Computation of power, force, and torque in SCUFF-EM

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1 Overview

The time-average power absorbed and scattered by, and the time-average force or torque on, material bodies in the presence of harmonically-varying electromagnetic radiation are common quantities of interest in computational electromagnetism. These quantities may be computed from knowledge of the electromagnetic fields $\boldsymbol{\mathcal{F}} = \binom{\mathbf{E}}{\mathbf{H}}$ in the presence of the bodies; more specifically, powers, forces, and torques (PFTs) depend quadratically (bilinearly) on $\boldsymbol{\mathcal{F}}$. On the other hand, in the surface-integral-equation (SIE) approach to electromagnetic scattering, one solves first for equivalent electric and magnetic *surface currents* $\boldsymbol{\mathcal{C}} = \binom{\mathbf{K}}{\mathbf{N}}$ flowing on the body surfaces; the fields are then obtained from the currents via linear convolution, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{G}} \star \boldsymbol{\mathcal{C}}$ (with $\boldsymbol{\mathcal{G}}$ a Green's dyadic). Since the PFTs are quadratic in the fields, and the fields are linear in the currents, it follows that the PFTs are quadratic in the currents, and thus that we may bypass the field-computation step and compute PFTs directly as quadratic (bilinear) functions of the surface currents.

There are (at least) three distinct ways to express these bilinear relationships, which originate from distinct starting points for evaluating PFTs from fields:

1. **DSIPFT:** By integrating the Poynting vector or Maxwell stress tensor over a bounding surface that encloses (but does not coincide with) the surface of the object. I will refer to this as the "displaced surface-integral PFT" (DSIPFT) approach, where the term "displaced" indicates that that surface over which we integrate does not coincide with the object surface. This method is conceptually straightforward but computationally expensive.

2. **OPFT:** By integrating the Poynting vector or Maxwell stress tensor over the surface of the object. I refer to this as the "overlap PFT" approach. A key distinction compared to the DSIPFT and EPPFT is that the matrix of the bilinear form it produces is *sparse.*

3. **EPPFT:** By considering the work done on, and the momentum transferred to, the equivalent surface currents by the fields. I refer to this as the "equivalence-principle PFT" approach. In contrast to the other two methods, this approach has the feature of being able to distinguish surface-current distributions induced by soures inside and outside the body.

All three of these methods are implemented in SCUFF-EM. In this note I first derive the three methods and discuss their implementation in the specific context of an SIE solver based on the PMCHWT formulation with RWG basis functions. Then I compare and contrast their relative strengths and weaknesses.

2 Fields and surface currents

To fix notation and ideas for the main text, in this section I review the procedure for computing the fields from the surface currents in an SIE solver.

In the surface-integral-equation (SIE) approach to classical electromagnetism problems, we solve for tangential *equivalent surface currents* (both electric currents $\mathbf{K}$ and magnetic currents $\mathbf{N}$) flowing on the surfaces of homogeneous material bodies. In numerical solvers, we approximate these as finite expansions in a discrete set of $N_{\rm B}$ basis functions; using a convenient 6-vector notation in which $\boldsymbol{\mathcal{C}} \equiv \binom{\mathbf{K}}{\mathbf{N}}$, we put

$$\boldsymbol{\mathcal{C}}(\mathbf{x}) = \sum_{\alpha=1}^{N_{\rm B}} c_\alpha \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x})$$

where $\{\boldsymbol{\mathcal{B}}_\alpha\}$ is a set of 6-vector basis functions[1] and $\{c_\alpha\}$ are scalar expansion coefficients. (We work at a fixed frequency ω and assume all fields and currents vary in time like $e^{-i\omega t}$.)

Scattered fields from surface currents The electric and magnetic fields produced by these currents are linear functions of the $\{c_\alpha\}$ coefficients. In 6-vector notation with $\boldsymbol{\mathcal{F}} \equiv \binom{\mathbf{E}}{\mathbf{H}}$, we have

$$\boldsymbol{\mathcal{F}}(\mathbf{x}) = \sum_{\alpha} c_\alpha \boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) \tag{1}$$

where $\boldsymbol{\mathcal{F}}_\alpha = \binom{\mathbf{E}_\alpha}{\mathbf{H}_\alpha}$ is the six-vector of fields produced by basis function $\boldsymbol{\mathcal{B}}_\alpha$ populated with unit strength. These may be calculated numerically as convolutions over basis functions,

$$\boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) = \int_{\sup \boldsymbol{\mathcal{B}}_\alpha} \boldsymbol{\mathcal{G}}^r(\mathbf{x},\mathbf{x}') \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}')\, d\mathbf{x}'$$

where $\boldsymbol{\mathcal{G}}^r$ is the 6×6 dyadic Green's function for the material properties of the region r in which $\mathbf{x}$ is located.

Surface currents from incident fields In the PMCHWT formulation of the SIE approach, the $\{c_\alpha\}$ coefficients are determined by solving a linear equation of the form

$$\mathbf{M}\cdot\mathbf{c} = \mathbf{v}$$

where the elements of the vector $\mathbf{c}$ are the surface-current expansion coefficients $\{c_\alpha\}$ and the elements of $\mathbf{v}$ are the (negatives of the) projections of the incident field onto the basis functions:

$$v_\alpha = -\left\langle \boldsymbol{\mathcal{B}}_\alpha \right| \boldsymbol{\mathcal{F}}^{\rm inc} \rangle$$

[1] In SCUFF-NEQ these take the form $\boldsymbol{\mathcal{B}}_\alpha = \binom{\mathbf{b}_\alpha}{0}$ or $\binom{0}{\mathbf{b}_\alpha}$ where $\mathbf{b}_\alpha$ is a three-vector RWG basis function. However, the FSC formalism is not specific to this choice.

Also, the α, β matrix element of the matrix $\mathbf{M}$ is a sum of 0, 1, or 2 terms depending on the relative locations of basis functions $\mathbf{b}_\alpha, \mathbf{b}_\beta$. For the simplest sitation of compact objects in vacuum (with no nesting of object), we have simply

$$M_{\alpha\beta} = \begin{cases} \langle \boldsymbol{\mathcal{F}}_\alpha | \, \boldsymbol{\mathcal{G}}^{\rm vac} \, | \boldsymbol{\mathcal{F}}_\beta \rangle \,, & \text{if } \mathbf{b}_\alpha, \mathbf{b}_\beta \text{ live on different surfaces} \\ \langle \boldsymbol{\mathcal{F}}_\alpha | \, \boldsymbol{\mathcal{G}}^{\rm vac} \, | \boldsymbol{\mathcal{F}}_\beta \rangle + \langle \boldsymbol{\mathcal{F}}_\alpha | \, \boldsymbol{\mathcal{G}}^{\mathcal{B}} \, | \boldsymbol{\mathcal{F}}_\beta \rangle \,, & \text{if } \mathbf{b}_\alpha, \mathbf{b}_\beta \text{ both live on the surface of body } \mathcal{B} \end{cases}$$

3 Displaced surface-integral PFT (DSIPFT)

Conceptually the simplest way to compute PFTs is simply to integrate the total time-average flux of energy (Poynting vector) or momentum (Maxwell stress tensor) over a bounding surface containing the body. In this section we review the surface-current bilinear product formula obtained in this approach.

Energy and momentum flux from field bilinears

The Poynting flux and Maxwell stress tensor are quadratic functions of the field components and may be conveniently written in the form of 6-dimensional vector-matrix-vector products. In particular, the time-average power flux in the direction of a unit vector $\hat{\mathbf{n}}$ is

$$\begin{aligned}\mathbf{P}(\mathbf{x})\cdot\hat{\mathbf{n}} &= \frac{1}{2}\,\mathrm{Re}\;\varepsilon_{ijk}\hat{\mathbf{n}}_i E_j^*(\mathbf{x})H_k(\mathbf{x}) \\ &= \frac{1}{4}\boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{\mathrm{P}}(\hat{\mathbf{n}})\,\boldsymbol{\mathcal{F}}(\mathbf{x}) \qquad (2)\end{aligned}$$

with

$$\boldsymbol{\mathcal{N}}^{\mathrm{P}} = \begin{pmatrix} 0 & \mathbf{N}^{\mathrm{P}} \\ -\mathbf{N}^{\mathrm{P}} & 0 \end{pmatrix}, \qquad \mathbf{N}^{\mathrm{P}} = \begin{pmatrix} 0 & \hat{n}_z & -\hat{n_y} \\ -\hat{n}_z & 0 & +\hat{n_x} \\ \hat{n}_y & -\hat{n}_x & 0 \end{pmatrix}.$$

Similarly, the flux of i-directed linear momentum is

$$\begin{aligned}\mathbf{T}_i(\mathbf{x})\cdot\hat{\mathbf{n}} &= \frac{1}{2}\,\mathrm{Re}\;\left[\epsilon E_i^*(\mathbf{x})E_j(\mathbf{x}) + \mu H_i^*(\mathbf{x})H_j(\mathbf{x}) - \frac{\delta_{ij}}{2}\Big(\epsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2\Big)\right]\hat{n}_j \\ &= \frac{1}{4}\boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{i\mathrm{F}}\,\boldsymbol{\mathcal{F}}(\mathbf{x}) \qquad (3)\end{aligned}$$

with

$$\boldsymbol{\mathcal{N}}^{i\mathrm{F}} = \begin{pmatrix} \epsilon\mathbf{N}^{i\mathrm{F}} & 0 \\ 0 & \mu\mathbf{N}^{i\mathrm{F}} \end{pmatrix}$$

where the 3×3 matrix $N^{i\mathrm{F}}$ has entries

$$N^{i\mathrm{F}}_{ab} = \delta_{ai}\hat{n}_b + \delta_{bi}\hat{n}_a - \hat{n}_i\delta_{ab}.$$

For example, if we are computing the x-force ($i = x$) we have

$$\mathbf{N}^{x\mathrm{F}} = \begin{pmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \hat{n}_y & -\hat{n}_x & 0 \\ \hat{n}_z & 0 & -\hat{n}_x \end{pmatrix}.$$

The flux of i-directed *angular* momentum, useful for computations of torque about an origin $\mathbf{x}_0$, is

$$\begin{aligned}\mathbf{t}_i(\mathbf{x})\cdot\hat{\mathbf{n}} &= \frac{1}{2}\,\mathrm{Re}\;\varepsilon_{ijk}(\mathbf{x}-\mathbf{x}_0)_j T_{k\ell}(\mathbf{x})\hat{n}_\ell \qquad (4)\\ &= \frac{1}{4}\boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{i\mathrm{T}}\,\boldsymbol{\mathcal{F}}(\mathbf{x}) \qquad (5)\end{aligned}$$

with

$$\boldsymbol{\mathcal{N}}^{i\mathrm{T}} = \begin{pmatrix} \epsilon\mathbf{N}^{i\mathrm{T}} & 0 \\ 0 & \mu\mathbf{N}^{i\mathrm{T}} \end{pmatrix}$$

where the 3×3 matrix $N^{i\mathrm{T}}$ has entries ($\mathbf{D} = \mathbf{x} - \mathbf{x}_0$)

$$N^{i\mathrm{T}}_{ab} = \varepsilon_{ija} D_j \hat{n}_b + \varepsilon_{ijb} D_j \hat{n}_a - \delta_{ab}\varepsilon_{ijk} D_j \hat{n}_k.$$

Energy and momentum flux from surface-current bilinears

Equations (2), (3), and (5) express the flux of power or momentum as bilinear products of the field six-vectors $\boldsymbol{\mathcal{F}}$. Using (1), we can turn these into bilinear products of the surface-current coefficient vectors $\mathbf{c}$. For example, the power flux (2) becomes

$$\begin{aligned} \mathbf{P}(\mathbf{x})\cdot\hat{\mathbf{n}} &= \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}(\mathbf{x})\boldsymbol{\mathcal{N}}^{\mathrm{P}}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}(\mathbf{x}) \\ &= \frac{1}{4}\sum_{\alpha\beta} c^*_\alpha \Big[\boldsymbol{\mathcal{F}}^{\dagger}_\alpha(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{\mathrm{P}}(\hat{\mathbf{n}})\,\boldsymbol{\mathcal{F}}_\beta(\mathbf{x})\Big] c_\beta \\ &= \mathbf{c}^{\dagger}\mathbf{M}^{\mathrm{PFLUX}}(\mathbf{x},\hat{\mathbf{n}})\mathbf{c} \end{aligned} \tag{6}$$

where $\mathbf{M}^{\mathrm{PFLUX}}(\mathbf{x},\hat{\mathbf{n}})$ is a matrix appropriate for $\hat{\mathbf{n}}$-directed power flux in at $\mathbf{x}$. The fluxes of i-directed linear and angular momentum read similarly

$$\mathbf{T}_i(\mathbf{x})\cdot\hat{\mathbf{n}} = \mathbf{c}^{\dagger}\mathbf{M}^{i\mathrm{FFLUX}}(\mathbf{x},\hat{\mathbf{n}})\mathbf{c} \tag{7}$$

$$\mathbf{t}_i(\mathbf{x})\cdot\hat{\mathbf{n}} = \mathbf{c}^{\dagger}\mathbf{M}^{i\mathrm{TFLUX}}(\mathbf{x},\hat{\mathbf{n}})\mathbf{c} \tag{8}$$

The $\mathbf{M}$ matrices in (6), (7), and (8) are $N_{\mathrm{B}}\times N_{\mathrm{B}}$ matrices whose entries are themselves 6-dimensional matrix-vector products:

$$\begin{aligned} M^{\mathrm{PFLUX}}_{\alpha\beta}(\mathbf{x},\hat{\mathbf{n}}) &= \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_\alpha(\mathbf{x})\boldsymbol{\mathcal{N}}^{\mathrm{P}}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}_\beta(\mathbf{x}) \\ M^{i\mathrm{FFLUX}}_{\alpha\beta}(\mathbf{x},\hat{\mathbf{n}}) &= \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_\alpha(\mathbf{x})\boldsymbol{\mathcal{N}}^{i\mathrm{F}}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}_\beta(\mathbf{x}) \\ M^{i\mathrm{TFLUX}}_{\alpha\beta}(\mathbf{x},\hat{\mathbf{n}}) &= \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_\alpha(\mathbf{x})\boldsymbol{\mathcal{N}}^{i\mathrm{T}}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}_\beta(\mathbf{x}). \end{aligned}$$

Power, force, and torque from surface-current bilinears

The simplest way to obtain surface-current bilinears for the total PFT on one or more bodies is simply to integrate the spatially-resolved fluxes of the previous section over a closed bounding surface $\mathcal{S}$ containing the bodies. Indeed, noting that the $\mathbf{x}$ and $\hat{\mathbf{n}}$ dependence of the flux expressions (6), (7), and (8) is entirely contained in the $\mathbf{M}$ matrices, it is easy to integrate those expressions over $\mathcal{S}$, then pull the surface-current vectors $\mathbf{c}$ outside the integral to identify what we shall call the *displaced surface-integral PFT* (DSIPFT) matrices.

For example, the total power absorbed by material bodies contained within a closed surface $\mathcal{S}$ is given by integrating the LHS of (6) over $\mathcal{S}$:

$$P_{\mathcal{S}}^{\rm abs} = \int_{\mathcal{S}} \mathbf{P}(\mathbf{x}) \cdot \hat{\mathbf{n}}\, d\mathbf{x} \tag{9}$$

with $\hat{\mathbf{n}}$ taken to be the inward-pointing surface normal. Insert the RHS of (6) and pull $\mathbf{c}^\dagger, \mathbf{c}$ outside the integral:

$$= \mathbf{c}^\dagger \mathbf{M}^{{\rm P}\mathcal{S}} \mathbf{c} \tag{10}$$

where the elements of $\mathbf{M}^{{\rm P}\mathcal{S}}$ involve integrals over $\mathcal{S}$:

$$M_{\alpha\beta}^{{\rm P}\mathcal{S}} = \frac{1}{4} \int_{\mathcal{S}} \boldsymbol{\mathcal{F}}_\alpha^\dagger(\mathbf{x}) \boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}}) \boldsymbol{\mathcal{F}}_\beta(\mathbf{x})\, d\mathbf{x}. \tag{11}$$

Similarly, the time-average i-directed force and torque on material bodies contained in $\mathcal{S}$ are

$$F_{i\mathcal{S}} = \mathbf{c}^\dagger \mathbf{M}^{i{\rm F}\mathcal{S}} \mathbf{c} \tag{12}$$

$$\mathcal{T}_{i\mathcal{S}} = \mathbf{c}^\dagger \mathbf{M}^{i{\rm T}\mathcal{S}} \mathbf{c} \tag{13}$$

where the entries of $\mathbf{M}^{i{\rm F}\mathcal{S}}$ and $\mathbf{M}^{i{\rm T}\mathcal{S}}$ are similar to (11) with $\boldsymbol{\mathcal{N}}^{\rm P} \to \boldsymbol{\mathcal{N}}^{i{\rm F}}, \boldsymbol{\mathcal{N}}^{i{\rm T}}$.

4 Overlap PFT (OPFT)

In the DSIPFT approach, the field components needed to compute the Poynting vector (PV) or Maxwell stress tensor (MST) at a point $\mathbf{x}$ on a bounding surface are obtained from the surface currents by linear convolution. In fact, since the fields enter these expressions quadratically, evaluating the PV or MST at a single point requires *two* separate convolutions. Thus, for example, equation (6) for the Poynting flux at $\mathbf{x}$ involves a double sum over surface-current expansion coefficients. This means that the DSIPFT approach is computationally costly.

We achieve a considerable simplification by allowing the bounding surface $\mathcal{S}$ to coincide with the body surface. In this case, computing the PV and MST requires only knowledge of the fields at the object surface. But the fields at the object surface may be obtained *directly* from the surface currents—with no convolution—simply by appealing to the definition of the surface currents:

$$\mathbf{E} = \hat{\mathbf{n}} \times \mathbf{N} + \frac{\nabla \cdot \mathbf{K}}{i\omega\epsilon}\hat{\mathbf{n}}, \qquad \mathbf{H} = -\hat{\mathbf{n}} \times \mathbf{K} + \frac{\nabla \cdot \mathbf{N}}{i\omega\mu}\hat{\mathbf{n}}.$$

Thus the double sum in e.g. equation (6) for the Poynting flux at a point $\mathbf{x}$ on the body surface collapses to just a few terms; indeed, only pairs of basis functions (α, β) that are both nonvanishing at $\mathbf{x}$ contribute. (For the RWG basis there are just 8 such pairs for each point, independent of the total number of functions in the discretization.)

(Insert OPFT formulas here)

5 Energy/momentum-transfer PFT (MTPFT)

In the EMTPFT approach, we compute the power, force and torque on a body $\mathcal{B}$ by considering the transfer of energy and momentum from the total fields to the equivalent surface currents:

$$P^{\rm abs} = \frac{1}{2}\mathrm{Re}\int \boldsymbol{\mathcal{C}}^* \cdot \boldsymbol{\mathcal{F}}\, dV \tag{14a}$$

$$F_i = \frac{1}{2\omega}\mathrm{Im}\int \boldsymbol{\mathcal{C}}^* \cdot \partial_i \boldsymbol{\mathcal{F}}\, dV \tag{14b}$$

$$\mathcal{T}_i = \frac{1}{2\omega}\mathrm{Im}\int \underbrace{\Big[\boldsymbol{\mathcal{C}}^* \times \boldsymbol{\mathcal{F}} + \boldsymbol{\mathcal{C}}^* \cdot \partial_{\theta_i}\mathbf{F}\Big]}_{\boldsymbol{\mathcal{C}}^*\cdot\widetilde{\partial}_i\boldsymbol{\mathcal{F}}} dV \tag{14c}$$

Equation (14a) is just the usual Joule heating $P = \frac{1}{2}\mathrm{Re}\left(\mathbf{J}^*\cdot\mathbf{E} + \mathbf{M}^*\cdot\mathbf{H}\right)$, while Equations (14b,c) follow from considering the time-average Lorentz force $d\mathbf{F} = \frac{1}{2}\mathrm{Re}\,(\rho^*_{\rm E}\mathbf{E} + \mu\,\mathbf{J}^*\times\mathbf{H} + \rho^*_{\rm M}\mathbf{H} - \epsilon\,\mathbf{M}^*\times\mathbf{E})dV$ and torque $\mathbf{r}\times d\mathbf{F}$ on the charges and currents in an infinitesimal volume dV; integrating over the volume and using integration by parts and Maxwell's equations yields (14b,c). [The symbol ∂_{θ_i} in (14c) denotes differentiation with respect to infinitesimal rotation about the ith coordinate axis. The symbol $\widetilde{\partial}_i$ is shorthand for the operation (cross product plus angular derivative) involved in (14c).]

In what follows it will be convenient to express equations (14) in terms of the following shorthand operator notation:

$$Q = \frac{1}{2}\int_{\partial\mathcal{B}} \boldsymbol{\mathcal{C}}^*\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{F}}\, dA \tag{15}$$

where $Q = \{P^{\rm abs}, F_i, \mathcal{T}_i\}$ and he operator $\boldsymbol{\mathcal{Q}}$ correspondingly operates on $\boldsymbol{\mathcal{F}}$ as in equations (14a,b,c).

Equations (14) involve the total fields at the body surface. Because SIE formulations allow surface fields to be computed in either of two distinct ways—namely, as the limiting values of the bulk fields as one approaches the surface from the exterior or the interior of the body—the EMTPFT strategy in fact bifurcates into two strategies, which we discuss separately below.

5.1 Exterior EMTPFT

In the exterior region, the total fields receive incident and scattered contributions, $\boldsymbol{\mathcal{F}} = \boldsymbol{\mathcal{F}}^{\rm inc} + \boldsymbol{\mathcal{F}}^{\rm scat}$, We identify the corresponding contributions to the PFT quantities (14) as respectively the extinction PFT and the (negative of

the) scattered PFT,

$$Q = Q^{\rm ext} - Q^{\rm scat} \tag{16a}$$

$$Q^{\rm ext} = \frac{1}{2}\int_{\partial\mathcal{B}} \boldsymbol{\mathcal{C}}^* \boldsymbol{\mathcal{Q}} \boldsymbol{\mathcal{F}}^{\rm inc}, \tag{16b}$$

$$Q^{\rm scat} = -\frac{1}{2}\int_{\partial\mathcal{B}} \boldsymbol{\mathcal{C}}^* \boldsymbol{\mathcal{Q}} \boldsymbol{\mathcal{F}}^{\rm scat} \tag{16c}$$

Extinction contributions to exterior EMTPFT

The discretized form of (16b) for the extinction reads

$$Q^{\rm ext} = \frac{1}{2}\sum_{\alpha} c_\alpha \int_{\sup \boldsymbol{\mathcal{B}}_\alpha} \boldsymbol{\mathcal{B}}_\alpha \boldsymbol{\mathcal{Q}} \boldsymbol{\mathcal{F}}^{\rm inc}\, dA$$

The integrals here are non-singular two-dimensional integrals over the basis-function supports, with the integrand involving values and derivatives of the incident fields. These are evaluated in SCUFF-EM by simple low-order numerical cubature.

Scattered contributions to exterior EMTPFT

The scattered-field contributions to the EMTPFT power and force follow from the discretized form of (16c) and read

$$P^{\rm scat} = -\frac{1}{2}{\rm Re}\sum_{ab} \begin{pmatrix} k_a \\ n_a \end{pmatrix}^\dagger \begin{pmatrix} i\omega\mu \mathbf{G}_{ab} & \widehat{\mathbf{C}}_{ab} \\ -\widehat{\mathbf{C}}_{ab} & i\omega\epsilon\mathbf{G}_{ab} \end{pmatrix} \begin{pmatrix} k_b \\ n_b \end{pmatrix} \tag{17a}$$

$$F_i^{\rm scat} = -\frac{1}{2\omega}{\rm Im}\sum_{ab} \begin{pmatrix} k_a \\ n_a \end{pmatrix}^\dagger \begin{pmatrix} i\omega\mu \partial_i\mathbf{G}_{ab} & \partial_i\widehat{\mathbf{C}}_{ab} \\ -\partial_i\widehat{\mathbf{C}}_{ab} & i\omega\epsilon\partial_i\mathbf{G}_{ab} \end{pmatrix} \begin{pmatrix} k_b \\ n_b \end{pmatrix} \tag{17b}$$

(The torque is similar to the force with $\partial_i \to \widetilde{\partial}_i$.) In these equations,

- a runs over all interior edges (RWG functions) on the surface bounding the region on which we are computing the power or force;
- b runs over all interior edges on all surfaces bounding the region exterior to the region on which we are computing the power or force;
- $\epsilon = \epsilon_0\epsilon^r$ and $\mu = \mu_0\mu^r$ are the material properties of the exterior region
- $\mathbf{G}_{ab} = \langle \mathbf{b}_a | \, \mathbb{G} \, | \mathbf{b}_b \rangle$ is the matrix element of the $\mathbb{G}$ kernel for the exterior medium between RWG basis functions.
- $\widehat{\mathbf{C}}_{ab} = \langle \mathbf{b}_a | \, ik\mathbb{C} \, | \mathbf{b}_b \rangle$.

Upon using the relations

$$\{\mathbf{G}_{ba},\mathbf{C}_{ba}\}=\{\mathbf{G}_{ab},\mathbf{C}_{ab}\},\qquad \partial_i\{\mathbf{G}_{ba},\mathbf{C}_{ba}\}=-\partial_i\{\mathbf{G}_{ab},\mathbf{C}_{ab}\},$$

equations (17) may be simplified to read

$$\begin{aligned}P^{\rm scat} &= +\frac{1}{2}\sum_{ab}\Big\{\omega\mu_0\Big[\text{Re }k_a^*k_b\Big]\Big[\text{Im }\mu^r\mathbf{G}_{ab}\Big]+\omega\epsilon_0\Big[\text{Re }n_a^*n_b\Big]\Big[\text{Im }\epsilon^r\mathbf{G}_{ab}\Big]\\ &\qquad +\Big[\text{Im }k_a^*n_b-n_a^*k_b\Big]\Big[\text{Im }\widehat{\mathbf{C}}_{ab}\Big]\Big\}\\ F_i^{\rm scat} &= +\frac{1}{2\omega}\sum_{ab}\Big\{\omega\mu_0\Big[\text{Im }k_a^*k_b\Big]\Big[\text{Im }\mu^r\partial_i\mathbf{G}_{ab}\Big]+\omega\epsilon_0\Big[\text{Im }n_a^*n_b\Big]\Big[\text{Im }\epsilon^r\partial_i\mathbf{G}_{ab}\Big]\\ &\qquad -\Big[\text{Re }k_a^*n_b-n_a^*k_b\Big]\Big[\text{Im }\partial_i\widehat{\mathbf{C}}_{ab}\Big]\Big\}\end{aligned}$$

EMTPFT matrix elements

$$\begin{aligned}\partial_i\mathbf{G}_{ab} &= \partial_i\mathbf{G}^1_{ab}+\partial_i\mathbf{G}^2_{ab}\\ \partial_i\mathbf{G}^1_{ab} &= \int_{\sup\mathbf{b}_a}d\mathbf{x}_a\int_{\sup\mathbf{b}_b}d\mathbf{x}_b\;(\mathbf{b}_a\cdot\mathbf{b}_b)\,\partial_iG_0(\mathbf{x}_a-\mathbf{x}_b)\\ \partial_i\mathbf{G}^2_{ab} &= -\frac{1}{k^2}\int_{\sup\mathbf{b}_a}d\mathbf{x}_a\int_{\sup\mathbf{b}_b}d\mathbf{x}_b\,(\nabla\cdot\mathbf{b}_a)(\nabla\cdot\mathbf{b}_b)\partial_iG_0(\mathbf{x}_a-\mathbf{x}_b)\end{aligned}$$

$$\text{Im }\partial_i\widehat{\mathbb{C}}_{jk}=\text{Im }\partial_i\left[\frac{e^{ikr}}{4\pi r^3}(ikr-1)\varepsilon_{jk\ell}r_\ell\right]$$

$$\partial_iG_0(\mathbf{r})=r_i(ikr-1)\frac{e^{ikr}}{4\pi r^3}\qquad\qquad =-\frac{r_i}{4\pi r^3}+\Big[\partial_iG_0\Big]^{\rm DS}$$

A Derivation of volume integrals for the force and torque

Consider a body throughout which exist volume distributions of electric and magnetic currents $\{\mathbf{J}(\mathbf{x}), \mathbf{M}(\mathbf{x})\}$ and electric and magnetic fields $\{\mathbf{E}(\mathbf{x}), \mathbf{H}(\mathbf{x})\}$. The time-average force experienced by just the *electric* currents in an infinitesimal volume dV is

$$d\mathbf{F}^{\mathbf{J}} = \frac{1}{2}\mathrm{Re}\,\Big[\rho^*\mathbf{E} + \mu_0\mathbf{J}^*\times\mathbf{H}\Big]\,dV$$

Use $\rho = \frac{1}{i\omega}(\nabla\cdot\mathbf{J})$ and[2] $\mathbf{H} = \frac{1}{i\omega\mu_0}(\nabla\times\mathbf{E}+\mathbf{M})$:

$$\begin{aligned}
&= \frac{1}{2}\mathrm{Re}\,\left\{\frac{1}{i\omega}\Big[-(\nabla\cdot\mathbf{J}^*)\mathbf{E} + \mathbf{J}^*\times(\nabla\times\mathbf{E}) + \mathbf{J}^*\times\mathbf{M}\Big]\,dV\right\}dV \\
&= \underbrace{\frac{1}{2\omega}\mathrm{Im}\,\Big[-(\nabla\cdot\mathbf{J}^*)\mathbf{E} + \mathbf{J}^*\times(\nabla\times\mathbf{E})\Big]\,dV}_{d\mathbf{F}^{\mathbf{J}1}} + \underbrace{\frac{1}{2\omega}\mathrm{Im}\,\Big[\mathbf{J}^*\times\mathbf{M}\Big]}_{d\mathbf{F}^{\mathbf{J}2}}. \qquad (18)
\end{aligned}$$

In component notation, the first term here reads

$$\begin{aligned}
dF_i^{\mathbf{J}1} &= -\frac{1}{2\omega}\mathrm{Im}\,\Big[(\partial_j J_j^*)E_i - \underbrace{\varepsilon_{ijk}\varepsilon_{k\ell m}}_{\delta_{i\ell}\delta_{jm}-\delta_{im}\delta_{j\ell}} J_j^*\partial_\ell E_m\Big]\,dV \\
&= -\frac{1}{2\omega}\mathrm{Im}\,\Big[(\partial_j J_j^*)E_i - J_j^*\partial_i E_j + J_j^*\partial_j E_i\Big]\,dV. \qquad (19)
\end{aligned}$$

The total force is given by integrating over the volume:

$$F_i^{\mathbf{J}1} = -\frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\Big[(\partial_j J_j^*)E_i - J_j^*\partial_i E_j + J_j^*\partial_j E_i\Big]dV \qquad (20)$$

The first and third terms here together read

$$\int\partial_j\big(J_j^*E_i\big)dV = \int\nabla\cdot(E_i\,\mathbf{J}^*)\,dV = \oint E_i\mathbf{J}^*\cdot d\mathbf{A} = 0 \qquad (21)$$

because $\mathbf{J}\cdot\hat{\mathbf{n}} = 0$ at the surface of the object (no current flows from the body into space). Thus only the middle term in (20) is nonvanishing, and we find simply

$$F_i^{\mathbf{J}1} = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n} J_j^*\partial_i E_j\,dV = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\mathbf{J}^*\cdot\partial_i\mathbf{E}\,dV.$$

Adding the contributions of the other term in (18), we find

$$F_i^{\mathbf{J}} = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\Big[\mathbf{J}^*\cdot\partial_i\mathbf{E} + (\mathbf{J}^*\times\mathbf{M})_i\Big]\,dV. \qquad (22)$$

On the other hand, the force on the *magnetic* currents in dV is

$$d\mathbf{F}^{\mathbf{M}} = \frac{1}{2}\mathrm{Re}\,\Big[\eta^*\mathbf{H} - \epsilon\mathbf{M}^*\times\mathbf{E}\Big]\,dV$$

Use $\eta = \frac{1}{i\omega}(\nabla\cdot\mathbf{M})$ and $\mathbf{E} = -\frac{1}{i\omega\epsilon}(\nabla\times\mathbf{H}-\mathbf{J})$:

$$\begin{aligned}
&= \frac{1}{2}\mathrm{Re}\,\left\{\frac{1}{i\omega}\Big[-(\nabla\cdot\mathbf{M}^*)\mathbf{H} + \mathbf{M}^*\times(\nabla\times\mathbf{H}) + \mathbf{M}^*\times\mathbf{J}\Big]\,dV\right\}dV \\
&= \underbrace{\frac{1}{2\omega}\mathrm{Im}\,\Big[-(\nabla\cdot\mathbf{M}^*)\mathbf{H} + \mathbf{M}^*\times(\nabla\times\mathbf{H})\Big]\,dV}_{d\mathbf{F}^{\mathbf{M}1}} + \underbrace{\frac{1}{2\omega}\mathrm{Im}\,\Big[\mathbf{M}^*\times\mathbf{J}\Big]}_{d\mathbf{F}^{\mathbf{M}2}}. \quad (23)
\end{aligned}$$

Now playing the same games as above allows us to rewrite this in a form analogous to that of (24):

$$F_i^{\mathbf{M}} = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\Big[\mathbf{M}^*\cdot\partial_i\mathbf{H} + (\mathbf{M}^*\times\mathbf{J})_i\Big]\,dV. \quad (24)$$

Torque

The contribution of currents in dV to the *torque* about an origin $\mathbf{r}_0$ is given by

$$d\boldsymbol{\mathcal{T}} = (\mathbf{r}-\mathbf{r}_0)\times d\mathbf{F}$$

or, in components,

$$d\mathcal{T}_i = \varepsilon_{ijk}(\mathbf{r}-\mathbf{r}_0)_j d\mathbf{F}_k.$$

Insert (19):

$$= -\frac{1}{2\omega}\mathrm{Im}\left\{\varepsilon_{ijk}(\mathbf{r}-\mathbf{r}_0)_j\Big[(\partial_\ell J_\ell^*)E_k - J_\ell^*\partial_k E_\ell + J_\ell^*\partial_\ell E_k\Big]\right\}dV.$$

The first and third terms here integrate to zero by an argument similar to (21), and we find

$$\mathcal{T}_i = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\epsilon_{ijk}(\mathbf{r}-\mathbf{r}_0)_j J_\ell^*\partial_k E_\ell\,dV = \frac{1}{2\omega}\mathrm{Im}\,\int_{\mathcal{B}_n}\mathbf{J}^*\cdot\partial_\theta\mathbf{E}\,dV \quad (25)$$

where the symbol $\partial_\theta\mathbf{E}$ denotes the derivative of $\mathbf{E}(\mathbf{r})$ with respect to an infinitesimal rotation of the point $\mathbf{r}$ about the ith coordinate axis with origin $\mathbf{r}_0$.

B Overlap integral between RWG basis functions

$$\mathbf{O}_{\alpha\beta} \equiv \int \mathbf{b}_\alpha(\mathbf{x}) \times \mathbf{b}_\beta(\mathbf{x})\, d\mathbf{x}.$$

This quantity vanishes unless $\mathbf{b}_\alpha, \mathbf{b}_\beta$ have exactly one panel in common. To evaluate the integral in that case, let $\mathcal{P}$ be the common panel, and suppose its vertices are $\{\mathbf{Q}_\alpha, \mathbf{V}_1, \mathbf{V}_2\}$ with $\mathbf{Q}_\alpha$ the source/sink vertex for $\mathbf{b}_\alpha$. Let $\mathbf{Q}_\beta$ be the source/sink vertex for $\mathbf{b}_\beta$. On $\mathcal{P}$ we can parametrize points according to

$$\mathbf{x} = \mathbf{Q}_\alpha + u\mathbf{A} + v\mathbf{B}, \qquad 0 \le u \le 1, \qquad 0 \le v \le u$$

$$\mathbf{A} \equiv (\mathbf{V}_1 - \mathbf{Q}_\alpha) \qquad \mathbf{B} \equiv (\mathbf{V}_2 - \mathbf{V}_1)$$

and the two RWG functions are

$$\begin{aligned}
\mathbf{b}_\alpha &= \pm\frac{\ell_\alpha}{2A}\Big(\mathbf{x} - \mathbf{Q}_\alpha\Big) \\
&= \pm\frac{\ell_\alpha}{2A}\Big(u\mathbf{A} + v\mathbf{B}\Big) \\
\mathbf{b}_\beta &= \pm\frac{\ell_\beta}{2A}\Big(\mathbf{x} - \mathbf{Q}_\beta\Big) \\
&= \pm\frac{\ell_\beta}{2A}\Big(u\mathbf{A} + v\mathbf{B} + \Delta\mathbf{Q}\Big)
\end{aligned}$$

where A is the area of the common panel, $\ell_{\alpha,\beta}$ are the lengths of the edges that define $\mathbf{b}_{\alpha,\beta}$, and $\Delta\mathbf{Q} = (\mathbf{Q}_a - \mathbf{Q}_b)$. The $\pm$ signs are determined by whether $\mathcal{P}$ is the positive or negative panel associated with $\mathbf{b}_{\alpha,\beta}$. The overlap integral then reads

$$\begin{aligned}
\mathbf{O}_{\alpha\beta} &= \pm\frac{\ell_\alpha\ell_\beta}{2A}\int_0^1 du \int_0^u dv\Big[u\mathbf{A} + v\mathbf{B}\Big] \times \Big[u\mathbf{A} + v\mathbf{B} + \Delta\mathbf{Q}\Big] \\
&= \pm\frac{\ell_\alpha\ell_\beta}{2A}\left[\frac{1}{3}\Big(\mathbf{A}\times\Delta\mathbf{Q}\Big) + \frac{1}{6}\Big(\mathbf{B}\times\Delta\mathbf{Q}\Big)\right] \\
&= \pm\frac{\ell_\alpha\ell_\beta}{12A}\Big[\Big(2\mathbf{A} + \mathbf{B}\Big) \times \Delta\mathbf{Q}\Big] \\
&= \pm\frac{\ell_\alpha\ell_\beta}{4A}\Big[\Big(\mathbf{X}_c - \mathbf{Q}_\alpha\Big) \times \Delta\mathbf{Q}\Big]
\end{aligned}$$

where $\mathbf{X}_c$ is the centroid of $\mathcal{P}$.

C PFT in Mie Scattering

It is instructive to compare the behavior of the various PFT algorithms in the simplest analytically-solvable case of scattering by a compact object, namely, namely, the scattering of a plane wave by a dielectric sphere (Mie scattering). Let the sphere have radius r and relative permittivity ϵ; for simplicity I consider the non-magnetic case, $\mu = 1$.

The incident fields are

$$\mathbf{E}^{\rm inc}(\mathbf{x}) = E_0 e^{ikx}\hat{\mathbf{x}}, \qquad \mathbf{H}^{\rm inc}(\mathbf{x}) = \frac{1}{Z_0} E_0 e^{ikx}\hat{\mathbf{y}}.$$

where k is the free-space wavenumber. (I work in units such that $c = 1$, so numerically I have $k = \omega$, although in the formulas below the quantities k and ω will both appear.

Interior fields

Retaining terms of zeroth and first order in k, the fields inside the sphere are

$$\mathbf{E}(\mathbf{x}) = E_0\Big[(C_0 + ikzC_1)\hat{\mathbf{x}} + ikx(C_1 - 1)\hat{\mathbf{z}}\Big]$$
$$\mathbf{H}(\mathbf{x}) = \frac{E_0}{Z_0}\Big[(1 + ikzC_2)\hat{\mathbf{y}} + iky(C_2 - \epsilon C_0)\hat{\mathbf{z}}\Big]$$

$$C_0 = \frac{3}{2+\epsilon}, \qquad C_1 = \frac{\epsilon+4}{3+2\epsilon}, \qquad C_2 = 2 - C_0.$$

Volume EPPFT

Volume current distribution:

$$\mathbf{J}(\mathbf{x}) = -\frac{k\,\mathrm{Im}\,\epsilon}{Z_0}\,\mathbf{E}(\mathbf{x})$$

Absorbed power per unit volume:

$$\frac{1}{2}\mathrm{Re}\;\mathbf{J}^*(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}) = -\frac{k\,(\mathrm{Im}\,\epsilon)E_0^2}{2Z_0}\mathrm{Re}\;\Big[\Big|C_0 + ikzC_1\Big|^2 + \Big|ikx(C_1-1)\Big|^2\Big]$$

z-force per unit volume:

$$\frac{1}{2\omega}\mathrm{Im}\;\mathbf{J}^*\cdot\partial_z\mathbf{E}(\mathbf{x}) = -\frac{(\mathrm{Im}\,\epsilon)E_0^2}{2Z_0}\mathrm{Im}\;\Big\{\Big(C_0^* - ikzC_1^*\Big)\Big(ikC_1\Big)\Big\}$$
$$= -\frac{k(\mathrm{Im}\,\epsilon)E_0^2}{2Z_0}\mathrm{Re}\;\Big[C_0^* C_1\Big]$$

Surface EPPFT, take 1

Surface currents:

$$\mathbf{K}(\mathbf{x}) = \hat{\mathbf{r}} \times \mathbf{H}(\mathbf{x}) = \frac{E_0}{Z_0}\Big[\Big(1 + ikzC_2\Big)(\hat{\mathbf{r}} \times \hat{\mathbf{y}}) + iky\Big(C_2 - \epsilon C_0\Big)(\hat{\mathbf{r}} \times \hat{\mathbf{z}})\Big]$$

$$\mathbf{N}(\mathbf{x}) = -\hat{\mathbf{r}} \times \mathbf{E}(\mathbf{x}) = -E_0\Big[\Big(C_0 + ikzC_1\Big)(\hat{\mathbf{r}} \times \hat{\mathbf{x}}) + ikx\Big(C_1 - 1\Big)(\hat{\mathbf{r}} \times \hat{\mathbf{z}})\Big]$$

$$\nabla \cdot \mathbf{K}(\mathbf{x}) = \frac{3i\epsilon E_0 kx}{Z_0(2+\epsilon)}, \qquad \nabla \cdot \mathbf{N}(\mathbf{x}) = ikyE_0.$$

Electric and magnetic surface charge densities:

$$\sigma^{\mathrm{E}} = \frac{1}{i\omega}\nabla \cdot \mathbf{K} = \frac{3\epsilon E_0 x}{Z_0(2+\epsilon)}, \qquad \sigma^{\mathrm{M}} = \frac{1}{i\omega}\nabla \cdot \mathbf{N} = yE_0.$$

z-directed force per unit area on surface currents:

$$F_z = \frac{1}{2}\mathrm{Re}\,\Big[\underbrace{\sigma^{*E}\mathbf{E}_z}_{F^1} + \underbrace{Z_0\big(\mathbf{K}^* \times \mathbf{H}\big)_z}_{F^2} + \underbrace{\sigma^{*M}\mathbf{H}_z}_{F^3} - \underbrace{\frac{1}{Z_0}\big(\mathbf{N}^* \times \mathbf{E}\big)_z}_{F^4}\Big]$$

$$\begin{aligned}
\frac{1}{2}\mathrm{Re}\;F^1 &= \frac{E_0^2}{2Z_0}kx^2\mathrm{Im}\,\left[\left(\frac{3\epsilon}{2+\epsilon}\right)^*\Big(C_1 - 1\Big)\right] \\
\frac{1}{2}\mathrm{Re}\;F^2 &= \frac{E_0^2}{2Z_0}k\Big[-2z^2\mathrm{Im}\;C_2 - y^2\mathrm{Im}\;(C_2 - \epsilon C_0)\Big] \\
\frac{1}{2}\mathrm{Re}\;F^3 &= \frac{E_0^2}{2Z_0}ky^2\mathrm{Im}\,\Big(C_2 - \epsilon C_0\Big)
\end{aligned}$$

Surface EPPFT, take 2

Force per unit surface area:

$$F_z = \frac{1}{2\omega}\mathrm{Im}\,\Big\{\underbrace{\mathbf{K}^* \cdot \partial_z\mathbf{E}}_{F^1} + \underbrace{\mathbf{N}^* \cdot \partial_z\mathbf{H}}_{F^2}\Big\}$$

$$\begin{aligned}
\frac{1}{2\omega}\mathrm{Im}\;F^1 &= \frac{kE_0^2}{2Z_0}\mathrm{Im}\,\Big[C_1(C_2^* - \epsilon^* C_0^*)y^2 - C_1C_2^*z^2\Big] \\
&\to \frac{kE_0^2}{2Z_0}y^2\mathrm{Im}\,\Big[-\epsilon^* C_1 C_0^*\Big] \\
\frac{1}{2\omega}\mathrm{Im}\;F^2 &= \frac{kE_0^2}{2Z_0}\mathrm{Im}\,\Big[C_2(C_1^* - 1)x^2 - C_1^*C_2z^2\Big] \\
&\to -\frac{kE_0^2}{2Z_0}x^2
\end{aligned}$$