Rules of Probability

Bayesian Modeling for Socio-Environmental Data

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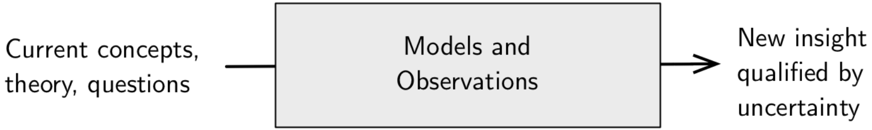
A line of inference



Insight in science

Probability model

Observations (data)

Deterministic model

Idea!

What is the probability that I would observe the data if my model is a faithful representation of the processes that gave rise to the data?

Road map for today

- Rules of probability
- Factoring joint probabilities
- Directed acyclic graphs (a.k.a. Bayesian networks)

All of Bayesian inference extends from three rules of probability

1. Conditional probability (and independence)
2. The law of total probability
3. The chain rule of probability

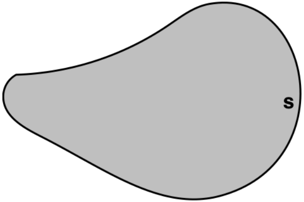
Random variables

The world can be divided into things that are observed and things that are unobserved.

1. Bayesians treat all unobserved quantities as *random varialbes*.
2. The values of random variables are governed by chance.
3. Probability distributions describe "governed by chance."
4. A specific value of a random variable is called an event or an outcome.

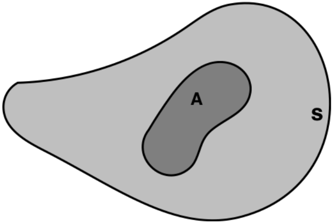
S=Sample Space

- The set of all possible values of a random variable.
- The sample space, S has a specific area.

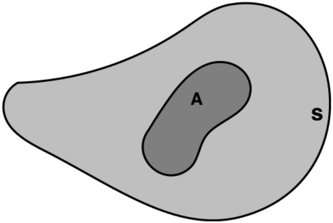


Events in S

- Can define and event, A.
- The area of event A is less than or equal to S.



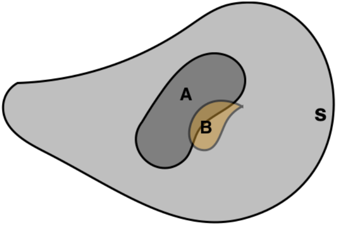
What is the probability of event A?



$\Pr(A) = \frac{\text{Area of } A}{\text{Area of } S}$

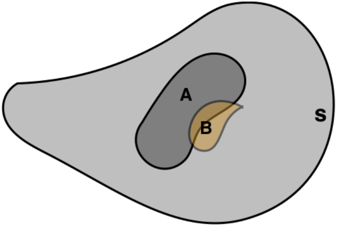
Conditional Probability

Conditional probability: the probability of an event given that *we know* another event has occurred.



Conditional Probability

What is the probability of event B, given that event A has occurred?

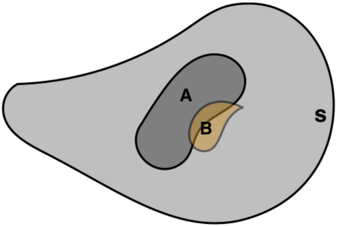


$\Pr(B|A) =$ probability of B conditional on knowing A has occurred

$$Pr(B|A) = \frac{\text{Joint Probability}}{\text{Probability of A}} = \frac{\Pr(A,B)}{\Pr(A)}$$

Conditional Probability

What is the probability of event A, given that event B has occurred?



Independence

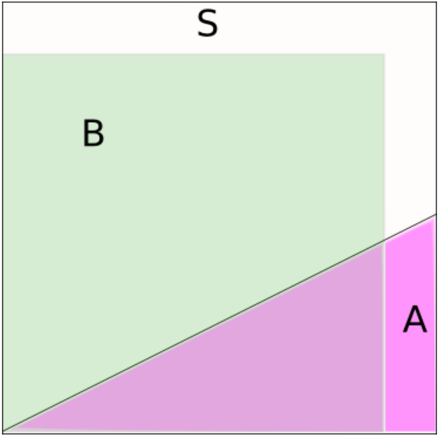
Event A and B are *independent* If the occurrence of event A does not tell us anything about event B.

Events are independent if and only if:

$\Pr(A|B) = \Pr(A)$

$\Pr(B|A) = \Pr(B)$

Independence

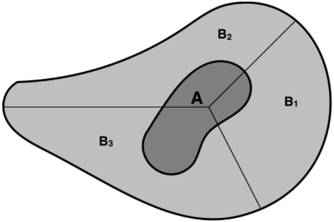


$$\Pr(A|B) = \frac{\text{area of A and B}}{\text{area of B}} = \frac{\text{area of A}}{\text{area of S}}$$

Independence

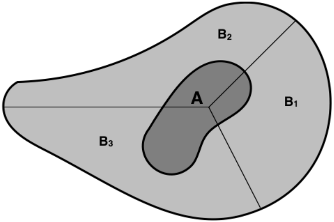
Show that $\Pr(A, B) = Pr(A)Pr(B)$ if A and B are indepdenent events.

The Law of Total Probability



We can define a set of events $\{B_n : n = 1, 2, 3, ...\}$, which taken together define the entire sample space, $\sum_n B_n = S$.

What is the probability of event A?



$\Pr(A) = \sum_n \Pr(A|B_n)\Pr(B_n)$ (discrete case)

$\Pr(A) = \int \Pr(A|B)\Pr(B)dB$ (continuous case)

Chain rule of probability

Board work

The Chain Rule of Probability

The chain rule of probability allows us to write joint distributions as a product of conditional distributions.

$$\Pr(z_1, z_2, ..., z_n) = \Pr(z_n | z_{n-1}, ..., z_1) ... \Pr(z_3 | z_2, z_1) \Pr(z_2 | z_1) \Pr(z_1)$$

Notice the pattern here.

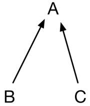
- z's can be scalars or vectors.
- Sequence of conditioning does not matter.
- When we build models, we choose a sequence that makes sense.

Factoring joint probabilities

Why is factoring useful?

- Factoring joint distributions is how we build Bayesian models.
- The rules of probability allow us to simplify complicated joint. distributions, breaking them down into chunks.
- Chunks can be analyzed one at a time.

Consider a factored joint distribution represented by a directed acyclic graph (DAG)



Represents [A | B, C] [B] [C]

The is one way to factor [A, B, C].

- Directed acyclic graphs (aka Bayesian networks) specify how joint distributions are factored into conditional distributions using nodes to represent RV's and arrows to represent dependencies.
- Nodes at the heads of arrows *must* be on the left hand side of the conditioning symbols;
- Nodes at the tails of arrows are on the right hand side of the conditioning symbols.
- Any node at the tail of an arrow without and arrow leading into it must be expressed unconditionally.
- Nodes at heads of arrows are called "children"; at tails, "parents."

Factoring joint probabilities

Illustrate with simple regression model on board.

Work on lab

Complete Probability Lab #1