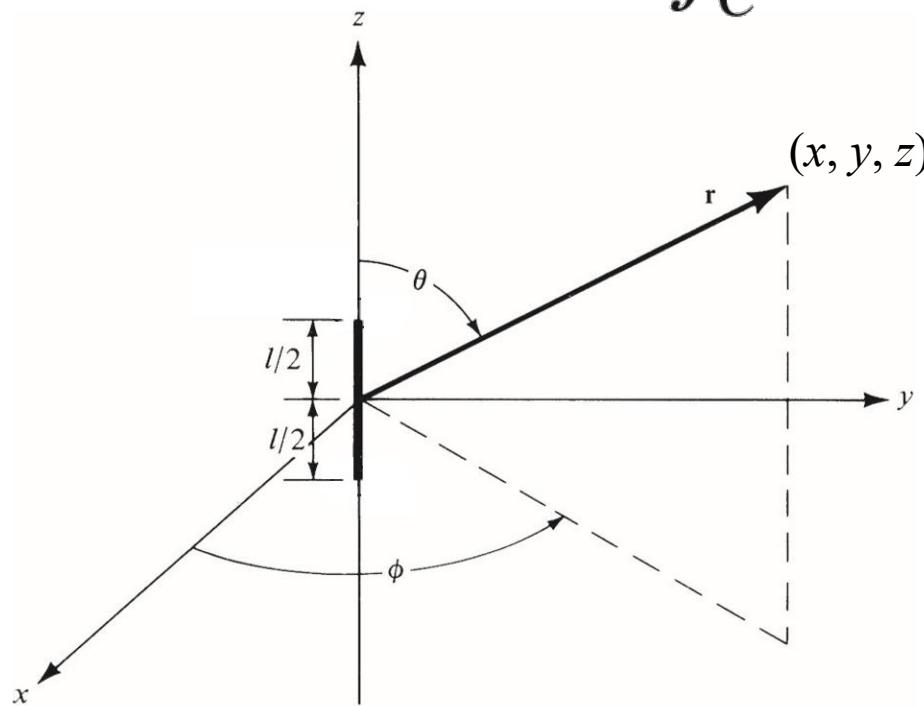
Infinitesimal Dipole Antenna

Vector Potential

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2} = \sqrt{x^2 + y^2 + z^2}$$

$$= r = \text{constant}$$

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$



Constant current

Infinitesimal Dipole Antenna Vector Potential

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$



$$\mathbf{A}(x, y, z) = \hat{\mathbf{a}}_z \frac{\mu I_0}{4\pi r} e^{-jkr} \int_{-l/2}^{+l/2} dz' = \hat{\mathbf{a}}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}$$



Convert to Spherical coordinates

$$A_r = A_z \cos \theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos \theta \quad A_\theta = -A_z \sin \theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r} \sin \theta \quad A_\phi = 0$$

Infinitesimal Dipole Antenna

Vector Potential

[Ref. Balanis 4th Ed.
P. 147]

$$A_r = A_z \cos \theta = \frac{\mu I_0 l e^{-jkr}}{4\pi r} \cos \theta \quad A_\theta = -A_z \sin \theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r} \sin \theta \quad A_\phi = 0$$

H field



$$\mathbf{H}_A = \frac{1}{\mu} \nabla \times \mathbf{A}$$

E field

$$\mathbf{E}_A = -\nabla \phi_e - j\omega \mathbf{A} = -j\omega \mathbf{A} - j \frac{1}{\omega \mu \epsilon} \nabla (\nabla \cdot \mathbf{A})$$

$$H_r = H_\theta = 0$$

$$H_\phi = j \frac{kI_0 l \sin \theta}{4\pi r} \left[1 + \frac{1}{jkr} \right] e^{-jkr}$$

$$E_r = \eta \frac{I_0 l \cos \theta}{2\pi r^2} \left[1 + \frac{1}{jkr} \right] e^{-jkr}$$

$$E_\theta = j\eta \frac{kI_0 l \sin \theta}{4\pi r} \left[1 + \frac{1}{jkr} - \frac{1}{(kr)^2} \right] e^{-jkr}$$

$$E_\phi = 0$$

Infinitesimal Dipole Antenna

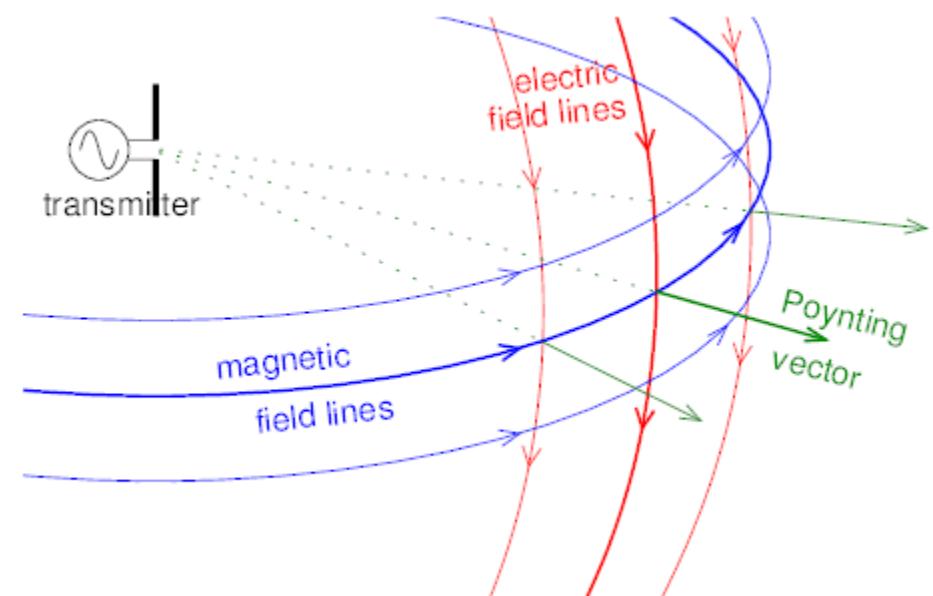
Power Density (Poynting Vector)

$$\begin{aligned}\mathbf{W} &= \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2}(\hat{\mathbf{a}}_r E_r + \hat{\mathbf{a}}_\theta E_\theta) \times (\hat{\mathbf{a}}_\phi H_\phi^*) \\ &= \frac{1}{2}(\hat{\mathbf{a}}_r E_\theta H_\phi^* - \hat{\mathbf{a}}_\theta E_r H_\phi^*)\end{aligned}$$



$$W_r = \frac{\eta}{8} \left| \frac{I_0 l}{\lambda} \right|^2 \frac{\sin^2 \theta}{r^2} \left[1 - j \frac{1}{(kr)^3} \right]$$

$$W_\theta = j\eta \frac{k |I_0 l|^2 \cos \theta \sin \theta}{16\pi^2 r^3} \left[1 + \frac{1}{(kr)^2} \right]$$



The transverse component W_θ of the power density does not contribute to the integral.
(Pure imaginary, no real radiated power!)

Infinitesimal Dipole Antenna

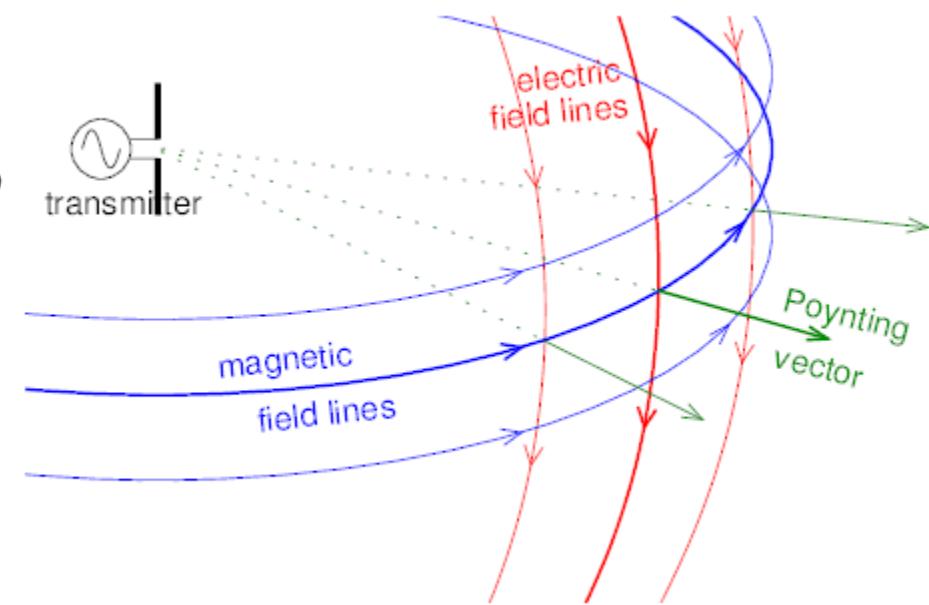
Radiated Power

$$P = \iint_S \mathbf{W} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^\pi (\hat{\mathbf{a}}_r W_r + \hat{\mathbf{a}}_\theta W_\theta) \cdot \hat{\mathbf{a}}_r r^2 \sin \theta \, d\theta \, d\phi$$



$$P = \int_0^{2\pi} \int_0^\pi W_r r^2 \sin \theta \, d\theta \, d\phi$$

$$= \eta \frac{\pi}{3} \left| \frac{I_0 l}{\lambda} \right|^2 \left[1 - j \frac{1}{(kr)^3} \right]$$



The transverse component W_θ of the power density does not contribute to the integral.
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Infinitesimal Dipole Antenna

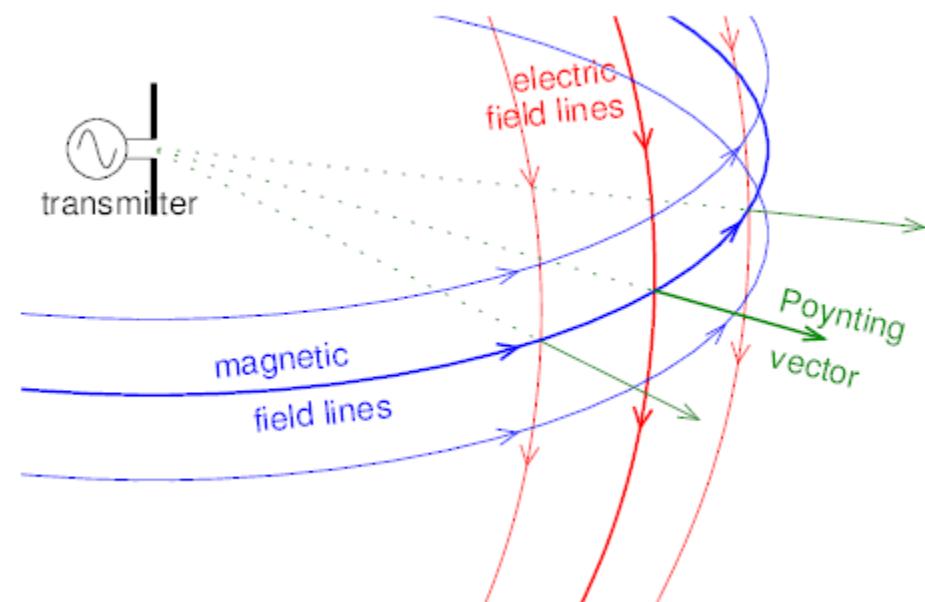
Radiated Power

$$P = \iint_S \mathbf{W} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^\pi (\hat{\mathbf{a}}_r W_r + \hat{\mathbf{a}}_\theta W_\theta) \cdot \hat{\mathbf{a}}_r r^2 \sin \theta \, d\theta \, d\phi$$



$$P = \int_0^{2\pi} \int_0^\pi W_r r^2 \sin \theta \, d\theta \, d\phi$$

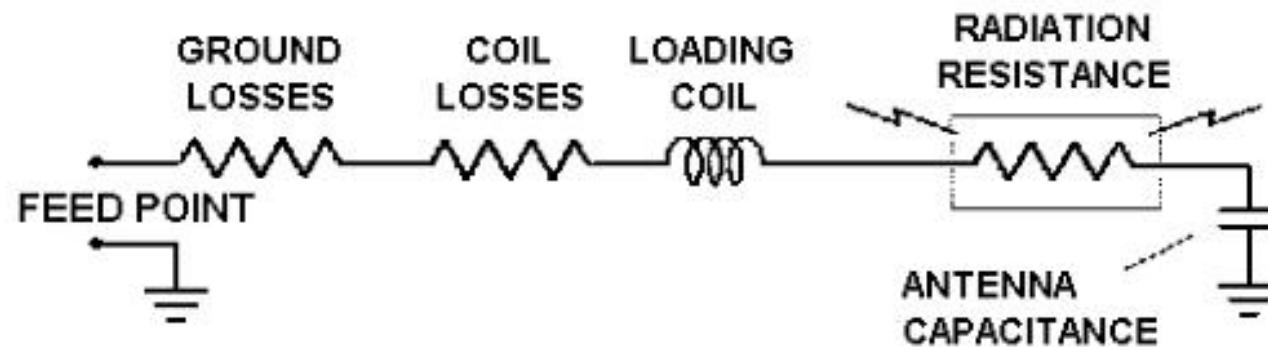
$$P_{\text{rad}} = \eta \frac{\pi}{3} \left| \frac{I_0 l}{\lambda} \right|^2 \left[1 - j \frac{1}{(kr)^3} \right]$$



The transverse component W_θ of the power density does not contribute to the integral.
(Pure imaginary, no real radiated power!)

Radiation Resistance

Antenna System Equivalent Circuit



An electrically short antenna looks like a small, lossy capacitor.

To make an electrically short antenna resonant, a loading coil is inserted in series with the antenna. The coil is built to have an inductive reactance equal and opposite to the capacitive reactance of the short antenna, so the combination of reactances cancels.

$$X = X_L + X_C = \omega L - \frac{1}{\omega C}$$

Radiation Resistance

an antenna's feedpoint resistance (R_{in})

$$R_{in} = R_R + R_L$$

R_R : **radiation resistance** caused by **the radiation of electromagnetic waves** from the antenna

R_L : **loss resistance** (also called ohmic resistance) caused by **ordinary electrical resistance** in the antenna, or **energy lost to nearby objects**, such as the earth, which dissipate RF energy as heat.

Radiation Resistance

R_L is the equivalent **resistance**

a **fictitious resistance** that would dissipate the **same amount of power** as if it is **radiated** by the **antenna**.

When substituted in series with the **antenna** will consume the **same power** as is **actually radiated**.

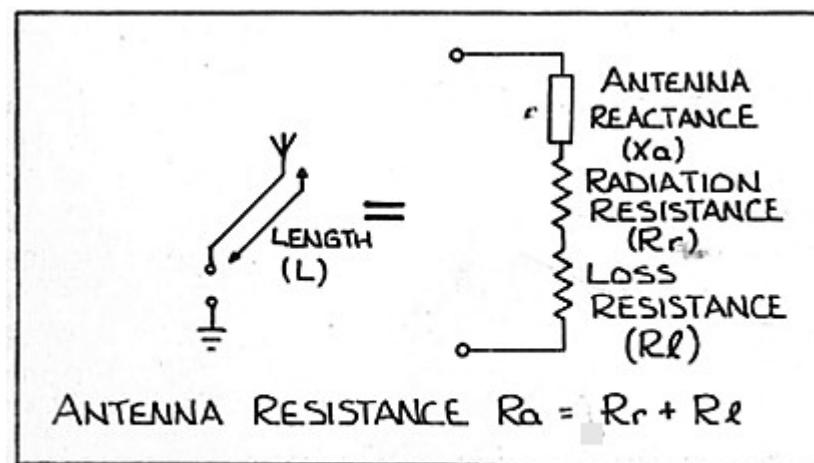


Figure 1 — Equivalent Antenna Electrical Circuit

Radiation Resistance

If radiated power is

$$P = |I|^2 R_r$$

where I is the electric current flowing into the feeds of the antenna. Usually I is the time average of the current (**root mean square value, RMS**)

Radiation resistance (effective resistance)

$$R_r = \frac{P}{|I|^2}$$

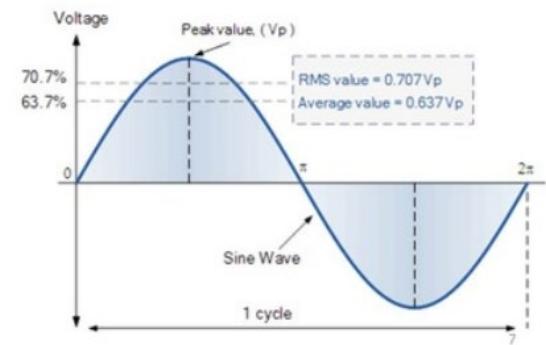
Radiation Resistance

root mean square of sine or cos wave

$$V_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} v^2(t) dt}$$

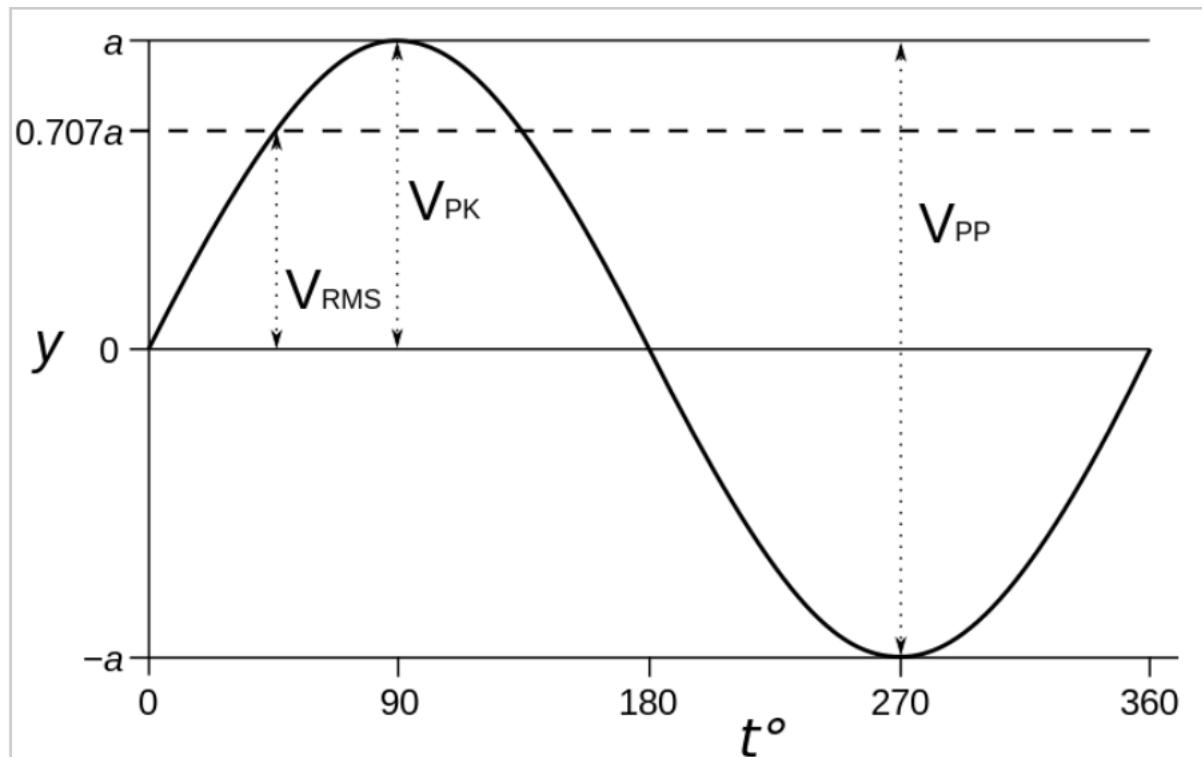
$$V_{rms} = \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \varphi) dt} = \frac{V_m}{\sqrt{2}}$$

$$= 0.707 V_m$$



Radiation Resistance

root mean square of sine or cos wave



Graph of a sine wave's voltage vs. time (in degrees), showing RMS, peak (PK), and peak-to-peak (PP) voltages.

Radiation Resistance (Infinitesimal Dipole Antenna)

Radiated Power

$$P_{\text{rad}} = \eta \left(\frac{\pi}{3} \right) \left| \frac{I_0 l}{\lambda} \right|^2 = \frac{1}{2} |I_0|^2 R_r$$

R_r is the radiation resistance

$$R_r = \eta \left(\frac{2\pi}{3} \right) \left(\frac{l}{\lambda} \right)^2 = 80\pi^2 \left(\frac{l}{\lambda} \right)^2$$

Radiation Resistance (Infinitesimal Dipole Antenna)

Example 4.1

Find the radiation resistance of an infinitesimal dipole whose overall length is $l = \lambda/50$.

Solution: Using (4-19)

$$R_r = 80\pi^2 \left(\frac{l}{\lambda}\right)^2 = 80\pi^2 \left(\frac{1}{50}\right)^2 = 0.316 \text{ ohms}$$

Since the radiation resistance of an infinitesimal dipole is about 0.3 ohms, it will present a very large mismatch when connected to practical transmission lines, many of which have characteristic impedances of 50 or 75 ohms. The reflection efficiency (e_r) and hence the overall efficiency (e_0) will be very small.

Infinitesimal Dipole Antenna

time average power density

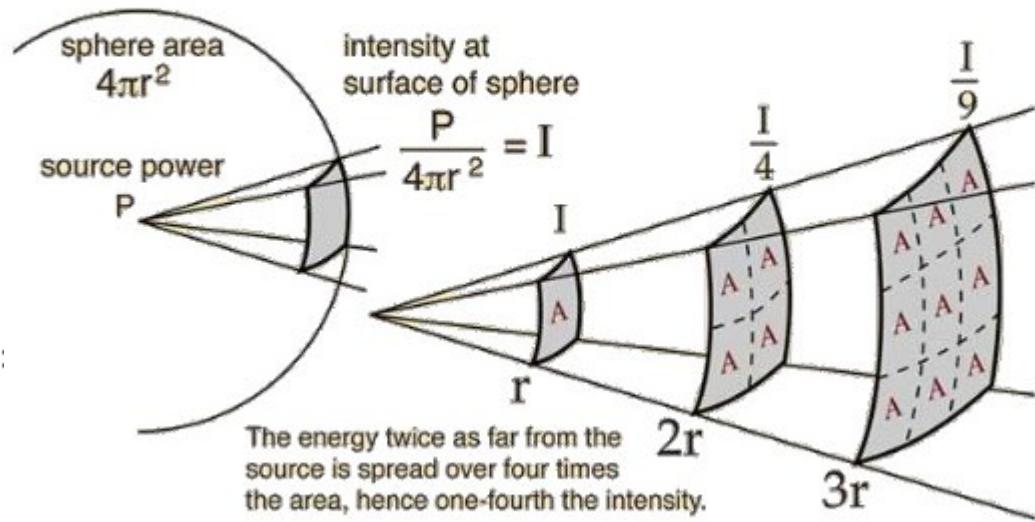
$$\mathbf{W}_{av} = \frac{1}{2} \operatorname{Re}(\mathbf{E} \times \mathbf{H}^*) = \hat{\mathbf{a}}_r \frac{1}{2\eta} |E_\theta|^2 = \hat{\mathbf{a}}_r \frac{\eta}{2} \left| \frac{kI_0 l}{4\pi} \right|^2 \frac{\sin^2 \theta}{r^2}$$

power intensity

- The intensity follows a $1/r^2$ relationship

$$U = r^2 W_{av}$$

$$= \frac{\eta}{2} \left(\frac{kI_0 l}{4\pi} \right)^2 \sin^2 \theta$$



Infinitesimal Dipole Antenna

maximum power intensity

$$U = r^2 \quad W_{\text{av}} = \frac{\eta}{2} \left(\frac{kI_0 l}{4\pi} \right)^2 \sin^2 \theta \quad \rightarrow \quad U_{\text{max}} = \frac{\eta}{2} \left(\frac{kI_0 l}{4\pi} \right)^2$$

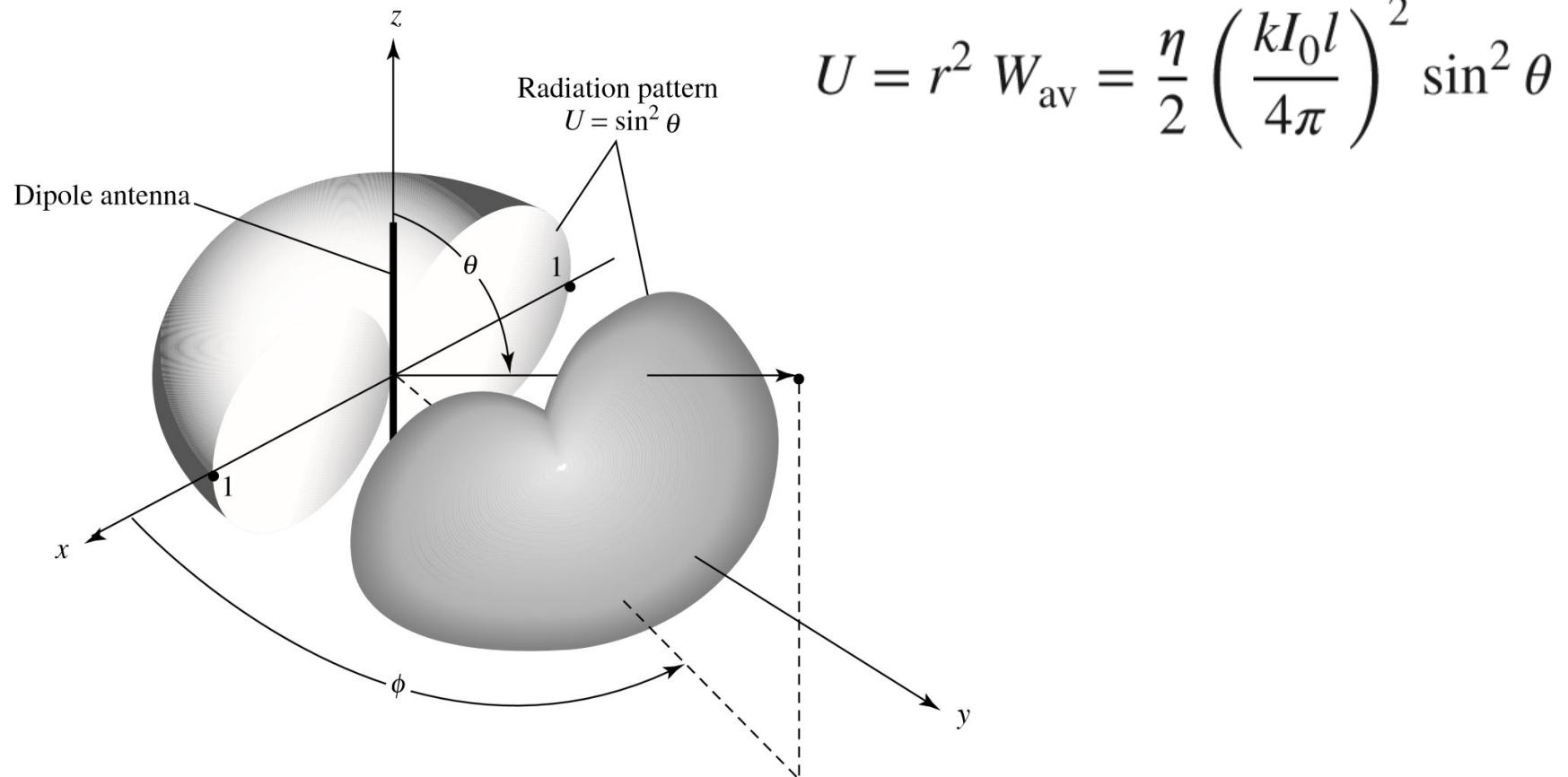
directivity

$$D_0 = 4\pi \frac{U_{\text{max}}}{P_{\text{rad}}} = \frac{3}{2}$$

$$P_{\text{rad}} = \eta \left(\frac{\pi}{3} \right) \left| \frac{I_0 l}{\lambda} \right|^2$$

Infinitesimal Dipole Antenna

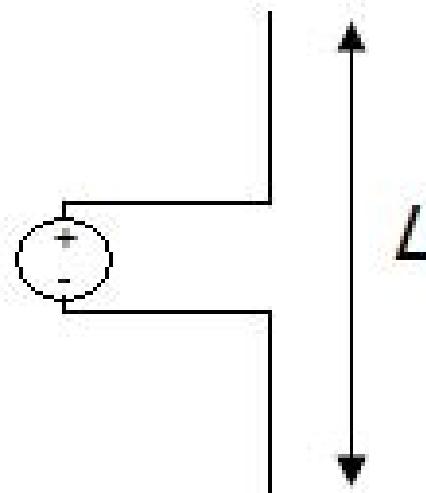
Radiation pattern



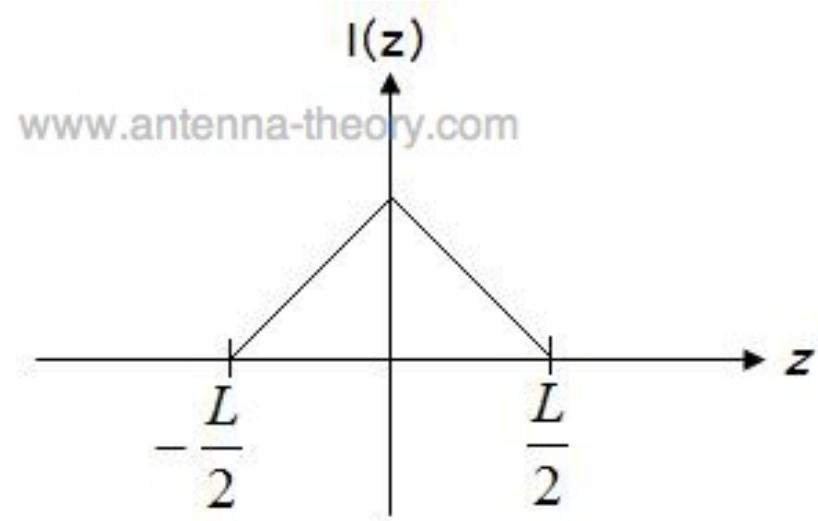
Short Dipole Antenna

the simplest of all antennas.

simply an open-circuited wire, fed at its center as shown (left), current distribution (right)



$$\lambda/50 < L \leq \lambda/10$$

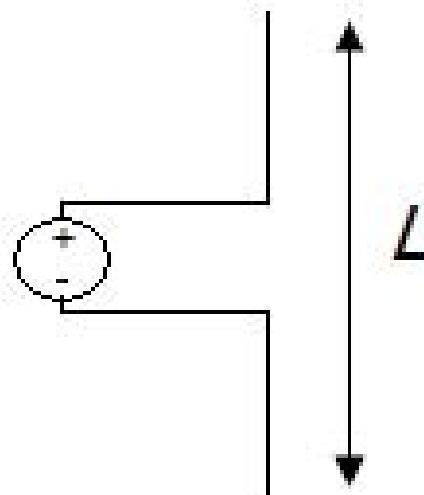


this is the amplitude of the current distribution; it is oscillating in time sinusoidally at frequency f .

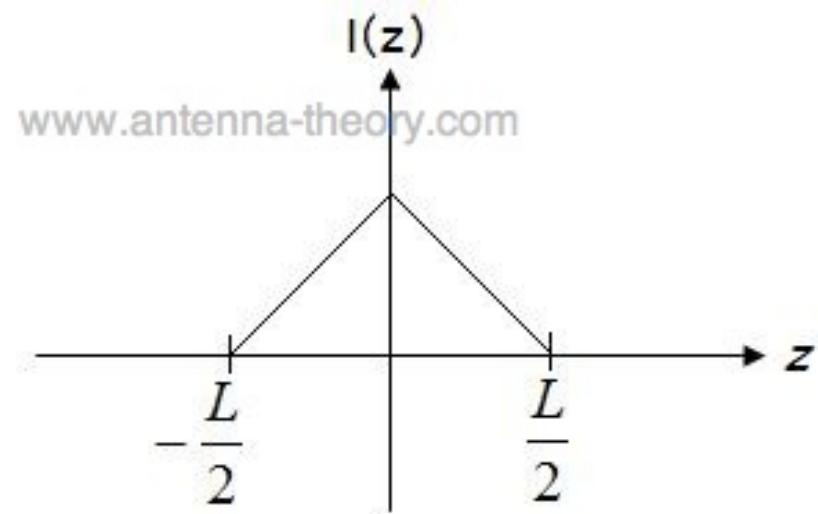
Short Dipole Antenna

the simplest of all antennas.

simply an open-circuited wire, fed at its center as shown (left), current distribution (right)



$$\lambda/50 < L \leq \lambda/10$$



$$\mathbf{I}_e(x', y', z') = \begin{cases} \hat{\mathbf{a}}_z I_0 \left(1 - \frac{2}{l} z'\right), & 0 \leq z' \leq l/2 \\ \hat{\mathbf{a}}_z I_0 \left(1 + \frac{2}{l} z'\right), & -l/2 \leq z' \leq 0 \end{cases}$$

Short Dipole Antenna

TABLE 4.1 Summary of Procedure to Determine the Far-Field Radiation Characteristics of an Antenna

1. Specify electric and/or magnetic current densities \mathbf{J} , \mathbf{M} [physical or equivalent (see Chapter 3, Figure 3.1)]
2. Determine vector potential components A_θ , A_ϕ and/or F_θ , F_ϕ using (3-46)–(3-54) in far field
3. Find far-zone \mathbf{E} and \mathbf{H} radiated fields (E_θ , E_ϕ ; H_θ , H_ϕ) using (3-58a)–(3-58b)
4. Form either

a.

$$\mathbf{W}_{\text{rad}}(r, \theta, \phi) = \mathbf{W}_{\text{av}}(r, \theta, \phi) = \frac{1}{2} \text{Re}[\mathbf{E} \times \mathbf{H}^*]$$

$$\simeq \frac{1}{2} \text{Re}[(\hat{\mathbf{a}}_\theta E_\theta + \hat{\mathbf{a}}_\phi E_\phi) \times (\hat{\mathbf{a}}_\theta H_\theta^* + \hat{\mathbf{a}}_\phi H_\phi^*)]$$

$$\mathbf{W}_{\text{rad}}(r, \theta, \phi) = \hat{\mathbf{a}}_r \frac{1}{2} \left[\frac{|E_\theta|^2 + |E_\phi|^2}{\eta} \right] = \hat{\mathbf{a}}_r \frac{1}{r^2} |f(\theta, \phi)|^2$$

or

$$\text{b. } U(\theta, \phi) = r^2 W_{\text{rad}}(r, \theta, \phi) = |f(\theta, \phi)|^2$$

5. Determine either

$$\text{a. } P_{\text{rad}} = \int_0^{2\pi} \int_0^\pi W_{\text{rad}}(r, \theta, \phi) r^2 \sin \theta \, d\theta \, d\phi$$

or

$$\text{b. } P_{\text{rad}} = \int_0^{2\pi} \int_0^\pi U(\theta, \phi) \sin \theta \, d\theta \, d\phi$$

6. Find directivity using

$$D(\theta, \phi) = \frac{U(\theta, \phi)}{U_0} = \frac{4\pi U(\theta, \phi)}{P_{\text{rad}}}$$

$$D_0 = D_{\max} = D(\theta, \phi)|_{\max} = \frac{U(\theta, \phi)|_{\max}}{U_0} = \frac{4\pi U(\theta, \phi)|_{\max}}{P_{\text{rad}}}$$

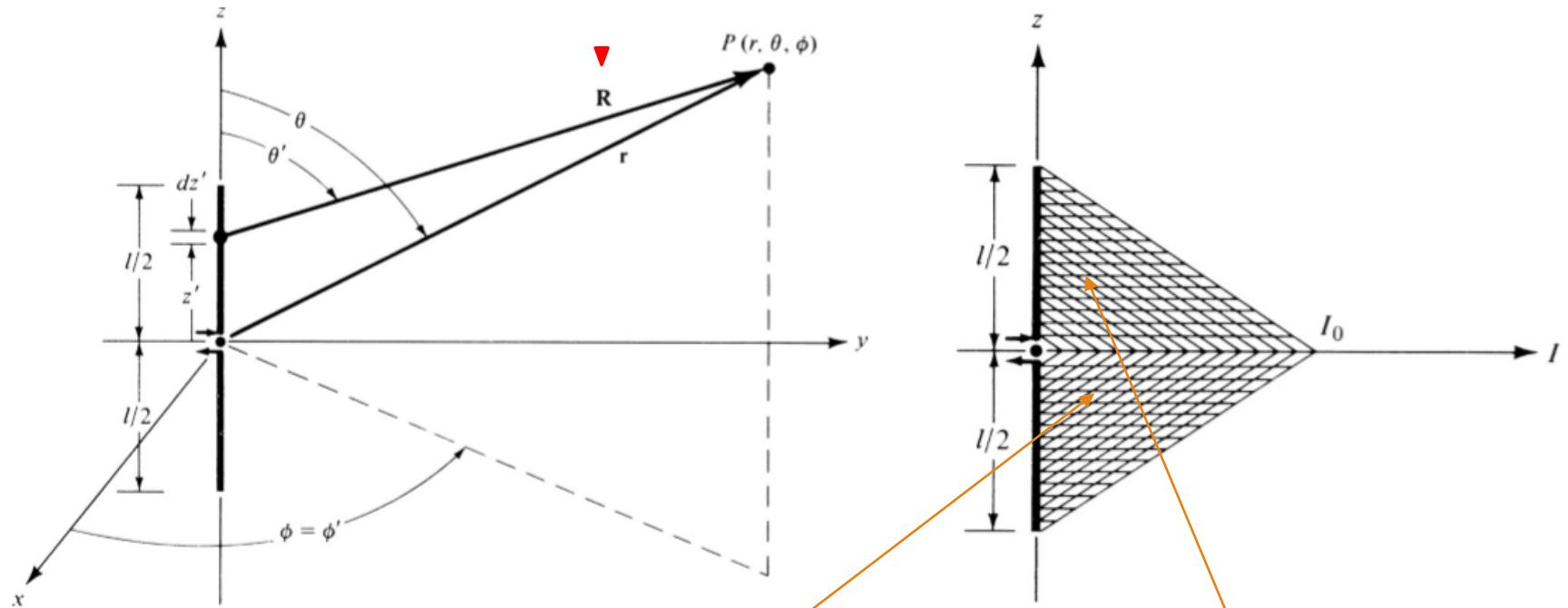
7. Form *normalized* power amplitude pattern:

$$P_n(\theta, \phi) = \frac{U(\theta, \phi)}{U_{\max}}$$

8. Determine radiation and input resistance:

$$R_r = \frac{2P_{\text{rad}}}{|I_0|^2}; \quad R_{in} = \frac{R_r}{\sin^2 \left(\frac{kl}{2} \right)}$$

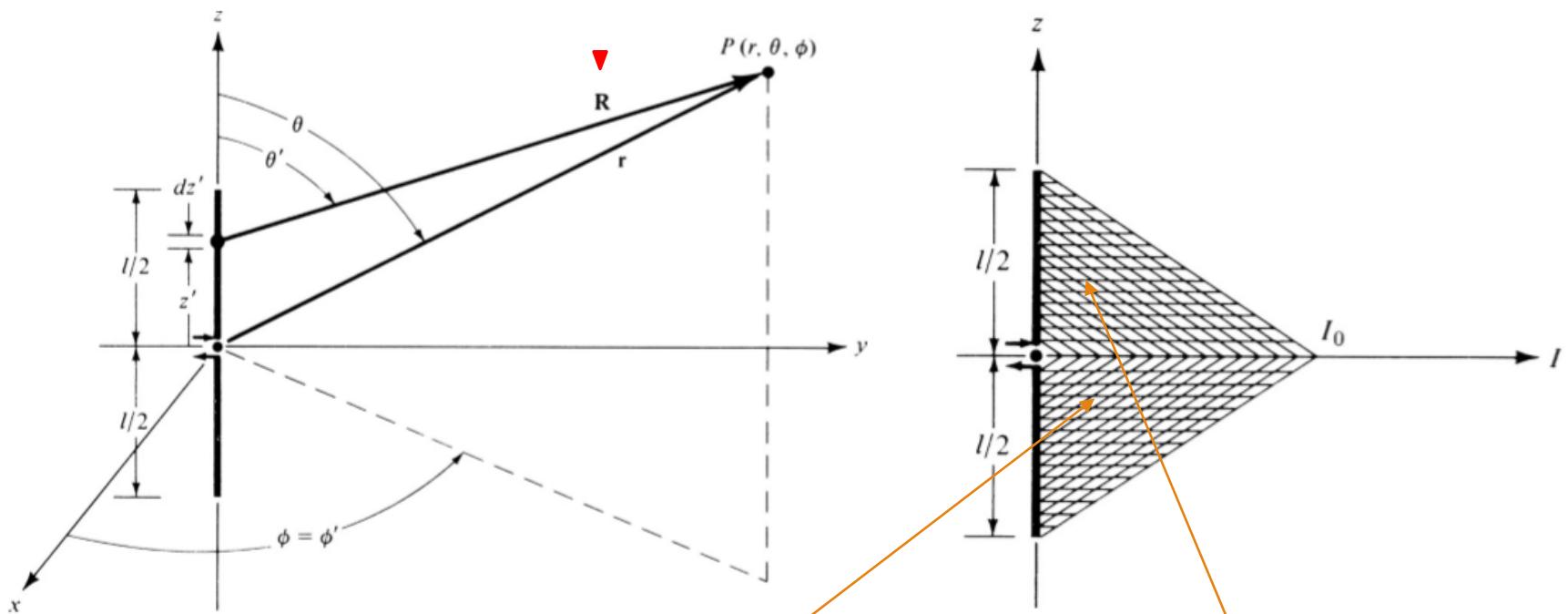
Short Dipole Antenna



Vector Potential

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \left[\hat{\mathbf{a}}_z \int_{-l/2}^0 I_0 \left(1 + \frac{2}{l} z' \right) \frac{e^{-jkR}}{R} dz' + \hat{\mathbf{a}}_z \int_0^{l/2} I_0 \left(1 - \frac{2}{l} z' \right) \frac{e^{-jkR}}{R} dz' \right]$$

Short Dipole Antenna



Vector Potential

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \left[\hat{\mathbf{a}}_z \int_{-l/2}^0 I_0 \left(1 + \frac{2}{l} z' \right) \frac{e^{-jkR}}{R} dz' + \hat{\mathbf{a}}_z \int_0^{l/2} I_0 \left(1 - \frac{2}{l} z' \right) \frac{e^{-jkR}}{R} dz' \right]$$

$R \approx r$ for small dipole $\lambda/50 < l \leq \lambda/10$

$$\mathbf{A} = \hat{\mathbf{a}}_z A_z = \hat{\mathbf{a}}_z \frac{1}{2} \left[\frac{\mu I_0 l e^{-jkr}}{4\pi r} \right]$$

Short Dipole Antenna

Fields solution

Vector Potential

$$\mathbf{A} = \hat{\mathbf{a}}_z A_z = \hat{\mathbf{a}}_z \frac{1}{2} \left[\frac{\mu I_0 l e^{-jkr}}{4\pi r} \right]$$



Ref. P. 18

$$\left. \begin{aligned} E_\theta &\simeq j\eta \frac{kI_0 l e^{-jkr}}{8\pi r} \sin \theta \\ H_\phi &\simeq j \frac{kI_0 l e^{-jkr}}{8\pi r} \sin \theta \end{aligned} \right\} kr \gg 1$$

Short Dipole Antenna

Radiation resistance (1/4 of infinitesimal dipole)

$$R_r = \frac{2P_{\text{rad}}}{|I_0|^2} = 20\pi^2 \left(\frac{l}{\lambda}\right)^2$$

Input impedance of short dipole

$$Z = R_r + R_{\text{loss}} + jX$$

R_{loss} and X depend on the radius a of the dipole, the conductivity σ and permeability of the conductor μ of the dipole and frequency f

$$R_{\text{loss}} = \frac{L}{6\pi a} \sqrt{\frac{\pi f \mu}{2\sigma}} \quad X = \frac{-120\lambda}{\pi L} \left(\ln\left(\frac{L}{2a}\right) - 1 \right)$$

$$\left. \begin{aligned} E_\theta &\simeq j\eta \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \\ E_r &\simeq E_\phi = H_r = H_\theta = 0 \\ H_\phi &\simeq j \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \end{aligned} \right\} kr \gg 1$$

Short Dipole Antenna

in the far-field, only the E_θ and H_ϕ fields are nonzero.

these fields are orthogonal and in-phase.

the fields are perpendicular to the direction of propagation \hat{r}
(away from the antenna).

ratio of the E-field to the H-field is given by the intrinsic
impedance of free space.

$$\eta = Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = \sqrt{\frac{4\pi 10^{-7}}{8.854e-12}} \approx 120\pi \approx 377 \text{ Ohms}$$

Short Dipole Antenna

- the fields die off as $1/r$, indicates the power falls off as

$$P(r) \propto \frac{1}{r^2}$$

- the fields are **proportional to** antenna length L , (a longer dipole will radiate more power as long as the short dipole assumption is valid, i.e., $\lambda/50 < L \leq \lambda/10$)
- the fields are **proportional to the current amplitude I_0** of current on short dipole

$$\left. \begin{aligned} E_\theta &\simeq j\eta \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \\ E_r &\simeq E_\phi = H_r = H_\theta = 0 \\ H_\phi &\simeq j \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \end{aligned} \right\} kr \gg 1$$

Short Dipole Antenna

The exponential term:

$$e^{-jkr}$$

$$\left. \begin{array}{l} E_\theta \simeq j\eta \frac{kI_0 l e^{-jkr}}{8\pi r} \sin \theta \\ E_r \simeq E_\phi = H_r = H_\theta = 0 \\ H_\phi \simeq j \frac{kI_0 l e^{-jkr}}{8\pi r} \sin \theta \end{array} \right\} kr \gg 1$$

describes the phase-variation of the wave versus distance. The parameter k is known as the **wavenumber**.

$$k = \frac{2\pi}{\lambda} = 2\pi f \sqrt{\mu\epsilon} = 2\pi f \sqrt{\mu_r \epsilon_r \mu_0 \epsilon_0}$$

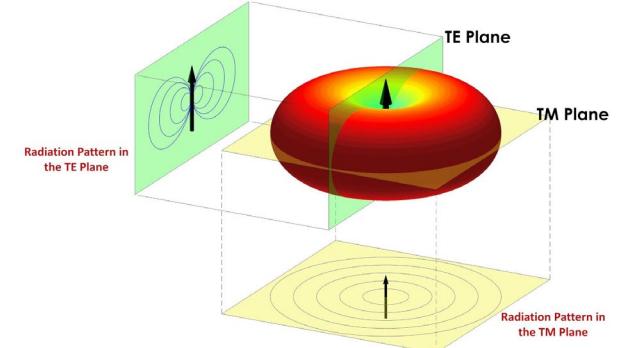
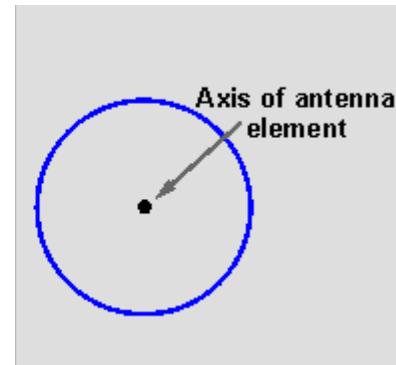
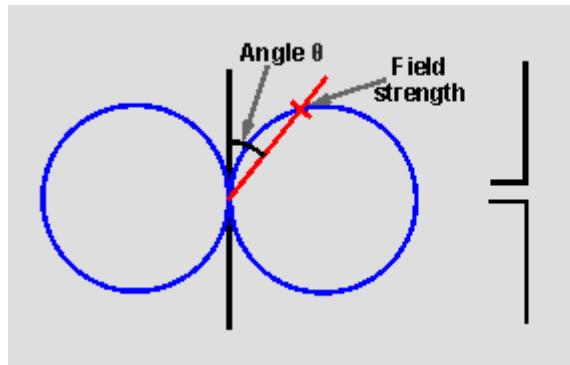
Note also that the fields are oscillating in time at a frequency f in addition to the above spatial variation.

Short Dipole Antenna

$$\left. \begin{aligned} E_\theta &\simeq j\eta \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \\ E_r &\simeq E_\phi = H_r = H_\theta = 0 \\ H_\phi &\simeq j \frac{kI_0 le^{-jkr}}{8\pi r} \sin \theta \end{aligned} \right\} kr \gg 1$$

the **spatial variation** of the **fields** as a function of **direction** from the antenna are given by radiation pattern $\sin \theta$ for short dipole. (**same as infinitesimal dipole**)

for a vertical antenna oriented along the z-axis, the radiation will be maximum in the x-y plane. Theoretically, **no radiation along the z-axis** far from the antenna.

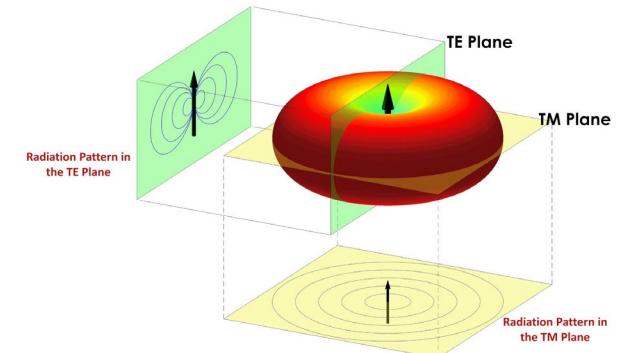
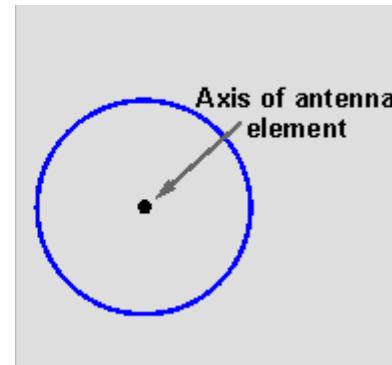
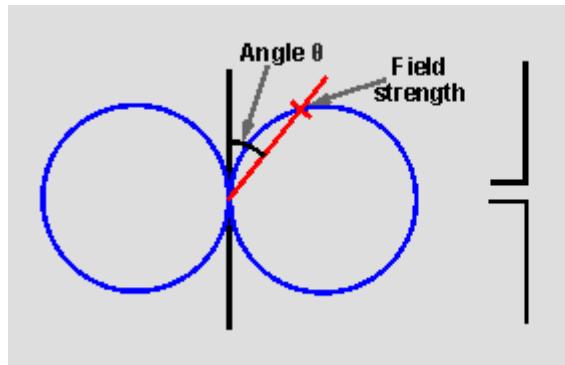


Short Dipole Antenna

the directivity $D = 1.5$ (1.76 dB)

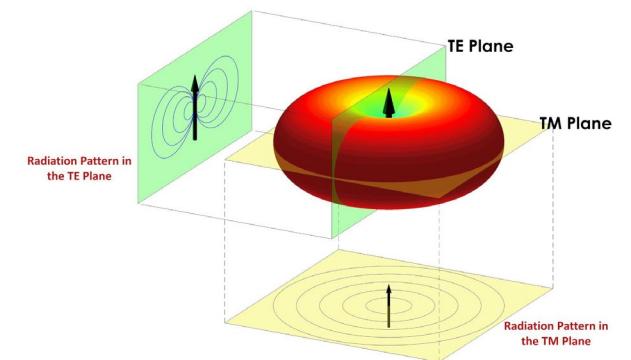
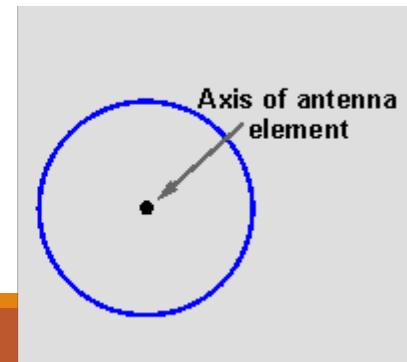
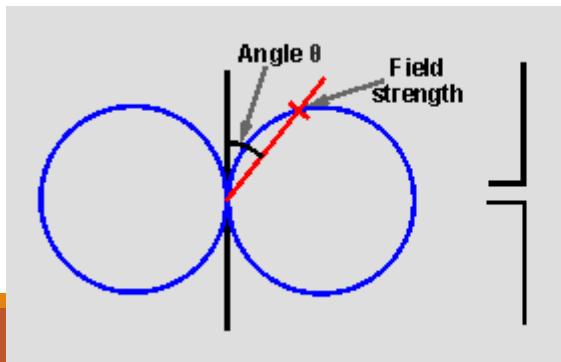
fields of the short dipole antenna are only a function of the polar angle, they have no azimuthal variation (omnidirectional).

the Half-Power Beamwidth is 90 degrees.



Short Dipole Antenna

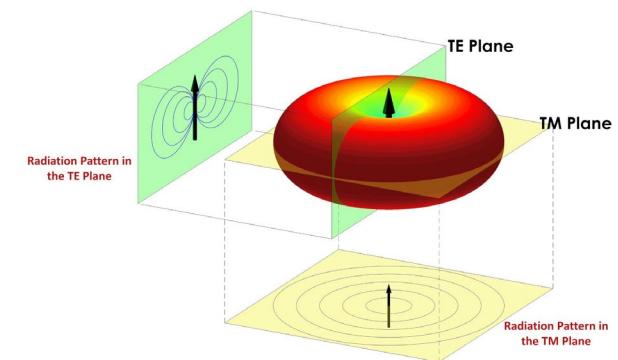
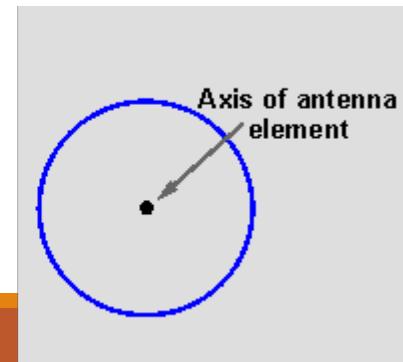
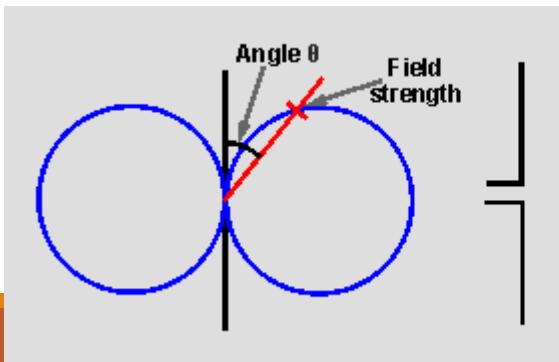
Antenna Type	Typical Directivity	Typical Directivity (dB)
Short Dipole Antenna	1.5	1.76
Half-Wave Dipole Antenna	1.64	2.15
Patch (Microstrip) Antenna	3.2-6.3	5-8
Horn Antenna	10-100	10-20
Dish Antenna	10-10,000	10-40



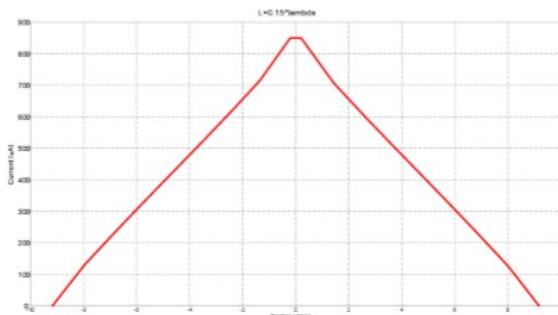
Short Dipole Antenna



Antenna Type	Directivity	HPBW (E-plane)
Short Dipole	1.5 dBi	120 degrees
Half-Wave Dipole	2.1 dBi	78 degrees
Patch (Microstrip)	4 - 9 dBi	30 - 90 degrees
Horn Antenna	10 - 25 dBi	10 - 60 degrees
Dish Antenna	20 - 40 dBi	1 - 10 degrees



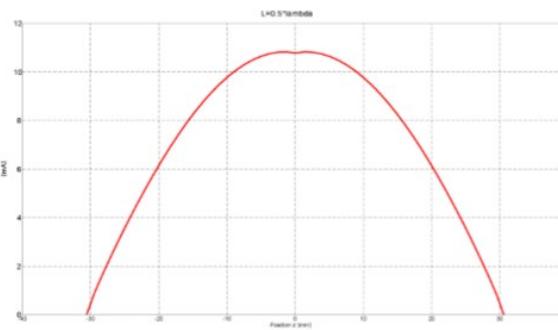
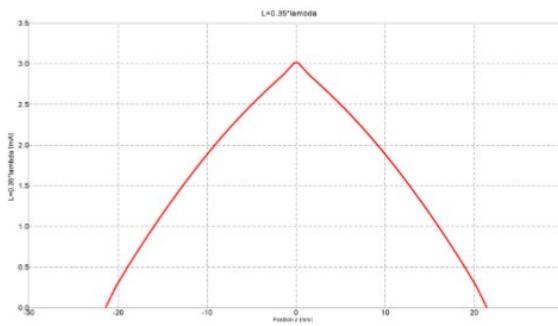
Linear Antenna (dipole) Current Distribution



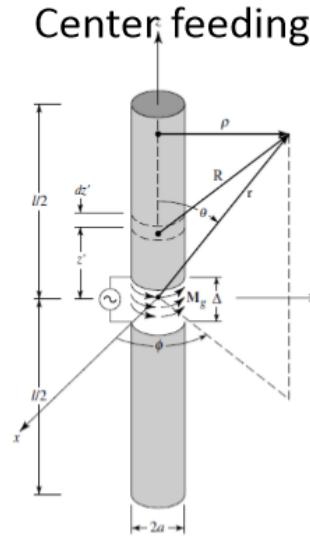
Whole dipole length

$$L = 0.15\lambda$$

small dipole



L = 0.5λ
half-wavelength dipole



Triangular distribution

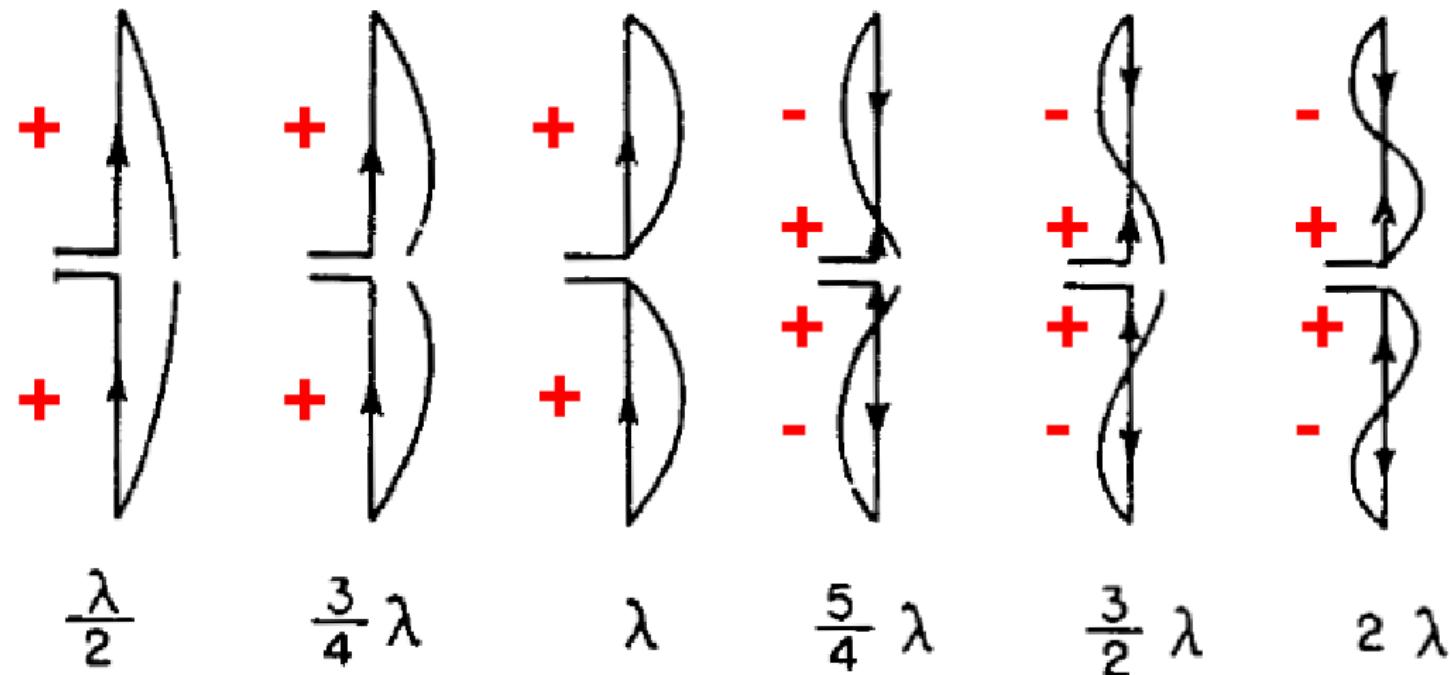
$$I(z) = I_0 \left(1 - \frac{2|z|}{l} \right)$$

Sine distribution

$$I(z) = I_0 \sin \left[k \left(\frac{l}{2} - |z| \right) \right]$$

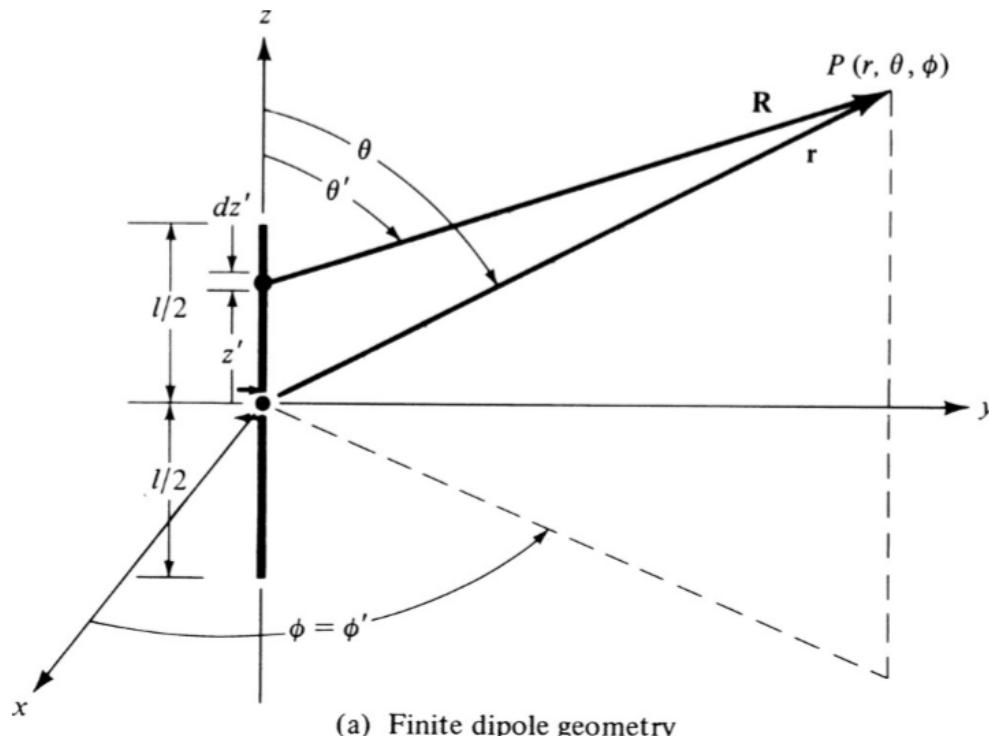
Linear Antenna (dipole)

Current Distribution



$L > \lambda \rightarrow$ out-of phase currents \rightarrow sidelobes

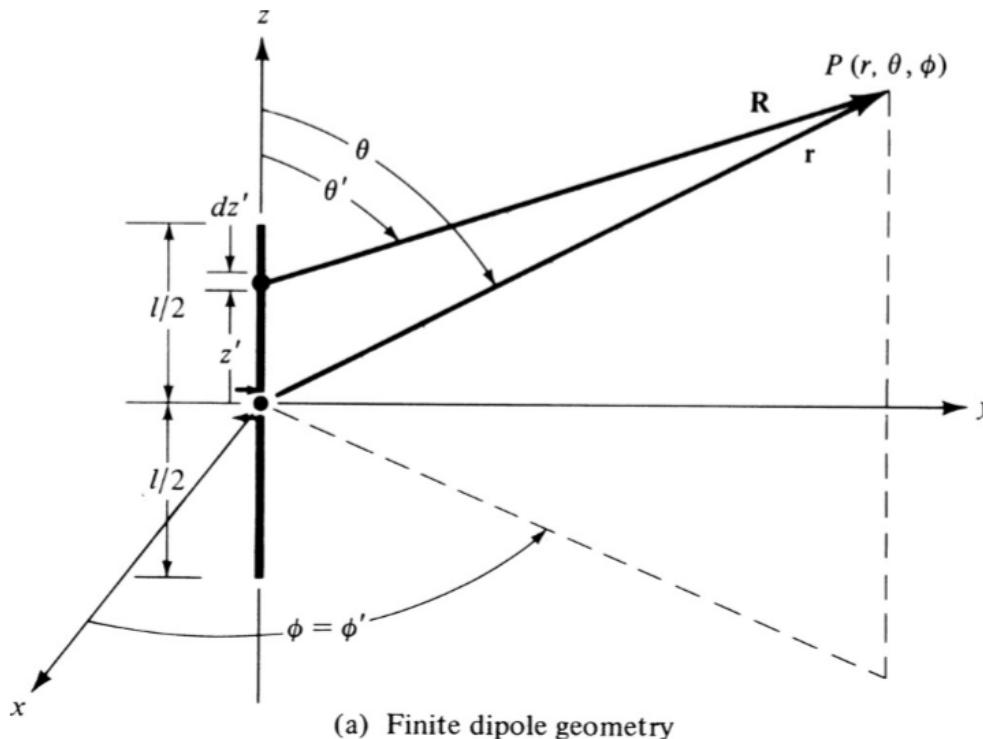
Linear Antenna (dipole) Vector Potential



$$\boxed{\begin{aligned} \mathbf{A}(x, y, z) \\ = \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl' \end{aligned}}$$

Linear Antenna (dipole)

Vector Potential



$$r^2 = x^2 + y^2 + z^2$$

$$z = r \cos \theta$$

$$\mathbf{A}(x, y, z)$$

$$= \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$

Thin wire

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$$

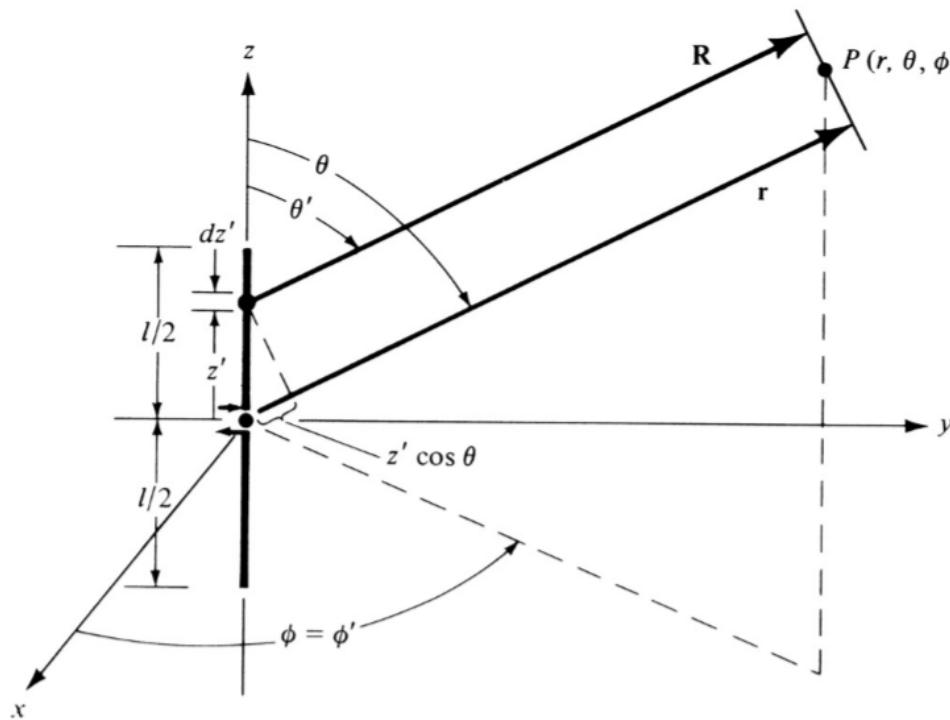
$$= \sqrt{x^2 + y^2 + (z - z')^2}$$

$$= \sqrt{(x^2 + y^2 + z^2) + (-2zz' + z'^2)} :$$

$$= \sqrt{r^2 + (-2rz' \cos \theta + z'^2)}$$

Linear Antenna (dipole)

Vector Potential



$$R \simeq r - z' \cos \theta$$

$$\mathbf{A}(x, y, z)$$

$$= \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$

Thin wire

$$R = \sqrt{r^2 + (-2rz' \cos \theta + z'^2)}$$

Binomial Expansion

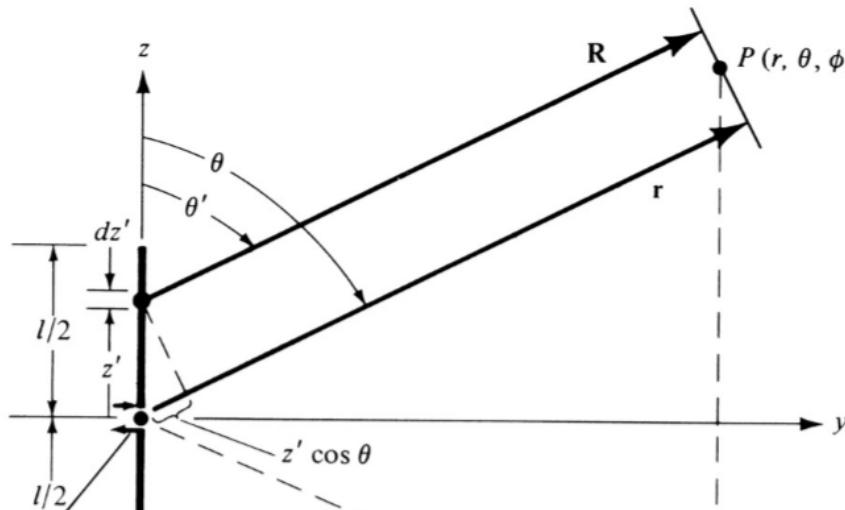
$$R = r - z' \cos \theta + \frac{1}{r} \left(\frac{z'^2}{2} \sin^2 \theta \right)$$

$$+ \frac{1}{r^2} \left(\frac{z'^3}{2} \cos \theta \sin^2 \theta \right) + \dots$$

higher order terms become less significant provided $r \gg z'$

Linear Antenna (dipole)

Vector Potential



$$(x+y)^{1/2} = x^{1/2} + \frac{1}{2}x^{-1/2}y - \frac{1}{8}x^{-3/2}y^2 + \frac{1}{16}x^{-5/2}y^3 - \text{etc.}$$

x

y

z

$$R \simeq r - z' \cos \theta$$

$$\mathbf{A}(x, y, z)$$

$$= \frac{\mu}{4\pi} \int_C \mathbf{I}_e(x', y', z') \frac{e^{-jkR}}{R} dl'$$

Thin wire

$$R = \sqrt{r^2 + (-2rz' \cos \theta + z'^2)}$$

Binomial Expansion

$$R = r - z' \cos \theta + \frac{1}{r} \left(\frac{z'^2}{2} \sin^2 \theta \right)$$

$$+ \frac{1}{r^2} \left(\frac{z'^3}{2} \cos \theta \sin^2 \theta \right) + \dots$$

higher order terms become less significant provided $r \gg z'$

Linear Antenna (dipole) Vector Potential

Far field approximations, i.e. $r \geq 2 \left(\frac{l^2}{\lambda} \right)$

$$R \simeq r - z' \cos \theta \quad \text{for phase terms}$$

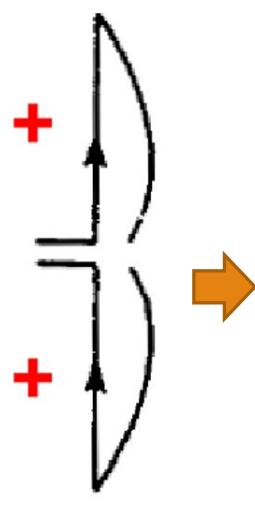
$$R \simeq r \quad \text{for amplitude terms}$$

Vector Potential has an analytical (closed form) solution

Linear Antenna (dipole) Decomposition

- A **standard approach** used to calculate the far field for an **arbitrary wire antenna**.
- Based on the solution for **the field of the infinitesimal dipole (Hertzian Dipole)**.
- The finite length dipole is **subdivided into an infinite number of infinitesimal dipoles** of length dz' .
- Each such dipole produces the **elementary far field given by infinitesimal dipole E and H field**.

Linear Antenna (dipole) Decomposition



$$dE_\theta \simeq j\eta \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$

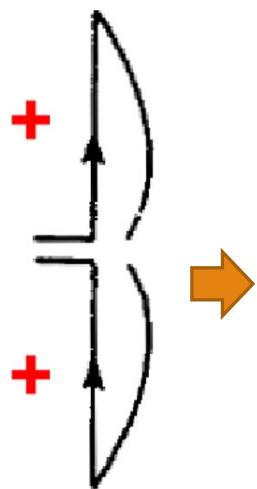
$$dE_r \simeq dE_\phi = dH_r = dH_\theta = 0$$

$$dH_\phi \simeq j \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$

infinitesimal dipole

infinitesimal dipole E and H field

Linear Antenna (dipole) Decomposition

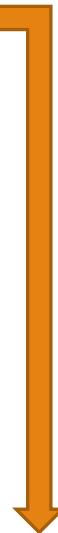


 dz'

$$dE_\theta \simeq j\eta \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$

$$dE_r \simeq dE_\phi = dH_r = dH_\theta = 0$$

$$dH_\phi \simeq j \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$



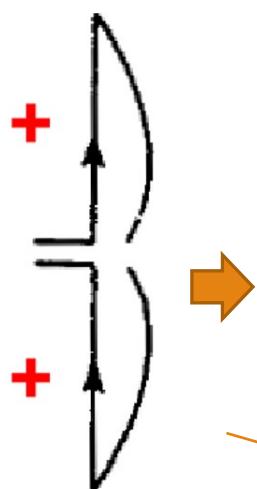
$$E_\theta = \int_{-l/2}^{+l/2} dE_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin \theta \left[\int_{-l/2}^{+l/2} I_e(x', y', z') e^{jkz' \cos \theta} dz' \right]$$

Linear Antenna (dipole)

Decomposition

$$R \simeq r - z' \cos \theta \quad \text{for phase terms}$$

$$R \simeq r \quad \text{for amplitude terms}$$



$\boxed{dz'}$

$$dE_\theta \simeq j\eta \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$

$$dE_r \simeq dE_\phi = dH_r = dH_\theta = 0$$

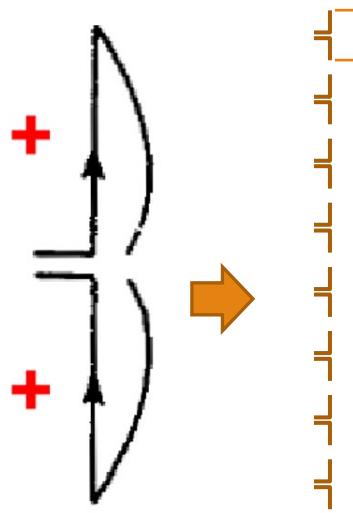
$$dH_\phi \simeq j \frac{kI_e(x', y', z')e^{-jkR}}{4\pi R} \sin \theta \ dz'$$

Current distribution

$$E_\theta = \int_{-l/2}^{+l/2} dE_\theta = j\eta \frac{ke^{-jkr}}{4\pi r} \sin \theta \left[\int_{-l/2}^{+l/2} I_e(x', y', z') e^{jkz' \cos \theta} dz' \right]$$

Linear Antenna (dipole) Decomposition

Sum (Integral) of all small infinitesimal dipole antennas



$$E_\theta \simeq j\eta \frac{I_0 e^{-jkr}}{2\pi r} \left[\frac{\cos\left(\frac{kl}{2} \cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]$$

$$H_\phi \simeq \frac{E_\theta}{\eta} \simeq j \frac{I_0 e^{-jkr}}{2\pi r} \left[\frac{\cos\left(\frac{kl}{2} \cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]$$

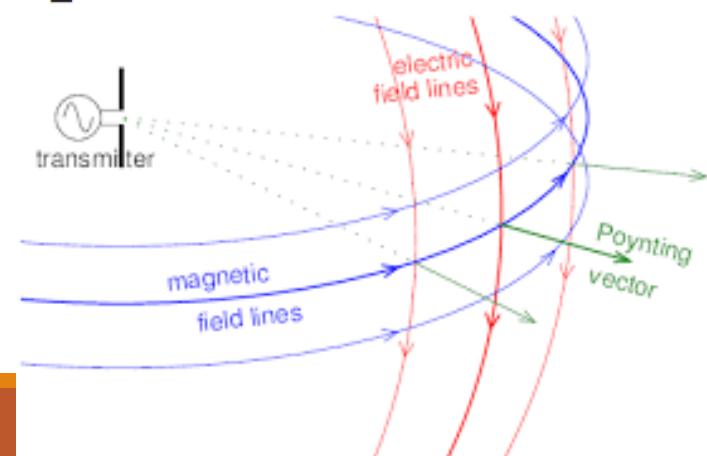
Linear Antenna (dipole)

Radiation Density

at a given direction

$$W_{av} = \frac{1}{2} \operatorname{Re}[\mathbf{E} \times \mathbf{H}^*] = \frac{1}{2} \operatorname{Re}[\hat{\mathbf{a}}_\theta E_\theta \times \hat{\mathbf{a}}_\phi H_\phi^*] = \frac{1}{2} \operatorname{Re} \left[\hat{\mathbf{a}}_\theta E_\theta \times \hat{\mathbf{a}}_\phi \frac{E_\theta^*}{\eta} \right]$$

$$W_{av} = \hat{\mathbf{a}}_r W_{av} = \hat{\mathbf{a}}_r \frac{1}{2\eta} |E_\theta|^2 = \hat{\mathbf{a}}_r \eta \frac{|I_0|^2}{8\pi^2 r^2} \left[\frac{\cos\left(\frac{kl}{2} \cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]^2$$

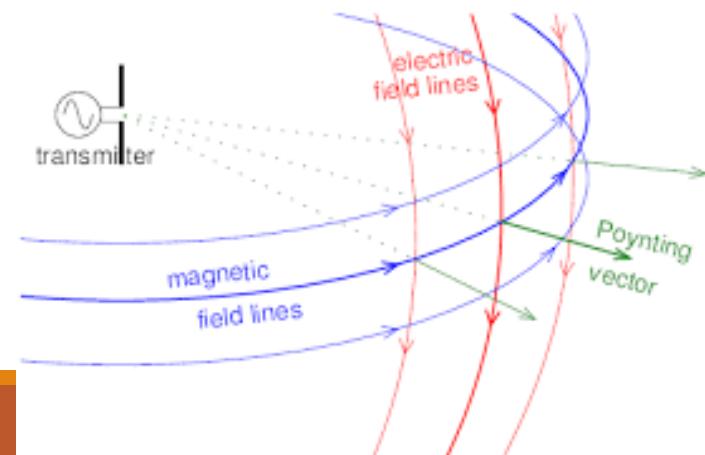


Linear Antenna (dipole)

Radiation Intensity

at a given direction

$$U = r^2 \quad W_{\text{av}} = \eta \frac{|I_0|^2}{8\pi^2} \left[\frac{\cos\left(\frac{kl}{2} \cos \theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin \theta} \right]^2$$



Linear Antenna (dipole)

Directivity

$$D_0 = D_{\max} = D(\theta, \phi)|_{\max} = \frac{U(\theta, \phi)|_{\max}}{U_0} = \frac{4\pi U(\theta, \phi)|_{\max}}{P_{\text{rad}}}$$

Total power radiated P_{rad}

$$\begin{aligned} P_{\text{rad}} &= \iint_S \mathbf{W}_{\text{av}} \cdot d\mathbf{s} = \int_0^{2\pi} \int_0^{\pi} \hat{\mathbf{a}}_r W_{\text{av}} \cdot \hat{\mathbf{a}}_r r^2 \sin \theta \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^{\pi} W_{\text{av}} r^2 \sin \theta \, d\theta \, d\phi \\ &= \eta \frac{|I_0|^2}{4\pi} \int_0^{\pi} \frac{\left[\cos\left(\frac{kl}{2} \cos \theta\right) - \cos\left(\frac{kl}{2}\right) \right]^2}{\sin \theta} \, d\theta \\ &= \eta \frac{|I_0|^2}{4\pi} \{ C + \ln(kl) - C_i(kl) + \frac{1}{2} \sin(kl)[S_i(2kl) - 2S_i(kl)] \\ &\quad + \frac{1}{2} \cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)] \} \end{aligned}$$

where $C = 0.5772$
 (Euler's constant) and
 $Ci(x)$ and $Si(x)$ are the
 cosine and sine integrals
 (see Appendix III)

Linear Antenna (dipole)

Directivity

Finite length dipole

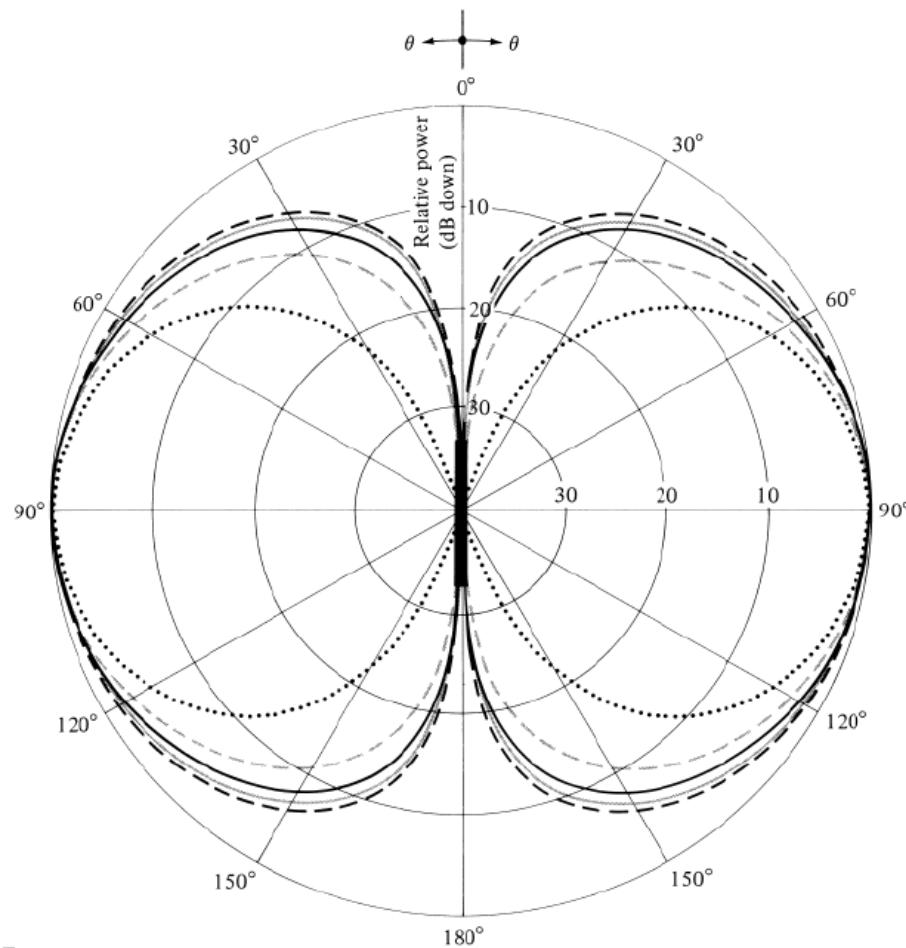
$$D_0 = \frac{2F(\theta)|_{\max}}{Q}$$

$$F(\theta, \phi) = F(\theta) = \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right]^2$$

where $C = 0.5772$
 (Euler's constant) and
 $Ci(x)$ and $Si(x)$ are the
 cosine and sine integrals
 (see Appendix III)

$$\begin{aligned} Q = & \{ C + \ln(kl) - C_i(kl) + \frac{1}{2} \sin(kl)[S_i(2kl) - 2S_i(kl)] \\ & + \frac{1}{2} \cos(kl)[C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)] \} \end{aligned}$$

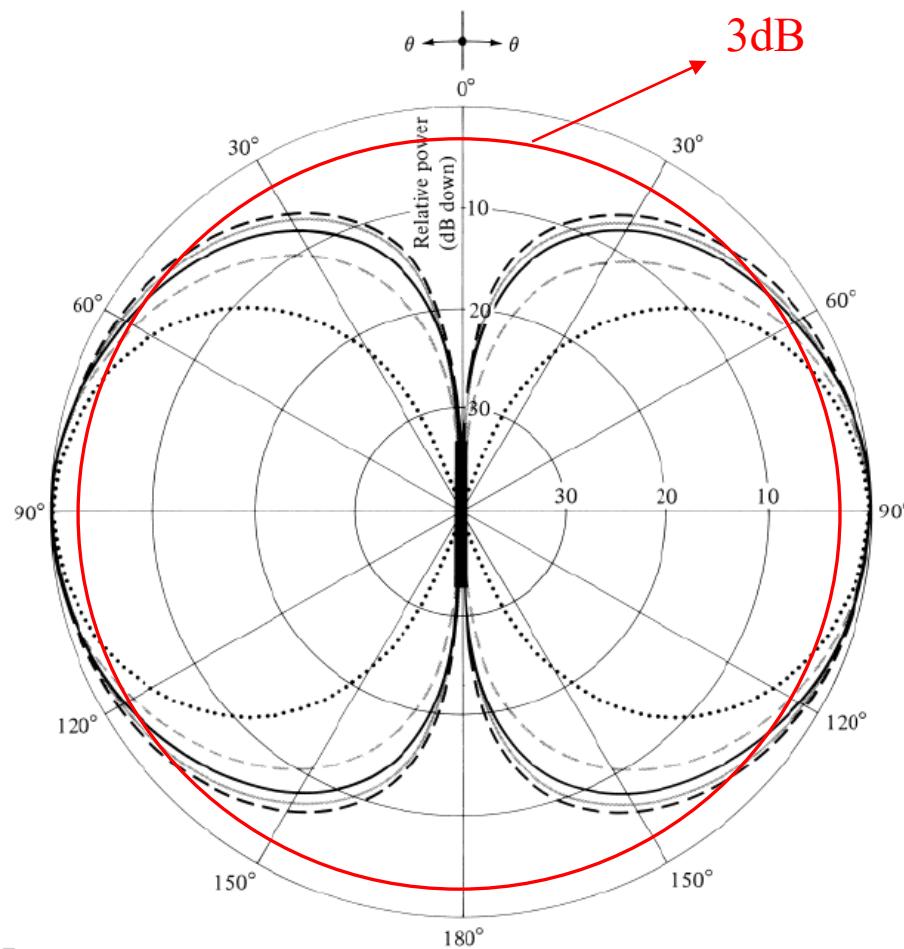
Linear Antenna (dipole) Radiation Pattern



Radiation pattern – changes with dipole length

- $l = \lambda/50$
- $l = \lambda/4$
- $l = \lambda/2$
- $l = 3\lambda/4$
- $l = \lambda$

Linear Antenna (dipole) Radiation Pattern



Radiation pattern – changes with dipole length

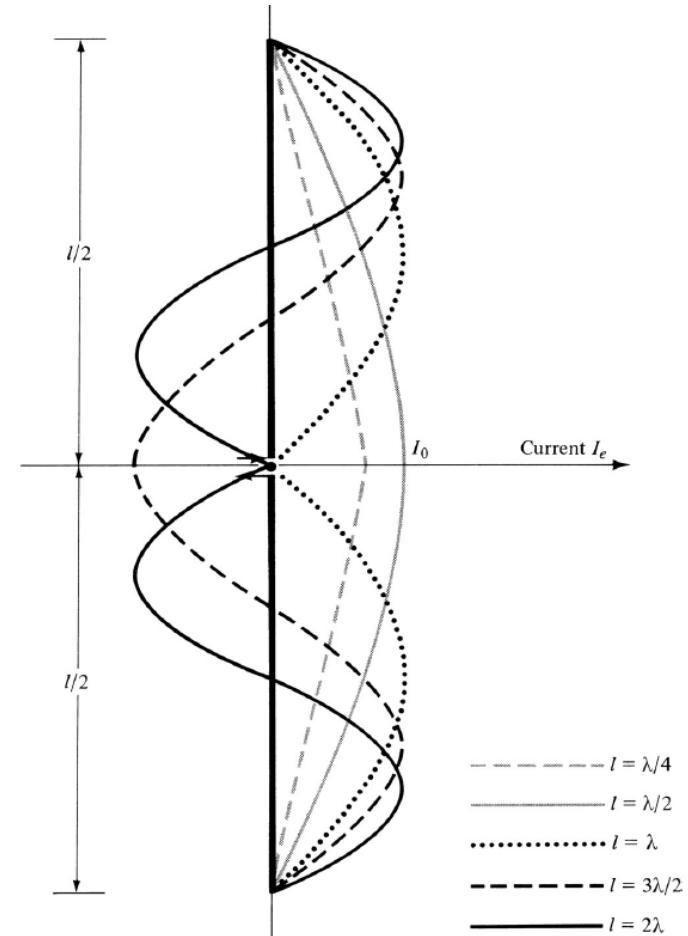
- $l = \lambda/50$
- $l = \lambda/4$
- $l = \lambda/2$
- $l = 3\lambda/4$
- $l = \lambda$

- 3-dB beamwidth = 90°
- 3-dB beamwidth = 87°
- 3-dB beamwidth = 78°
- 3-dB beamwidth = 64°
- 3-dB beamwidth = 47.8°

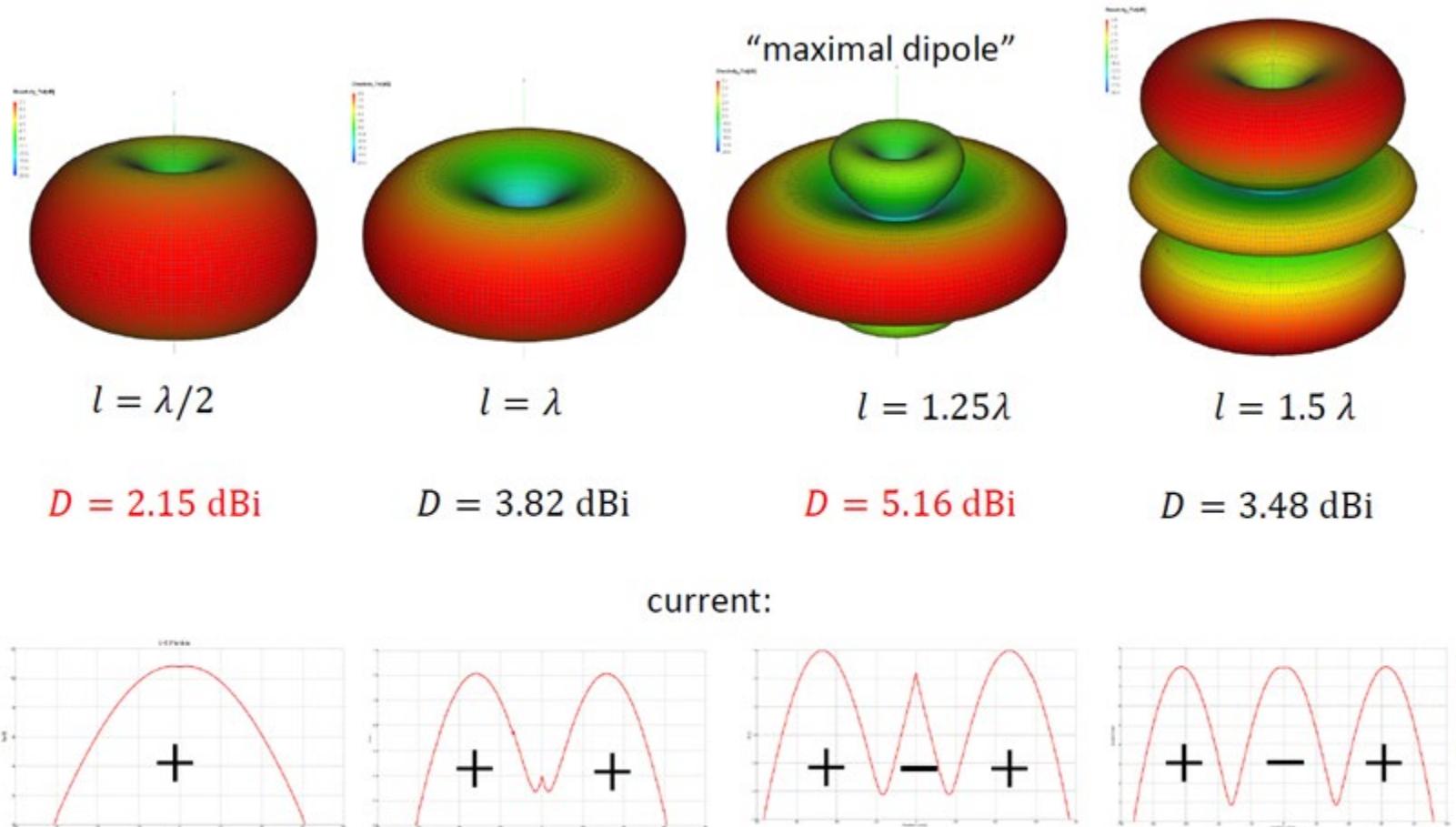
Linear Antenna (dipole) Radiation Pattern

Current distributions along the length of a linear wire antenna

- $l = \lambda/4$
- $l = \lambda/2$
- $l = \lambda$
- - - $l = 3\lambda/2$
- $l = 2\lambda$



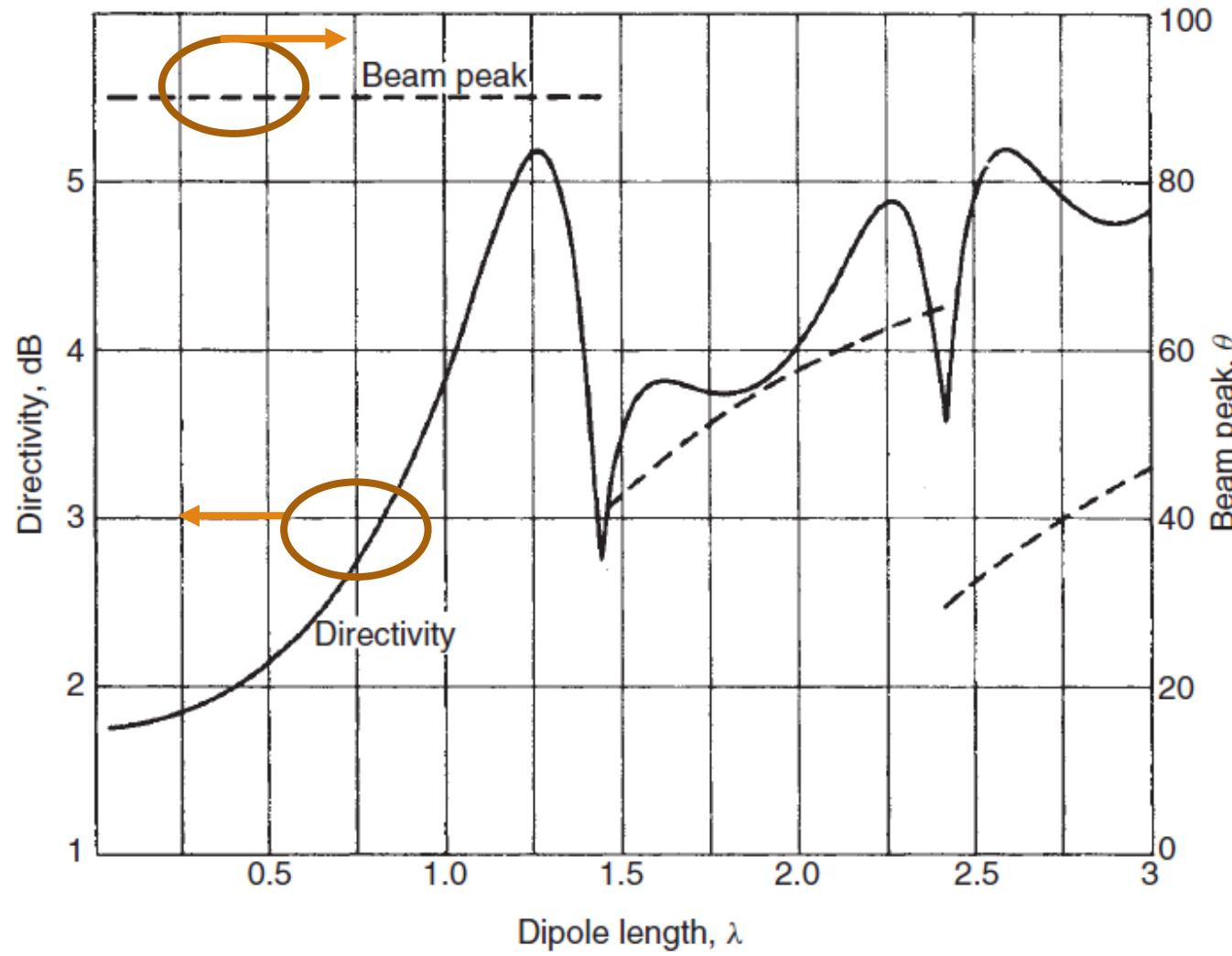
Linear Antenna (dipole) Radiation Pattern



"One of the goals of antenna design is to place lobes at the desired angles."

For dipole antenna, length of 1.25λ , directivity is maximum

Linear Antenna (dipole) Directivity

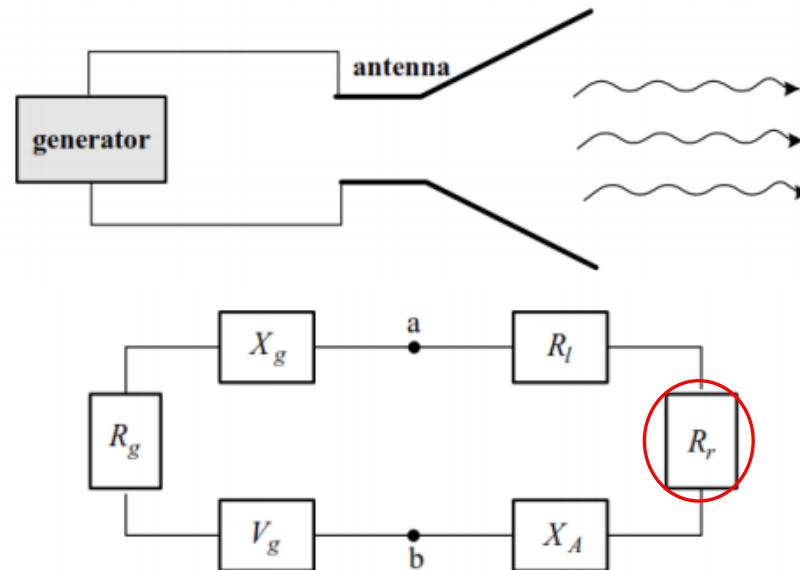


Linear Antenna (dipole)

Radiation Resistance

$$\begin{aligned}
 R_r &= \frac{2P_{\text{rad}}}{|I_0|^2} = \frac{\eta}{2\pi} \{ C + \ln(kl) - C_i(kl) \\
 &\quad + \frac{1}{2} \sin(kl) \times [S_i(2kl) - 2S_i(kl)] \\
 &\quad + \frac{1}{2} \cos(kl) \times [C + \ln(kl/2) + C_i(2kl) - 2C_i(kl)] \}
 \end{aligned}$$

where $C = 0.5772$ (Euler's constant) and $Ci(x)$ and $Si(x)$ are the cosine and sine integrals (see Appendix III)



Linear Antenna (dipole)

Input Impedance The $\lambda/2$ dipole

input resistance

$$R_{rad} = \frac{2W_{rad}}{I_m^2} = 73.1280 \Omega$$

input impedance

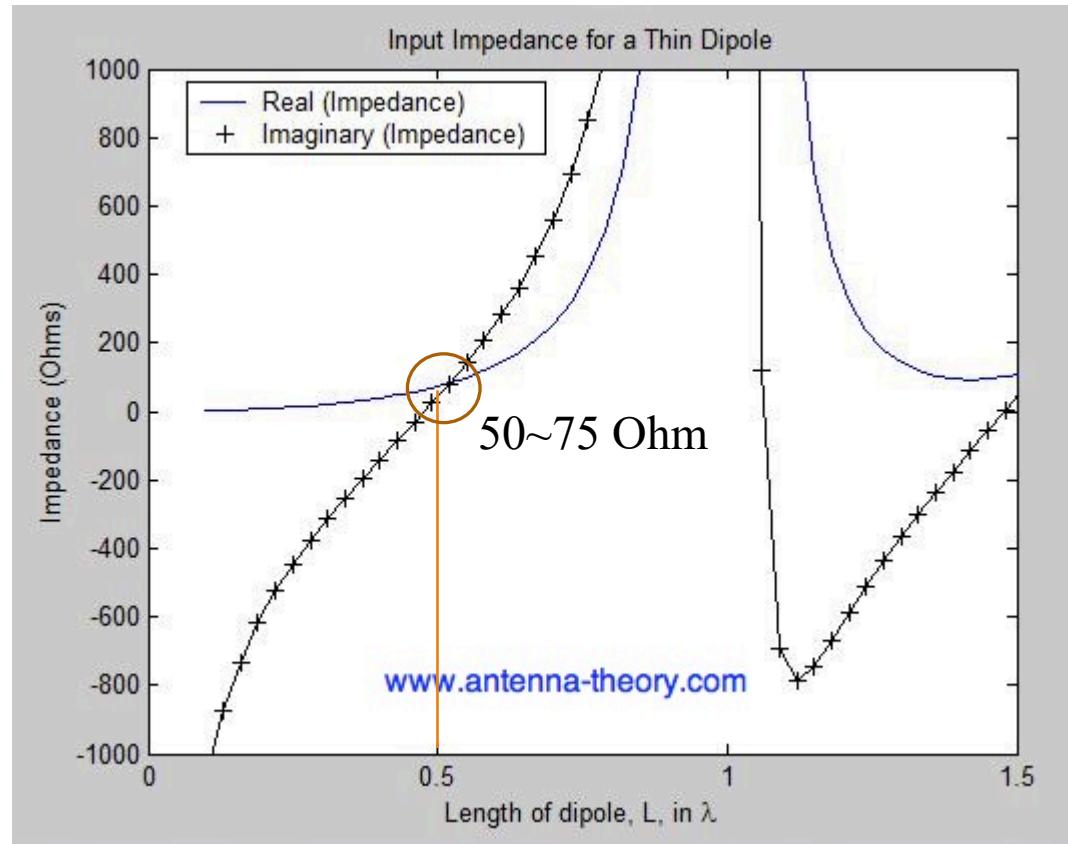
$$Z_{dipole} = 73 + j42.5 \Omega.$$

at a slightly lower frequency the inductive reactance can be cancelled out

$$Z_{dipole} = 70 + j0 \Omega,$$

Linear Antenna (dipole)

Input Impedance The $\lambda/2$ dipole

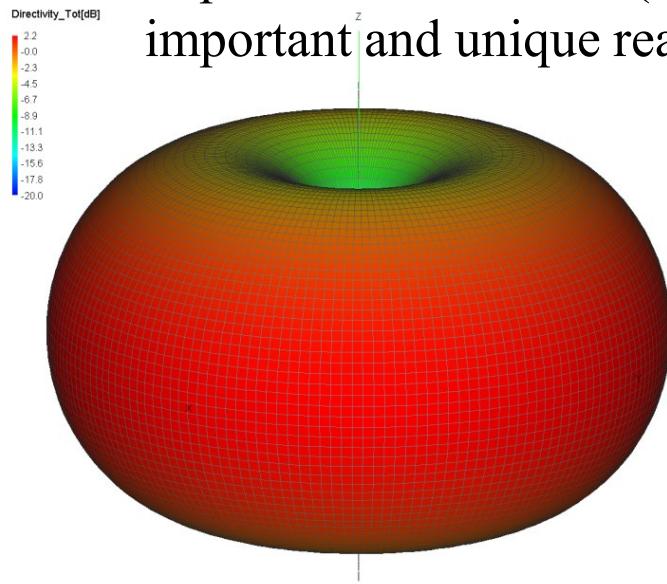


Input impedance as a function of the length (L) of a dipole antenna.

Linear Antenna (dipole)

The $\lambda/2$ dipole

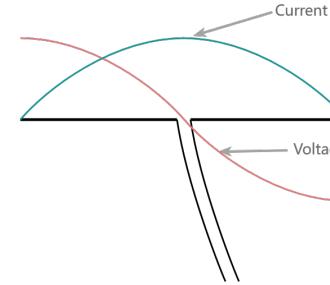
- most common linear antenna
- omnidirectional in the H-plane, good for applications as mobile communications. reasonable directivity (2.15 dBi) - larger than short dipoles
- input impedance $73\ \Omega$ (not sensitive to the radius)
- well matched with a standard transmission line of characteristic impedance 75 or $50\ \Omega$ (with $VSWR < 2$). This is probably the most important and unique reason to use it.



Video – Dipole Antenna
3'56"

Practical Tips

The $\lambda/2$ dipole



Use balanced feeder or balun: The dipole antenna is a balanced antenna. It is therefore necessary to **use a balanced feeder**, or if coaxial feeder needs to be used, then a balun must be used – there are several types that can easily be constructed.

Half wave dipole is not a half wave: A half wave dipole antenna is not the same length as a half wavelength in free space. End effects mean that the **actual length required is slightly shorter**.

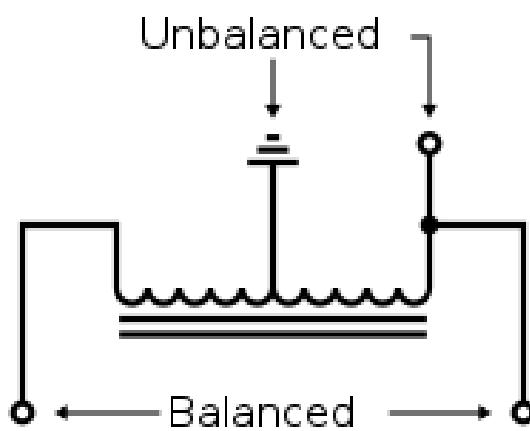
Current maximum sections: It can be shown that the areas of the antenna where the current is a maximum contribute most to the radiation / reception. To ensure the most effective operation, these areas should **be free from obstacles and be given the optimum position for radiation**. This is most applicable for antennas used for lower frequencies where lengths are much longer. For VHF and UHF antennas, where antennas are very much shorter, the whole length of the dipole is likely to have a similar ‘view’. For HF antennas, sometimes the centre of the antenna may be kept higher than the ends and therefore have a better chance of radiating a signal better.

Voltage maxima at the antenna ends: The points of maximum voltage are at the ends of the antenna. If used for transmitting make sure these cannot be accidentally touched, and also ensure they are **adequately insulated**. This is important when using wire antennas where the ends are used as anchor points. These should **also be away from nearby objects** that can act to absorb power and detune the antenna.

Feed a Dipole: Balun

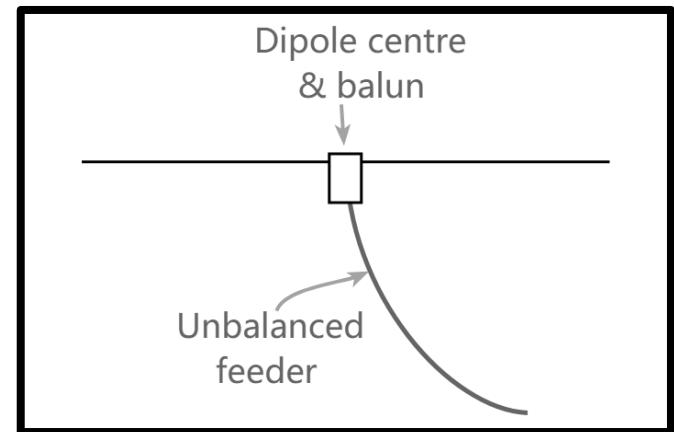
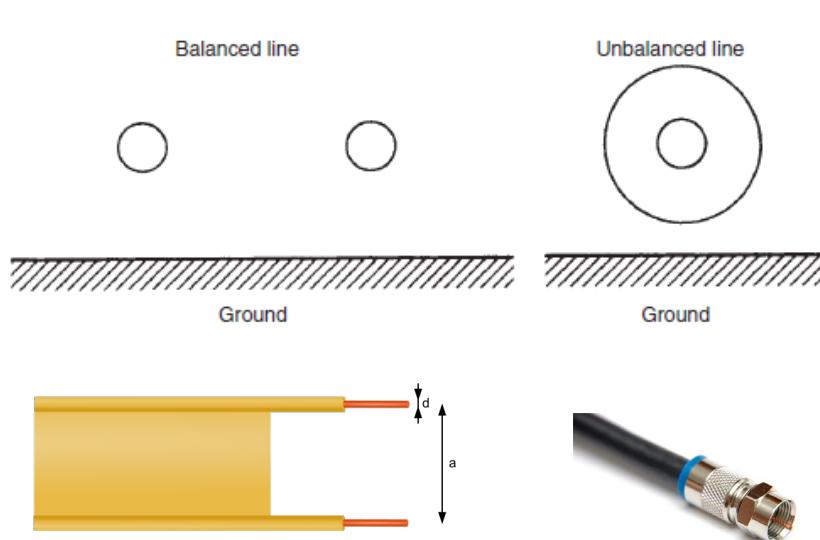
“balance” unbalanced systems

those where power flows from an unbalanced line to a balanced line (hence, balun derives from *balance* to *unbalanced*).



- two conductors of the same type
- both of which have equal impedances along their lengths
- equal impedances to ground and to other circuits
- good rejection of common-mode noise
- reduction of interference when fed to a differential device such as a transformer or differential amplifier

Feed a Dipole: Balun

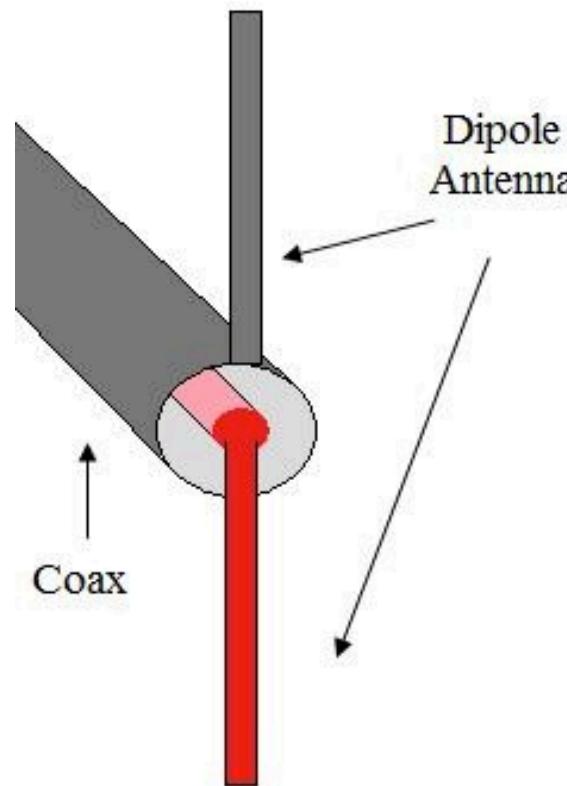


Note: Voltage to the ground is different

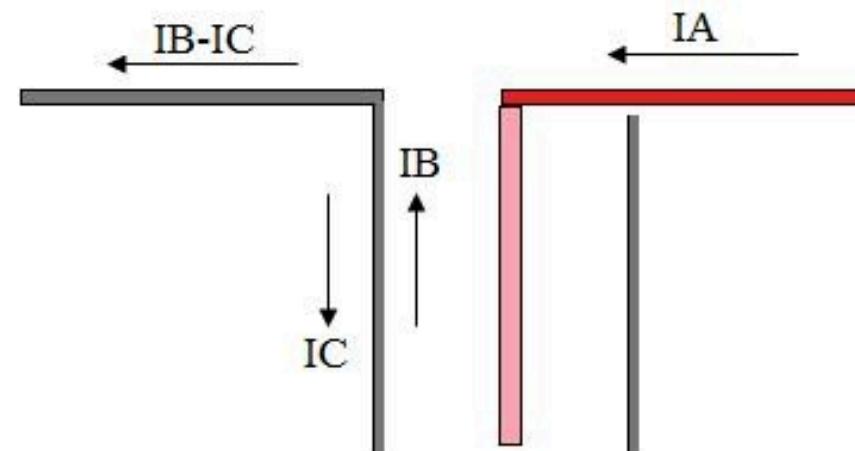


Feed a Dipole: Balun

www.antenna-theory.com



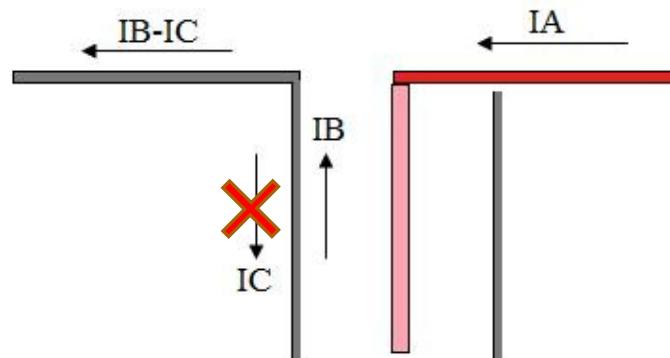
(a) Physical Model



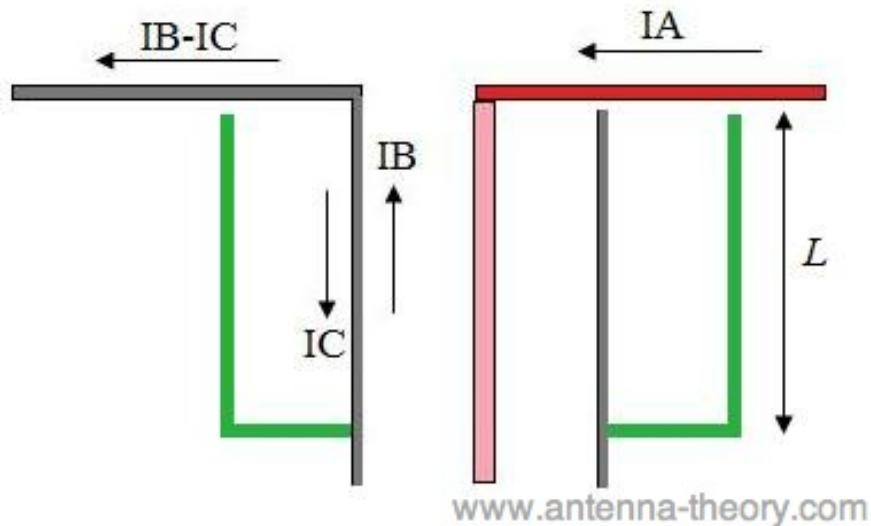
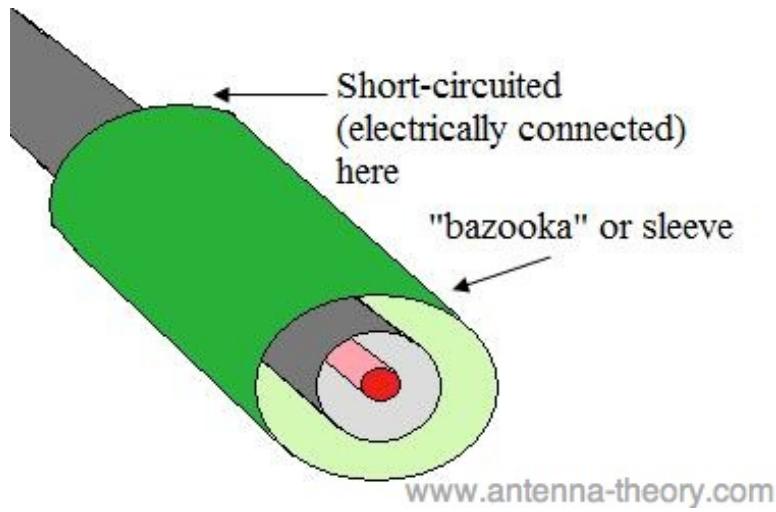
(b) Equivalent View Showing
Current Paths

Feed a Dipole: Balun

A balun forces an unbalanced transmission line to properly feed a balanced component. This would be done by forcing IC to be zero somehow - this is often called choking the current or a current choke.



Bazooka or Sleeve Balun.

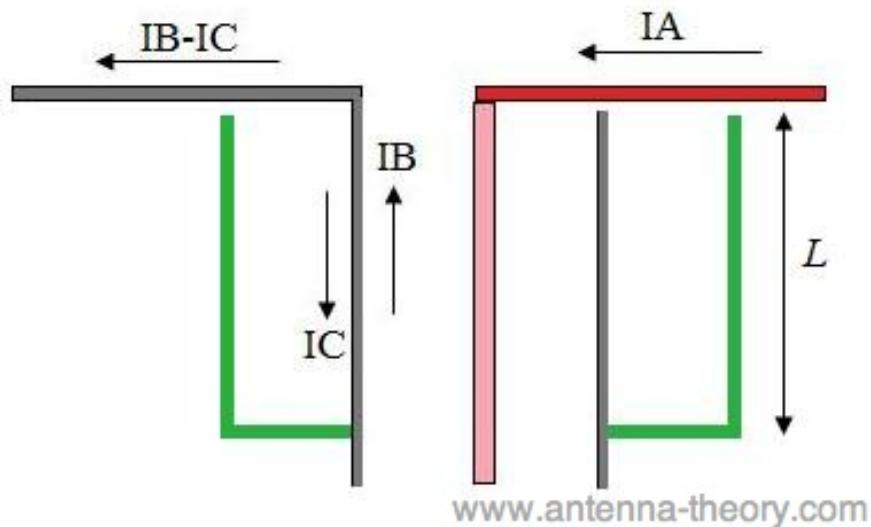
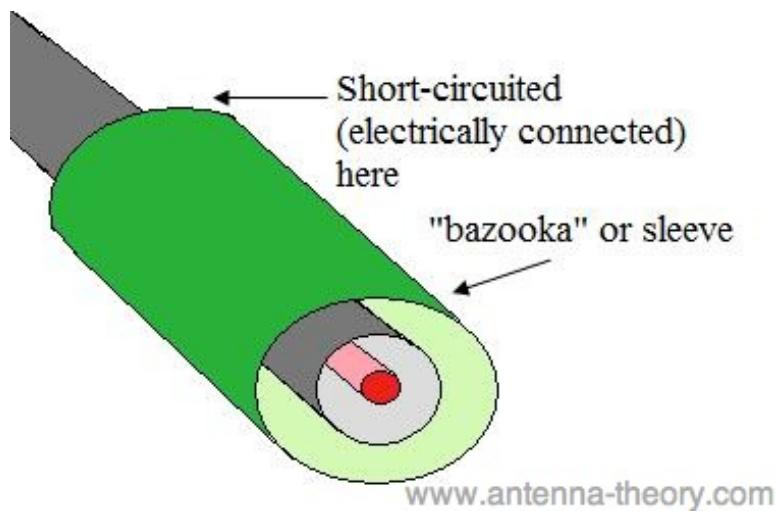


$\lambda_0/4$ parallel line – very large impedance:

- Suppression of the currents IC on the outer shield
- No interference with the antenna input impedance

"Balun transforms the balanced input impedance of the dipole to the unbalanced impedance of the coaxial line such that there is no net current on the outer conductor of the coax."

Bazooka or Sleeve Balun.

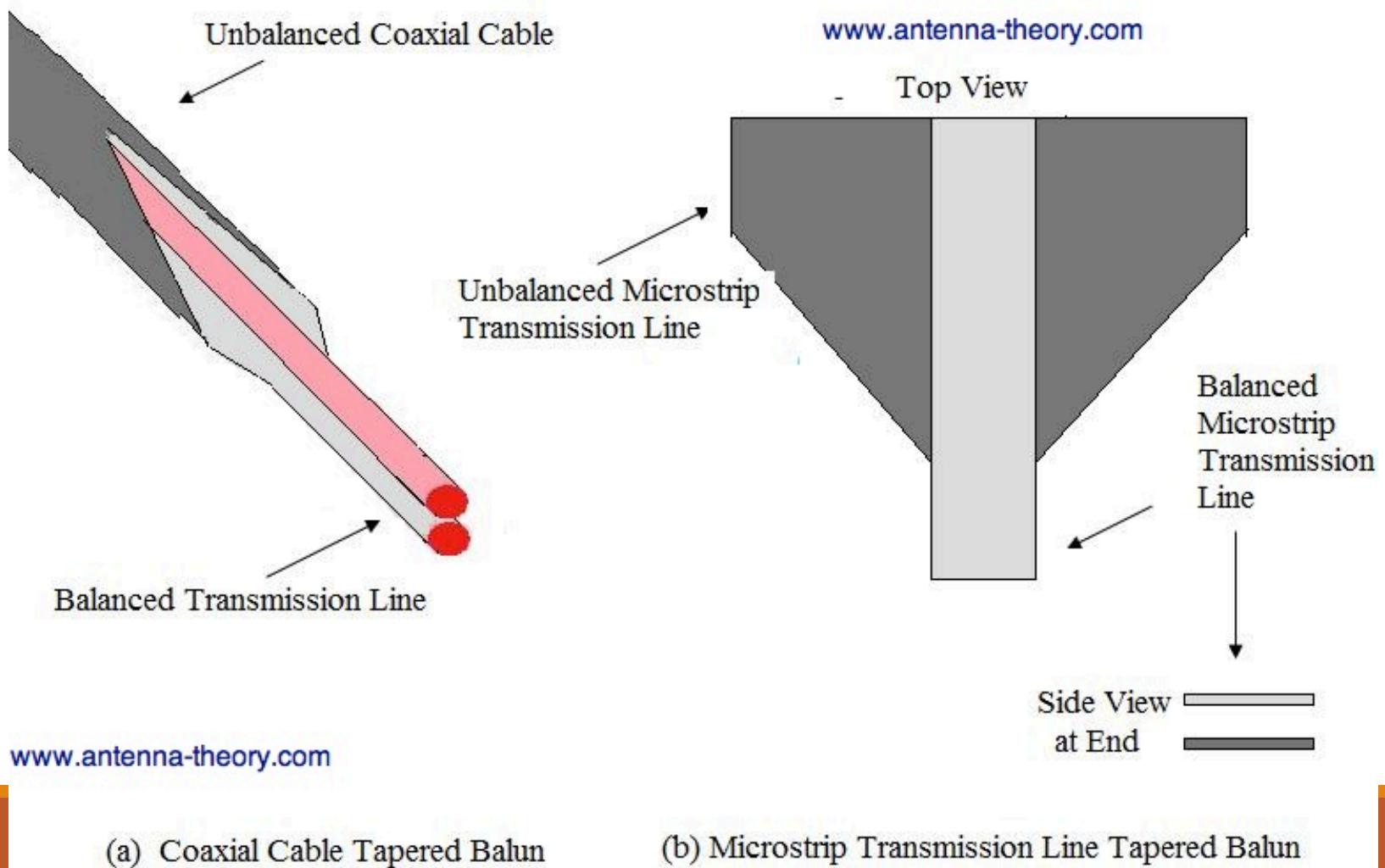


$\lambda_0/4$ parallel line – very large impedance:

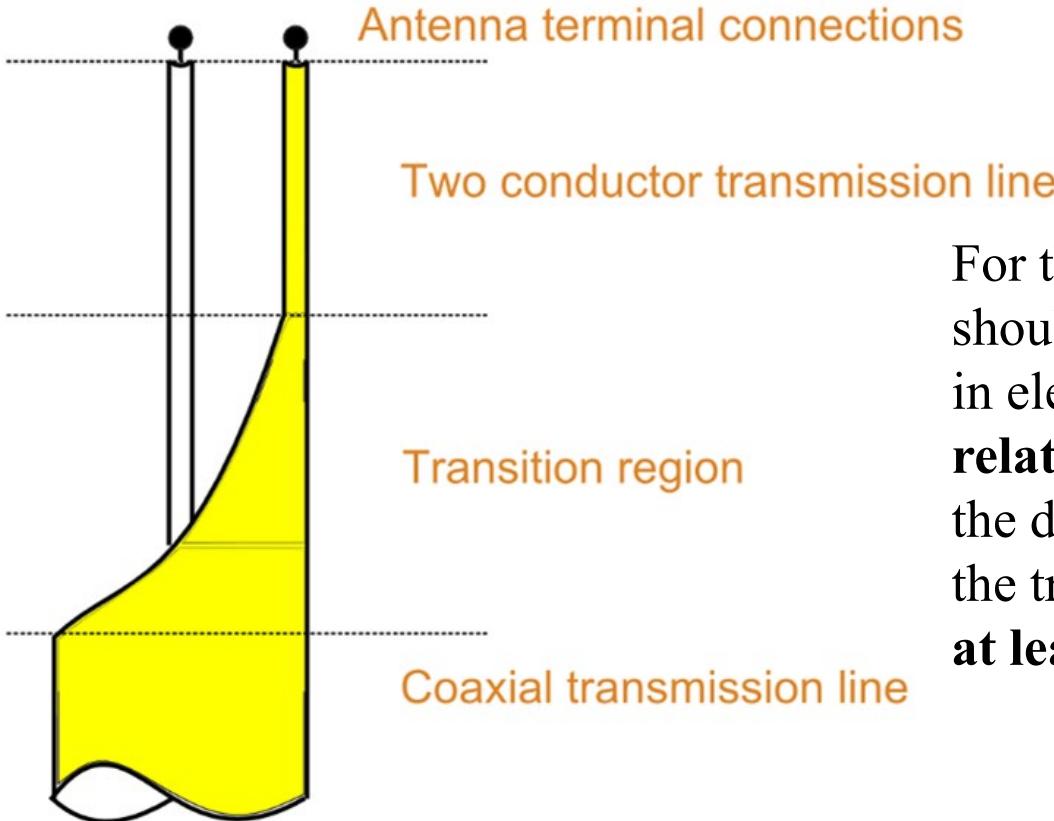
- Suppression of the currents IC on the outer shield
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"Balun transforms the balanced input impedance of the dipole to the unbalanced impedance of the coaxial line such that there is no net current on the outer conductor of the coax."

Tapered Balun.

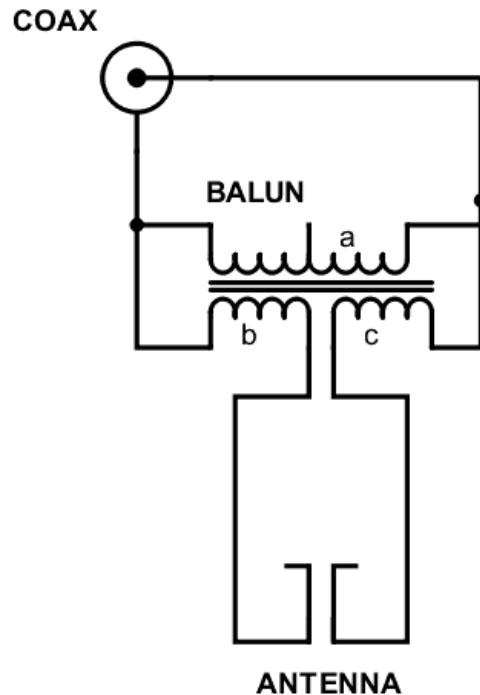


Tapered Balun.



For the **tapered transition** to work, it should **vary slowly**. The term “slowly” in electronics always mean “**slow relative to a wavelength**”. Hence, for the desired frequency of operation, the transition should take place over **at least a few wavelengths in length**.

Balun (75Ohm-300Ohm)



© Azega.com

The early antennas used in radio and television systems were often designed to work with a 300 ohm impedance. This was because at the time, the cables used to transmit signals from the antenna to the receiver had a characteristic impedance of 300 ohms.

Linear Antenna (Monopole)

monopole antenna

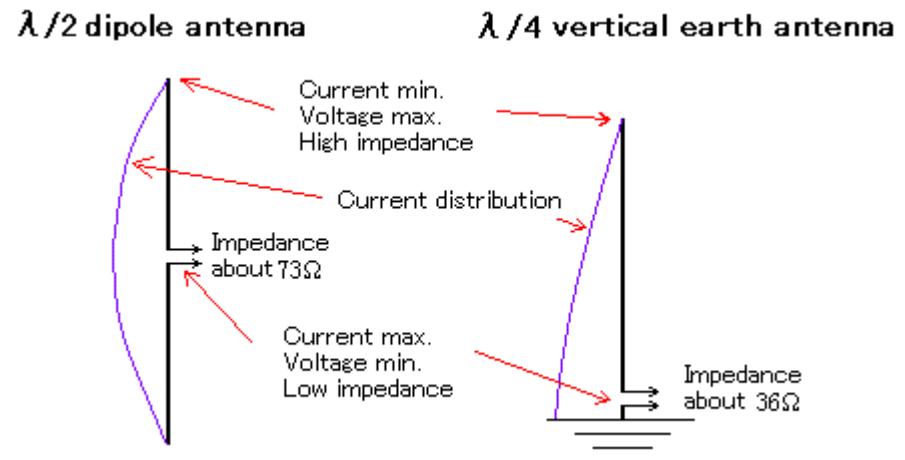
a class of radio antenna consisting of a **straight rod-shaped conductor**, often mounted perpendicularly over some type of **conductive surface**, called a *ground plane*.



[Mast radiator](#) monopole antenna used for broadcasting. AM radio station WARE, Warren, Massachusetts, US.

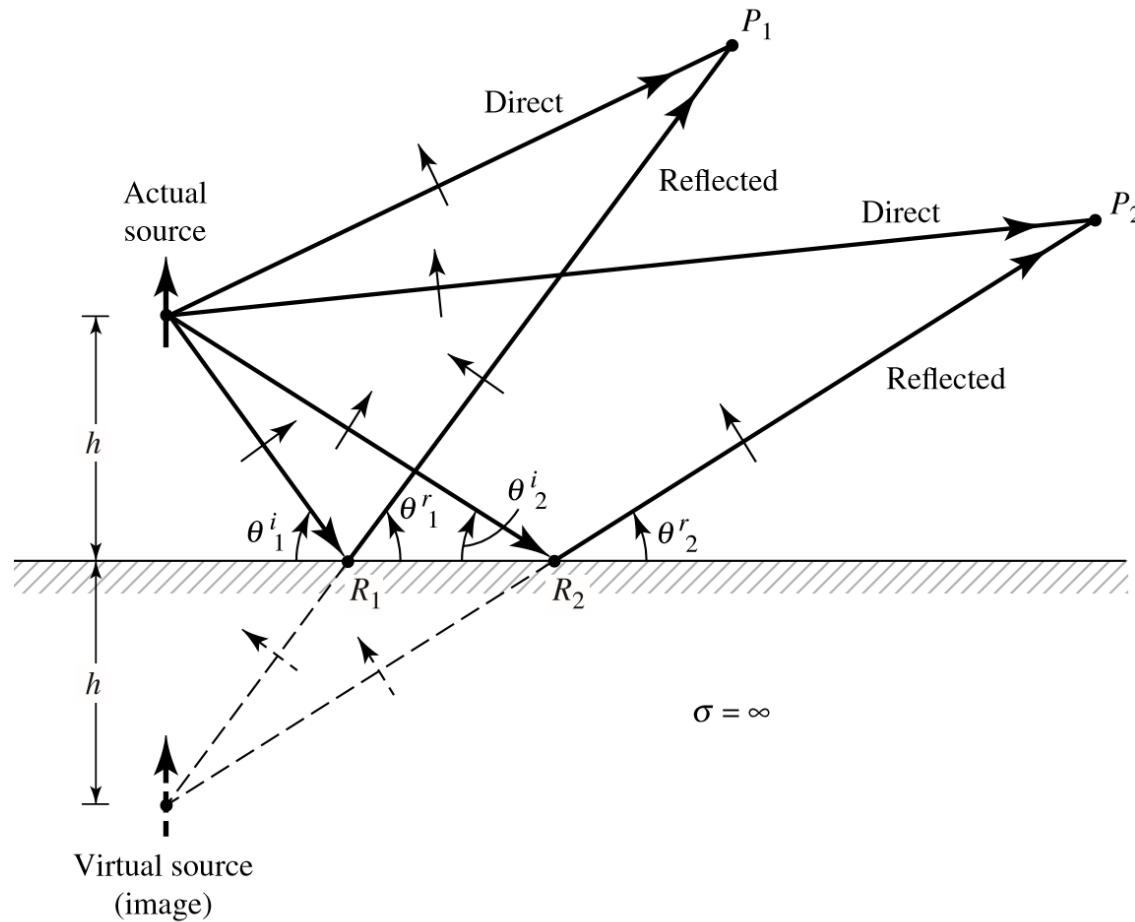
Linear Antenna (Monopole)

- The **driving signal** (from the transmitter) or **output signal** to the receiver **between the lower end of the monopole and the ground plane**.
- One side of the antenna feedline attached to **the lower end of the monopole**, and the other side attached to the **ground plane** (often the Earth).
- This contrasts with a **dipole antenna**:
 - Consists of two identical rod conductors
 - **signal** from the transmitter applied **between the two halves of the antenna**.



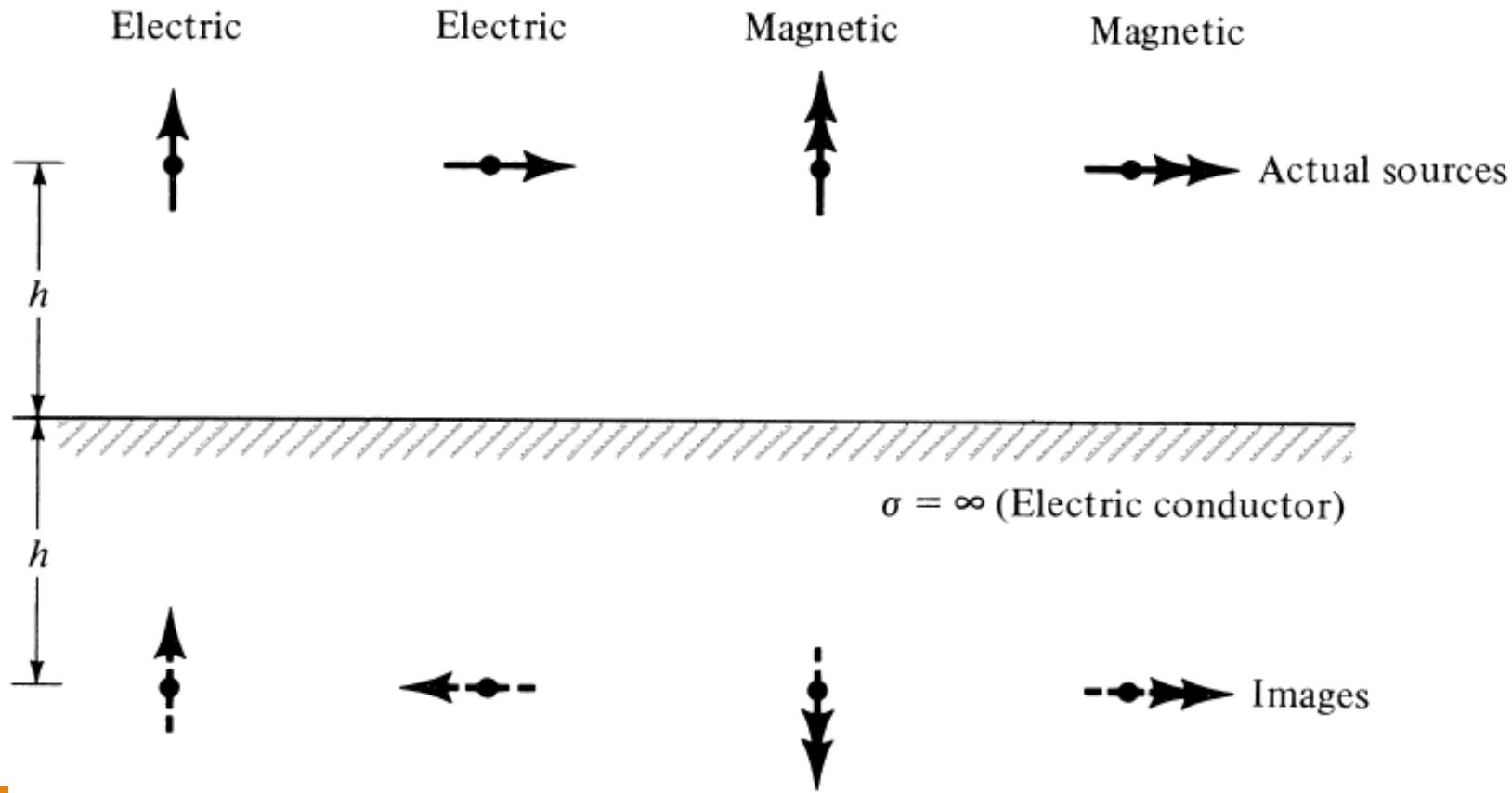
Linear Antenna (Monopole)

vertical current above PEC ground • Method of images



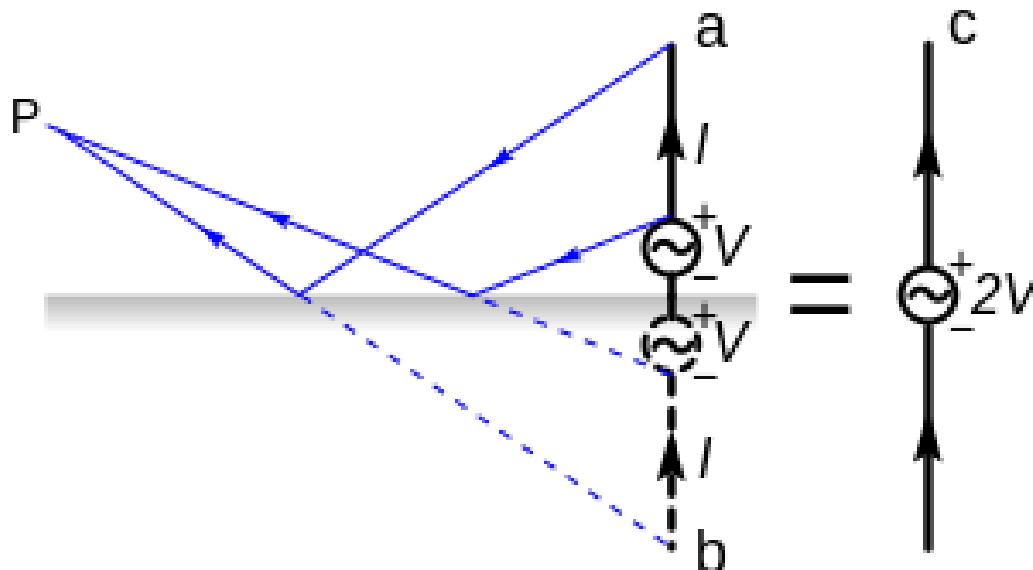
Linear Antenna (Monopole)

vertical current above PEC ground • Method of images



Linear Antenna (Monopole)

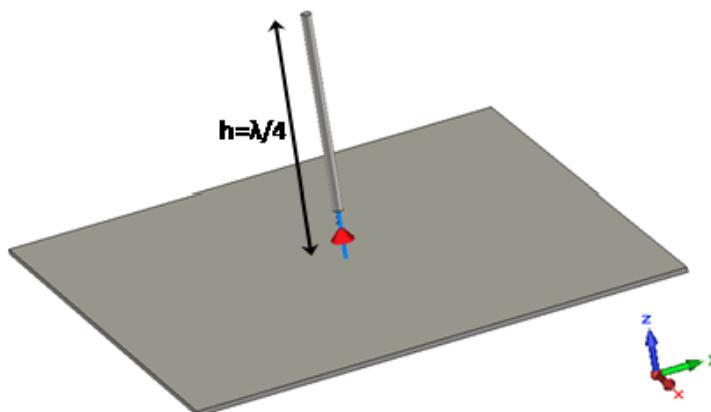
vertical current above PEC ground • Method of images



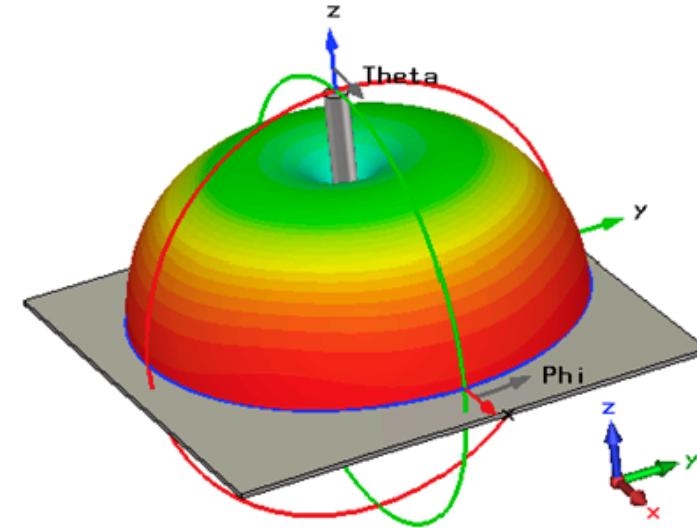
the monopole antenna has the **same radiation pattern over perfect ground** as a **dipole** in free space with **twice the voltage** above the ground plane

The monopole antenna fields **below the ground plane** are **zero**.

Linear Antenna (Monopole)



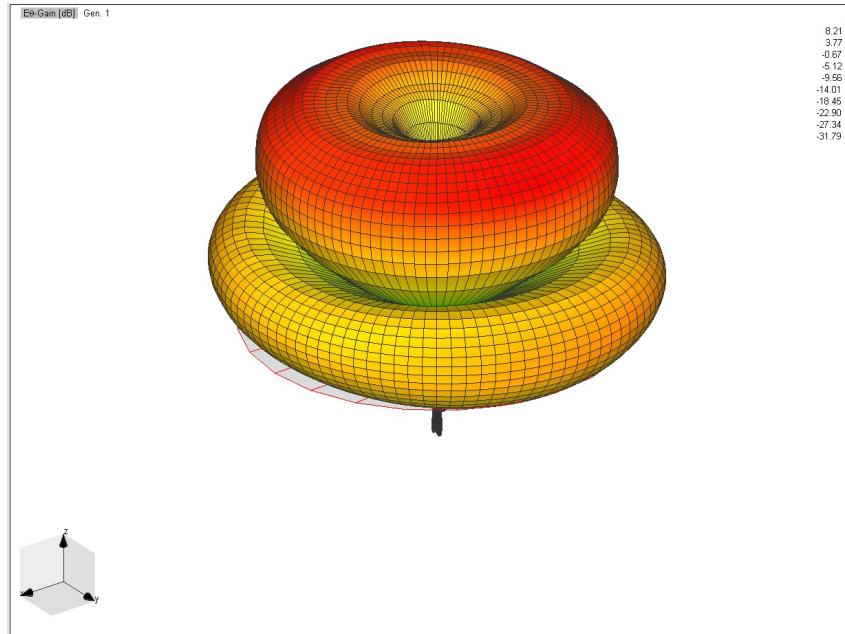
(a)



(b)

Monopole antennas up to $1/4$ wavelength long have a single “lobe”, with field strength declining monotonically from a maximum in the horizontal direction to zero in the vertical direction.

Linear Antenna (Monopole)



Radiation pattern of 3/2 wavelength monopole. **Longer monopoles ($> 1/4$ wavelength) have more complicated patterns with several conical “lobes” (radiation maxima) directed at angles into the sky.**

Linear Antenna (Monopole)

Monopole length l	$\lambda/20$	$\lambda/4$	$\lambda/2$	$3/4$
Current distribution				
Radiation pattern				
Directivity	3.0 or 4.76 dBi	3.28 or 5.15 dBi	4.8 or 6.8 dBi	About 4.6
HPBW	45°	39°	23.5°	NA
Input impedance	$R: \text{very small } (\sim 1\Omega)$ $jX: \text{capacitive}$	$R: \sim 37 \Omega$ $jX: \sim 0 \Omega$	$R: \text{very large}$ $jX: \sim 0 \Omega$ for thin dipole	$R: \sim 50 \Omega$ $jX: \sim 0 \Omega$ for thin dipole
Note	jX sensitive to the radius	$R+jX$ not sensitive to the radius	$R+jX$ sensitive to the radius	$R+jX$ sensitive to the radius

radiates only into **the space above the ground plane** (half the space of a dipole antenna)
a gain of twice (3 dBi over) the gain of a similar dipole antenna
a radiation resistance half that of a dipole

For infinite ground plane

$$Z_{monopole} = \frac{1}{2} Z_{dipole} = 36.5 + j21.25 \Omega$$

Antenna can be tuned by shortening the radiator.

Linear Antenna (Monopole)



Using arms instead of real ground, R_{in} could be made close to 50Ω and thus matched to coaxial cable.

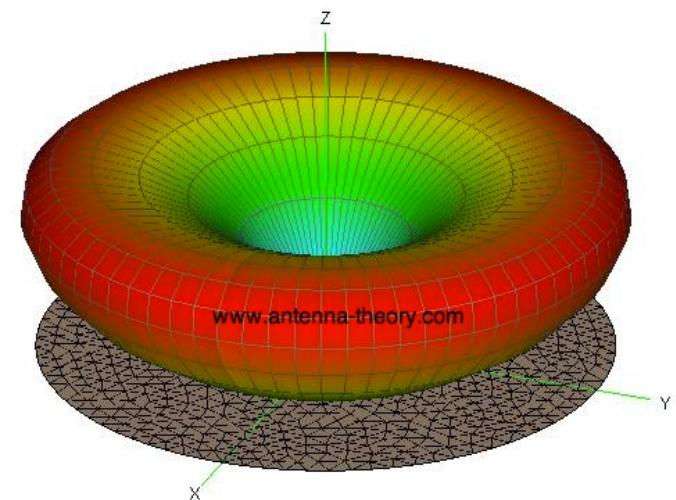
VHF ground plane antenna, a type of monopole antenna used at high frequencies. The three conductors projecting downward are the ground plane

Linear Antenna (Monopole)

Effects of a Finite Size Ground Plane on the Pattern

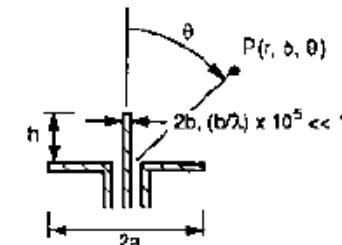
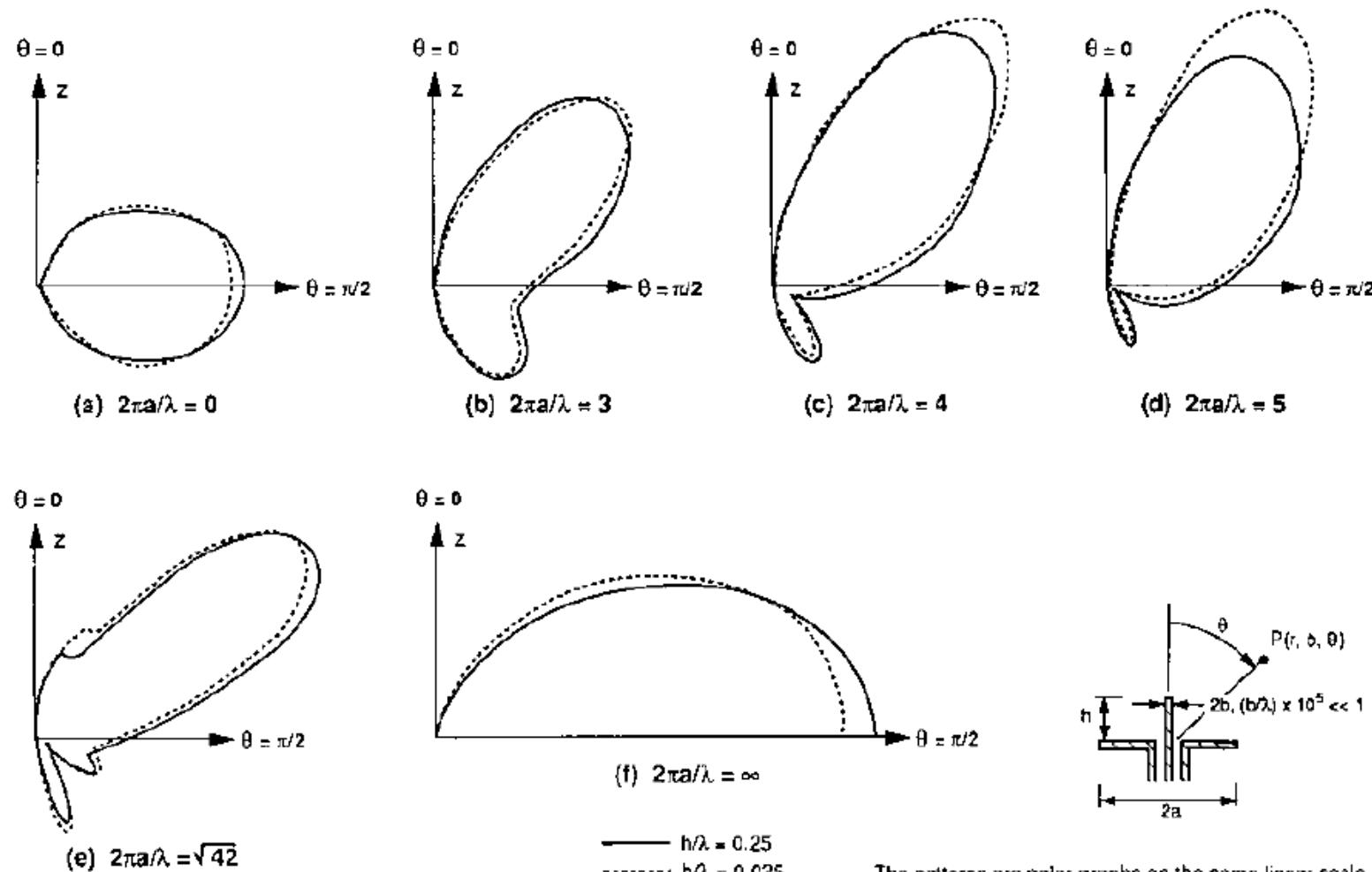
radiation pattern radiates in a “skewed” direction, away from the horizontal plane

An example of the radiation pattern for a quarter-wavelength monopole antenna (oriented in the +z-direction) on a ground plane with a diameter of 3 wavelengths



Linear Antenna (Monopole)

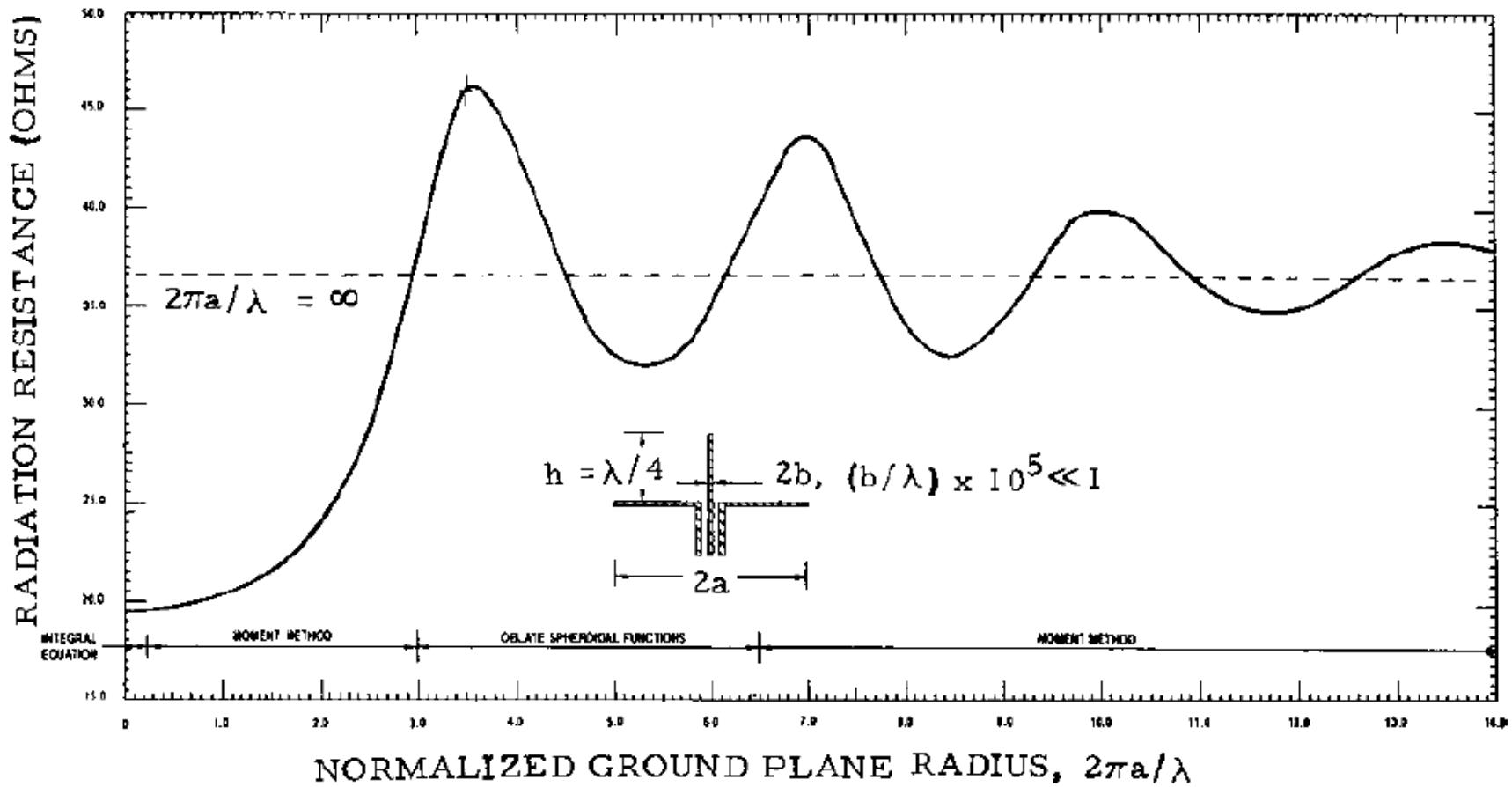
Effects of a Finite Size Ground Plane

 λ 

In general, the larger the ground plane is, the lower this direction of maximum radiation; as the ground plane approaches infinite size, the radiation pattern approaches a maximum in the x-y plane.

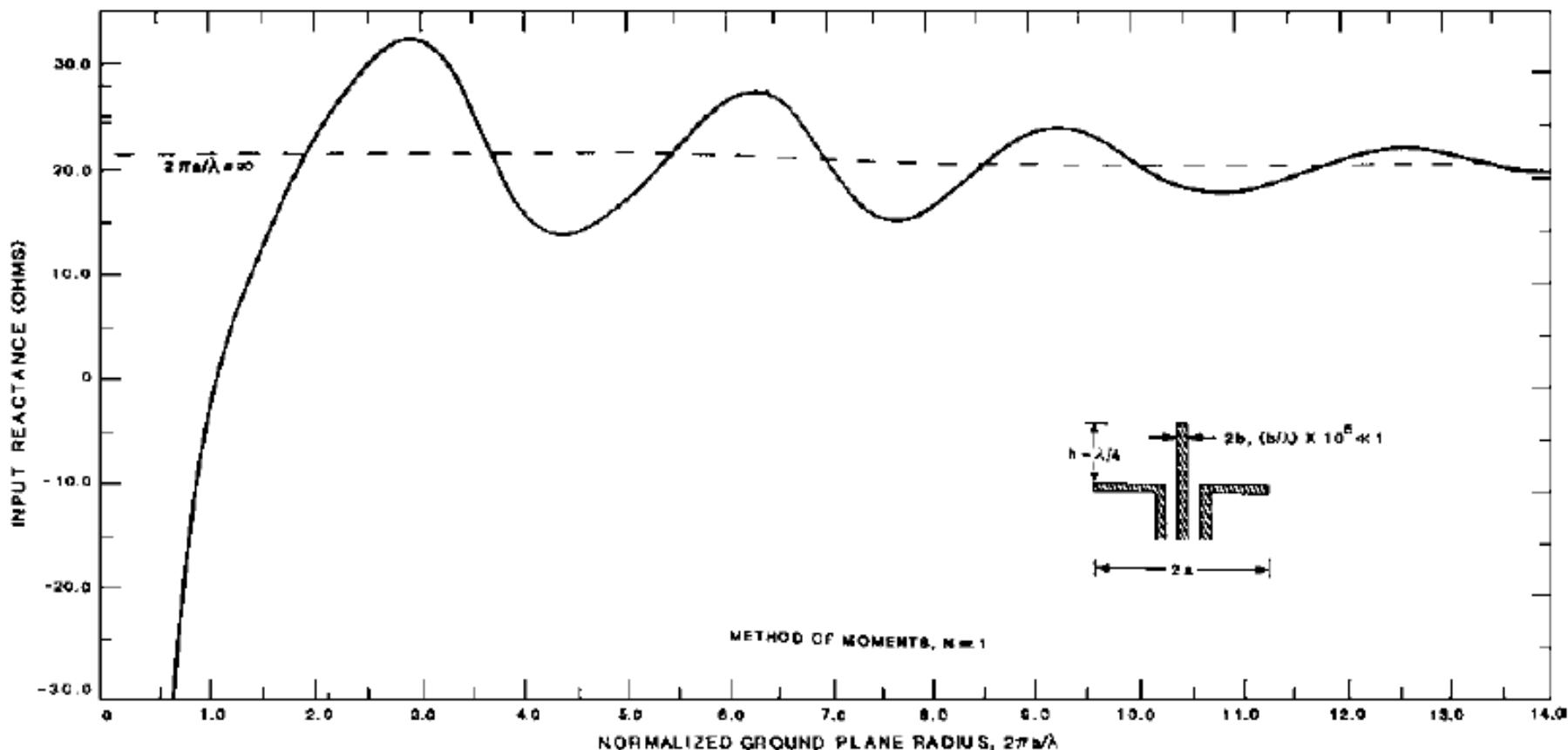
Linear Antenna (Monopole)

Effects of a Finite Size Ground Plane



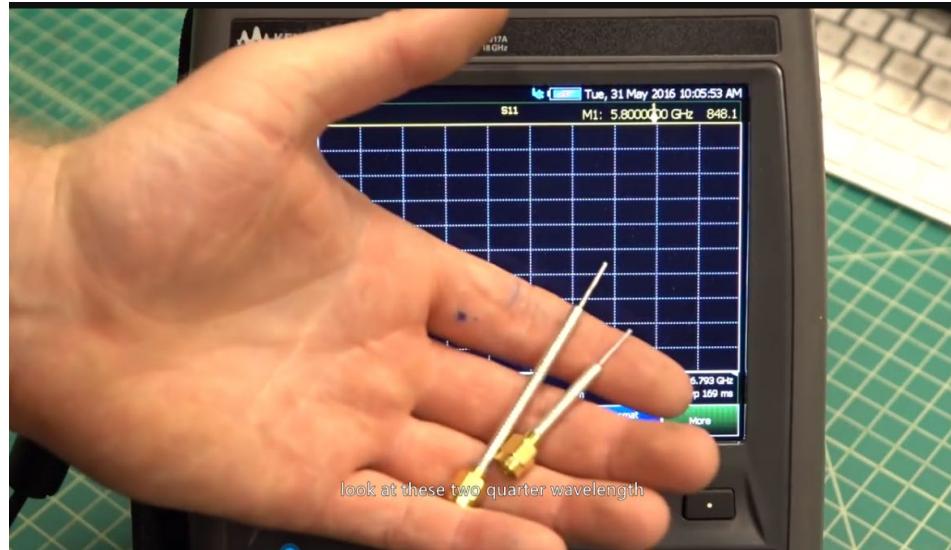
Linear Antenna (Monopole)

Effects of a Finite Size Ground Plane



Linear Antenna (Monopole)

[Monopole Antenna Test video](#) 5'42



Homework

Derive H and E field from Vector Potential on [page 19](#)

A 1-m-long dipole is excited by a 1-MHz current with an amplitude of 12 A. What is the average power density radiated by the dipole at a distance of 5 km in a direction that is 45° from the dipole axis?

Calculate radiation resistance of a 0.25-meter dipole antenna working at 600MHz

Plot radiation pattern for

half wavelength dipole

quarter wavelength monopole (infinite ground)

Calculate directivity and HPBW for each antenna

Homework 2

Please discuss if the following Chat GPT answer has any problem.



Is half-wave dipole antenna efficiency around 73%?



Yes, the efficiency of a half-wave dipole antenna is typically around 73%. This is a theoretical limit for the efficiency of a dipole antenna and is based on the calculation of the radiation resistance of the antenna.



The radiation resistance is a measure of the power that is radiated by the antenna, and it depends on the physical dimensions of the antenna. For a half-wave dipole antenna, the radiation resistance is approximately 73 ohms. This means that if the dipole is fed with 100 watts of power, only about 73 watts will be radiated by the antenna, and the remaining 27 watts will be lost due to various factors such as resistive losses in the antenna and ground losses.

The 73% efficiency of a half-wave dipole antenna is often referred to as the "dipole's fundamental efficiency" or the "antenna factor". Achieving this efficiency in practice requires careful design and construction of the antenna and its matching network, as well as proper installation and placement to minimize losses.

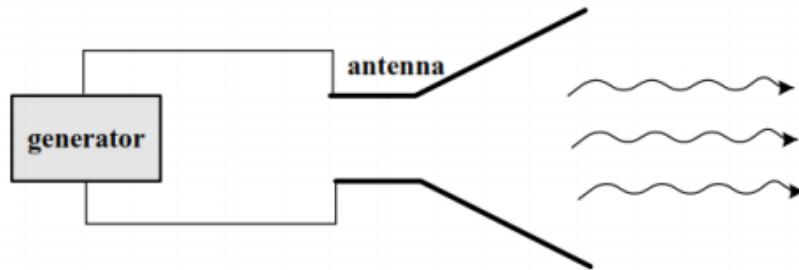
It's worth noting that the actual efficiency of a half-wave dipole antenna can vary depending on a number of factors, including the diameter of the wire, the height of the antenna above the ground, the type of feedline used, and the frequency of operation. Nevertheless, a well-designed half-wave dipole antenna can typically achieve an efficiency of around 73%.

Input Resistance

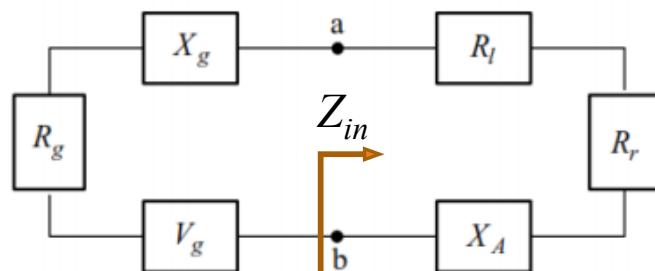
Assume lossless ($R_l=0$) and the imaginary part of the antenna impedance

$$X_A = \frac{\eta}{4\pi} \left\{ 2S_i(kl) + \cos(kl)[2S_i(kl) - S_i(2kl)] - \sin(kl) \left[2C_i(kl) - C_i(2kl) - C_i\left(\frac{2ka^2}{l}\right) \right] \right\}$$

where $C = 0.5772$ (Euler's constant) and $Ci(x)$ and $Si(x)$ are the cosine and sine integrals (see Appendix III)



$$\frac{|I_{in}|^2}{2} R_{in} = \frac{|I_0|^2}{2} R_r$$



$$R_{in} = \left[\frac{I_0}{I_{in}} \right]^2 R_r$$

Infinitesimal Dipole Antenna Vector Potential

Spherical Coordinates

$$\nabla \psi = \hat{\mathbf{a}}_r \frac{\partial \psi}{\partial r} + \hat{\mathbf{a}}_\theta \frac{1}{r} \frac{\partial \psi}{\partial \theta} + \hat{\mathbf{a}}_\phi \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi}$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\nabla \times \mathbf{A} = \frac{\hat{\mathbf{a}}_r}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right] + \frac{\hat{\mathbf{a}}_\theta}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right]$$

$$+ \frac{\hat{\mathbf{a}}_\phi}{r} \left[\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right]$$