# Analytical solutions to the Heat Equation for validating numerical solvers

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## 1 Introduction

This document provides a number of Heat Equation solutions that can be used to validate numerical heat solvers.

## 2 Eigen Modes

The homogeneous heat equation can be written generally (without reference to a specific coordinate system) as:

$$\rho c \partial_t T = \nabla \cdot k \nabla T. \tag{1}$$

For a material with uniform properties (actually, only the conductivity needs to be uniform), this can be simplified:

$$\rho c \partial_t T = k \nabla^2 T. \tag{2}$$

This is similar to the wave equation, except that the time derivative is first order rather than second order. The eigen functions of the $\nabla^2$ operator are also useful here. We can in fact determine the general solution by transforming to the eigen basis of the Laplace operator, but here we want to find a specific solution suitable for comparing numerical heat solvers to.
The eigen functions of the Laplace operator have the interesting property that they decay in time without changing shape. So, if the initial temperature distribution is an eigen function of $\nabla^2$, then the temperature distribution will not change shape in

time, but simply scale instead. These special eigen temperature distributions can be used to validate a heat solver in much the same way that a beam propagator can be validated by propagating the eigen mode of a waveguide. We note that the eigen function equation for the Laplace operator is typically written as

$$\nabla^2\phi = -\lambda^2\phi \tag{3}$$

Consider a temperature distribution which is an eigen function of the Laplace operator with eigenvalue $\lambda$. The time-evolution of this temperature distribution is given by equation 2:

$$\rho c\partial_t\phi = k\nabla^2\phi = -k\lambda^2\phi, \tag{4}$$

$$\partial_t\phi = \frac{-k\lambda^2}{\rho c}\phi. \tag{5}$$

This is just an ordinary first-order differential equations in time, which has a simple exponential solution in time. Let $\alpha = \frac{k\lambda^2}{\rho c}$, then

$$\phi(t, \vec{r}) = e^{-\alpha t}\phi(0, \vec{r}). \tag{6}$$

## 2.1 1D Cartesian Coordinates

In one-dimensional Cartesian coordinates, equation 2 becomes:

$$\rho c\partial_t T = k\partial_{xx}T. \tag{7}$$

The eigen equation for the conduction operator is

$$\partial_{xx}\phi = -\lambda^2\phi, \tag{8}$$

which has solutions

$$\phi \propto \sin(\lambda x), \cos(\lambda x). \tag{9}$$

The eigenvalues, $\lambda$, are determined by the boundary conditions.

### 2.1.1 Dirichlet Boundary Conditions

Consider a uniform material over the domain $(0, L)$ with boundary conditions $T(0) = T(L) = 0$. The eigen functions satisfying these boundary conditions are $A\sin(\lambda_m x)$ where $\lambda_m = \frac{m\pi}{L}$. So, for an initial temperature distribution $T(0, x) = A\sin(\frac{m\pi}{L}x)$ we will have

$$T(t, x) = e^{-\alpha t}A\sin\left(\frac{m\pi}{L}x\right) = e^{-\frac{km^2\pi^2}{\rho cL^2}t}A\sin\left(\frac{m\pi}{L}x\right) \tag{10}$$

### 2.1.2 Neumann Boundary Conditions

Consider a uniform material over the domain $(0, L)$ with boundary conditions $\partial_x T|_{x=0} = \partial_x T|_{x=L} = 0$. The eigen functions satisfying these boundary conditions are $A\cos(\lambda_m x)$ where $\lambda_m = \frac{m\pi}{L}$. So, for an initial temperature distribution $T(0, x) = A\cos(\frac{m\pi}{L}x)$ we will have

$$T(t, x) = e^{-\frac{km^2\pi^2}{\rho cL^2}t} A\cos\left(\frac{m\pi}{L}x\right) \tag{11}$$

## 2.2 2D Cylindrical Coordinates

For problems exhibiting azimuthal symmetry, we work in two-dimensional cylindrical coordinates, $r$ and $z$. The eigen equation is

$$\frac{1}{r}\partial_r(r\partial_r\phi) + \partial_{zz}\phi = -\lambda^2\phi. \tag{12}$$

The eigenfunctions $\phi(r, z)$ can be written as the product of a function of $r$ and a function of $z$, $\phi(r, z) = \phi_r(r)\phi_z(z)$. The eigen equation can be rewritten

$$\phi_z\frac{1}{r}\partial_r(r\partial_r\phi_r) + \phi_r\partial_{zz}\phi_z = -\lambda^2\phi_r\phi_z \tag{13}$$

The above equation implies that

$$\frac{1}{r}\partial_r(r\partial_r\phi_r) = -\lambda_r^2\phi_r \tag{14}$$

$$\partial_{zz}\phi_z = -\lambda_z^2\phi_z \tag{15}$$

Equation 15 is just equation 7, so the eigen functions $\phi_z$ will also be sine and cosine functions. Equation 14 can be rewritten as

$$\partial_{rr}\phi_r + \frac{1}{r}\partial_r\phi_r = -\lambda_r^2\phi_r. \tag{16}$$

Multiplying by $r^2$ and substituting $\rho = \lambda_r r$ shows that this is just Bessel's differential equation with $\alpha = 0$,

$$r^2\partial_{rr}\phi_r + r\partial_r\phi_r = -r^2\lambda_r^2\phi_r \tag{17}$$

$$\frac{\rho^2}{\lambda^2}\partial_{\rho\rho}\phi_r + \frac{\rho}{\lambda}\partial_\rho\phi_r = -\rho^2\phi_r \tag{18}$$

$$\frac{\rho^2}{\lambda^2}\partial_{rr}\phi_r + \frac{\rho}{\lambda}\partial_r\phi_r + \rho^2\phi_r = 0 \tag{19}$$

$$\rho^2\partial_{\rho\rho}\phi_r + \rho\partial_\rho\phi_r + \rho^2\phi_r = 0 \tag{20}$$

The solutions are Bessel functions, we require the temperature to be finite at $r = 0$, so they are of the first kind,

$$\phi_r(r \propto J_0(\rho) = J_0(\lambda_r r). \tag{21}$$

Again, the eigenvalues are determined by the boundary conditions.
Once both $\lambda_z$ and $\lambda_r$ are set, the value of $\lambda$ will be $\lambda^2 = \lambda_z^2 + \lambda_r^2$.

### 2.2.1 Dirichlet Boundary Conditions

Consider a uniform material over the domain $(r, z) \in (0, R) \times (0, L)$ with boundary conditions $T(r, 0) = T(r, L) = T(R, z) = 0$. The $z$ eigen functions satisfying these boundary conditions are again $A \sin(\lambda_{zm} z)$ where $\lambda_{zm} = \frac{m\pi}{L}$. The $r$ eigen functions will be Bessel function, $AJ_0(\lambda_{rn} r)$ with $\lambda_{rn} = \frac{\alpha_n}{R}$ where $\alpha_n$ is the $n$'th zero of $J_0(\rho)$.

$$T(t, r, x) = e^{-\alpha t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) = e^{-\frac{k}{\rho c}\left(\frac{m^2\pi^2}{L^2} + \frac{\alpha_n^2}{R^2}\right)t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) \tag{22}$$

In order to compare this solution to a numerical solution, we will need to have a value for $\alpha_n$. An analytical expression for these zeros does not exist, but there are several numerical libraries that provide functions to calculate them.

### 2.2.2 Neumann Boundary Conditions

Consider a uniform material over the domain $(r, z) \in (0, R) \times (0, L)$ with boundary conditions $\partial_z T|_{(r,0)} = \partial_z T|_{(r,L)} = \partial_r T|_{(R,z)} = 0$. The $z$ eigen functions satisfying these boundary conditions are $A \cos(\lambda_{zm} z)$ where $\lambda_{zm} = \frac{m\pi}{L}$. The $r$ eigen functions will again be Bessel function, $AJ_0(\lambda_{rn} r)$, but now the derivative must be zero $r = R$. This means that one of the peaks (or valleys) must be at the boundary. Let $\lambda_{rn} = \frac{\beta_n}{R}$ where $\beta_n$ is the coordinates of the $n$'th peak or valley of $J_0(\rho)$. This gives

$$T(t, r, x) = e^{-\alpha t} A \cos\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) = e^{-\frac{k}{\rho c}\left(\frac{m^2\pi^2}{L^2} + \frac{\beta_n^2}{R^2}\right)t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\beta_n}{R} r\right) \tag{23}$$

The zero's $\beta_n$ are again not known analytically, and numerical libraries do not calculate them, but we can determine them by using the properties of Bessesl functions. The derivative of $J_0(\rho)$ is itself a Bessel function,

$$\partial_\rho J_0(\rho) = -J_1(\rho) \tag{24}$$

So, the peaks and valleys of $J_0$ can be determined by computing the zeros of $J_1$.