A Green's Function based model for retinal temperature rise from laser exposure

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The heat equation describing heat conduction in a material with source term is

$$\rho c \frac{\partial T}{\partial t} = \boldsymbol{\nabla} \cdot k \boldsymbol{\nabla} T + A(\vec{r}, t).$$

Here, $T(\vec{r}, t)$ is the temperature field and $A(\vec{r}, t)$ is the source term. The thermal properties, ρ, c, and k (density, specific heat, and thermal conductivity) are in general function of space and time, but for a homogeneous media, this can be rewritten as:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{A(\vec{r}, t)}{\rho c}$$

where $\alpha = \frac{\kappa}{\rho c}$ is the thermal diffusivity. The Green's function for the heat equation, expressed in Cartesian coordinates, is

$$G(x, y, z, t, x', y', z', t') = \left(\frac{1}{4\pi\alpha(t - t')}\right)^{3/2} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\alpha(t-t')}}.$$

This function describes the thermal response of the media to an instantaneous point source delivered at the point (x', y', z') and time t', and it can be used to compute the response of the media to the source term A,

$$T(x, y, z, t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, z, t; x', y', z', t') \frac{A(x', y', z', t')}{\rho c} \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \mathrm{d}t'$$

A laser beam incident on a thin, linear absorbing layer of tissue will produce a spatial source term

$$A(x', y', z') = \mu_a E(x', y') e^{-\mu_a z'} = \mu_a E_0 \bar{E}(x', y') e^{-\mu_a z'}$$

while the laser is on, where $\bar{E}$ denotes the normalized beam profile ($\bar{E}(0, 0,) = 0$) and E_0 is the irradiance at the center of the beam. For a circular flat top beam of radius R, this is

$$\bar{E}(x', y') = \begin{cases} 1 & x'^2 + y'^2 \leq R^2 \\ 0 & x'^2 + y'^2 > R^2 \end{cases}$$

For a circular Gaussian beam with 1/e radius σ, this is

$$\bar{E}(x', y') = e^{-\frac{x'^2 + y'^2}{\sigma^2}}$$

Let the absorbing layer have a thickness d and span from z_0 to $z_0 + d$. Then the temperature by the absorbing layer while the laser is on will be

$$T(x, y, z, t) = \int_0^t \int_{z_0}^{z_0+d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\alpha t'}} \frac{\mu_a E(x', y') e^{-\mu_a (z' - z_0)}}{\rho c} \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \mathrm{d}t',$$

where we have used the fact that A does not depend on time to simplify the time-dependence. The axial integral can be carried out analytically,

$$\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a(z'-z_0)} \mathrm{d}z'$$

Let $\beta = \frac{1}{\sqrt{4\alpha t'}}$. Then,

$$\int_{z_0}^{z_0+d} e^{-\beta^2(z-z')^2-\mu_a(z'-z_0)} \mathrm{d}z' = e^{\mu_a z_0} \int_{z_0}^{z_0+d} e^{-\left(\beta^2 z^2+\beta^2 z'^2-2\beta^2 zz'+\mu_a z'\right)} \mathrm{d}z'$$

Completing the square for the exponent

$$\beta^2 z^2+\beta^2 z'^2-2\beta^2 zz'+\mu_a z' = \left(\beta z' + \frac{\mu_a}{2\beta}\right)^2 - \left(\frac{\mu_a}{2\beta}\right)^2 + \beta^2 z^2 - 2\beta^2 zz' = \left(\beta z' + \frac{\mu_a}{2\beta} - \beta z\right)^2 - \left(\frac{\mu_a}{2\beta}\right)^2 + \mu_a z$$

gives

$$e^{\mu_a z_0} e^{-\mu_a z} e^{\mu_a^2/2\beta^2} \int_{z_0}^{z_0+d} e^{-\left(\beta z' + \frac{\mu_a}{2\beta} - \beta z\right)^2} \mathrm{d}z' = e^{-\mu_a(z-z_0)} e^{\mu_a^2/2\beta^2} \int_{\beta z_0+\mu_a/2\beta-\beta z}^{\beta(z_0+d)+\mu_a/2\beta-\beta z} e^{-u^2} \frac{\mathrm{d}u'}{\beta}$$

with $u = \beta z' + \mu_a/2\beta - \beta z$, which can be integrated using definition of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \mathrm{d}u,$$

to give

$$\begin{aligned}
&\frac{\sqrt{\pi}}{2\beta} e^{-\mu_a(z-z_0)} e^{\mu_a^2/2\beta^2} \left[\mathrm{erf}\left(\beta(z_0+d-z) + \frac{\mu_a}{2\beta}\right) - \mathrm{erf}\left(\beta(z_0-z) + \frac{\mu_a}{2\beta}\right)\right] \\
&= \sqrt{\pi\alpha t'} e^{-\mu_a(z-z_0)} e^{\alpha t' \mu_a^2} \left[\mathrm{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \mathrm{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right]
\end{aligned}$$

To simplify transverse integrals we consider only the temperature along the z axis. Then we need to evaluate

$$\int\int \bar{E}(x', y') e^{-\frac{x'^2+y'^2}{4\alpha t'}} \mathrm{d}x' \mathrm{d}y'$$

For circularly symmetric beams, we can switch to polar coordinates and integrate the azimuthal angle,

$$\int_0^{2\pi} \int_0^{\infty} \bar{E}(r') e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' \mathrm{d}\theta' = 2\pi \int_0^{\infty} \bar{E}(r') e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r'$$

To evaluate radial integral, we need to specify the beam profile. For a flat top beam of radius R, we will have

$$2\pi \int_0^R e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' = 4\pi\alpha t' \left[1 - e^{-R^2/4\alpha t'}\right]$$

For a Gaussian beam with 1/e radius σ clipped by a circular aperture of radius R, we will have

$$2\pi \int_0^R e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' = \frac{\pi}{1/4\alpha t' + 1/\sigma^2} \left[1 - e^{-\frac{R^2}{4\alpha t'+\sigma^2}}\right] = \frac{4\pi\alpha t' \sigma^2}{4\alpha t' + \sigma^2} \left[1 - e^{-\frac{R^2}{4\alpha t'+\sigma^2}}\right].$$

For an unclipped beam, $R = \infty$.

Putting this all together, for a flat top beam we have

$$\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2}\left\{\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a(z'-z_0)}\mathrm{d}z'\right\}\left\{2\pi\int_0^R E_0 e^{-\frac{r'^2}{4\alpha t'}} r'\mathrm{d}r'\right\}\mathrm{d}t' \\
=& \frac{\mu_a E_0}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \\
&\times\left\{\sqrt{\pi\alpha t'} e^{-\mu_a(z-z_0)} e^{\alpha t'\mu_a^2}\left[\mathrm{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\mathrm{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right]\right\} \\
&\times\left\{4\pi\alpha t'\left[1-e^{-R^2/4\alpha t'}\right]\right\}\mathrm{d}t' \\
=& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a(z-z_0)}\int_0^t e^{\alpha t'\mu_a^2} \\
&\times\left[\mathrm{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\mathrm{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right] \\
&\times\left[1-e^{-R^2/4\alpha t'}\right]\mathrm{d}t' \qquad (1)
\end{aligned}$$

For a Gaussian beam we have

$$\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2}\left\{\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a(z'-z_0)}\mathrm{d}z'\right\}\left\{2\pi\int_0^R E_0 e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r'^2}{4\alpha t'}} r'\mathrm{d}r'\right\}\mathrm{d}t' \\
=& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a(z-z_0)}\int_0^t e^{\alpha t'\mu_a^2} \\
&\times\left[\mathrm{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\mathrm{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right] \\
&\times\frac{1}{1+4\alpha t'/\sigma^2}\left[1-e^{-R^2/4\alpha t'}\right]\mathrm{d}t' \qquad (2)
\end{aligned}$$

0.1 Approximations

In theory, Equation 1 and 2 can be integrated numerically to calculate the temperature at $r = 0$ at any time t. However, in practice the calculation is difficult. One of the issues is that the terms arising from the z integral become very large or small individually and exceed the precision of standard floating point numbers for long time. This is especially an issue when the absorption coefficient is large.

0.1.1 For long time

The z portion of the integral is

$$\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a(z'-z_0)}\mathrm{d}z' = e^{\mu_a z_0}\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'}\mathrm{d}z'$$

The Gaussian term can be written as a power series

$$e^{-\frac{(z-z')^2}{4\alpha t'}} = 1-\frac{(z-z')^2}{4\alpha t'}+\frac{1}{2}\left(\frac{(z-z')^2}{4\alpha t'}\right)^2+\ldots$$

If $(z-z')^2/4\alpha t'$ is small, which will be the case for positions z close to the source at long times, then

$$e^{\mu_a z_0}\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'}\mathrm{d}z' \approx e^{\mu_a z_0}\int_{z_0}^{z_0+d}\left[1-\frac{(z-z')^2}{4\alpha t'}\right] e^{-\mu_a z'}\mathrm{d}z'$$

$$\left[1-\frac{(z-z')^2}{4\alpha t'}\right]e^{-\mu_a z'}=\left[1-\frac{z^2}{4\alpha t'}+\frac{2zz'}{4\alpha t'}-\frac{z'^2}{4\alpha t'}\right]e^{-\mu_a z'}$$

There are three integrals to evaluate

$$e^{\mu_a z_0}\int_{z_0}^{z_0+d}\left(1-\frac{z^2}{4\alpha t'}\right)e^{-\mu_a z'}\mathrm{d}z'=e^{\mu_a z_0}\left(1-\frac{z^2}{4\alpha t'}\right)\frac{e^{-\mu_a z'}}{-\mu_a}\Bigg|_{z_0}^{z_0+d}=\left(1-\frac{z^2}{4\alpha t'}\right)\frac{1-e^{-\mu_a d}}{\mu_a}$$

$$\begin{aligned}e^{\mu_a z_0}\int_{z_0}^{z_0+d}\frac{2zz'}{4\alpha t'}e^{-\mu_a z'}\mathrm{d}z' &= e^{\mu_a z_0}\frac{2z}{4\alpha t'}\left(\frac{-\mu_a z'-1}{\mu_a^2}\right)e^{-\mu_a z'}\Bigg|_{z_0}^{z_0+d}\\ &= e^{\mu_a z_0}\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}e^{-\mu_a z_0}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a(z_0+d)}\right)\\ &= \frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a d}\right)\end{aligned}$$

$$\begin{aligned}e^{\mu_a z_0}\int_{z_0}^{z_0+d}\frac{z'^2}{4\alpha t'}e^{-\mu_a z'}\mathrm{d}z' &= e^{\mu_a z_0}\frac{1}{4\alpha t'}\left(\frac{z'^2}{-\mu_a}-\frac{2z'}{\mu_a^2}+\frac{2}{-\mu_a^3}\right)e^{-\mu_a z'}\Bigg|_{z_0}^{z_0+d}\\ &= e^{\mu_a z_0}\frac{1}{4\alpha t'}\left(\frac{z'^2}{\mu_a}+\frac{2z'}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z'}\Bigg|_{z_0+d}^{z_0}\\ &= e^{\mu_a z_0}\frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z_0}-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a(z_0+d)}\right]\\ &= \frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a d}\right]\end{aligned}$$

$$\begin{aligned}e^{\mu_a z_0}\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z' \approx &\left(1-\frac{z^2}{4\alpha t'}\right)\frac{1-e^{-\mu_a d}}{\mu_a}\\ &+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a d}\right)\\ &-\frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a d}\right]\end{aligned}$$

For $z=z_0=0$, this simplifies to

$$e^{\mu_a z_0}\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z'\approx\frac{1-e^{-\mu_a d}}{\mu_a}-\frac{1}{4\alpha t'}\left[\left(\frac{2}{\mu_a^3}\right)-\left(\frac{d^2}{\mu_a}+\frac{2d}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a d}\right]$$

For “long” time, $4\alpha t' >> z$, we can keep just the first term,

$$e^{\mu_a z_0}\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z'\approx\frac{1-e^{-\mu_a d}}{\mu_a}$$

Inserting the approximation into Equation 1

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a E_0}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \\
&\times\Bigg\{\left(1-\frac{z^2}{4\alpha t'}\right)\frac{1-e^{-\mu_a d}}{\mu_a} \\
&+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a d}\right) \\
&-\frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a d}\right]\Bigg\} \\
&\times\Big\{4\pi\alpha t'\Big[1-e^{-R^2/4\alpha t'}\Big]\Big\}\mathrm{d}t' \qquad (3) \\
=& \frac{E_0}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{1/2} \\
&\times\Bigg\{\left[1-e^{-\mu_a d}\right]-\frac{z^2}{4\alpha t'}\left[1-e^{-\mu_a d}\right] \\
&+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a}-\frac{\mu_a(z_0+d)+1}{\mu_a}e^{-\mu_a d}\right) \\
&-\frac{1}{4\alpha t'}\left[\left(z_0^2+\frac{2z_0}{\mu_a}+\frac{2}{\mu_a^2}\right)-\left((z_0+d)^2+\frac{2(z_0+d)}{\mu_a}+\frac{2}{\mu_a^2}\right)e^{-\mu_a d}\right]\Bigg\} \\
&\times\Big[1-e^{-R^2/4\alpha t'}\Big]\mathrm{d}t' \qquad (4)
\end{aligned}
$$

0.2 For large absorption

The approximation in the previous section for long times has been implemented, and it works, however it does not solve the problem. Evaluating the z integral without approximation leads to

$$
e^{\alpha t'\mu_a^2}\left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right].
$$

The issue is that as t' increases, $e^{\alpha t'\mu_a^2}$ becomes very large. The difference between the two error functions becomes small, so the product of the two remains finite (it actually decreases with time). However, trying to compute the two terms separately leads to floating point overflow. In Python, both the standard math module and numpy overflow when passing an argument greater than about 700 to the exp function. For visible light, the absorption coefficient of the RPE can be on the order of $1000\,\mathrm{cm}^{-1}$, which means (assuming the thermal properties of water, $\alpha = 0.0015\,\mathrm{cm}^2\,\mathrm{s}^{-1}$) that we will get an overflow when $t' > 0.46\,\mathrm{s}$.This is much too short a time to simulate a laser exposure. But even before an overflow, we will get a loss of precision.

The problem with the long-time approximation is that it is valid when $(z'-z_0)/4\alpha t' \ll 1$, which does not depend on the absorption coefficient at all. So its possible to overflow with a large absorption coefficient before the approximation can be employed.

Instead, it is possible to approximate the result of the integral using an asymptotic expansion for the error function

$$
\operatorname{erf}(x) = 1-\operatorname{erfc}(x) = 1-\frac{e^{-x^2}}{x\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2x^2)^n} = 1-\frac{e^{-x^2}}{x\sqrt{\pi}}+\frac{e^{-x^2}}{x\sqrt{\pi}}\frac{1}{2x^2}-\mathcal{O}(1/x^5). \qquad (5)
$$

This approximation of the error function works for "large" values, however, the argument does not need to be that "large". Keeping only the first two terms gives less than 0.1 error for $x > 0.88$. $x > 1.38$ gives an error less than 0.01, $x > 1.86$ gives and error less than 0.001, and $x > 2.28$ give an error less than 0.0001. Keeping the third term reduces the values of x to 0.92, 1.30, 1.69, and 2.07 to reach these error threshold. Looking at the arguments of the two error functions, there are two terms. The first gets smaller with time, the second gets larger. The second term is the only one that depends on μ_a. If we assume that the first term is small, then we can estimate the time at which the 2 term asymptotic expansion would give less than 0.001 error as $1.86 = \sqrt{\alpha t'}\mu_a$. For $\mu_a = 1000\,\mathrm{cm}^{-1}$, this gives $t' = 0.001\,\mathrm{s}$.

To simplify the algebra, let us define the following variables:

$$
\begin{aligned}
A &= \sqrt{\alpha t'}\mu_a \\
B &= \frac{z_0 - z}{\sqrt{4\alpha t'}} \\
C &= \frac{d}{\sqrt{4\alpha t'}}
\end{aligned}
$$

Then the product we want to approximate can be written

$$e^{A^2}[\operatorname{erf}(C + B + A) - \operatorname{erf}(B + A)].$$

The issue is e^{A^2}, it will overflow at short times. If we only had e^{A}, then we would not get an overflow until $t' = 326\,\mathrm{s}$, which should be sufficient for most laser exposures, sine we would expect to reach steady-state long before that time.

$$
\begin{aligned}
(A + B + C)^2 &= A^2 + B^2 + C^2 + 2AB + 2AC + 2BC \\
(A + B)^2 &= A^2 + B^2 + 2AB \\
e^{A^2}[\operatorname{erf}(C + B + A) - \operatorname{erf}(B + A)] &= e^{A^2}\left[\left(1 - \frac{e^{-(A^2+B^2+C^2+2AB+2AC+2BC)}}{(A + B + C)\sqrt{\pi}}\sum\cdots\right) - \left(1 - \frac{e^{-(A^2+B^2+2AB)}}{(A + B)\sqrt{\pi}}\sum\cdots\right)\right] \\
&= \frac{-e^{-(B^2+C^2+2AB+2AC+2BC)}}{(A + B + C)\sqrt{\pi}}\sum\cdots + \frac{e^{-(B^2+2AB)}}{(A + B)\sqrt{\pi}}\sum\cdots \\
&= \frac{e^{-(B^2+2AB)}}{(A + B)\sqrt{\pi}}\left[1 - \frac{1}{2(A + B)^2} + \ldots\right] - \frac{e^{-(B^2+C^2+2AB+2AC+2BC)}}{(A + B + C)\sqrt{\pi}}\left[1 - \frac{1}{2(A + B + C)^2} +\right.
\end{aligned}
$$

This gives us way to calculate the product without having to evaluate e^{A^2} and cause an overflow. We have the following

$$
\begin{aligned}
A^2 &= \alpha t' \mu_a^2 \\
B^2 &= \frac{(z_0 - z)^2}{4\alpha t'} \\
C^2 &= \frac{d^2}{4\alpha t'} \\
2AB &= 2\sqrt{\alpha t'}\mu_a \frac{z_0 - z}{\sqrt{4\alpha t'}} = (z_0 - z)\mu_a \\
2AC &= 2\sqrt{\alpha t'}\mu_a \frac{d}{\sqrt{4\alpha t'}} = d\mu_a \\
2BC &= 2\frac{z_0 - z}{\sqrt{4\alpha t'}}\frac{d}{\sqrt{4\alpha t'}} = \frac{d(z_0 - z)}{2\alpha t'}
\end{aligned}
$$

substituting in

$$e^{\alpha t'\mu_a^2}\left[\text{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\text{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right]=\frac{e^{-\left(\frac{(z_0-z)^2}{4\alpha t'}+\frac{d^2}{4\alpha t'}+(z_0-z)\mu_a+d\mu_a+\frac{d(z_0-z)}{2\alpha t'}\right)}}{(\sqrt{\alpha t'}\mu_a+\frac{z_0-z}{\sqrt{4\alpha t'}}+\frac{d}{\sqrt{4\alpha t'}})\sqrt{\pi}}\sum\cdots$$
$$-\frac{e^{-\left(\frac{(z_0-z)^2}{4\alpha t'}+(z_0-z)\mu_a\right)}}{(\sqrt{\alpha t'}\mu_a+\frac{z_0-z}{\sqrt{4\alpha t'}})\sqrt{\pi}}\sum\cdots$$

0.3 Off Axis Temperatures

The derivation of Equations 1 and 2 assumed $x=y=0$, i.e., it is only valid for the temperature rise on the z axis. To compute the temperature off axis, we need to evaluate

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{E}(x',y')e^{-\frac{(x-x')^2+(y-y')^2}{4\alpha t'}}\mathrm{d}x'\mathrm{d}y'.$$

We can switch to polar coordinates by noting that $(x-x')^2+(y-y')^2=|\vec{r}-\vec{r}'|^2=(\vec{r}-\vec{r}')\cdot(\vec{r}-\vec{r}')=\vec{r}\cdot\vec{r}+\vec{r}'\cdot\vec{r}'-2rr'\cos(\phi-\phi')$

$$\int_0^{\infty}\int_0^{2\pi}\bar{E}(r',\phi')e^{-\frac{r^2}{4\alpha t'}}e^{-\frac{r'^2}{4\alpha t'}}e^{\frac{2rr'\cos(\phi-\phi')}{4\alpha t'}}r'\mathrm{d}\phi'\mathrm{d}r'.$$

If the source term is symmetric about the z axis, we can evaluate the integral at $\phi=0$ without loss of generality.

$$\int_0^{\infty}\bar{E}(r')r'e^{-\frac{r^2}{4\alpha t'}}e^{-\frac{r'^2}{4\alpha t'}}\int_0^{2\pi}e^{2rr'\cos(\phi')/4\alpha t'}\mathrm{d}\phi'\mathrm{d}r'.$$

That the zero'th order Modified Bessel Function of the First Kind, $I_0(x)$ has an integral representation

$$I_0(x)=\frac{1}{\pi}\int_0^{\pi}e^{x\cos(\phi)}\mathrm{d}\phi. \tag{6}$$

The azimuthal integral then can be carried out by noting that the integrand is symmetric about $\phi'=\pi$

$$\int_0^{2\pi}e^{2rr'\cos(\phi')/4\alpha t'}\mathrm{d}\phi'=2\int_0^{\pi}e^{2rr'\cos(\phi')/4\alpha t'}\mathrm{d}\phi'=2\pi I_0\left(\frac{2rr'}{4\alpha t'}\right).$$

The radian integral is then

$$2\pi\int_0^{\infty}\bar{E}(r')e^{-\frac{r^2}{4\alpha t'}}e^{-\frac{r'^2}{4\alpha t'}}I_0\left(\frac{2rr'}{4\alpha t'}\right)r'\mathrm{d}r'.$$

0.3.1 Flat Top Beams

For a flat top beam with radius R, we need to evaluate the integral

$$2\pi\int_0^{R}e^{-\frac{r^2}{4\alpha t'}}e^{-\frac{r'^2}{4\alpha t'}}I_0\left(\frac{2rr'}{4\alpha t'}\right)r'\mathrm{d}r'.$$

The integral can be carried out using the the Marcum Q-function, which is defined as

$$Q_\nu(a,b)=1-\frac{1}{a^{\nu-1}}\int_0^{b}x^\nu e^{-\frac{x^2+a^2}{2}}I_{\nu-1}(ax)\mathrm{d}x.$$

In our case, $\nu = 1$. To cast our integral into this form, let $x = r'/\sqrt{2\alpha t'}$, $a = r/\sqrt{2\alpha t'}$, which gives

$$2\pi 2\alpha t' \int_0^{R/\sqrt{2\alpha t'}} e^{-a^2/2} e^{-x^2/2} I_0(ax) x \mathrm{d}x = 4\pi\alpha t' \Big(1 - Q_1(r/\sqrt{2\alpha t'}, R/\sqrt{2\alpha t'})\Big)$$

Plugging this into Equation 1 gives

$$\begin{aligned} T(r=0,z,t) = &\frac{\mu_a E_0}{2\rho c} e^{-\mu_a(z-z_0)} \int_0^t e^{\alpha t' \mu_a^2} \\ &\times \left[\mathrm{erf}\left(\frac{z_0 + d - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) - \mathrm{erf}\left(\frac{z_0 - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) \right] \\ &\times \Big[1 - Q_1\Big(r/\sqrt{2\alpha t'}, R/\sqrt{2\alpha t'} \Big) \Big] \mathrm{d}t' \end{aligned} \tag{7}$$

0.3.2 Gaussian Beams

For a Gaussian beam with 1/e radius σ we need to evaluate

$$2\pi \int_0^\infty e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r' = 2\pi e^{-\frac{r^2}{4\alpha t'}} \int_0^\infty e^{-\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right) r'^2} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r'.$$

The integral can be cast into a standard form and evaluated [“Table of Integrals, Series, and Products”, Gradshteyn and Ryzhik pg. 707],

$$\int_0^\infty x e^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x) \mathrm{d}x = \frac{1}{2\alpha} \exp\left(\frac{\beta^2 - \gamma^2}{4\alpha} \right) J_\nu\left(\frac{\beta\gamma}{2\alpha} \right)$$

In our case, $\nu = \gamma = 0$, $\beta = 2r/4\alpha t'$, and $\alpha = \left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right)$, which gives

$$\begin{aligned} 2\pi e^{-\frac{r^2}{4\alpha t'}} \int_0^\infty e^{-\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right) r'^2} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r' &= 2\pi e^{-\frac{r^2}{4\alpha t'}} \frac{1}{2\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right)} e^{\left(2r/4\alpha t'\right)^2 / 4\left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right)} \\ &= 4\pi\alpha t' e^{-\frac{r^2}{4\alpha t'}} \frac{1}{1 + \frac{4\alpha t'}{\sigma^2}} e^{\frac{r^2}{4\alpha t'} \frac{1}{1 + \frac{4\alpha t'}{\sigma^2}}} \\ &= 4\pi\alpha t' \frac{1}{1 + \frac{4\alpha t'}{\sigma^2}} e^{\frac{r^2}{4\alpha t'} \left(\frac{1}{1 + \frac{4\alpha t'}{\sigma^2}} - 1 \right)} \\ &= 4\pi\alpha t' \frac{\sigma^2}{\sigma^2 + 4\alpha t'} e^{\frac{-r^2}{\sigma^2 + 4\alpha t'}} \end{aligned}$$

Plugging this into Equation 2 gives

$$\begin{aligned} T(r=0,z,t) = &\frac{\mu_a E_0}{2\rho c} e^{-\mu_a(z-z_0)} \int_0^t e^{\alpha t' \mu_a^2} \\ &\times \left[\mathrm{erf}\left(\frac{z_0 + d - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) - \mathrm{erf}\left(\frac{z_0 - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) \right] \\ &\times \left[\frac{\sigma^2}{\sigma^2 + 4\alpha t'} e^{\frac{-r^2}{\sigma^2 + 4\alpha t'}} \right] \mathrm{d}t' \end{aligned} \tag{8}$$

0.4 Pulsed Exposures

The temperature rises caused by a CW exposure is given by integrating the Green’s Function,

$$\Delta T(t) = \int_0^t G(t') \mathrm{d}t'. \tag{9}$$

Because the temperature rise is linear, we can use the CW temperature rise to compute the temperature rise caused by a pulsed exposure,

$$\Delta T(t) = \begin{cases} \int_0^t G(t')\mathrm{d}t' & t \leq \tau \\ \int_0^t G(t')\mathrm{d}t' - \int_0^{t-\tau} G(t')\mathrm{d}t' & t > \tau. \end{cases} \tag{10}$$

After the pulse, the temperature rise is given by the difference between the CW temperature rise at the times t and $t-\tau$. We can either calculate $\Delta T(t)$ and $\Delta(t-\tau)$, or evaluate the integral over a different set of limits,

$$\int_0^t G(t')\mathrm{d}t' - \int_0^{t-\tau} G(t')\mathrm{d}t' = \Delta T(t) - \Delta T(t-\tau) = \int_{t-\tau}^t G(t')\mathrm{d}t' \tag{11}$$