Scattering Amplitudes from Surface Currents in SCUFF-EM

Homer Reid

January 22, 2014

Contents

1 Compact objects

First consider a compact scattering geometry (consisting of one or more objects) localized near the origin of coordinates and illuminated by an arbitrary unspecified external sources at frequency $\omega = ck_0$ (i.e. k_0 is the free-space wavenumber corresponding to the frequency at which we work). At large distances, we write the scattered electric field in the form

$$\mathbf{E}^{\text{scat}}(\mathbf{x}) = \mathbf{E}^{\text{scat}}(r, \Omega) = \mathbf{F}(\Omega)\frac{e^{ikr}}{4\pi r} \tag{1}$$

which defines the *scattering amplitude* $\mathbf{F}(\Omega)$; it is a function of solid angle only, not of the distance from the target to the evaluation point.

On the other hand, we also have the usual expression for the scattered electric field in terms of the surface currents,

$$\begin{aligned}\mathbf{E}^{\text{scat}}(\mathbf{x}) &= \oint \Big\{\mathbf{\Gamma}^{\text{EE}}(\mathbf{x},\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}') + \mathbf{\Gamma}^{\text{EM}}(\mathbf{x},\mathbf{x}')\cdot\mathbf{N}(\mathbf{x}')\Big\}d\mathbf{x}' \\ &= ik_0\oint\Big\{Z_0 Z^r\mathbf{G}(\mathbf{x},\mathbf{x}')\cdot\mathbf{K}(\mathbf{x}') + \mathbf{C}(\mathbf{x},\mathbf{x}')\cdot\mathbf{N}(\mathbf{x}')\Big\}d\mathbf{x}' \end{aligned} \tag{2}$$

where k_0 and Z^r are the wavenumber and relative wave impedance of the exterior medium.

To make equation (2) look like (1), we use the asymptotic expressions for the $\mathbf{G}$ and $\mathbf{C}$ dyadics:

$$\mathbf{G}(\mathbf{x}-\mathbf{x}') \rightarrow$$

2 Extended objects

Next we consider an extended structure described by Bloch-periodic boundary conditions. We will derive expressions for the plane-wave reflection and transmission coefficients in terms of the surface-current distribution in the unit cell of the structure.

Definition of scattering coefficients

For definiteness, I suppose that the structure is infinitely extended in the x and y directions with finite extent in the z direction, and I suppose the structure is illuminated from below by a plane wave with propagation vector $\mathbf{k}$ confined to the xz plane. More general situations could easily be handled by extending the methods of this section.

The incident, reflected, and transmitted fields may be written in the form

$$\begin{aligned}
\mathbf{E}^{\text{inc}}(\mathbf{x}) &= E_0\boldsymbol{\epsilon}_0 e^{i(k_x x+k_z z)} & \mathbf{H}^{\text{inc}}(\mathbf{x}) &= H_0\overline{\boldsymbol{\epsilon}}_0 e^{i(k_x x+k_z z)} \\
\mathbf{E}^{\text{refl}}(\mathbf{x}) &= rE_0\boldsymbol{\epsilon}_0^{\text{refl}} e^{i(k_x x-k_z z)} & \mathbf{H}^{\text{refl}}(\mathbf{x}) &= rH_0\overline{\boldsymbol{\epsilon}}_0^{\text{refl}} e^{i(k_x x+k_z z)} \\
\mathbf{E}^{\text{trans}}(\mathbf{x}) &= tE_0\boldsymbol{\epsilon}_0^{\text{trans}} e^{i(k_x x+k'_z z)} & \mathbf{H}^{\text{trans}}(\mathbf{x}) &= \frac{tH_0}{Z'}\overline{\boldsymbol{\epsilon}}_0^{\text{trans}} e^{i(k_x x+k'_z z)}
\end{aligned} \tag{3}$$

where E_0 is the incident field magnitude, $\boldsymbol{\epsilon}_0$ is the incident-field polarization vector, and

$$H_0 \equiv \frac{i|\mathbf{k}|E_0}{Z_0}, \qquad \overline{\boldsymbol{\epsilon}} = \hat{\mathbf{k}}\times\boldsymbol{\epsilon}, \qquad Z' = \sqrt{\frac{\mu'}{\epsilon'}}$$

Equations (3) define the reflection and transmission coefficients r and t. Also, the components of the propagation vector in the lower medium are

$$k_x = k_0\sin\theta, \qquad k_z = k_0\cos\theta$$

(where θ is the incident angle of the illuminating radiation). The components of the propagation vector in the upper medium are

$$k'_x = k_x \tag{4}$$
$$= k'_0\sin\theta' \tag{5}$$
$$k'_z = \sqrt{k_0'^2 - k_x^2} \qquad = k'_0\cos\theta' \tag{6}$$

where

$$k'_0 = \sqrt{\epsilon'\mu'}\cdot k_0, \qquad \sin\theta' = \sqrt{\frac{1}{\epsilon'\mu'}}\sin\theta.$$

is the wavenumber in the upper medium.

Polarization vectors for TE and TM polarizations

TE:

$$\begin{aligned}
\boldsymbol{\epsilon}_0 &= \hat{\mathbf{y}} & \overline{\boldsymbol{\epsilon}}_0 &= -\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{z}} \\
\boldsymbol{\epsilon}_0^{\text{refl}} &= \hat{\mathbf{y}} & \overline{\boldsymbol{\epsilon}}_0^{\text{refl}} &= -\cos\theta'\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{z}} \\
\boldsymbol{\epsilon}_0^{\text{trans}} &= \hat{\mathbf{y}} & \overline{\boldsymbol{\epsilon}}_0^{\text{trans}} &= -\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{z}}
\end{aligned} \tag{7}$$

TM:

$$\begin{aligned}
\boldsymbol{\epsilon}_0 &= \cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}} & \overline{\boldsymbol{\epsilon}}_0 &= \hat{\mathbf{y}} \\
\boldsymbol{\epsilon}_0 &= \cos\theta'\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}} & \overline{\boldsymbol{\epsilon}}_0 &= \hat{\mathbf{y}} \\
\boldsymbol{\epsilon}_0 &= \cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{z}} & \overline{\boldsymbol{\epsilon}}_0 &= \hat{\mathbf{y}}.
\end{aligned} \tag{8}$$

Bloch periodicity

Notice that all fields satisfy Bloch-periodic boundary conditions,

$$\mathbf{E}(\mathbf{x}+\mathbf{L}) = e^{i\mathbf{k}\cdot\mathbf{L}}\,\mathbf{E}(\mathbf{x}) \tag{9}$$

where the Bloch wavevector is

$$\mathbf{k} = k_0 \sin\theta\,\hat{\mathbf{x}}.$$

For the plane waves (3), equation (9) actually holds for any arbitrary vector $\mathbf{L}$; for our purposes we will only need to use it for certain special vectors $\mathbf{L}$ determined by the structure of the lattice in our PBC geometry.

Fields from surface currents

On the other hand, the scattered $\mathbf{E}$ fields in the uppermost and lowermost regions may be obtained in the usual way from the surface-current distributions on the surfaces bounding those regions. For example, at points in the upper medium we have

$$\begin{aligned}
\mathbf{E}^{\text{trans}}(\mathbf{x}) &= \oint_{\mathcal{S}} \Big\{ \boldsymbol{\Gamma}^{\text{EE}}(\mathbf{x},\mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') + \boldsymbol{\Gamma}^{\text{EM}}(\mathbf{x},\mathbf{x}') \cdot \mathbf{N}(\mathbf{x}') \Big\} d\mathbf{x}' \\
&= ik_0' \oint_{\mathcal{S}} \Big\{ Z_0 Z' \mathbf{G}(k_0';\mathbf{x},\mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') + \mathbf{C}(k_0';\mathbf{x},\mathbf{x}') \cdot \mathbf{N}(\mathbf{x}') \Big\} d\mathbf{x}'
\end{aligned} \tag{10}$$

where $\mathcal{S}$ is the surface bounding the uppermost medium and k_0', Z' are the wavenumber and relative wave impedance of the uppermost region:

$$k_0' = \sqrt{\epsilon'\mu'} \cdot \mathbf{k}_0, \qquad Z' = \sqrt{\frac{\mu'}{\epsilon'}}.$$

Using the Bloch-periodicity of the surface currents, i.e.

$$\left\{ \begin{array}{c} \mathbf{K}(\mathbf{x}+\mathbf{L}) \\ \mathbf{N}(\mathbf{x}+\mathbf{L}) \end{array} \right\} = e^{i\mathbf{k}\cdot\mathbf{L}} \left\{ \begin{array}{c} \mathbf{K}(\mathbf{x}) \\ \mathbf{N}(\mathbf{x}) \end{array} \right\}$$

we can restrict the surface integral in (10) to just the lattice unit cell:

$$\mathbf{E}^{\text{trans}}(\mathbf{x}) = ik_0' \int_U \Big\{ Z_0 Z' \overline{\mathbf{G}}(k_0'; \mathbf{x}, \mathbf{x}') \cdot \mathbf{K}(\mathbf{x}') + \overline{\mathbf{C}}(k_0'; \mathbf{x}, \mathbf{x}') \cdot \mathbf{N}(\mathbf{x}') \Big\} \, d\mathbf{x}' \tag{11}$$

where the periodic Green's functions are

$$\left\{ \begin{array}{c} \overline{\mathbf{G}}(\mathbf{x}, \mathbf{x}') \\ \overline{\mathbf{C}}(\mathbf{x}, \mathbf{x}') \end{array} \right\} \equiv \sum_{\mathbf{L}} e^{i\mathbf{k}\cdot\mathbf{L}} \left\{ \begin{array}{c} \mathbf{G}(\mathbf{x}, \mathbf{x}' + \mathbf{L}) \\ \mathbf{C}(\mathbf{x}, \mathbf{x}' + \mathbf{L}) \end{array} \right\} \tag{12}$$

I now invoke the following representation of the dyadic Green's functions (Chew, 1995): for $z > z'$,

$$\mathbf{G}(k_0'; \boldsymbol{\rho}, z; \boldsymbol{\rho}', z') = \frac{i}{2} \int \frac{d\mathbf{q}}{(2\pi)^2 q_z} \hat{\mathbf{G}}(\mathbf{q}) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} e^{iq_z(z-z')} \tag{13a}$$

$$\mathbf{C}(k_0'; \boldsymbol{\rho}, z; \boldsymbol{\rho}', z') = \frac{i}{2} \int \frac{d\mathbf{q}}{(2\pi)^2 q_z} \hat{\mathbf{C}}(\mathbf{q}) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} e^{iq_z(z-z')} \tag{13b}$$

where $\mathbf{q} = (q_x, q_y)$ is a two-dimensional Fourier wavevector, $d\mathbf{q} = dq_x dq_y$, $q_z = \sqrt{k_0'^2 - |\mathbf{q}|^2}$, and

$$\hat{\mathbf{G}}(\mathbf{q}) = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) - \frac{1}{k_0'^2} \left(\begin{array}{ccc} q_x^2 & q_x q_y & q_x q_z \\ q_y q_x & q_y^2 & q_y q_z \\ q_z q_x & q_z q_y & q_z^2 \end{array} \right)$$

$$\hat{\mathbf{C}}(\mathbf{q}) = \frac{1}{k_0'} \left(\begin{array}{ccc} 0 & q_z & -q_y \\ -q_z & 0 & q_x \\ q_y & -q_x & 0 \end{array} \right).$$

Inserting (13) into (12), I obtain, for the periodic version of e.g. the $\mathbf{G}$ kernel,

$$\overline{\mathbf{G}}(k_0'; \boldsymbol{\rho}, z; \boldsymbol{\rho}', z) = \frac{i}{2} \int \frac{d\mathbf{q}}{(2\pi)^2 q_z} \hat{\mathbf{G}}(\mathbf{q}) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} e^{iq_z(z-z')} \underbrace{\sum_{\mathbf{L}} e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{L}}}_{\mathcal{V}^{-1}\delta(\mathbf{k}-\mathbf{q})}$$

The sum over lattice vectors $\mathbf{L}$ yields a two-dimensional δ function whose prefactor is $\mathcal{V}$, the volume of the lattice unit cell.[1] Using this to evaluate the $\mathbf{q}$ integrals in (13) yields

$$\left\{ \begin{array}{c} \overline{\mathbf{G}}(k_0'; \boldsymbol{\rho}, z; \boldsymbol{\rho}', z) \\ \overline{\mathbf{C}}(k_0'; \boldsymbol{\rho}, z; \boldsymbol{\rho}', z) \end{array} \right\} = \frac{i}{2\mathcal{V}k_z'} \left\{ \begin{array}{c} \hat{\mathbf{G}}(\mathbf{k}) \\ \hat{\mathbf{C}}(\mathbf{k}) \end{array} \right\} e^{i\mathbf{k}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')} e^{ik_z'(z-z')} \tag{14}$$

[1] A more precise statement would be that the sum over lattice vectors defines a *train* of δ functions,

$$\sum_{\mathbf{L}} e^{i(\mathbf{k}-\mathbf{q})\cdot\mathbf{L}} = \frac{1}{\mathcal{V}} \sum_{\boldsymbol{\Gamma}} \delta(\mathbf{k} - \mathbf{q} - \boldsymbol{\Gamma})$$

where the sum is over reciprocal lattice vectors. In the above we have truncated this sum at the term $\boldsymbol{\Gamma} = 0$, which is justified for evaluating fields at large distance from the surface because the $\boldsymbol{\Gamma} \neq 0$ terms will typically correspond to evanescent waves.

where k'_z is defined in terms of the wavenumber k'_0 and the Bloch wavevector $\mathbf{k}$ by

$$k'_z = \sqrt{k_0'^2 - |\mathbf{k}|^2}.$$

[This is the same equation as (6).] Inserting (14) into (11), the scattered field above the surface takes the form

$$\begin{aligned}
&\mathbf{E}^{\text{trans}}(\mathbf{x}) \\
&= e^{i(k_x x + k'_z z)} \left\{ -\frac{k'_0}{2\mathcal{V} k'_z} \int_U e^{-i(\mathbf{k}\cdot\boldsymbol{\rho}' + k_z z')} \Big[Z_0 Z' \hat{\mathbf{G}}(\mathbf{k}) \cdot \mathbf{K}(\mathbf{x}') + \hat{\mathbf{C}}(\mathbf{k}) \cdot \mathbf{N}(\mathbf{x}') \Big] \, d\mathbf{x}' \right\}
\end{aligned} \tag{15}$$

Comparing this to (3c), we see that the quantity in curly braces is to be identified as $tE_0 \boldsymbol{\epsilon}_0^{\text{trans}}$, and thus we can extract the transmission coefficient t in the form

$$t = -\frac{k'_0}{2\mathcal{V} k'_z E_0} \int_U e^{-i(\mathbf{k}\cdot\boldsymbol{\rho}' + k_z z')} \Big[Z_0 Z' (\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{G}}(\mathbf{k}) \cdot \mathbf{K}(\mathbf{x}') + (\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{C}}(\mathbf{k}) \cdot \mathbf{N}(\mathbf{x}') \Big] \, d\mathbf{x}' \tag{16}$$

Transmission coefficient for TE polarization

In this case we have

$$\begin{aligned}
&(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{G}}(\mathbf{k}) \cdot \mathbf{K}(\mathbf{x}') \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{k_0'^2} \begin{pmatrix} k_x^2 & k_x k_y & k_x k_z \\ k_y k_x & k_y^2 & k_y k_z \\ k_z k_x & k_z k_y & k_z^2 \end{pmatrix} \right] \begin{pmatrix} K_x \\ K_y \\ K_z \end{pmatrix} \\
&= \frac{1}{k_0'^2} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} -k_y k_x K_x \\ (k_0'^2 - k_y^2) K_y \\ -k_y k_z K_z \end{pmatrix} \\
&= K_y
\end{aligned} \tag{17}$$

since, in our setup, the y component of the Bloch wavevector is $k_y = 0$.

Similarly,

$$\begin{aligned}
&(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{C}}(\mathbf{k}) \cdot \mathbf{N}(\mathbf{x}') \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \left[\frac{1}{k'_0} \begin{pmatrix} 0 & k_z & -k_y \\ -k_z & 0 & k_x \\ k_y & -k_x & 0 \end{pmatrix} \right] \begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} \\
&= \frac{1}{k'_0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \cdots \\ -k_z N_x + k_x N_z \\ \cdots \end{pmatrix} \\
&= -\cos\theta' N_x + \sin\theta' N_z.
\end{aligned} \tag{18}$$

Comparing (17) and (24) to (7), we see that the vector-matrix-vector sandwiches that enter the definition of the transmission coefficient (16) wind up computing simply the inner products of the $\mathbf{K}$ and $\mathbf{N}$ surface currents with the field polarization vectors:

$$(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{G}}(\mathbf{k}) \cdot \mathbf{K}(\mathbf{x}') = \boldsymbol{\epsilon}_0^{\text{trans}} \cdot \mathbf{K}, \qquad (\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger \cdot \hat{\mathbf{C}}(\mathbf{k}) \cdot \mathbf{N}(\mathbf{x}') = \overline{\boldsymbol{\epsilon}_0}^{\text{trans}} \cdot \mathbf{N} \tag{19}$$

Putting it all together,

$$\begin{aligned} t &= -\frac{k_0'}{2\mathcal{V}k_z'E_0}\int_U e^{-i(\mathbf{k}\cdot\boldsymbol{\rho}'+k_z'z')}\Big[Z_0Z'\boldsymbol{\epsilon}_0^{\text{trans}}\cdot\mathbf{K}(\mathbf{x}') + \overline{\boldsymbol{\epsilon}}_0^{\text{trans}}\cdot\mathbf{N}(\mathbf{x}')\Big]\,d\mathbf{x}' && (20)\\ &= -\frac{Z_0Z'}{2E_0\cos\theta'}\cdot\frac{1}{\mathcal{V}}\int_U \Big(\mathbf{E}^{\text{ref}*}\cdot\mathbf{K} + \mathbf{H}^{\text{ref}*}\cdot\mathbf{N}\Big)\,d\mathbf{x}'. && (21)\end{aligned}$$

Here $\{\mathbf{E},\mathbf{H}\}^{\text{ref}}$ are the fields of a "reference" plane wave with E-field magnitude $\mathbf{E}_0$ traveling in the upper medium. If the upper medium is the same as the lower medium (for example, in a thin-film geometry where both the upper and lower media are vacuum) then the reference fields are just the same as the incident fields. If the upper medium is a different medium than the lower medium (for example, in a Fresnel scattering problem) then the reference fields are similar to the incident fields but with propagation and polarization vectors modified appropriately.

Transmission coefficient for TM polarization

$$t^{\text{TM}} = -\frac{k_0'}{2\mathcal{V}k_z'}\int_U e^{-i(\mathbf{k}\cdot\boldsymbol{\rho}'+k_zz')}\Big[Z_0Z'(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger\cdot\hat{\mathbf{G}}(\mathbf{k})\cdot\mathbf{K}(\mathbf{x}') + (\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger\cdot\hat{\mathbf{C}}(\mathbf{k})\cdot\mathbf{N}(\mathbf{x}')\Big]\,d\mathbf{x}'$$

We have

$$\begin{aligned}&(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger\cdot\hat{\mathbf{G}}(\mathbf{k})\cdot\mathbf{K}(\mathbf{x}')\\ &= \begin{pmatrix}\cos\theta'\\ 0\\ -\sin\theta'\end{pmatrix}^\dagger\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{1}{k_0'^2}\begin{pmatrix}k_x^2 & k_xk_y & k_xk_z\\ k_yk_x & k_y^2 & k_yk_z\\ k_zk_x & k_zk_y & k_z^2\end{pmatrix}\right]\begin{pmatrix}K_x\\K_y\\K_z\end{pmatrix}\end{aligned} \tag{22}$$

Use $\cos\theta' = k_z/k_0'$, $\sin\theta' = k_x/k_0'$:

$$= \cos\theta' K_x - \sin\theta' K_z \tag{23}$$

$$\begin{aligned}&(\boldsymbol{\epsilon}_0^{\text{trans}})^\dagger\cdot\hat{\mathbf{C}}(\mathbf{k})\cdot\mathbf{N}(\mathbf{x}')\\ &= \begin{pmatrix}\cos\theta'\\ 0\\ -\sin\theta'\end{pmatrix}^\dagger\left[\frac{1}{k_0'}\begin{pmatrix}0 & k_z & -k_y\\ -k_z & 0 & k_x\\ k_y & -k_x & 0\end{pmatrix}\right]\begin{pmatrix}N_x\\N_y\\N_z\end{pmatrix}\\ &= N_y.\end{aligned} \tag{24}$$

Comparing (**??**) and (24) to (8) we see that once again equation (**??**) holds and we simply recover equation (21) for the TM polarization as well.

Transmission coefficients from surface-current coefficients

$$\mathbf{K}(\mathbf{x}) = \sum k_\alpha \mathbf{b}_\alpha(\mathbf{x}), \qquad \mathbf{N}(\mathbf{x}) = -Z_0 \sum n_\alpha \mathbf{b}_\alpha(\mathbf{x})$$

$$t = -\frac{Z_0 Z'}{2E_0 \cos\theta' \mathcal{V}} \sum_\alpha \left(k_\alpha \left\langle \mathbf{E}^{\text{ref}} \middle| \mathbf{b}_\alpha \right\rangle - Z_0 n_\alpha \left\langle \mathbf{H}^{\text{ref}} \middle| \mathbf{b}_\alpha \right\rangle \right)$$

$$\left\langle \mathbf{E}^{\text{ref}} \middle| \mathbf{b}_\alpha \right\rangle = \int_{\text{sup } \mathbf{b}_\alpha} \boldsymbol{\epsilon} \cdot \mathbf{b}_\alpha(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \, d\mathbf{x}$$

For RWG functions this integral may be evaluated in closed form:

$$\begin{aligned}
&\int_{\text{sup } \mathbf{b}_\alpha} \boldsymbol{\epsilon} \cdot \mathbf{b}_\alpha(\mathbf{x}) e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= \ell_\alpha \boldsymbol{\epsilon} \cdot \int_0^1 du \int_0^u dv \, \Big\{ e^{-i\mathbf{q}\cdot[\mathbf{Q}^+ + u\mathbf{A}^+ + v\mathbf{B}]} \Big(u\mathbf{A}^+ + v\mathbf{B} \Big) \\
&\qquad\qquad - e^{-i\mathbf{q}\cdot[\mathbf{Q}^- + u\mathbf{A}^- + v\mathbf{B}]} \Big(u\mathbf{A}^- + v\mathbf{B} \Big) \Big\} \\
&= \ell_\alpha \Big\{ e^{-i\mathbf{q}\cdot\mathbf{Q}^+} \Big[f_1\big(\mathbf{q}\cdot\mathbf{A}^+, \mathbf{q}\cdot\mathbf{B}\big) \boldsymbol{\epsilon}\cdot\mathbf{A}^+ + f_2\big(\mathbf{q}\cdot\mathbf{A}^+, \mathbf{q}\cdot\mathbf{B}\big) \boldsymbol{\epsilon}\cdot\mathbf{B} \Big] \\
&\qquad - e^{-i\mathbf{q}\cdot\mathbf{Q}^-} \Big[f_1\big(\mathbf{q}\cdot\mathbf{A}^-, \mathbf{q}\cdot\mathbf{B}\big) \boldsymbol{\epsilon}\cdot\mathbf{A}^- + f_2\big(\mathbf{q}\cdot\mathbf{A}^-, \mathbf{q}\cdot\mathbf{B}\big) \boldsymbol{\epsilon}\cdot\mathbf{B} \Big] \Big\}
\end{aligned}$$

where

$$\begin{aligned}
f_1(x,y) &= \int_0^1 \int_0^u u e^{-i(ux+vy)} \, dv \, du \\
f_2(x,y) &= \int_0^1 \int_0^u v e^{-i(ux+vy)} \, dv \, du
\end{aligned}$$