Implicit handling of multilayered material substrates in full-wave SCUFF-EM calculations

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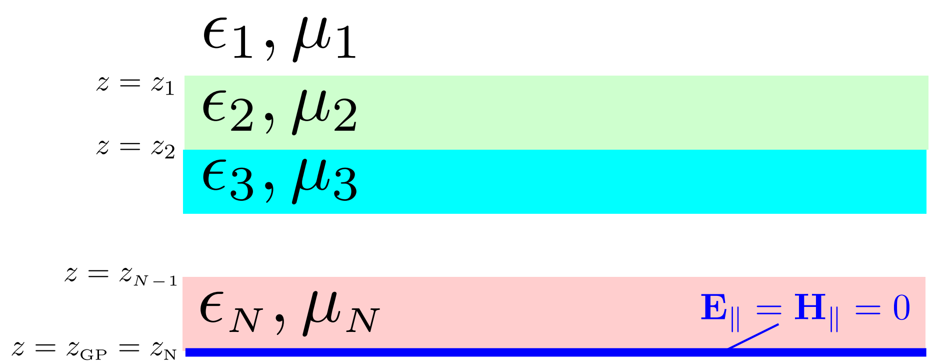

Figure 1: Geometry of the layered substrate. The nth layer has relative permittivity and permeability ϵ_n, μ_n, and its lower surface lies at $z = z_n$. The ground plane, if present, lies at $z = z_{\mathrm{GP}}$.

1 Overview

In a previous memo[1] I considered SCUFF-STATIC electrostatics calculations in the presence of a multilayered dielectric substrate. In this memo I extend that discussion to the case of *full-wave* (i.e. nonzero frequencies beyond the quasistatic regime) scattering calculations in the SCUFF-EM core library.

Substrate geometry

As shown in Figure 1, I consider a multilayered substrate consisting of N material layers possibly terminated by a perfectly-conducting ground plane. The uppermost layer (layer 1) is the infinite half-space above the substrate. The nth layer has relative permittivity and permeability ϵ_n, μ_n, and its lower surface lies at $z = z_n$. The ground plane, if present, lies at $z \equiv z_{\mathrm{N}} \equiv z_{\mathrm{GP}}$. If the ground plane is absent, layer N is an infinite half-space[2] ($z_{\mathrm{N}} = -\infty$).

Definition of the substrate DGF

For given source and evaluation (or "destination") points $\{\mathbf{x}_s, \mathbf{x}_d\}$ at a given angular frequency ω in the presence of a multilayer substrate, the **E** and **H** fields at $\mathbf{x}_d$ due to point sources at $\mathbf{x}_s$ receive corrections (relative to their free-space values) due to the presence of the substrate. These are described by the substrate dyadic Green's function $\boldsymbol{\mathcal{G}}(\omega; \mathbf{x}_d, \mathbf{x}_s)$, a 6×6 matrix with a 2×2 block

[1] "Implicit handling of multilayered dielectric substrates in SCUFF-STATIC"

[2] As in the electrostatic case, this means that a finite-thickness substrate consisting of N material layers is described as a stack of $N + 1$ layers in which the bottommost layer is an infinite vacuum half-space.

structure:

$$\boldsymbol{\mathcal{G}}(\omega;\mathbf{x}_d,\mathbf{x}_s)=\left(\begin{array}{cc}\boldsymbol{\mathcal{G}}^{\text{EE}} & \boldsymbol{\mathcal{G}}^{\text{EM}}\\ \boldsymbol{\mathcal{G}}^{\text{ME}} & \boldsymbol{\mathcal{G}}^{\text{MM}}\end{array}\right) \tag{1a}$$

with the 3×3 subblocks defined by

$$\mathcal{G}_{ij}^{\text{PQ}}=\left(\begin{array}{l}\text{substrate contribution to } i\text{-component of P-type}\\ \text{field at } \mathbf{x}_d \text{ due to } j\text{-directed Q-type source at } \mathbf{x}_s\\ \text{with angular frequency } \omega\end{array}\right) \tag{1b}$$

Iff $\mathbf{x}_s,\mathbf{x}_d$ lie in the same layer of the multilayer substrate, then to get the *total* fields (1) must be augmented by the contribution of the homogeneous DGF of the medium::

$$\boldsymbol{\mathcal{G}}^{\text{total}}(\omega,\mathbf{x}_s,\mathbf{x}_d)=\left(\begin{array}{cc}\boldsymbol{\mathcal{G}}^{\text{EE}} & \boldsymbol{\mathcal{G}}^{\text{EM}}\\ \boldsymbol{\mathcal{G}}^{\text{ME}} & \boldsymbol{\mathcal{G}}^{\text{MM}}\end{array}\right)+\left(\begin{array}{cc}ik_rZ_0Z^r\mathbf{G} & ik_r\mathbf{C}\\ -ik_r\mathbf{C} & \frac{ik_r}{Z_0Z^r}\mathbf{C}\end{array}\right),\quad \mathbf{x}_s,\mathbf{x}_d\in\text{layer }\#r \tag{2}$$

where k_r, Z^r are the wavevector and relative wave impedance of substrate layer r. On the other hand, if $\mathbf{x}_s,\mathbf{x}_d$ lie in different layers then $\boldsymbol{\mathcal{G}}$ in (1) already gives the total field at $\mathbf{x}_d$.

Mechanics of implementation in SCUFF-EM

The full-wave substrate implementation in SCUFF-EM consists of multiple working parts that fit together in a somewhat modular fashion. Roughly speaking, the problem may be divided into two parts:

(a) For given source and evaluation (or "destination") points $\{\mathbf{x}_s,\mathbf{x}_d\}$ at a given angular frequency ω in the presence of a multilayer substrate, numerically compute the substrate DGF correction $\boldsymbol{\mathcal{G}}(\omega,\mathbf{x}_d,\mathbf{x}_s)$. This task is independent of SCUFF-EM. (Section 2.)

(b) For a SCUFF-EM geometry in the presence of a substrate, compute the substrate corrections to the BEM system matrix $\mathbf{M}$ and RHS vector $\mathbf{v}$, as well as the substrate corrections to post-processing quantities such as scattered fields. (Section **??**.)

2 LIBSUBSTRATE: Numerical computation of $\boldsymbol{\mathcal{G}}$

The LIBSUBSTRATE code that implements step **(a)** above (numerical computation of the substrate DGF $\boldsymbol{\mathcal{G}}$) divides the problem into several steps:

(a1) Solve a linear system to obtain the Fourier-space representation $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$. Here $\mathbf{q} = (q_x, q_y)$ is a 2D Fourier variable. (Section 2.1.)

(a2) Reduce the two-dimensional integral over $\mathbf{q}$ to a one-dimensional integral over $|\mathbf{q}| \equiv q$. (Section 2.2.)

(a3) Evaluate the q integral using known methods for evaluating Sommerfeld integrals. (Section 2.3.)

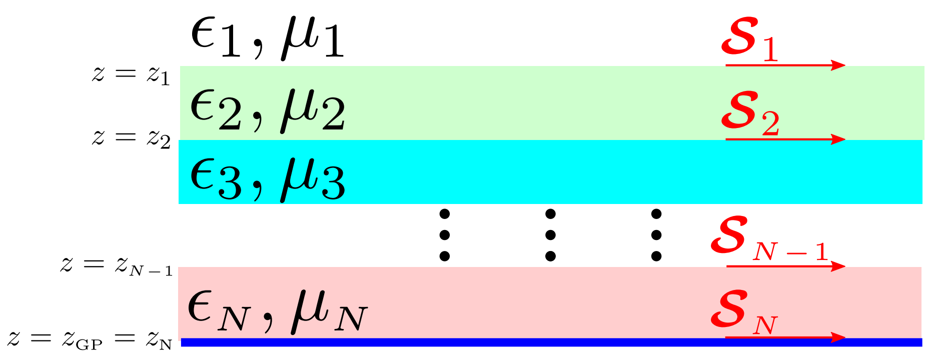

Figure 2: Effective surface-current approach to treatment of multilayer substrate. External field sources induce a distribution of electric and magnetic surface currents $\boldsymbol{\mathcal{S}}_n = \binom{\mathbf{K}_n}{\mathbf{N}_n}$ on the nth material interface, and the fields radiated by these effective currents account for the disturbance presented by the substrate.

2.1 Computation of Fourier-space DGF $\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q})$

To compute the substrate correction to the fields of external sources, I consider the effective tangential electric and magnetic surface currents $\mathbf{K}$ and $\mathbf{N}$ induced on the interfacial ayers by the external field sources (Figure 2). This is the direct extension to full-wave problems of the formalism I used in the electrostatic case, and it comports well with the spirit of surface-integral-equation methods.

More specifically, on the material interface layer at $z = z_n$ I have a four-vector surface-current density $\boldsymbol{\mathcal{S}}_n(\boldsymbol{\rho})$, where $\boldsymbol{\rho} = (x, y)$ and the components of $\boldsymbol{\mathcal{S}}$ are

$$\boldsymbol{\mathcal{S}}_n(\boldsymbol{\rho}) = \left(\begin{array}{c} K_x(\boldsymbol{\rho}) \\ K_y(\boldsymbol{\rho}) \\ N_x(\boldsymbol{\rho}) \\ N_y(\boldsymbol{\rho}) \end{array}\right). \tag{3}$$

Fields in layer interiors. I will adopt the convention that the lower (upper) bounding surface for each region is the positive (negative) bounding surface for that region in the usual sense of SCUFF-EM regions and surfaces (in which the sign of a {surface,region} pair $\{\mathcal{S}, \mathcal{R}\}$ is the sign with which surface currents on $\mathcal{S}$ contribute to fields in $\mathcal{R}$). Thus, at a point $\mathbf{x} = (\boldsymbol{\rho}, z)$ in the interior of layer n ($z_{n-1} > z > z_n$), the six-vector of total fields $\boldsymbol{\mathcal{F}} = \left(\begin{array}{c} \mathbf{E} \\ \mathbf{H} \end{array}\right)$ reads

$$\boldsymbol{\mathcal{F}}_n(\boldsymbol{\rho}, z) = -\boldsymbol{\mathcal{G}}^{0n}(z_{n-1}) \star \boldsymbol{\mathcal{S}}_{n-1} + \boldsymbol{\mathcal{G}}^{0n}(z_n) \star \boldsymbol{\mathcal{S}}_n + \boldsymbol{\mathcal{F}}_n^{\mathrm{ext}}(\boldsymbol{\rho}, z) \tag{4}$$

where $\boldsymbol{\mathcal{F}}_n^{\mathrm{ext}}$ are the externally-sourced (incident) fields due to sources in layer n, $\boldsymbol{\mathcal{G}}^{0n}$ is the homogeneous dyadic Green's function for material layer n, and $\star$

is shorthand for the convolution operation

$$\text{“}\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)\equiv\boldsymbol{\mathcal{G}}(z')\star\boldsymbol{\mathcal{S}}\text{”}\qquad\Longrightarrow\qquad\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)=\int\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}-\boldsymbol{\rho}',z-z')\cdot\boldsymbol{\mathcal{S}}(\boldsymbol{\rho}')d\boldsymbol{\rho}'\tag{5}$$

where the integral extends over the entire interfacial plane. I will evaluate convolutions of this form using the 2D Fourier representation of $\boldsymbol{\mathcal{G}}^{0n}$:

$$\boldsymbol{\mathcal{G}}^{0n}(\boldsymbol{\rho},z)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\widetilde{\boldsymbol{\mathcal{G}}^{0n}}(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\tag{6a}$$

$$\widetilde{\boldsymbol{\mathcal{G}}^{0n}}(\mathbf{q},z)=\frac{1}{2}\left(\begin{array}{cc}-\frac{\omega\mu_0\mu_n}{q_{zn}}\widetilde{\mathbf{G}}^{\pm} & +\widetilde{\mathbf{C}}^{\pm}\\ -\widetilde{\mathbf{C}}^{\pm} & -\frac{\omega\epsilon_0\epsilon_n}{q_{zn}}\widetilde{\mathbf{G}}^{\pm}\end{array}\right)e^{iq_z|z|}\tag{6b}$$

$$\widetilde{\mathbf{G}}^{\pm}(\mathbf{q},k)=\left(\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right)-\frac{1}{k^2}\left(\begin{array}{ccc}q_x^2 & q_xq_y & \pm q_xq_z\\ q_yq_x & q_y^2 & \pm q_yq_z\\ \pm q_zq_x & \pm q_zq_y & q_z^2\end{array}\right)\tag{6c}$$

$$\widetilde{\mathbf{C}}^{\pm}(\mathbf{q},k)=\left(\begin{array}{ccc}0 & \mp 1 & +q_y/q_z\\ \pm 1 & 0 & -q_x/q_z\\ -q_y/q_z & +q_x/q_z & 0\end{array}\right)\tag{6d}$$

$$k_n\equiv\sqrt{\epsilon_0\epsilon_n\mu_0\mu_n}\cdot\omega,\qquad q_z\equiv\sqrt{k^2-|\mathbf{q}|^2},\qquad \pm=\text{sign }z.\tag{6e}$$

With this representation, convolutions like (5) become products in Fourier space:

$$\boldsymbol{\mathcal{G}}(z')\star\boldsymbol{\mathcal{S}}=\boldsymbol{\mathcal{F}}(\boldsymbol{\rho},z)=\int\frac{d^2\mathbf{q}}{(2\pi)^2}\widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{q},z)e^{i\mathbf{q}\cdot\boldsymbol{\rho}},\quad\text{with}\quad\widetilde{\boldsymbol{\mathcal{F}}(\mathbf{q},z)}=\widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q},z-z')\widetilde{\boldsymbol{\mathcal{S}}}(\mathbf{q})$$

Surface currents from incident fields. To determine the surface currents induced by given incident-field sources, I apply boundary conditions. The boundary condition at $z=z_n$ is that the tangential $\mathbf{E},\mathbf{H}$ fields be continuous: in Fourier space, we have

$$\widetilde{\boldsymbol{\mathcal{F}}}_{\parallel}(\mathbf{q},z=z_n^+)=\widetilde{\boldsymbol{\mathcal{F}}}_{\parallel}(\mathbf{q},z=z_n^-)\tag{7}$$

The fields just **above** the interface ($z\to z_n^+$) receive contributions from three sources:

- Surface currents at $z=z_{n-1}$, which contribute with a minus sign and via the Green's function for region n;
- Surface currents at $z=z_n$, which contribute with a plus sign and via the Green's function for region n; and
- external field sources in region n.

The fields just **below** the interface ($z = z_n^-$) receive contributions from three sources:

- Surface currents at $z = z_n$, which contribute with a minus sign and via the Green's function for region $n+1$;
- Surface currents at $z = z_{n+1}$, which contribute with a plus sign and via the Green's function for region $n+1$; and
- external field sources in region $n+1$.

Then equation (7) reads

$$\begin{aligned}&-\widetilde{\boldsymbol{\mathcal{G}}^{0n}}_{\|}(z_n - z_{n-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n-1} + \widetilde{\boldsymbol{\mathcal{G}}^{0n}}_{\|}(0^+)\cdot\widetilde{\boldsymbol{\mathcal{S}}}_n + \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n\|}(z_n)\\ &\qquad = -\widetilde{\boldsymbol{\mathcal{G}}^{0,n+1}}\|(0^-)\cdot\widetilde{\boldsymbol{\mathcal{S}}}_n + \widetilde{\boldsymbol{\mathcal{G}}^{0,n+1}}\|(z_n - z_{n+1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n+1} + \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n+1\|}(z_n)\end{aligned}$$

or

$$\mathbf{M}_{n,n-1}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n-1} + \mathbf{M}_{n,n}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_n + \mathbf{M}_{n,n+1}\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{n+1} = \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n+1\|}(z_n) - \widetilde{\boldsymbol{\mathcal{F}}}^{\text{ext}}_{n\|}(z_n) \tag{8}$$

with the 4×4 matrix blocks[3]

$$\mathbf{M}_{n,n-1} = -\widetilde{\boldsymbol{\mathcal{G}}^{0n}}_{\|}(z_n - z_{n-1}) \tag{11a}$$

$$\mathbf{M}_{n,n} = +\widetilde{\boldsymbol{\mathcal{G}}^{0n}}_{\|}(0^+) + \widetilde{\boldsymbol{\mathcal{G}}^{0,n+1}}_{\|}(0^-) \tag{11b}$$

$$\mathbf{M}_{n,n+1} = -\widetilde{\boldsymbol{\mathcal{G}}^{0,n+1}}_{\|}(z_n - z_{n+1}) \tag{11c}$$

Writing down equation (8) equation for all N dielectric interfaces yields a $4N\times 4N$ system of linear equations, with triadiagonal 4×4 block form, relating the surface currents on all layers to the external fields due to sources in all regions:

$$\mathbf{M}\cdot\mathbf{s} = \mathbf{f} \tag{12}$$

[3]The 4×4 $\mathbf{M}$ blocks here have 2×2 block structure:

$$\mathbf{M}_{n,n} = \sum_{r\in\{n,n+1\}} \frac{1}{2}\begin{pmatrix} -\frac{\omega\epsilon_r}{Z_0 q_{zr}}\mathbf{g}(k_r,\mathbf{q}) & 0 \\ 0 & -\frac{\omega\mu_r Z_0}{q_{zr}}\mathbf{g}(k_r,\mathbf{q})\end{pmatrix} \tag{9}$$

$$\mathbf{M}_{n,n\pm1} = \frac{1}{2}\begin{pmatrix} -\frac{\omega\epsilon_r}{Z_0 q_{zr}}\mathbf{g}(k_r,\mathbf{q}) & \mathbf{c}^{\pm} \\ -\mathbf{c}^{\pm} & -\frac{\omega\mu_r Z_0}{q_{zn^*}}\mathbf{g}(k_r,\mathbf{q})\end{pmatrix} e^{iq_{zr}|z_n - z_{n\pm1}|} \tag{10}$$

where I put $r \equiv \begin{cases} n, & \text{for } \mathbf{M}_{n,n-1} \\ n+1, & \text{for } \mathbf{M}_{n,n+1}\end{cases}$. and

$$\mathbf{g}(k;\mathbf{q}) = \mathbf{1} - \frac{\mathbf{q}\mathbf{q}^\dagger}{k^2}, \qquad \mathbf{c}^{\pm} = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0\end{pmatrix}$$

where $\mathbf{M}$ is the $4N\times 4N$ block-tridiagonal matrix (11) and where the $4N$-vectors $\mathbf{s}$, $\mathbf{f}$ read

$$\mathbf{s}=\left(\begin{array}{c}\widetilde{\boldsymbol{\mathcal{S}}}_1\\ \widetilde{\boldsymbol{\mathcal{S}}}_2\\ \widetilde{\boldsymbol{\mathcal{S}}}_3\\ \vdots\\ \widetilde{\boldsymbol{\mathcal{S}}}_N\end{array}\right),\qquad \mathbf{f}=\left(\begin{array}{c}-\widetilde{\boldsymbol{\mathcal{F}}}_{1\|}(z_1)+\widetilde{\boldsymbol{\mathcal{F}}}_{2\|}(z_1)\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{2\|}(z_2)+\widetilde{\boldsymbol{\mathcal{F}}}_{3\|}(z_2)\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{3\|}(z_3)+\widetilde{\boldsymbol{\mathcal{F}}}_{3\|}(z_4)\\ \vdots\\ -\widetilde{\boldsymbol{\mathcal{F}}}_{N-1,\|}(z_{N-1})+\widetilde{\boldsymbol{\mathcal{F}}}_{N\|}(z_{N-1})\end{array}\right).$$

Solving (12) yields the induced surface currents on all layers in terms of the incident fields:

$$\mathbf{s}=\mathbf{W}\cdot\mathbf{f}\quad\text{where}\quad \mathbf{W}\equiv\mathbf{M}^{-1}$$

or, more explicitly,

$$\widetilde{\boldsymbol{\mathcal{S}}}_n=\sum_m W_{nm}\mathbf{f}_m \tag{13}$$

Surface currents induced by point sources

For DGF computations the incident fields arise from a single point source—say, a j-directed source in region s. Then the only nonzero length-4 blocks of the RHS vector in (12) are $\mathbf{f}_{s-1},\mathbf{f}_{\text{S}}$ with components ($\ell=\{1,2,4,5\}$)

$$\Big(\mathbf{f}_{s-1}\Big)_\ell=-\widetilde{\mathcal{G}}^{0s}_{\ell j}(z_{s-1}-z_{\text{S}}),\qquad \Big(\mathbf{f}_s\Big)_\ell=+\widetilde{\mathcal{G}}^{0s}_{\ell j}(z_{\text{S}}-z_{\text{S}}) \tag{14}$$

and the surface currents on interface layer n are obtained by solving (13):

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{S}}}_n&=\mathbf{W}_{n,s-1}\,\mathbf{f}_{s-1}+\mathbf{W}_{n,s}\,\mathbf{f}_s\\ &=\sum_{p=0}^{1}(-1)^{p+1}\mathbf{W}_{n,s-1+p}\cdot\widetilde{\boldsymbol{\mathcal{G}}^{0s}}_{\|,j}(z_{\text{S}}-z_{s-1+p})\end{aligned} \tag{15}$$

Fields due to surface currents

Given the surface currents induced by a j-directed point source at $\mathbf{x}_{\text{S}}$, I evaluate the fields due to these currents to get DGF components. If the evaluation point $\mathbf{x}_{\text{D}}$ lies in region d, then the fields receive contributions from the surface currents at z_{d-1} and z_{D}, propagated by the homogeneous DGF for region d:

$$\begin{aligned}\widetilde{\boldsymbol{\mathcal{F}}}(z_{\text{D}})&=-\widetilde{\boldsymbol{\mathcal{G}}^{0d}}(z_{\text{D}}-z_{d-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{d-1}+\widetilde{\boldsymbol{\mathcal{G}}^{0d}}(z_{\text{D}}-z_{\text{D}})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_d\\ &=\sum_{q=0}^{1}(-1)^{q+1}\widetilde{\boldsymbol{\mathcal{G}}^{0d}}(z_{\text{D}}-z_{d+q-1})\cdot\widetilde{\boldsymbol{\mathcal{S}}}_{d+q-1}\end{aligned}$$

(The minus sign in the first term arises because, in my convention, surface currents on the upper surface of a region contribute to the fields in that region with a minus sign). Inserting (15), the i component here—which is the ij component of the substrate DGF—is

$$\widetilde{\mathcal{G}}_{ij}(z_{\text{D}}, z_{\text{S}}) \equiv \widetilde{\mathcal{F}}_i(z_{\text{D}}) = \sum_{p,q=0}^{1} (-1)^{p+q} \widetilde{\boldsymbol{\mathcal{G}}^{0d}}_{i,\parallel}(z_{\text{D}} - z_{d-1+q}) \mathbf{W}_{d-1+q,s-1+p} \widetilde{\boldsymbol{\mathcal{G}}^{0s}}_{\parallel,j}(z_{s-1+p} - z_{\text{S}}) \tag{16}$$

2.2 Reduction of 2D integrals over q to 1D (Sommerfeld) integrals over q

The real-space DGF is the inverse Fourier transform of (16):

$$\boldsymbol{\mathcal{G}}(\boldsymbol{\rho}_{\mathrm{D}}, z_{\mathrm{D}}; \boldsymbol{\rho}_{\mathrm{S}}, z_{\mathrm{S}}) = \int \frac{d^2\mathbf{q}}{(2\pi)^2} \widetilde{\boldsymbol{\mathcal{G}}}(\mathbf{q}; z_{\mathrm{D}}; z_{\mathrm{S}}) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}_{\mathrm{D}}-\boldsymbol{\rho}_{\mathrm{S}})}. \tag{17}$$

I will evaluate the $\mathbf{q}$ integral here in polar coordinates, $\mathbf{q} = (q_x, q_y) = \big(q\cos\theta_{\mathbf{q}}, q\sin\theta_{\mathbf{q}}\big)$ with $q = |\mathbf{q}|$. Although $\widetilde{G}(\mathbf{q})$ has 36 Cartesian components, these may be expressed in terms of just 18 scalar functions of q times cosines and sines of $\theta_{\mathbf{q}}$:

$$\begin{aligned}
\widetilde{\boldsymbol{\mathcal{G}}^{\mathrm{EE}}}(\mathbf{q}) =& \widetilde{g}^{\mathrm{EE}0\parallel}(q) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{0\parallel}} + \widetilde{g}^{\mathrm{EE}0z}(q) \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{\Lambda}^{0z}} \\
&+ \widetilde{g}^{\mathrm{EE1A}}(q) \underbrace{\begin{pmatrix} 0 & 0 & \cos\theta_{\mathbf{q}} \\ 0 & 0 & \sin\theta_{\mathbf{q}} \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{1}(\theta_{\mathbf{q}})} + \widetilde{g}^{\mathrm{EE1B}}(q) \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cos\theta_{\mathbf{q}} & \sin\theta_{\mathbf{q}} & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{1\dagger}(\theta_{\mathbf{q}})} \\
&+ \widetilde{g}^{\mathrm{EE2}}(q) \underbrace{\begin{pmatrix} \cos^2\theta_{\mathbf{q}} & \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & 0 \\ \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & \sin^2\theta_{\mathbf{q}} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{2A}(\theta_{\mathbf{q}})}
\end{aligned}$$

$$\begin{aligned}
\widetilde{\boldsymbol{\mathcal{G}}^{\mathrm{EM}}}(\mathbf{q}) =& g^{\mathrm{EM}0\parallel}(q) \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{0\times}} + g^{\mathrm{EM2}}(q) \underbrace{\begin{pmatrix} \cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & \sin^2\theta_{\mathbf{q}} & 0 \\ -\cos^2\theta_{\mathbf{q}} & -\cos\theta_{\mathbf{q}}\sin\theta_{\mathbf{q}} & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\boldsymbol{\Lambda}^{2\times}} \\
&+ g^{\mathrm{EM1A}}(q) \underbrace{\begin{pmatrix} 0 & 0 & -\sin\theta_{\mathbf{q}} \\ 0 & 0 & +\cos\theta_{\mathbf{q}} \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{\Lambda}^{1\times}} + g^{\mathrm{EM1B}}(q) \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\sin\theta_{\mathbf{q}} & \cos\theta_{\mathbf{q}} & 1 \end{pmatrix}}_{\boldsymbol{\Lambda}^{1\times\dagger}}
\end{aligned}$$

(The expressions for $\widetilde{\boldsymbol{\mathcal{G}}}^{\text{ME}}$ and $\widetilde{\boldsymbol{\mathcal{G}}}^{\text{MM}}$ are similar.)

$$\int \frac{d\mathbf{q}}{(2\pi)^2}\widetilde{g}(q)e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\boldsymbol{\Lambda}^{(0)} = \int \frac{qdq}{2\pi}\widetilde{g}(q)J_0(q\rho)\boldsymbol{\Lambda}^{(0)}$$

$$\int \frac{d\mathbf{q}}{(2\pi)^2}\widetilde{g}(q)e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\boldsymbol{\Lambda}^{(1)}(\theta_{\mathbf{q}}) = i\int \frac{qdq}{2\pi}\widetilde{g}(q)J_1(q\rho)\boldsymbol{\Lambda}^{(1)}(\theta_{\boldsymbol{\rho}})$$

$$\int \frac{d\mathbf{q}}{(2\pi)^2}\widetilde{g}(q)e^{i\mathbf{q}\cdot\boldsymbol{\rho}}\boldsymbol{\Lambda}^{(2)}(\theta_{\mathbf{q}}) = -\int \frac{qdq}{2\pi}\widetilde{g}(q)J_2(q\rho)\boldsymbol{\Lambda}^2(\theta_{\boldsymbol{\rho}}) + \int \frac{qdq}{2\pi}\widetilde{g}(q)\frac{J_1(q\rho)}{q\rho}\boldsymbol{\Lambda}^2$$

2.3 Evaluation of Sommerfeld integrals

3 Substrate contributions to panel and panel-panel integrals

$$\begin{aligned}
\boldsymbol{\mathcal{G}}^{\text{PQ}}(\rho,\theta,z_{\text{D}},z_{\text{S}}) &= \sum_{\nu p} g^{\text{PQ}\nu p}(\rho,z_{\text{D}},z_{\text{S}})\boldsymbol{\Lambda}^{\text{PQ}\nu p}(\theta)\\
g^{\text{PQ}\nu p}(\rho) &= \int \frac{q\,dq}{(2\pi)}\widetilde{g}^{\text{PQ}\nu p}(q)J_\nu(q\rho)e^{i\alpha(q,z_{\text{D}},z_{\text{S}})}\boldsymbol{\Lambda}^{\text{PQ}\nu p}(\theta)
\end{aligned}$$

$$M^{\text{PQ}}_{\alpha\beta}\Big\langle \mathbf{b}_\alpha \Big| \boldsymbol{\mathcal{G}}^{\text{PQ}} \Big| \mathbf{b}_\beta \Big\rangle$$