Computing fields near RWG panels in SCUFF-EM

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August 27, 2014

In this note I discuss the evaluation of the **E** and **H** fields due to RWG currents on a single triangular source panel $\mathcal{P}$ in the case where the evaluation point lies on or near the source panel. My method is essentially that of Graglia[1]; this is a desingularization scheme in which the first few terms in the low-frequency series expansion of the Green's function are subtracted off and evaluated analytically. However, in SCUFF-EM I need also to compute the *derivatives* of the **E** and **H** fields, which requires an extension of Graglia's methods.

The algorithm described here is implemented in the file `GetNearFields.cc` in the SCUFF-EM source distribution.

Contents

[1] Graglia, *IEEE Trans. Ant. Prop.* **41** 1448 (1993); see also Wilton et al., *IEEE Trans. Ant. Prop.* **32** 276 (1984).

1 Fields from reduced potentials

1.1 Reduced vector and scalar potentials

Consider a triangular panel $\mathcal{P}$ on which we have a flow of surface current described by an RWG basis function $\mathbf{b}_\alpha(\mathbf{x})$ with source vertex $\mathbf{Q}$. I begin by defining "reduced" vector and scalar potentials produced by this current at a point $\mathbf{x}$:

$$\mathbf{a}(\mathcal{P};\mathbf{Q};\mathbf{x}) = \frac{\ell}{2A}\int_{\mathcal{P}}(\mathbf{x}'-\mathbf{Q})G(\mathbf{x},\mathbf{x}')\,d\mathbf{x}', \qquad p(\mathcal{P};\mathbf{x}) = 2\cdot\frac{\ell}{2A}\int_{\mathcal{P}}G(\mathbf{x}-\mathbf{x}')\,d\mathbf{x}', \tag{1}$$

where ℓ is the length of the edge associated with basis function $\mathbf{b}$, A is the area of $\mathcal{P}$, and

$$G(\mathbf{r}) = \frac{e^{ik|\mathbf{r}|}}{4\pi|\mathbf{r}|}$$

is the scalar Helmholtz Green's function.

The total reduced vector and scalar potentials of $\mathbf{b}_\alpha$ involve contributions from two panels:

$$\mathbf{a}_\alpha(\mathbf{x}) = \mathbf{a}(\mathcal{P}^+,\mathbf{Q}^+,\mathbf{x})-\mathbf{a}(\mathcal{P}^-,\mathbf{Q}^-,\mathbf{x}), \qquad p_\alpha(\mathbf{x}) = p(\mathcal{P}^+,\mathbf{Q}^+,\mathbf{x})-p(\mathcal{P}^-,\mathbf{Q}^-,\mathbf{x}).$$

In terms of $\mathbf{a}_\alpha$ and p_α, I can define the "reduced fields" of basis function $\mathbf{b}_\alpha$ according to

$$\mathbf{e}_\alpha(\mathbf{x}) = \mathbf{a}_\alpha(\mathbf{x}) + \frac{1}{k^2}\nabla p_\alpha(\mathbf{x}), \qquad \mathbf{h}_\alpha(\mathbf{x}) = \nabla\times\mathbf{a}_\alpha(\mathbf{x}) \tag{2}$$

Given a set of RWG functions $\{\mathbf{b}_\alpha\}$ populated with electric and magnetic surface-current coefficients $\{k_\alpha, n_\alpha\}$, the full $\mathbf{E}$ and $\mathbf{H}$ fields at $\mathbf{x}$ are

$$\begin{aligned}\mathbf{E}(\mathbf{x}) &= \sum_\alpha \Big\{ ikZk_\alpha\mathbf{e}_\alpha(\mathbf{x}) - n_\alpha\mathbf{h}_\alpha(\mathbf{x})\Big\},\\ \mathbf{H}(\mathbf{x}) &= \sum_\alpha \Big\{ k_\alpha\mathbf{h}_\alpha(\mathbf{x}) + \frac{ik}{Z}n_\alpha\mathbf{e}_\alpha(\mathbf{x})\Big\}\end{aligned}$$

where k is the wavenumber in the material region containing $\mathbf{x}$ and $Z = Z_0Z^r$ with Z_0 the impedance of free space and Z^r the relative wave impedance of the material.

To compute the first derivatives of the reduced fields we need the second derivatives of the reduced potentials:

$$\partial_i e_j(\mathcal{P}^\pm;\mathbf{x}) = \partial_i a_j(\mathcal{P}^\pm;\mathbf{x}) + \frac{1}{k^2}\partial_i\partial_j p(\mathcal{P}^\pm,\mathbf{c}), \qquad \partial_i h_j(\mathcal{P}^\pm;\mathbf{x}) = \varepsilon_{jk\ell}\partial_i\partial_k a_\ell(\mathcal{P}^\pm;\mathbf{x}).$$

1.2 Desingularized reduced potentials

To compute $\mathbf{a}$ and p at evaluation points $\mathbf{x}$ on or near the source panel $\mathcal{P}$ it is convenient to invoke the expansion

$$G(r)=\frac{e^{ikr}}{4\pi r}=\frac{1}{4\pi r}+(ik)\frac{1}{4\pi}+(ik)^2\frac{r}{8\pi}+\frac{\texttt{ExpRelBar}(ikr,3)}{4\pi r}$$

where `ExpRelBar` is just the usual exponential minus the first few terms in its series expansion:[2]

$$\texttt{ExpRelBar}(x,N)=e^x-\sum_{n=0}^{N-1}\frac{x^n}{n!}=\sum_{n=N}^{\infty}\frac{x^n}{n!}.$$

The reduced potentials due to $\mathcal{P}$ are

$$p(\mathbf{x})=2\frac{\ell}{2A}\left[\sum_{n=-1}^{1}\frac{(ik)^{n+1}}{4\pi}p^{(n)}(\mathbf{x})+p^{\text{DS}}(\mathbf{x})\right]$$
$$\mathbf{a}(\mathbf{x})=\frac{\ell}{2A}\left[\sum_{n=-1}^{1}\frac{(ik)^{n+1}}{4\pi}\mathbf{a}^{(n)}(\mathbf{x})+\mathbf{a}^{\text{DS}}(\mathbf{x})\right]$$

where[3]

$$p^{(n)}(\mathbf{x})=\int_{\mathcal{P}}r^n\,dA \tag{3}$$
$$\mathbf{a}^{(n)}(\mathbf{x})=\int_{\mathcal{P}}(\mathbf{x}'-\mathbf{Q})r^n\,dA. \tag{4}$$

(here $r=|\mathbf{x}'-\mathbf{x}|$). To write $\mathbf{a}^{(n)}$ in a more convenient form, let $\overline{\mathbf{x}}$ be the projection of the evaluation point into the plane of the panel and write

$$\begin{aligned}\mathbf{a}^{(n)}(\mathbf{x})&=\int_{\mathcal{P}}\Big(\mathbf{x}'-\overline{\mathbf{x}}+\overline{\mathbf{x}}-\mathbf{Q}\Big)r^n\,dA\\&=\int_{\mathcal{P}}\Big(\mathbf{x}'-\overline{\mathbf{x}}\Big)r^n\,dA-(\mathbf{Q}-\overline{\mathbf{x}})\int_{\mathcal{P}}r^n dA.\end{aligned}$$

To proceed I now define scalar and vector-valued functions according to[4]

$$I^p(\mathcal{P},\mathbf{x})\equiv\int_{\mathcal{P}}r^p\,dA \tag{5a}$$
$$\boldsymbol{\mathcal{J}}^p(\mathcal{P},\mathbf{x})\equiv\int_{\mathcal{P}}\overline{\mathbf{r}}\cdot r^p\,dA. \tag{5b}$$

[2] `ExpRelBar` is similar to, but distinct from, the function named `ExpRel` in the GNU SCIENTIFIC LIBRARY.

[3] Two special cases that may be computed analytically are $p^{(0)}=A,\mathbf{a}^{(0)}=A(\mathbf{x}_c-\mathbf{Q})$ where A is the panel area and $\mathbf{x}_c$ is its centroid. These are independent of the evaluation point $\mathbf{x}$ and thus do not contribute to derivatives.

[4] Note that I use a p superscript for $\mathcal{I}$ and $\boldsymbol{\mathcal{J}}$ but an (n) superscript for $\mathbf{a}$ and p. This is because the indices p and n have different ranges: The n values for which I need $\mathbf{a}^{(n)}$ and $p^{(n)}$ and their derivatives are $n=-1,0,1$, but computing all of these quantities turns out to require $\mathcal{I}^p$ and $\boldsymbol{\mathcal{J}}^p$ for $p=-5,-3,-1,1$.

where $\overline{\mathbf{r}} = \mathbf{x}' - \overline{\mathbf{x}}$ is a vector that has nonzero components only in the plane of $\mathcal{P}$. In terms of $\mathcal{I}$ and $\mathcal{J}$, the reduced potentials are

$$p^{(n)}(\mathcal{P};\mathbf{x}) = \mathcal{I}^n(\mathcal{P};\mathbf{x}), \qquad \mathbf{a}^{(n)}(\mathcal{P};\mathbf{x}) = \boldsymbol{\mathcal{J}}^n(\mathcal{P};\mathbf{x}) - \big(\mathbf{Q} - \overline{\mathbf{x}}\big)\mathcal{I}^n(\mathcal{P};\mathbf{x}) \tag{6}$$

Similarly, derivatives of $\mathbf{a}^{(n)}$ and $p^{(n)}$ are related to derivatives of $\mathcal{I}^p$ and $\boldsymbol{\mathcal{J}}^p$:

$$\begin{aligned} &\partial_i p^p(\mathbf{x}) = d_i \mathcal{I}^p(\mathcal{P},\mathbf{x}) \\ &\partial_i a_j^p(\mathbf{x}) = \partial_i \mathcal{J}_j^p(\mathcal{P},\mathbf{x}) - (\mathbf{Q} - \overline{\mathbf{x}})_j \partial_i \mathcal{I}^p - \delta_{ij}\mathcal{I}^p(\mathcal{P},\mathbf{x}) \end{aligned}$$

The two-dimensional integrals in the quantities $\mathcal{I}^p$ and $\boldsymbol{\mathcal{J}}^p$ defined by Equation (5b), as well as their derivatives, may be evaluated analytically in closed form, as discussed in the following section.

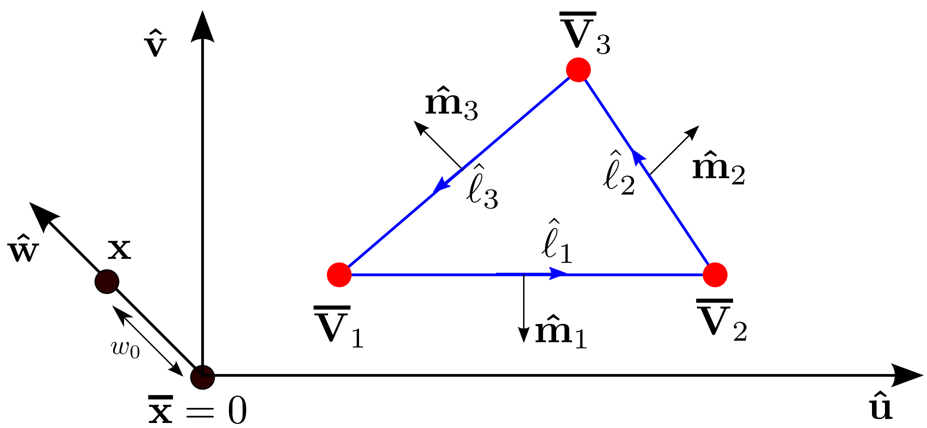

Figure 1: Rotated and translated coordinate system $(\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}})$ for evaluation of $\mathcal{I}, \mathcal{J}$ integrals. The panel lies in the uv plane with one edge parallel to the $\hat{\mathbf{u}}$ axis. The origin of the uv plane is the projection of the evaluation point into the plane of the panel. The perpendicular distance from the plane of the panel to the evaluation point is w_0.

2 Evaluation of $\mathcal{I}, \mathcal{J}$ integrals

2.1 Modified coordinate system

Let the panel $\mathcal{P}$ have vertices $\{\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3\}$. For convenience in what follows, we introduce a rotated and translated coordinate system with cartesian coordinates (u, v, w), in which $\mathcal{P}$ lies entirely in the uv plane (the panel normal $\hat{\mathbf{n}}$ defines the $\hat{\mathbf{w}}$ axis) and edge $\mathbf{V}_1\mathbf{V}_2$ lies parallel to the $\hat{\mathbf{u}}$ axis (Figure 1). The unit vectors of this system are

$$\hat{\mathbf{u}} = \frac{\mathbf{V}_2 - \mathbf{V}_1}{|\mathbf{V}_2 - \mathbf{V}_1|}, \qquad \hat{\mathbf{v}} = \hat{\mathbf{w}} \times \hat{\mathbf{u}}, \qquad \hat{\mathbf{w}} = \frac{(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)}{|(\mathbf{V}_2 - \mathbf{V}_1) \times (\mathbf{V}_3 - \mathbf{V}_1)|}$$

In the modified coordinate system, the evaluation point has coordinates $\mathbf{x} = (\boldsymbol{\rho}_0, w_0)$ with

$$w_0 = (\mathbf{x}_0 - \mathbf{x}_c) \cdot \hat{\mathbf{w}}, \qquad \boldsymbol{\rho}_0 = (\mathbf{x}_0 - \mathbf{x}_c) - w_0 \hat{\mathbf{w}}$$

(where $\mathbf{x}_c$ is the centroid of $\mathcal{P}$). It is convenient to choose the origin of the (u, v) plane to be the point $\boldsymbol{\rho}_0$. The (u, v, w) coordinates of the ith panel vertex are then $\overline{\mathbf{V}}_i = (u_i, v_i, 0)$ where

$$u_i = (\mathbf{V}_i - \mathbf{x}_c) \cdot \hat{\mathbf{u}}, \qquad v_i = (\mathbf{V}_i - \mathbf{x}_c) \cdot \hat{\mathbf{v}}.$$

I also define a two-dimensional unit vector $\hat{\boldsymbol{\ell}}_i$ pointing in the direction of the ith panel edge according to

$$\hat{\boldsymbol{\ell}}_i = \frac{\overline{\mathbf{V}}_{i+1} - \overline{\mathbf{V}}_i}{|\overline{\mathbf{V}}_{i+1} - \overline{\mathbf{V}}_i|}$$

and a two-dimensional unit vector $\hat{\mathbf{m}}_i$ normal to the ith panel edge according to

$$\hat{\mathbf{m}}_i = \hat{\mathbf{w}} \times \hat{\boldsymbol{\ell}}_i.$$

2.2 Reduction of surface integrals to line integrals

I now convert the two-dimensional (surface) integrals $\{\mathcal{I}, \boldsymbol{\mathcal{J}}\}^p$ defined by (5b) into one-dimensional (line) integrals by using Stokes' theorem in the forms

$$\int_{\mathcal{P}} \nabla \cdot \mathbf{f}(\boldsymbol{\rho})\, dA = \oint_{\partial\mathcal{P}} \mathbf{f}(\boldsymbol{\rho}) \cdot \hat{\mathbf{m}}\, d\ell \tag{7a}$$

$$\int_{\mathcal{P}} \nabla f(\boldsymbol{\rho})\, dA = \oint_{\partial\mathcal{P}} f(\boldsymbol{\rho})\, \hat{\mathbf{m}}\, d\ell. \tag{7b}$$

[Here $\boldsymbol{\rho} = (u, v)$.]

First consider the vector-valued function

$$\mathbf{f}(\boldsymbol{\rho}) = \frac{1}{p+2} \frac{[\rho^2 + w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}}$$

with divergence

$$\nabla \cdot \mathbf{f}(\boldsymbol{\rho}) = \frac{1}{\rho} \frac{\partial}{\partial \rho}\Big(\rho f_\rho(\boldsymbol{\rho})\Big) = [\rho^2 + w^2]^{p/2}.$$

Applying (7a) to this function yields

$$\mathcal{I}^p \equiv \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2} dA = \frac{1}{(p+2)} \int_{\partial\mathcal{P}} \frac{[\rho^2 + w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}}\, d\ell.$$

Next consider the scalar function

$$f(\rho) = \frac{1}{p+2} [\rho^2 + w^2]^{(p+2)/2}$$

with gradient

$$\nabla f(\rho) = \boldsymbol{\rho} [\rho^2 + w^2]^{p/2}.$$

Applying (7b) to this function yields

$$\boldsymbol{\mathcal{J}}^p \equiv \int_{\mathcal{P}} \boldsymbol{\rho} \cdot r^p dA = \frac{1}{(p+2)} \int_{\partial\mathcal{P}} [\rho^2 + w^2]^{(p+2)/2} \hat{\mathbf{m}}\, d\ell.$$

2.3 Evaluation of line integrals

Line integrals are sums of integrals over line segments (edges). On edge i, we have

$$\boldsymbol{\rho}(s, t_i) = s\hat{\boldsymbol{\ell}}_i - t_i \hat{\mathbf{m}}_i, \qquad \rho(s, t_i) = \sqrt{s^2 + t_i^2}, \qquad \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}} = -\frac{t_i}{\rho}.$$

Here t_i is the perpendicular distance from edge $\overline{\mathbf{V}}_i\overline{\mathbf{V}}_{i+1}$ to the origin and s runs from s_i^- to s_i^+, where

$$\begin{aligned}
t_i &= -\mathbf{V}_i \cdot \hat{\mathbf{m}}_i = -\mathbf{V}_{i+1} \cdot \hat{\mathbf{m}}_i \\
s_i^- &= \mathbf{V}_i \cdot \hat{\boldsymbol{\ell}}_i \\
s_i^+ &= \mathbf{V}_{i+1} \cdot \hat{\boldsymbol{\ell}}_i
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathcal{I}^p &= \frac{1}{(p+2)} \int_{\partial\mathcal{P}} \frac{[\rho^2+w^2]^{(p+2)/2}}{\rho} \hat{\boldsymbol{\rho}} \cdot \hat{\mathbf{m}}\, d\ell \\
&= -\frac{1}{(p+2)} \sum_i \underbrace{t_i \int_{s_i^-}^{s_i^+} \frac{(s^2+t_i^2+w^2)^{(p+2)/2}}{(s^2+t_i^2)} ds}_{I^p(s_i^-,s_i^+,t_i,w)} \\
\boldsymbol{\mathcal{J}}^p &= \frac{1}{(p+2)} \int_{\partial\mathcal{P}} [\rho^2+w^2]^{(p+2)/2} \hat{\mathbf{m}}\, d\hat{\boldsymbol{\ell}} \\
&= \frac{1}{(p+2)} \sum_i \hat{\mathbf{m}}_i \underbrace{\int_{s_i^-}^{s_i^+} [s^2+t_i^2+w^2]^{(p+2)/2} ds}_{J^p(s_i^-,s_i^+,t_i,w)} .
\end{aligned}$$

The one-dimensional integrals defining I^p and J^p may be evaluated in closed form:

p	$I^p(s^-,s^+,t,w)$	$J^p(s^-,s^+,t,w)$
-5	$\frac{t}{w^2X^2}\left(\frac{s^-}{R^-}-\frac{s^+}{R^+}\right)+\frac{\zeta}{w^3}$	$\frac{1}{X^2}\left(\frac{s^+}{R^+}-\frac{s^-}{R^-}\right)$
-3	$\frac{\zeta}{w}$	Λ
-1	$w\zeta+t\Lambda$	$\frac{1}{2}\Big(R^+s^+-R^-s^-+X^2\Lambda\Big)$
$+1$	$\frac{t}{2}\Big[R^+s^+-R^-s^-+\Lambda(t^2+3w^2)\Big]+w^3\zeta$	$\frac{1}{8}\Big[2R^+s^{+3}-2R^-s^{-3}+5X^2(R^+s^+-R^-s^-)+3\Lambda X^4\Big]$

In this table, we have used the following shorthand:

$$\begin{aligned}
X &= \sqrt{t^2+w^2}, \qquad R^+ = \sqrt{s^{+2}+X^2}, \qquad R^- = \sqrt{s^{-2}+X^2} \\
Z^+ &= R^+ + s^+ \qquad Z^- = R^- + s^- \\
\Lambda &= \log\frac{Z^+}{Z^-}, \qquad \zeta = \text{atan}\left(\frac{ws^+}{tR^+}\right) - \text{atan}\left(\frac{ws^-}{tR^-}\right)
\end{aligned}$$

2.4 Derivatives of reduced potentials

2.4.1 Potential derivatives from $\mathcal{I}, J$ derivatives

When computing derivatives it is easiest to work first in the $\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$ coordinate system and later rotate back to the original coordinate system. In this case, in-plane derivatives (derivatives with respect to the u, v coordinates) are distinguished from normal derivatives (derivatives with respect to the w coordinate). To highlight this distinction, in what follows the subscripts α, β, γ will refer only to the in-plane coordinates (u, v coordinates), so that e.g. $\partial_\alpha \mathcal{I}$ refers to an in-plane derivative, while $\partial_w \mathcal{I}$ is a normal derivative. Note that the vector potential $\mathbf{a}$, like the quantities $\boldsymbol{\mathcal{J}}$ and $\mathbf{Q}$, has only in-plane components.

The starting point is equation (6):

$$p^{(n)} = \mathcal{I}^n, \qquad a_\gamma^{(n)} = \mathcal{J}_\gamma^n - \overline{\mathbf{Q}}_\gamma \mathcal{I}^n$$

Derivatives of p are just derivatives of $\mathcal{I}$, computed as discussed below.

First derivatives of $\mathbf{a}$ take the form

$$\begin{aligned}
\partial_\beta a_\gamma^{(n)} &= \partial_\beta \mathcal{J}_\gamma^n + n\overline{\mathbf{Q}}_\gamma \mathcal{J}_\beta^{n-2} + \delta_{\beta\gamma}\mathcal{I}^n \\
\partial_w a_\gamma^{(n)} &= \partial_w \mathcal{J}_\gamma^n - \overline{\mathbf{Q}}_\gamma \partial_w \mathcal{I}^n \\
&= nw\big[\mathcal{J}_\gamma^{n-2} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-2}\big]
\end{aligned}$$

When computing second derivatives of $\mathbf{a}$, it turns out that the double normal derivative $\partial_w^2 a_\gamma$ and the mixed second partial $\partial_w \partial_\alpha \mathbf{a}_\gamma$ are straightforward to compute in terms of $\mathcal{I}, \boldsymbol{\mathcal{J}}$ and their first derivatives:

$$\begin{aligned}
\partial_w \partial_\beta a_\gamma^{(n)} &= (n-2)w\Big[\partial_\beta \boldsymbol{\mathcal{J}}_\gamma^{(n-2)} + n\overline{\mathbf{Q}}_\gamma \mathcal{J}_\beta^{n-4}\Big] + nw\delta_{\beta\gamma}\mathcal{I}^{n-2} \\
\partial_w^2 a_\gamma^{(n)} &= n\big[\mathcal{J}_\gamma^{n-2} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-2}\big] \\
&\quad + n(n-2)w^2\Big[\mathcal{J}_\gamma^{n-4} - \overline{\mathbf{Q}}_\gamma \mathcal{I}^{n-4}\Big]
\end{aligned}$$

The double in-plane derivative $\partial_\alpha \partial_\beta a_\gamma$ is more difficult to compute. However, it turns out we don't need to compute this quantity as long as we are only interested in first derivatives of the $\mathbf{E}$ and $\mathbf{H}$ fields. To see this, note that second derivatives of $\mathbf{a}$ only enter in the computation of first derivatives of $\mathbf{H}$, which involves differentiating the curl of $\mathbf{a}$. In the (uvw) system, the curl of $\mathbf{a}$ reads

$$\nabla \times \mathbf{a} = -\partial_w a_v \hat{\mathbf{u}} + \partial_w a_u \hat{\mathbf{v}} + (\partial_u a_v - \partial_v a_u)\hat{\mathbf{w}}$$

The w derivative of this is

$$\partial_w(\nabla \times \mathbf{a}) = -\partial_w^2 a_v \hat{\mathbf{u}} + \partial_w^2 a_u \hat{\mathbf{v}} + (\partial_w \partial_u a_v - \partial_w \partial_v a_u)\hat{\mathbf{w}}$$

which does not require the double in-plane derivative. The in-plane derivative of $\nabla \times a$ is

$$\partial_\alpha(\nabla \times \mathbf{a}) = -\partial_\alpha \partial_w a_v \hat{\mathbf{u}} + \partial_\alpha \partial_w a_u \hat{\mathbf{v}} + (\partial_\alpha \partial_u a_v - \partial_\alpha \partial_v a_u)\hat{\mathbf{w}}$$

To elucidate the structure of the w component of this, I write

$$\begin{aligned}
\partial_u a_v^{(n)} - \partial_v a_u^{(n)} &= -n \int \underbrace{\Big[\big(\overline{\mathbf{x}}\big)_u\big(\overline{\mathbf{x}} - \overline{\mathbf{Q}}\big)_v - \big(\overline{\mathbf{x}}\big)_v\big(\overline{\mathbf{x}} - \overline{\mathbf{Q}}\big)_u\Big]}_{\left(\overline{\mathbf{Q}}\right)_u\left(\overline{\mathbf{x}}\right)_v - \left(\overline{\mathbf{Q}}\right)_v\left(\overline{\mathbf{x}}\right)_u} r^{n-2}\,dA \\
&= -n\Big[\big(\overline{\mathbf{Q}}\big)_u \int \big(\overline{\mathbf{x}}\big)_v r^{n-2}\,dA - \big(\overline{\mathbf{Q}}\big)_v \int \big(\overline{\mathbf{x}}\big)_u r^{n-2}\,dA\Big] \\
&= -n\Big[\big(\overline{\mathbf{Q}}\big)_u \mathcal{J}_v^{n-2} - \big(\overline{\mathbf{Q}}\big)_v \mathcal{J}_u^{n-2}\Big]
\end{aligned}$$

and thus

$$\partial_\alpha(\nabla \times \mathbf{a})_w = -n\Big[\big(\overline{\mathbf{Q}}\big)_u \partial_\alpha \mathcal{J}_v^{n-2} - \big(\overline{\mathbf{Q}}\big)_v \partial_\alpha \mathcal{J}_u^{n-2}\Big].$$

The upshot is that all quantities needed to compute first and second derivatives of the potentials may be obtained from the $\mathcal{I}, J$ integrals and their first derivatives.

2.4.2 Derivatives of $\mathcal{I}, J$ integrals

(In what follows, subscripts μ, ν refer to derivatives with respect to coordinates in the plane of the panel [u, v derivatives in the (u, v, w) system], as distinct from w derivatives, which are directional derivatives in the direction normal to the panel.)

Derivatives of the $\mathcal{I}$ integrals, and the normal derivative of the $\mathcal{J}$ integrals, may be carried out at the level of surface integrals:

$$\begin{aligned}
\partial_\mu \mathcal{I}^p(\mathbf{x}) &= \partial_\mu \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2}\,dA = -p \int_{\mathcal{P}} \rho_\mu [\rho^2 + w^2]^{(p-2)/2}\,dA \\
&= -p\mathcal{J}_\mu^{p-2}(\mathbf{x}) \\
\partial_w \mathcal{I}^p(\mathbf{x}) &= \partial_w \int_{\mathcal{P}} [\rho^2 + w^2]^{p/2}\,dA = pw\partial_w \int_{\mathcal{P}} [\rho^2 + w^2]^{(p-2)/2}\,dA \\
&= pw\mathcal{I}^{p-2}(\mathbf{x})
\end{aligned}$$

and similarly

$$\partial_w \boldsymbol{\mathcal{J}}^p(\mathbf{x}) = pw\boldsymbol{\mathcal{J}}^{p-2}(\mathbf{x})$$

In-plane derivatives of the $\mathcal{J}$ integrals are easiest to carry out at the level of line integrals:

$$\partial_\mu \mathcal{J}^p_\nu(\mathbf{x}) = \frac{1}{(p+2)} \sum_i \Big\{ \Big(\partial_\mu J^p\Big) \hat{m}_{i\nu} \Big\}$$

$$\partial_\mu J^p(s_i^-, s_i^+, t_i, w) = \left[\frac{\partial J^p}{\partial \hat{\boldsymbol{\ell}}_i} \hat{\boldsymbol{\ell}}_i + \frac{\partial J^p}{\partial \hat{\mathbf{m}}_i} \hat{\mathbf{m}}_i \right]$$

$$\frac{\partial J^p}{\partial \hat{\boldsymbol{\ell}}_i} = -(s_i^{-2} + t_i^2 + w^2)^{(p+2)/2} - (s_i^{+2} + t_i^2 + w^2)^{(p+2)/2}$$

$$\frac{\partial J^p}{\partial \hat{\boldsymbol{m}}_i} = (p+2) t_i J^{p-2}$$

2.4.3 Derivatives of desingularized terms

We have

$$G^{\text{DS}}(r) = \frac{\texttt{ExpRelBar}(ikr, 3)}{4\pi r}$$

and thus

$$\partial_i G^{\text{DS}}(r) = r_i \Big[ik \frac{\texttt{ExpRelBar}(ikr,2)}{4\pi r^2} - \frac{\texttt{ExpRelBar}(ikr,3)}{4\pi r^3} \Big]$$

$$\begin{aligned} \partial_i \partial_j G^{\text{DS}}(r) &= \delta_{ij} \Big[ik \frac{\texttt{ExpRelBar}(ikr,2)}{4\pi r^2} - \frac{\texttt{ExpRelBar}(ikr,3)}{4\pi r^3} \Big] \\ &\quad + r_i r_j \Big[(ik)^2 \frac{\texttt{ExpRelBar}(ikr,1)}{4\pi r^3} - 3ik \frac{\texttt{ExpRelBar}(ikr,2)}{4\pi r^4} + 3 \frac{\texttt{ExpRelBar}(ikr,3)}{4\pi r^5} \Big] \end{aligned}$$

3 Far fields at nearby points

The contribution of a single panel $\mathcal{P}$ to the reduced fields may be written in an alternative way using the dyadic Green's functions $\mathbf{G}(\mathbf{r}), \mathbf{C}(\mathbf{r})$

$$e_i(\mathbf{x}) = \int G_{ij}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')d\mathbf{x}', \qquad h_i(\mathbf{x}) = -ik\int C_{ij}(\mathbf{x},\mathbf{x}')b_j(\mathbf{x}')d\mathbf{x}'$$

Retaining only far-field contributions,

$$G_{ij} = \left(\delta_{ij} + \frac{r_i r_j}{r^2}\right)\frac{e^{ikr}}{4\pi r}, \qquad -ikC_{ij} = -ik\varepsilon_{ijk}\frac{r_k}{r}\frac{e^{ikr}}{4\pi r}$$

Separate e_i into singular and non-singular contributions:

$$\begin{aligned}
\mathbf{e}(\mathbf{x}) &= \frac{\ell}{8\pi A}\Big[\mathbf{e}^{(-1)}(\mathbf{x}) + \mathbf{e}^{\text{DS}}(\mathbf{x})\Big] \\
e_i^{(-1)}(\mathbf{x}) &= \int \left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right)(\mathbf{x}-\mathbf{Q})_j\, d\mathbf{r} \\
e_i^{\text{DS}}(\mathbf{x}) &= \int \left(\delta_{ij} + \frac{r_i r_j}{r^2}\right)\frac{\texttt{ExpRelBar}(ikr,1)}{r}(\mathbf{x}-\mathbf{Q})_j\, d\mathbf{r}
\end{aligned}$$

The contributions to $\mathbf{e}^{(-1)}$ are easiest to work out in the coordinate system of $\mathcal{P}$. The first term is

$$\begin{aligned}
e_\mu^{(-1)a}(\mathbf{x}) &= \int \frac{(\mathbf{x}'-\mathbf{Q})_\mu}{r}\, dA \\
&= a_\mu^{(-1)}
\end{aligned}$$

The second term is

$$\begin{aligned}
e_\mu^{(-1)b}(\mathbf{x}) &= \int \frac{(\mathbf{x}'-\mathbf{x})_\mu(\mathbf{x}'-\mathbf{x})_\nu(\mathbf{x}'-\mathbf{Q})_\nu}{r^3} dA \\
&= \int \frac{(\mathbf{x}'-\mathbf{x})_\mu}{r}\, dA - \big(\overline{\mathbf{Q}}\big)_\nu \int \frac{(\mathbf{x}'-\mathbf{x})_\mu(\mathbf{x}'-\mathbf{x})_\nu}{r^3} dA
\end{aligned}$$

The w component of this is

$$e_w^{(-1)b}(\mathbf{x}) = w\mathcal{I}^{-1} - w\big(\overline{\mathbf{Q}}\big)_\nu \mathcal{J}_\nu^{-3}$$