A Green's Function based model for retinal temperature rise from laser exposure

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The heat equation describing heat conduction in a material with source term is

$$\rho c \frac{\partial T}{\partial t} = \boldsymbol{\nabla} \cdot k \boldsymbol{\nabla} T + A(\vec{r}, t).$$

Here, $T(\vec{r}, t)$ is the temperature field and $A(\vec{r}, t)$ is the source term. The thermal properties, ρ, c, and k (density, specific heat, and thermal conductivity) are in general function of space and time, but for a homogeneous media, this can be rewritten as:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{A(\vec{r}, t)}{\rho c}$$

where $\alpha = \frac{\kappa}{\rho c}$ is the thermal diffusivity. The Green's function for the heat equation, expressed in Cartesian coordinates, is

$$G(x, y, z, t, x', y', z', t') = \left(\frac{1}{4\pi\alpha(t - t')} \right)^{3/2} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\alpha(t-t')}}.$$

This function describes the thermal response of the media to an instantaneous point source delivered at the point (x', y', z') and time t', and it can be used to compute the response of the media to the source term A,

$$T(x, y, z, t) = \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, z, t; x', y', z', t') \frac{A(x', y', z', t')}{\rho c} \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \mathrm{d}t'$$

A laser beam incident on a thin, linear absorbing layer of tissue will produce a spatial source term

$$A(x', y', z') = \mu_a E(x', y') e^{-\mu_a z'} = \mu_a E_0 E(x', \bar{y'}) e^{-\mu_a z'}$$

while the laser is on, where $\bar{E}$ denotes the normalized beam profile ($\bar{E}(0, 0,) = 0$) and E_0 is the irradiance at the center of the beam. For a circular flat top beam of radius R, this is

$$\bar{E}(x', y') = \begin{cases} 1 & x'^2 + y'^2 \leq R^2 \\ 0 & x'^2 + y'^2 > R^2 \end{cases}$$

For a circular Gaussian beam with 1/e radius σ, this

$$\bar{E}(x', y') = e^{-\frac{x'^2 + y'^2}{\sigma^2}}$$

Let the absorbing layer have a thickness d and span from z_0 to $z_0 + d$. Then the temperature by the absorbing layer while the laser is on will be

$$T(x, y, z, t) = \int_0^t \int_{z_0}^{z_0+d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{4\pi\alpha t'} \right)^{3/2} e^{-\frac{(x-x')^2+(y-y')^2+(z-z')^2}{4\alpha t'}} \frac{\mu_a E(x', y') e^{-\mu_a z'}}{\rho c} \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \mathrm{d}t',$$

where we have used the fact that A does not depend on time to simplify the time-dependence. The axial integral can be carried out analytically,

$$\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'} \mathrm{d}z'$$

Let $\beta = \frac{1}{\sqrt{4\alpha t'}}$. Then,

$$\int_{z_0}^{z_0+d} e^{-\beta^2(z-z')^2-\mu_a z'} \mathrm{d}z' = \int_{z_0}^{z_0+d} e^{-\left(\beta^2 z^2+\beta^2 z'^2-2\beta^2 zz'+\mu_a z'\right)} \mathrm{d}z'$$

Completing the square for the exponent

$$\beta^2 z^2 + \beta^2 z'^2 - 2\beta^2 zz' + \mu_a z' = \left(\beta z' + \frac{\mu_a}{2\beta}\right)^2 - \left(\frac{\mu_a}{2\beta}\right)^2 = \left(\beta z' + \frac{\mu_a}{2\beta} - \beta z\right)^2 - \left(\frac{\mu_a}{2\beta}\right)^2 + \mu_a z$$

gives

$$e^{-\mu_a z} e^{\mu_a^2/2\beta^2} \int_{z_0}^{z_0+d} e^{-\left(\beta z' + \frac{\mu_a}{2\beta} - \beta z\right)^2} \mathrm{d}z' = e^{-\mu_a z} e^{\mu_a^2/2\beta^2} \int_{\beta z_0+\mu_a/2\beta-\beta z}^{\beta(z_0+d)+\mu_a/2\beta-\beta z} e^{-u^2} \frac{\mathrm{d}u'}{\beta}$$

with $u = \beta z' + \mu_a/2\beta - \beta z$, which can be integrated using definition of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} \mathrm{d}u,$$

to give

$$\frac{\sqrt{\pi}}{2\beta} e^{-\mu_a z} e^{\mu_a^2/2\beta^2} \left[\mathrm{erf}\left(\beta(z_0+d-z) + \frac{\mu_a}{2\beta}\right) - \mathrm{erf}\left(\beta(z_0-z) + \frac{\mu_a}{2\beta}\right)\right]$$
$$= \sqrt{\pi\alpha t'} e^{-\mu_a z} e^{\alpha t' \mu_a^2} \left[\mathrm{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \mathrm{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right]$$

To simplify transverse integrals we consider only the temperature along the z axis. Then we need to evaluate

$$\int\int \bar{E}(x', y') e^{-\frac{x'^2+y'^2}{4\alpha t'}} \mathrm{d}x' \mathrm{d}y'$$

For circularly symmetric beams, we can switch to polar coordinates and integrate the azimuthal angle,

$$\int_0^{2\pi} \int_0^{\infty} \bar{E}(r') e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' \mathrm{d}\theta' = 2\pi \int_0^{\infty} \bar{E}(r') e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r'$$

To evaluate radial integral, we need to specify the beam profile. For a flat top beam of radius R, we will have

$$2\pi \int_0^R e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' = 4\pi\alpha t' \left[1 - e^{-R^2/4\alpha t'}\right]$$

For a Gaussian beam with 1/e radius σ clipped by a circular aperture of radius R, we will have

$$2\pi \int_0^R e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' = \frac{\pi}{1/4\alpha t' + 1/\sigma^2} \left[1 - e^{-\frac{R^2}{4\alpha t'+\sigma^2}}\right] = \frac{4\pi\alpha t' \sigma^2}{4\alpha t' + \sigma^2} \left[1 - e^{-\frac{R^2}{4\alpha t'+\sigma^2}}\right].$$

For an unclipped beam, $R = \infty$.

Putting this all together, for a flat top beam we have

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a E_0}{\rho c} \int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \left\{ \int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'} \mathrm{d}z' \right\} \left\{ 2\pi \int_0^R e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' \right\} \mathrm{d}t' \\
=& \frac{\mu_a E_0}{\rho c} \int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \\
\times& \left\{ \sqrt{\pi\alpha t'} e^{-\mu_a z} e^{\alpha t' \mu_a^2} \left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right]\right\} \\
\times& \left\{ 4\pi\alpha t' \left[1 - e^{-R^2/4\alpha t'}\right]\right\} \mathrm{d}t' \\
=& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a z} \int_0^t e^{\alpha t' \mu_a^2} \\
\times& \left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right] \\
\times& \left[1 - e^{-R^2/4\alpha t'}\right] \mathrm{d}t' \qquad (1)
\end{aligned}
$$

For a Gaussian beam we have

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a}{\rho c} \int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \left\{ \int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'} \mathrm{d}z' \right\} \left\{ 2\pi \int_0^R E_0 e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r'^2}{4\alpha t'}} r' \mathrm{d}r' \right\} \mathrm{d}t' \\
=& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a z} \int_0^t e^{\alpha t' \mu_a^2} \\
\times& \left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right] \\
\times& \frac{1}{1 + 4\alpha t'/\sigma^2} \left[1 - e^{-R^2/4\alpha t'}\right] \mathrm{d}t' \qquad (2)
\end{aligned}
$$

0.1 Approximations

In theory, Equation 1 and 2 can be integrated numerically to calculate the temperature at $r = 0$ at any time t. However, in practice the calculation is difficult. One of the issues is that the terms arising from the z integral become very large or small individually and exceed the precision of standard floating point numbers for long time. This is especially an issue when the absorption coefficient is large.

0.1.1 For long time

The z portion of the integral is

$$\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'} \mathrm{d}z'$$

The Gaussian term can be written as a power series

$$e^{-\frac{(z-z')^2}{4\alpha t'}} = 1 - \frac{(z-z')^2}{4\alpha t'} + \frac{1}{2}\left(\frac{(z-z')^2}{4\alpha t'}\right)^2 + \dots$$

If $(z-z')^2/4\alpha t'$ is small, which will be the case for positions z close to the source at long times, then

$$\int_{z_0}^{z_0+d} e^{-\frac{(z-z')^2}{4\alpha t'}} e^{-\mu_a z'} \mathrm{d}z' \approx \int_{z_0}^{z_0+d} \left[1 - \frac{(z-z')^2}{4\alpha t'}\right] e^{-\mu_a z'} \mathrm{d}z'$$

$$\left[1-\frac{(z-z')^2}{4\alpha t'}\right]e^{-\mu_a z'}=\left[1-\frac{z^2}{4\alpha t'}+\frac{2zz'}{4\alpha t'}-\frac{z'^2}{4\alpha t'}\right]e^{-\mu_a z'}$$

There are three integrals to evaluate

$$\int_{z_0}^{z_0+d}\left(1-\frac{z^2}{4\alpha t'}\right)e^{-\mu_a z'}\mathrm{d}z'=\left(1-\frac{z^2}{4\alpha t'}\right)\frac{e^{-\mu_a z'}}{-\mu_a}\Bigg|_{z_0}^{z_0+d}=\left(1-\frac{z^2}{4\alpha t'}\right)\frac{e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}}{\mu_a}$$

$$\int_{z_0}^{z_0+d}\frac{2zz'}{4\alpha t'}e^{-\mu_a z'}\mathrm{d}z'=\frac{2z}{4\alpha t'}\left(\frac{-\mu_a z'-1}{\mu_a^2}\right)e^{-\mu_a z'}\Bigg|_{z_0}^{z_0+d}=\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}e^{-\mu_a z_0}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a(z_0+d)}\right)$$

$$\begin{aligned}
\int_{z_0}^{z_0+d}\frac{z'^2}{4\alpha t'}e^{-\mu_a z'}\mathrm{d}z' &= \frac{1}{4\alpha t'}\left(\frac{z'^2}{-\mu_a}-\frac{2z'}{\mu_a^2}+\frac{2}{-\mu_a^3}\right)e^{-\mu_a z'}\Bigg|_{z_0}^{z_0+d}\\
&= \frac{1}{4\alpha t'}\left(\frac{z'^2}{\mu_a}+\frac{2z'}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z'}\Bigg|_{z_0+d}^{z_0}\\
&= \frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z_0}-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a(z_0+d)}\right]
\end{aligned}$$

$$\begin{aligned}
\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z' \approx{}& \left(1-\frac{z^2}{4\alpha t'}\right)\frac{e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}}{\mu_a}\\
&+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}e^{-\mu_a z_0}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a(z_0+d)}\right)\\
&-\frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z_0}-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a(z_0+d)}\right]
\end{aligned}$$

For $z=z_0=0$, this simplifies to

$$\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z'\approx\frac{1-e^{-\mu_a d}}{\mu_a}-\frac{1}{4\alpha t'}\left[\left(\frac{2}{\mu_a^3}\right)-\left(\frac{d^2}{\mu_a}+\frac{2d}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a d}\right]$$

For "long" time, $4\alpha t' >> z$, we can keep just the first term,

$$\int_{z_0}^{z_0+d}e^{-\frac{(z-z')^2}{4\alpha t'}}e^{-\mu_a z'}\mathrm{d}z'\approx\frac{e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}}{\mu_a}$$

Inserting the approximation into Equation 1

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a E_0}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{3/2} \\
&\times\Bigg\{\left(1-\frac{z^2}{4\alpha t'}\right)\frac{e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}}{\mu_a} \\
&+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a^2}e^{-\mu_a z_0}-\frac{\mu_a(z_0+d)+1}{\mu_a^2}e^{-\mu_a(z_0+d)}\right) \\
&-\frac{1}{4\alpha t'}\left[\left(\frac{z_0^2}{\mu_a}+\frac{2z_0}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a z_0}-\left(\frac{(z_0+d)^2}{\mu_a}+\frac{2(z_0+d)}{\mu_a^2}+\frac{2}{\mu_a^3}\right)e^{-\mu_a(z_0+d)}\right]\Bigg\} \\
&\times\left\{4\pi\alpha t'\left[1-e^{-R^2/4\alpha t'}\right]\right\}\mathrm{d}t' \qquad (3) \\
=& \frac{E_0}{\rho c}\int_0^t \left(\frac{1}{4\pi\alpha t'}\right)^{1/2} \\
&\times\Bigg\{\left[e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}\right]-\frac{z^2}{4\alpha t'}\left[e^{-\mu_a z_0}-e^{-\mu_a(z_0+d)}\right] \\
&+\frac{2z}{4\alpha t'}\left(\frac{\mu_a z_0+1}{\mu_a}e^{-\mu_a z_0}-\frac{\mu_a(z_0+d)+1}{\mu_a}e^{-\mu_a(z_0+d)}\right) \\
&-\frac{1}{4\alpha t'}\left[\left(z_0^2+\frac{2z_0}{\mu_a}+\frac{2}{\mu_a^2}\right)e^{-\mu_a z_0}-\left((z_0+d)^2+\frac{2(z_0+d)}{\mu_a}+\frac{2}{\mu_a^2}\right)e^{-\mu_a(z_0+d)}\right]\Bigg\} \\
&\times\left[1-e^{-R^2/4\alpha t'}\right]\mathrm{d}t' \qquad (4)
\end{aligned}
$$

0.1.2 For short time

When t' is small, then $1/t'$ becomes very large. For flat top beams, the temperature rise is given by

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a E_0}{2\rho c}e^{-\mu_a z}\int_0^t e^{\alpha t'\mu_a^2} \\
&\times\left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)-\operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}}+\sqrt{\alpha t'}\mu_a\right)\right] \\
&\times\left[1-e^{-R^2/4\alpha t'}\right]\mathrm{d}t'
\end{aligned}
$$

The first term in the integral can be expanded in a Taylor series for the exponential function,

$$
e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n = 1+x+\frac{1}{2}x^2+\mathcal{O}(x^n) \qquad (5)
$$

where $x = \alpha t'\mu_a^2$.

The time at which we can truncate this series to approximate the exponential depends on α and μ_a. α is the thermal diffusivity. When this parameter is large, heat flows through the tissue quickly. When the absorption coefficient μ_a is large, there will be a steep thermal gradient in the axial direction in the absorbing tissue, which will cause a faster heat flow. Physically then, this series can be truncated when heat has not had significant time to conduct out of the absorption volume.

Keeping only the first two terms will give an error of about 1 % or less when $\alpha t'\mu_a^2 \leq 0.15$. Keeping the first three terms will give an error of about 1 % or less when $\alpha t'\mu_a^2 \leq 0.44$.

The second term in the integral can be expanded using a Taylor series approximation if the argument to the error function is small, or the asymptotic expansion of $\operatorname{erfc}(x)$ if the argument is large.

The Taylor series expansion of $\mathrm{erf}(x)$ is

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n x^{2n+1}}{(2n+1)!!} \tag{6}$$

where $N!!$ is the double factorial, the product of all odd numbers up to N. The first form gives

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}} - \frac{2x^3}{3\sqrt{\pi}} + \frac{2x^5}{10\sqrt{\pi}} + \mathcal{O}(x^7). \tag{7}$$

The second from gives

$$\mathrm{erf}(x) = \frac{2x}{\sqrt{\pi}} e^{-x^2} + \frac{4x^3}{3\sqrt{\pi}} e^{-x^2} + \frac{8x^5}{15\sqrt{\pi}} e^{-x^2} + \mathcal{O}(x^7). \tag{8}$$

The first form gives a more accurate approximation for larger arguments. Keeping only the first term gives and error of about 1 % or less for $x \leq 0.17$ using the first form and $x \leq 0.12$ for the second. Keeping the first two terms give an error of about 1 % or less for $x \leq 0.56$ for the first form and $x \leq 0.45$ for the second.

The asymptotic expansion of $\mathrm{erfc}(x)$ is

$$\mathrm{erfc}(x) = \frac{e^{-x^2}}{x\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2x^2)^n}. \tag{9}$$

For the error function,

$$\mathrm{erf}(x) = 1 - \mathrm{erfc}(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} + \frac{e^{-x^2}}{x\sqrt{\pi}} \frac{1}{2x^2} - \mathcal{O}(1/x^5). \tag{10}$$

Keeping the first two terms in this series gives an error of about 1 % or less when $x \geq 1.39$. Keeping the first three gives a 1 % error or less around $x \geq 1.3$.

Using the lowest order approximations for the expon

0.2 Off Axis Temperatures

The derivation of Equations 1 and 2 assumed $x = y = 0$, i.e., it is only valid for the temperature rise on the z axis. To compute the temperature off axis, we need to evaluate

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{E}(x', y') e^{-\frac{(x-x')^2 + (y-y')^2}{4\alpha t'}} \mathrm{d}x' \mathrm{d}y'.$$

We can switch to polar coordinates by noting that $(x - x')^2 + (y - y')^2 = |\vec{r} - \vec{r}'|^2 = (\vec{r} - \vec{r}') \cdot (\vec{r} - \vec{r}') = \vec{r} \cdot \vec{r} + \vec{r}' \cdot \vec{r}' - 2rr' \cos(\phi - \phi')$

$$\int_{0}^{\infty} \int_{0}^{2\pi} \bar{E}(r', \phi') e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} e^{\frac{2rr' \cos(\phi - \phi')}{4\alpha t'}} r' \mathrm{d}\phi' \mathrm{d}r'.$$

If the source term is symmetric about the z axis, we can evaluate the integral at $\phi = 0$ without loss of generality.

$$\int_{0}^{\infty} \bar{E}(r') r' e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} \int_{0}^{2\pi} e^{2rr' \cos(\phi')/4\alpha t'} \mathrm{d}\phi' \mathrm{d}r'.$$

That the zero'th order Modified Bessel Function of the First Kind, $I_0(x)$ has an integral representation

$$I_0(x) = \frac{1}{\pi} \int_{0}^{\pi} e^{x \cos(\phi)} \mathrm{d}\phi. \tag{11}$$

The azimuthal integral then can be carried out by noting that the integrand is symmetric about $\phi' = \pi$

$$\int_0^{2\pi} e^{2rr'\cos(\phi')/4\alpha t'} \mathrm{d}\phi' = 2\int_0^{\pi} e^{2rr'\cos(\phi')/4\alpha t'} \mathrm{d}\phi' = 2\pi I_0\left(\frac{2rr'}{4\alpha t'}\right).$$

The radian integral is then

$$2\pi \int_0^{\infty} \bar{E}(r') e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r'.$$

0.2.1 Flat Top Beams

For a flat top beam with radius R, we need to evaluate the integral

$$2\pi \int_0^{R} e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r'.$$

The integral can be carried out using the the Marcum Q-function, which is defined as

$$Q_\nu(a, b) = 1 - \frac{1}{a^{\nu-1}} \int_0^b x^\nu e^{-\frac{x^2+a^2}{2}} I_{\nu-1}(ax) \mathrm{d}x.$$

In our case, $\nu = 1$. To cast our integral into this form, let $x = r'/\sqrt{2\alpha t'}, a = r/\sqrt{2\alpha t'}$, which gives

$$2\pi 2\alpha t' \int_0^{R/\sqrt{2\alpha t'}} e^{-a^2/2} e^{-x^2/2} I_0(ax) x \mathrm{d}x = 4\pi\alpha t' \Big(1 - Q_1(r/\sqrt{2\alpha t'}, R/\sqrt{2\alpha t'})\Big)$$

Plugging this into Equation 1 gives

$$\begin{aligned} T(r=0, z, t) =& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a z} \int_0^t e^{\alpha t' \mu_a^2} \\ &\times \left[\operatorname{erf}\left(\frac{z_0 + d - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) - \operatorname{erf}\left(\frac{z_0 - z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a \right) \right] \\ &\times \Big[1 - Q_1\Big(r/\sqrt{2\alpha t'}, R/\sqrt{2\alpha t'}\Big) \Big] \mathrm{d}t' \end{aligned} \tag{12}$$

0.2.2 Gaussian Beams

For a Gaussian beam with 1/e radius σ we need to evaluate

$$2\pi \int_0^{\infty} e^{-\frac{r'^2}{\sigma^2}} e^{-\frac{r^2}{4\alpha t'}} e^{-\frac{r'^2}{4\alpha t'}} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r' = 2\pi e^{-\frac{r^2}{4\alpha t'}} \int_0^{\infty} e^{-(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}) r'^2} I_0\left(\frac{2rr'}{4\alpha t'}\right) r' \mathrm{d}r'.$$

The integral can be cast into a standard form and evaluated ["Table of Integrals, Series, and Products", Gradshteyn and Ryzhik pg. 707],

$$\int_0^{\infty} x e^{-\alpha x^2} I_\nu(\beta x) J_\nu(\gamma x) \mathrm{d}x = \frac{1}{2\alpha} \exp\left(\frac{\beta^2 - \gamma^2}{4\alpha}\right) J_\nu\left(\frac{\beta\gamma}{2\alpha}\right)$$

In our case, $\nu = \gamma = 0$, $\beta = 2r/4\alpha t'$, and $\alpha = \left(\frac{1}{\sigma^2} + \frac{1}{4\alpha t'}\right)$, which gives

$$
\begin{aligned}
2\pi e^{-\frac{r^2}{4\alpha t'}} \int_0^\infty e^{-\left(\frac{1}{\sigma^2}+\frac{1}{4\alpha t'}\right)r'^2} I_0\left(\frac{2rr'}{4\alpha t'}\right) r'\mathrm{d}r' &= 2\pi e^{-\frac{r^2}{4\alpha t'}} \frac{1}{2\left(\frac{1}{\sigma^2}+\frac{1}{4\alpha t'}\right)} e^{\left(2r/4\alpha t'\right)^2/4\left(\frac{1}{\sigma^2}+\frac{1}{4\alpha t'}\right)} \\
&= 4\pi\alpha t' e^{-\frac{r^2}{4\alpha t'}} \frac{1}{1+\frac{4\alpha t'}{\sigma^2}} e^{\frac{r^2}{4\alpha t'}\frac{1}{1+\frac{4\alpha t'}{\sigma^2}}} \\
&= 4\pi\alpha t' \frac{1}{1+\frac{4\alpha t'}{\sigma^2}} e^{\frac{r^2}{4\alpha t'}\left(\frac{1}{1+\frac{4\alpha t'}{\sigma^2}}-1\right)} \\
&= 4\pi\alpha t' \frac{\sigma^2}{\sigma^2+4\alpha t'} e^{\frac{-r^2}{\sigma^2+4\alpha t'}}
\end{aligned}
$$

Plugging this into Equation 2 gives

$$
\begin{aligned}
T(r=0,z,t) =& \frac{\mu_a E_0}{2\rho c} e^{-\mu_a z} \int_0^t e^{\alpha t' \mu_a^2} \\
&\times \left[\operatorname{erf}\left(\frac{z_0+d-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right) - \operatorname{erf}\left(\frac{z_0-z}{\sqrt{4\alpha t'}} + \sqrt{\alpha t'}\mu_a\right)\right] \\
&\times \left[\frac{\sigma^2}{\sigma^2+4\alpha t'} e^{\frac{-r^2}{\sigma^2+4\alpha t'}}\right] \mathrm{d}t'
\end{aligned} \tag{13}
$$

0.3 Pulsed Exposures

The temperature rises caused by a CW exposure is given by integrating the Green's Function,

$$
\Delta T(t) = \int_0^t G(t')\mathrm{d}t'. \tag{14}
$$

Because the temperature rise is linear, we can use the CW temperature rise to compute the temperature rise caused by a pulsed exposure,

$$
\Delta T(t) = \begin{cases} \int_0^t G(t')\mathrm{d}t' & t \leq \tau \\ \int_0^t G(t')\mathrm{d}t' - \int_0^{t-\tau} G(t')\mathrm{d}t' & t > \tau. \end{cases} \tag{15}
$$

After the pulse, the temperature rise is given by the difference between the CW temperature rise at the times t and $t-\tau$. We can either calculate $\Delta T(t)$ and $\Delta(t-\tau)$, or evaluate the integral over a different set of limits,

$$
\int_0^t G(t')\mathrm{d}t' - \int_0^{t-\tau} G(t')\mathrm{d}t' = \Delta T(t) - \Delta T(t-\tau) = \int_{t-\tau}^t G(t')\mathrm{d}t' \tag{16}
$$