More About Priors I

Models for Socio-Environmental Data

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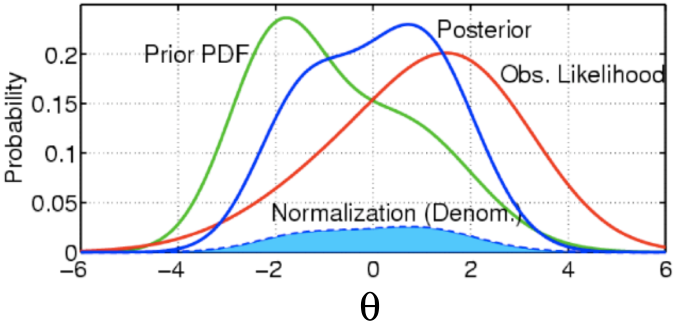
The posterior distribution uses information from both the likelihood and prior distributions. *An informative prior is more influential than an vague prior*.



Outline

- Informative priors
- Vague priors
- Conjugate priors

Informative priors

Informative priors are distributions that are not difuse relative to the posterior. These distributions may be based on

- statistics reported in the literature
- posterior distributions from previous studies
- meta-analyses
- "plausible" assumptions

Why use informative priors?

- They speed up convergence
- They reduce problems with identifiability
- They can allow you to estimate difficult/impossible quantities

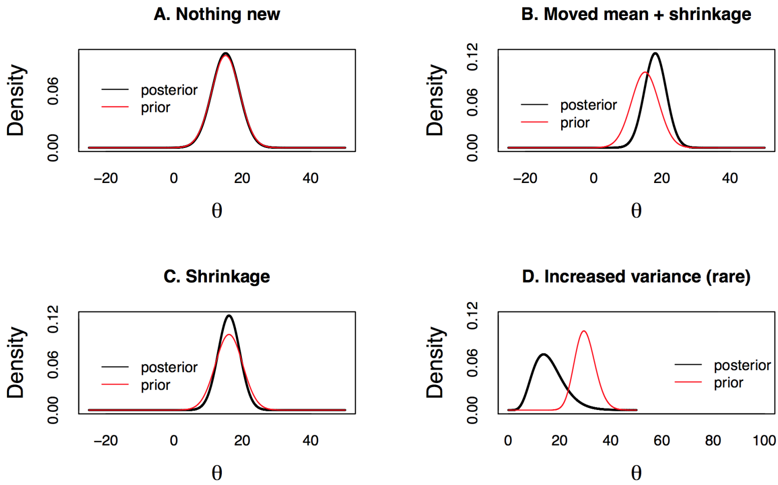
Why don't we find people using informative priors more often?

- *Cultural*: "All studies stand alone" argument
- Texts often use vague priors (including H&H)
- Hard work!
- Concerns about "excessive subjectivity"

If you wanted to use an informative prior, how would you do it?

- Strong scholarship is the basis for strong priors
- Moment match, converting means and standard deviations to usable parameters
- Pilot studies
- Allometric relationships
- Deterministic models with parameters that have specific meaning

How much does an informative prior influence the posterior?



Communicating your use of informative priors

Ecological Monographs, 85(4), 2015, pp. 525–556

State-space modeling to support management of brucellosis in the Yellowstone bison population

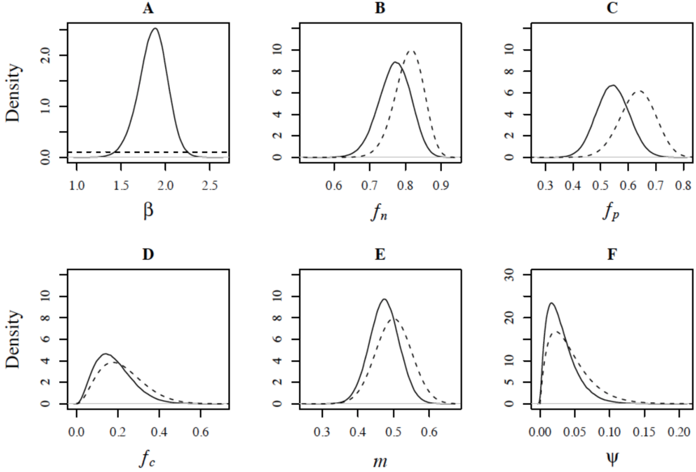
N. Thompson Hobbs,[1,5] Chris Geremia,[2] John Treanor,[2] Rick Wallen,[2] P. J. White,[2] Mevin B. Hooten,[3] and Jack C. Rhyan[4]

Communicating your use of informative priors

Table 3: Prior distributions for parameters in model of brucellosis in the Yellowstone bison population. Sources are given for informative priors.

Parameter	Definition	Distribution	Mean	SD	Source
β	Rate of transmission (yr^{-1})	uniform(0,50)	25	14.3	vague
f_n	Number of offspring recruited per seronegative (susceptible) female	beta(77,18)	.81	.04	Fuller et al., 2007
f_p	Number of offspring recruited per seropositive (recovered) female	beta(37,20)	.64	.06	Fuller et al., 2007
f_c	Number of offspring recruited per seroconverting	beta(3.2,11)	.22	.10	Fuller et al., 2007

Communicating your use of informative priors



Vague Priors

A vague prior is a distribution with a range of uncertainty that is clearly wider than the range of reasonable values for the parameter (Gelman and Hill 2007:347).

Vague Priors

- Avoid calling a prior “uninformative” or “non-informative” rather:
  - difuse
  - flat
  - automatic
  - nonsubjective
  - locally uniform
  - objective

Commonly used vague priors

- gamma(.001, .001) for strictly non-negative quantities
- inverse gamma(.001, .001) for variances
- uniform(0, some large number) for variances
- normal(0, a variance much greater than the mean) regression coefficients

The uniform and normal must be scaled properly!! For example $\beta_0 \sim \text{normal}(0, 1000)$ is extremely informative if $\beta_0 = 10000$)

Issues With Vague Priors

- Computational: failure to converge, slicer errors, failure to calculate log density, etc.
- Sensitivity: changes in parameters of "vague" priors meaningfully changes the posterior when data sets are small or when they have high variance (e.g. $\tau \sim \text{gamma}(.001, .001)$ can really be problematic, this will come up in the multilevel modeling lab)

Conjugacy

- In special cases the posterior, $[\theta|y]$, has the same distributional form as the prior, $[\theta]$.
  - For example, if you had a prior, gamma(α, β), your posterior would be gamma$(\alpha_{new}, \beta_{new})$
- In these cases, the prior and the posterior are said to be *conjugate*.

Conjugacy is important for two reasons:

1. Conjugacy minimizes computational work and, in more complicated cases, allows us to break down calculations into manageable chunks.
2. Conjugacy plays an important role in Markov chain Monte Carlo (more on this later).

Deriving the Beta-Binomial Conjugacy Relationship

We know that the beta distribution is a conjugate prior for the binomial likelihood.

- Consider calculating the posterior distribution for the parameter θ.
- θ is the probability of a success conditional on n trials and y observed successes.

Deriving the Beta-Binomial Conjugacy Relationship

Using Bayes theorem:

$$[\phi \mid y, n] \propto \underbrace{\binom{n}{y} \phi^{y}(1-\phi)^{n-y}}_{\text{binomial likelihood}} \underbrace{\frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \phi^{\alpha-1}(1-\phi)^{\beta-1}}_{\text{beta prior}}$$

where α and β are beta prior parameters.

Deriving the Beta-Binomial Conjugacy Relationship

$$[\phi \mid y, n] \propto \binom{n}{y} \phi^{y}(1-\phi)^{n-y} \frac{\gamma(\alpha+\beta)}{\gamma(\alpha)\gamma(\beta)} \phi^{\alpha-1}(1-\phi)^{\beta-1}$$

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$$[\phi \mid y, n] \propto \phi^{y} (1-\phi)^{n-y} \phi^{\alpha-1} (1-\phi)^{\beta-1}$$

$$[\phi \mid y, n] \propto \phi^{y+\alpha-1} (1-\phi)^{\beta+n-y-1}$$

Deriving the Beta-Binomial Conjugacy Relationship

$$[\phi \mid y, n] \propto \phi^{y+\alpha-1}(1-\phi)^{\beta+n-y-1}$$

Let $\alpha_{new} = y + \alpha$ and $\beta_{new} = \beta + n - y$. Multiply by normalizing constant:

$$\frac{\gamma(\alpha_{new} + \beta_{new})}{\gamma(\alpha_{new})\gamma(\beta_{new})},$$

and the posterior of ϕ is a beta distribution:

$$[\phi \mid y, n] = \frac{\gamma(\alpha_{new} + \beta_{new})}{\gamma(\alpha_{new})\gamma(\beta_{new})}\phi^{\alpha_{new}-1}(1-\phi)^{\beta_{new}-1},$$

with parameters α_{new} and β_{new}.

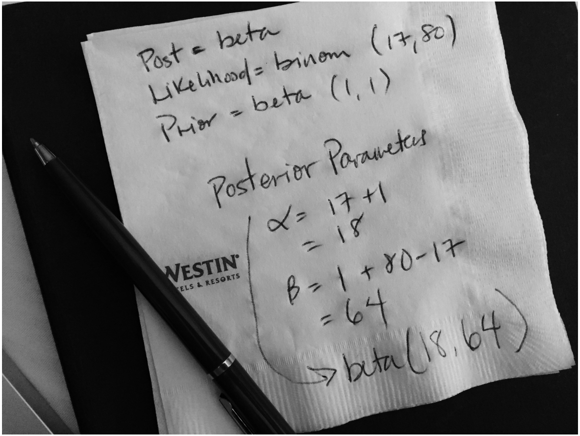
Conjugate prior for normal mean with variance known

$$\mu \sim \text{normal}\left(\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^{n} y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}, \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$$

Conjugate priors

Table A.3: Table of conjugate distributions

Likelihood	Prior distribution	Posterior distribution
$y_i \sim \text{binomial}\,(n, \phi)$	$\phi \sim \text{beta}\,(\alpha, \beta)$	$\phi \sim \text{beta}\,(\sum y_i + \alpha, n - \sum y_i + \beta)$
$y_i \sim \text{Bernoulli}(\phi)$	$\phi \sim \text{beta}\,(\alpha, \beta)$	$\phi \sim \text{beta}\,(\sum_{i=1}^n y_i + \alpha, \sum_{i=1}^n (1 - y_i) + \beta)$
$y_i \sim \text{Poisson}\,(\lambda)$	$\lambda \sim \text{gamma}\,(\alpha, \beta)$	$\lambda \sim \text{gamma}\,(\alpha + \sum_{i=1}^n y_i,\ \beta + n)$
$y_i \sim \text{normal}(\mu, \sigma^2)$ σ^2 is known.	$\mu \sim \text{normal}\,(\mu_0, \sigma_0^2)$	$\mu \sim \text{normal}\left(\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}, \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$
$y_i \sim \text{normal}(\mu, \sigma^2)$ μ is known.	$\sigma^2 \sim$ inverse gamma(α, β)	$\sigma^2 \sim$ inverse gamma $\left(\alpha + \frac{n}{2}, \beta + \frac{\sum_{i=1}^n (y_i - \mu)^2}{2}\right)$
$y_i \sim \text{lognormal}(\mu, \sigma^2)$, μ is known	$\sigma^2 \sim$ inverse gamma(α, β) ,	$\sigma^2 \sim$ inverse gamma $\left(n/2 + \alpha, \frac{(\log(y_i) - \mu)^2}{2} + \beta\right)$
$y_i \sim \text{lognormal}\,(\mu, \sigma^2)$ σ^2 is known	$\mu \sim \text{normal}\,(\mu_0, \sigma_0^2)$	$\mu \sim \text{normal}\left(\frac{\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^n \log y_i}{\sigma^2}\right)}{\left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)}, \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}\right)$


Post = beta
Likelihood = binom (17, 80)
Prior = beta (1, 1)
Posterior Parameters
α = 17 + 1
= 18
β = 1 + 80 - 17
= 64
→ beta(18, 64)
WESTIN
HOTELS & RESORTS

Why Use Conjugacy

- It is not necessary, conjugate priors will accelerate MCMC.
- For simple models, you can use conjugate priors to obtain the posterior distribution in closed form, without any simulation.

Things to remember

- There is no such thing as a uninformative prior, but certain priors influence the posterior distribution more than others.
- Informative priors, when properly justified, can be useful.
- Strong data overwhelms a prior.
- Priors represent current knowledge (or lack of), which is updated with data.
- We encourage you to think of vague priors as a provisional starting point.

Lab exercises.