

2D Cylindrical Finite Difference Time Dependent Heat Equation for Non-Constant Thermal Conductivity with Constant Spacing

Emily Cranwell and Daniel Huantes

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1 Introduction

2 The Problem

3 Derivation

$$\rho c \frac{dT}{dt} = \vec{\nabla} \cdot \kappa \vec{\nabla} T + A \quad (1)$$

$$\rho c \frac{dT}{dt} = \vec{\nabla} \cdot \kappa \left[\frac{\partial T}{\partial r} \hat{r} + \frac{\partial T}{\partial z} \hat{z} \right] \quad (2)$$

$$\rho c \frac{dT}{dt} = \vec{\nabla} \cdot \left[\frac{\partial T}{\partial r} \kappa(r, z) \hat{r} + \frac{\partial T}{\partial z} \kappa(r, z) \hat{z} \right] \quad (3)$$

$$\rho c \frac{dT}{dt} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial T}{\partial r} \kappa(r, z) \right] + \frac{\partial}{\partial z} \left[\frac{\partial T}{\partial z} \kappa(r, z) \right] \quad (4)$$

$$\rho c \frac{dT}{dt} = \frac{1}{r} \left[\frac{\partial T}{\partial r} \kappa(r, z) + r \frac{\partial^2 T}{\partial r^2} \kappa(r, z) + r \frac{\partial T}{\partial r} \frac{\partial \kappa}{\partial r} \right] + \frac{\partial^2 T}{\partial z^2} \kappa(r, z) + \frac{\partial T}{\partial z} \frac{\partial \kappa}{\partial z} \quad (5)$$

$$\rho c \frac{dT}{dt} = \frac{1}{r} \frac{\partial T}{\partial r} \kappa(r, z) + \frac{\partial^2 T}{\partial r^2} \kappa(r, z) + \frac{\partial T}{\partial r} \frac{\partial \kappa}{\partial r} + \frac{\partial^2 T}{\partial z^2} \kappa(r, z) + \frac{\partial T}{\partial z} \frac{\partial \kappa}{\partial z} \quad (6)$$

$$\begin{aligned} \rho c \frac{dT}{dt} = & \frac{1}{r} \left[\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right] \kappa_{r,z}^n + \left[\frac{T_{r+1,z}^n - 2T_{r,z}^n + T_{r-1,z}^n}{(\Delta r)^2} \right] \kappa_{r,z}^n + \left(\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right) \left(\frac{\kappa_{r+1,z}^n - \kappa_{r-1,z}^n}{2\Delta r} \right) \\ & + \left[\frac{T_{r,z+1}^n - 2T_{r,z}^n + T_{r,z-1}^n}{(\Delta z)^2} \right] \kappa_{r,z}^n + \left(\frac{T_{r,z+1}^n - T_{r,z-1}^n}{2\Delta z} \right) \left(\frac{\kappa_{r,z+1}^n - \kappa_{r,z-1}^n}{2\Delta z} \right) \end{aligned} \quad (7)$$

$$\begin{aligned} \rho c \frac{dT}{dt} = & \frac{1}{r} \left[\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right] \kappa_{r,z}^n + \left[\frac{T_{r+1,z}^n - 2T_{r,z}^n + T_{r-1,z}^n}{(\Delta r)^2} \right] \kappa_{r,z}^n + \left(\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right) \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n \\ & + \left[\frac{T_{r,z+1}^n - 2T_{r,z}^n + T_{r,z-1}^n}{(\Delta z)^2} \right] \kappa_{r,z}^n + \left(\frac{T_{r,z+1}^n - T_{r,z-1}^n}{2\Delta z} \right) \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n \end{aligned} \quad (8)$$

$$\begin{aligned}
\rho c \frac{T_{r,z}^{n+1} - T_{r,z}^n}{\Delta t} &= \frac{1}{r} \left[\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right] \kappa_{r,z}^n + \left[\frac{T_{r+1,z}^n - 2T_{r,z}^n + T_{r-1,z}^n}{(\Delta r)^2} \right] \kappa_{r,z}^n \\
&+ \left(\frac{T_{r+1,z}^n - T_{r-1,z}^n}{2\Delta r} \right) \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n + \left[\frac{T_{r,z+1}^n - 2T_{r,z}^n + T_{r,z-1}^n}{(\Delta z)^2} \right] \kappa_{r,z}^n + \left(\frac{T_{r,z+1}^n - T_{r,z-1}^n}{2\Delta z} \right) \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n
\end{aligned} \tag{9}$$

$$\begin{aligned}
T_{r,z}^{n+1} - T_{r,z}^n &= \frac{\Delta t}{\rho c} \left(T_{r,z}^n \left[\frac{-2\kappa_{r,z}^n}{(\Delta r)^2} - \frac{2\kappa_{r,z}^n}{(\Delta z)^2} \right] + T_{r+1,z}^n \left[\frac{\kappa_{r,z}^n}{2r\Delta r} + \frac{\kappa_{r,z}^n}{(\Delta r)^2} + \frac{1}{2\Delta r} \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n \right] \right. \\
&+ T_{r-1,z}^n \left[\frac{-\kappa_{r,z}^n}{2r\Delta r} + \frac{\kappa_{r,z}^n}{(\Delta r)^2} - \frac{1}{2\Delta r} \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n \right] + T_{r,z-1}^n \left[\frac{\kappa_{r,z}^n}{(\Delta z)^2} - \frac{1}{2\Delta z} \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n \right] \\
&\left. + T_{r,z+1}^n \left[\frac{\kappa_{r,z}^n}{(\Delta z)^2} + \frac{1}{2\Delta z} \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n \right] \right)
\end{aligned} \tag{10}$$

$$A = \left[\frac{-2\kappa_{r,z}^n}{(\Delta r)^2} - \frac{2\kappa_{r,z}^n}{(\Delta z)^2} \right] \tag{11}$$

$$B = \left[\frac{\kappa_{r,z}^n}{2r\Delta r} + \frac{\kappa_{r,z}^n}{(\Delta r)^2} + \frac{1}{2\Delta r} \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n \right] \tag{12}$$

$$C = \left[\frac{-\kappa_{r,z}^n}{2r\Delta r} + \frac{\kappa_{r,z}^n}{(\Delta r)^2} - \frac{1}{2\Delta r} \left(\frac{\Delta \kappa}{\Delta r} \right)_{r,z}^n \right] \tag{13}$$

$$D = \left[\frac{\kappa_{r,z}^n}{(\Delta z)^2} - \frac{1}{2\Delta z} \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n \right] \tag{14}$$

$$E = \left[\frac{\kappa_{r,z}^n}{(\Delta z)^2} + \frac{1}{2\Delta z} \left(\frac{\Delta \kappa}{\Delta z} \right)_{r,z}^n \right] \tag{15}$$