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## Random experiment(随机试验), Sample points(样本点) and sample spaces(样本空间)

- We roll a dice and the possible outcomes are 1, 2, 3, 4, 5, 6 corresponding to the side that turns up.
- We toss a coin with possible outcomes H (heads) and T (tails).

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Let $\omega_i = \{i \text{ comes up}\}$ in the proceeding example of dice throwing, then $\Omega = \{\omega_1, \omega_2, \cdots, \omega_6\}$.

## Example

A bag contains 10 balls, 3 of which are red, 3 white and 4 black. If a ball is drawn at random, then the sample space may be taken as

$$\Omega_1 = \{\text{a red ball, a white ball, a black ball}\}.$$

## Example

If we label these 10 balls with, red–$1, 2, 3$, white– $4, 5, 6$, black–$7, 8, 9, 10$, and take a ball randomly, then the sample space may be taken as

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If we label these 10 balls with, red–$1, 2, 3$, white– $4, 5, 6$, black–$7, 8, 9, 10$, and take a ball randomly, then the sample space may be taken as

$$\Omega_2 = \{\omega_1, \cdots, \omega_{10}\}, \quad \omega_i = \{\text{ the } i\text{-th ball }\}$$

## Example

In the above Example 1, if two balls are taken at a time, then each sample point is expressed as $(i, j)$, where $i$ and $j$ are the numbers attached to balls,

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In the above Example 1, if two balls are taken at a time, then each sample point is expressed as $(i, j)$, where $i$ and $j$ are the numbers attached to balls, and so the sample space is equal to

$$\begin{aligned}\Omega = \{&(1,2),(1,3),\cdots,(1,10),\\ &(2,3),\cdots,(2,10),\cdots,(9,10)\},\end{aligned}$$

which consists of $\binom{10}{2} = 45$ sample points in sum.

## Example

The distance between the target and the projectile is a non-negative real number. In this case the sample space may be taken as

$$\Omega = [0, \alpha],$$

a 1-dimensional continuous interval, where $\alpha$ is a positive real constant.

Classical probability models: possesses the following two fundamental features.

1. The sample space is finite, i.e., $\Omega=\{\omega_1,\omega_2,\cdots,\omega_n\}$, where $\omega_i, i=1,2,\cdots,n$ are elementary events.
2. Each elementary event comes up with equal possibility, i.e., their probabilities are identical.

## Definition

If a random experiment possesses $n$ elementary events of equal possibility and event $A$ contains $m$ of these elementary events, then the probability $P(A)$ that event $A$ comes up is defined as

$$P(A) = \frac{m}{n} = \frac{\#A}{\#\Omega}.$$

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2. Normalization(规范性): $P(\Omega) = 1$, where $\Omega$ is a sure event;
3. Additivity(可加性): Suppose that $A$ and $B$ will never come up simultaneously (i.e., $A \cap B = \emptyset$) and that $A + B$ stands for the event that $A$, or $B$, or both come up(i.e., $A \cup B$), then $P(A + B) = P(A) + P(B)$.

In particular, if event $A$ and $B$ will never come up simultaneously (i.e., $A \cap B = \emptyset$), and, at least one of $A$ and $B$ will come up (i.e., $A \cup B = \Omega$), then

$$P(A) = 1 - P(B).$$

### Example

有 $n$ 个球, $N$ 个格子 ($n \le N$), 球与格子都是可以区分的. 每个球落在各格子内的概率相同(设格子足够大, 可以容纳任意多个球). 将这 $n$ 个球随机地放入 $N$ 个格子, 求:

(1) 指定的 $n$ 格各有一球的概率;

(2) 有 $n$ 格各有一球的概率.

**Solution.**

把球编号为$1 \sim n$, $n$个球的每一种放法是一个样本点，每一种放法是等可能的，这属于古典概率模型，

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$$P(A)=n!/N^n.$$

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$$\begin{aligned}P(B)=&\frac{P_N^n}{N^n}=\frac{N!}{N^n(N-n)!}\\=&\frac{N(N-1)\cdots(N-n+1)}{N^n}\\=&(1-\frac{1}{N})(1-\frac{2}{N})\cdots(1-\frac{n-1}{N}).\end{aligned}$$

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$$\log(1-x) = -x + O(x^2);\ x \to 0.$$

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$$\begin{aligned}
&\log\Big(1-\frac{1}{N}\Big)\Big(1-\frac{2}{N}\Big)\cdots\Big(1-\frac{n-1}{N}\Big)\\
=&\sum_{k=1}^{n-1}\log\Big(1-\frac{k}{N}\Big) = -\sum_{k=1}^{n-1}\frac{k}{N} + O\Big(\sum_{k=1}^{n-1}\frac{k^2}{N^2}\Big)\\
=&-\frac{n(n-1)}{2N} + O\Big(\frac{n^3}{N^2}\Big) \ \text{ if } \ \frac{n}{N} \to 0.
\end{aligned}$$

所以

$$P(B) \approx \exp\left\{-\frac{n(n-1)}{2N}\right\}.$$

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$$
\begin{aligned}
p_n =& P(A) = 1 - P(\overline{A}) \\
=& 1 - \frac{P_{365}^n}{365^n} = 1 - \frac{365!}{(365-n)!365^n}.
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计算这样的概率，要计算阶乘数$n!$. 随$n$的增大，$n!$增长非常快，例如

$$10! = 3,628,000, \quad 15! = 1,307,674,368,700,$$

而100!包含158位数字.

在实际计算中，常常用Stirling 公式进行近似计算:

$$
\begin{aligned}
n! =& \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \exp\left\{\frac{\theta_n}{12n}\right\}, \quad 0 < \theta_n < 1 \\
\approx& \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.
\end{aligned}
$$

另一方面,

$$
\begin{aligned}
p_n =&1-\frac{P_{365}^n}{365^n}\\
=&1-\Big(1-\frac{1}{365}\Big)\Big(1-\frac{2}{365}\Big)\cdots\Big(1-\frac{n-1}{365}\Big)\\
\approx&1-\exp\Big\{-\frac{n(n-1)}{2\times 365}\Big\}\widehat{=}\widetilde{p}_n.
\end{aligned}
$$

可以计算出如下结果:

| $n$ | 20 | 30 | 40 | 50 | 60 | 70 | 80 |
|---|---|---|---|---|---|---|---|
| $p_n$ | 0.411 | 0.706 | 0.891 | 0.970 | 0.994 | 0.9992 | 0.9999 |
| $\widetilde{p}_n$ | 0.406 | 0.696 | 0.882 | 0.965 | 0.992 | 0.9987 | 0.9998 |

| $n$ | 22 | 23 |
|---|---|---|
| $p_n$ | 0.4757 | 0.5073 |
| $\widetilde{p}_n$ | 0.4689 | 0.5000 |

$$p_n - \widetilde{p}_n \le 0.01.$$

### Example

A bag contains $a$ white balls and $b$ black balls. These balls are drawn one by one randomly and without replacement. Find the probability that the $k$-th ball drawn is a white one.

**Solution 1.** 把球编号，按摸球的次序把球排成一列，直到$(a+b)$个球都摸完，每种排法为一个样本点，样本点总数为

**Solution 1.** 把球编号，按摸球的次序把球排成一列，直到$(a+b)$个球都摸完，每种排法为一个样本点，样本点总数为$(a+b)$个球的排列数$(a+b)!$.

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所考察的事件相当于在第$k$个位置放白球，共有$a$种放法，每种放法又对应其它$(a+b-1)$个球的$(a+b-1)!$放法,故该事件包含的样本点数为$a(a+b-1)!$，所求的概率为

$$P=\frac{a(a+b-1)!}{(a+b)!}=\frac{a}{a+b}.$$

**Solution 2.** 各球不编号，即所有白球都看成相同，这时相当于在$(a+b)$个位置中取$a$个位置放白球，共有

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故所求的概率为

$$P = \frac{\binom{a+b-1}{a-1}}{\binom{a+b}{a}} = \frac{a}{a+b}.$$

### Example

There are $a$ defective products and $b$ nondefective products and they are indistinguishable. If $n$ $(n \leq a)$ products are sampled from them, find the probability that the $n$ products sampled contain $k$ defective ones.

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**Solution.** 在$(a+b)$件产品中取$n$件有$\binom{a+b}{n}$种取法, 而在$a$件次品中取$k$件, $b$件正品中取$n-k$件共有$\binom{a}{k}\binom{b}{n-k}$种取法. 故所求的概率为

$$P=\frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}}.$$

## Example

One has two boxes of matches, each having $n$ matches, in his pocket. Each time he wants to use match, he will randomly take out a box and draw one match from it. When he finds the box he takes out is empty, find the probability that the other box has just $m$ matches.

 

**Solution.** It is obvious that the event

{ When one box he takes out is empty, the other box has just $m$ matches }

is equal to the event

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And also, it is equal to $A+B$, where

$$A = \{ \text{ at the } (2n+1-m)\text{-th draw, he finds box A is empty}\},$$

$$B = \{ \text{ at the } (2n+1-m)\text{-th draw, he finds box B is empty}\}.$$

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\begin{aligned}
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& \text{box A is drawn at the } (2n+1-m)\text{-th draw;} \\
& \text{and in other } (2n-m) \text{ draws , box A is drawn } n \text{ times,} \\
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$$\text{box B is drawn } n-m \text{ times}\}.$$

We have totally $2^{2n+1-m}$ ways to take the first $(2n+1-m)$ draws in which $\binom{2n-m}{n}$ ways satisfy the condition in event $A$.

Similarly, in the totally $2^{2n+1-m}$ ways to take the first $(2n+1-m)$ draws, $\binom{2n-m}{n}$ ways satisfy the condition in event $B$. Therefore, the desired probability is

$$\frac{2\binom{2n-m}{n}}{2^{2n+1-m}} = \frac{\binom{2n-m}{n}}{2^{2n-m}}.$$

## 古典概率模型的推广

在古典概率模型中, 样本空间$\Omega = \{\omega_1, \cdots, \omega_n\}$是有限的且每个样本点出现是等可能的.

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一般地，如果样本空间$\Omega=\{\omega_1,\omega_2,\cdots\}$含有可列个元素，样本点$\omega_i$出现的可能性为$p(\omega_i)$，其中$p(\omega_i)\geq 0$，$\sum_{i=1}^{\infty} p(\omega_i)=1$. 这时事件$A$的概率为

$$P(A)=\sum_{i:\omega_i\in A} p(\omega_i).$$

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$$P(A)=\sum_{i:\omega_i\in A}p(\omega_i).$$

对这样的概率模型,容易验证有如下性质:

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4. 可列可加性:若事件$A_i$, $i = 1, 2, \cdots$中任何两个都不会同时发生(即,$A_i \cap A_j = \emptyset$, $i \neq j$)，用$\sum_{i=1}^{\infty} A_i$表示它们中至少有一个发生(即$\bigcup_{i=1}^{\infty} A_i$)，则

$$P(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

In fact,

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## Geometrical probability models(几何概率)

Sample space $\Omega$—a region in $\mathbf{R}^n$.

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Sample space $\Omega$—a region in $\mathbf{R}^n$.

"equal possibility": measure of $A$ = measure of $B$ $\Longrightarrow$ $P(A) = P(B)$.

## Definition

Event $A_g = \{$ a sample point falls into region $g \subset \Omega\}$. The probability of $A_g$ is defined as

$$P(A_g) = \frac{\text{Measure of } g}{\text{Measure of } \Omega}.$$

This is called the geometric probability.

### Example

(The arrangement problem). Two people make an appointment to meet at a park between 7 o'clock and 8 o'clock and the person who first arrives at the park will keep waiting for another for 20 minutes. Find the probability that they can meet.

**Solution.**.....

Take 7 o'clock as the beginning time and assume that one people arrives at $x$ and the other arrives at $y$. The sample ponit is $(x, y)$ and

Take 7 o'clock as the beginning time and assume that one people arrives at $x$ and the other arrives at $y$. The sample ponit is $(x, y)$ and the sample space is

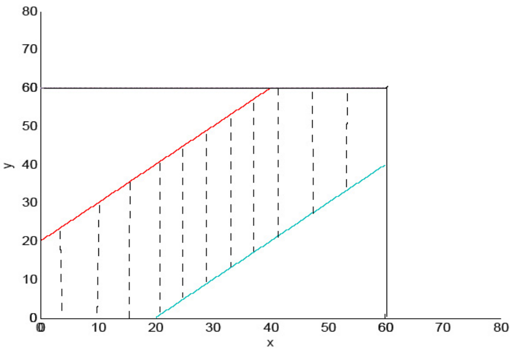
$$\Omega = \{(x, y) \big| 0 \leq x \leq 60, 0 \leq y \leq 60\}.$$

Take 7 o'clock as the beginning time and assume that one people arrives at $x$ and the other arrives at $y$. The sample ponit is $(x, y)$ and the sample space is

$$\Omega = \{(x, y) \big| 0 \le x \le 60, 0 \le y \le 60\}.$$

The two people meet each other if and only if $|x - y| \le 20$. Therefore the sample points such that event $A = \{\text{they meet each other}\}$ happens constitute the area

$$g = \{(x, y) \big| |x - y| \le 20, 0 \le x, y \le 60\}.$$


80
70
60
50
40
30
20
10
0
y
0
10
20
30
40
50
60
70
80
x

So we have

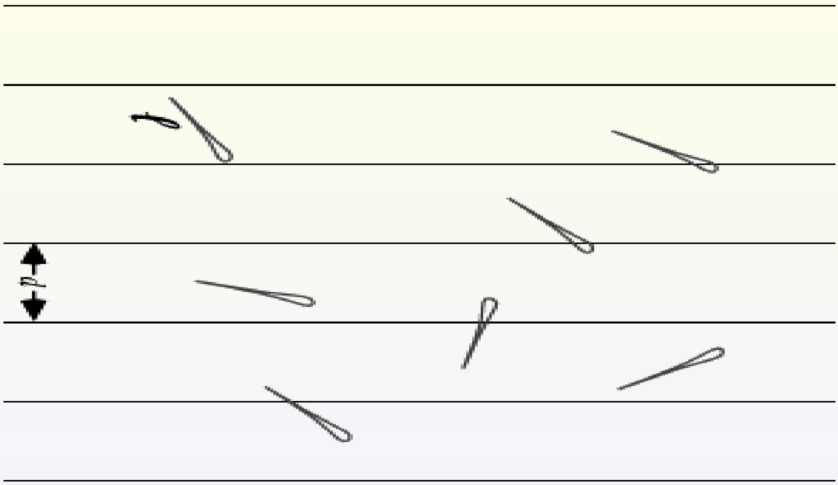
$$P(A) = \frac{\text{the area of } g}{\text{the area of } \Omega}$$

So we have

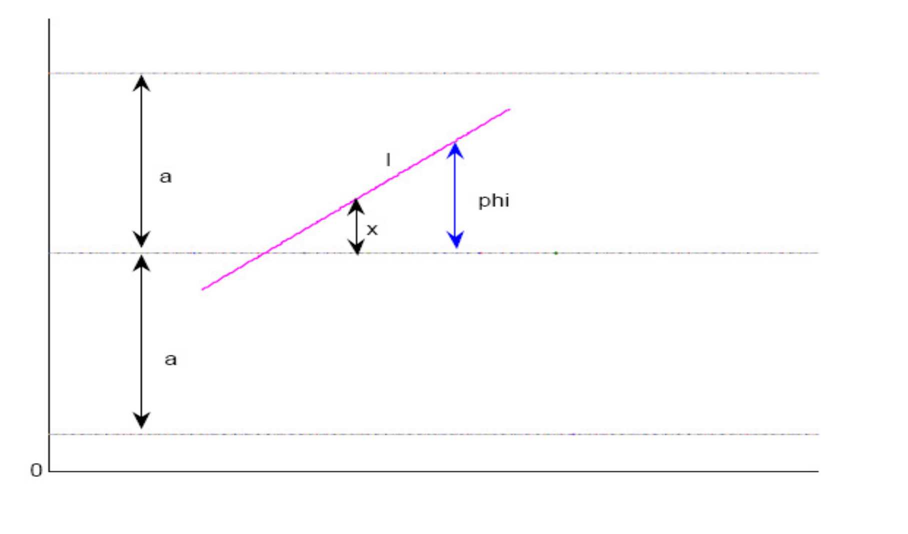
$$P(A) = \frac{\text{the area of } g}{\text{the area of } \Omega} = \frac{60^2 - (60-20)^2}{60^2} = \frac{5}{9}.$$

### Example

(The problem of Buffon's needles) If a needle of length $l$ is dropped at random on the middle of a horizontal surface ruled with parallel lines a distance $a > l$ apart, what is the probability that the needle will cross one of the lines?



# Solution.



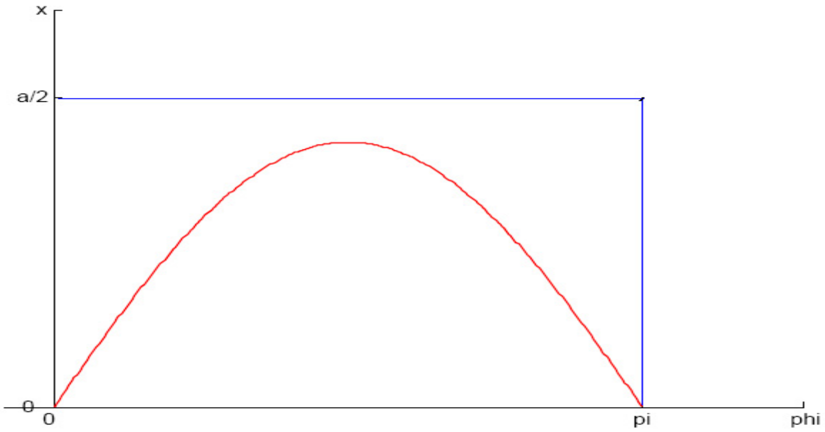
The position the needle lies in (the sample point) is decided by two parameters, the distance $x$ between needle's midpoint and the line closest to it , and the angle $\varphi$ between the needle and parallel lines.

The position the needle lies in (the sample point) is decided by two parameters, the distance $x$ between needle's midpoint and the line closest to it , and the angle $\varphi$ between the needle and parallel lines. So the sample space is

$$\Omega = \{(\varphi, x)\big|0 \le \varphi \le \pi, 0 \le x \le \frac{a}{2}\}.$$

The needle crosses one of the parallel lines if and only if $x \le \frac{l}{2}\sin\varphi$ (denote this area by $g$).


x
a/2
0
0
pi
phi

Hence the desired probability is

$$P = \frac{\text{the area of } g}{\text{the area of } \Omega}$$

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$$P = \frac{\text{the area of } g}{\text{the area of } \Omega} = \frac{\int_0^{\pi} \frac{l}{2} \sin\varphi d\varphi}{\pi a/2} = \frac{2l}{a\pi}.$$

Monte Carlo method If we know the value of $P$, then we can obtain

$$\pi = \frac{2l}{a}P.$$

Since

$$\text{probability}P \approx \text{frequency } \frac{n}{N},$$

the latter can be obtained from a large number of repeated independent experiments. Then

$$\pi \approx \frac{2l}{a}\frac{n}{N}.$$

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Monte Carlo method is very popular in computation.

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1. Non-negativity(非负性): $P(A) \geq 0$;
2. Normalization(规范性): $P(\Omega) = 1$, where $\Omega$ is a sure event;
3. Additivity(可加性): Suppose that $A$ and $B$ will never happen simultaneously and that $A + B$ stands for the event that $A$, or $B$, or both happen, then $P(A + B) = P(A) + P(B)$.

4. **Countable Additivity(可数可加性)**: Suppose that any two of $A_i$, $i = 1, 2, \cdots$, will never happen simultaneously (i.e., $A_i \cap A_j = \emptyset$, $i \neq j$) and that $\sum_{i=1}^{\infty} A_i$ stands for the event that at least one of $A_i$, $i = 1, 2, \cdots$, happens (i.e., $\bigcup_{i=1}^{\infty} A_i$), then

$$P(\sum_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$