SCUFF-NEQ Technical Reference

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July 6, 2014

Contents

1 Theoretical background

SCUFF-NEQ implements the "fluctuating-surface-current" (FSC) approach to computational fluctuation physics. The FSC method was originally developed for equilibrium Casimir computations [1, 2, 3] and later extended to non-equilibrium phenomena in Refs. 4 and 5. Here we present a quick summary of the key equations in this approach; slightly more detail may be found in the Appendices.

1.1 Time-average quantities from surface-current bilinears

The FSC approach begins with the observation that, in a classical, deterministic frequency-domain scattering problem involving the fields of known time-harmonic external sources impinging upon a collection of material bodies, the time-average values of many physical quantities of interest may be expressed as bilinear (quadratic) products of the *surface currents* induced on the body surfaces by the external fields. The quantities whose time-average values may be be expressed in this way include both **(a)** spatially-resolved fluxes of energy and momentum at individual points in space, and **(b)** total (spatially-integrated) power-transfer rates, forces, and torques on individual bodies.

More specifically, let the six-vector of surface electric and magnetic currents (flowing on the surfaces of *all* bodies in a geometry) be

$$\boldsymbol{\mathcal{C}}(\mathbf{x}) \equiv \begin{pmatrix} \mathbf{K}(\mathbf{x}) \\ \mathbf{N}(\mathbf{x}) \end{pmatrix} \approx \sum_{\alpha} c_\alpha \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}) \tag{1}$$

where we have discretized by approximating $\boldsymbol{\mathcal{C}}(\mathbf{x})$ as an expansion in a discrete set of six-vector basis functions[1] $\{\boldsymbol{\mathcal{B}}_\alpha(\mathbf{x})\}$; here $\{c_\alpha\}$ are scalar expansion coefficients[2] typically obtained by inverting a linear system to solve a scattering problem. (We work at a fixed frequency ω and assume all fields and currents vary in time like $e^{-i\omega t}$.)

Then the time-average values of physical quantities associated with power, force and torque (PFT)—including both spatially-resolved PFT fluxes at individual points in space as well as spatially-integrated quantities representing the total PFT on compact bodies—may generally be expressed as vector-matrix-vector products of the form

$$Q = \mathbf{c}^\dagger \mathbf{Q} \mathbf{c} \tag{2}$$

where Q is the quantity of interest, $\mathbf{c}$ is the vector of surface-current coefficients in (1), and $\mathbf{Q}$ is a matrix that depends on the quantity we are computing and the method we use to compute it.

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[1] In SCUFF-NEQ these take the form $\boldsymbol{\mathcal{B}}_\alpha = \left(\begin{smallmatrix} \mathbf{b}_\alpha \\ 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 \\ \mathbf{b}_\alpha \end{smallmatrix}\right)$ where $\mathbf{b}_\alpha$ is a three-vector RWG basis function. However, the FSC formalism is not specific to this choice.

[2] In SCUFF-EM the basis functions are dimensionless and the expansion coefficients have units of

$$[c_{2\alpha}] = \frac{\text{Amperes}}{\text{meters}} \quad \text{(electric surface currents)}, \qquad [c_{2\alpha+1}] = \frac{\text{Volts}}{\text{meters}} \quad \text{(magnetic surface currents)}.$$

Spatially-resolved PFT fluxes For example, the time-average Poynting vector and Maxwell stress tensor at a point $\mathbf{x}$—that is, the time-average fluxes, in the $\hat{\mathbf{n}}$ direction, of energy, of $\hat{\mathbf{u}}$-directed linear momentum, and of $\hat{\mathbf{u}}$-directed angular momentum—are given by

$$\begin{array}{lll} \text{Energy flux:} & \mathbf{P}(\mathbf{x})\cdot\hat{\mathbf{n}} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{PFLUX}}(\mathbf{x};\hat{\mathbf{n}})\mathbf{c} \\ \text{Linear momentum flux:} & \hat{\mathbf{u}}\cdot\mathbf{T}(\mathbf{x})\cdot\hat{\mathbf{n}} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{FFLUX}}(\mathbf{x};\hat{\mathbf{u}},\hat{\mathbf{n}})\mathbf{c} \\ \text{Angular momentum flux:} & \hat{\mathbf{u}}\cdot\Big[(\mathbf{x}-\mathbf{x}_0)\times\mathbf{T}(\mathbf{x})\Big]\cdot\hat{\mathbf{n}} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{TFLUX}}(\mathbf{x};\mathbf{x}_0,\hat{\mathbf{u}},\hat{\mathbf{n}})\mathbf{c} \end{array}$$

where the form of the $\mathbf{Q}^{\text{FLUX}}$ matrices is derived in an Appendix. (In the last line, $\mathbf{x}_0$ is the origin about which we are computing the torque).

Spatially-integrated PFT fluxes On the other hand, the *total* (spatially-integrated) power absorbed by, and the $\hat{\mathbf{u}}$-directed force and torque exerted on, one or more compact bodies contained within a closed bounding surface $\mathcal{S}$ are given by

$$\begin{array}{lll} \text{Absorbed power:} & P^{\text{abs}} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{PTOT}}(\mathcal{S})\mathbf{c} \\ \text{Force:} & \hat{\mathbf{u}}\cdot\mathbf{F} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{FTOT}}(\mathcal{S};\hat{\mathbf{u}})\mathbf{c} \\ \text{Torque:} & \hat{\mathbf{u}}\cdot\boldsymbol{\mathcal{T}} & = \frac{1}{2}\text{Re }\mathbf{c}^\dagger\mathbf{Q}^{\text{TTOT}}(\mathcal{S};\mathbf{x}_0,\hat{\mathbf{u}})\mathbf{c} \end{array}$$

Explicit expressions for the $\mathbf{Q}^{\text{TOT}}$ matrices are given in the Appendices.

1.2 Statistical averages of surface-current bilinears

Equation (2) gives the time-average value of a PFT quantity in a classical, deterministic scattering problem in which the surface currents $\mathbf{c}_\alpha$ are known. The heart of the FSC approach to nonequilibrium phenomena is a concise formula for the *statistical average* of such surface-current bilinears, where the averaging is performed over thermal and quantum-mechanical fluctuations of sources inside material bodies at fixed temperatures. A precise statement of the FSC equivalence is as follows.

- If Q is a physical quantity whose time-average value (in a classical, deterministic scattering problem involving time-harmonic fields at frequency ω) may be expressed as a bilinear function of the surface currents with a matrix $\mathbf{Q}(\omega)$, i.e

$$\text{if} \qquad Q = \mathbf{c}^\dagger\mathbf{Q}\mathbf{c} \tag{3a}$$

- then the statistical *average* of the quantity Q—where the averaging is over quantum and thermal fluctuations—is given by a sum over the contributions of fluctuating sources in all regions of space:

$$\text{then} \qquad \big\langle Q \big\rangle = \int_0^\infty \sum_r \Theta(T_r,\omega)\Phi_r(\omega)\,d\omega \tag{3b}$$

  where $\Theta(T_r,\omega) = \frac{\hbar\omega}{e^{\hbar\omega/kT_r}-1}$ is the Bose-Einstein factor for the temperature T_r of region r, and where the "generalized flux" Φ_r is the trace of a four-matrix product involving the $\mathbf{Q}$ matrix:

$$\Phi_s = \frac{2}{\pi}\mathrm{Tr}\,\left\{\mathbf{Q}\underbrace{\mathbf{W}\overline{\mathbf{G}}_r\mathbf{W}^\dagger}_{\mathbf{R}_s}\right\}. \tag{3c}$$

  Here $\mathbf{W}$ is the inverse of the BEM matrix for the *entire* collection of bodies and $\overline{\mathbf{G}}_r$ is a symmetrized version of a portion of the BEM matrix for region r alone.

The matrix $\mathbf{R}_s$ in (3)—the "Rytov" matrix—furnishes a concise description of the fluctuating sources inside body s. For a given output quantity Q, the generalized flux Φ_s is the contribution to $\langle Q\rangle$ of just the fluctuating currents that lie within body s (here s stands for "source"), although these contributions are computed with proper consideration paid to the scattering response of the other bodies in the geometry. In other words, $\Theta(T_s,\omega)\Phi_s$ is the contribution to $\langle Q\rangle$ obtained by setting body s to temperature T_s and setting all other bodies to *zero* temperature, so that those bodies are present as electromagnetic scatterers but not as sources of radiation.

Separating equilibrium from non-equilibrium contributions

The set of regions over which we sum in (3a) includes the *exterior* region, i.e. the environment in which our interacting bodies are embedded. It is convenient to decompose this sum into two terms:

$$\big\langle Q \big\rangle = \big\langle Q \big\rangle^{\text{EQ}} + \big\langle Q \big\rangle^{\text{NEQ}} \tag{4}$$

$$\big\langle Q \big\rangle^{\text{EQ}} \equiv \int_0^\infty \Theta(T_{\text{env}},\omega)\sum_r \Phi_r(\omega)\,d\omega \tag{5}$$

$$\big\langle Q \big\rangle^{\text{NEQ}} \equiv \int_0^\infty \sum_s \Big[\Theta(T_s,\omega) - \Theta(T_{\text{env}},\omega)\Big]\Phi_s(\omega)\,d\omega \tag{6}$$

$$= \int_0^\infty \sum_s \Delta\Theta(T_s,\omega)\Phi_s(\omega)\,d\omega \tag{7}$$

where

$$\Delta\Theta(T_s,\omega) \equiv \Theta(T_s,\omega) - \Theta(T_{\text{env}},\omega).$$

The quantity $\langle Q \rangle^{\text{EQ}}$ is the average value of Q that would obtain if the temperature in all material regions were equal to the environment temperature T_{env}—that is, it is the *equilibrium* value of $\langle Q \rangle$ at temperature T_{env}. The equilibrium value of PFT quantities may be computed by methods that are less costly than SCUFF-NEQ. (For example, if Q is a spatially-integrated force or torque, then $\langle Q \rangle^{\text{EQ}}$ is just the equilibrium Casimir force, for which it is far easier to Wick-rotate the ω integral to the positive imaginary frequency axis, as is done in the equilibrium Casimir code SCUFF-CAS3D. On the other hand, if Q is a spatially-integrated power transfer quantity, then $\langle Q \rangle^{\text{EQ}} = 0$ identically.) Thus this contribution is not computed by SCUFF-NEQ.

The quantity $\langle Q \rangle^{\text{NEQ}}$ is the extent to which $\langle Q \rangle$ *deviates* from its equilibrium value, and the sum in (7) now ranges only over the source bodies in the geometry, not including the environment contribution. $\langle Q \rangle^{\text{NEQ}}$ is the quantity that is computed by SCUFF-NEQ.

2 Output files and units

2.1 General comments about units in SCUFF-NEQ

The numerical values specified as inputs to, or reported as outputs by, SCUFF-NEQ are interpreted as multiples of the following base units.

Length	Microns ($L_0 = 1\ \mu$m)
Angular frequency	$\omega_0 = \frac{c}{1\ \mu\text{m}} = 3 \cdot 10^{14}$ rad/sec. (This corresponds to a free-space wavelength of $\lambda \approx 6.28\ \mu$m.)
Power	Watts
Force	Nanonewtons
Torque	Nanonewtons $\times$ microns
Power flux	Watts / microns^2
Linear momentum flux	Nanonewtons / microns^2
Angular momentum flux	Nanonewtons / microns

2.2 Output files for spatially-integrated quantities

Let $\langle Q_d \rangle$ be the total time-average power, force, or torque (PFT) on body d. (d stands for a "destination" body, as distinct from a "source" body whose fluctuating sources contribute to Q_d). From equation (3b), we have

$$\langle Q_d \rangle = \int_0^\infty \sum_s \Delta\Theta_s(\omega)\Phi_{s\to d}(\omega)\, d\omega \tag{8}$$

where $\Delta\Theta_s = \Theta_s - \Theta_{\rm env}$ (Θ_s and $\Theta_{\rm env}$ are the Bose-Einstein factors for the temperature of body s and the temperature of the environment) and $\Phi_{s\to d}(\omega)$ is the (frequency-resolved, spatially-integrated) flux of power or momentum from body s to body d. To evaluate this integral, it is convenient to agree to measure angular frequency in the default SCUFF units, whereupon we introduce a dimensionless angular frequency u according to[3]

$$\omega = u \cdot \omega_0 \qquad \text{where} \qquad \omega_0 \equiv 3 \cdot 10^{14}\ \text{rad/sec.}$$

Then the total time-average PFT on body d may be expressed as an integral over the dimensionless variable u:

$$\langle Q_d \rangle = \hbar\omega_0^2 \int_0^\infty \sum_s \Delta\widehat{\Theta}_s(u)\Phi^{\rm PFT}_{s\to d}(u)\, du \tag{9}$$

where $\widehat{\Theta}_s(u)$ is a dimensionless version of the Bose-Einstein factor:

$$\widehat{\Theta}_s(u) \equiv \frac{u}{e^{\beta_s u} - 1}, \qquad \beta_s \equiv \frac{\hbar\omega_0}{kT_s} \approx \frac{2278}{T_s \text{ in Kelvin.}}$$

The various constituents of equation (9) are written by SCUFF-NEQ to three distinct output files, as follows:

$$\langle Q_d \rangle = \underbrace{\left[\int_0^\infty \underbrace{\left\{\hbar\omega_0^2 \sum_s \Delta\widehat{\Theta}_s(u)\, \underbrace{\Phi^{\rm PFT}_{s\to d}(u)}_{\texttt{.SIFlux}}\right\}}_{\texttt{.SIIntegrand}} du\right]}_{\texttt{.NEQPFT}} \tag{10}$$

More specifically,

- The quantity $\Phi_{s\to d}(u)$ is written to the `.SIFlux` file. This quantity may be interpreted as the PFT per unit frequency per unit temperature (which, dimensionally, amounts to the PFT per unit power, i.e.

[3] Note that u is the numerical quantity referred to as `Omega` on the SCUFF-NEQ command line and in the SCUFF-NEQ source code. Some convenient dimensionful quantities involving ω_0 are:

$$\hbar\omega_0 = 0.197\ \text{eV} = 31.6 \cdot 10^{-21}\ \text{Joules}, \qquad \hbar\omega_0^2 = 9.49115\,\mu\text{W (microWatts)}.$$

the PFT per watt) exerted on body d due to sources in body s. This is a temperature-independent quantity. For `PFT`=power, this quantity is *dimensionless*. For `PFT`=force, this quantity has units of *nanoNewtons/watts*. For `PFT`=torque, this quantity has units of *nanoNewtons* $\times$ *microns/watts*.

- The quantity written to the `.SIIntegrand` file is the PFT per unit frequency exerted on body d due to thermally-weighted sources in body s. (The `.SIIntegrand` file records, for each destination body d, both the individual contributions of each source body s and the total contributions of all source bodies.) Plotting this quantity versus the frequency yields spectrally-resolved PFT information showing the contributions of fluctuations at each frequency to the power, force, or torque on the destination body from the souce body.

  When we say "per unit frequency" here, we mean "per unit *dimensionless* frequency" (i.e. per unit interval of the dimensionless frequency u), which means that the quantity reported in the `.SIIntegrand` file has the same units as the final PFT output quantity: *watts* for power, *nanonewtons* for force, or *nanonewtons* $\times$ *microns* for torque.

- The quantity written to the `.NEQPFT` file is the total time-average PFT exerted on body d due to thermally-weighted sources in body s. (The `.NEQPFT` file records, for each destination body d, both the individual contributions of each source body s and the total contributions of all source bodies.) This quantity has units of *watts* for power, *nanonewtons* for force, *nanonewtons* $\times$ *microns* for torque.

2.3 Output files for spatially-resolved quantities

Let $Q_{\mathbf{x}}$ be a spatially-resolved PFT quantity: one of the 3 Cartesian components of the (time-average) Poynting vector (PV, units of watts/microns^2) or one of the 9 Cartesian components of the (time-average) Maxwell stress tensor (MST, units of nanoNewtons/microns^2). Then the thermal average of $Q_{\mathbf{x}}$ may be expressed in a form similar to that of (9):

$$\langle Q_{\mathbf{x}} \rangle = \hbar\omega_0^2 \int_0^\infty \sum_s \Delta\widehat{\Theta}_s(u) \Phi^{\rm PFT}_{s\to\mathbf{x}}(u)\, du \tag{11}$$

The various constituents of equation (11) are written by SCUFF-NEQ to three distinct output files, as follows:

$$\langle Q_{\mathbf{x}} \rangle = \underbrace{\left[\int_0^\infty \underbrace{\left\{ \hbar\omega_0^2 \sum_s \Delta\widehat{\Theta}_s(u) \underbrace{\Phi^{\rm PFT}_{s\to\mathbf{x}}(u)}_{\texttt{.SRFlux}} \right\}}_{\texttt{.SRIntegrand}} du \right]}_{\texttt{.PVMST}} \tag{12}$$

More specifically,

- The quantity $\Phi_{s\to\mathbf{x}}(u)$ is written to the `.SRFlux` file. This quantity may be interpreted as the spatially-resolved PFT flux, per unit frequency per unit temperature (which, dimensionally, amounts to the PFT per unit power, i.e. the PFT per watt) at the point $\mathbf{x}$ due to sources in body s. This is a temperature-independent quantity. For PV components, this quantity has units of *microns*$^{-2}$. For MST components, this quantity has units of *nanoNewtons/(watts* $\cdot$ *microns*$^{-2}$*)*.

- The quantity written to the `.SRIntegrand` file is the PFT flux per unit frequency at $\mathbf{x}$ due to thermally-weighted sources in body s. (The `.SRIntegrand` file records, for each evaluation point $\mathbf{x}$, both the individual contributions of each source body s and the total contributions of all source bodies.) Plotting this quantity versus the frequency yields spectrally-resolved PFT information showing the contributions of fluctuations at each frequency to the power/force flux at $\mathbf{x}$ from the souce body.

  When we say "per unit frequency" here, we mean "per unit *dimensionless* frequency" (i.e. per unit interval of the dimensionless frequency u), which means that the quantity reported in the `.SRIntegrand` file has the same units as the final PFT output quantity: *watts/microns*^{2} for PV components and *nanoNewtons/microns*^{2} for MST components.

- The quantity written to the `.PVMST` file is the total (frequency-integrated) time-average PV or MST at $\mathbf{x}$. (The `.PVMST` file records, for each evaluation point $\mathbf{x}$, both the individual contributions of each source body s and the total contributions of all source bodies.) This quantity has units of *watts/microns*^{2} for PV components and *nanoNewtons/microns*^{2} for MST components.

2.4 Visualization: Flux plots

As discussed above, the power, force, and torque quantities computed by SCUFF-NEQ are traces over products of matrices whose individual elements describe the interactions between RWG basis functions.[4] Each diagonal element of these matrices may thus be associated with a single localized RWG basis function, and the matrix trace may thus be decomposed into a sum of contributions associated with individual RWG functions. Thus the flux defined by (3)may be written in the form

$$\Phi = \sum_{\alpha} \Phi_{\alpha}$$

where Φ_α, the contribution of basis function $\boldsymbol{\mathcal{B}}_\alpha$ to Φ, is just the (α,α) diagonal element of the matrix in (3b).

Since the supports of the basis functions are spatially localized, we can think of Φ_α as the surface integral of a certain spatially-constant surface density ρ_Φ. over the support of $\boldsymbol{\mathcal{B}}_\alpha$. If A_α is the total surface area over which $\mathcal{B}_\alpha$ is supported,[5], then the contribution of $\boldsymbol{\mathcal{B}}_\alpha$ to ρ_Φ is simply Q_α/A_α.

[4] For example, the matrix $\mathbf{W}$ in equation (3a) is the inverse of the BEM matrix $\mathbf{M}$; the (α,β) entry of this matrix ($M_{\alpha\beta}$) is the inner product of basis functions $\boldsymbol{\mathcal{B}}_\alpha$ with the electromagnetic field due to basis function $\boldsymbol{\mathcal{B}}_\beta$.

[5] A_α will generally be the sum of the areas of two triangles in the surface mesh; for half-RWG functions it will be the area of just one triangle.

3 Tests and Demonstrations

As a test and demonstration of the use of SCUFF-NEQ, in this section we reproduce the results of Krüger et al. [6] for non-equilibrium fluctuation-induced phenomena involving individual and paired spheres.

3.1 Radiation of a single sphere

Equation (122) of Ref. 6, giving the rate of power emission from a warm (temperature T) sphere into a cold environment, may be written in the form

$$P^{\rm rad} = \int_0^\infty d\omega\, \Theta(T,\omega)\Phi(\omega),$$
$$\Phi(\omega) = -\frac{2}{\pi}\sum_{P\ell m}\Big[\text{Re }\mathcal{T}_{P\ell m} + |\mathcal{T}_{P\ell m}|^2\Big]$$

where $\{\mathcal{T}_{P\ell m}(\omega)\}$ are the diagonal $\mathbb{T}$-matrix elements of the sphere. For a sphere of small radius R, we may restrict the sum to just the $\ell = 1$ contributions, for which we have [6]

$$\mathcal{T}_{N,\ell=1,m}(\omega) = i\frac{2(\epsilon-1)}{3(\epsilon+2)}(kR)^3 + 2i\frac{2-3\epsilon+\epsilon^2(1+\mu)}{5(2+\epsilon^2)}(kR)^5 - \frac{4(\epsilon-1)^2}{9(2+\epsilon)^2}(kR)^6 + \cdots,$$

and $\mathcal{T}_{M1m}$ is given by the same formula but with $\epsilon \leftrightarrow \mu$. (Here ϵ, μ are the relative permittivity and permeability of the sphere and $k = \omega/c$ is the free-space wavelength).

3.2 Heat transfer between two spheres

For two spheres separated by a center-center distance d, the power absorbed by sphere 2 due to fluctuating sources in sphere 1 is, in the large-d limit [6],

$$P^{1\to 2} = \int_0^\infty d\omega\, \Theta(T_1,\omega)\Phi_{\rm P}^{1\to 2}(\omega),$$
$$\Phi_{\rm P}^{1\to 2}(\omega) = \frac{2}{\pi}\sum_{PP'}\Big[\text{Re }\mathcal{T}_P^1 + |\mathcal{T}_P^1|^2\Big]\Big[\text{Re }\mathcal{T}_{P'}^2 + |\mathcal{T}_{P'}^2|^2\Big]U_{PP'}(kd),$$
$$U_{PP'}(x) = \frac{9}{2x^2} + \frac{9}{2x^4} + \frac{27}{2x^6}\delta_{PP'}$$

where we have suppressed the $\ell = 1, m$ subscript on $\mathcal{T}$.

3.3 Non-equilibrium forces between spheres

For two spheres separated by a center-center distance d, the force on sphere 2 due to fluctuating sources in sphere 1 is, in the large-d limit [6],

$$
\begin{aligned}
F^{1\to 2} &= \int_0^\infty d\omega\, \Theta(T_1,\omega)\Phi_{\rm F}^{1\to 2}(\omega), \\
\Phi_{\rm F}^{1\to 2}(\omega) &= \frac{1}{c}\cdot\frac{1}{\pi}\sum_{PP'}\Big[\text{Re }\mathcal{T}_P^1 + |\mathcal{T}_P^1|^2\Big]\Big[C_{PP'}^1 + C_{PP'}^2 + C_{PP'}^3\Big] \\
C_{PP'}^1 &= \frac{9}{(kd)^2}\Big[\text{Re }\mathcal{T}_{P'}^2 + \text{Re }\left(\mathcal{T}_2^P T_2^{\overline{P}*}\right)\delta_{PP'}\Big] \\
C_{PP'}^2 &= \text{Im }T_{P'}^2\Big[\frac{9}{(kd)^3} + \frac{18}{(kd)^5} + \frac{81}{(kd)^7}\delta_{PP'}\Big] \\
C_{PP'}^3 &= -\frac{9}{(kd)^5}\text{Im }\left(\mathcal{T}_2^P T_2^{\overline{P}*}\right)\delta_{PP'}
\end{aligned}
$$

The force on sphere 2 due to sources in sphere 2 is

$$
\begin{aligned}
F^{2\to 2} &= \int_0^\infty d\omega\, \Theta(T_2,\omega)\Phi_{\rm F}^{2\to 2}(\omega), \\
\Phi_{\rm F}^{2\to 2}(\omega) &= \frac{1}{c}\cdot\frac{1}{\pi}\sum_{PP'}\Big[\text{Re }\mathcal{T}_P^2 + |\mathcal{T}_P^2|^2\Big]\Big\{\text{Re }\Big[D_{PP'}^1 + D_{PP'}^2 + D_{PP'}^3\Big]\Big\} \\
D_{PP'}^1 &= \text{Re }\Big[\left(\mathcal{T}_P^1 - \mathcal{T}_{\overline{P}}^1\right)\left(\frac{9}{(kd)^2} + i\frac{27}{(kd)^3}\right)e^{2ikd}\Big] \\
D_{PP'}^2 &= \text{Re }\Big[\left(\mathcal{T}_P^1 - \frac{1}{2}\mathcal{T}_{\overline{P}}^1\right)\left(\frac{72}{(kd)^4}\right)e^{2ikd}\Big] \\
D_{PP'}^3 &= \Re\Big[(\mathcal{T}_P^1\left(\frac{162}{(kd)^6} + i\frac{81}{(kd)^7}\right)e^{2ikd}\Big]
\end{aligned}
$$

3.4 Spatial distribution of poynting flux in the sphere–sphere case

4 SCUFF-NEQ for Region-and-Surface Geometries

- Surface 1: vacuum-metal interface
- Surface 2: metal-dielectric interface
- Surface 3: vacuum-dielectric interface

Structure of surface-current vector:

$$\mathbf{c} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \mathbf{c}_3 \end{pmatrix},$$

Structure of BEM matrix:

$$\mathbf{M} = \begin{pmatrix} \mathbf{T}_1^{\rm vacuum} + \mathbf{T}_1^{\rm metal} & \mathbf{U}_{12}^{\rm metal} & \mathbf{U}_{13}^{\rm vacuum} \\ \mathbf{U}_{21}^{\rm metal} & \mathbf{T}_2^{\rm metal} + \mathbf{T}_2^{\rm dielectric} & \mathbf{U}_{23}^{\rm dielectric} \\ \mathbf{U}_{31}^{\rm vac} & \mathbf{U}_{31}^{\rm dielectric} & \mathbf{T}_3^{\rm vacuum} + \mathbf{T}_3^{\rm dielectric} \end{pmatrix}$$

Rytov matrix for metal:

$$\mathbf{R}^{\rm metal} = \mathbf{W} \cdot \left[{\rm sym} \begin{pmatrix} \mathbf{T}_1^{\rm metal} & 0 & 0 \\ 0 & \mathbf{T}_2^{\rm metal} & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \mathbf{W}^\dagger$$

Rytov matrix for dielectric:

$$\mathbf{R}^{\rm dielectric} = \mathbf{W} \cdot \left[{\rm sym} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbf{T}_2^{\rm dielectric} & 0 \\ 0 & 0 & \mathbf{T}_3^{\rm dielectric} \end{pmatrix} \right] \mathbf{W}^\dagger$$

Structure of $\mathbf{Q}$ matrices for PFT transfer between the composite particle and the exterior (vacuum) region, as computed using the EPPFT / OPFT formalisms or the DSIPFT formalism:

$$\mathbf{Q}^{\rm EP/O} = \begin{pmatrix} \mathbf{Q}_1^{\rm EP/O} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{Q}_3^{\rm EP/O} \end{pmatrix}, \qquad \mathbf{Q}^{\rm DSI} = \begin{pmatrix} \mathbf{Q}_{11}^{\rm DSI} & 0 & \mathbf{Q}_{13}^{\rm DSI} \\ 0 & 0 & 0 \\ \mathbf{Q}_{31}^{\rm DSI} & 0 & \mathbf{Q}_{33}^{\rm DSI} \end{pmatrix}$$

A Derivation of surface-current bilinears

In Section 1.1 we noted that the fluxes of power, force, and torque (PFT) at individual points in space, as well as the total PFT absorbed by or exerted on full bodies, may be expressed as vector-matrix-vector product formulas of the form

$$Q = \mathbf{c}^\dagger \mathbf{Q} \mathbf{c}$$

where Q is the quantity of interest (a PFT flux or a total PFT), $\mathbf{c}$ is the vector of surface-current coefficients, and $\mathbf{Q}$ is a matrix that depends on the quantity we are computing and the method we use to compute it. In this Appendix we discuss the specific forms of the $\mathbf{Q}$ matrix appropriate for various quantities.

Fields from surface currents

Recall that in the surface-integral-equation (SIE) approach to classical electromagnetism problems, we solve for tangential *surface currents* (both electric currents $\mathbf{K}$ and magnetic currents $\mathbf{N}$) flowing on the surfaces of homogeneous material bodies. In numerical solvers, we approximate these as finite expansions in a discrete set of N_B basis functions; using a convenient 6-vector notation in which $\boldsymbol{\mathcal{C}} \equiv \binom{\mathbf{K}}{\mathbf{N}}$, we put

$$\boldsymbol{\mathcal{C}}(\mathbf{x}) = \sum_{\alpha=1}^{N_\mathrm{B}} c_\alpha \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x})$$

where $\{\boldsymbol{\mathcal{B}}_\alpha\}$ is a set of 6-vector basis functions[6] and $\{c_\alpha\}$ are scalar expansion coefficients. (We work at a fixed frequency ω and assume all fields and currents vary in time like $e^{-i\omega t}$.)

The electric and magnetic fields produced by these currents are linear functions of the $\{c_\alpha\}$ coefficients. In 6-vector notation with $\boldsymbol{\mathcal{F}} \equiv \binom{\mathbf{E}}{\mathbf{H}}$, we have

$$\boldsymbol{\mathcal{F}}(\mathbf{x}) = \sum_\alpha c_\alpha \boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) \tag{13}$$

where $\boldsymbol{\mathcal{F}}_\alpha = \binom{\mathbf{E}_\alpha}{\mathbf{H}_\alpha}$ is the six-vector of fields produced by basis function $\boldsymbol{\mathcal{B}}_\alpha$ populated with unit strength. These may be calculated numerically as convolutions over basis functions,

$$\boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) = \int_{\sup \boldsymbol{\mathcal{B}}_\alpha} \boldsymbol{\mathcal{G}}(\mathbf{x}, \mathbf{x}') \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}')\, d\mathbf{x}' \tag{14}$$

where $\boldsymbol{\mathcal{G}}$ is an appropriate 6×6 dyadic Green's function.

[6] In SCUFF-NEQ these take the form $\boldsymbol{\mathcal{B}}_\alpha = \binom{\mathbf{b}_\alpha}{0}$ or $\binom{0}{\mathbf{b}_\alpha}$ where $\mathbf{b}_\alpha$ is a three-vector RWG basis function. However, the FSC formalism is not specific to this choice.

Regions and surfaces

Equations (13) and (14) gloss over some subtleties regarding the location of the evaluation point $\mathbf{x}$. To get these straight, consider a SCUFF-EM geometry like that depicted in Figure 1, in which we have multiple homogeneous regions $\{\mathcal{R}_i\}$ and multiple interface surfaces $\{\mathcal{S}_i\}$.

If the evaluation point $\mathbf{x}$ in (13) lies in region $\mathcal{R}_i$, then the summation in equation (13) runs only over the contributions of basis functions lying on the surface bounding $\mathcal{R}_i$:

$$\mathbf{x} \in \mathcal{R}_i \qquad \Longrightarrow \qquad \boldsymbol{\mathcal{F}}(\mathbf{x}) = \sum_{\boldsymbol{\mathcal{B}}_\alpha \in \partial\mathcal{R}_i} c_\alpha \boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) \tag{15}$$

In this case, the dyadic Green's function that enters (14) is the DGF for region $\mathcal{R}_i$:

$$\mathbf{x} \in \mathcal{R}_i \qquad \Longrightarrow \qquad \boldsymbol{\mathcal{F}}_\alpha(\mathbf{x}) = \int_{\sup \boldsymbol{\mathcal{B}}_\alpha} \boldsymbol{\mathcal{G}}(\mathcal{R}_i; \mathbf{x}, \mathbf{x}') \boldsymbol{\mathcal{B}}_\alpha(\mathbf{x}')\, d\mathbf{x}' \tag{16}$$

Energy and momentum flux from field bilinears

The Poynting flux and Maxwell stress tensor are quadratic functions of the field components and may be conveniently written in the form of 6-dimensional vector-matrix-vector products. In particular, the power flux in the direction of a unit vector $\hat{\mathbf{n}}$ is

$$\begin{aligned} \mathbf{P}(\mathbf{x}) \cdot \hat{\mathbf{n}} &= \frac{1}{2}\ \mathrm{Re}\ \varepsilon_{ijk} \hat{\mathbf{n}}_i E_j^*(\mathbf{x}) H_k(\mathbf{x}) \\ &= \frac{1}{4} \boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\, \boldsymbol{\mathcal{N}}^{\mathrm{P}}(\hat{\mathbf{n}})\, \boldsymbol{\mathcal{F}}(\mathbf{x}) \end{aligned} \tag{17}$$

with

$$\boldsymbol{\mathcal{N}}^{\mathrm{P}} = \begin{pmatrix} 0 & \mathbf{N}^{\mathrm{P}} \\ -\mathbf{N}^{\mathrm{P}} & 0 \end{pmatrix}, \qquad \mathbf{N}^{\mathrm{P}} = \begin{pmatrix} 0 & \hat{n}_z & -\hat{n_y} \\ -\hat{n}_z & 0 & +\hat{n_x} \\ \hat{n}_y & -\hat{n}_x & 0 \end{pmatrix}.$$

Similarly, the flux of $\hat{\mathbf{u}}$-directed linear momentum is

$$\begin{aligned} \hat{\mathbf{u}} \cdot \mathbf{T}(\mathbf{x}) \cdot \hat{\mathbf{n}} &= \frac{1}{2}\ \mathrm{Re}\ u_i \left[\epsilon E_i^*(\mathbf{x}) E_j(\mathbf{x}) + \mu H_i^*(\mathbf{x}) H_j(\mathbf{x}) - \frac{\delta_{ij}}{2} \Big(\epsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2 \Big) \right] \hat{n}_j \\ &= \frac{1}{4} \boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\, \boldsymbol{\mathcal{N}}^{\mathrm{F}}\, \boldsymbol{\mathcal{F}}(\mathbf{x}) \end{aligned} \tag{18}$$

with

$$\boldsymbol{\mathcal{N}}^{\mathrm{F}} = \begin{pmatrix} \epsilon \mathbf{N}^{\mathrm{F}} & 0 \\ 0 & \mu \mathbf{N}^{\mathrm{F}} \end{pmatrix}$$

where the 3×3 matrix $\mathbf{N}^{\mathrm{F}}$ involves the outer product of $\hat{\mathbf{u}}$ and $\hat{\mathbf{n}}$:

$$\mathbf{N}^{\mathrm{F}}(\hat{\mathbf{u}}, \hat{\mathbf{n}}) = \hat{\mathbf{u}}\hat{\mathbf{n}}^T + \hat{\mathbf{n}}\hat{\mathbf{u}}^T + (\hat{\mathbf{u}} \cdot \hat{\mathbf{n}})\mathbf{1}$$

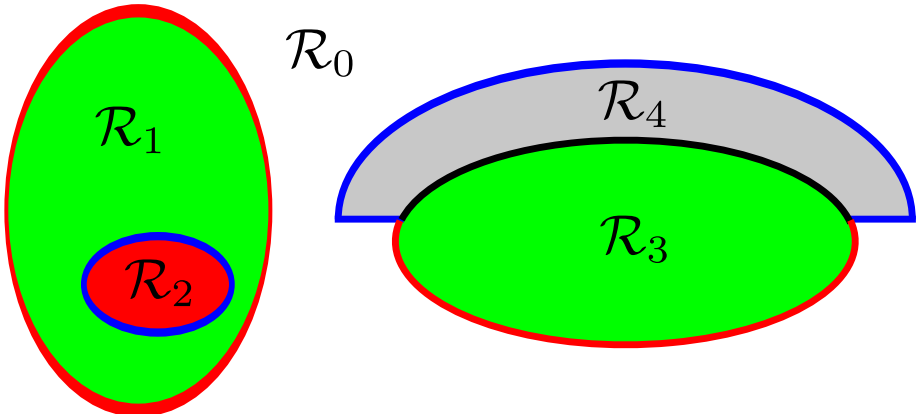

Figure 1: Schematic depiction of a SCUFF-EM geometry containing nested surfaces and three-material junctions. The body at left is a silicon ellipsoid ($\mathcal{R}_1$) that fully contains an SiO_2 subregion ($\mathcal{R}_2$). The object at right is a silicon ellipsoid partially coated with gold; the interior of the ellipsoid and the interior of the gold coating layer are regions $\mathcal{R}_3$ and $\mathcal{R}_4$ respectively.

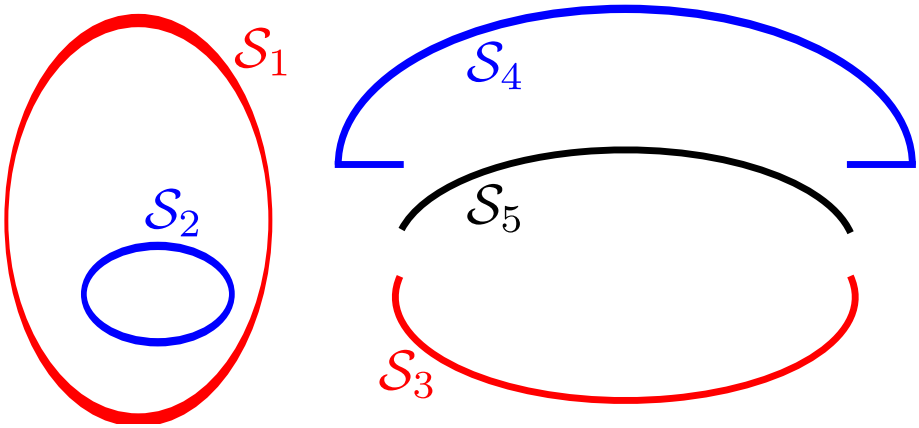

Figure 2: The surfaces for the geometry of Figure 1. Regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are bounded by the closed surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$, while regions $\mathcal{R}_3$ and $\mathcal{R}_4$ are bounded by appropriate unions of the open surfaces $\mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5$. More specifically, the region boundaries are $\partial R_1 = S_1, \partial R_2 = S_2, \partial R_3 = \mathcal{S}_3 \cup \mathcal{S}_5, \partial R_4 = \mathcal{S}_4 \cup \mathcal{S}_5$.

```
REGION R3 MATERIAL Silicon
REGION R4 MATERIAL Gold

OBJECT  S1
      MESHFILE S1.msh
      MATERIAL Silicon
ENDOBJECT

OBJECT  S2
      MESHFILE S2.msh
      MATERIAL SiO2
ENDOBJECT

SURFACE S3
      MESHFILE S3.msh
      REGIONS EXTERIOR R3
ENDSURFACE

SURFACE S4
      MESHFILE S4.msh
      REGIONS EXTERIOR R4
ENDSURFACE

SURFACE S5
      MESHFILE S5.msh
      REGIONS R3 R4
ENDSURFACE
```

Figure 3: Sample `.scuffgeo` file for the geometry of Figure 1.

or

$$N^{\rm F}_{ab}(\hat{\mathbf{u}},\hat{\mathbf{n}}) = \hat{u}_a\hat{n}_b + \hat{n}_a\hat{u}_b - (\hat{\mathbf{u}}\cdot\hat{\mathbf{n}})\delta_{ab}.$$

For example, if we are computing the x-force ($\hat{\mathbf{u}} = \hat{\mathbf{x}}$) we have

$$\mathbf{N}^{\rm F}(\hat{\mathbf{x}},\hat{\mathbf{n}}) = \begin{pmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ \hat{n}_y & -\hat{n}_x & 0 \\ \hat{n}_z & 0 & -\hat{n}_x \end{pmatrix}.$$

The flux of $\hat{\mathbf{u}}$-directed *angular* momentum, useful for computations of torque about an origin $\mathbf{x}_0$, is

$$\hat{\mathbf{u}}\cdot\Big[(\mathbf{x}-\mathbf{x}_0)\times\mathbf{T}\Big]\cdot\hat{\mathbf{n}} = \frac{1}{2}\ \mathrm{Re}\ \hat{u}_i\varepsilon_{ijk}(\mathbf{x}-\mathbf{x}_0)_j T_{k\ell}(\mathbf{x})\hat{n}_\ell \tag{19}$$

$$= \frac{1}{4}\boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{\rm T}\,\boldsymbol{\mathcal{F}}(\mathbf{x}) \tag{20}$$

with

$$\boldsymbol{\mathcal{N}}^{\rm T} = \begin{pmatrix} \epsilon\mathbf{N}^{\rm T} & 0 \\ 0 & \mu\mathbf{N}^{\rm T} \end{pmatrix}$$

where the 3×3 matrix $N^{\rm T}$ has entries ($\mathbf{D} = \mathbf{x}-\mathbf{x}_0$)

$$N^{\rm T}_{ab}(\hat{\mathbf{u}},\hat{\mathbf{n}}) = \varepsilon_{ija}\hat{u}_i D_j\hat{n}_b + \varepsilon_{ijb}\hat{u}_i D_j\hat{n}_a - \delta_{ab}\varepsilon_{ijk}\hat{u}_i D_j\hat{n}_k.$$

Energy and momentum flux from surface-current bilinears

Equations (17), (18), and (20) express the flux of power or momentum as bilinear products of the field six-vectors $\boldsymbol{\mathcal{F}}$. Using (13), we can turn these into bilinear products of the surface-current coefficient vectors $\mathbf{c}$. For example, the power flux in the $\hat{\mathbf{n}}$ direction, equation (17), becomes

$$\mathbf{P}(\mathbf{x})\cdot\hat{\mathbf{n}} = \frac{1}{4}\boldsymbol{\mathcal{F}}^\dagger(\mathbf{x})\boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}(\mathbf{x})$$

$$= \frac{1}{4}\sum_{\alpha\beta} c^*_\alpha\Big[\boldsymbol{\mathcal{F}}^\dagger_\alpha(\mathbf{x})\,\boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}})\,\boldsymbol{\mathcal{F}}_\beta(\mathbf{x})\Big]c_\beta \tag{21}$$

$$= \mathbf{c}^\dagger\mathbf{Q}^{\rm PFLUX}(\hat{\mathbf{n}},\mathbf{x})\mathbf{c} \tag{22}$$

where $\mathbf{Q}^{\rm PFLUX}(\hat{\mathbf{n}},\mathbf{x})$ is a matrix appropriate for $\hat{\mathbf{n}}$-directed power flux at $\mathbf{x}$. [As in equation (15), the sum over α,β in (21) runs only over basis functions lying on the boundary of the region containing the evaluation point $\mathbf{x}$.] The fluxes of $\hat{\mathbf{u}}$-directed linear and angular momentum read similarly

$$\hat{\mathbf{u}}\cdot\mathbf{T}(\mathbf{x})\cdot\hat{\mathbf{n}} = \mathbf{c}^\dagger\mathbf{Q}^{\rm FFLUX}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x})\mathbf{c} \tag{23}$$

$$\hat{\mathbf{u}}\cdot\Big[(\mathbf{x}-\mathbf{x}_0)\times\mathbf{T}(\mathbf{x})\Big]\cdot\hat{\mathbf{n}} = \mathbf{c}^\dagger\mathbf{Q}^{\rm TFLUX}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x})\mathbf{c} \tag{24}$$

The $\mathbf{Q}$ matrices in (22), (23), and (24) are $N_{\rm B} \times N_{\rm B}$ matrices whose entries are themselves 6-dimensional matrix-vector products:

$$Q^{\rm PFLUX}_{\alpha\beta}(\hat{\mathbf{n}},\mathbf{x}) = \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_{\alpha}(\mathbf{x})\boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}},\mathbf{x})\boldsymbol{\mathcal{F}}_{\beta}(\mathbf{x}) \tag{25a}$$

$$Q^{\rm FFLUX}_{\alpha\beta}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x}) = \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_{\alpha}(\mathbf{x})\boldsymbol{\mathcal{N}}^{\rm F}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x})\boldsymbol{\mathcal{F}}_{\beta}(\mathbf{x}) \tag{25b}$$

$$Q^{\rm TFLUX}_{\alpha\beta}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x}) = \frac{1}{4}\boldsymbol{\mathcal{F}}^{\dagger}_{\alpha}(\mathbf{x})\boldsymbol{\mathcal{N}}^{\rm T}(\hat{\mathbf{u}},\hat{\mathbf{n}},\mathbf{x})\boldsymbol{\mathcal{F}}_{\beta}(\mathbf{x}). \tag{25c}$$

Power, force, and torque from surface-current bilinears

The main quantities of interest in SCUFF-NEQ are the power, force, or torque (PFT) on individual bodies in a geometry. There are actually three separate ways to derive surface-current bilinear expressions for these quantities, which we here consider in turn.

Dense surface-integral PFT bilinears

The simplest way to obtain surface-current bilinears for the total PFT on one or more bodies is simply to integrate the spatially-resolved fluxes of the previous section over a closed bounding surface $\mathcal{S}$ containing the bodies. Indeed, noting that the $\mathbf{x}$ and $\hat{\mathbf{n}}$ dependence of the flux expressions (22), (23), and (24) is entirely contained in the $\mathbf{Q}$ matrices, it is easy to integrate those expressions over $\mathcal{S}$, then pull the surface-current vectors $\mathbf{c}$ outside the integral to identify what we shall call the *displaced surface-integral PFT* (DSIPFT) matrices.

For example, the total power absorbed by material bodies contained within a closed surface $\mathcal{S}$ is given by integrating the LHS of (22) over $\mathcal{S}$:

$$P^{\rm abs}_{\mathcal{S}} = \int_{\mathcal{S}} \mathbf{P}(\mathbf{x})\cdot\hat{\mathbf{n}}\,d\mathbf{x} \tag{26}$$

with $\hat{\mathbf{n}}$ taken to be the inward-pointing surface normal. Insert the RHS of (22) and pull $\mathbf{c}^{\dagger},\mathbf{c}$ outside the integral:

$$= \mathbf{c}^{\dagger}\mathbf{Q}^{\rm PTOT}\mathbf{c} \tag{27}$$

where the elements of $\mathbf{Q}^{\rm PTOT}$ involve integrals over $\mathcal{S}$:

$$Q^{\rm PTOT}_{\alpha\beta}(\mathcal{S}) = \frac{1}{4}\int_{\mathcal{S}} \boldsymbol{\mathcal{F}}^{\dagger}_{\alpha}(\mathbf{x})\boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}})\boldsymbol{\mathcal{F}}_{\beta}(\mathbf{x})\,d\mathbf{x} \tag{28}$$

If $\{w_p,\mathbf{x}_p\}$ is an $N_{\rm C}$-point cubature rule for integrating over $\mathcal{S}$, this may be approximated as

$$\approx \frac{1}{4}\sum_{p=1}^{N^{\rm C}} w_p\boldsymbol{\mathcal{F}}^{\dagger}_{\alpha}(\mathbf{x}_p)\boldsymbol{\mathcal{N}}^{\rm P}(\hat{\mathbf{n}}_p)\boldsymbol{\mathcal{F}}_{\beta}(\mathbf{x}_p) \tag{29}$$

where $\hat{\mathbf{n}}_p$ is the normal to $\mathcal{S}$ at $\mathbf{x}_p$. Similarly, the time-average $\hat{\mathbf{u}}$-directed force and torque on material bodies contained in $\mathcal{S}$ are

$$F_{\mathcal{S}}(\hat{\mathbf{u}}) = \mathbf{c}^\dagger \mathbf{Q}^{\text{FTOT}} \mathbf{c} \tag{30}$$

$$\mathcal{T}_{\mathcal{S}}(\hat{\mathbf{u}}) = \mathbf{c}^\dagger \mathbf{Q}^{\text{TTOT}} \mathbf{c} \tag{31}$$

where the entries of $\mathbf{Q}^{\text{FTOT}}$ and $\mathbf{Q}^{\text{TTOT}}$ are similar to (29) with $\boldsymbol{\mathcal{N}}^{\text{P}} \to \boldsymbol{\mathcal{N}}^{\text{F}}, \boldsymbol{\mathcal{N}}^{\text{T}}$.

Other Q matrices for spatially-integrated PFTs

For some quantities of interest it is possible to write alternative surface-current bilinears that compute the same quantities as (27), (30), and (31) using different matrices. This is discussed in Ref. 7.

Simplification of 6×6 matrix-vector products

The preceding derivation was agnostic regarding the choice of basis functions $\boldsymbol{\mathcal{B}}_\alpha$. For the particular basis functions used in SCUFF-EM, we can write slightly more explicit expressions for the matrix entries in equation (25).

In SCUFF-EM, each internal edge on a meshed surface of dielectric interface corresponds to a single RWG basis function and two surface-current coefficients, one for electric surface current and one for magnetic surface current. If the edge index is a, then the RWG function is $\mathbf{b}_a(\mathbf{x})$ and the electric and magnetic basis function indices are $\alpha = \{2a, 2a+1\}$.

Then $\boldsymbol{\mathcal{F}}_{2a}$ and $\boldsymbol{\mathcal{F}}_{2a+1}$ in equation (25) are the six-vector fields due respectively to unit-strength electric and magnetic surface currents in region $\mathcal{R}_i$ (the region containing the evaluation point):

$$\boldsymbol{\mathcal{F}}_{2a}(\mathbf{x}) = \begin{pmatrix} ikZ\mathbf{e}_a(\mathbf{x}) \\ \mathbf{h}_a(\mathbf{x}) \end{pmatrix}, \qquad \boldsymbol{\mathcal{F}}_{2a+1} = \begin{pmatrix} -\mathbf{h}_a(\mathbf{x}) \\ \frac{ik}{Z}\mathbf{e}_a(\mathbf{x}) \end{pmatrix}$$

Here k and Z are the wavenumber and (absolute) wave impedance in region $\mathcal{R}_i$ and $\mathbf{e}_a$ and $\mathbf{h}_a$ are the "reduced fields" due to basis function $\mathbf{b}_a$:

$$\mathbf{e}_a(\mathbf{x}) = \int_{\sup \mathbf{b}_a} \mathbf{G}(k;\mathbf{x},\mathbf{x}')\cdot\mathbf{b}_a(\mathbf{x}')\,d\mathbf{x}', \quad \mathbf{h}_a(\mathbf{x}) = -ik \int_{\sup \mathbf{b}_a} \mathbf{C}(k;\mathbf{x},\mathbf{x}')\cdot\mathbf{b}_a(\mathbf{x}')\,d\mathbf{x}'.$$

Power flux at $\mathbf{x}$:

$$\begin{aligned}
P_i(\mathbf{x}) &= \frac{1}{2}\epsilon_{ijk}\text{Re}\,\Big[E_j^*(\mathbf{x})H_k(\mathbf{x})\Big] \\
&= \frac{1}{4}\epsilon_{ijk}\Big[E_j^*(\mathbf{x})H_k(\mathbf{x}) + E_j(\mathbf{x})H_k^*(\mathbf{x})\Big] \\
&= \frac{1}{4}\epsilon_{ijk}\Big[E_j^*(\mathbf{x})H_k(\mathbf{x}) - H_j^*(\mathbf{x})E_k(\mathbf{x})\Big] \\
&= \frac{1}{4}\epsilon_{ijk}\sum_{\alpha\beta}\Big\{\Big(k_\alpha\cdot ikZ\mathbf{e}_\alpha - n_\alpha\mathbf{h}_\alpha\Big)_j^*\Big(k_\beta\mathbf{h}_\beta + n_\beta\cdot\frac{ik}{Z}\mathbf{h}_\beta\Big)_k \\
&\qquad\qquad - \Big(k_\alpha\mathbf{h}_\alpha + n_\alpha\cdot\frac{ik}{Z}\mathbf{h}_\alpha\Big)_j^*\Big(k_\beta\cdot ikZ\mathbf{e}_\alpha - n_\alpha\mathbf{h}_\alpha\Big)_k\Big\}
\end{aligned}$$

The contributions of a single pair of basis functions $\mathbf{b}_a, \mathbf{b}_b$ to the power and momentum flux at $\mathbf{x}$ are

$$\Delta P_i = \frac{1}{2}\mathrm{Re}\ \hat{\mathbf{n}}_i \cdot \Big(ikZk_\alpha \mathbf{e}_\alpha - n_\alpha \mathbf{h}_\alpha\Big)^* \times \Big(k_\beta \mathbf{h}_\beta + \frac{ik}{Z} n_\beta \mathbf{e}_\beta\Big) \tag{32}$$

$$= \frac{1}{2}\mathrm{Re}\ \begin{pmatrix} k_\alpha \\ n_\alpha \end{pmatrix}^\dagger \begin{pmatrix} (ikZ)^* \mathbf{e}_\alpha^* \times \mathbf{h}_\beta & |k|^2 \frac{Z^*}{Z} \mathbf{e}_\alpha^* \times \mathbf{e}_\beta \\ -\mathbf{h}_\alpha^* \times \mathbf{h}_\beta & -\frac{ik}{Z} \mathbf{h}_\alpha^* \times \mathbf{e}_\beta \end{pmatrix} \begin{pmatrix} k_\beta \\ n_\beta \end{pmatrix} \tag{33}$$

$$\Delta T_{ij} = \frac{1}{2}\mathrm{Re}\ \begin{pmatrix} ikZk_\alpha \mathbf{e}_\alpha - n_\alpha \mathbf{h}_\alpha \\ k_\alpha \mathbf{h}_\alpha + \frac{ik}{Z} n_\alpha \mathbf{e}_\alpha \end{pmatrix} \tag{34}$$

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