Research Note

An Application of the Empirical Bayes Approach to Directly Adjusted Rates: A Note on Suicide Mapping in California

Kung-Jong Lui, PhD,
San Diego State University
Beverly Martinez, PhD, and James Mercy, PhD
Centers for Disease Control

ABSTRACT: A simple, reliable, and comparable measure for suicide mapping and other health problems is needed. Because standardized mortality ratios (SMRs) may not indicate the relative meaning of their magnitudes when compared with one another, and statistical significance levels of tests for SMRs overlook the areas that have small populations, neither of these approaches provides a satisfactory index. The results using directly adjusted rates can be ordered directly according to their magnitudes. However, because of the lack of reliable estimates of local age-specific rates, the usefulness of directly adjusted rates in mapping suicide is also limited. To extend the usefulness of directly adjusted rates, an empirical Bayes approach whereby information from other areas is borrowed to improve the precision of the estimates of local age-specific rates in calculating directly adjusted rates—especially in the areas with small population sizes—is proposed. When an empirical Bayes approach was applied to the 1983 suicide data for California counties, a more reasonable conclusion than could be obtained by using directly adjusted rates was reached.

Suicide is recognized today as an important public health problem in the United States. Each year, approximately 30,000 people take their

Requests for reprints should be sent to Dr. Kung-Jong Lui, Associate Professor, Department of Mathematical Sciences, San Diego State University, San Diego, CA 92182. Beverly Martinez is a Mathematical Statistician and James Mercy is Assistant Director for Sciences, Division of Injury Epidemiology and Control, Center for Environmental Health and Injury Control, Centers for Disease Control, Public Health Service, U.S. Department of Health and Human Services, Atlanta, GA 30333. We wish to thank the three referees for reviewing the work; D. McGee, R. Waxweiler, and L. Annest for their suggestions and administrative support; and M. Kirk for graphic assistance.

own lives, making suicide the eighth leading cause of death in the United States (Centers for Disease Control, 1987). There is a growing need for strategies that can help public and mental health decision makers appropriately target scarce resources for suicide prevention toward those persons and places in greatest need of intervention. Suicide mapping can be a valuable tool for this task (Ashford & Lawrence, 1976; Boor, 1981; Farmer, Preston, & O'Brien, 1977; Jarvis, Ferrence, Whitehead, & Johnson, 1982; Koller & Cotgrove, 1976; Lester, 1980; Miller, 1980; Moens, 1984; Morgan, Pocock, & Pottle, 1975). However, it has been difficult to choose an appropriate statistical measure that is reliable and correctly indicates the relative magnitudes of a suicide rate across relatively small geopolitical units such as counties. In this paper, we apply an empirical Bayes approach to map county suicide data in California and compare findings using this approach to those based on directly adjusted and Louis's empirical Bayes adjusted rates (Louis, 1984).

Background

Direct and indirect adjustment are the most commonly used methods for comparing mortality (or morbidity) rates across geopolitical units. These methods, however, are subject to certain limitations, particularly when applied to relatively small geopolitical units such as counties. Using the direct-adjustment method allows investigators to make valid comparisons of rates among counties in less restricted situations than using the indirect-adjustment method does (Freeman & Holford, 1980). Because the population sizes in counties are often found to be small in age-specific categories, however, the general inaccessibility of reliable local age-specific rates limits the usefulness of direct adjustment. On the other hand, using the indirect-adjustment method produces standardized mortality (or morbidity) ratios (SMRs) derived from more reliable standard age-specific rates than those of local areas. The resulting SMRs, however, may not be compared with one another directly (Freeman & Holford, 1980; Tsai & Wen, 1986). Presenting SMRs in mapping might give a misleading impression. Furthermore, using statistical significance levels based on local deviations of risk from the overall rates on the map would tend to overlook the local areas that might have moderately large numbers of suicides but only small population sizes. Areas with large populations would have better chances of being statistically significant than would areas with small populations, even if they have the same relative risk to the standard population.

The empirical Bayes approach to calculating age-adjusted mortality (or morbidity) rates provides an alternative that overcomes some of the limitations of the direct- and indirect-adjustment procedures. Based on ideas similar to those by James and Stein (1961), Efron and Morris (1973), Clayton and Kaldor (1987), Lui and Cumberland (1989), Tsutakawa, Shoop, and Marienfeld (1985), Manton et al. (1987), and Chiang (1968), the local age-specific rate can be estimated by borrowing information from other relative counties to improve the precision of the calculation. Furthermore, because Bayes or empirical Bayes adjustment using the posterior means may sometimes shrink the observed data too far toward the prior mean, to avoid this shrinkage we also have applied Louis's empirical Bayes adjustment (Louis, 1984) to the suicide data in this report. Note that Tsutakawa et al. (1985) assumed that the logit of the local age-specific rate followed a normal distribution. Because a rate is not a probability (Chiang, 1961) and can be larger than 1, however, we need to be careful in applying their model when the diseases under study may have rates larger than 1. The Bayes model proposed in this report is similar to that discussed by Manton et al. (1987). They did not, however, discuss the directly adjusted rate of suicide.

Statistical Methods

For a given u county ($u = 1, 2, \ldots, N$) and x age category, we assume that the number of suicides D_{ux} follows a Poisson distribution with parameter $P_{ux}\, M_{ux}$, where P_{ux} is the midyear population and M_{ux} is the age-specific rate. The maximum likelihood estimate of M_{ux} is $\hat{M}_{ux} = D_{ux}/P_{ux}$, which is the usual estimate of the age-specific rate, suggested by Chiang (1961). To borrow information from other counties to improve the estimate $\hat{M}_{ux}$ when P_{ux} is small, we assume that M_{ux} follows a conjugate prior distribution (Cox & Hinkely, 1974; Morris, 1983)—a gamma distribution, with shape parameter α_x and scale parameter β_x. In other words, the probability density of M_{ux} is given by the following equation:

$$f(M_{ux}) = \frac{\beta_x{}^{\alpha_x}\, e^{-\beta_x M_{ux}}\, M_{ux}^{\alpha_x - 1}}{\Gamma(\alpha_x)}$$

Some theoretical considerations and practical justification for choosing the conjugate prior distribution to be a gamma distribution can be found elsewhere (Manton et al., 1987; Morris, 1983). As shown in Appendix I, the posterior distribution of M_{ux} is a gamma distribution with shape parameter $D_{ux} + \alpha_x$ and scale parameter $P_{ux} + \beta_x$, and therefore the posterior mean of M_{ux} is given by this equation:

$$E(M_{ux}|D_{ux}) = W_{ux}\,\hat{M}_{ux} + (1 - W_{ux})\,(\alpha_x/\beta_x) \qquad (1)$$

where $W_{ux} = P_{ux}/(P_{ux} + \beta_x)$. Note that when P_{ux} is small, we will put more weight to the prior mean (α_x/β_x) in Equation 1. Particularly, if P_{ux} is equal to 0, then we will estimate M_{ux} by the prior mean α_x/β_x instead of 0. Similarly, if β_x is close to 0, this implies that the prior distribution has a very large variance; the posterior mean would be close to the local age-specific rate $\hat{M}_{ux}$.

Because α_x and β_x are generally unknown, we cannot apply the posterior mean immediately in calculating directly adjusted rates. The log-likelihood of the marginal density of D_{ux} can be, however, used to obtain the maximum likelihood estimates $\hat{\alpha}_x$ and $\hat{\beta}_x$. We use the Newton–Raphson technique (Gross & Clark, 1975) to obtain the maximum likelihood estimates of α_x and β_x. Substituting these estimates $\hat{\alpha}_x$ and $\hat{\beta}_x$ for the corresponding parameters in Equation 1 gives the empirical Bayes estimate of the posterior mean of M_{ux} (Cox & Hinkely, 1974). Then we can apply the empirical Bayes estimate of M_{ux} to calculate the directly adjusted rates (Mausner & Bahn, 1974). The details of the procedure for deriving the maximum likelihood estimates of α_x and β_x are briefly discussed in Appendix I as well. Furthermore, to avoid Bayes or empirical Bayes estimates by using posterior means that may shrink the observed data too far toward the prior mean, we have also applied Louis's procedure, which adjusts this shrinkage so that the sample mean and variance of the estimates match the posterior expected mean and variance for parameters. The details of Louis's estimates under our model assumption—Poisson gamma instead of the normal distribution (Louis, 1984)—are briefly presented in Appendix II. The appropriateness of the assumed model is checked by applying the goodness-of-fit test (Hasselblad, Stead, & Anderson, 1981) for D_{ux} for each age-specific category.

Data and Results

The suicide data were taken from the Mortality Surveillance System (MSS), prepared by the National Center for Health Statistics (NCHS), for 58 counties of California in 1983. After excluding five cases with unknown ages from our consideration, we identified a total of 3,630 suicides from a population of 25 million in 1983. Table 1 gives the age distribution of the California population and the number of suicides in all counties.

Taking the California population as the standard population, we calculated the directly adjusted rates and the empirical Bayes directly

TABLE 1. Age Distribution of Suicides and of Population in California, 1983

Age	Suicides		Population		Rate of Suicides in Population (per 100,000)	Prior Mean Estimate (per 100,000)	Age Distribution
	Total	Average	Total	Average			
0–14	24	0.1	5,404,667	23,296.0	0.4	0.6	0.22
15–24	535	9.2	4,376,788	75,461.9	12.5	12.5	0.17
25–34	854	14.7	4,691,678	80,891.0	18.2	18.6	0.19
35–44	591	10.2	3,320,576	57,251.3	17.8	17.8	0.13
34–44	437	7.5	2,385,373	41,127.1	18.3	19.1	0.10
55–64	462	8.0	2,285,626	39,407.3	20.2	20.2	0.09
65–74	361	6.2	1,505,247	27,331.8	22.8	22.8	0.06
75–84	283	4.9	790,166	13,623.6	35.8	38.1	0.03
Over 85	83	1.4	247,436	4,266.1	33.5	37.1	0.01
Total	3,630	62.6	25,087,557	432,544			

adjusted rates for 58 counties; these are summarized in Table 2. The counties were then ranked according to these resulting rates in descending order. The counties with the five highest directly adjusted suicide rates were Lassen, Shasta, Yuba, Trinity, and Tehama, while San Francisco, Riverside, Shasta, Sonoma, and Santa Cruz Counties were the five highest after either the empirical Bayes method or Louis's empirical Bayes method was used to adjust local age-specific rates. The differences in the locations of these counties, based on the directly adjusted and empirical Bayes adjusted rates, are presented in Figure1. Shasta County is the only county that belongs to the hot spot on the basis of any of the three methods (Figure 1). In general, the ranks on the basis of the empirical Bayes and Louis's empirical Bayes methods are quite similar (Table 2). The goodness-of-fit test showed that negative-binomial distributions adequately described D_{ux} for each age category except for ages less than 15, in which the number of counties with suicides was relatively small.

Discussion

The directly adjusted rates of suicide in California in 1983 ranged from 0 to 33.7 deaths per 100,000 population, while the empirical Bayes estimates ranged from 12.8 to 15.7. The higher variability in directly adjusted rates comes from smaller counties, where the occurrence of a few deaths magnifies directly adjusted suicide rates. In Table 2, for example, based on directly adjusted rates, the counties with the five highest rates were Lassen, Shasta, Yuba, Trinity, and Tehama. These counties, except for Shasta County, all had relatively small populations (about 50,000 or less) as compared with the average county population of approximately 433,000 in California. Similarly, the lowest directly adjusted rates were also found in counties with very small populations; both Alpine and Sierra Counties, which had directly adjusted rates equal to 0, have populations of less than 3,500 people. Using directly adjusted rates would have indicated that Lassen, Shasta, Yuba, Trinity, and Tehama Counties, which are all suburban counties located in northern California, should be targeted for suicide intervention. The directly adjusted rates in these counties, however, were certainly not so reliable as those in counties with large populations. Based on the resulting rates from the empirical Bayes approach, the counties of San Francisco, Riverside, Shasta, Sonoma, and Santa Cruz had the five highest suicide rates. All these counties are urban areas and have populations larger than 100,000. This conclusion seems to be more reasonable, because suicides are more likely to occur in urban than in

TABLE 2. Population Sizes, Number of Suicides, Direct Age-Adjusted Rates, Empirical Bayes Age-Adjusted Rates, and Louis's Bayes Age-Adjusted Rates by Counties (per 100,000) in California, 1983

County Name	Population Size	Number of Suicides	Direct Age-Adjusted Rates	Ranks[a]	Empirical Bayes Age-Adjusted Rates	Ranks[b]	Louis's Bayes Age-Adjusted Rates	Ranks[c]
Alpine	1,273	0	0.0000	57	14.7760	31	14.7631	31
Sierra	3,450	0	0.0000	57	14.7663	32	14.7349	34
Modoc	9,524	3	24.9082	6	14.8355	20	14.9401	19
Mono	9,775	1	11.5954	47	14.7806	30	14.7813	29
Mariposa	12,404	3	16.7682	16	14.7233	41	14.6336	41
Trinity	12,696	3	25.8232	4	14.8831	13	15.0690	11
Calusa	14,210	2	13.6768	38	14.8271	21	14.8978	21
Del Norte	18,468	4	20.7442	8	14.8045	24	14.8479	23
Plumas	18,606	2	9.7038	53	14.6965	45	14.5486	45
Inyo	18,622	3	10.0693	52	14.7427	38	14.6745	38
Amador	21,244	3	16.6091	18	14.7427	39	14.6658	39
Glenn	22,422	3	14.5869	29	14.7975	27	14.8111	28
Lassen	24,063	8	33.7424	1	14.8132	22	14.8965	22
Calaveras	24,467	3	8.9671	54	14.7163	42	14.6048	42
San Benito	27,634	3	11.5784	48	14.7660	33	14.7285	35
Tuolumne	37,117	4	10.8617	51	14.7127	43	14.5849	44
Tehama	41,993	11	24.9485	5	14.8523	18	14.9798	17
Siskiyou	41,998	9	20.1416	9	14.8595	17	15.0048	15
Lake	42,742	10	23.9498	7	14.9796	7	15.3129	7
Yuba	51,355	13	27.0643	3	14.7517	36	14.7401	33
Sutter	56,530	6	11.2115	49	14.8501	19	14.9383	20
Nevada	63,481	6	8.5076	55	14.6347	46	14.3885	46
Mendocino	70,457	10	13.9120	35	14.8025	26	14.8136	27
Medera	71,359	11	16.4067	19	14.7957	28	14.8180	26
Kings	80,277	11	14.5445	30	14.8764	14	15.0023	16
El Dorado	97,129	8	8.3873	56	14.7350	40	14.6473	40
Imperial	100,368	14	15.1351	25	14.7110	44	14.6027	43

Napa	101,619	15	13.5255	41	14.8060	23	14.8428	24
Humboldt	110,519	18	15.9205	21	14.9048	11	15.0880	10
Yolo	118,305	22	19.9084	10	14.9055	10	15.1620	9
Shasta	123,750	34	27.3140	2	15.2728	3	16.1623	3
Placer	129,734	24	17.9671	14	14.8854	12	15.0398	13
Merced	146,204	19	14.3224	31	14.5794	50	14.2446	52
Butte	155,682	21	12.1462	46	14.5946	49	14.2693	50
San Luis Obispo	171,535	26	14.1359	34	14.4869	53	14.0099	54
Santa Cruz	199,150	40	18.6600	12	15.1303	5	15.7471	5
Marin	223,219	41	16.9316	15	14.7618	34	14.7804	30
Solano	262,145	39	15.6075	22	14.7541	35	14.6905	37
Tulare	264,029	32	12.9542	44	14.6276	48	14.3745	48
Stanislaus	286,795	30	10.8620	50	14.5397	52	14.0952	53
Monterey	311,394	37	12.7570	45	14.2688	57	13.3972	58
Santa Barbara	313,217	50	15.4981	23	14.6291	47	14.3793	47
Sonoma	318,472	55	16.7368	17	15.2602	4	16.0799	4
San Joaquin	383,828	52	13.6486	39	14.8744	16	15.0384	14
Kern	448,882	68	15.9833	20	14.8039	25	14.8412	25
Fresno	545,043	72	13.5311	40	14.9081	9	14.9535	18
Ventura	569,934	80	14.9810	27	14.9538	8	15.2289	8
San Mateo	592,312	87	14.1757	33	14.7879	29	14.6941	36
Contra Costa	686,614	105	15.1582	24	14.7488	37	14.7553	32
San Francisco	701,288	156	19.2477	11	15.7480	1	17.5259	1
Riverside	737,690	134	18.1485	13	15.5309	2	16.8346	2
Sacramento	850,100	127	14.9551	28	15.0332	6	15.4495	6
San Bernardino	997,413	131	13.8799	36	14.3936	55	13.8444	55
Alameda	1,157,259	164	13.8328	37	14.2837	56	13.5381	56
Santa Clara	1,344,883	178	13.3061	42	14.5767	51	14.2584	51
San Diego	2,007,338	301	15.0582	26	14.8746	15	15.0526	12
Orange	2,050,989	288	14.2099	32	14.4619	54	14.2702	49
Los Angeles	7,784,551	1,030	13.2879	43	13.8035	58	13.4024	57

[a] Ranks based on direct age-adjusted rates in descending order.
[b] Ranks based on empirical Bayes age-adjusted rates in descending order.
[c] Ranks based on Louis's Bayes -adjusted rates in descending order.

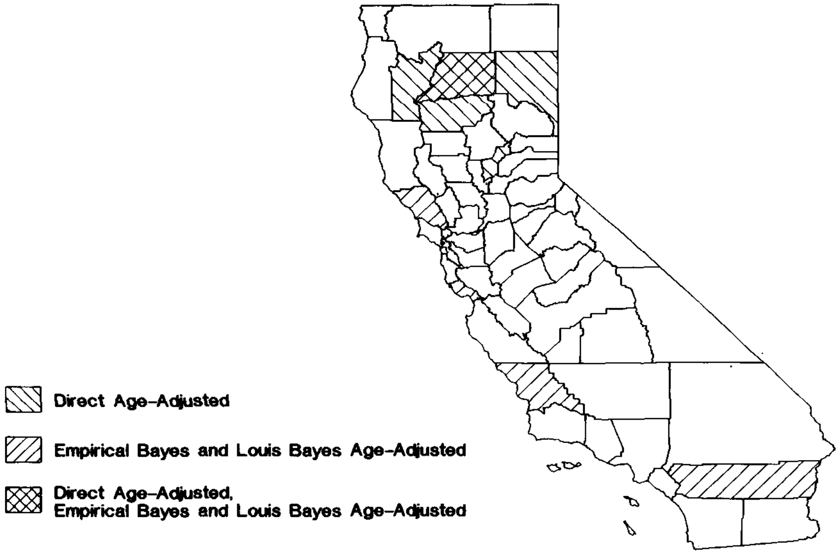

Figure 1. The counties with the five largest suicide rates in California, based on the direct-adjustment, empirical Bayes, and Louis's empirical Bayes methods, 1983.

rural areas (Jarvis et al., 1982; Koller & Cotgrove, 1976; Morgan et al., 1975).

To allocate the limited resources of a society to hot spots of suicides, we must identify which area has an unusually high rate. Suicide mapping is certainly useful for this purpose, and the empirical Bayes approach is superior to the corresponding SMR and direct-adjustment approaches because it allows direct and more reliable comparisons of age-adjusted suicide rates across local areas. This means that when the Bayes approach is used, decisions for the allocation of scarce resources across geopolitical units will be less likely to be unduly influenced by inappropriate comparisons or by spuriously high or low age-adjusted rates for small population areas. The results presented here will surely be helpful to sociologists and epidemiologists in planning interventions for suicide in the future.

APPENDIX I

For a given u county and x age category, under the model assumptions as described in the "Statistical Methods" section of this paper, the joint density of D_{ux} and M_{ux} is given by this equation:

$$f(D_{ux}, M_{ux}) = f(D_{ux}|M_{ux})\, f(M_{ux})$$

$$= \frac{e^{-P_{ux}M_{ux}}\,(P_{ux}\,M_{ux})^{D_{ux}}}{D_{ux}!}\;\frac{\beta_x^{\alpha_x}\, e^{-\beta_x M_{ux}}\, M_{ux}^{\alpha_x-1}}{\Gamma(\alpha_x)}$$

Therefore, the marginal probability density of D_{ux} equals the following:

$$f(D_{ux}) = \int f(D_{ux}, M_{ux})\, dM_{ux}$$

$$= \frac{\Gamma(D_{ux} + \alpha_x)}{\Gamma(\alpha_x)\, D_{ux}!}\left(\frac{\beta_x}{P_{ux} + \beta_x}\right)^{\alpha_x}\left(\frac{P_{ux}}{P_{ux} + \beta_x}\right)^{D_{ux}}$$

Then the posterior density of M_{ux} for a given D_{ux} is as follows:

$$f(M_{ux}|D_{ux}) = \frac{f(D_{ux}, M_{ux})}{f(D_{ux})}$$

$$= \frac{(P_{ux} + \beta_x)^{\alpha_x+D_{ux}}\, e^{-(P_{ux}+\beta_x)M_{ux}}\, M_{ux}^{\alpha_x+D_{ux}-1}}{\Gamma(\alpha_x + D_{ux})}$$

This is a gamma distribution with shape parameter $D_{ux} + \alpha_x$ and scale parameter $P_{ux} + \beta_x$. Thus, the posterior mean of M_{ux} for given D_{ux} equals

$$E(M_{ux} \mid D_{ux}) = \frac{\alpha_x + D_{ux}}{P_{ux} + \beta_x} = W_{ux}\,\hat{M}_{ux} + (1 - W_{ux})(\alpha_x/\beta_x)$$

where $\hat{M}_{ux} = D_{ux}/P_{ux}$ and where $W_{ux} = P_{ux}/(P_{ux} + \beta_x)$.

If α_x and β_x were known, then the Bayes approach by using the posterior mean $E(M_{ux} \mid D_{ux})$, as given above, could be used to estimate M_{ux}. However, we often do not know what the values of α_x and β_x are in practice and are required to estimate these parameters from the data. This thought leads to the empirical Bayes approach, in which we substitute the estimates $\hat{\alpha}_x$ and $\hat{\beta}_x$ for α_x and β_x in the formula $W_{ux}\,\hat{M}_{ux} + (1 - W_{ux})(\alpha_x/\beta_x)$ to estimate M_{ux}.

As suggested by Cox and Hinkley (1974), we can use the log-posterior likelihood on the basis of the data on D_{ux} to derive the maximum likelihood estimates of the parameters α_x and β_x. The details are summarized as follows:

For a given x age category, the log-posterior likelihood is the following:

$$L = \log\left(\prod_u f(D_{ux})\right)$$

$$= \log\left(\prod_u \frac{\Gamma(D_{ux} + \alpha_x)}{\Gamma(\alpha_x)\, D_{ux}!}\left(\frac{\beta_x}{P_{ux} + \beta_x}\right)^{\alpha_x}\left(\frac{P_{ux}}{P_{ux} + \beta_x}\right)^{D_{ux}}\right)$$

Furthermore, the likelihood above can be shown to equal

$$C + \sum_u [\log \Gamma(D_{ux} + \alpha_x) - \log \Gamma(\alpha_x) + \alpha_x \log(\beta_x)$$

$$- (D_{ux} + \alpha_x) \log(P_{ux} + \beta_x)]$$

where C is a constant that does not involve the parameters of α_x and β_x. Therefore, to obtain the maximum likelihood estimates of α_x and β_x, we want to solve the following two equations:

$$\delta L/\delta\alpha_x = \sum_u \sum_{j=0}^{D_{ux}-1} 1/(\alpha_x + j) + N \log(\beta_x)$$

$$- \sum_u \log(P_{ux} + \beta_x) = 0, \text{ and}$$

$$\delta L/\delta\beta_x = \sum_u [\alpha_x/\beta_x - (D_{ux} + \alpha_x)/(P_{ux} + \beta_x)] = 0$$

Note that if $D_{ux} = 0$, then

$$\sum_{j=0}^{D_{ux}-1} 1/(\alpha_x + j) = 0.$$

We use the Newton–Raphson numerical method (Gross & Clark, 1975) with the following results to obtain the maximum likelihood estimates of α_x and β_x:

$$\delta^2 L/\delta\alpha_x^2 = -\sum_u \sum_{j=0}^{D_{ux}-1} 1/(\alpha_x + j)^2$$

$$\delta^2 L/\delta\beta_x^2 = \sum_u [-\alpha_x/\beta_x^2 + (D_{ux} + \alpha_x)/(P_{ux} + \beta_x)^2]$$

$$\delta^2 L/\delta\alpha_x\delta\beta_x = \sum_u [1/\beta_x - 1/(P_{ux} + \beta_x)]$$

We have used several different initial estimates, including the moment estimates of α_x and β_x, to assure that the final estimates are the maximum likelihood estimates.

APPENDIX II

For a fixed x age category, let $u_x = (\alpha_x / \beta_x)$. Then the posterior mean of M_{ux} for a given D_{ux} (see Appendix I) can be rewritten as

$$M^{B}_{ux} = E(M_{ux} \mid D_{ux}) = u_x + W_{ux} (\hat{M}_{ux} - u_x)$$

where $W_{ux} = P_{ux}/(P_{ux} + \beta_x)$. Let $M^{B}_{x.} = \sum_u M^{B}_{ux} / N$ and $M_{x.} = \sum_u M_{ux}/N$. Note that $E(M_{x.} \mid D_{ux}) = M^{B}_{x.}$. The Louis's modified estimate, M^{L}_{ux}, is defined as $\xi_x + F_x W_{ux} (\hat{M}_{ux} - \xi_x)$ such that $E(M_{x.} | D_{ux}) = M^{L}_{x.}$ and $E(\sum_u (M_{ux} - M_{x.})^2 | D_{ux}) = \sum_u (M^{L}_{ux} - M^{L}_{x.})^2$ where $M^{L}_{x.} = \sum_u M^{L}_{ux}/N$. Therefore, we need to find ξ_x and F_x. However, it is easy to show that

$$E(\sum_u (M_{ux} - M_{x.})^2 | D_{ux}) = E(\sum_u M^2_{ux} | D_{ux}) - E(\sum_u M_{ux})^2/N | D_{ux})$$

where $E(\sum_u M^2_{ux} | D_{ux}) = \sum_u (\mathrm{Var}(M_{ux} | D_{ux}) + (E(M_{ux} | D_{ux}))^2)$ and where $E((\sum_u M_{ux})^2 | D_{ux}) = \sum_u \mathrm{Var}(M_{ux} | D_{ux}) + (\sum_u E(M_{ux} | D_{ux}))^2$. As shown in Appendix I, $f(M_{ux} | D_{ux})$ is distributed as a gamma $(D_{ux} + \alpha_x, P_{ux} + \beta_x)$, and therefore $\mathrm{Var}(M_{ux} | D_{ux})$ and $E(M_{ux} | D_{ux})$ can then be easily obtained.

Furthermore, note that the marginal probability density $f(D_{ux})$ of D_{ux} is as follows:

$$\frac{\Gamma(D_{ux} + \alpha_x)}{\Gamma(\alpha_x)\, D_{ux}!} \left(\frac{\beta_x}{P_{ux} + \beta_x}\right)^{\alpha_x} \left(\frac{P_{ux}}{P_{ux} + \beta_x}\right)^{D_{ux}}$$

Therefore, we have $\mathrm{Var}(D_{ux}/P_{ux}) = (\alpha_x/B^2_x)/W_{ux}$ (i.e., the variance of $\hat{M}_{ux}$ is proportional to the inverse of the weight W_{ux}). On the basis of this result, as suggested by Louis (1984), we can use the optimal weighted average

$$\hat{u}_x = C_x \sum_u \hat{W}_{ux} \hat{M}_{ux})$$

where $C_x^{-1} = \sum_u \hat{W}_{ux}$, and where $\hat{W}_{ux} + P_{ux}/(P_{ux} + \hat{\beta}_x)$, to estimate u_x.

After some algebraic manipulation, we have obtained the following results:

$$\hat{\xi}_x = \hat{u}_x \text{ and } F_x^2 = 1 + \frac{(N-1)}{N} \sum_u [\hat{\xi}_x + Y_u(\hat{\xi}_x)]/(P_{ux} + \hat{\beta}_x)/\sum_u Y_u(\hat{\xi}_x)^2$$

where $Y_u(\hat{\xi}_x) = \hat{W}_{ux}(\hat{M}_{ux} - \hat{\xi}_x)$. Note that F_x is always greater than 1. Note also that in the Louis's estimator of $\xi_x + F_x W_{ux} (M_{ux} - \xi_x)$, if $F_x W_{ux}$ is greater than 1, we will set $M^L_{ux} = \hat{M}_{ux}$.

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