Likelihood

Models for Socio-Environmental Data

Chris Che-Castaldo, Mary B. Collins, N. Thompson Hobbs

December 10, 2020

Why Likelihood?

- Maximum likelihood is an important statistical approach.
- Likelihood is a component of all Bayesian models.

Learning objectives for lecture and lab

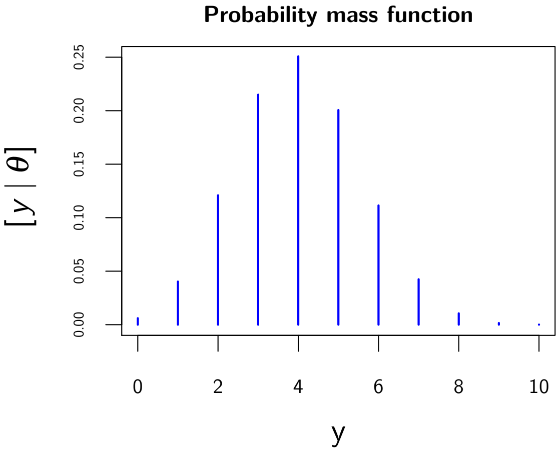
- Understand the concept of likelihood and its relationship to the probability of data conditional on parameters.
- Describe a likelihood profile and how it differs from a probability density function.
- Compose likelihoods for multiple parameters and multiple observations.

Course progression so far...

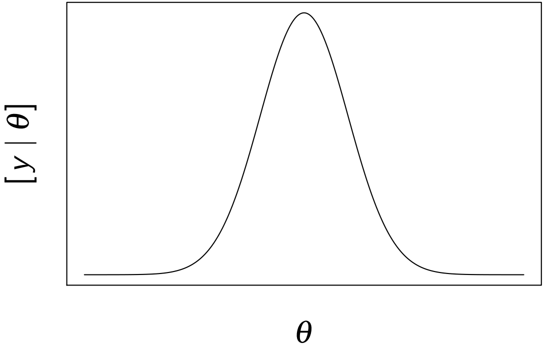
$$[\,y\,]$$

$$[\,y \mid \theta\,]$$

$$y \sim \text{binomial}(\,\theta, 10\,)$$



Inference from likelihood is based on $[y \mid \theta]$



- Likelihood allows us to compare alternative parameter values by calculating the probability (or probability density) of the data conditional on parameters $[y \mid \theta]$.

The key idea in likelihood

- In a probability mass or density function, the parameter θ is constant (fixed) and the outcomes $\boldsymbol{y}$ vary (these outcomes represent data we may observe). The function sums or integrates to 1 over its support.
- In a likelihood function, the data $\boldsymbol{y}$ are constant (fixed) and the parameter θ varies. We use $[y \mid \theta]$ to assess the likelihood of different values of θ in light of the data. In this case, the function does not sum or integrate to one over all possible values of the parameter.
- As you will see, all evidence based on likelihood is relative.

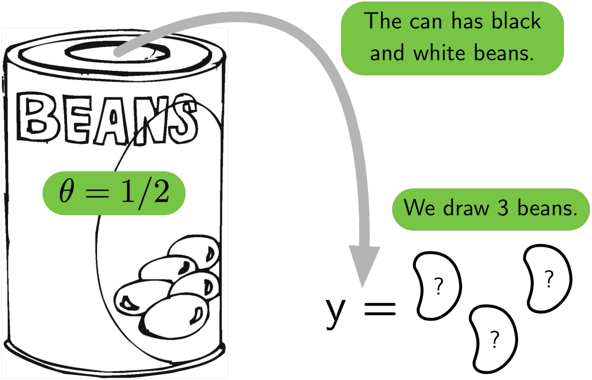
$$\underbrace{L(\theta \mid y)}_{\text{likelihood function}} \propto \underbrace{[y \mid \theta]}_{\text{PDF or PMF}}$$

Likelihood is *proportional* to probability or probability density

Discuss notation

$$
\begin{aligned}
L(\theta \mid y) &\propto [y \mid \theta] && (1)\\
L(\theta \mid y) &= c[y \mid \theta] && (2)\\
L(\theta \mid y) &= [y \mid \theta] && (3)
\end{aligned}
$$

The parameter is known and the data are random



- What are the possible outcomes?
- What probability mass function would you use to model theses data?
- What is the probability of each outcome?
- What is the sum of the individual probabilities?

The parameter is known and the data are random

We draw three beans from a can with equal numbers of white and black beans. The possible outcomes are:

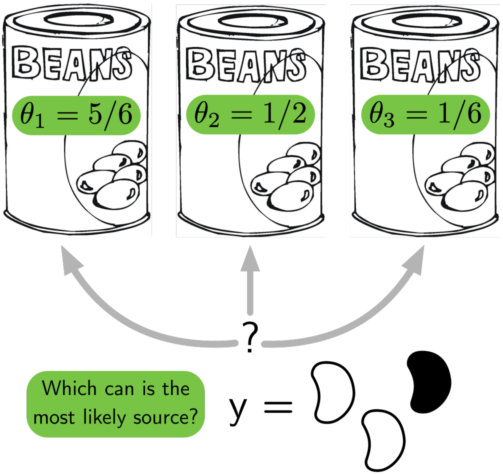
θ	number of white beans, y_i	$[y_i \mid \theta = ?]$
	$\sum_{i=1}^{4}[y_i \mid \theta = ?]$	?

The parameter is known and the data are random

We draw three beans from a can with equal numbers of white and black beans. The possible outcomes are:

θ	number of white beans, y_i	$[y_i \mid \theta = .5]$
.5	0	.125
.5	1	.375
.5	2	.375
.5	3	.125
	$\sum_{i=1}^{4}[y_i \mid \theta = .5]$	1

The data are known and the parameter is random



- What is the likelihood of each parameter value?

The data are known and the parameter is random

We have three hypothesized parameter values, $(5/6, 1/2, 1/6)$. Data in hand are 2 white beans on three draws. The likelihood of each parameter value is:

hypothesis, θ_i	number of white beans, y	$[y = ? \mid \theta_i]$
	$\sum_{i=1}^{3}[y = ? \mid \theta_i]$	?

The data are known and the parameter is random

We have three hypothesized parameter values, $(5/6, 1/2, 1/6)$. Data in hand are 2 white beans on three draws. The likelihood of each parameter value is:

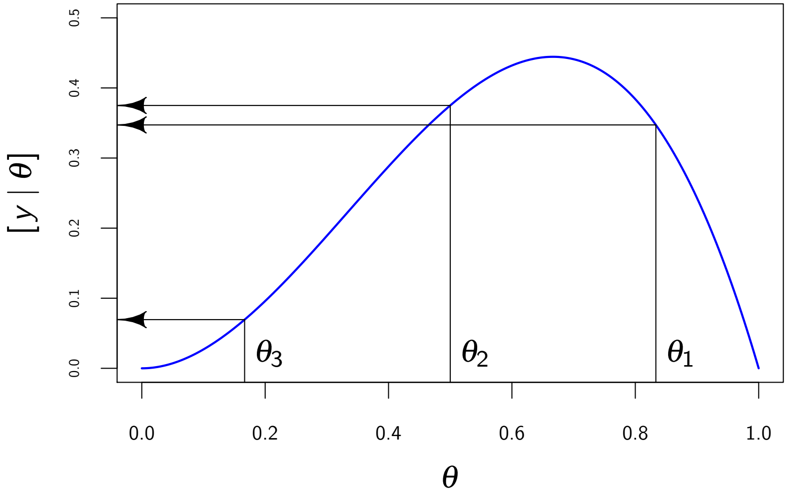
hypothesis, θ_i	number of white beans, y	$[y = 2 \mid \theta_i]$
$\theta_1 = 5/6$	2	0.35
$\theta_2 = 1/2$	2	0.38
$\theta_3 = 1/6$	2	0.069
	$\sum_{i=1}^{3}[y = 2 \mid \theta_i]$	0.79

The data are known and the parameter is random (rescaled)

We have three hypothesized parameter values, $(5/6, 1/2, 1/6)$. Data in hand are 2 white beans on three draws. The likelihood of each parameter value is:

hypothesis, θ_i	number of white beans, y	$[y = 2 \mid \theta_i]$
$\theta_1 = 5/6$	2	$0.35 \ / \ 0.38 = 0.92$
$\theta_2 = 1/2$	2	$0.38 \ / \ 0.38 = 1.0$
$\theta_3 = 1/6$	2	$0.069 \ / \ 0.38 = 0.18$
	$\sum_{i=1}^{3} [y = 2 \mid \theta_i]$	2.1

A likelihood profile: 2 white beans on 3 draws

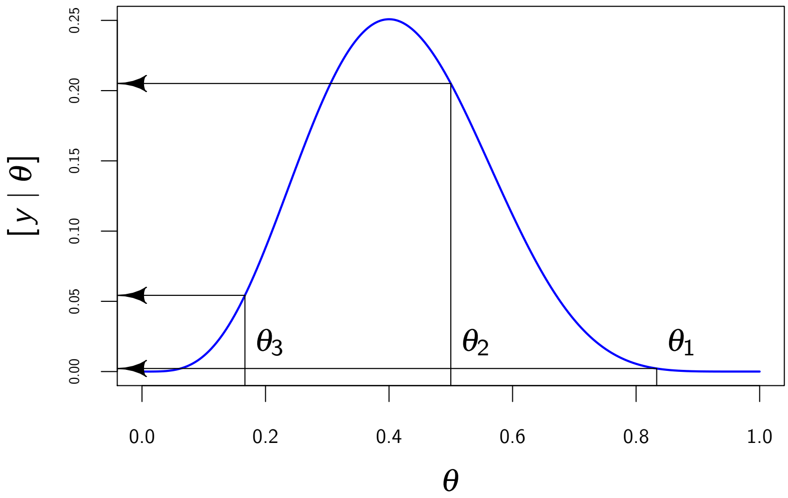


Class exercise: Can of beans

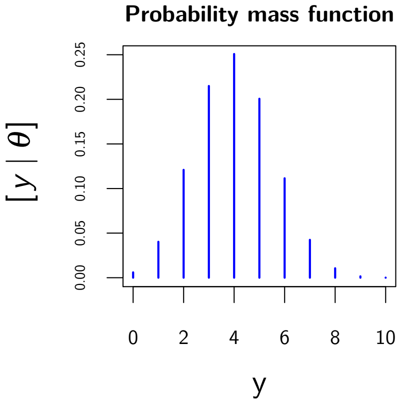
An adventurous person takes a draw of 10 beans from one of the cans where the identity of the can is unknown. Of the 10 beans drawn from the mystery can, 4 are white.

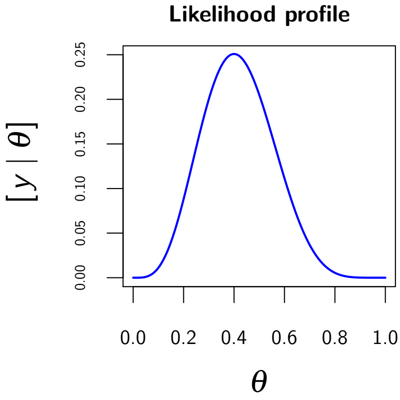
- Which of the three cans is the most likely to have produced this draw?
- How much more likely is this can than the other two?

A likelihood profile: 4 white beans on 10 draws



Likelihood vs Probability:





$$[y \mid \theta] = \binom{10}{y} \theta^y (1-\theta)^{n-y}$$

How do we compute likelihoods for multiple parameters?

- In the example we had a single parameter, θ, one set of observation, 4 successes on 10 draws, and a binomial likelihood.
- However, we could have θ a function of covariates. Procedurally, we could replace θ with $g(\beta, x_i)$, which would allow θ to vary with respect to x_i.
- More generally, we often wish to predict the mean response of an RV and then embed this mean function in a probability mass or density function, either directly or using moment matching.

$$
\begin{aligned}
\mu_i &= g(\alpha, x_i) \\
\mathrm{L}(\alpha, \sigma^2 \mid y_i) &\propto \underbrace{[y_i \mid \mu_i, \sigma^2]}_{\text{PDF or PMF}}
\end{aligned}
$$

Likelihood Surfaces

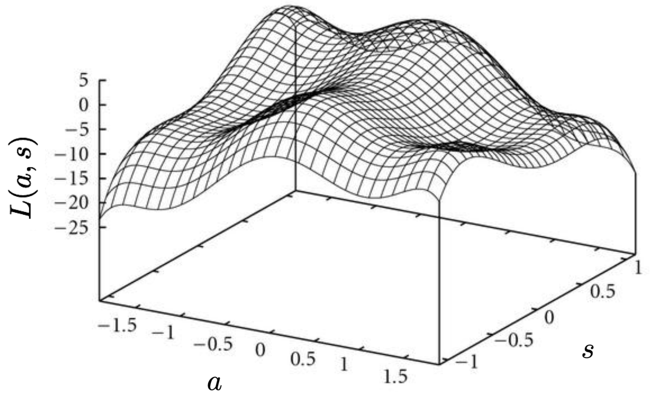


Figure courtesy of Sergiy Nesterko.

How do we compute likelihoods for multiple parameters and multiple data points?

The total likelihood is the product of the individual likelihoods, assuming the data are *conditionally independent*:

$$\mathsf{L}(\boldsymbol{\mu}, \sigma^2 \mid \mathbf{y}) \quad = \quad c \prod_{i=1}^{n} [y_i \mid g(\boldsymbol{\theta}, x_i), \sigma^2]$$

What does conditionally independent mean?

Independence is an assumption! Remember from the chain rule:

$$Pr(y_1, \ldots, y_n) = Pr(y_1 \mid y_2 \ldots y_n) Pr(y_2 \mid y_3 \ldots y_n) \ldots Pr(y_n).$$

However, by assuming that these random variables are independent, you can simplyify the joint probability into:

$$Pr(y_1, \ldots, y_n) = Pr(y_1) Pr(y_2) \ldots Pr(y_n),$$

such that the total likelihood a product of the individual likelihoods.

Log likelihoods:

We often use the sum of the log likelihoods to get the total log likelihood as a basis for fitting models:

$$\log(\mathsf{L}(\boldsymbol{\theta},\sigma^2 \mid y)) = \log(c) + \sum_{i=1}^{n} \log([y_i \mid g(\boldsymbol{\theta},x_i),\sigma^2])$$

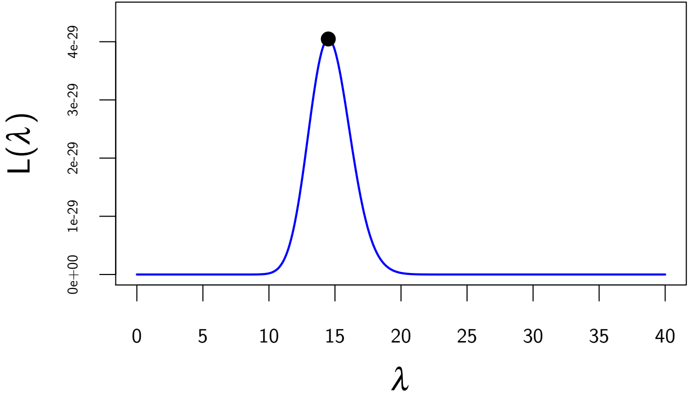
Class Exercise: Likelihood profile of λ for tadpole counts

Assume we have observed the following counts of tadpoles in funnel traps from 6 different ponds: 7, 11, 54, 12, 2, 1 individuals.

Assuming these counts were generated using a Poisson process governed by λ, compute a likelihood profile using R.

Eyeball this profile to determine the MLE.

Likelihood profile of λ for tadpole counts




```r
lambda <- seq(0, 40, length = 1000)
y <- NA
for(i in 1:length(lambda)) {y[i] <- prod(dpois(c(7, 11, 54, 12, 2, 1), lambda[i]))}
```

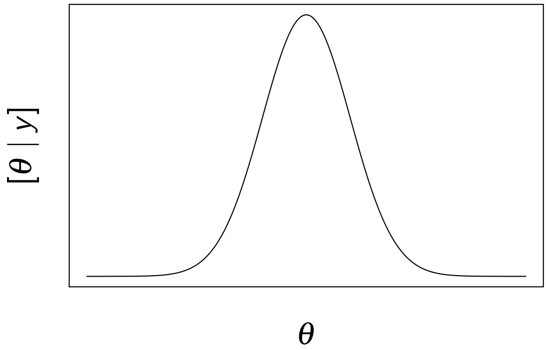
Things to remember

- Likelihood allows us to evaluate the relative strength of evidence for one parameter or model relative to another.
- Likelihood is a component of all Bayesian models.
- The data are fixed and the parameters are variable in likelihood functions. These functions do not integrate or sum to one over the range of values of the parameter.
- The outcomes are variable and the parameters are fixed in probability mass functions and density functions. These functions sum or integrate to one over the support of the random variable.

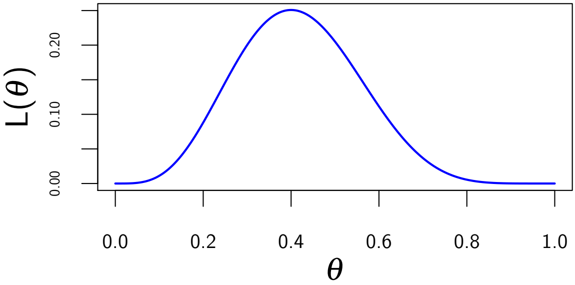
End of Lecture

The remaining slides are in anticipation of questions or to provide additional information.

Looking ahead: Bayesian inference is based on $[\theta \mid y]$



Looking ahead: Getting to $[\theta \mid y]$ from a likelihood



- What must be done to insure that the area under the curve $= 1$?
- If you do this, is this now a probability density function for θ?

What does conditionally independent mean?

We evaluate the independence assumption by examining the residuals (ε) from a model, where ($\varepsilon_i = y_i - g(\boldsymbol{\theta}, x_i)$).

The independence assumption holds if knowing a residual tells you nothing about the other residuals.

We assess this by ensuring that the residuals:

- do not show a trend, meaning they should be centered on 0 throughout the range of fitted values,
- and are not be autocorrelated.

Likelihood ratio confidence intervals

Find the upper and lower bounds of an interval where all λ values within that interval are as consistent with the data as λ_{MLE}.

We compute the likelihood ratio statistic:

$$\mathrm{R} = 2\log\left(\frac{\mathrm{L}(\lambda_{MLE} \mid y)}{\mathrm{L}(\lambda_0 \mid y)}\right) \sim \chi^2_{k=1}$$

which is distributed χ^2 with 1 degree of freedom. Note that we fail to reject H_0 that $\lambda = \lambda_0$ if $\mathrm{R} < \chi^2_{k=1}(1-\alpha)$.

Likelihood ratio confidence intervals

We determine the $(1-\alpha=0.95)$ likelihood ratio confidence interval by finding the upper and lower bounds for all values of λ_0 where we would fail to reject H_0.

$$
\begin{aligned}
2\log\left(\frac{\mathrm{L}(\lambda_{MLE}\mid y)}{\mathrm{L}(\lambda\mid y)}\right) &< \chi^2_{k=1}(0.95) \\
L(\lambda_{MLE})-\frac{3.84}{2} &< L(\lambda\mid y) \\
L(\lambda\mid y) &> L(\lambda_{MLE})-1.92
\end{aligned}
$$

Likelihood ratio confidence intervals

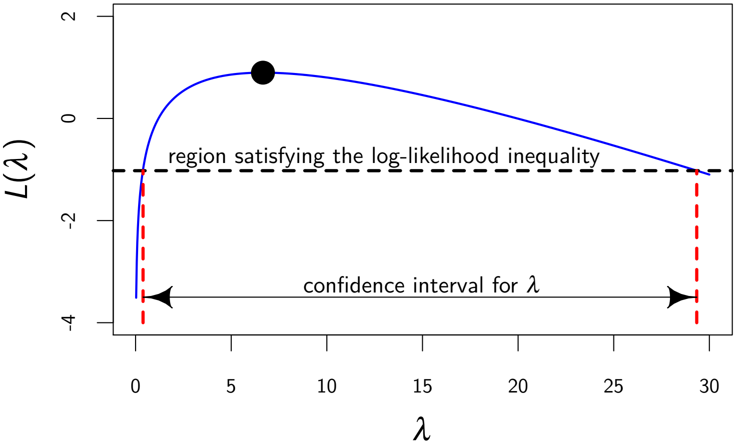
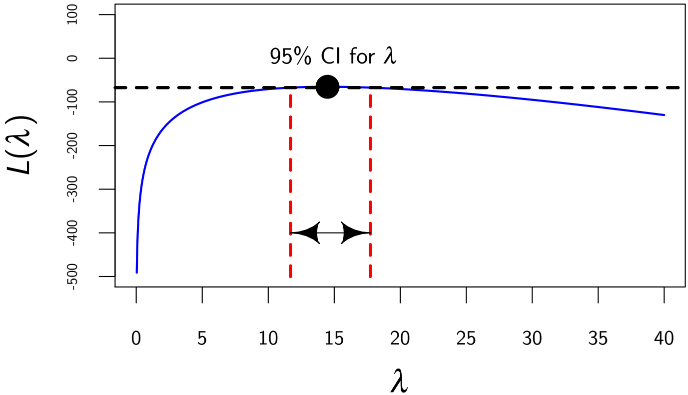


Figure courtesy of the UNC Biology Department.

Likelihood profile of λ for tadpoles




```r
lambda <- seq(0, 40, length = 1000)
y <- NA
for(i in 1:length(lambda)) {y[i] <- log(prod(dpois(c(7, 11, 54, 12, 2, 1), lambda[i])))}
```