2-D Cylindrical Finite Difference Time Dependent Heat Equation for Non-Constant Thermal Conductivity with Constant Spacing

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1 Introduction

2 The Problem

3 Derivation

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \kappa \vec{\nabla} \mathrm{T} + A \tag{1}$$

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \kappa \left[\frac{\partial \mathrm{T}}{\partial r} \hat{r} + \frac{\partial \mathrm{T}}{\partial z} \hat{z} \right] \tag{2}$$

$$\rho c \frac{d\mathrm{T}}{dt} = \vec{\nabla} \cdot \left[\frac{\partial \mathrm{T}}{\partial r} \kappa(r,z) \hat{r} + \frac{\partial \mathrm{T}}{\partial z} \kappa(r,z) \hat{z} \right] \tag{3}$$

$$\rho c \frac{d\mathrm{T}}{dt} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \frac{\partial \mathrm{T}}{\partial r} \kappa(r,z) \right] + \frac{\partial}{\partial z} \left[\frac{\partial \mathrm{T}}{\partial z} \kappa(r,z) \right] \tag{4}$$

$$\rho c \frac{d\mathrm{T}}{dt} = \frac{1}{r} \left[\frac{\partial \mathrm{T}}{\partial r} \kappa(r,z) + r \frac{\partial^2 \mathrm{T}}{\partial r^2} \kappa(r,z) + r \frac{\partial \mathrm{T}}{\partial r} \frac{\partial \kappa}{\partial r} \right] + \frac{\partial^2 \mathrm{T}}{\partial z^2} \kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z} \frac{\partial \kappa}{\partial z} \tag{5}$$

$$\rho c \frac{d\mathrm{T}}{dt} = \frac{1}{r} \frac{\partial \mathrm{T}}{\partial r} \kappa(r,z) + \frac{\partial^2 \mathrm{T}}{\partial r^2} \kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r} \frac{\partial \kappa}{\partial r} + \frac{\partial^2 \mathrm{T}}{\partial z^2} \kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z} \frac{\partial \kappa}{\partial z} \tag{6}$$

$$\begin{aligned} \rho c \frac{d\mathrm{T}}{dt} = {} & \frac{1}{r} \left[\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r} \right] \kappa^n_{r,z} + \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2} \right] \kappa^n_{r,z} + \left(\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r} \right) \left(\frac{\kappa^n_{r+1,z} - \kappa^n_{r-1,z}}{2\Delta r} \right) \\ & + \left[\frac{\mathrm{T}^n_{r,z+1} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r,z-1}}{(\Delta z)^2} \right] \kappa^n_{r,z} + \left(\frac{\mathrm{T}^n_{r,z+1} - \mathrm{T}^n_{r,z-1}}{2\Delta z} \right) \left(\frac{\kappa^n_{r,z+1} - \kappa^n_{r,z-1}}{2\Delta z} \right) \end{aligned} \tag{7}$$

$$\begin{aligned} \rho c \frac{d\mathrm{T}}{dt} = {} & \frac{1}{r} \left[\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r} \right] \kappa^n_{r,z} + \left[\frac{\mathrm{T}^n_{r+1,z} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r-1,z}}{(\Delta r)^2} \right] \kappa^n_{r,z} + \left(\frac{\mathrm{T}^n_{r+1,z} - \mathrm{T}^n_{r-1,z}}{2\Delta r} \right) \left(\frac{\Delta \kappa}{\Delta r} \right)^n_{r,z} \\ & + \left[\frac{\mathrm{T}^n_{r,z+1} - 2\mathrm{T}^n_{r,z} + \mathrm{T}^n_{r,z-1}}{(\Delta z)^2} \right] \kappa^n_{r,z} + \left(\frac{\mathrm{T}^n_{r,z+1} - \mathrm{T}^n_{r,z-1}}{2\Delta r} \right) \left(\frac{\Delta \kappa}{\Delta z} \right)^n_{r,z} \end{aligned} \tag{8}$$

$$
\begin{aligned}
\rho c \frac{\mathrm{T}^{n+1}_{r,z} - \mathrm{T}^{n}_{r,z}}{\Delta t} &= \frac{1}{r}\left[\frac{\mathrm{T}^{n}_{r+1,z} - \mathrm{T}^{n}_{r-1,z}}{2\Delta r}\right]\kappa^{n}_{r,z} + \left[\frac{\mathrm{T}^{n}_{r+1,z} - 2\mathrm{T}^{n}_{r,z} + \mathrm{T}^{n}_{r-1,z}}{(\Delta r)^2}\right]\kappa^{n}_{r,z} \\
&+ (\frac{\mathrm{T}^{n}_{r+1,z} - \mathrm{T}^{n}_{r-1,z}}{2\Delta r})(\frac{\Delta\kappa}{\Delta r})^{n}_{r,z} + \left[\frac{\mathrm{T}^{n}_{r,z+1} - 2\mathrm{T}^{n}_{r,z} + \mathrm{T}^{n}_{r,z-1}}{(\Delta z)^2}\right]\kappa^{n}_{r,z} + (\frac{\mathrm{T}^{n}_{r,z+1} - \mathrm{T}^{n}_{r,z-1}}{2\Delta r})(\frac{\Delta\kappa}{\Delta z})^{n}_{r,z}
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\mathrm{T}^{n+1}_{r,z} - \mathrm{T}^{n}_{r,z} = \frac{\Delta t}{\rho c}(\mathrm{T}^{n}_{r,z}&\left[\frac{-2\kappa^{n}_{r,z}}{(\Delta r)^2} - \frac{2\kappa^{n}_{r,z}}{(\Delta z)^2}\right] + \mathrm{T}^{n}_{r+1,z}\left[\frac{\kappa^{n}_{r,z}}{2r\Delta r} + \frac{\kappa^{n}_{r,z}}{(\Delta r)^2} + \frac{1}{2\Delta r}(\frac{\Delta\kappa}{\Delta r})^{n}_{r,z}\right] \\
&+ \mathrm{T}^{n}_{r-1,z}\left[\frac{-\kappa^{n}_{r,z}}{2r\Delta r} + \frac{\kappa^{n}_{r,z}}{(\Delta r)^2} - \frac{1}{2\Delta r}(\frac{\Delta\kappa}{\Delta r})^{n}_{r,z}\right] + \mathrm{T}^{n}_{r,z-1}\left[\frac{\kappa^{n}_{r,z}}{(\Delta z)^2} - \frac{1}{2\Delta z}(\frac{\Delta\kappa}{\Delta z})^{n}_{r,z}\right] \\
&+ \mathrm{T}^{n}_{r,z+1}\left[\frac{\kappa^{n}_{r,z}}{(\Delta z)^2} + \frac{1}{2\Delta z}(\frac{\Delta\kappa}{\Delta z})^{n}_{r,z}\right])
\end{aligned}
\tag{10}
$$

$$
A = \left[\frac{-2\kappa^{n}_{r,z}}{(\Delta r)^2} - \frac{2\kappa^{n}_{r,z}}{(\Delta z)^2}\right] \tag{11}
$$

$$
B = \left[\frac{\kappa^{n}_{r,z}}{2r\Delta r} + \frac{\kappa^{n}_{r,z}}{(\Delta r)^2} + \frac{1}{2\Delta r}(\frac{\Delta\kappa}{\Delta r})^{n}_{r,z}\right] \tag{12}
$$

$$
C = \left[\frac{-\kappa^{n}_{r,z}}{2r\Delta r} + \frac{\kappa^{n}_{r,z}}{(\Delta r)^2} - \frac{1}{2\Delta r}(\frac{\Delta\kappa}{\Delta r})^{n}_{r,z}\right] \tag{13}
$$

$$
D = \left[\frac{\kappa^{n}_{r,z}}{(\Delta z)^2} - \frac{1}{2\Delta z}(\frac{\Delta\kappa}{\Delta z})^{n}_{r,z}\right] \tag{14}
$$

$$
E = \left[\frac{\kappa^{n}_{r,z}}{(\Delta z)^2} + \frac{1}{2\Delta z}(\frac{\Delta\kappa}{\Delta z})^{n}_{r,z}\right] \tag{15}
$$

3.1 Case for $r = 0$

$$
\rho c \frac{d\mathrm{T}}{dt} = \frac{1}{r}\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z) + \frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa}{\partial r} + \frac{\partial^2 \mathrm{T}}{\partial z^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial z}\frac{\partial \kappa}{\partial z} \tag{16}
$$

$$
\begin{aligned}
\lim_{r\to 0}\left[\frac{\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)}{r}\right] &= \frac{0}{0} \\
\lim_{r\to 0}\left[\frac{\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)}{r}\right] &\overset{\textcircled{L}}{=} \lim_{r\to 0}\left[\frac{\frac{\partial}{\partial r}\left[\frac{\partial \mathrm{T}}{\partial r}\kappa(r,z)\right]}{\frac{\partial}{\partial r}\left[r\right]}\right] \\
&= \lim_{r\to 0}\left[\frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) + \frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa(r,z)}{\partial r}\right]
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
\frac{\partial^2 \mathrm{T}}{\partial r^2}\kappa(r,z) &= \left[\frac{\mathrm{T}^{n}_{r+1,z} - 2\mathrm{T}^{n}_{r,z} + \mathrm{T}^{n}_{r-1,z}}{(\Delta r)^2}\right]\kappa^{n}_{r,z} \\
&= \left[\frac{\mathrm{T}^{n}_{r+1,z} - 2\mathrm{T}^{n}_{r,z} + \mathrm{T}^{n}_{r-1,z}}{(\Delta r)^2}\right]\kappa^{n}_{r,z}
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
\frac{\partial \mathrm{T}}{\partial r}\frac{\partial \kappa(r,z)}{\partial r} &= \left[\frac{\mathrm{T}^n_{r+1,z}-\mathrm{T}^n_{r-1,z}}{2\Delta r}\right](\frac{\Delta \kappa}{\Delta r})^n_{r,z} \\
&= \left[\frac{\mathrm{T}^n_{r+1,z}-\mathrm{T}^n_{r-1,z}}{2\Delta r}\right](\frac{\Delta \kappa}{\Delta r})^n_{r,z}
\end{aligned} \tag{19}
$$