4-11. Consider a time-invariant block fading channel with frequency response

$$H(f) = \begin{cases} 1 & f_c - 20\text{ MHz} \le f < f_c - 10\text{ MHz}, \\ .5 & f_c - 10\text{ MHz} \le f < f_c, \\ 2 & f_c \le f < f_c + 10\text{ MHz}, \\ .25 & f_c + 10\text{ MHz} \le f < f_c + 20\text{ MHz}, \\ 0 & \text{else}, \end{cases}$$

for $f > 0$ and $H(-f) = H(f)$. For a transmit power of 10 mW and a noise PSD of .001 μW per Hertz, find the optimal power allocation and corresponding Shannon capacity of this channel.

$\bar{S} = 10\,\text{mW}$. $N_0 = 0.001\,\mu W/Hz$, $B = 10\,MHz$.

$\therefore \gamma_i = \frac{|H_i|^2 \bar{S}}{N_0 B}$

$$\therefore \begin{cases} \gamma_1 = 1 \\ \gamma_2 = 0.25 \\ \gamma_3 = 4 \\ \gamma_4 = 0.0625 \end{cases}$$

$$\because \sum_{j=1}^{N-1} \frac{1}{\gamma_0} - \frac{1}{\gamma_j} = 1 \Rightarrow \frac{N-1}{\gamma_0} = 1 + \sum_{j=1}^{N-1} \frac{1}{\gamma_j}$$

① assume $\gamma_0 < 0.0625$

$$\frac{4}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{0.25} + \frac{1}{4} + \frac{1}{0.0625}\right) = 22.25$$

$\Rightarrow \gamma_0 = 0.1798 > 0.0625$. Wrong.

② assume $0.0625 < \gamma_0 < 0.25$

$$\frac{3}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{0.25} + \frac{1}{4}\right) = 6.25$$

$\Rightarrow \gamma_0 = \frac{3}{6.25} = 0.48 > 0.25$. Wrong

③ assume $0.25 < \gamma_0 < 1$.

$$\frac{2}{\gamma_0} = 1 + \left(\frac{1}{1} + \frac{1}{4}\right) = 2.25 \Rightarrow \gamma_0 = \frac{8}{9} < 1$$

$$\therefore C = \sum_{i: \gamma_i \ge \gamma_0} B \log_2\left(\frac{\gamma_i}{\gamma_0}\right) = 10\left(\log_2\left(\frac{9}{2}\right) + \log_2\left(\frac{9}{8}\right)\right) = 23.4\,\text{Mbps}.$$

5-1 Using properties of orthonormal basis functions, show that if $s_i(t)$ and $s_j(t)$ have constellation points $\mathbf{s}_i$ and $\mathbf{s}_j$ (respectively) then

$$\|\mathbf{s}_i - \mathbf{s}_j\|^2 = \int_0^T (s_i(t) - s_j(t))^2 \, dt.$$

$\because$ $s_i(t) = \sum_{k=0}^{\infty} s_{ik} \phi_k(t)$

$s_j(t) = \sum_{k=0}^{\infty} s_{jk} \phi_k(t)$

$\therefore \int_0^T (s_i(t) - s_j(t))^2 dt = \int_0^T \left(\sum_{k=0}^{\infty} (s_{ik} - s_{jk}) \phi_k(t) \right)^2 dt = \int_0^T \sum_{k=0}^{\infty} (s_{ik} - s_{jk})^2 \phi_k^2(t) \, dt$

$= \sum_{k=0}^{\infty} (s_{ik} - s_{jk})^2 = \| s_i - s_j \|^2$

5-3. Consider a set of M orthogonal signal waveforms $s_m(t)$, for $1 \le m \le M$ and $0 \le t \le T$, where each waveform has the same energy $\mathcal{E}$. Define a new set of M waveforms as

$$s'_m(t) = s_m(t) - \frac{1}{M} \sum_{i=1}^{M} s_i(t), \quad 1 \le m \le M, \ 0 \le t \le T.$$

Show that the M signal waveforms $\{s'_m(t)\}$ have equal energy, given by

$$\mathcal{E}' = (M-1)\mathcal{E}/M. \qquad \frac{\mathcal{E}'}{\mathcal{E}} = \frac{M-1}{M}$$

What is the inner product between any two waveforms?

$\because \mathcal{E}' = \int_0^T s_m'^2(t) \, dt = \int_0^T \left(s_m(t) - \frac{1}{M} \sum_{i=1}^{M} s_i(t) \right)^2 dt$

$= \int_0^T s_m^2(t) \, dt - \int_0^T \frac{1}{M} s_m^2(t) \, dt = \mathcal{E} - \frac{1}{M} \mathcal{E} = \frac{M-1}{M} \mathcal{E}$

$\langle s_m'(t) . s_n'(t) \rangle = \int_0^T s_m'(t) s_n'(t) \, dt = \int_0^T \left(s_m(t) - \frac{1}{M} \sum_{i=1}^{M} s_i(t) \right) \left(s_n(t) - \frac{1}{M} \sum_{i=1}^{M} s_i(t) \right) dt$

$= -\int_0^T \frac{1}{M} s_m^2(t) \, dt = -\frac{\mathcal{E}}{M}$