分支过程

模型介绍

1. 设ξ是取值非负整数的随机变量,
$P(\xi = k) = p_k, k = 0, 1, \cdots$, 这里$p_0 < 1$.

2. 一个祖先$Z_0 = 1$.

3. 第一代个体数Z_1与ξ同分布.

4. Z_1个个体独立繁衍, 方式与祖先一致. 用$\xi_{1,j}$表示第1代第j个个体的后代数, 则$\{\xi_{1,j} : j = 1, 2, \cdots\}$独立同分布, 与$\xi$同分布, 且与$Z_1$独立.第二代个体数为:

$$Z_2 = \sum_{j=1}^{Z_1} \xi_{1,j}.$$

5. 令Z_n为第n代个体总数. 用$\xi_{n,j}$表示第n代第j个个体的后代数, 则$\{\xi_{n,j}: j=1,2,\cdots\}$独立同分布, 与$\xi$同分布, 且与$(Z_1,\cdots,Z_n)$独立.第$n+1$代个体数为:

$$Z_{n+1}=\sum_{j=2}^{Z_n}\xi_{n,j}.$$

...

Example

设$P(\xi=1)=p,\ P(\xi=0)=1-p,\ 0<p<1.$

则$P(Z_n=1)=p^n,\ P(Z_n=0)=1-p^n.$

Theorem

$\{Z_n; n \geq 0\}$ 是时齐$Markov$链，状态空间为$\{0, 1, \cdots\}$，

$$p_{ij} = P(\sum_{l=1}^{i} \xi_l = j),\ i, j \geq 0,$$

其中$\xi_1, \xi_2, \cdots$独立同分布且与ξ分布相同.

证明:

$$
\begin{aligned}
& P(Z_{n+1}=j|Z_0=1,Z_1=i_1,\cdots,Z_{n-1}=i_{n-1},Z_n=i) \\
= \ & P(\sum_{l=1}^{i}\xi_{n,l}=j|Z_0=1,Z_1=i_1,\cdots,Z_{n-1}=i_{n-1},Z_n=i) \\
= \ & P(\sum_{l=1}^{i}\xi_{n,l}=j).(\because(\xi_{n,1},\cdots,\xi_{n,i})\text{与}(Z_0,\cdots,Z_n)\text{独立}.)
\end{aligned}
$$

Example

设$\xi \sim B(n,p)$. 则在$Z_n = i$条件下,
$Z_{n+1} = \sum_{l=1}^{i} \xi_{n,l} \sim B(ni,p)$. 所以

$$p_{ij} = \binom{ni}{j} p^j (1-p)^{ni-j}, j = 0, 1, \cdots, ni.$$

Example

设$\xi \sim \pi(\lambda)$. 则在$Z_n = i$条件下,
$Z_{n+1} = \sum_{l=1}^{i} \xi_{n,l} \sim \pi(i\lambda)$. 所以

$$p_{ij} = e^{-i\lambda}\frac{(i\lambda)^j}{j!}, j = 0, 1, \cdots$$

Example

设

$$P(\xi = k) = (1-p)^k p, k = 0, 1, \cdots.$$

则

$$p_{ij} = P(\sum_{l=1}^{i} \xi_{n,l} = j) = \binom{i+j-1}{i-1}(1-p)^j p^i.$$

Example

设$p_0=0.2,\ p_1=0.3,\ p_2=0.2,\ p_3=0.2,\ p_4=0.1.$
求$P(Z_2=0)$和$P(Z_2=1)$.

解：

$$\begin{aligned}
P(Z_2=0) &= \sum_{k=0}^{4} P(Z_2=0|Z_1=k)P(Z_1=k)=\sum_{k=0}^{4} p_0^k p_k \\
&= 0.2+0.2\times 0.3+0.2^2\times 0.2 \\
&\quad +0.2^3\times 0.2+0.2^4\times 0.1 \\
&= 0.26976.
\end{aligned}$$

$$
\begin{aligned}
P(Z_2=1) &= \sum_{k=0}^{4} P(Z_2=1|Z_1=k)P(Z_1=k) \\
&= \sum_{k=1}^{4} k p_1 p_0^{k-1} p_k \\
&= p_1^2 + 2p_0p_1p_2 + 3p_0^2p_1p_3 + 4p_0^3p_1p_4 \\
&= 0.12216.
\end{aligned}
$$

Theorem

设$E\xi=\mu$, $\mathrm{Var}(\xi)=\sigma^2$. 那么对$n\geq 1$,

(1) $E(Z_n)=\mu^n$;

(2) $\mathrm{Var}(Z_n)=\sigma^2\mu^{n-1}(1+\mu+\cdots+\mu^{n-1})$.

证明: (1) $E(Z_1)=E\xi=\mu$. 若已证得$EZ_n=\mu^n$, 则

$$\begin{aligned} EZ_{n+1} &= E\Big(\sum_{k=1}^{Z_n}\xi_{n,j}\Big) \\ &= E\Big[E\Big(\sum_{k=1}^{Z_n}\xi_{n,j}\Big|Z_n\Big)\Big] \\ &= E(Z_n\mu)=\mu E(Z_n)=\mu^{n+1}. \end{aligned}$$

由归纳法, (1)对所有$n\geq 1$成立.

证明: (2) $\mathrm{Var}(Z_1) = \mathrm{Var}(\xi) = \sigma^2$. 若已证得(2)对$n$成立,
则

$$
\begin{aligned}
E(Z_{n+1}^2) &= E[E([\sum_{k=1}^{Z_n} \xi_{n,j}]^2 | Z_n)] \\
&= E(Z_n \sigma^2 + Z_n^2 \mu^2) = \sigma^2 E(Z_n) + \mu^2 E(Z_n^2)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(Z_{n+1}) &= E(Z_{n+1}^2) - [E(Z_{n+1})]^2 \\
&= \sigma^2 E(Z_n) + \mu^2 E(Z_n^2) - \mu^2 (EZ_n)^2 \\
&= \sigma^2 E(Z_n) + \mu^2 \mathrm{Var}(Z_n) \\
&= \sigma^2 \mu^n + \mu^2 \sigma^2 \mu^{n-1}(1 + \mu + \cdots + \mu^{n-1}) \\
&= \sigma^2 \mu^n (1 + \mu + \cdots + \mu^n).
\end{aligned}
$$

由归纳法, (2)对所有$n \geq 1$成立.

注：

(1) 当$\mu > 1$时，平均人数几何级数增长；

(2) 当$\mu = 1$时，平均人数恒为1；

(3) 当$\mu < 1$时，平均人数几何级数递减.

注： Z_n的分布律一般很难算.

生成函数

设ξ的生成函数为:

$$\phi(s) := E(s^\xi) = \sum_{k=0}^{\infty} p_k s^k, 0 \le s \le 1.$$

生成函数性质:(i) $0 \le \phi(s) \le 1$, $\phi(0) = p_0$, $\phi(1) = 1$, $\phi(s)$在$[0,1]$ 单调递增且一致连续.

(ii) 对正整数k, 若$E(\xi^k) < \infty$, 则

$$E[\xi(\xi-1)\cdots(\xi-k+1)] = \phi^{(k)}(1).$$

特别地若$E\xi < \infty$, 则$E\xi = \phi'(1)$;

(iii) 非负整数随机变量的分布律与生成函数一一对应:

$$p_k = \frac{\phi^{(k)}(0)}{k!}.$$

Example

设$\xi \sim Poi(\lambda)$, 则

$$\phi(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} s^k = e^{\lambda(s-1)},$$

$$E(\xi) = \phi'(1) = \lambda.$$

Example

设$P(\xi = k) = (1-p)^{k-1}p,\ k \geq 1$, 则

$$\phi(s) = \sum_{k=1}^{\infty}(1-p)^{k-1}ps^k = \frac{ps}{1-(1-p)s},$$

$$E(\xi) = \phi'(1) = 1/p.$$

令Z_n的生成函数为$\phi_n(s) = E(s^{Z_n})$.

Theorem

$$\phi_0(s) = s,$$

$$\phi_1(s) = \phi(s),$$

对$n \geq 1$,

$$\phi_{n+1}(s) = \phi_n(\phi(s)) = \phi(\phi_n(s)).$$

证明：

$$\phi_0(s) = E(s^{Z_0}) = E(s^1) = s,$$

$$\phi_1(s) = E(s^{Z_1}) = E(s^{\xi}) = \phi(s),$$

对$n \geq 1$，
$$\begin{aligned}
\phi_{n+1}(s) &= E(s^{Z_{n+1}}) = E[E(s^{\sum_{j=1}^{Z_n} \xi_{n,j}} | Z_n)] \\
&= E[(\phi(s))^{Z_n}] = \phi_n(\phi(s)) \\
&= \underbrace{\phi(\phi \cdots \phi}_{n+1}(s)) \\
&= \phi(\underbrace{\phi(\phi \cdots \phi}_{n}(s)) = \phi(\phi_n(s)).
\end{aligned}$$

Example

设$P(\xi=k)=\frac{1}{2^{k+1}}$, $k=0,1,\cdots$, 对$n\geq 1$, 计算(1) $\phi_n(s)$, (2) Z_n的分布律.

解:

$$\phi(s)=\sum_{k=0}^{\infty}\frac{1}{2^{k+1}}s^k=\frac{1}{2-s},\ 0\leq s\leq 1.$$

所以$\phi_1(s)=\phi(s)=\frac{1}{2-s}$,

$$\phi_2(s)=\phi(\phi_1(s))=\frac{1}{2-\phi_1(s)}=\frac{1}{2-\frac{1}{2-s}}=\frac{2-s}{3-2s}.$$

若已算得$\phi_n(s)=\frac{n-(n-1)s}{n+1-ns}$, 则

$$\begin{aligned}\phi_{n+1}(s) &= \phi(\phi_n(s))=\frac{1}{2-\phi_n(s)}\\ &= \frac{1}{2-\frac{n-(n-1)s}{n+1-ns}}\\ &= \frac{(n+1)-ns}{(n+2)-(n+1)s}.\end{aligned}$$

由归纳法知对$n\geq 1$, $\phi_n(s)=\frac{n-(n-1)s}{n+1-ns}$.

$$
\begin{aligned}
\phi_n(s) &= \frac{n-(n-1)s}{n+1-ns} \\
&= \frac{n-(n-1)s}{n+1}\frac{1}{1-\frac{n}{n+1}s} \\
&= \frac{n-(n-1)s}{n+1}\sum_{k=0}^{\infty}(\frac{n}{n+1}s)^k \\
&= \frac{n}{n+1}+\sum_{k=0}^{\infty}(\frac{n}{n+1}\frac{n^{k+1}s^{k+1}}{(n+1)^{k+1}}-\frac{n-1}{n+1}\frac{n^k s^{k+1}}{(n+1)^k}) \\
&= \frac{n}{n+1}+\sum_{k=1}^{\infty}\frac{n^{k-1}}{(n+1)^{k+1}}s^k.
\end{aligned}
$$

所以 $P(Z_n=0)=\frac{n}{n+1},\ P(Z_n=k)=\frac{n^{k-1}}{(n+1)^{k+1}}, k\geq 1.$

灭绝概率

设$0 < p_0 < 1$. 令$\alpha_n = P(Z_n = 0)$.

由于0是吸收态，所以α_n单调递增.

令$\tau := \lim_{n\to\infty} P(Z_n = 0)$，则$\tau = P(Z_n = 0 \text{ for some } n)$.

问题： 灭绝概率τ 为多少?

若$\mu < 1$, 则由Markov不等式,

$$P(Z_n \geq 1) \leq E(Z_n) = \mu^n \to 0.$$

所以$\lim_{n\to\infty} P(Z_n = 0) = 1$, 即$\tau = 1$.

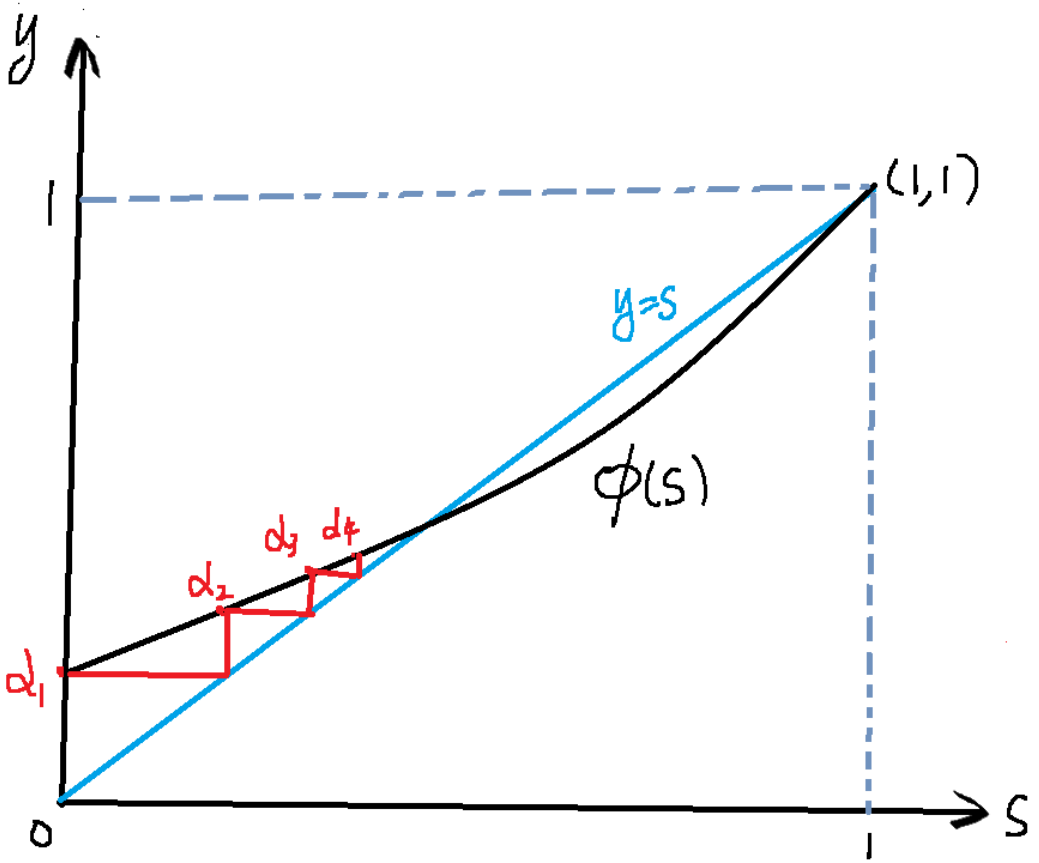
Theorem

(1) τ 是方程$s=\phi(s)$的最小正解.
(2) $\tau=1$当且仅当$\mu\leq 1$.

证明：(1)

$$\alpha_{n+1}=\phi_{n+1}(0)=\phi(\phi_n(0))=\phi(\alpha_n).$$

令$n\to\infty$，由$\phi(s)$在$[0,1]$连续推得$\tau=\phi(\tau)$.



令s_0是方程$s = \phi(s)$的最小非负解, 则$\tau \geq s_0$.

由于$\phi(0) = p_0 > 0$和$\phi(1) = 1$, 所以$0 < s_0 \leq 1$, s_0是最小正解, 1是方程的解. 因为$s_0 \geq 0$, 所以$s_0 = \phi(s_0) \geq \phi(0) = \alpha_1$. 若已证得$s_0 \geq \alpha_n$,

则$s_0 = \phi(s_0) \geq \phi(\alpha_n) = \alpha_{n+1}$. 因此$s_0 \geq \alpha_n$对所有$n$成立. 所以$s_0 \geq \lim_{n\to\infty} \alpha_n = \tau$.

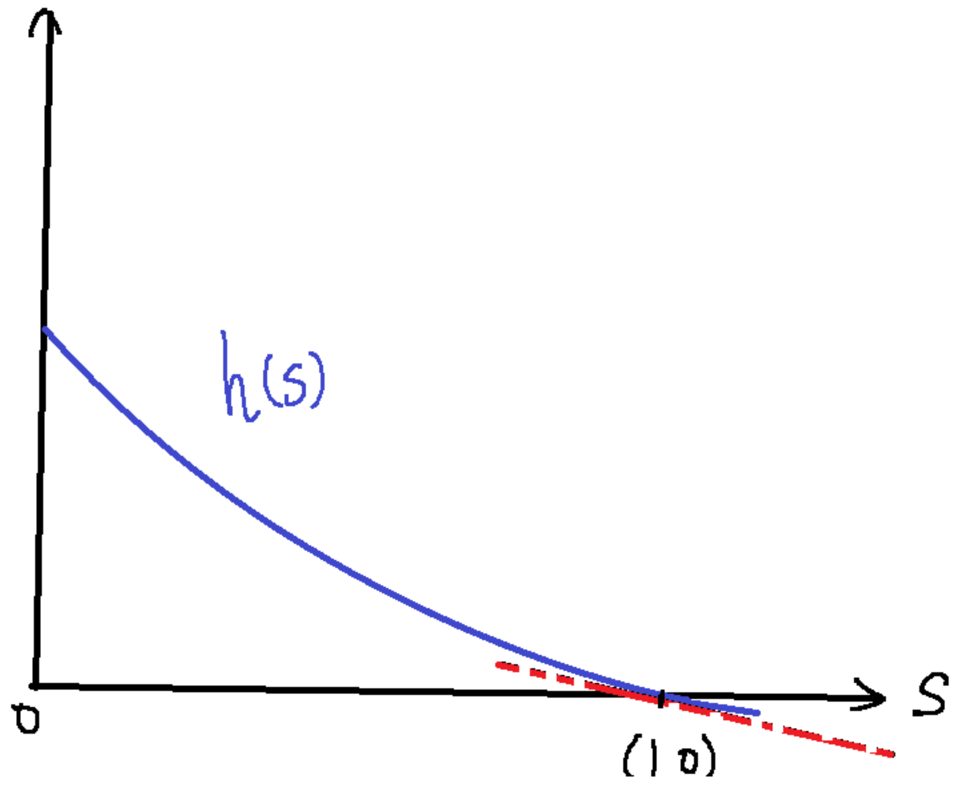
这就证明了$\tau = s_0$.

(2) 若$p_0+p_1=1$, 则$\mu=p_1<1$, 所以$\tau=1$.

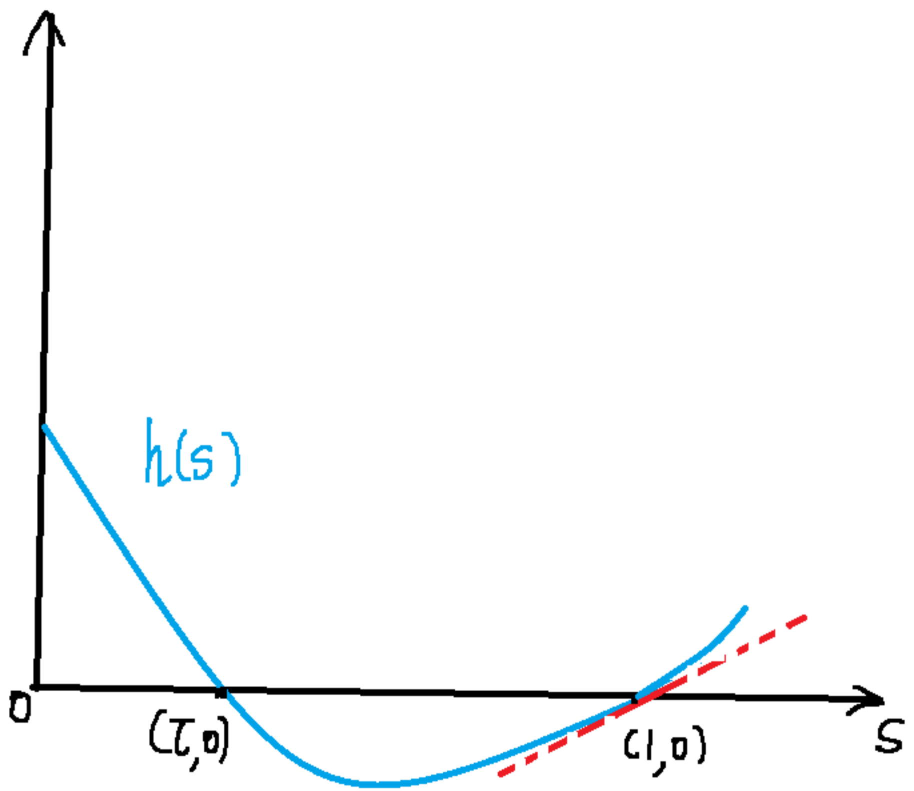
下设$p_0+p_1<1$. 令$h(s)=\phi(s)-s, 0\le s\le 1$.
则$h'(s)=\sum_{k=1}^{\infty} kp_k s^{k-1}-1$在$[0,1]$严格递增,
$h(0)=p_0>0,\ h(1)=0.$

若$\mu \le 1$, 则对$0 \le s < 1$, $h'(s) < h'(1) = \mu - 1 \le 0$,

所以h在$[0,1]$严格递减, 方程$h(s) = 0$在$[0,1]$上有唯一解$s = 1$, 即1是方程$s = \phi(s)$的最小正解, 所以$\tau = 1$.



若$\mu > 1$, 则$h'(0) = p_1 - 1 < 0$ 而$h'(1) = \mu - 1 > 0$. 所以h在$[0,1]$先严格递减后严格递增,因此h在$[0,1]$存在两个零点. 即$\tau < 1$.



Example

设$P(\xi=k)=\frac{1}{2^{k+1}}, k=0,1,\cdots.$

则$\phi(s)=\frac{1}{2-s}$, $\mu=\phi'(1)=1$, 因此$\tau=1$. 计算T_0分布律

已算得$\alpha_n=P(Z_n=0)=\frac{n}{n+1}$.

令$T_0=\min\{n\geq 1: Z_n=0\}$,首次灭绝的时刻.

则对$n\geq 1$,

$$\begin{aligned} P(T_0=n) &= P(Z_n=0, Z_{n-1}\neq 0) \\ &= P(Z_n=0)-P(Z_{n-1}=0)=\frac{1}{n(n+1)}. \end{aligned}$$

Example

设$P(\xi = k) = (1-p)^k p, k = 0, 1, \cdots$.

则$\phi(s) = \frac{p}{1-(1-p)s}$, $\mu = \phi'(1) = \frac{1-p}{p}$.

$$\begin{aligned} \phi(s) = s \quad & \Leftrightarrow (1-p)s^2 - s + p = 0 \\ & \Leftrightarrow (s-1)((1-p)s - p) = 0 \end{aligned}$$

所以当$p \geq 1/2$时, $\tau = 1$.

当$p < 1/2$时, $\tau = p/(1-p)$.