

Modeling Ordinal Categorical Variables

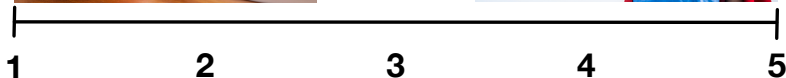
Models for Socio-Environmental Data

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How confident are you in your ability use Bayesian models?



We use *ordinal regression* to deal with data where the dependent variable is measured in ordered categories. Examples of such variables include:

- Psychometric Likert scales
- Tumor grading
- General quantities (i.e. insurance level: none, adequate, full; index of environmental concern: none, low, moderate, high)
- Cover classes (i.e., Daubenmire classes)

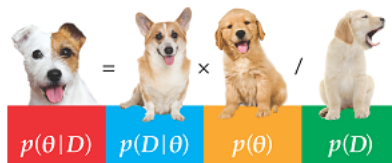
Ordered categorical data can be

- unscaled (e.g. attitudes/opinions, etc.)
- scaled (e.g. cover/size classes, etc.)

Useful reference

Doing Bayesian Data Analysis

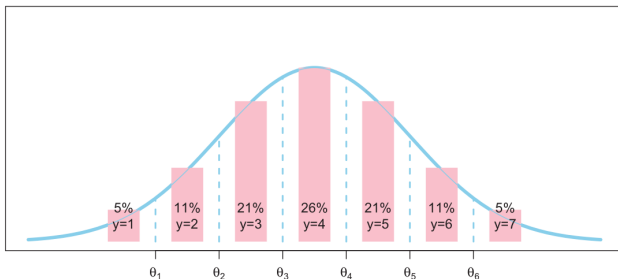
A Tutorial with R, JAGS, and Stan



Kruschke, J. (2014). Doing Bayesian data analysis: A tutorial with R, JAGS, and Stan. Academic Press.

“How do people generate a discrete ordered response?”

- Imagine that your true Bayesian abilities vary on a continuous scale, but you also have some sense of which categorical threshold you would report
- **Central idea:** there is a latent continuous metric that underlies the observed ordinal response
- Categories or *thresholds* partition regions of this continuous metric



Crutial bit: *the probability of a particular ordinal outcome is the area under the normal curve between the thresholds of that outcome.*

Therefore, the probability of outcome 2 is the area under the normal curve between thresholds θ_1 and θ_2 . How?

A general, Bayesian model for ordinal data

$$[\theta, \beta, \sigma^2 | \mathbf{y}] \propto \prod_{i=1}^n [y_i | p_i] \beta_1 [\beta_2] \prod_{k=2}^{K-1} [\theta] [\sigma^2]$$

$$y_i \sim \text{categorical}(y_i | p_i)$$

$$p_i = \int_{-\infty}^{\theta_{k=1}} [z_i | g(\beta, x_i), \sigma^2] dz_i, \int_{\theta_{k=k+1}}^{\theta_{k=k+2}} [z_i | g(\beta, x_i), \sigma^2] dz_i, \dots, \int_{\theta_{k=K}}^{\infty} [z_i | g(\beta, x_i), \sigma^2] dz_i$$

$$\beta \sim \text{normal}(0, 0.001)$$

$$\sigma^2 \sim \text{inversegamma}(0.001, 0.001)$$

$$\theta_j \sim \text{uniform}(0, 10)$$

- y_i is i th observation in categories = $k = 1, \dots, K$, θ is an *ordered* vector of cutpoints, $\theta_0 = -\infty$, $\theta_K = +\infty$

Why is \mathbf{z} missing from the posterior?

What is $Pr(\theta_{k_i-1} < z_i < \theta_{k_i})$?

What is the quantity between the large brackets?

An general algorithm for implementation

Let $F(\theta_k, \mu, \sigma^2)$ be a properly moment matched, cumulative distribution function for the distribution of the latent quantity z_i . The function $F()$ returns the probability that $z_i < \theta_k$. For notational convenience, we let $\mu_i = g(\beta, \mathbf{x}_i)$. Compute:

$$p[1, i] = F(\theta_1, \mu_i, \sigma^2) \quad (1)$$

$$p[2, i] = F(\theta_2, \mu_i, \sigma^2) - F(\theta_1, \mu, \sigma^2) \quad (2)$$

$$\cdot \quad (3)$$

$$\cdot \quad (4)$$

$$p[K-1] = F(\theta_{K-1}, \mu, \sigma^2) - F(\theta_{K-2}, \mu, \sigma^2) \quad (5)$$

$$p[K] = 1 - F(\theta_K, \mu, \sigma^2) \quad (6)$$

The likelihood of the data conditional on the parameters is then:

$$y_i \sim \text{categorical}(\mathbf{p}_i)$$

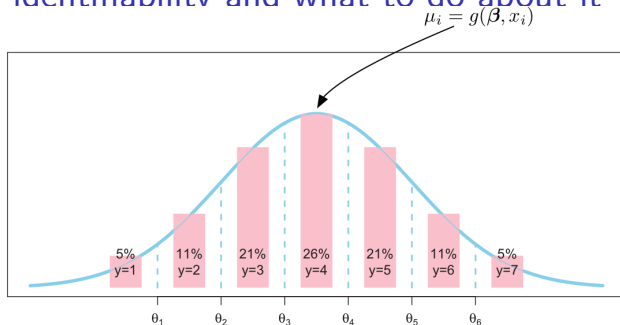
The categorical distribution

$$y_i \sim \text{categorical}(\mathbf{p}_i)$$

Let y_i be an observation that can take on values $k = 1, \dots, K$. \mathbf{p} is a vector of length K with elements $p_i = \Pr(y_i = k_i)$, which is the same as $\Pr(y_i = i)$.

You can use *any continuous distribution* appropriate to the support of the random variable, y_i .

Issues of identifiability and what to do about it

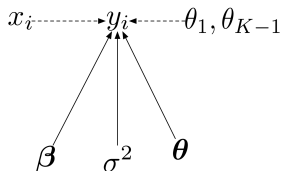


- The likelihood will not result in a unique solution.
- Both β and θ are “location” parameters that calibrate the mapping from what is observed, y_i to the latent z_i .
- In other words, there is no unique combination of θ and β that produce equally informative posterior distributions.
- Put differently, for any given β there exists a θ that produces a likelihood equal to that obtained from at least one other β and θ .

Potential Identification Constraints to Apply

Options	β	σ	θ
1	unconstrained	fixed	fix one of θ_j
2	drop intercept, β_0	fixed	unconstrained
3	unconstrained	unconstrained	fix two of θ_j

Example: Predicting A *Unscaled* Ordinal Quantity



$$[\theta, \beta, \sigma^2 | \mathbf{y}] \propto \prod_{i=1}^n \left[y_i \mid \int_{\theta_{k-1}}^{\theta_k} [z_i \mid g(\beta, x_i), \sigma^2] dz_i \right]$$

$$\times [\beta_1][\beta_2] \prod_{j=2}^{K-2} [\theta_j][\sigma]$$

$$y_i \sim \left[y_i \mid \int_{\theta_{k-1}}^{\theta_k} [z_i \mid g(\beta, x_i), \sigma^2] dz_i \right]$$

$$\beta \sim \text{normal}(0, 0.001)$$

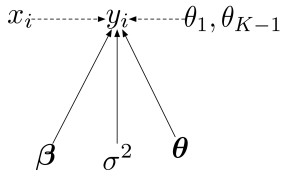
$$\sigma \sim \text{uniform}(0, 100)$$

$$\theta_j \sim \text{uniform}(0, 10)$$

```
for (i in 1:length(y)) {
  mu[i] = beta[1] + beta[2]*x[i]
  y[i] ~ dcat( pr[i,1:nYlevels])
  y.sim[i] ~ dcat( pr[i,1:nYlevels])
  pr[i,1] <- pnorm( thresh[1], mu[i], tau)

  for ( k in 2:(nYlevels-1) ) {
    pr[i,k] <- max(.00001, pnorm( thresh[ k ], mu[i], tau) - pnorm( thresh[k-1], mu[i], tau ))
  }
  pr[i,nYlevels] <- 1 - pnorm( thresh[nYlevels-1], mu[i], tau )
}
```

Example: Predicting A Scaled Ordinal Quantity



$$\mu = \frac{e^{\beta_1 + \beta_2 x_i}}{1 + e^{\beta_1 + \beta_2 x_i}} = g(\beta, x_i)$$

$$[\theta, \beta, \sigma^2 | \mathbf{y}] \propto \prod_{i=1}^n \left[y_i \mid \int_{\theta_{k-1}}^{\theta_k} [z_i \mid m(g(\beta, x_i), \sigma^2)] dz_i \right]$$

$$\times [\beta_1][\beta_2] \prod_{j=2}^{K-2} [\theta_j][\sigma]$$

$$y_i \sim \left[y_i \mid \int_{\theta_{k-1}}^{\theta_k} [z_i \mid m(g(\beta, x_i), \sigma^2)] dz_i \right]$$

$$\beta \sim \text{normal}(0, 0.0001)$$

$$\sigma \sim \text{uniform}(0.01, .5)$$

$$\theta_j \sim \text{uniform}(0, 1)$$

```
for (i in 1:length(y)) {
  mu[i] = ilogit(beta[1] + beta[2]*x[i])
  a[i] <- max(.00001, (mu[i]^2 - mu[i]^3 - mu[i]*sigma^2)/sigma^2)
  b[i] <- max(.00001, (mu[i] - 2*mu[i]^2 + mu[i]^3 - sigma^2 + mu[i]*sigma^2)/sigma^2)
  y[i] ~ dcat( pr[i, 1:nYlevels])
  pr[i, 1] <- pbeta( theta[1], a[i], b[i])
  for ( k in 2:(nYlevels-1) ) {
    pr[i, k] <- max(.00001, pbeta( theta[ k ], a[i], b[i]) - pbeta( theta[k-1], a[i], b[i] ))
  }
  pr[i, nYlevels] <- 1 - pbeta( theta[nYlevels-1], a[i], b[i] )
}
```

Other notables

- Referred to as *ordinal regression* or *ordered probit regression*.
- Cut points are often specified using τ .
- The latent quantity that we are calling z_i is also specified as y_i^*
- Often in the unscaled case, the standard normal is used ($\beta_0 = 0$ and $\sigma = 1$) with the probability of outcome θ_k being:

$$p(\tau = k \mid \mu, \sigma, \theta_j) = \Phi((\theta_k - \mu)/\sigma) - \Phi((\theta_{k-1} - \mu)/\sigma)$$

Table 15.2: For the generalized linear model: typical noise distributions and inverse-link functions for describing various scale types of the predicted variable y . The value μ is a central tendency of the predicted data (not necessarily the mean). The predictor variable is x , and $\text{lin}(x)$ is a linear function of x , such as those shown in Table 15.1. Copyright © Kruschke, J. K. (2014). *Doing Bayesian Data Analysis: A Tutorial with R, JAGS, and Stan. 2nd Edition*. Academic Press / Elsevier.

Scale Type of Predicted y	Typical Noise Distribution $y \sim \text{pdf}(\mu, [\text{parameters}])$	Typical Inverse-Link Function $\mu = f(\text{lin}(x), [\text{parameters}])$
Metric	$y \sim \text{normal}(\mu, \sigma)$	$\mu = \text{lin}(x)$
Dichotomous	$y \sim \text{bernoulli}(\mu)$	$\mu = \text{logistic}(\text{lin}(x))$
Nominal	$y \sim \text{categorical}(\dots, \mu_k, \dots)$	$\mu_k = \frac{\exp(\text{lin}_k(x))}{\sum_c \exp(\text{lin}_c(x))}$
Ordinal	$y \sim \text{categorical}(\dots, \mu_k, \dots)$	$\mu_k = \frac{\Phi((\theta_k - \text{lin}(x)) / \sigma)}{\Phi((\theta_k - \text{lin}(x)) / \sigma) - \Phi((\theta_{k-1} - \text{lin}(x)) / \sigma)}$
Count	$y \sim \text{poisson}(\mu)$	$\mu = \exp(\text{lin}(x))$