2-D Cylindrical Finite Difference Time Dependent Heat Equation for Non-Constant Thermal Conductivity with Constant Spacing

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February 21, 2021

1 Introduction

Solving the heat equation in cylindrical coordinates is of great interest to applications where heat energy is deposited normal to a surface via a laser exposure. With Dr. Clark's work in solving the heat equation with thermal conductivity, κ as a function of z allowed for better simulating the weakly homogenous media of tissue. The following derivation aims to take this work a step further, solving with $\kappa = \kappa(r, z)$, in order to extend our weak homogeny to the radial direction, especially valuable in situations with irregularities such as a hair follicle in the center of the beam path.

2 The Problem

The problem is first expanding the differential equation to reflect the radial dependence of thermal conductivity. We implement a finite difference method in order to substitute our derivatives for the values of T and κ that are easily implemented in a computational setting. In the derivation, our result is discontinuous at $r = 0$ and so we utilize L'Hospital's rule and create a solution valid for the case where $r \to 0$.

3 Derivation

We begin with the heat transfer equation

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \kappa \vec{\nabla} \mathrm{T} + A \tag{1}$$

We ignore the source term, A for now and proceed by expanding the gradient, considering no phi dependence, so $\frac{\partial \mathrm{T}}{\partial \phi} = 0$

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \kappa \left[\frac{\partial \mathrm{T}}{\partial r} \hat{r} + \frac{\partial \mathrm{T}}{\partial z} \hat{z} \right] \tag{2}$$

We rewrite κ to state it's radial and z dependence

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \left[\frac{\partial \mathrm{T}}{\partial r} \kappa(r, z) \hat{r} + \frac{\partial \mathrm{T}}{\partial z} \kappa(r, z) \hat{z} \right] \tag{3}$$

$$\rho c\frac{d\mathrm{T}}{dt} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)\right] + \frac{\partial}{\partial z}\left[\frac{\partial \mathrm{T}}{\partial z}\kappa(r,z)\right] \tag{4}$$

$$\rho c\frac{d\mathrm{T}}{dt} = \frac{1}{r}\left[\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z) + r\frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + r\frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa}{\partial r}\right] + \frac{\partial^2 \mathrm{T}}{\partial z^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z}\frac{\partial \kappa}{\partial z} \tag{5}$$

$$\rho c\frac{d\mathrm{T}}{dt} = \frac{1}{r}\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z) + \frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa}{\partial r} + \frac{\partial^2 \mathrm{T}}{\partial z^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z}\frac{\partial \kappa}{\partial z} \tag{6}$$

$$\begin{split}\rho c\frac{d\mathrm{T}}{dt} = \frac{1}{r}\left[\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r}\right]\kappa^n_{r,z} + \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2}\right]\kappa^n_{r,z} + (\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r})(\frac{\kappa^n_{r+1,z} - \kappa^n_{r-1,z}}{2\Delta r}) \\ + \left[\frac{\mathrm{T}^n_{r,z+1} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r,z-1}}{(\Delta z)^2}\right]\kappa^n_{r,z} + (\frac{\mathrm{T}^n_{r,z+1} - \mathrm{T}^n_{r,z-1}}{2\Delta z})(\frac{\kappa^n_{r,z+1} - \kappa^n_{r,z-1}}{2\Delta z})\end{split} \tag{7}$$

$$\begin{split}\rho c\frac{d\mathrm{T}}{dt} = \frac{1}{r}\left[\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r}\right]\kappa^n_{r,z} + \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2}\right]\kappa^n_{r,z} + (\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r})(\frac{\Delta \kappa}{\Delta r})^n_{r,z} \\ + \left[\frac{\mathrm{T}^n_{r,z+1} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r,z-1}}{(\Delta z)^2}\right]\kappa^n_{r,z} + (\frac{\mathrm{T}^n_{r,z+1} - \mathrm{T}^n_{r,z-1}}{2\Delta r})(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\end{split} \tag{8}$$

$$\begin{split}\rho c\frac{\mathrm{T}^{n+1}_{r,z} - \mathrm{T}^n_{r,z}}{\Delta t} = \frac{1}{r}\left[\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r}\right]\kappa^n_{r,z} + \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2}\right]\kappa^n_{r,z} \\ + (\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r})(\frac{\Delta \kappa}{\Delta r})^n_{r,z} + \left[\frac{\mathrm{T}^n_{r,z+1} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r,z-1}}{(\Delta z)^2}\right]\kappa^n_{r,z} + (\frac{\mathrm{T}^n_{r,z+1} - \mathrm{T}^n_{r,z-1}}{2\Delta r})(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\end{split} \tag{9}$$

$$\begin{split}\mathrm{T}^{n+1}_{r,z} - \mathrm{T}^n_{r,z} = \frac{\Delta t}{\rho c}(\mathrm{T}^n_{r,z}\left[\frac{-2\kappa^n_{r,z}}{(\Delta r)^2} - \frac{2\kappa^n_{r,z}}{(\Delta z)^2}\right] + \mathrm{T}^n_{r+1,z}\left[\frac{\kappa^n_{r,z}}{2r\Delta r} + \frac{\kappa^n_{r,z}}{(\Delta r)^2} + \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{r,z}\right] \\ + \mathrm{T}^n_{r-1,z}\left[\frac{-\kappa^n_{r,z}}{2r\Delta r} + \frac{\kappa^n_{r,z}}{(\Delta r)^2} - \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{r,z}\right] + \mathrm{T}^n_{r,z-1}\left[\frac{\kappa^n_{r,z}}{(\Delta z)^2} - \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\right] \\ + \mathrm{T}^n_{r,z+1}\left[\frac{\kappa^n_{r,z}}{(\Delta z)^2} + \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\right])\end{split} \tag{10}$$

$$A^n_{r,z} = \left[\frac{-2\kappa^n_{r,z}}{(\Delta r)^2} - \frac{2\kappa^n_{r,z}}{(\Delta z)^2}\right] \tag{11}$$

$$B^n_{r,z} = \left[\frac{\kappa^n_{r,z}}{2r\Delta r} + \frac{\kappa^n_{r,z}}{(\Delta r)^2} + \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{r,z}\right] \tag{12}$$

$$C^n_{r,z} = \left[\frac{-\kappa^n_{r,z}}{2r\Delta r} + \frac{\kappa^n_{r,z}}{(\Delta r)^2} - \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{r,z}\right] \tag{13}$$

$$D^n_{r,z} = \left[\frac{\kappa^n_{r,z}}{(\Delta z)^2} - \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\right] \tag{14}$$

$$E^n_{r,z} = \left[\frac{\kappa^n_{r,z}}{(\Delta z)^2} + \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{r,z}\right] \tag{15}$$

3.1 Case for $r = 0$

$$\rho c \frac{d\mathrm{T}}{dt} = \frac{1}{r}\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z) + \frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa}{\partial r} + \frac{\partial^2 \mathrm{T}}{\partial z^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z}\frac{\partial \kappa}{\partial z} \tag{16}$$

$$\lim_{r\to 0}\left[\frac{\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)}{r}\right] = \frac{0}{0}$$

$$\begin{aligned} \lim_{r\to 0}\left[\frac{\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)}{r}\right] &\overset{L}{=} \lim_{r\to 0}\left[\frac{\frac{\partial}{\partial r}\left[\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)\right]}{\frac{\partial}{\partial r}\left[r\right]}\right] \\ &= \lim_{r\to 0}\left[\frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa(r,z)}{\partial r}\right] \end{aligned} \tag{17}$$

$$\lim_{r\to 0}\left[\frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa(r,z)}{\partial r}\right] = \lim_{r\to 0}\left[\frac{\partial \mathrm{T}}{\partial r}\right]\lim_{r\to 0}\left[\frac{\partial \kappa(r,z)}{\partial r}\right] = 0 \tag{18}$$

since we have symmetry across the origin

$$\begin{aligned} \frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) &= \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2}\right]\kappa^n_{r,z} \\ &= \frac{\mathrm{T}^n_{r+1,z}\kappa^n_{r,z}}{(\Delta r)^2} - \frac{2\mathrm{T}^n_{r,z}\kappa^n_{r,z}}{(\Delta r)^2} + \frac{\mathrm{T}^n_{r-1,z}\kappa^n_{r,z}}{(\Delta r)^2} \end{aligned} \tag{19}$$

$$A^n_{r=0,z} = \left[\frac{-4\kappa^n_{0,z}}{(\Delta r)^2} - \frac{2\kappa^n_{0,z}}{(\Delta z)^2}\right] \tag{20}$$

$$B^n_{r=0,z} = \left[2\frac{\kappa^n_{0,z}}{(\Delta r)^2} + \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{0,z}\right] \tag{21}$$

$$C^n_{r=0,z} = \left[-2\frac{\kappa^n_{0,z}}{(\Delta r)^2} - \frac{1}{2\Delta r}(\frac{\Delta \kappa}{\Delta r})^n_{0,z}\right] \tag{22}$$

$$D^n_{r=0,z} = \left[\frac{\kappa^n_{0,z}}{(\Delta z)^2} - \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{0,z}\right] \tag{23}$$

$$E^n_{r=0,z} = \left[\frac{\kappa^n_{0,z}}{(\Delta z)^2} + \frac{1}{2\Delta z}(\frac{\Delta \kappa}{\Delta z})^n_{0,z}\right] \tag{24}$$