

Bayesian Regression

Models for Socio-Environmental Data

Chris Che-Castaldo, Mary B. Collins, N. Thompson Hobbs

December 16, 2020



Outline

- Be able to write models for different types of data.
- Appreciate one-to-one relationship between math and JAGS code.
- Be able to interpret coefficients of general linear models.

The general Bayesian set-up

Recall that the posterior distribution is proportional to their joint distribution:

$$[\theta|y] \propto [\theta, y].$$

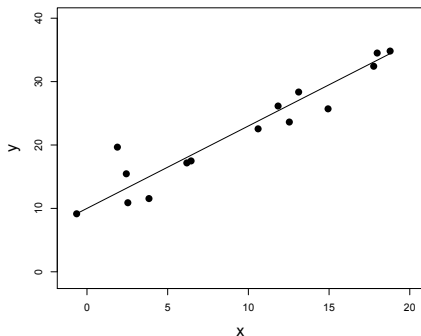
The joint distribution can be factored into a likelihood and priors:

$$[\theta, \sigma^2] = [y | \theta, \sigma^2] [\theta] [\sigma^2]$$

We often wish to model the mean response to y . To do this we embed some ecological or socio-environmental deterministic function $g(\beta, x)$ into the likelihood like this. . . .

$$[\theta, \sigma^2] \propto [y | g(\beta, x), \sigma^2] [\beta] [\sigma^2]$$

A simple example with an old friend



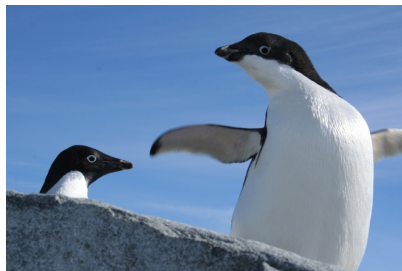
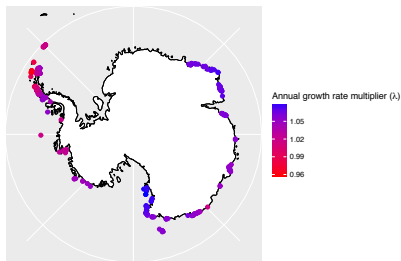
$$g(\beta, x_i) = \beta_0 + \beta_1 x_i$$

$$\begin{aligned} [\beta, \sigma^2 | \mathbf{y}] &\propto \prod_{i=1}^{15} \text{normal}(y_i | g(\beta, x_i), \sigma^2) \text{normal}(\beta_0 | 0, 1000) \times \\ &\quad \text{normal}(\beta_1 | 0, 1000) \times \\ &\quad \text{inverse gamma}(\sigma^2 | 0.001, 0.001) \end{aligned}$$

Lots of flexibility as a modeler

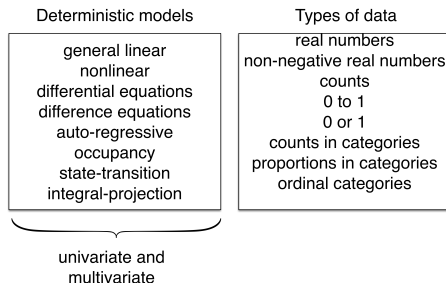
Bayesian regression models are building blocks for more complex models. For example, the continent-wide Adélie penguin population dynamics:

$$z_{s,y} \sim \text{lognormal}(z_{s,y} \mid g(\beta_1, \beta_2, z_{s,y-1}), \sigma^2)$$
$$g(\theta) = \log(z_{s,y-1} e^{\beta_1 + \beta_2 \text{wsic}_{s,y} + \beta_3 \text{ssic}_{s,y}})$$



{Photo c/o Heather J. Lynch}

What about other kinds of data?



Two considerations:

- Moment matching (which you are familiar with)
- Mapping our deterministic model $g(\beta, x) = \beta_0 + \dots + \beta_n x_n$ from $(-\infty, \infty)$ to a range appropriate for the likelihood in our model

You don't have to be normal!

| Data (y-values) | Distribution | "Mean" function | Link |
|---|--------------|---|--|
| continuous, real valued | normal | $\mu = \beta_0 + \beta_1 x$ | NA |
| discrete, strictly positive | Poisson | $\mu = e^{\beta_0 + \beta_1 x}$ | $\log(\mu) = \beta_0 + \beta_1 x$ |
| 0 or 1 | Bernoulli | $\mu = \frac{\exp(\beta_0 + \beta_1 x)}{\exp(\beta_0 + \beta_1 x) + 1}$ | $\text{logit}(\mu) = \log\left(\frac{\mu}{1-\mu}\right) = \beta_0 + \beta_1 x$ |
| [0, 1] | beta | $\mu = \frac{\exp(\beta_0 + \beta_1 x)}{\exp(\beta_0 + \beta_1 x) + 1}$ | $\text{logit}(\mu) = \log\left(\frac{\mu}{1-\mu}\right) = \beta_0 + \beta_1 x$ |
| continuous, strictly positive, variance \uparrow as a f(mean) | lognormal | $\mu = e^{\beta_0 + \beta_1 x}$ | $\log(\mu) = \beta_0 + \beta_1 x$ |
| continuous, strictly positive, constant variance | gamma | $\mu = e^{\beta_0 + \beta_1 x}$ | $\log(\mu) = \beta_0 + \beta_1 x$ |

Normal data, continuous and real valued

$$\begin{aligned} [\beta_0, \beta_1, \sigma \mid \mathbf{y}] &\propto \prod_{i=1}^N \text{normal}(y_i \mid g(\beta_0, \beta_1, x_i), \sigma^2) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 1000) \text{normal}(\beta_1 \mid 0, 1000) \\ &\quad \times \text{uniform}(\sigma \mid 0, 100) \\ g(\beta_0, \beta_1, x_i) &= \beta_0 + \beta_1 x_i \end{aligned}$$

```
b0 ~ dnorm(0, .001)
b1 ~ dnorm(0, .001)
sigma ~ dunif(0, 100)
tau <- 1/sigma^2
for (i in 1:length(y)){
  mu[i] <- b0 + b1 * x[i]
  y[i] ~ dnorm(mu[i], tau)
}
```


Counts, discrete and non-negative

You have collected some count data ($y = 12, 17, 1, 0, 31, \dots, 25$) at n sites, along with a covariate (x) which you believe is likely to affect these counts. Write a model regressing y on x .

- Choose a specific deterministic and stochastic model.
- Write out the DAG, posterior distribution, and joint distribution for your model.
- Interpret the coefficients of your model.

Poisson, discrete and non-negative

$$\begin{aligned} [\beta_0, \beta_1 \mid \mathbf{y}] &\propto \prod_{i=1}^N \text{Poisson}(y_i \mid g(\beta_0, \beta_1, x_i)) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 1000) \text{normal}(\beta_1 \mid 0, 1000) \\ g(\beta_0, \beta_1, x_i) &= e^{\beta_0 + \beta_1 x_i} \end{aligned}$$

```
b0 ~ dnorm(0, .001)
b1 ~ dnorm(0, .001)
for(i in 1:length(y)){
  log(mu[i]) <- b0 + b1 * x[i]
  y[i] ~ dpois(mu[i])
}
```

or

```
mu[i] <- exp(b0 + b1 * x[i])
y[i] ~ dpois(mu[i])
```

Poisson, discrete and non-negative

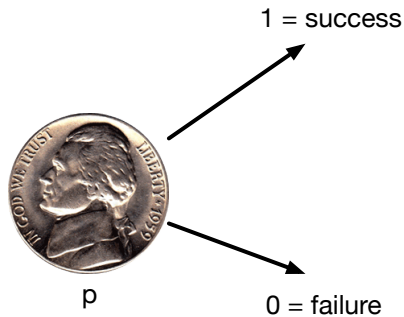
$$\begin{aligned} [\beta_0, \beta_1 \mid \mathbf{y}] &\propto \prod_{i=1}^N \text{Poisson}(y_i \mid g(\beta_0, \beta_1, x_i)) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 1000) \text{normal}(\beta_1 \mid 0, 1000) \\ g(\beta_0, \beta_1, x_i) &= e^{\beta_0 + \beta_1 x_i} \end{aligned}$$

Exponentiate β_0 and β_1 and report counts or multiplicative changes in counts.

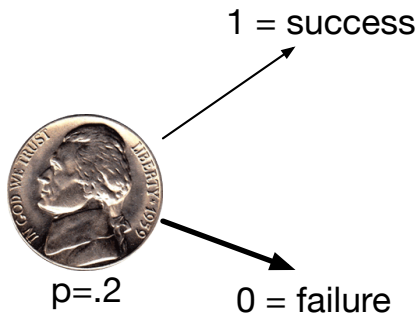
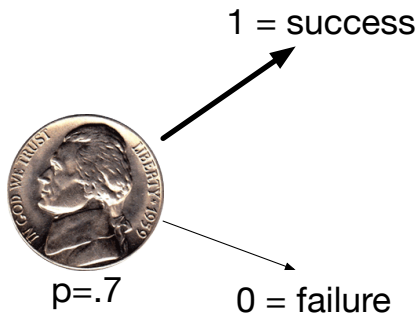
$e^{\beta_1} = \frac{e^{\beta_0 + \beta_1 x}}{e^{\beta_0}}$ which is the multiplicative change in the mean count per unit change in x .

For example: “Mean western toad juvenile abundance is reduced by a factor of 5.1 (95% CI: 3.4, 10.8) per unit change in UV-B radiation.”

Binary data, 0 or 1 (aka logistic)



Binary data, 0 or 1 (aka logistic)



Binary data, 0 or 1 (aka logistic)

What happens when we want to make p a function of a continuous predictor?

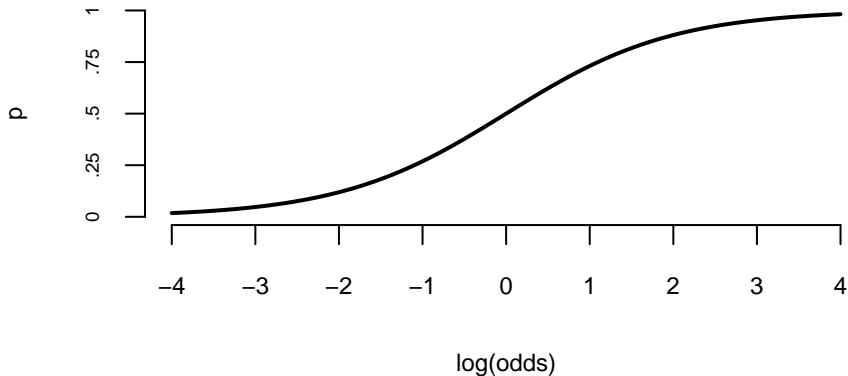
- probability: p $[0, 1]$
- odds: $\frac{p}{1-p}$ $[0, \infty)$
- log odds: $\log(\text{odds})$ $(-\infty, \infty)$

Moving between probability and log odds

- $\text{logit}() = \log\left(\frac{p}{1-p}\right)$: input is probability, output is log odds
- $\text{inverse logit}() = \frac{e^{\log(\text{odds})}}{e^{\log(\text{odds})} + 1}$: input is $\log(\text{odds})$, output is probability

Binary data, 0 or 1 (aka logistic)

inverse logit mapping: input is $\log(\text{odds})$, output is probability



Binary data, 0 or 1 (aka logistic)

You have collected some binary data ($y = 1, 0, 0, 1, 1, 0, 1, \dots, 1$) at n sites, along with a covariate (x) which you believe is likely to affect these counts. Write a model regressing y on x .

- Choose a specific deterministic and stochastic model.
- Write out the DAG, posterior distribution, and joint distribution for your model.
- Interpret the coefficients of your model.

Bernoulli, 0 or 1 (aka logistic)

$$\begin{aligned} [\beta_0, \beta_1 \mid \mathbf{y}] &\propto \prod_{i=1}^N \text{Bernoulli}(y_i \mid g(\beta_0, \beta_1, x_i)) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 2.7) \text{normal}(\beta_1 \mid 0, 2.7) \\ g(\beta_0, \beta_1, x_i) &= \frac{e^{\beta_0 + \beta_1 x_i}}{e^{\beta_0 + \beta_1 x_i} + 1} \end{aligned}$$

```
b0 ~ dnorm(0, .37)
b1 ~ dnorm(0, .37)
for(i in 1:length(y)){
  logit(p[i]) <- b0 + b1 * x[i]
  y[i] ~ dbern(p[i])
}
```

or

```
p[i] <- inv.logit(b0 + b1 * x[i])
y[i] ~ dbern(p[i])
```

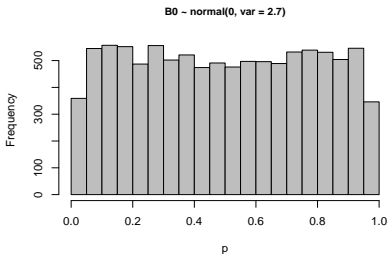
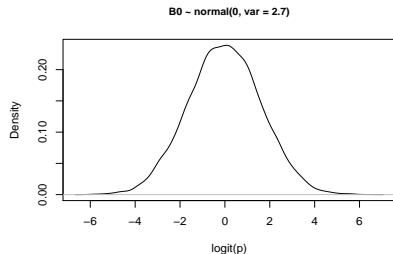
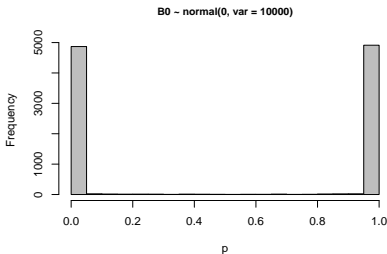
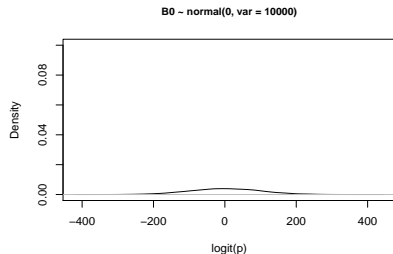
Interpreting logit coefficients

Exponentiate β_0 and β_1 and report odds and odds ratios.

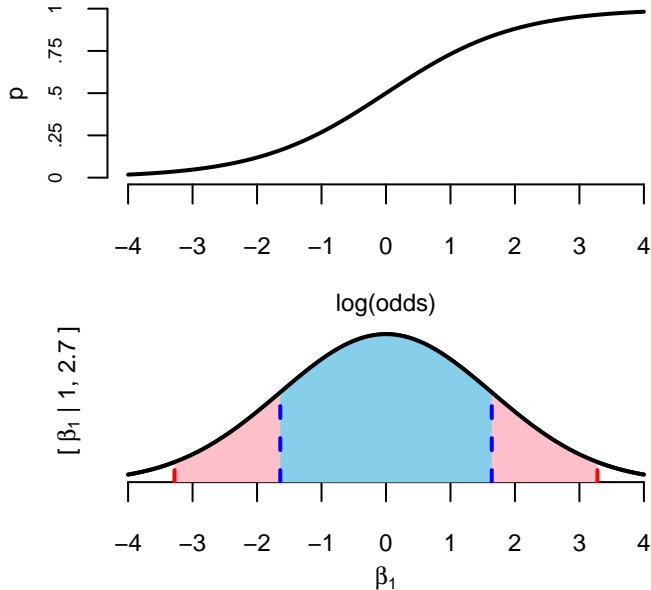
$e^{\beta_1} = \frac{e^{\beta_0 + \beta_1 x}}{e^{\beta_0}}$ which is an odds ratio, i.e. the change in the odds.

For example: “The odds of detecting weevils in upland willow stems were 3.2 (95% CI: 2.3, 4.8) times greater than detecting them in riparian willow stems.”

Choosing reasonable flat priors on logit intercept



Choosing reasonable flat priors on logit effects



lognormal, data continuous and > 0 (log link)

$$\begin{aligned} [\beta_0, \beta_1, \sigma \mid \mathbf{y}] &\propto \prod_{i=1}^N \text{lognormal}(y_i \mid \log(g(\beta_0, \beta_1, x_i)), \sigma^2) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 1000) \text{normal}(\beta_1 \mid 0, 1000) \\ &\quad \times \text{uniform}(\sigma \mid 0, 5) \\ g(\beta_0, \beta_1, x_i) &= e^{\beta_0 + \beta_1 x_i} \end{aligned}$$

Talk about the interpretation of σ .

```
b0 ~ dnorm(0, .001)
b1 ~ dnorm(0, .001)
sigma ~ dunif(0, 5)
tau <- 1/sigma^2
for(i in 1:length(y)){
  mu[i] <- exp(b0 + b1 * x[i])
  y[i] ~ dlnorm(log(mu[i]), tau)
}
```

lognormal, data continuous and > 0 (not log link)

$$\begin{aligned} [\beta_0, \beta_1, \sigma \mid \mathbf{y}] &\propto \prod_{i=2}^N \text{lognormal}(y_i \mid \log(g(\beta_0, \beta_1, y_{i-1})), \sigma^2) \\ &\quad \times \text{normal}(\beta_0 \mid 0, 1000) \text{normal}(\beta_1 \mid 0, 1000) \\ &\quad \times \text{uniform}(\sigma \mid 0, 5) \text{uniform}(y_1 \mid 1, 1E6) \end{aligned}$$

$$g(\beta_0, \beta_1, y_{i-1}) = y_{i-1} e^{\beta_0 + \beta_1 y_{i-1}}$$

```
b0 ~ dnorm(0, .001)
b1 ~ dnorm(0, .001)
sigma ~ dunif(0, 5); tau <- 1/sigma^2
y[1] ~ dunif(1, 1E6)
for(i in 2:length(y)){
  mu[i] <- y[i-1] * exp(b0 + b1 * y[i-1])
  y[i] ~ dlnorm(log(mu[i]), tau)
}
```