Analytical solutions to the Heat Equation for validating numerical solvers

C.D. Clark III

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1   Introduction

This document provides a number of Heat Equation solutions that can be used to validate numerical heat solvers.

2   Time-Dependent Solutions

2.1   Homogeneous Heat Equation

The homogeneous heat equation can be written generally (without reference to a specific coordinate system) as:

$$\rho c \partial_t T = \nabla \cdot k \nabla T. \tag{1}$$

For a material with uniform properties (actually, only the conductivity needs to be uniform), this can be simplified:

$$\rho c \partial_t T = k \nabla^2 T. \tag{2}$$

This is similar to the wave equation, except that the time derivative is first order rather than second order. The eigen functions of the ∇^2 operator are also useful here. We can in fact determine the general solution by transforming to the eigen basis of the Laplace operator, but here we want to find a specific solution suitable for comparing numerical heat solvers to.

The eigen functions of the Laplace operator have the interesting property that they decay in time without changing shape. So, if the initial temperature distribution is

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an eigen function of ∇^2, then the temperature distribution will not change shape in time, but simply scale instead. These special eigen temperature distributions can be used to validate a heat solver in much the same way that a beam propagator can be validated by propagating the eigen mode of a waveguide. We note that the eigen function equation for the Laplace operator is typically written as

$$\nabla^2 \phi = -\lambda^2 \phi \tag{3}$$

Consider a temperature distribution which is an eigen function of the Laplace operator with eigenvalue $-\lambda^2$ (this is just a convinient way to write the eigenvalue). We can write the temperature as a product of a time-dependent function and the eigenfunction $T = f(t)\phi(\vec{r})$. The time-evolution of this temperature distribution is given by equation 2:

$$\rho c \partial_t f \phi = k \nabla^2 f \phi = -k\lambda^2 f \phi, \tag{4}$$

$$\phi \partial_t f = \frac{-k\lambda^2}{\rho c} \phi f, \tag{5}$$

$$\partial_t f = \frac{-k\lambda^2}{\rho c} f. \tag{6}$$

This is just an ordinary first-order differential equations in time, which has a simple exponential solution in time. Let $\alpha = \frac{k\lambda^2}{\rho c}$, then

$$T(t, \vec{r}) = f\phi = e^{-\alpha t} \phi(\vec{r}). \tag{7}$$

2.2 Inhomogeneous Heat Equation

If the problem includes a source term, then the Inhomogeneous heat equation is required.

$$\rho c \partial_t T = \nabla \cdot k \nabla T + A, \tag{8}$$

which can be simplifed for uniform media

$$\rho c \partial_t T = k \nabla^2 T + A. \tag{9}$$

Consider the special case when $A = \beta\phi$, where ϕ is the eigenfunction introduced above. Then Equation 4 will be

$$\rho c \phi \partial_t f = -k\lambda^2 \phi f + \beta\phi, \tag{10}$$

$$\partial_t f = -\alpha f + \frac{\beta}{\rho c} \tag{11}$$

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Let $u = \alpha f - \frac{\beta}{\rho c}$. Then $\mathrm{d}u = \alpha \mathrm{d}f$ and 11 can be rewritten as

$$\frac{1}{\alpha}\partial_t u = -u \tag{12}$$

which is the equation for an exponential,

$$u(t) = u(0)e^{-\alpha t} = \left(\alpha f(0) - \frac{\beta}{\rho c}\right)e^{-\alpha t} = \alpha f(t) - \frac{\beta}{\rho c}. \tag{13}$$

Solving for $f(t)$

$$f(t) = f(0)e^{-\alpha t} - \frac{\beta}{\rho c \alpha}e^{-\alpha t} + \frac{\beta}{\rho c \alpha} = f(0)e^{-\alpha t} + (1 - e^{-\alpha t})\frac{\beta}{k\lambda^2} \tag{14}$$

The temperature will therefore be

$$T(t, \vec{r}) = f(0)e^{-\alpha t}\phi(\vec{r}) + (1 - e^{-\alpha t})\frac{\beta}{k\lambda^2}\phi(\vec{r}) \tag{15}$$

which is the same as Equation 7 when $\beta = 0$.

2.3   1D Cartesian Coordinates

In one-dimensional Cartesian coordinates, equation 2 becomes:

$$\rho c \partial_t T = k \partial_{xx} T. \tag{16}$$

The eigen equation for the conduction operator is

$$\partial_{xx}\phi = -\lambda^2\phi, \tag{17}$$

which has solutions

$$\phi \propto \sin(\lambda x), \cos(\lambda x). \tag{18}$$

The eigenvalues, λ, are determined by the boundary conditions.

2.3.1   Dirichlet Boundary Conditions

Consider a uniform material over the domain $(0, L)$ with boundary conditions $T(0) = T(L) = 0$. The eigen functions satisfying these boundary conditions are $A \sin(\lambda_m x)$ where $\lambda_m = \frac{m\pi}{L}$. So, for an initial temperature distribution $T(0, x) = A \sin\left(\frac{m\pi}{L}x\right)$ we will have

$$T(t, x) = e^{-\alpha t}A \sin\left(\frac{m\pi}{L}x\right) = e^{-\frac{km^2\pi^2}{\rho c L^2}t}A \sin\left(\frac{m\pi}{L}x\right) \tag{19}$$

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2.3.2   Neumann Boundary Conditions

Consider a uniform material over the domain $(0, L)$ with boundary conditions $\partial_x T|_{x=0} = \partial_x T|_{x=L} = 0$. The eigen functions satisfying these boundary conditions are $A \cos(\lambda_m x)$ where $\lambda_m = \frac{m\pi}{L}$. So, for an initial temperature distribution $T(0, x) = A \cos\left(\frac{m\pi}{L} x\right)$ we will have

$$T(t, x) = e^{-\frac{km^2\pi^2}{\rho c L^2} t} A \cos\left(\frac{m\pi}{L} x\right) \tag{20}$$

2.4   2D Cylindrical Coordinates

For problems exhibiting azimuthal symmetry, we work in two-dimensional cylindrical coordinates, r and z. The eigen equation is

$$\frac{1}{r}\partial_r(r\partial_r\phi) + \partial_{zz}\phi = -\lambda^2\phi. \tag{21}$$

The eigenfunctions $\phi(r, z)$ can be written as the product of a function of r and a function of z, $\phi(r, z) = \phi_r(r)\phi_z(z)$. The eigen equation can be rewritten

$$\phi_z\frac{1}{r}\partial_r(r\partial_r\phi_r) + \phi_r\partial_{zz}\phi_z = -\lambda^2\phi_r\phi_z \tag{22}$$

The above equation implies that

$$\frac{1}{r}\partial_r(r\partial_r\phi_r) = -\lambda_r^2\phi_r \tag{23}$$

$$\partial_{zz}\phi_z = -\lambda_z^2\phi_z \tag{24}$$

Equation 24 is just equation 16, so the eigen functions ϕ_z will also be sine and cosine functions. Equation 23 can be rewritten as

$$\partial_{rr}\phi_r + \frac{1}{r}\partial_r\phi_r = -\lambda_r^2\phi_r. \tag{25}$$

Multiplying by r^2 and substituting $\rho = \lambda_r r$ shows that this is just Bessel's differential equation with $\alpha = 0$, (note that $\partial_r = \frac{\partial\rho}{\partial r}\partial_\rho = \lambda_r\partial_\rho$)

$$r^2\partial_{rr}\phi_r + r\partial_r\phi_r = -r^2\lambda_r^2\phi_r \tag{26}$$

$$\frac{\rho^2}{\lambda_r^2}\partial_{rr}\phi_r + \frac{\rho}{\lambda_r}\partial_r\phi_r = -\rho^2\phi_r \tag{27}$$

$$\frac{\rho^2}{\lambda_r^2}\partial_{rr}\phi_r + \frac{\rho}{\lambda_r}\partial_r\phi_r + \rho^2\phi_r = 0 \tag{28}$$

$$\rho^2\partial_{\rho\rho}\phi_r + \rho\partial_\rho\phi_r + \rho^2\phi_r = 0 \tag{29}$$

The solutions are Bessel functions, we require the temperature to be finite at $r = 0$, so they are of the first kind,

$$\phi_r(r) \propto J_0(\rho) = J_0(\lambda_r r). \tag{30}$$

Again, the eigenvalues are determined by the boundary conditions.
Once both λ_z and λ_r are set, the value of λ will be $\lambda^2 = \lambda_z^2 + \lambda_r^2$.

2.4.1 Dirichlet Boundary Conditions

Consider a uniform material over the domain $(r, z) \in (0, R) \times (0, L)$ with boundary conditions $T(r, 0) = T(r, L) = T(R, z) = 0$. The z eigen functions satisfying these boundary conditions are again $A \sin(\lambda_{zm} z)$ where $\lambda_{zm} = \frac{m\pi}{L}$. The r eigen functions will be Bessel function, $A J_0(\lambda_{rn} r)$ with $\lambda_{rn} = \frac{\alpha_n}{R}$ where α_n is the n'th zero of $J_0(\rho)$.

$$T(t, r, x) = e^{-\alpha t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) = e^{-\frac{k}{\rho c}\left(\frac{m^2 \pi^2}{L^2} + \frac{\alpha_n^2}{R^2}\right) t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) \tag{31}$$

In order to compare this solution to a numerical solution, we will need to have a value for α_n. An analytical expression for these zeros does not exist, but they have been tabulated, and there are several numerical libraries that provide functions to calculate them. The first five zeros are listed on the Wolfram Math-World page for Bessesl Function Zeros `https://mathworld.wolfram.com/BesselFunctionZeros.html`

n	α_n
1	2.4048
2	5.5201
3	8.6537
4	11.7915
5	14.9309

2.4.2 Neumann Boundary Conditions

Consider a uniform material over the domain $(r, z) \in (0, R) \times (0, L)$ with boundary conditions $\partial_z T|_{(r,0)} = \partial_z T|_{(r,L)} = \partial_r T|_{(R,z)} = 0$. The z eigen functions satisfying these boundary conditions are $A \cos(\lambda_{zm} z)$ where $\lambda_{zm} = \frac{m\pi}{L}$. The r eigen functions will again be Bessel function, $A J_0(\lambda_{rn} r)$, but now the derivative must be zero $r = R$. This means that one of the peaks (or valleys) must be at the boundary. Let

$\lambda_{rn} = \frac{\beta_n}{R}$ where β_n is the coordinates of the n'th peak or valley of $J_0(\rho)$. This gives

$$T(t, r, x) = e^{-\alpha t} A \cos\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\alpha_n}{R} r\right) = e^{-\frac{k}{\rho c}\left(\frac{m^2 \pi^2}{L^2} + \frac{\beta_n^2}{R^2}\right)t} A \sin\left(\frac{m\pi}{L} z\right) J_0\left(\frac{\beta_n}{R} r\right) \quad (32)$$

The zero's β_n are again not known analytically, and numerical libraries do not calculate them, but we can determine them by using the properties of Bessesl functions. The derivative of $J_0(\rho)$ is itself a Bessel function,

$$\partial_\rho J_0(\rho) = -J_1(\rho) \quad (33)$$

So, the peaks and valleys of J_0 can be determined by computing the zeros of J_1. The first five zeros are listed on the Wolfram MathWorld page for Bessesl Function Zeros `https://mathworld.wolfram.com/BesselFunctionZeros.html`

n	α_n
1	3.8317
2	7.0156
3	10.1735
4	13.3237
5	16.4706

3 Surface Boundary Conditions

We can model a surface by specifying a heat flux at the boundary. The simplest example is a constant heat flux, which is used to model heat loss do to evaporation for example. At the upper boundary, we have

$$k\frac{dT}{dx} = -Q \quad (34)$$

This simply says that the heat flux $\left(k\frac{dT}{dx}\right)$ at the bounary is equal to a constant value. The negative sign indicates that heat is *lost* at the boundary. At the lower boundary, there is no negative sign, because the heat flux points in the opposite direction.

$$k\frac{dT}{dx} = Q \quad (35)$$

These are just examples of Neumann boundary conditions.

Consider the steady-state temperature distribution with a constant heat flux. The time-indepdent heat equation is

$$\nabla \cdot k\nabla T = 0 \quad (36)$$

which can be simplified if k is uniform,

$$k\nabla^2 T = 0 \tag{37}$$

In one-dimensional Cartesian coordinates, this is just the equation for a line

$$k\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} \tag{38}$$

So the temperature will be linear with x between the boundaries,

$$T(x) = mx + b \tag{39}$$

The constants m and b will be determined by the boundary conditions.
Let $T(0) = 0$, and $k\frac{\mathrm{d}T}{\mathrm{d}x}\big|_{x=L} = -Q$. Applying the boundary conditions gives $b = 0$
and $m = -\frac{Q}{k}$. The temperature distribution after long times then will be

$$T(x) = -\frac{Q}{k}x. \tag{40}$$

and the surface temperature at $x = L$ will be $T(L) = -\frac{QL}{k}$.
The most common surface we need to model is a convective boundary. Here, the heat flux depends on the temperature difference between the surface, and the ambient medium, which is at T_∞.

$$k\frac{\mathrm{d}T}{\mathrm{d}x} = -h_e(T - T_\infty). \tag{41}$$

Again, at the lower boundary, the negative sign is removed.
At long times, the temperature distribution will again be linear with x, however, slope will depend on the boundary temperature according to 41. So, we need to solve for T. Consider a constant temperature boundary with $T = 0$ at $x = 0$, and a convective boundary at $x = L$. Let

$$T(x) = mx + b. \tag{42}$$

Then

$$\frac{\mathrm{d}T}{\mathrm{d}x} = m. \tag{43}$$

Applying the lower boundary condition gives $b = 0$. Applying the upper boundary gives $m = -\frac{h_e}{k}(T(L) - T_\infty)$. We also have $T(L) = mL$, so

$$m = -\frac{h_e}{k}(T(L) - T_\infty), \tag{44}$$

$$T(L) = mL, \tag{45}$$

$$m = -\frac{h_e}{k}(mL - T_\infty) = -\frac{h_e}{k}mL + \frac{h_e}{k}T_\infty, \tag{46}$$

$$m + \frac{h_e}{k}mL = \frac{h_e}{k}T_\infty, \tag{47}$$

$$m = \frac{\frac{h_e}{k}T_\infty}{1 + \frac{h_e}{k}L}. \tag{48}$$

The temperature distribution will then be

$$T(x) = \frac{\frac{h_e}{k}T_\infty}{1 + \frac{h_e}{k}L}x = \frac{h_e T_\infty}{k + h_e L}x. \tag{49}$$