Numerical Evaluation of the Arrhenius Integral

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The Arrhenius Integral model for thermal damage requires the evaluation of the integral[2]

$$\Omega(\tau) = \int_0^\tau A e^{\frac{-Ea}{RT(t)}} dt. \tag{1}$$

If the temperature, $T(t)$, is predicted using a numerical simulation, i.e. a finite-difference or finite-element model, then the temperature is calculated at a discrete set of times, $(t_0, t_1, \ldots)$. This limits the methods that can be used to evaluate Eq. 1 numerically to those that work on predefined nodes (Gaussian quadrature, for example, cannot be used) numerically to those that work on predefined nodes (Gaussian quadrature, for example, cannot be used).

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1 Recasting the integral

If we assume that the temperature between two times, t_0 and t_1 can be written as a linear function of time,

$$T(t) = mt + T_0 \tag{2}$$

and inserting into this into Eq. 1 gives,

$$\Omega = \int_{t_0}^{t_1} A e^{\frac{-Ea}{R(mt+T_0)}} dt. \tag{3}$$

Changing the integration variable to T,

$$dT = mdt \tag{4}$$

$$\Omega = \int_{T_0}^{T_1} \frac{A}{m} e^{\frac{-Ea}{RT}} dT. \tag{5}$$

Where T_0 and T_1 are the temperature at t_0 and t_1. Next, we substitute $u = \frac{E_a}{RT}$ and change the integration variable again to u.

$$u = \frac{E_a}{R}\frac{1}{T} \tag{6}$$

$$du = -\frac{E_a}{R}\frac{1}{T^2}dT \tag{7}$$

$$dT = -\frac{E_a}{R}\frac{1}{u^2}du \tag{8}$$

$$\Omega = -\frac{A}{m}\frac{E_a}{R}\int_{E_a/RT_0}^{E_a/RT_1}\frac{e^{-u}}{u^2}du \tag{9}$$

$$= \frac{A}{m}\frac{E_a}{R}\int_{E_a/RT_1}^{E_a/RT_0}\frac{e^{-u}}{u^2}du. \tag{10}$$

The integrand can be evaluated "analytically" using the *exponential integral*, E_n, which is defined as[1]

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}dt \tag{11}$$

The variable substitution $y = xt$ gives

$$dy = xdt \tag{12}$$

$$dt = \frac{1}{x}dy \tag{13}$$

$$E_n(x) = x\int_x^\infty \frac{e^{-y}}{y^n}dy \tag{14}$$

Comparing this to Eq. 10, it is simple to show that

$$\Omega = \frac{A}{m}\frac{E_a}{R}\left[\frac{E_2\{E_a/RT_1\}}{E_a/RT_1} - \frac{E_2\{E_a/RT_0\}}{E_a/RT_0}\right] \tag{15}$$

$$= \frac{A}{m}\left[T_1E_2\{E_a/RT_1\} - T_0E_2\{E_a/RT_0\}\right] \tag{16}$$

Finally, we note that the slope, m, can be determined from T_0 and T_1 if t_1 and t_0 are known (rise over run),

$$\Omega = A\frac{t_1 - t_0}{T_1 - T_0}\left[T_1E_2\{E_a/RT_1\} - T_0E_2\{E_a/RT_0\}\right] \tag{17}$$

This gives the accumulated damage over a time $t_1 - t_0$, when the temperature rise is linear. If the actual

References

[1] Milton Abramowitz and Erene A. Stegun, editors. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* U.S. Department of Commerce, 1970.

[2] A.J. Welch and M.J.C van Germert, editors. *Optical-Thermal Response of Laser-Irradiated Tissue.* Plenum Press, 2nd edition, 2011.