偏微分方程（甲）笔记

1　一些基本概念

有界开集： $\Omega \subseteq R^n$ ($\partial\Omega$ 光滑：C^1, C^k)
梯度：

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \cdots, \partial_{x_n} u) = Du$$

Hessian 矩阵：

$$\nabla^2 u = (\partial_{x_i} \partial_{x_j} u)_{1 \le i,j \le n}$$

u 的**Laplacian：**

$$tr\nabla^2 u = \partial^2_{x_1} u + \cdots + \partial^2_{x_n} u = \Delta u$$

$F = (F_1, \cdots, F_n)$. 定义 F 的**散度：**

$$\nabla \cdot F = \partial_{x_1} F_1 + \partial_{x_2} F_2 + \cdots + \partial_{x_n} F_n$$

k 阶偏微分方程：

$$F[D^k u(x), D^{k-1} u(x), \cdots, Du(x), u(x), x] = 0, x \in \Omega$$

其中 $F : R^{n^k} \times R^{n^{k-1}} \times \cdots \times R^n \times R \times \Omega \to R$
多重指标： $\alpha = (\alpha_1, \cdots, \alpha_n)$, 有：

$$|\alpha| = \alpha_1 + \cdots + \alpha_n \ , \ D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial {x_1}^{\alpha_1} \partial {x_2}^{\alpha_2} \cdots \partial {x_n}^{\alpha_n}} \ , \ \alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$$

$C^k(\Omega)$： Ω 上 k 阶偏导数存在且连续的函数构成的函数空间。

古典解（经典解）： 若 $u \in C^k(\Omega) \Rightarrow F(D^k u, \cdots, Du, u, x)$ 连续。$F = 0 \Rightarrow \forall x \in \Omega, F$ 存在且为 0. 称 u 是方程的古典解（经典解），满足：(1) $u \in C^k(\Omega)$　(2)$F = 0$

函数的**模：**

$$\|u\|_{C^k(\Omega)} = \sum_{|\alpha|=0}^{\infty} \sup_{x \in \Omega} |\partial^\alpha u|$$

线性 PDE：

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x)$$

其中 a_α, f 为给定的函数。
非线性 PDE：

(1)**半线性 PDE:**

$$\sum_{|\alpha|\le k} a_\alpha(x)D^\alpha u = f(x,u,Du,\cdots,D^{k-1})$$

(2)**拟线性 PDE:**

$$\sum_{|\alpha|\le k} a_\alpha(x,u,Du,\cdots,D^{k-1})D^\alpha u = f(x,u,Du,\cdots,D^{k-1})$$

(3)**完全非线性 PDE:** F 关于 $D^k u$ 是非线性的。

线性 PDE 举例:

(1)**输运方程:**

$$\partial_x u + a\partial_y u = 0$$

(2)**Possion 方程:**

$$\Delta u = f(x) \quad (u:\Omega\to R)$$

$f=0 \Rightarrow$ **Laplace 方程**

(3)**热方程:**

$$\partial_t u - \Delta u = f \quad (u:R_+\times\Omega\to R)$$

(4)**波动方程:**

$$\partial_t^2 u - \Delta u = f \quad (u:R_+\times\Omega\to R)$$

(5)**Maxwell 方程:**

$$\begin{cases} \frac{1}{c}\partial_t E = \nabla\times B \quad (B\text{ 的旋度}) \\ \frac{1}{c}\partial_t B = -\nabla\times E \quad (E\times B: R_+\times R^3\to R^3) \\ \nabla\cdot E = \nabla\cdot B = 0 \end{cases}$$

其中

$$\nabla\times B = (\partial_2 B_3 - \partial_3 B_2, \partial_3 B_1 - \partial_1 B_3, \partial_1 B_2 - \partial_2 B_1)$$

并且:

$$\begin{cases} \partial_t^2 B = c^2\Delta B \\ \partial_t^2 E = c^2\Delta E \end{cases}$$

非线性 PDE 举例:

(1)**Burgers 方程:**

$$\partial_t u + u\partial_x u = 0$$

(2)**守恒律方程:**

$$\partial_t u + \partial_x(F(u)) = 0$$

(3)**Navier-Stokes 方程:** $u: R_+ \times R^3 \to R^3$, $p = p(t,x) \in R$

$$\begin{cases} \partial_t u + (u\cdot\nabla)u - v\Delta u + \nabla p = 0 \quad \text{动量守恒} \\ \nabla\cdot u = 0 \quad \text{不可压条件，} v>0\text{为}\textbf{粘性系数} \end{cases}$$

$v = 0 \Rightarrow$ **Euler 方程.**（理想流体）

(4)**Schrodinger 方程:**

$$i\partial_t u + \Delta u = f(x,u,Du)\ ,\ u \in C$$

(5)**极小曲面方程:** $u = u(x_1, x_2, \cdots, x_n)$, 有:

$$\nabla\cdot(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0$$

(6)**Einstain 方程:**

$$R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} + \lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T\mu\nu$$

(7)**Monge-Ampere 方程:**

$$\det\left(\nabla^2 u\right) = f(x,u,\nabla u)\quad ,\quad k = C$$

2 波动方程

Gauss-Green 公式/散度公式:

$$\int_{\Omega} \nabla \cdot F \ dx = \int_{\partial\Omega} F \cdot \nu \ dr(x)$$

其中 ν 是 $\partial\Omega$ 的单位外法向量.

波方程公式:

$$\begin{cases} \partial_t^2 u - a^2 \partial_x^2 u = f(t,x) \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = \psi(x) \end{cases}$$

通解为:

$$u = \frac{1}{2}(\phi(x+at) + \phi(x-at)) + \frac{1}{2a}\int_{x-at}^{x+at} \psi(y)dy + \frac{1}{2a}\int_0^t \int_{x-a(t-s)}^{x+a(t-s)} f(y,s)dyds \quad (*)$$

特殊情况: $\partial_t^2 u - \partial_x^2 u = 0$

不难发现:

$$\left.\begin{array}{l} F(x-t) \text{ 是 } \partial_t u + \partial_x u = 0 \text{ 的解} \\ G(x+t) \text{ 是 } \partial_t u - \partial_x u = 0 \text{ 的解} \end{array}\right\} \Rightarrow \text{都是 } \partial_t^2 u - \partial_x^2 u = 0 \text{ 的解}$$

$$\Rightarrow u(x,t) = F(x-t) + G(x+t) \quad \begin{cases} F(x) + G(x) = \phi(x) \\ -F'(x) + G'(x) = \psi(x) \end{cases} \quad \text{从而解出 } F, G.$$

这里我们将 $F(x-t)$ 称为**右行波**, 将 $G(x+t)$ 称为**左行波**

Burgers 方程: $\partial_t u + u\partial_x u = 0$ (满足 $\partial_{xx} u = 0$)

对 x 求偏导:

$$\partial_t(\partial_x u) + (\partial_x u)^2 + u\partial_x^2 u = 0$$

记 $v = \partial_x u$, 则

$$\partial_t v + u\partial_x v + v^2 = 0 \quad \Rightarrow \quad \frac{d}{dt}v + v^2 = 0, \frac{d}{dt}(-v) = (-v)^2$$

解 ODE 即可。

Duhamel 原理: 对于波动方程

$$\begin{cases} \partial_t^2 u - a^2 \Delta u = f(t,x) \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = \psi(x) \end{cases} \qquad x \in R^n \ , \ t \geq 0$$

可以分解成下面三个方程组:

$$(1) \begin{cases} \partial_t^2 u - a^2 \Delta u = 0 \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = 0 \end{cases} \quad (2) \begin{cases} \partial_t^2 u - a^2 \Delta u = 0 \\ u(x,0) = 0 \\ \partial_t u(x,0) = \psi(x) \end{cases} \quad (3) \begin{cases} \partial_t^2 u - a^2 \Delta u = f(t,x) \\ u(x,0) = 0 \\ \partial_t u(x,0) = 0 \end{cases}$$

设 $u_2 = M_\psi(x,t)$ 为初值问题 (2) 的解。同理，(1)(3) 的解分别为：

$$u_1 = \frac{\partial}{\partial t} M_\phi(x,t)\ ,\ u_3 = \int_0^t M_{f_\tau}(x,t-\tau)d\tau \quad f_\tau = f(x,\tau)$$

下面证明 u_1, u_3 定义的正确性：

$$u_1(x,0) = (\partial_t M_\phi)(x,0) = \phi(x)$$

$$\partial_t^2 u_1 - a^2\Delta u_1 = (\partial_t^2 - a^2\Delta)(\partial_t M_\phi) = \partial_t((\partial_t^2 - a^2\Delta))M_\phi = 0$$

$$\partial_t u_1(x,0) = \partial_t^2 M_\phi(x,t) = a^2\Delta M_\phi(x,0). \quad \Rightarrow\ u_1\text{正确。}$$

$M_{f(\tau)}(x,t)$ 满足：$\begin{cases} (\partial_t^2 - a^2\Delta)M_{f(\tau)}(x,t) = 0 \\ M_{f(\tau)}(x,0) = 0 \\ \partial_t M_{f(\tau)}(x,0) = f(x,t) \end{cases}$

固定 τ ： 令 $\omega_\tau(x,t) = M_{f_{(\tau)}}(x,t-\tau)$，则

$$u_3 = \int_0^t \omega_\tau(x,t)dt\ ,\ u_3(x,0) = 0\ ,\ \partial_t u_3(x,0) = \omega_t(x,t) + \int_0^t \partial_t\omega_\tau(x,t)d\tau$$

$t = 0$ 时，

$$\partial_t u_3(x,0) = \omega_0(x,0) = 0\ ,\ \partial_t u_3 = \int_0^t \partial_t\omega_\tau(x,t)d\tau$$

$$\partial_t^2 u_3 = \partial_t\omega_\tau(x,t) + \int_0^t \partial_t^2\omega_\tau(x,t)d\tau = f(x,t) + \int_0^t a^2\Delta\omega_\tau(x,t)d\tau = f(x,t) + a^2\Delta u_3$$

$\Rightarrow\ (\partial_t^2 - a^2\Delta)u_3 = f(x,t) \quad \Rightarrow\ u_3$正确。

定理：若 $\phi(x) \in C^2(R)$ ，$f \in C^1(R \times \bar{R}_+)$ ，则由（*）给出的 u 为原波方程的古典解。
推论：若 f, ϕ, ψ 同为奇（偶，周期）函数，那么 u 为奇（偶，周期）函数。

半无界问题：

$$\begin{cases} (\partial_t^2 - a^2\Delta)M_{f(\tau)}(x,t) = 0 \\ M_{f(\tau)}(x,0) = 0 \\ \partial_t M_{f(\tau)}(x,0) = f(x,t) \\ u(0,t) = g(t) \longrightarrow \text{边界条件} \end{cases}$$

（第一类）Dirichlet 边界条件：

$$u|_{\partial\Omega} = g$$

（第二类）Neumann 边界条件：

$$\partial_v u|_{\partial\Omega} = g$$

（第三类）混合边界条件：

$$\partial_v u + \alpha u|_{\partial\Omega} = g$$

齐次边界条件： $g = 0$

$\partial_t^2 u - a^2\Delta u = f(x,t)$. 考虑 $x=0, t=0 \;\Rightarrow\; 0 - a^2\phi''(0) = f(0,0)$

相容性条件（必要条件）： $\begin{cases} -a^2\phi''(0) = f(0,0) \\ u(0,0) = \phi(0) = 0 \\ \partial_t u(0,0) = \psi(0) \end{cases}$

构造函数：

$\tilde{\phi} = \begin{cases} \phi(x)\ ,\ x>0 \\ -\phi(-x)\ ,\ x\le 0 \end{cases}$ $\tilde{\psi} = \begin{cases} \psi(x)\ ,\ x>0 \\ -\psi(-x)\ ,\ x\le 0 \end{cases}$ $\tilde{f}(x,t) = \begin{cases} f(x,t)\ ,\ x>0 \\ -f(-x,t)\ ,\ x\le 0 \end{cases}$ $f(0,t)=0$

$\tilde{\phi}, \tilde{\psi}, \tilde{f}$ 为奇函数 $\Rightarrow \tilde{u}(s,t)$ 为奇函数。

(1) 齐次边界条件：$g \equiv 0$

(2) $g \not\equiv 0$ 时，$u(0,t) = g(t)$, 构造 $v(x,t) = u(x,t) - g(t)$ 即可。

(3) $\partial_x u(0,t) = g(t)$. 令 $vx, t = u(x,t) - xg(t)$. 则 $\partial_x v(0,t) = 0$

高维初值问题：

$$\begin{cases} \partial_t^2 u - a^2\Delta u = f(t,x) \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = \psi(x) \end{cases} \qquad x\in R^n\ ,\ t\ge 0$$

先考虑三维情况：定义

$$u(r,t;x) = \fint_{\partial B(x,r)} u(y,t)d\sigma(y) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} u(y,t)d\sigma(y)$$

其中 $\boldsymbol{B(x,at)}$ 是 x 为圆心，at 为半径的圆盘。

令 $y = x + rz$, $u(r,t;x) = \frac{1}{4\pi}\int_{\partial B(0,1)} u(x+rz,t)d\sigma(z)$

$$\partial_r u(r,t;x) = \frac{1}{4\pi}\int_{\partial B(0,1)} z\cdot Du(x+rz,t)dS(z) = \frac{1}{4\pi r^2}\int_{\partial B(x,r)} \frac{y-x}{r}\cdot Du(y,t)dS(y)$$

$$= \frac{1}{4\pi r^2}\int_{\partial B(x,r)} \partial_\nu u(y,t)dS(y) = \frac{1}{4\pi r^2}\int_{B(x,r)} \Delta u(y,t)dy = \frac{1}{a^2 4\pi r^2}\partial_t^2\int_{B(x,r)} u(y,t)dy$$

$$\Rightarrow\; a^2 4\pi r^2 \partial_r u(r,t;x) = \partial_t^2 \int_{B(x,r)} u(y,t)dy$$

$$\text{则}\;\; \partial_r(a^2 4\pi r^2 \partial_r u) = \partial_t^2\int_{\partial B(x,r)} u(y,t)dS(y) = \partial_t^2(4\pi r^2 u)$$

$r^2\partial_t^2 u = \partial(r^2\partial_r u) \;\Rightarrow\; \partial_t^2(ru) = r\partial_r^2 u + 2\partial_r u = \partial_r^2(ru) \;\Rightarrow\; \partial_t^2(ru) = \partial_r^2(ru)$

构造 $\tilde{u} = ru(r,t;x)$, 显然 $r=0$ 时，$\tilde{u}=0$

$$\tilde{u}(r,0;x) = ru(r,0;x) = \frac{1}{4\pi r}\int_{\partial B(x,r)} \phi(y)dS(y) = \tilde{\Phi}(r,x)$$

$$\partial_t\tilde{u}(r,0;x) = \frac{1}{4\pi r}\int_{\partial B(x,r)} \Psi(y)dS(y) = \tilde{\psi}(r,x)$$

$$u(s,t) = \lim_{r\to 0} u(r,t,x) = \lim_{r\to 0}\frac{\tilde{u}(r,t,x)}{r}$$

当 r 充分小时，

$$\tilde{u}(r,t,x)=\frac{1}{2}(\tilde{\Phi}(r+t)-\tilde{\Phi}(r-t))+\frac{1}{2}\int_{t-r}^{t+r}\tilde{\psi}(y)dy \;\Rightarrow\; u(x,t)=\tilde{\Phi}'(t;x)+\tilde{\psi}(t;x)$$

$$=\frac{d}{dt}\left(\frac{1}{4\pi t}\int_{\partial B(x,t)}\phi(y)dS(y)\right)+\frac{1}{4\pi t}\int_{\partial B(x,t)}\psi(y)dS(y)=\frac{1}{4\pi t^2}\int_{\partial B(x,t)}[\phi(y)+(y-x)D\phi(y)+\psi(y)]dS(y)$$

总结： 对于三维问题

$$\begin{cases}\partial_t^2 u-a^2\Delta u=f(t,x) & \\ u(x,0)=\phi(x) & x\in R^3\ ,\ t\geq 0\\ \partial_t u(x,0)=\psi(x) & \end{cases}$$

其解为：

$$u=u_1+u_2+u_3=\frac{\partial}{\partial t}\left[\frac{1}{4\pi a^2 t}\int_{\partial B(x,at)}\phi(y)dS(y)\right]+\frac{1}{4\pi a^2 t}\cdot\int_{\partial B(x,at)}\psi(y)dS(y)$$
$$+\int_0^t\frac{1}{4\pi a^2(t-\tau)}\int_{\partial B(x,a(t-\tau))}f(y,\tau)dS(y)d\tau$$

简化：

$$u(x,t)=\frac{1}{4\pi a^2t^2}\int_{\partial B(x,at)}[\phi(y)+D\phi(y)\cdot(y-x)+t\psi(y)]dS(y)+\frac{1}{4\pi a^2}\int_{B(x,at)}\frac{f\left(y,t-\frac{|y-x|}{a}\right)}{|y-x|}dy$$

当 $f\equiv 0$ 时，可以得出**Kirchhoff 公式：**

$$u(x,t)=\frac{1}{4\pi a^2t^2}\int_{\partial B(x,at)}[\phi(y)+D\phi(y)\cdot(y-x)+t\psi(y)]dS(y)$$

接下来考虑**二维情况：**

$$\begin{cases}\partial_t^2 u-a^2\Delta u=0 & \\ u(x,0)=\phi(x) & x\in R^2\ ,\ t\geq 0\\ \partial_t u(x,0)=\psi(x) & \end{cases}$$

设 $x=(x_1,x_2)$, $\tilde{x}=(x_1,x_2,x_3)$，考虑 $\tilde{\phi}(\tilde{x})=\phi(x_1,x_2)$, $\tilde{\psi}(\tilde{x})=\psi(x_1,x_2)$，得到如下方程：

$$\begin{cases}\partial_t^2 \tilde{u}-a^2\Delta \tilde{u}=0 & \\ \tilde{u}(\tilde{x},0)=\tilde{\phi}(\tilde{x}) & \tilde{x}\in R^3\ ,\ t\geq 0\\ \partial_t \tilde{u}(\tilde{x},0)=\tilde{\psi}(\tilde{x}) & \end{cases}$$

$$\Rightarrow\quad \tilde{u}(\tilde{x},t)=\frac{1}{4\pi a^2t^2}\int_{\partial B(\tilde{x},at)}[\tilde{\phi}(\tilde{y})+D\tilde{\phi}(\tilde{y})\cdot(\tilde{y}-\tilde{x})+t\tilde{\psi}(\tilde{y})]dS(\tilde{y})\ ,\ \text{与 } x_3 \text{ 无关}$$

故 $\tilde{u}$ 满足

$$\partial_t^2\tilde{u}-\partial_{x_1}^2\tilde{u}-\partial_{x_2}^2\tilde{u}=0$$

令 $u(x,t)=\tilde{u}(x_1,x_2,0,t)$，原式

$$= \frac{1}{4\pi a^2 t^2}\int_{B(x,at)}[\phi(y)+(y-x)D\phi(y)+t\psi(y)]dS(\tilde{y})\text{，其中 } dS(\tilde{y})=\frac{1}{\sqrt{t^2-(y-x)^2}}dy$$

总结： 对于二维问题

$$\begin{cases} \partial_t^2 u - a^2\Delta u = 0 \\ u(x,0)=\phi(x) & x\in R^2\ ,\ t\geq 0 \\ \partial_t u(x,0)=\psi(x) \end{cases}$$

其解为：$u=u_1+u_2$

$$= \frac{1}{2\pi at}\int_{B(x,at)}\frac{\phi(y)+D\phi(y)\cdot(y-x)+t\psi(y)}{\sqrt{(at)^2-|y-x|^2}}dy+\frac{1}{2\pi a}\iint_{C(x,t)}\frac{f(y,\tau)}{\sqrt{a^2(t-\tau)^2-|y-x|^2}}dyd\tau$$

其中 $\boldsymbol{C(x,t)}$=$\{(y,\tau)\in R^3:0\leq\tau\leq t,|y-x|\leq a(t-\tau)\}$，即 R^3 中以 (x,t) 为顶点，圆盘 $\{(y,0):|y-x|\leq at\}$ 为底面的锥。

当 $f\equiv 0$ 时，可以得出**Poisson 公式：**

$$u=\frac{1}{2\pi at}\int_{B(x,at)}\frac{\phi(y)+D\phi(y)\cdot(y-x)+t\psi(y)}{\sqrt{(at)^2-|y-x|^2}}dy$$

定理：若 $\phi\in C^3(R^3)$，$\psi\in C^3(R^3)$，$f\equiv 0$. 则由 Kirchhoff 公式给定的解是初值问题的古典解。

3 特征锥相关

特征锥： 给定 (x_0,t_0) ，定义

$$C(x_0,t_0)=\{(x,t):|x-x_0|\le a(t_0-t),0\le t\le t_0\}$$

称为以 (x_0,t_0) 为顶点的特征锥。

依赖区域：

$$D(x_0,t_0)=\{x\in R^n:|x-x_0|\le at_0\}$$

$u(x_0,t_0)$ 的值只依赖于 $D(x_0,t_0)$ 上的初值。

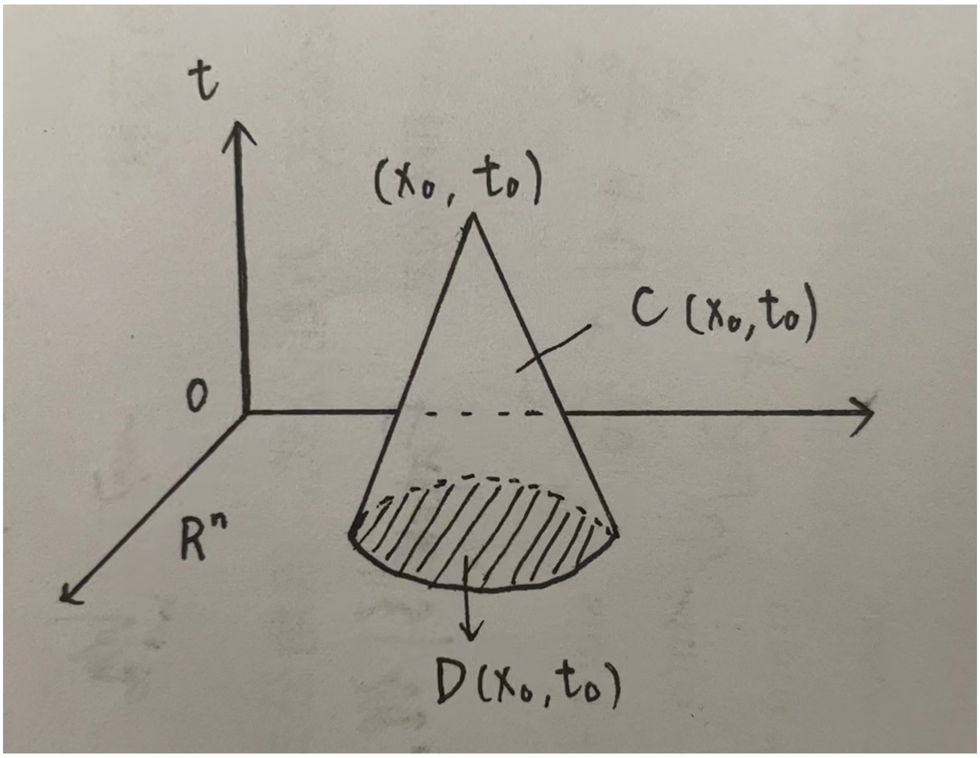

图 1: 特征锥 (画的太丑，请见谅...)

影响区域：

$$J_{x_0}=\{(x,t):|x-x_0|\le at\}$$

称为 $(x_0,0)$ 的影响区域.

对于 $D_0\subset R^n$ ，

$$J_{D_0}=\bigcup_{x_0\in D_0}J_{x_0}$$

称为 D_0 的影响区域.

决定区域： 给定 $D_0\subset R^n$ ，$u(x,t)$ 完全由 D_0 中的初值决定的点 (x,t) 构成的集合，称为 D_0 的决定区域.

Huygens 原理/无后效现象： 三维空间的波既有**波前**，也有**波后**.

波的弥漫/有后效现象： 两维空间的波只有波前，没有波后.

4 调和函数及其相关性质

位势方程：

$$\begin{cases} -\Delta u = f\ ,\ u \in \Omega \subset R^n \\ u = g\ ,\ u \in \partial\Omega \end{cases}$$

取 u_1, u_2 为两个解，则 $u_1 - u_2$ 满足 $\Delta u = 0$. 定解条件如下：

（第一类）Dirichlet 边界条件： $u|_{\partial\Omega} = g$

（第二类）Neumann 边界条件： $\partial_v u|_{\partial\Omega} = g$

（第三类）混合边界条件： $\partial_v u + \alpha u|_{\partial\Omega} = g$

调和函数： $u \in C^2(\Omega)$, 且 $\Delta u = 0$

平均值性质：

$$0 = \int_{B_r(x)} \Delta u dx = \int_{B_r(x)} \nabla \cdot (Du) = \int_{\partial B_r(x)} \partial_\nu u(x) dx$$

给定 x，定义

$$\phi(r) = \fint_{\partial B_r(x)} u(y) dS(y) \quad \phi'(r) = \left(\frac{\int_{\partial B_r(x)} u(y) dS(y)}{\omega_n \cdot r^{n-1}} \right)'$$

ω_n 为单位 S^{n-1} 的面积.

令 $y = x + rz$，原式

$$= \frac{\int_{\partial B_1(0)} (u(x+rz))' dS(\varepsilon) \cdot r^{n-1}}{\omega_n \cdot r^{n-1}} = \frac{\int_{\partial B_1(0)} (u(x+rz))' dS(\varepsilon)}{\omega_n} = \frac{\int_{\partial B_1(0)} z \cdot Du(x+rz) dS(z)}{\omega_n}$$

代入 $z = \frac{y-x}{r}$，原式

$$= \frac{\int_{\partial B_r(x)} \frac{y-x}{r} D' u(y) dS(y)}{\omega_n \cdot r^{n-1}} = \frac{\int_{\partial B_1(x)} \partial_\nu u(y) dS(y)}{\omega_n \cdot r^{n-1}} = \frac{\int_{B_r(x)} \Delta u dy}{\omega_n \cdot r^{n-1}} = 0$$

$\Rightarrow$ $\phi(r)$ 与 r 无关，$\phi(x) = u(x)$

第一平均值性质：

$$u(x) = \fint_{\partial B_r(x)} u(y) dS(y)$$

第二平均值性质：

$$u(x) = \fint_{B_r(x)} u(y) dy \equiv \psi(r)$$

$$\int_{B(x,r)} u(y) dy = \int_0^r \left(\int_{\partial B(x,s)} u(y) dS(y) \right) ds = \int_0^r (u(x) \omega_n S^{n-1}) ds = u(x) \omega_n \frac{r^n}{n} \ \rightarrow\ |B_r(x)|$$

定理：若 $u \in C^2$ 且 u 满足第一或第二平均值性质，则 u 调和。

定理：平均值性质 $\Longrightarrow u \in C^2(C^\infty)$

证明：取 η 为镜面对称，非负 C^∞ 函数是 supp $\eta \subset B_1(0)$

$$\int_{R^n} \eta(x) dx = 1$$

令 $\eta_\varepsilon(x)=\frac{1}{\varepsilon^n}\eta(\frac{x}{\varepsilon})$，则同样有

$$\int_{R^n}\eta_\varepsilon(x)dx=1$$

定义

$$u_\varepsilon(x)=\int_{R^n}u(x-y)\underbrace{\eta_\varepsilon(y)}_{|y|\leq\varepsilon}dy\ ,\ x\in\Omega_\varepsilon\ \Rightarrow x-y\in\Omega$$

其中 $\Omega_\varepsilon=\{x\in\Omega|\mathrm{dist}(x,\partial\Omega)>\varepsilon\}$

$$u_\varepsilon(x)=\int_{R^n}u(x-y)\frac{1}{\varepsilon^n}\eta\left(\frac{y}{\varepsilon}\right)dy=\int_0^\infty\left[\int_{\partial B_r(0)}u(x-y)\frac{1}{\varepsilon}\eta\left(\frac{r}{\varepsilon}\right)dy\right]dr$$

$$=\int_0^\infty\frac{1}{\varepsilon^n}\eta(\frac{r}{\varepsilon})\left[\int_{\partial B_r(0)}u(x-y)dS(y)\right]dr.$$

$$\text{平均值}=u(x)\omega_nr^{n-1}=u(x)\int_0^\infty\frac{1}{\varepsilon^n}\eta(\frac{r}{\varepsilon})\omega_nr^{n-1}dr=u(x)$$

推论：调和函数一定 $C^\infty(\Omega)$

定理（**梯度估计**）：u 为 Ω 上调和函数，则 $\forall$ 球 $B_r(x)\subset\Omega$ 以及多重指标 $\alpha(|\alpha|=k>1)$，有

$$|D^\alpha u(x)|\leqslant\frac{n^ke^{k-1}k!}{r^k}\max_{B_r(x)}|u(y)|$$

证明：$k=1$ 时，$\partial_{x_1}u$ 调和。 $(\partial_{x_1}u=\nabla\cdot F=\nabla\cdot(0,0,\cdots,u,0,\cdots,0)$，第 i 个为 $u)$

$$|\partial x_1u|=\left|⨍_{\partial B_r(x)}\partial_{x_i}udy\right|=\underbrace{\frac{1}{\alpha_nr^n}}_{\text{单位球体积}}\left|\int_{B_r(x)}\partial_{x_i}udy\right|=\frac{1}{\alpha_nr^n}\left|\int_{\partial B_r(x)}u\nu_idS(y)\right|$$

$$\leqslant\frac{1}{\alpha_nr^{n-1}}\int_{\partial B_r(x)}|u|dS(y)\leqslant\frac{1}{\alpha_nr^{n-1}}\max_{B_r(x)}|u(y)|=\frac{n}{r}\cdot\frac{\max|u(y)|}{B_r(x)}$$

$k=2$ 时，

$$\left|\partial_{x_i}\partial_{x_j}u\right|\leqslant\frac{n}{r}\max_{B_r(x)}\left|\partial_{x_j}u(y)\right|\leqslant\frac{n}{r}\frac{\max|u(z)|}{B_r(y)}=\frac{n^2}{\left(\frac{r}{2}\right)^2}\max_{z\in B_r(x)}|u(z)|=\frac{4n^2}{r^2}\max_{z\in B_r(x)}|u(z)|$$

作修改：

$$\left|\partial_{x_i}\partial_{x_j}u\right|\leqslant\frac{n}{s}\max_{\eta\in B_1(x)}\left|\partial_{x_j}u(y)\right|\leqslant\frac{n}{r-s}\max_{z\in B_{(r-s)}(y)}|u(\varepsilon)|\leqslant\frac{n^2}{s(r-s)}\max_{z\in B_r(x)}|u(z)|$$

取 $s=\frac{r}{2}$，原式

$$=\frac{4n^2}{r^2}\max_{z\in B_r(x)}|u(z)|$$

对于 $k\geq2$，设 $D^\alpha=\partial_{x_i}D^\beta$，有

$$|D^\alpha u(x)|\leqslant\frac{n}{s}\max_{y\in B_s(x)}\left|D^\beta u(y)\right|\leqslant\frac{n}{s}\max_{y\in B_s(x)}\frac{n^{k-1}e^{k-2}(k-1)!}{t^{k-1}}\max_{z\in B_r(y)}|u(z)|$$

$$=\frac{n^ke^{k-2}(k-1)!}{s\cdot t^{k-1}}\max_{y\in B_s(x)}\max_{z\in B_t(y)}|u(z)|$$

取 $t=r-s$，原式

$$=\frac{n^k e^{k-2}(k-1)!}{s(r-s)^{k-1}}\max_{z\in B_r(x)}|u(z)|$$

由基本不等式，代入 $s=\frac{r}{k}$ ，即得结果.

Liouwille 定理： 若全空间上的调和函数 u 有界，则 u 为常数。

注：u 为上有界/下有界 $\implies u\equiv C$

解析性： 若 u 是 Ω 上调和函数，则 u 在 Ω 上解析。

解析函数： $\forall x_0\in\Omega$, $\exists r>0$，使得当 $x\in B_r(x_0)$ 时，有

$$u(x)=\sum_{|\alpha|=0}^{+\infty}\frac{D^\alpha u\left(x_0\right)}{\alpha!}\left(x-x_0\right)^\alpha\quad(\text{其中}x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}\ ,\ \alpha!=\alpha_1!\alpha_2!\cdots\alpha_n!)$$

右侧假设在 $B_r(x_0)$ 上收敛到 $u(x)$.

证明：$\forall x_0\in\Omega$,取 $R=\frac{1}{3}\operatorname{dist}\left(x_0,\partial\Omega\right)\Rightarrow B_{2R}\left(x_0\right)\subset\Omega$.

$\exists\theta\in(0,1)$，满足

$$u(x_0+h)=u(x_0)+\sum_{k=1}^{m-1}\left(\sum_{|\alpha|=k}\frac{D^\alpha u\left(x_0\right)h^\alpha}{\alpha!}\right)+R_m(h)\ ,\ R_m(h)=\sum_{|\alpha|=m}\frac{D^\alpha u\left(x_0+\theta h\right)}{\alpha!}h^\alpha$$

要证：$R_m(h)\rightrightarrows 0\ (m\to+\infty)$ 关于 h 一致。

$$|R_m(h)|\leqslant\sum_{|\alpha|=m}\frac{\left|D^\alpha u\left(x_0+\theta h\right)\right|}{\alpha!}|h^\alpha|\leq\sum_{|\alpha|=m}\frac{|h^\alpha|}{\alpha!}\frac{n^m e^{m-1}m!}{R^m}\max_{y\in B_R(x_0+\theta h)}|u(y)|$$

$$\leq\max_{B_{2R}(x_0)}|u(y)|\frac{n^m e^{m-1}\cdot m_m\cdot|h^\alpha|}{R^m}\sum_{|x|=m}\frac{1}{\alpha!}=\frac{n^{2m}e^{m-1}\cdot m!\,|h^\alpha|}{m!R^m}\max_{y\in B_{2R}(x_0)}|u(y)|=\frac{1}{e}\left(\frac{n^2|h|}{R}\right)^m\cdot M$$

取 $t_0=\frac{R}{2n^2}$，则当 $|h|\leq r_0$ 时，$|R_m(h)|\leq\frac{M}{2}(\frac{1}{2})^m\rightrightarrows 0$ （与 h 无关）

从而 u 在 $B_r(x_0)$ 上解析 $\implies u$ 在 Ω 上解析.

强极值原理： 设 $\Omega\subseteq R^n$ 中有界开集，u 为 Ω 上调和函数，则有：

(1)

$$\max_{\Omega}u=\max_{\partial\Omega}u\ ,\ u\text{在}\bar{\Omega}\text{上的最大值在边界上取到}$$

(2) 若 Ω 连通，且存在 $x_0\in\Omega$，使得 $u(x_0)=\max\limits_{\Omega}u$，则 u 一定是常数.

证明：(1) 设 $M=\max\limits_{\Omega}u>\max\limits_{\partial\Omega}u$，则存在 $x_0\in\Omega$, 使得$u\left(x_0\right)=M$. 令$A=\{x\in\bar{\Omega}\mid u(x)=M\}\subset\Omega,A\neq\emptyset$

① A 为 Ω 的相对开集 (若 $\{x_i\}\subset A$, $x_i\to\tilde{x}_0\in\Omega\ \Rightarrow\ \tilde{x}_0\in A$)

若 $x_i\in A\Rightarrow u\left(x_i\right)=M$, $x_i\to\tilde{x}_0\in\Omega\ \Rightarrow\ u\left(\tilde{x}_0\right)=M$

② A 为开集 ($x_0\in A$, $\exists r_0$, $B_{r_0}(x_0)\subset A$)

$\forall x_0\in A$，取 $r_1>0$, $B_{r_1}(x_0)\subset\Omega$

$$M=u\left(x_0\right)=\fint_{B_{r_1}(x_0)}u(y)dy\leqslant M$$

$\Rightarrow u(y)=M,\ B_{r_1}(x_0)\subset A \xrightarrow{\Omega\text{连通}} A=\emptyset$ 或 Ω.

(2) 对于 (1) 考察 Ω 的每一个连通分支 Ω_i，若 $x_i\in\Omega_i \Rightarrow u$ 在 Ω_i 上为常数

$$\Rightarrow \max_{x\in\partial x_i} u \equiv M \ \Rightarrow \ \max_{\partial\Omega} u = M.$$

Harnack 不等式： $u\geq 0$ 在 Ω 调和，则 $\forall$ 连通紧集 $V\subset\Omega$ 存在常数 $C=C(V)$，使得

$$\max_V u \leq C \min_V u$$

证明：对 $x,y\in\Omega.|x-y|=r<\frac{1}{2}\operatorname{dist}(x,\Omega)$

$$u(y)=⨍_{B_r(y)} u(z)dz=\frac{\int_{B_r(y)}u(z)dz}{|B_r(y)|}\leqslant\frac{\int_{B_{2r}(x)}u(z)dz}{|B_r(y)|}=\frac{u(x)\,|B_{2r}(x)|}{|B_r(y)|}=2^n\cdot u(x)$$

设 $x_0\in V$, $u(x_0)=\max\limits_V u$, $y_0\in V$, $u(y_0)=\min\limits_V u$

取连接 x_0,y_0 的道路 γ，取 $\gamma_0>0$，使得 $\bigcup\limits_{z\in\gamma} B_{2\gamma_0}(z)\subset\Omega$. 在 γ 上取一个序列，$\bigcup\limits_{z\in\gamma} B_{\gamma_0}(z)$ 为 γ 的开覆盖。

从而存在有限子覆盖 $B_{\gamma_0}(x_1),B_{\gamma_1}(x_2),\cdots,B_{\gamma_{n-1}}(x_n)$. 因此 $B_{\gamma_0}(x_1)\bigcup B_{\gamma_0}(x_2)\bigcup\cdots\bigcup B_{\gamma_0}(x_N)$ 包含 V，从而包含 γ.

则 $u(x_0)\leq(2^n)^N u(y_0)$，取 $C=(2^n)^N$ 即可.

定理： u 是调和函数，则函数

$$H(r)=\int_{\partial B_r} u^2\ \mathrm{d}S\ ,\ D(r)=r^2\int_{B_r}|\nabla u|^2\ \mathrm{d}y$$

都是关于 r 的单增函数，且 $f(r)=\dfrac{H(r)}{D(r)}$ 也是关于 r 的单增函数.

5 基本解与 Green 函数

Laplace 方程:

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$$

考虑径向对称特解：$u(x) = v(r), \Delta u = 0$

$$\Delta u(x) = \Delta v(r) = \nabla \cdot (\nabla v(r)) = \nabla \cdot \left(v'(r)\frac{v}{r}\right) = \nabla \cdot v'(r)\frac{x}{r} + v'(r)\nabla\left(\frac{x}{r}\right)$$

$$\left(\nabla v(r) = v(r)\nabla r = v(r)\frac{x}{r} \text{ , } r = \sqrt{x_1^2 + \cdots + x_n^2}\right)$$

$$= v''\frac{x}{r}\frac{x}{r} + v'(r)\frac{n}{r} + v'(r)x \cdot \left(-\frac{1}{r^2}\right)\frac{x}{r}\nabla r \ = v'' + v'(r)\frac{n}{r} - v'(r)\frac{1}{r} = v'' + v'(r)\frac{n-1}{r} = 0$$

$$\Rightarrow r^{n-1}v' = c \ \Rightarrow \ v' = \frac{c}{r^{n-1}},\ v = \frac{c_1}{r^{n-2}} + c_2\ (n \geqslant 3)\ rv'' + (n-1)v' = 0 \text{ , } (r^\alpha v')' = r^\alpha\left(v'' + \frac{\alpha}{r}v'\right).$$

取 $\alpha = n-1$ 即为 0 .

$$v(r) = \begin{cases} \dfrac{c_1}{r^{n-2}} + c_2 & n \geqslant 3 \\ a\ln r + c_2 & ,n = 2 \end{cases}$$

由计算可知:

$$\int_{\partial B_r} \frac{\partial v}{\partial r} dS(x) = \int_{\partial B_r} -C_1(n-2)\frac{1}{r^{n-1}}dS(x) \quad (n \geqslant 3)$$

$$= -c_1(n-2)\underline{\int_{\partial B_r}\frac{1}{r^{n-1}}dS(x)} = |\partial B_1| = \omega_n$$

$$= -c_1(n-2)\omega_n = -1 \quad , \quad c_1 = \frac{1}{(n-2)\omega_n}$$

$n = 2$ 时，$c_1 = -\frac{1}{2r}$ $(n = 2)$.

基本解:

$$v(r) = \begin{cases} \dfrac{1}{(n-2)\omega_n \cdot r^{n-2}} \text{ , } n \geq 3 \\ -\dfrac{1}{2\pi}\ln r \text{ , } n = 2 \end{cases} \qquad \Gamma(x) = \begin{cases} \dfrac{1}{(n-2)\omega_n \cdot |x|^{n-2}} \text{ , } n \geq 3 \\ -\dfrac{1}{2\pi}\ln|x| \text{ , } n = 2 \end{cases}$$

性质：① $\Delta\Gamma(x) = 0\ (x \neq 0)$

② $\forall |x| > 0$, $\Gamma(x) \in L^1(B_r)$ 但 $\Gamma \notin L^1(R^n)$

③ $\int_{\partial\omega} \frac{\partial v}{\partial r} dS(x) = 1 \quad (0 \notin \Omega)$

④

$$|D\Gamma(x)| \leq \frac{C}{r^{n-1}} \text{ , } |D^2\Gamma(x)| \leq \frac{C}{r^n} \text{ . } |D^3\Gamma(x)| \in L^1(R \backslash B_r)$$

其中 C 是只依赖于空间的维数 n 的正常数.

定理：设 $f \in C_l^\infty(R^n)$，则 $u = \int_{R^n}\Gamma(x-y)f(y)dy$ 是 $-\Delta u = f(x \in R^n)$ 的经典解.

$$\int_{R^n}(-\Delta r(x))\phi(x)dx = \int_{R^n} -r(x)\Delta\phi(x)dx$$

定理：设 $f \in C_0^2(R^n)$，则 $u = \int_{R^n} \Gamma(x-y)f(y)dy \in C^2(R^n)$ 是 $-\Delta u = f$ 的解.
引理：$u, v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \subseteq R^n$ 为有界区域, 且 $\partial\Omega$ 为 C^1. 则有

$$\int_\Omega (u\Delta v - v\Delta u)dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial \nu} - v\frac{\partial u}{\partial \nu}\right) dS(x)$$

定理：设 $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega}), \Omega$ 同上，则 $\forall x \in \Omega$，有

$$u(x) = -\int_\Omega \Gamma(x-y)\Delta u(y)dy + \int_{\partial\Omega} \left[\Gamma(x-y)\frac{\partial}{\partial \nu}u(y) - u(y)\frac{\partial}{\partial \nu}\Gamma(x-y)\right] dS(y)$$

证明：考虑 $\Omega_\varepsilon = \Omega \backslash B_\varepsilon(x)$, $u, v \in C^2(\Omega_\varepsilon) \cap C^1(\bar{\Omega}_\varepsilon)$. 由引理有：

$$\underbrace{\int_{\Omega_\varepsilon} (u(y)\Delta\Gamma(x-y) - \Gamma(x-y)\Delta u(y))dy}_{①} = \underbrace{\int_{\partial\Omega} \left(u(y)\frac{\partial \Gamma}{\partial \nu}(x-y) - \frac{\partial u}{\partial \nu}(y)\Gamma(x-y)\right) dS(y)}_{②} -$$

$$\underbrace{\int_{\partial B_\varepsilon(x)} \left(u(y)\frac{\partial \Gamma}{\partial \nu}(x-y) - \frac{\partial u}{\partial \nu}(y)\Gamma(x-y)\right) dS(y)}_{③}$$

当 $\varepsilon \to 0$ 时，$|B_\varepsilon(0)| \to 0$，而 $\Gamma(x-y)\Delta u(y) \in L'\left(\overline{B_\varepsilon(x)}\right) \Rightarrow \int_{B_s(x)} \Gamma(x-y)\Delta u dy \to 0$
分别考虑 ①②③ 三个部分：

$$① \xrightarrow{\varepsilon \to 0} \int_{\partial\Omega} \Gamma(x-y)\Delta u(y)dy$$

$$② = -\int_{\partial B_\varepsilon(x)} \frac{\partial \Gamma(x-y)}{\partial \nu}u(y)dS(y) = -\int_{\partial B_\varepsilon(0)} \frac{\partial \Gamma(y)}{\partial \nu}u(x-y)dS(y) = \int_{\partial B_\varepsilon(0)} \frac{1}{\omega_n r^{n-1}}(u(x-y) - u(x))dS(y)$$

$$+ \int_{\partial B_\varepsilon(0)} \frac{u(x)}{\omega_n r^{n-1}}dS(y) \leqslant \underbrace{c\int_{\partial B_\varepsilon(0)} \frac{r}{\omega_n \cdot r^{n-1}}dS(y)}_{= c\varepsilon^{n-1}\omega_n \frac{1}{\varepsilon^{n-2}} = c\omega_n\varepsilon \to 0} + \underbrace{u(x)\varepsilon^{n-1} \cdot \varepsilon^{-(n-1)}}_{\to u(x)} \to u(x)$$

$$③ \leqslant \left|\int_{\partial B_\varepsilon(x)} \frac{\partial u}{\partial \nu}\Gamma(x-y)dS(y)\right| \leqslant c\int_{\partial B_\varepsilon(x)} |\Gamma(x-y)|dS(y) = \begin{cases} c\omega_n\varepsilon^{n-1}\dfrac{1}{\omega_n(n-2)\varepsilon^{n-2}} \to 0, \ n \geq 3 \\ c\omega_2\dfrac{\varepsilon}{2\pi}\ln 2 \to 0 \ , \ n = 2 \end{cases}$$

要说明 $u = \int_{R^n} \Gamma(x-y)f(y)dy$ 满足 $-\Delta u = f$
证明：(1) 证 $u \in C^2$.

$$\partial x_1 u = \lim_{\varepsilon \to 0} \frac{u(x+\varepsilon e_i) - u(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \int_{R^n} r(y)\frac{f(x+\varepsilon e_i - y) - f(x-y)}{\varepsilon}dy \to \int_{R^n} r(y)\partial x_i f(x-y)dy$$

$\partial_{x_i} u$ 存在且有界，同理 $\partial_{x_i}\partial_{x_j} u$ 存在且有界 $\to u \in C^2(R^n)$
(2) 计算 Δu：

$$\int_{R^n} \Gamma(y)\Delta f(x-y)dy = \int_{R^n} \Delta\Gamma(y)f(x-y)dy + \int_{\partial R^n} \left(\Gamma(y)\frac{\partial f(x-y)}{\partial \nu} - f(x-y)\frac{\partial \Gamma(y)}{\partial \nu}\right) dS(y) = -f(x)$$

取 $\Omega_k = B_k(0)$，当 $x \notin \Omega_k$ 时，$f = 0$. 由前述定理：

$$f(x) = -\int_{B_k(0)} \Gamma(x-y)\Delta f(y)dy + \int_{\partial B_k(0)} \left[\Gamma(x-y)\frac{\partial f}{\partial \nu} - \frac{\partial \Gamma}{\partial \nu}(x-y)f(y)\right] dS(y)$$

$$= -\int_{R^n} \Gamma(x-y)\Delta f(y)dy = \Delta x\left(\int_{R^n} \Gamma(y)f(x-y)dy\right) = -\int_{R^n} \Gamma(y)\Delta x f(x-y)dy$$

$$= -\int_{R^n} \Gamma(y)\Delta y f(x-y)dy = -\int_{R^n} \Gamma(x-y)\Delta y f(y)dy = -\Delta u$$

$-\Delta u = f$ 所有有界解为：$u^* = \int_{R^n} \Gamma(x-y)f(y)dy + C$ （有界 + 调和 $\Rightarrow$ 常数）

$$\begin{cases} -\Delta u = f(x) & x \in \Omega \\ u = g & x \in \partial\Omega \end{cases} \qquad u, v \in C^2(\Omega) \cap C^1(\bar{\Omega}) \text{ , } v(x,y) = \Gamma(x-y)$$

$$u(x) = -\int_\Omega \Gamma(x-y)\Delta u(y)dy (= \int_\Omega \Gamma(x-y)f(y)dy) + \int_{\partial\Omega} \left[\frac{\partial u(x)}{\partial \nu}\Gamma(x-y) - \frac{\partial \Gamma(x-y)}{\partial \nu}u(y)\right] dS(y)$$

$$\int u\Delta v dy = \int v\Delta u dy + \int 2n\frac{\partial u}{\partial \nu}v - u\frac{\partial v}{\partial \nu}dS(y) \text{ , } v \in C^2 \cap C^1(\bar{\Omega})$$

令 v 满足：

$$\begin{cases} -\Delta u = 0 & x \in \Omega \\ v(y) = \Gamma(x-y) & y \in \partial\Omega \end{cases}$$

$$\Rightarrow u(x) = -\int_\Omega (\Gamma(x-y) - v)f(y)dy + \int_{\partial\Omega} \frac{\partial}{\partial \nu}(\Gamma(x-y) - v(y))g(y)dy$$

Green 函数:

给定有界区域 $\Omega \subseteq R^n$ ，定义调和函数 $\phi^x(y)$ 如下：

$$\begin{cases} -\Delta_y \phi^x(y) = 0 & y \in \Omega \\ \phi^x(y) = \Gamma(x-y) & y \in \partial\Omega \end{cases}$$

令 $G(x,y) = \Gamma(x-y) - \phi^x(y)$ （不依赖于 u, f, g，只依赖于 Ω 以及 x, y）

$$\Rightarrow u(x) = \int_\Omega G(x,y)f(y)dy - \int_{\partial\Omega} g(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) \quad (x \in \Omega)$$

$G(x,y)$ 称为 **Green 函数**.

验证：$(1)u \in C^2 \quad (2) - \Delta u = f \quad (3)u = g\ (x \in \partial\Omega)$

定理：$G(x,y) = G(y,x) \ \Leftrightarrow\ \phi^x(y) = \phi^y(x)$

证明：取 $x_1, x_2 \in \Omega, x_1 \neq x_2$. 令 $G_1(x) = G(x, x_1)$ ，$G_2(x) = G(x_2, x)$. 下证 $G_1(x_2) = G_2(x_1)$

令 $\Omega_\varepsilon = \Omega \backslash (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$，则 G_1, G_2 在 Ω_ε 上光滑.

$$\int_{\Omega\varepsilon} (G_1\Delta G_2 - G_2\Delta G_1)\, dx = \int_{\partial\Omega\varepsilon} \left(\frac{\partial G_2}{\partial \nu}G_1 - \frac{\partial G_1}{\partial \nu}G_2\right) dS(x)$$

$$\left.\begin{array}{l}\Delta G_1(x)=\Delta\left(G\left(x_1,x\right)\right)=\Delta\left(\Gamma\left(x_1-x\right)-\phi^{x_1}(x)\right)=0\\ \Delta G_2(x)=\Delta\left(G\left(x_2,x\right)\right)=\Delta\left(\Gamma\left(x_2-x\right)-\phi^{x_2}(x)\right)=0\end{array}\right\}\quad \text{左边}=0$$

右边：

$$\int_{\partial\Omega}\left(\frac{\partial G_2}{\partial\nu}G_1-\frac{\partial G_1}{\partial\nu}G_2\right)dS(x)=-\int_{\partial B_\varepsilon(x_1)}\left(\frac{\partial G_2}{\partial\nu}G_1-\frac{\partial G_1}{\partial\nu}G_2\right)dS(x)-\int_{\partial B_\varepsilon(x_2)}\left(\frac{\partial G_2}{\partial\nu}G_1-\frac{\partial G_1}{\partial\nu}G_2\right)dS(x)$$

当 $x\in\partial\Omega$，$G_1(x)=\Gamma(x_1-x)-\phi^{x_1}(x)=0$，同理 $G_2(x)=0$

在 $B_\varepsilon(x_1)$ 中，$G_2(x)=\Gamma(x_2-x)-\phi^{x_2}(x)=0$ 光滑。

$$G_1(x)=\Gamma\left(x_1-x\right)-\phi^{x_1}(x)\leqslant\frac{c}{|x-x_1|^{n-2}}+c$$

$$\int_{\partial B_\varepsilon(x)}\frac{\partial G_2}{\partial\nu}G_1dx\leqslant c\int_{\partial B_\varepsilon(x)}\left(\frac{c}{|x-x_1|^{n-2}}+c\right)dS(x)\le c\varepsilon^{n-1}\left(c\varepsilon^{-(n-2)}+c\right)\le c\varepsilon\to 0$$

$$\int_{\partial B_\varepsilon(x_1)}\frac{\partial G_1}{\partial\nu}G_2=\int_{\partial B_\varepsilon(x_1)}\left(\frac{\partial\Gamma\left(x_1-x\right)}{\partial\nu}G_2(x)-\frac{\partial\phi^{x_1}(x)}{\partial\nu}G_2(x)\right)dS(x)$$

$$=\int_{\partial B_\varepsilon(x_1)}\frac{\partial\Gamma\left(x_1-x\right)}{\partial\nu}G_2\left(x_1\right)dS(x)+\int_{\partial B_\varepsilon(x)}\frac{\partial\Gamma\left(x_1-x\right)}{\partial\nu}(G_2(x)-G_2(x_1)+c\varepsilon^{n-1})dS(x),n\ge 3$$

$$\leqslant -G_2\left(x_1\right)+c\varepsilon^{n-1}\varepsilon^{-(n-1)}\varepsilon+c\varepsilon^n\to -G_2\left(x_1\right)$$

$\therefore -G_2(x_1)+G_1(x_2)=0$，$G_2(x_1)=G_1(x_2)$

$G(x,y)$ 在 $x-y=0$ 处有奇性，在 $x-y\neq 0$ 处光滑.

当 $y\to x$ 时，

$$G(x,y)\sim\Gamma(x-y)\sim\begin{cases}-\dfrac{1}{|x-y|^{n-2}} & n\ge 3\\ \ln|x-y| & n=2\end{cases}\qquad \partial_xG\sim\partial_x\Gamma(x-y)$$

定理：设 $u\in C^2(\Omega)\cap C^1(\bar\Omega)$ 为 Dirichlet 问题 $\begin{cases}-\Delta u=f\\ u=g\end{cases}$ 的解。则有

$$u(x)=\int_\Omega G(x,y)f(y)dy-\int_{\partial\Omega}\frac{\partial G}{\partial\nu}(x,y)g(y)dS(y)$$

特殊情况：

①上半平面：$\Omega=R^n_+$，$G(x,y)=r(x-y)-\phi^x(y)$，求 $\phi^x(y)=\begin{cases}-\Delta_y\phi^x(y)=0, & x\in\Omega\\ \phi^x(y)=\Gamma(x-y), & x\in\partial\Omega\end{cases}$

令 $\widetilde{x}=(x_1,x_2,\cdots,x_{n-1},-x_n)$，当 $y_n=0$ 时，$\Gamma(x-y)=\Gamma(\widetilde{x}-y)$

取 $\phi^x(y)=\Gamma(\widetilde{x}-y)$，即满足要求

$$G(x,y)=\Gamma(x-y)-\Gamma(\tilde{x}-y)=\frac{1}{\omega_n(n-2)}\left(\frac{1}{|x-y|^{n-1}}-\frac{1}{|\tilde{x}-y|^{n-1}}\right)\qquad n\geqslant 3$$

$$\frac{\partial G}{\partial\nu}=-\partial_{y_n}G\quad\left(y\in\partial R^n_+\text{ 时}\right)=-\frac{1}{w_n}\left(\frac{y_n-x_n}{|y-x|^n}-\frac{y_n-\tilde{x}_n}{|y-\tilde{x}|^n}\right)=-\frac{2x_n}{\omega_n|x-y|^n}$$

$$u(x)=\int_{R^n_+}G(x,y)f(y)dy+\int_{\partial R^n_+}\frac{2x_ng(y)}{\omega_n|x-y|^n}dy$$

$$\begin{cases} -\Delta u = 0 & x \in R_+^n \\ u = g & x \in \partial R_+^n \end{cases} \quad \Rightarrow u = \frac{2x_n}{\omega_n} \int_{R^{n-1}} \frac{g(y)}{|x-y|^n} dy$$

② $\Omega = B_1(0)$. 同理，考虑 $G(x,y) = r(x-y) - \phi^x(y)$, 求 $\phi^x(y) = \begin{cases} -\Delta_y \phi^x(y) = 0, & x \in \Omega \\ \phi^x(y) = \Gamma(x-y), & x \in \partial\Omega \end{cases}$

考虑关于 $\partial B_1(0)$ 的**对偶点:** $x^* = \frac{x}{|x|^2}$，有 $|x^*| \cdot |x| = 1$

$\Delta\Gamma(y - x^*) = 0$ $(y \in B_1(0))$. 取 $\phi^*(y) = \Gamma(|x|(y - x^*))$，满足要求.

$$G(x,y) = \Gamma(x-y) - \Gamma(|x|(y-x^*))$$

当 $|y| = 1$ 时，有 $|x| \cdot |y - x^*| = |y - x|$. （“相似三角形”性质）

定理：设 g 为 ∂R_+^n 上的有界函数.

$$u(x) = \int_{\partial R_+^n} K(x,y) g(y) dy$$

其中

$$\int_{\partial R_+^n} K(x,y) dy = 1 \text{ , } K(x,y) = \frac{2x_n}{n\alpha_n |x-y|^n}$$

则 (1) $u \in C^\infty(R_+^n)$，且有界.

(2) $\Delta u = 0$ $(x \in R_+^n)$

(3) $\forall x_0 \in \partial R_+^n$, $\lim_{x \to x_0} u(x) = g(x_0)$

证明：(1) $|g| \le M \;\Rightarrow\; |u| \le M$，由 $K(x,y)$ 光滑，$x_n > 0 \;\Rightarrow\; u(x)$ 光滑

(2) $\Delta_x K(x,y) = 0$ 由 $K(x,y) = \partial_{y_n} G(x,y)$, $\Delta_x K = \partial_{y_n} \Delta_x G(x,y) = 0$

(3) 即证

$$\lim_{x \to x_0} \int_{\partial R_+^n} K(x,y) \left(g(y) - g\left(x_0\right)\right) dy = 0$$

$$\Rightarrow \int_{|y-x_0|<\delta_1} K(x,y) \left(g(y) - g\left(x_0\right)\right) dy + \int_{|y-x_1|>\delta_1} K(x,y) \left(g(y) - g\left(x_0\right)\right) dy = \mathrm{I} + \mathrm{II}$$

$\forall \varepsilon_0 > 0$, $\exists \delta, > 0$，使得 $|g(y) - g\left(x_0\right)| < \frac{\varepsilon_0}{2}$ （当 $|y - x_0| < \delta_1$ 时）

因此，

$$\mathrm{I} \le \int_{(y-x_0)<\delta_1} K(x,y) \frac{\varepsilon_0}{2} dy \le \frac{\varepsilon_0}{2} II \leqslant 2M \int_{|y-x_0|>\delta_1} K(x,y) dy = \frac{2M \cdot 2x_n}{n\alpha(n)} \int_{|y-x_0|>\delta_1} \frac{1}{|x-y|^n} dy$$

若 $|x - x_0| < \frac{\delta_1}{2}$, $x_n < \frac{\delta_1}{2}$, $\mathrm{II} \le \tilde{M} \delta_1$

取 $\delta_0 = \min\{\delta_1, \frac{\varepsilon_0}{2\tilde{M}}\}$. 当 $\delta < \delta_0$ 时，$\mathrm{I} \le \frac{\delta_0}{2}$, $\mathrm{II} \le \frac{\delta_0}{2} \quad \Rightarrow \quad \mathrm{I} + \mathrm{II} \le \varepsilon_0 \;\to 0$

6 分离变量法

$$\begin{cases} \partial_t^2 u - \Delta u = f \\ u(x,0) = \phi(x) \qquad x \in \Omega, t > 0 \\ \partial_t u(x,0) = \psi(x) \\ u(x,t) = g(x,t)\ ,\ x \in \partial\Omega\ , t > 0 \end{cases}$$

不妨设 $g \equiv 0$ （若不为 0，则构造 $v(x,t) = u(x,t) - g(x,t)$ ）

考虑 $u(x,t) = T(t) \cdot X(x)$，代入得 $T''(t)X(x) - T(t)\Delta X(x) = 0 \quad \Rightarrow \quad \frac{T''(t)}{T(t)} = \frac{\Delta X(x)}{X(x)} = \lambda$

特征值问题:

$$(1)\begin{cases} \Delta X(x) = \lambda X(x) \\ X(x) = 0 \quad , x \in \partial\Omega \end{cases} \qquad (2)\ T''(t) = \lambda T(t)$$

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = X(1) = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} \lambda_n = -\left(\frac{n\pi}{\lambda}\right)^2 \\ X_n(x) = \sin\left(\frac{n\pi}{\lambda}x\right) \end{cases}$$

S-L(Sturn-Liouville) 问题

$$\begin{cases} (p(x)X'(x))' - q(x)X(x) + \lambda X(x) = 0 \\ -\alpha_1 X'(a) + \beta_1 X(a) = 0 \qquad\qquad x \in [a,b] \\ \alpha_2 X'(b) + \beta_2 X(b) = 0 \end{cases}$$

$$p(x) \geqslant c_0 > 0\ ,\ q(x) \geqslant 0\ \alpha_i, \beta_i \geqslant 0\ ,\ \alpha_i + \beta_i > 0$$

性质 1：S-L 问题的所有解 (λ, X_λ) 满足 $\lambda \geq 0$，且当 $\beta_1 + \beta_2 > 0$ 时，有 $\lambda > 0$.

性质 2：$(\lambda, X_\lambda, (\mu, X_\mu))$ 是两个解 $(\lambda \neq \mu)$，则有 $\int_a^b X_\lambda X_\mu dx = 0$

性质 3：特征值均为单的. $\Leftrightarrow$ X_1, X_2 均为 λ 对应的特征函数列，则 $X_1 = cX_2$.

性质 4：所有特征值均构成一个单增且趋于无穷的序列：

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\ ,\ \lim_{n\to+\infty} \lambda_n = +\infty$$

性质 5：$\{X_\lambda(x)\}$ 在 $L^2([a,b])$ 中完备：$\forall f \in L^2$，有

$$f(x) = \sum_{n=1} c_n X_n(x)\ ,\ c_n = \frac{\langle f(x), X_n(x)\rangle}{\|X_n\|^2}$$

分离变量法/特征函数展开法:

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t)X_n(x) \text{ 由条件可以得到：} \begin{cases} T_n''(t) + \lambda_n T_n(t) = 0 \\ T_n(0) = \phi_n\ ,\ T_n'(0) = \psi_n \end{cases}$$

相容性条件:

$$\begin{cases} \partial_t^2 u - \partial_x^2 u = 0 \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = \psi(x) \\ u(0,t) = u(l,t) = 0 \end{cases} \quad \text{在角点}(0,0),(l,0)\text{满足 } \phi(0) = \phi(l) = 0\ ,\ \phi''(0) = \phi''(l) = 0\ ,\ \psi(0) = \psi(l) = 0$$

定理：$\phi(x)\in C^3([0,l]),\psi(x)\in C^2([0,l])$ 且 $\phi(x),\psi(x)$ 满足相容性条件，则

$$u(x,t)=\sum_{n=1}^{+\infty}u_n(x,t)=\sum_{n=1}^{+\infty}\left(A_n\cos\frac{n\pi}{l}t+B_n\sin\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x$$

满足边值问题. $u\in C^2([0,l]\times[0,T])$ ，其中

$$A_n=\frac{2}{l}\int_0^l\phi(x)\sin\frac{n\pi}{l}xdx\ ,\ B_n=\frac{2}{n\pi}\int_0^l\psi(x)\sin\frac{n\pi}{l}xdx$$

要证 $u\in C^2$，只要证：①$\sum\limits_{n=1}^{\infty}u_n(x,t)$ 一致连续. ②$\sum\limits_{n=1}^{\infty}Du_n(x,t)$ 一致连续. ③$\sum\limits_{n=1}^{\infty}D^2u_n(x,t)$ 一致连续.

不难看出 $|u_n(x,t)|\le|A_n|+|B_n|$

$$A_n=\frac{2}{l}\int_0^l\phi(x)\sin\frac{n\pi}{l}xdx=\frac{2}{l}\cdot\frac{l}{n\pi}\int_0^l\partial_x\phi(x)\cos\frac{n\pi}{l}xdx=\frac{2l}{n^2\pi^2}\int_0^l-\partial_x^2\phi(x)\cdot\sin\frac{n\pi}{l}xdx$$

$$=\frac{2l}{n^2\pi^2}\cdot\int_0^l-\partial_x^2\phi\cdot\partial_x\left(-\cos\frac{n\pi}{l}x\right)dx=\frac{2l^2}{n^3\pi^3}\int_0^l-\partial_x^3\phi(x)\cos\frac{n\pi}{l}xdx=\frac{1}{n^3}a_n,\ a_n=-\frac{2l^2}{\pi^3}\int_0^l\phi'''(x)\cos\frac{n\pi}{l}xdx$$

$$\text{同理，}B_n=\frac{2}{n\pi}\int_0^1\psi(x)\sin\frac{n\pi}{l}xdx=\frac{1}{n^3}b_n\ ,\ b_n=-\frac{2l^2}{\pi^3}\int_0^l\psi'''(x)\sin\frac{n\pi}{l}xdx$$

从而有

$$|u_n(x,t)|\leqslant\frac{c_1}{n^3}\ ,\ |Du_n(x,t)|\leqslant\frac{c_2}{n^2}\ ,\ \left|D^2u_n(x,t)\right|\leqslant a_n^2+b_n^2+\frac{c_3}{n^2},\ n=1,2,\cdots$$

均一致收敛.

$$u_n(x,t)=\left(A_n\cos\frac{n\pi}{l}t+B_n\sin\frac{n\pi}{l}t\right)\sin\frac{n\pi}{l}x=\underbrace{\sqrt{A_n^2+B_n^2}}_{\text{振幅}}\sin(\underbrace{\frac{n\pi}{l}}_{\text{频率}}t+\underbrace{\theta_n}_{\text{相位}})\sin\frac{n\pi}{l}x$$

$$\Rightarrow\quad D_n\sin\left(\omega_nt+\theta_n\right)\sin\left(\zeta_n\right)$$

共振现象：

$$\begin{cases}\partial_t^2u-a^2\partial_x^2u=A(x)\sin\omega t\\u(x,0)=0\\\partial_tu(x,0)=0\\u(0,t)=u(l,t)=0\end{cases}\qquad f_n(z)=\int_0^1f(x,t)X_n(x)dx$$

$$u(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{l}x\left(\frac{1}{na\pi}\int_0^tf_n(z)\sin\frac{na\pi(t-\tau)}{l}d\tau\right)=\sum_{n=1}^{\infty}\sin\frac{n\pi}{l}x\frac{a_n}{\omega_n}\int_0^t\sin\omega\tau\sin\omega_n(t-\tau)d\tau$$

$$\omega_n=\frac{na\pi}{l}\qquad a_n=\frac{2}{l}\int_0^lA(x)\sin\frac{n\pi}{l}xdx<C$$

$$\int_0^t\sin\omega t\sin\left(\omega_n(t-\tau)\right)d\tau=\int_0^t\frac{1}{2}\left[\cos\left(\omega\tau-\omega_n(t-\tau)\right)-\cos\left(\omega\tau+\omega_n(t-\tau)\right)\right]d\tau$$

$$\int_0^t\cos\left(\left(\omega-\omega_n\right)\tau-\omega_nt\right)d\tau=\int_0^t\frac{1}{\omega-\omega_n}\partial_\tau\left(\sin\left(\omega+\omega_n\right)\tau-\omega_nt\right)d\tau=\frac{1}{\omega+\omega_n}\left(\sin\omega t+\sin\omega_nt\right)<C$$

若 $\omega=\omega_n,\int_0^t\cos(\omega t)dt=t\cos(\omega t)$ 无界！（**共振现象**）

7 Fourier 变换

Fourier 变换：

对 $u \in L^1(R^n)$，定义：

$$Fu(\xi) = \frac{1}{(2\pi)^{\frac{\pi}{2}}} \int_{R^n} e^{-i\xi x} u(x)dx \in L^\infty(R^n)$$

称为 u 的 Fourier 变换，记作 $\hat{u}(\xi)$

例：

$$F(e^{-t|x|^2})(\xi) = \frac{1}{(2t)^{\frac{n}{2}}} e^{-\frac{|\xi|^2}{4t}}.\text{取}t = \frac{1}{2}\text{，对应}e^{-\frac{1}{2}|x|^2}\text{,其 Fourier 变换为自身.}$$

Fourier 逆变换：

$$F^{-1}[u(\xi)](x) = \frac{1}{(2\pi)^{\frac{\pi}{2}}} \int R^n u(\xi) e^{i\xi x} d\xi, x \in R^n$$

考虑卷积的 Fourier 变换：$\omega = u * v$

$$\hat{\omega}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{-i\xi \cdot x} \cdot \left(\int_{R^n} u(x-y)v(y)dy \right) dx = (2\pi)^{-\frac{n}{2}} \int_{R^n} \int_{R^n} e^{-i\xi(x-y+y)} u(x-y)v(y)dxdy$$

令 $x - y - z$. 原式

$$= (2\pi)^{-\frac{n}{2}} \int_{R^n} \int_{R^n} e^{-i\xi(y+z)} u(z)v(y)dydz = (2\pi)^{\frac{n}{2}} \hat{u}(\xi) \cdot \hat{v}(\xi)$$

定理：设 $u \in L'(R^n) \cap L^2(R^n)$，则 $\hat{u} \in L^2(R^n)$，且 $\|\hat{u}\|_{L^2} = \|u\|_{L^2}$

证明：(1)

$$\langle v, Fw \rangle_{L^1 \times L^\infty} = \langle Fv, w \rangle_{L^1 \times L^\infty} = \int_{R^n} v(x)Fw(x)dx = \int_{R^n} Fv(\xi)w(\xi)d\xi = \int_{R^n \times R^n} e^{-i\xi x} v(x)\omega(\xi)dxd\xi$$

(2) 令 $v = e^{-\varepsilon|x|^2}$

$$\int_{R^n} e^{-\varepsilon|x|^2} \omega(x)dx = \int_{R^n} \omega(x) \underbrace{\frac{1}{(2\varepsilon)^{\frac{n}{2}}} e^{-\frac{|x|^2}{4\varepsilon}}}_{\to \eta_\varepsilon(x)} dx$$

引理：$\lim\limits_{\varepsilon \to 0}$ 右边 $= (2\pi)^{\frac{n}{2}} w(0)$，$\forall w \in L'(R^n) \cap C(R^n)$

$$\varepsilon \to 0 \begin{cases} \int_{R^n} \eta_\varepsilon(x)dx \to (2\pi)^{\frac{n}{2}} \\ \int_{R^n} (w(x) - \omega(0))\eta_\varepsilon(x)dx \to 0 \end{cases}$$

(3) $v(x) = \overline{u(-x)}$ $\quad w(x) = (u * v)(x)$ 且 $w \in C(R^n)$

$$\hat{v}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-ix\xi} \bar{u}(-x)dx = \overline{\hat{u}(\xi)} = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R_n} e^{-iy\xi} u(y)dy$$

$\therefore \hat{\omega}(\xi) = \hat{u}(\xi)\hat{v}(\xi)(2\pi)^{\frac{n}{2}} = |\hat{u}(\xi)|^2 (2\pi)^{\frac{n}{2}}$

$$(2\pi)^{\frac{n}{2}} \int_{R^n} |\hat{u}(\xi)|^2 d\xi = \int_{R^n} \hat{\omega}(\xi)d\xi = (2\pi)^{\frac{n}{2}} \omega(0) = (2\pi)^{\frac{n}{2}} \int_{R^n} u^2(x)dx$$

$f: L^2 \to L^2$? $\forall u \in L^2(R^n)$. 设 $u_n \in L^2(R^n) \cap L^1(R^n)$ $C_l^\infty(R^n)$ $u_n \to u$ $(L^2(R^n))$，那么 Fu_n 可定义，且 $Fu_n \in L^\infty \cap L^2(R^n)$，$\{Fu_n\}$ 是 L^2 中 Cauchy 列，$\|Fu_m - Fu_n\|_2 = \|u_m - u_n\|_2$

$\therefore \exists u(x) \in L^2(R^n)$ 使得 $\|Fu_n - v\|_{L^2} \to 0$. 定义 $Fu = v, v$ 不依赖于 $\{u_n\}$ 的选取，从而得到 $F: L^2 \to L^2$ $(A: u \to Y$ 有界 $A: X \to Y$，u 在 X 中稠密)

$$Fu(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-iA\xi} u(x) dx$$

定理：设 $u, v \in L^2(R^n)$，则

(1)$\int_{R^n} u\bar{v} dx = \int_{R^n} \hat{u}\bar{\hat{v}} dx \Leftrightarrow \langle u, v\rangle_{L^2} = \langle \hat{u}, \hat{v}\rangle_{L^2}$

(2) 若 $D^\alpha u \in L^2$ $(u \in C^\infty(R^n))$ 则 $(D^\alpha u)^\wedge(\xi) = (i\xi)^\alpha \hat{u}(\xi)$

证明：

$$(D^\alpha u)^\wedge(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-ix\xi} D^\alpha u(x) dx$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} D\left[\left(e^{-i\xi x} D^{\alpha-1} u(x)\right)\right] dx - \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} D\left(e^{-ix\xi}\right) D^{\alpha-1} u(x) dx$$

$$= i\xi \int_{R^n} e^{-ix\xi} D^{\alpha-1} u(x) dx = \cdots = (i\xi)^\alpha \hat{u}(\xi)$$

(3)$u, v \in L^2 \cap L^1 \Rightarrow (u * v)^\wedge = (2\pi)^{\frac{n}{2}} \hat{u} \cdot \hat{v}$

(4) **（Fourier 逆变换）** $u \in L^2 \cap L^1$，定义

$$\tilde{F}u(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{ix\xi} u(x) dx$$

则 $u = \tilde{F}(\hat{u})$

(5)F 线性：$(u(x - x_0))^\wedge = e^{-ix_0\xi}\hat{u}(\xi)$，$(u(\lambda x))^\wedge = \frac{1}{|\lambda|}\hat{u}\left(\frac{\xi}{\lambda}\right)$

例：$\int_\Omega Du dx = \int_{\partial\Omega} u \cdot n dx$ $(Du, \Omega) = (u \cdot n, \partial\Omega)$，$(du, \Omega) = (u, \partial\Omega)$ 外微分

热方程：

$$\begin{cases} \partial_t u - a^2 \Delta u = 0 \\ u(x, 0) = \phi(x) \end{cases} \quad \Delta\left(F^{-1}\hat{u}\right) = -|\xi|^2 F^{-1}\hat{u}$$

设 $u = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{ix\xi} \hat{u}(\xi, t) d\xi$，代入热方程，有：

$$(2\pi)^{-\frac{n}{2}} \int_{R^n} e^{i\pi\xi} \partial_t \hat{u}(\xi, t) d\xi + (2\pi)^{-\frac{n}{2}} a^2|\xi|^2 \int_{R^n} e^{i\pi\xi} \hat{u}(\xi, t) d\xi = 0$$

$$\Rightarrow \partial_t \hat{u}(\xi, t) + a^2|\xi|^2 \hat{u}(\xi, t) = 0 \quad \left(\partial_x\left(e^{ix\xi}\right) = i\xi e^{ix\xi}\right)$$

$$\hat{u}(\xi, t) = e^{-a^2|\xi|^2 t}\hat{u}(\xi, 0), \quad u(x, t) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{ix\cdot\xi} e^{-a^2|\xi|^2 t} \hat{u}(\xi, 0) d\xi$$

其中

$$\hat{u}(\xi, 0) = (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{-ix\xi} \phi(x) dx$$

从而

$$u(x,t)=F^{-1}\left(e^{-a^2|\xi|^2t}\hat{u}(\xi,0)\right)=(2\pi)^{-\frac{n}{2}}F^{-1}\left(e^{-a^2|\xi|^2t}\right)*F^{-1}\hat{u}(\varphi,v)$$
$$=F^{-1}\left((2\pi)^{-\frac{n}{2}}e^{-a^2|\xi|^2t}\right)*\phi(x)=\frac{1}{(4\pi t)^{\frac{n}{2}}}\int_{R^n}e^{-\frac{(x-y)^2}{4t}}\phi(y)dy=k*\phi$$

Poisson 核：

$$K(x,t)=\begin{cases}\frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{|x|^2}{4t}} & ,t>0\\ 0,\quad t\leqslant 0\end{cases}$$

若 $\phi\notin C^c$，$u(x,t)\notin C^c$ （无限传播速度）

对于热方程

$$\begin{cases}\partial_t u-a^2\Delta u=f\\ u(x,0)=\phi(x)\end{cases}\qquad u(x,t)=\int_{[0,t]\times R^n}K(x-y,t-\tau)f(y,t)dyd\tau$$

$$=\int_{R^n}K(x-y,t)\phi(y)dy+\int_0^t\left(\int_{R^n}K(x-y,t-\tau)f(\tau-y)dy\right)d\tau(*)$$

$K(x-y,t-\tau)$ 称为热方程的**基本解**.

$\Gamma(x,t:y,\tau)=k(x-y;t-\tau)$，有如下性质：

(1)$t>\tau$ 时，$\tau>0$

(2)$\Gamma(x,t;y,\tau)=\Gamma(y,t;x,\tau)$

(3)$t>\tau$ 时，$\int_{R^n}\Gamma(x,t;y,z)dy=1$

(4)$t>\tau$ 时，Γ 关于 $(x,t;y,\tau)$ 是 C^∞，$\begin{cases}(\partial_t-\Delta_x)\tau=0\\(\partial_t+\Delta_y)\tau=0\end{cases}$

(5)$t>\tau$ 时，$|\Gamma(x,t;y,\tau)|\leqslant\frac{1}{(4\pi(t-\tau))^{\frac{n}{2}}}$

(6) 若 $g\in C(R^n)\cap L^\infty(R^n)$，则 $\forall x\in R^n$

$$\lim_{t\to 0^+}\int_{R^n}\Gamma(x,t;y,0)g(y)dy=g(x)$$

从而有

$$\lim_{t\to 0^+}\int_{R^n}K(x-y,t)\phi(y)dy=\phi(x)$$

定理：设 $\phi\in C(R^n)\cap L^\infty(R^n),f\equiv 0$，则 (*) 得出的解满足：

(1)$u\in C^\infty(R^n\times(0,+\infty))$ (2)$\partial_t u-\Delta u=0$ (3) $\lim\limits_{t\to 0^+}u(x,t)=\phi(x)$

$\Rightarrow$ 是原问题的古典解.

$$\begin{cases}\partial_t u-a^2\Delta u=f\\ u(x,0)=\phi(x)\end{cases}\qquad F(x-t):\partial_t^2u-\partial_x^2u=0$$

$$u(x,0)=F(x)\in C^2 \quad u(x,t)\in C^2 \quad \partial_t u(x,0)=-F'(x)\in C^1$$

$$\begin{cases} \partial_t^2 u - a^2\Delta u = 0 & x\in R^n, t>0 \\ u(x,0)=\phi(x) & x\in R^n \\ \partial_t u(x,0)=\psi(x) & \end{cases} \qquad \phi < e^{A|x|}, \forall t>0 \qquad \Leftrightarrow a=\frac{F}{m}\propto X(t)$$

$$F^{-1}\hat{u}(\xi,t) \qquad \begin{cases} i\partial_t u - \Delta u = f \\ u(x,0)=\phi(x) \end{cases}$$

几个典型的 Fourier 变换：

(1)$f=e^{-|x|}$ ， $\hat{f}(\lambda)=\frac{2}{\sqrt{2\pi(1+\lambda^2)}}$

(2)$f=e^{-x^2}$ ， $\hat{f}(\lambda)=\frac{1}{\sqrt{2}}e^{-\frac{\lambda^2}{4}}$

(3)$f=e^{-Ax^2}$ ， $\hat{f}(\lambda)=\frac{1}{\sqrt{2A}}e^{-\frac{\lambda^2}{4A}}$

8 能量不等式

考虑方程:

$$\begin{cases} -\Delta u + c(x)u = f \ , \ x \in \Omega \\ u = 0 \ , \ x \in \partial\Omega \end{cases}$$

有

$$\int_\Omega -\Delta u u + c(x)u^2 = \int_\Omega fu$$

$$\int_\Omega -\Delta u \cdot u = \int_\Omega -\nabla\cdot(\nabla u u) + |\nabla u|^2 = -\int_\Omega v\cdot(\nabla u\cdot u)dS(x) + \int_\Omega |\nabla u|^2 = \int_\Omega |\nabla u|^2$$

$$\Rightarrow \int_\Omega |\nabla u|^2 + c(x)u^2 dx \leqslant \int_\Omega fudx$$

$(1)c(x) \geqslant c_0 > 0:$ 左边 $\geqslant \int_\Omega |\nabla u|^2 + c_0 u^2 dx$ 右边 $\leqslant \dfrac{c_0}{2}\int_\Omega u^2dx + \dfrac{1}{2c_0}\int_\Omega f^2 dx$

$$\Rightarrow \int_\Omega |\nabla u|^2 + \frac{c_0}{2}u^2 dx \leqslant \frac{1}{2c_0}\int_\Omega f^2 dx$$

若 $f \in L^2(\Omega) \Rightarrow u$, $\nabla u \in L^2(\Omega)$

$(2)c(x) \geqslant 0:$

$$\int_\Omega |\nabla u|^2 dx \leqslant \int_\Omega fudx \leqslant \frac{c_0}{4}\int_\Omega u^2 + \frac{1}{c_0}\int_\Omega f^2 dx$$

引理: 若 $\Omega \subseteq R^n$ 有界开区域, $u \in C_0^1(\Omega)$ $(u \in C^1(n), u|_{\partial\Omega} = 0)$, 则存在常数 $c_0 = c_0(\Omega)$, 使得

$$c_0\int_\Omega |\nabla u|^2 dx \geqslant \int_\Omega u^2 dx$$

设 Ω 的直径为 d , 则 c_0 可取 $4d^2$, 对应 **Friedrichs 不等式**

证明: 不妨设 $\Omega \in Q$, 令 $Q = \{x \in R^n \mid 0 \leqslant x \leqslant 2d\}$ $d(1,1,\cdots,1) \in \Omega$

令

$$\tilde{u} = \begin{cases} 0, & x \notin \Omega \\ u, & x \in \Omega \end{cases} \qquad \tilde{u}(x_1,\cdots,x_n) = \tilde{u}(0,x_2,\cdots,x_n) + \int_0^{x_1} \partial_1\tilde{u}(s,x_2,\cdots,x_n)\,ds$$

$$\leqslant x_1^{\frac{1}{2}}\cdot\left(\int_0^{x_1}|\partial_1\tilde{u}|^2 ds\right)^{\frac{1}{2}} \leqslant x_1^{\frac{1}{2}}\cdot\left(\int_0^{2d}|\partial_1\tilde{u}|^2 ds\right)^{\frac{1}{2}}$$

$$\therefore \int_\Omega u^2dx = \int_Q \tilde{u}^2 dx \leqslant \int_0^{2d} x_1 dx_1 \int_0^{2d}\cdots\int_0^{2d}|\nabla\tilde{u}|^2 ds dx_2\cdots dx_n \leqslant 4d^2\int_Q |\nabla\tilde{u}|^2 dx = 4d^2\int_Q|\nabla u|^2 dx$$

$$\text{左边} \geqslant \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{c_0}{2}\int_\Omega u^2 dx \quad \Rightarrow \frac{1}{2}\int_\Omega|\nabla u|^2 + \frac{c_0}{4}\int_\Omega u^2dx \leqslant \frac{1}{c_0}\int_\Omega f^2dx \quad \textbf{能量模估计}$$

$-\Delta u+c(x)u=f \quad c(x)\geqslant 0 \quad (u|_{\partial\Omega}=0)$

$$\int -\Delta u\cdot udx=\int -\nabla(\nabla u\cdot u)dx+\int_n |\nabla u|^2dx$$

$$\langle -\Delta u,u\rangle_{L^2(\Omega)}=\|\nabla u\|_L^2\geqslant c_0\|u\|^2\geqslant 0\ ,\ \langle Au,u\rangle_{L^2}\geqslant c_0\|u\|_{L^2}^2$$

A 非负定。若有特征值 λ，则 $\lambda>0$

考虑

$$-\sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right)+c(x)u=f \quad (c(x)\geqslant 0)$$

$$\int_\Omega -\sum_{i,j=1}^n \partial_i\left(a_{ij}\partial_j u\right)u+c(x)u^2=\int_\Omega fudx=\int_\Omega -\sum_{i=1}^n\partial_i(\underbrace{\sum_{j=1}^n a_{ij}(x)\partial_j\, uu}_{F_i})+\sum_{i=1}^n\sum_{j=1}^n a_{ij}(x)\partial_j u\partial_i u$$

$$=-\int_{\partial\Omega}F\cdot rdS(x)+\int_\Omega a_{ij}\partial_i u\partial_j u\cdot udx\geqslant\lambda\int_\Omega|\nabla u|^2dx \quad (\lambda|\xi|^2\leqslant a_{ij}(x)\xi_i\xi_j\leqslant\lambda|\xi|^2)$$

第二边值问题：

$$\begin{cases}-\Delta u+c(x)u=f\\ \partial y u=0 \quad x\in\partial\Omega\end{cases} \qquad \int -\Delta u\cdot u+c(x)u^2dx=\int_\Omega fudx=\int_{\partial\Omega}\cancel{-\partial_\nu uu}dx+\int_\Omega\left(|\nabla u|^2+c(x)u^2\right)dx$$

$$\int_\Omega|\nabla u|^2+c(x)u^2dx=\int_\Omega f\cdot udx\ ,\ \int_\Omega|\nabla u|^2\geqslant c_0\int_\Omega u^2dx$$

$$\int_\Omega|\nabla u|^2+c(x)u^2dx=0\ ,\ \int_\Omega|\nabla u|^2dx=0\ ,\ u\equiv c.\ c\neq 0,\ 则c(x)\equiv 0.$$

$$\int_{\partial\Omega}-\partial_\nu uudS(x)=-\int_{\partial\Omega}\alpha(x)u^2dS(x)\geqslant 0$$

$$\int_{\partial\Omega}\alpha(x)u^2dx+\int_\Omega|\nabla u|^2+c(x)u^2dx\leqslant\int_\Omega fudx$$

考虑方程：

$$\begin{cases}\partial_t u-\Delta u=f\ ,\ x\in\Omega \quad t\geqslant 0\\ u(x,0)=\phi(x)\ ,\ x\in\Omega\\ u|_{\partial\Omega}=0\end{cases} \qquad (\partial_t u-\Delta u)\,u=fu$$

$$\int_\Omega\partial_t uu-\Delta u\cdot udx=\int_\Omega f\cdot u=\frac{d}{dt}\frac{1}{2}\int_\Omega u^2dx+\int_\Omega|\nabla u|^2dx\geq\frac{1}{2}\frac{d}{dt}\int_\Omega u^2dx+c_0\int_\Omega u^2dx$$

且

$$\int_\Omega fudx\leq\frac{c_0}{2}\int_\Omega u^2dx+\frac{1}{2c_0}\int_\Omega f^2dx$$

若 $f\equiv 0$，则

$$\frac{d}{dt}\underbrace{\frac{1}{2}\int_\Omega u^2dx}_{A}+\underbrace{\int_\Omega|\nabla u|^2dx}_{B}=0 \quad A'+B=0\ ,\ A\downarrow\ ,\ B\geq c_0\int_\Omega u^2dx$$

$$\frac{1}{2}\int_\Omega u^2dx\leqslant\frac{1}{2}\int_\Omega\phi^2dx,\quad \frac{d}{dt}\left(\frac{1}{2}A\right)+c_0A\leqslant 0\ ,\ \left(e^{2c_0t}A\right)'\leqslant 0\ ,\ A\leqslant e^{-2c_0t}\int_\Omega\phi^2dx$$

若 $f\not\equiv 0$

$$\frac{d}{dt}\underbrace{\int_\Omega u^2dx}_{A}+c_0\int_\Omega u^2dx\leqslant\underbrace{\frac{1}{c_0}\int_\Omega f^2dx}_{B}\quad A'+c_0A\leqslant B\ ,\ \left(e^{c_0t}A\right)'\leqslant e^{c_0t}B$$

$$\Rightarrow e^{c_0t}A(t)-A(0)=\int_0^t e^{c_0t}B(\tau)d\tau.$$

由此可以得出**Gronwall 不等式：**

$$A(t)\leqslant e^{-c_0t}A(0)+e^{-c_0t}\int_0^t e^{c_0\tau}B(\tau)d\tau$$

定理：

$$\int_\Omega u^2dx\leqslant e^{-c_0t}\int_\Omega\phi^2(x)dx+\int_0^t e^{-c_0(t-\tau)}\left(\int_\Omega f^2(x,\tau)dx\right)d\tau$$

第二类边界条件：

$$\underbrace{\frac{d}{dt}\frac{1}{2}\int_\Omega u^2dx}_{E'(t)}+\underbrace{\int_\Omega|\nabla u|^2dx}_{H(t)}=\int_\Omega u\cdot fdx\leqslant\underbrace{\frac{1}{2}\int_\Omega u^2dx}_{E(t)}+\underbrace{\frac{1}{2}\int_\Omega f^2dx}_{F(t)}$$

$$E'(t)+H(t)\leqslant E(t)+F(t)\ \Rightarrow\ E(t)+e^t\int_0^t e^{-\tau}H(\tau)d\tau\leqslant e^tE(0)+e^t\int_0^t e^{-\tau}F(\tau)d\tau$$

$$\frac{1}{2}\int_\Omega u^2+\int_0^t e^{t-\tau}\left(\int_\Omega|\nabla u|^2dx\right)d\tau\leqslant e^t\int_\Omega\phi^2dx+\frac{1}{2}\int_0^t e^{t-\tau}\int_\Omega f^2dxd\tau$$

定理：

$$\begin{cases}\partial_t u-\Delta u=0 & x\in\Omega\ ,\ 0\leq t\leq T\\ u(x,T)=0 & \\ u|_{\partial\Omega}=0 & 0\leq t\leq T\end{cases}\qquad \text{仅有零解.}$$

上述方程 $\Longleftrightarrow\begin{cases}\partial_t u+\Delta u=0\\ u(x,0)=0\\ u|_{\partial a}=0\end{cases}$ 反向热方程解的唯一性：适定性、存在性、唯一性，“连续依赖性”

证明：令 $e(t)=2\int_\Omega u^2(x,t)dx$，$e'(t)=2\int_\Omega \partial_t u\cdot udx=2\int_\Omega \Delta u_x dx=-2\int_\Omega |\nabla u|^2dx$

$$e''(t)=-2\int_\Omega \nabla u\cdot\nabla u_t dx'=-2\int_\Omega [\nabla(\nabla u u_t)-\Delta u u_t]\,dx=2\int_\Omega (u_t)^2dx \quad \left(\int_{\partial\Omega}\nu\nabla u\cdot u_t dS_{(x)}\equiv 0\right)$$

$$\Rightarrow (e'(t))^2=4\left(\int_\Omega \partial_t u\cdot udx\right)^2\le 4\int_\Omega(\partial_t u)^2dx\int_\Omega u^2dx=e''(t)\,e(t)$$

$$\left(\frac{e}{e'}\right)'=\frac{(e')^2-e''e}{(e')^2}\le 0\ ,\ \left(\frac{e'}{e}\right)'=\frac{e''e-(e')^2}{e^2}\ge 0\iff (\ln e(t))''\ge 0$$

$e(T)=0$，要证 $e(t)\equiv 0$ $(\forall 0\le t\le T)$

设 $e(t)\not\equiv 0$，则存在 $t_1,t_2\in[0,T]$ 使得 $e(t)>0$ $\quad(t\in[t_1,t_2)$ 时$)\quad e(t_0)=0$

(取 $e(t_1)>0$，再取 $t_2=t_2=\inf\limits_{\zeta>t_1}\{e(\zeta)=0\}$)

$$\ln e(s_1)+\ln e(s_2)\ge 2\ln e\left(\frac{s_1+s_2}{2}\right)\ ,\ e(s_1)\,e(s_2)\ge\left(e\left(\frac{s_1+s_2}{2}\right)\right)^2$$

取 $s_1=t_1,s_2=t_2-\varepsilon$，$e(t_1)\,e(t_2-\varepsilon)\ge\left(e\left(\frac{s_1+s_2-\varepsilon}{2}\right)\right)^2$

令 $\varepsilon\to 0$，有：左边 $\to 0$，右边 >0，矛盾.

$$\Omega=[0,2\pi]\quad u_n(x,t)=ne^{n^2(t-T)}\sin nx\ \partial_t u_n+\Delta u_n=n^2u_n-n^2u_n=0$$

$$u_n(x,0)=ne^{-n^2T}\sin nx\ ,\ \int_0^1 n^2e^{-n^2T}\left(\sin^2 nx\right)dx\xrightarrow{n\to+\infty}0$$

$$u_n(x,T)=\int_0^1 n^2\sin^2 nxdx=\int_0^1 n^2\left(\frac{1+\cos 2nx}{2}\right)dx=\frac{1}{2}n^2\xrightarrow{n\to\infty}+\infty$$

波方程初值问题的能量不等式：

$$\begin{cases}\partial_t^2 u-\Delta u=0 & x\in R^n\ ,\ t>0\\ u(x,0)=\phi(x) & x\in R^n\\ \partial_t u(x,0)=\psi(x) & x\in R^n\end{cases}$$

$$\text{左边}=\frac{d}{dt}\frac{1}{2}\int_{R^n}(\partial_t u)^2dx-\int_{R^n}\Delta u\partial_t udx=\frac{d}{dt}\frac{1}{2}\int_{R^n}(\partial_t u)^2+\frac{d}{dt}\frac{1}{2}\int_{R^n}|\nabla u|^2dx$$

$$=\frac{d}{dt}\frac{1}{2}\int_{R^n}\left(\underbrace{|\partial_t u|^2}_{\text{动能}}+\underbrace{|\nabla u|^2}_{\text{势能}}\right)dx=0$$

$$E(t)=\frac{1}{2}\int_{R^n}|\partial_t u|^2+|\nabla u|^2dx=\frac{1}{2}\int_{R^n}|\nabla\phi|^2+|\psi|^2dx$$

$$C(x_0,t_0)=\{(x,t)\in R^n\times R_+;|x-x_0|\le t_0-t\}\quad C_t=\{x\in R^n\mid(x,t)\in C(x_0,t_0)\}$$

定理：

$$\int_{C_r}\frac{1}{2}\left((\partial_t u)^2+|\nabla u|^2\right)dx\le M_1\left(\int_C\frac{1}{2}\left(\psi^2+|\nabla\phi|^2\right)dx+M\int_0^t\int_{C_\tau}f^2(y,\tau)\,dyd\tau\right)$$

其中 M 仅依赖于 t （若 $f\equiv 0$，M 可取 1）

(1) $f\equiv 0$ 时，

$$\frac{1}{2}\int_{C_t}(\partial_x u)^2+|\nabla u|^2dx\le\frac{1}{2}\int_{C_0}\left(\psi^2+|\nabla\phi|^2\right)dx.\quad\left(\int fu_tdx\le\varepsilon\int_\Omega(u_x)^2+\frac{1}{\varepsilon}F^2\right)$$

（雷诺）输运定理： $g\in C^1(R^n\times R_+)$，则有

$$\frac{d}{dt}\int_{C_t}gdx=\int_{C_t}g_tdx-\int_{\partial C_t}g(x,t)\,dS(x)$$

证明：$C_t=\{x\,||x-x_0|\le t_0-t\}$.

令 $y=\frac{x-x_0}{t_0-t}$，$x=x_0+(t-t_0)\,y\Leftrightarrow x\in C_t\Leftrightarrow y\in\{y||y|\le 1\}$

$$\int_{c_t}gdx=\frac{d}{dt}\int_{B_1}g\left(x_0+(t-t_0)\,y_1t\right)(t-t_0)^ndy$$

$$=\int_{B_1}\partial_tg\left(x_0+(t-t_0)\,yt\right)(t-t_0)^n+y\nabla g\left(x_0+(t-t_0)\,yt\right)(t-t_0)^n+\underline{\int_{B_1}gn(t-t_0)^{n-1}dy}\rightarrow\int_{C_t}g\frac{n}{t-t_0}$$

$$\int_{B_1}y\nabla g(t-t_0)^ndy=\int_{C_t}\frac{x-x_0}{t-t_0}\nabla_xgdx=\int_{C_t}\left[\underbrace{v_x\left(\frac{x-x_0}{t-t_0}g\right)}_{\mathrm{I}}-\underbrace{\nabla_x\left(\frac{x-x_0}{t-t_0}\right)g}_{\mathrm{II}}\right]dx$$

$$=\int_{\partial C_t}\frac{x-x_0}{t_0-t}\cdot\frac{x-x_0}{t-t_0}gdS_{(x)}-\int_{C_t}\frac{n}{t-t_0}gdx=-\int_{\partial C_t}gdS_{(x)}-\int_{C_t}\frac{n}{t-t_0}gdx$$

$$\frac{d}{dt}\int_{C_t}\frac{1}{2}\left\{(\partial_tu)^2+|\nabla u|^2\right\}dx=\int_{C_t}\left(\partial_tu\partial_tu+\nabla u\cdot\nabla u_0\right)dx-\int_{\partial C_t}\frac{1}{2}\left[(\partial_tu)^2+|\nabla u|^2\right]dS(x)$$

$$=\int_{C_t}\partial_tu\left(\partial_tu-\Delta u\right)+\partial_tu\Delta_u+\nabla u\cdot\nabla u_t-\int_{\partial C_t}\frac{1}{2}\left[(\partial_tu)^2+|\nabla u|^2\right]dS(x)$$

$$=\int_{C_x}\partial_tufdx+\int_{\partial C_x}\nu\nabla uu_tdS(x)-\int_{\partial C_x}\frac{1}{2}\left(u_t^2+|\nabla u|^2\right)dS(x)\le\frac{1}{2}\int_{C_t}\left[(\partial_tu)^2+f^2\right]dx$$

其中 $\frac{1}{2}\left({u_t}^2+|\nabla u|^2\right)\ge\frac{1}{2}\left({u_t}^2+|\nu\cdot\nabla u|^2\right)\ge u_t\nu\cdot\nabla u.$

$\Rightarrow E'(t)\le E(t)+F(t)$，$e^{t_0}=M$

$$\left(e^{-t}E(t)\right)'\le e^{-t}F(t)\,,\;E(t)\le e^tE(0)+e^t\int_0^te^{-\tau}F(\tau)\,d\tau\le M\left(E(0)+\frac{1}{2}\int_0^t\int_{c_\tau}f^2dyd\tau\right)$$

$$\frac{d}{dt}\int_{R^n}u^2+\int_{R^n}|\nabla u|^2dx=\int_{R^n}u\cdot fdx$$

$$\begin{cases} \partial_t^2 u - \Delta u = f \\ u(x,0) = \phi(x) \\ \partial_t u(x,0) = \psi(x) \\ u(x,t) = 0\ ,\ x \in \partial\Omega \quad (\partial_t u \equiv 0\ ,\ x \in \partial\Omega) \end{cases}$$

$$\int_\Omega \frac{1}{2}\left((\partial_t u)^2 + |\nabla u|^2\right) dx \le M\left(\int_\Omega \frac{1}{2}\left(\psi^2 + |\nabla\phi|^2\right) dx + \int_0^t \int_\Omega \frac{1}{2} f^2 dx dr\right)$$

$\int_{\partial\Omega} \partial_\nu u u_t dx = 0$ (第一类、第二类边界条件都成立)

$$\partial_\nu u + \alpha u = 0\ ,\ \alpha > 0\ \rightarrow\ \int_{\partial\Omega} \partial_\nu u u_t + d\,\mathrm{S}(x) = \int_{\partial\Omega} -\alpha u u_t\,\mathrm{S}(x) = \frac{d}{dt} - \int_{\partial\Omega} \frac{\alpha}{2} u^2 dS(x)$$

$$E(t) = \frac{1}{2}\int_\Omega u_t^2 + |\nabla u|^2 dx + \frac{1}{2}\int_{\partial\Omega} \alpha u^2 dS(x)\ ,\ E'(t) \le E(t) + F(t)$$

$u|_{\partial\Omega} = g$ 时，令 $v = u - g$ 可证.

9 最大模估计

位势方程最大模估计:

$$\begin{cases} Lu := -\Delta u + c(x) u = f \quad x \in \Omega \subseteq R^n \\ u|_{\partial\Omega} = g \quad c(x) \geq 0 \end{cases} \quad \| u \|_{L^\infty(\Omega)} \leq C \left(\| f \|_{L^\infty(\Omega)} + \| g \|_{L^\infty(\partial\Omega)} \right)$$

$$\begin{cases} \Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases} \Rightarrow \| u\|_{L^\infty} \leq C \| g\|_{L^\infty} \quad -\Delta u + c(x)u = 0$$

$$\text{设 } u(x_0) = \max_{\bar{\Omega}} u(x) \text{ , } \nabla^2 u(x_0) \text{ 非正定的，} \Delta_\alpha(x_0) = \operatorname{tr} \nabla^2 u(x_0) \leq 0$$

若 $x_0 \in \partial\Omega$, $u(x_0) = \| g \|_{L^\infty(\partial\Omega)}$

若 u 在内部取到非负最大值，一定有 $\Delta u(x_0) = 0$, $c(x_0)u(x_0) = 0$

若 $c(x) > 0$, $u(x) > 0 \Rightarrow u$ 不能在内部取到非负最大值.

引理: $u \in C^2(\bar{\Omega}) \cap C'(\bar{\Omega}), C(x) \geq 0$. 满足 $-\Delta u + c(x)u < 0$. 则 u 不可能在内部取到非负最大值.

弱极值原理: $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ 满足 $-\Delta u + c(x) u \leq 0$，则

$$\max_{x\in\Omega} u \leq \max_{x\in\partial\Omega} u^+, u^+ = \max\{0, u\}$$

证明：设 $0 \in \Omega$，令 $\tilde{u}(x) = u(x) + \varepsilon(|x|^2 - \alpha^2)\, d$ 为 Ω 直径

$$-\Delta\tilde{u}+c(x)\tilde{u} = -\Delta u+c(x)u+\varepsilon\left(-\Delta|x|^2 + c(x)\left(|x|^2 - d^2\right)\right) \leq \varepsilon\left(-2n + c(x)\left(|x|^2 - d^2\right)\right) \leq -2n\varepsilon < 0$$

$$\max_{\Omega} \tilde{u} \leq \max_{\partial\Omega} \tilde{u}^+ \leq \max_{\partial\Omega} u^+ + \varepsilon \max_{\partial\Omega} \left(\left(|x|^2 - d'\right)^+\right)$$

$$\max \tilde{u} \geq \max_{\Omega} u - \varepsilon d^2 \Rightarrow \max_{\Omega} u \leq \max_{\partial\Omega} u^+ + \varepsilon d^2. \text{ 令}\varepsilon \to 0\text{即可}$$

Hopf 引理: 设 B 为 R^n 中开球，$u(x) =\in C^2(B) \cap C^1(\bar{B})$ 满足

(1) $Lu = -\Delta u + c(x) u \leq 0 \quad (c(x) \geq 0\text{在}B\text{上有界})$

(2) (2) $\exists x_0 \in \partial B$ 使得 $u(x_0) \geq 0$ 且 $u(x) < u(x_0)$ $(\forall x \in B)$

则 $\partial_\nu u(x_0) > 0$

证明：设 $B = B_r(0)$ ，考察 $\omega(x) = u(x) + \varepsilon v(x)$ 以及区域 $u(r) = \{|x| \in \left(\frac{r}{2}, r\right)\}$

$Lw = Lu + \varepsilon Lv \leq \varepsilon Lv < 0$ ，w 在 ∂u_r 上取到 u_r 的非负最大值，记为 x_1

(1) $x_1 \in \partial B_{\frac{r}{2}}$, $u(x_1) < u(x_0)$

$$w(x_1) = u(x_1) + \varepsilon v(x_1) \overset{?}{<} w(x_1) = u(x_0) + \varepsilon v(x_0) \quad \begin{cases} ① - \Delta v + c(x) v \leq 0 \\ ② \begin{cases} v(x_1) \leq 0 \\ v(x_0) = 0 \end{cases} \end{cases}$$

矛盾. 取 $v(x) = v(|x|) = (|x|^\alpha - r^\alpha) \leq 0 \;\; \alpha < 0, n - 2r\alpha < 0$

$$\nabla(|x|^\alpha) = \alpha|x|^{\alpha-1}\cdot|x| \text{ , } -\Delta v = -\alpha\nabla(|x|^{\alpha-2}x) = -\alpha n|x|^{\alpha-2} - \alpha(\alpha-2)\, x|x|^{\alpha-4}x = -|x|^{\alpha-2}\left(\alpha n + \alpha^2 - 2\alpha\right)$$

$$= -\alpha|x|^{\alpha-2}(n-2+\alpha)$$

(2) $x_1 \in \partial B_r$，若 $x_1 \neq x_0$

$$w(x_1) = u(x_1) + \varepsilon v(x_1) < u(x_0) + \varepsilon v(x_0) = w(x_0)$$

$$\Rightarrow x_1 = x_0 \quad \Rightarrow \partial_\nu w(x_0) \geq 0 \quad \Rightarrow \partial_\nu u(x_0) + \varepsilon \partial_\nu v(x_0) \geq 0$$

回顾：$w = u + \varepsilon v$，v 待定，$u(r) = \{\frac{r}{2} \leq |x| \leq r\}$ 上用弱极值原理.

$$\left.\begin{array}{l} \text{①排除 } x_1 \in \partial B_{\frac{r}{2}} \\ \text{②排除 } x_1 \neq x_0 \end{array}\right\} x_1 = x_0 \text{ (} w \text{ 在 } x_0 \text{ 取到最大值) } \frac{\partial w}{\partial \nu} \geq 0 \Rightarrow \frac{\partial u}{\partial \nu} \geq -\varepsilon \frac{\partial v}{\partial \nu}$$

强极值原理： $\Omega \subseteq R^n$ 有界连通开集，$c(x) \geq 0$ 有界. $u \in C^2(x) \cap C^1(\bar{\Omega})$ 在 Ω 上满足 $Lu \leq 0$. 若 u 在 Ω 中取到非负最大值，则 u 为常数.

证明：记 $A = \{x \in \Omega \mid u(x) = \max\limits_{\bar{\Omega}} u\}$

①$u \in C(\bar{\Omega})$ u 为相对闭集.

②u 为 Ω 中开集，若不然，设 $x_0 \notin A$ 且 $\forall u_k, \exists x_k \in u_k$，$x_k \notin A$. $\exists r > 0$ 使得 $B_{2r}(x_0) \subset \Omega$，$\exists \widetilde{x} \notin A$，$\widetilde{x} \in B_r(x_0)$. $\widetilde{x}$ 不是 A 的聚点，因此

$$d = \mathrm{dist}\,(\widetilde{x}, A) = \min\{|x - \widetilde{x}|, x \in A\} \text{ 存在}$$

显然 $d \leq r \quad \Rightarrow B_d(\widetilde{x}) \subset B_{2r}(x_0) \subset \Omega$. 设 $y_0 \in \partial B_d(\widetilde{x}) \cap A$

$$\forall y \in B_d(x),\ y \neq y_0 \quad \Rightarrow u(y) < u(y_0) = M = \max_{\bar{\Omega}} u \quad \Rightarrow \left.\frac{\partial u}{\partial x_1}\right|_{x=y_0} = 0 \ (u \text{ 在 } x_0 \text{ 极大})$$

由 Hopf 引理，$\exists r\ (r(y, -\widetilde{x}) > 0)$，$\frac{\partial y}{\partial \nu}|_{x=y_0} > 0$，矛盾.

比较原理： $\mathcal{L}u_1 \leq \mathcal{L}u_2$，$(u_1 - u_2)|\partial\Omega \leq 0$

$$\begin{cases} -\Delta u = f \\ u|_{\partial\Omega} = g \end{cases}$$

定理：设 $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ 是该方程的解，则

$$\max_{\bar{\Omega}} |u| \leq G + C(n,d)\,F \quad \left(G = \max_{\partial\Omega} |g| \ ,\ F = \max_{\bar{\Omega}} |f| \ ,\ d \text{ 为 } \Omega \text{ 直径}\right)$$

证明：$0 \in \Omega$，$|x| \leq d(x \in \bar{\Omega})$

令 $w(x) = u(x) - v(x)$

目标：(1) $-\Delta w \leq 0$ $(-\Delta u + \Delta v \leq 0)$ (2)$w|_{\partial\Omega} \leq 0$，$f + \Delta v \leq 0$ $\Rightarrow w|_\Omega \leq 0$ $\Rightarrow u(x)|_\Omega \leq v(x)|_\Omega$

$$v(x) = G + \frac{F}{2n}\left(-|x|^2 + d^2\right) \ ,\ \Delta v + f = \frac{F}{2n}\Delta\left(-|x|^2\right) + f = -F + f \leq 0$$

$$u(x) \leq v(x) \leq G + \frac{Fd^2}{2n} \quad \Rightarrow C(n,d) \text{ 取 } \frac{d^2}{2n} v(x) \text{ 称为 } \textbf{barrier function（闸函数）}$$

$$\begin{cases} -\Delta u_i = f_i \\ u_i \mid_{\partial n} = g_i \end{cases} i = 1, 2 \quad |u_1 - u_2| \leq c\left(|f_1 - f_2| + |g_1 - g_2|\right)$$

第三边值问题：

$$\begin{cases} -\Delta u + c(x)\, u = f \\ \frac{\partial u}{\partial \nu} + \alpha(x)\, u\big|_{\partial\Omega} = g(x) \;\; \alpha(x) \geq \alpha_0 > 0 \end{cases}$$

定理：

$$\max_{\bar{\Omega}} |a| \leq C\left(F + G\right), C = C\left(n, d, \alpha_0\right)$$

证明：引理：

$$\begin{cases} -\Delta w + c(x)\, w \leq 0 \\ \dfrac{\partial w}{\partial \nu} + \alpha w \leq 0 \end{cases} \Rightarrow w \text{ 在 } \bar{\Omega} \text{ 上 } \leq 0 \qquad v(x) = \frac{F}{2n}\left(d^2 - |x|^2\right) + \widetilde{G}$$

令 $w(x) = u(x) - v(x)$ ，$\mathcal{L}w = \mathcal{L}u - \mathcal{L}v = f + \Delta v - c(x)\, v \leq 0$

$$\frac{\partial w}{\partial \nu} + \alpha w = \frac{\partial u}{\partial \nu} + \alpha u - \frac{\partial v}{\partial \nu} - \alpha v = g - \frac{\partial v}{\partial \nu} - \alpha v \leq 0 \;\Leftrightarrow\; g \leq \frac{\partial v}{\partial \nu} + \alpha v$$

$$\frac{\partial}{\partial \nu}\left(\frac{F}{2n}\left(d^2 - |x|^2\right) + \widetilde{G}\right) + \alpha\left(\frac{F}{2n}\left(d^2 - |x|^2\right) + \widetilde{G}\right) = -\frac{F}{n} x \cdot \nu + \alpha\left(\frac{F}{2n}\left(d^2 - |x|^2\right) + \widetilde{G}\right) \widetilde{G} \geq \frac{E}{\alpha_0}\left(G + \frac{Fd}{n}\right)$$

热方程最大模估计：

$$Q_T = \{(x, t) | x \in \Omega\ ,\ t \in (0, T]\}$$

抛物边界：

$$\partial_P Q_T = \{\bar{\Omega} \times \{0\}\} \cup \{\partial\Omega \times (0, T]\} = \partial\Omega \backslash (\Omega \times \{T\})$$

$$\partial_t u - \Delta u = 0 \text{ 考虑 } u(x_0, t_0) = \max_{(x,t) \in Q} u(x, t) \text{ 则}$$

$$\Delta u|_{(x_0, t_0)} \leq 0\ ,\ \partial_t u|_{(x_0, t_0)} \geq 0\ ,\ (\partial_t u - \Delta u)|_{(x_0, t_0)} \geq 0$$

$$v = u - \varepsilon t\ ,\ \partial_t v - \Delta v = -\varepsilon < 0$$

在 $\bar{Q}_T$ 中 v 的最大值不可能在 Q_T 中取到.

$$\max_{\bar{Q}_T} (u - \varepsilon T) \leqslant \max_{\bar{Q}_T} v = \max_{\partial_p Q_T} v < \max_{\partial_p Q_T} u$$

$$\Rightarrow \max_{\bar{Q}_T} u \leqslant \max_{\partial_p Q_T} u - \varepsilon T. \quad \text{令 } \varepsilon \to 0\ ,\ \max_{\bar{Q}_T} u = \max_{\partial_p Q_T} u$$

设 u 满足 $\mathcal{L}u = \partial_t u - \Delta u \leq 0 (\geq 0)$，则称 u 是热方程 $\partial_t u - \Delta u = 0$ 的**下解（上解）**

(热方程最大模估计) 定理：$u \in C^{2,1}(Q_T) \cap C(Q_T)$ 是热方程的一个下解，则 u 在 $\bar{Q}_T$ 上的最大值一定在抛物边界取到，即

$$\max_{\bar{Q}_T} u = \max_{\partial_p Q_T} u$$

比较原理： $u,v\in C^{2,1}(Q_T)\cap C(Q_T)$ 满足 $\mathcal{L}u\le\mathcal{L}v$，且对 $(x,t)\in\partial_P Q_T$，则在 $\bar{Q}$ 上，有 $u\le v$ **第一边值问题的最大模估计：**

$$\begin{cases}\mathcal{L}u=\partial_t u-\Delta u=f \quad x\in\Omega \quad 0<t\leqslant T\\ u|_{t=0}=\varphi(x) \quad x\in\Omega\\ u(x,t)=g(x,t) \quad x\in\partial\Omega\end{cases} \qquad \text{闸函数：}v=\begin{cases}\mathcal{L}u\in\geqslant f\\ v|_{\partial_P Q_T}\geqslant u|_{\partial_P Q_T}\end{cases} \quad u\leqslant v$$

定理：设 $u\in C^{2,1}(Q_T)\cap C(Q_T)$ 满足上述方程，则：

$$\max_{\bar{Q}_T}|u|\le T\sup_{Q_T}\|\ f\ \|+\max\left\{\max_{\bar{\Omega}}|\varphi|,\max_{\partial\Omega\times[0,T]}|g|\right\}=TF+B$$

$$v=B+tF\ (\partial_t v-\Delta v\ge F)\ ,\ \partial_t v-\Delta u=F\ge\pm f=\pm\mathcal{L}u$$

$$v|_{\partial_P Q_T}\ge B\ge(\pm u)|_{\partial_P Q_T}\quad\Rightarrow\max_{\bar{Q}_T}(\pm u)\le\max_{\bar{Q}_T}v\le B+TF.$$

推论：(1) 唯一性 (2) 解对初边值，外力项的连续依赖性。

$$\begin{cases}Lu_1=f_1\\ Lu_2=f_2\end{cases}\quad\Rightarrow L(u_1-u_2)=f_1-f_2$$

第二、第三边值问题的最大模估计：

$$\begin{cases}\partial_t u-\Delta u=fx\in\Omega\ t>0\\ u(x,0)=\varphi(x)\ \ x\in\Omega\\ \dfrac{\partial u}{\partial\nu}+\alpha(x,t)u=g(x,t)\ \ x\in\Omega\ t>0\end{cases}\qquad \alpha=0:\ \text{第二}\quad \alpha\ne0:\ \text{第三}$$

定理：设 $u\in C^{2,1}(Q_T)\cap C^{1,0}(\bar{Q}_T)$ 是混合问题的解，则有：

$$\max_{\bar{Q}_T}|u|\le C(T)(F+B)$$

引理：（二、三边值比较定理）

$$\begin{cases}\mathcal{L}u\ge0\\ u(x,0)\ge o\\ \dfrac{\partial u}{\partial\nu}+\alpha(x,t)u=g(x,t)\ \ x\in\Omega\ t>0\end{cases}$$

且 $\alpha\ge0$ ，$u\in C^{2,1}(Q_T)\cap C^{1,0}(\bar{Q}_T)\ \Rightarrow u\ge0\ \ \left((x,t)\in\bar{Q}_T\right)$

证明：①$\mathcal{L}u\ge0$ 时，$\partial_\nu+\alpha u>0$

(1) 证明 u 的最小值在抛物边界上取到. 设 u 在 (x_0,t_0) 取到最小值, 则 $(x_0,t_0)\in\Omega\times(0,T]$, $(\partial_t u-\Delta u)(x_0,t_0)\ge0$，矛盾. $\partial_t u|_{(x_0,t_0)}\ge0\quad\Delta u|_{(x_0,t_0)}\le0$

(2) 若 $(x_0,T_0)\in\Omega\times\{0\}\ \Rightarrow\ u(x,t)\ge u(x_0,T_0)\ge0$

(3) 若 $(x_0.t_0)\in\partial\Omega\times(0,T)$，则 $\partial_\nu u\ _{(x_0,t_0)}\le0$ ，$\alpha u>0$ 而 $\alpha>0\ \Rightarrow u>0$

②$\mathcal{L}u\ge0$ ，$\partial_\nu u+\alpha u\ge0$ 时，令 $w=u+\varepsilon v$

$\mathcal{L}u>0\quad\partial_\nu u+\alpha u>0\quad v(x,0)\ge0\quad v=t+2nt+|x|^2$ ，$\Omega=B_R(0)$. $\mathcal{L}v=1+2n-\Delta\left(|x|^2\right)=1>0$

$$\underline{v\nabla v}+\alpha v=|x|+\alpha\left((2n+1)\,t+|x|^2\right)>0\ \left(v\nabla v=\frac{x}{|x|}\cdot x\right),\ v\,(x,0)>0\,,\ \mathcal{L}w>0\,,\ \partial_\nu w+\alpha w>0$$

$$w(x,0)\geq 0\,,\ w(x,t)\geq 0\ \ \left((x,t)\in\bar{Q}_T\right)\ \Rightarrow u\geq -\varepsilon\left((2n+1)t+|x|^2\right)$$

令 $\varepsilon\to 0$，$u\geq 0\ \ \left((x,t)\in\bar{Q}_T\right)$

定理证明：考虑辅助函数 $\Omega=B_R(0)$

$$w\,(x,t)=\left(Ft+B\left(1+\frac{v}{R}\right)\right)\pm u\quad v=2nt+|x|^2$$

$$\mathcal{L}w=\partial_t w-\Delta w=F\pm f\quad \mathcal{L}v=0\quad w(x,0)=B+B\frac{|x|^2}{R}\pm\varphi\geq 0$$

$$\partial_\nu w+\alpha w=B\frac{x\frac{x}{|x|}}{R}+\alpha v\pm(\partial_\nu u+\alpha u)\geq B\pm(\partial_\nu u+\alpha u)\geq 0\quad\ \Rightarrow w\geq 0\ (\text{ 对 }(x,t)\in\bar{Q}_T)$$

$$\Rightarrow|u|\leq FT+B(1+\frac{2nT}{R}+R)\leq C(T)(F+B)$$

定理：设 $u\in C^{2,1}(R^n\times(0,T])\cap C(R^n\times(0,T])$. 满足：

$$\begin{cases}\partial_t u-\Delta u=0 & x\in R^n\,,\ t>0\\ u\,(x,0)=g & x\in R^n\end{cases}$$

设 $\exists A,a$ 使得 $|u\,(x,t)|\leq Ae^{a|x|^2}$ **（增长性条件）** $(\forall(x,t)\in R^n\times[0,T])$，则

$$\sup_{R^n\times[0,T]}|u|\leq\sup_{R^n}|u|$$

证明：先假设 $4aT<1$，则存在 $\varepsilon>0$，$4a(T+\varepsilon)<1$.

令

$$v\,(x,t)=u\,(x,t)-\frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}}e\frac{|x-x_0|^2}{4\,(T+\varepsilon-t)}\quad\partial_t v-\Delta v=\partial_t n-\Delta u=0$$

取 $U_T=B_r\,(x_0)\times(0,T]$，$\max\limits_{\bar{U}_T}v\leq\max\limits_{\partial_P U_T}v$

①$v(x,0)\leq u(x,0)\leq g(x)$

②$x\in\partial B_r(x_0)$

$$v(x,t)=u(x,t)-\frac{\mu}{(T+\varepsilon-t)^{\frac{n}{2}}}e^{\frac{r^2}{4(T+\varepsilon-t)}}\leq Ae^{a(|x_0|+r)^2}-\frac{\mu}{(T+\varepsilon)^{\frac{n}{2}}}e^{\frac{r^2}{4(T+\varepsilon)}}\quad(\frac{1}{4(T+\varepsilon)}>a)$$

由于 $a<\frac{1}{4(T+\varepsilon)}$，可取 r 充分大使得 $v(x,t)\leq\sup\limits_{R^n}g$

$$(x,t)\in\bar{U}_T,v(x,t)\leq\sup_{R^n}g\quad r\to\infty\,,\ U_T\to R^n\times[0,T]\quad\ \Rightarrow\sup_{x\in R^n\ ,\ t\in(0,T)}v(x,t)\leq\sup_{R^n}g$$

当 T 足够大时，取 $\widetilde{T}=\dfrac{1}{8a}[0,T],[T,2T],\cdots,[kT,(k+1)T]$ 分别利用第一部结论.

定理：设 $g\in C(R^n)$ $f\in C(R^n\times[0,T])$，则初值问题

$$\begin{cases}\partial_t u-\Delta u=f\\ u\,(x,0)=g\end{cases}$$ 至多存在一个解满足**增长性条件：** $|u\,(x,t)|\leq Ae^{a|x|^2}(x\in R^n\,,\ 0\leq t\leq T)$

$\mathcal{L}(u_1-u_2)=0$, $u_1-u_2|_{t=0}=0 \quad \Rightarrow u_1-u_2=0$

定理：假设 $u\in C^2(Q_T)\cap C(\bar{Q}_T)$ 是初值问题的有界解，则

$$\sup_{\bar{Q}_T}|u|\leq T\sup_{\bar{Q}_T}|f|+\sup_{R^n}|g|:=F+G \quad |u|\leq Ae^{a|x|^2}$$

证明：令 $v(x,t)=Ft+G+v_R(x,t)\pm u$, $v_R(x,t)=\frac{M}{R^2}(2nt+|x|^2)$

$$\mathcal{L}v=F+f\geq 0 \quad v|_{t=0}=G+\frac{M}{R^2}|x|^2\pm g\geq 0 \quad v|_{|x|=R}=Ft+G+M(1+\frac{2nt}{R^2})\pm u\geq 0$$

$$\Rightarrow |u|\leq Ft+G+\frac{M}{R^2}(2nt+|x|^2) \quad (\text{对 } |x|\leq R\text{ , } t\geq 0)$$

对任一 $(x,t)\in Q_T$，取 $R\to\infty$ 则有 $|u(x,t)|\leq Ft+G\leq FT+G$.

定理：若 $u\in C^{2,1}(R^n\times[0,T])\cap C(R^n\times[0,T])$ 满足 $\mathcal{L}u\leq 0$, $u(x,0)=g$，且 $|u(x,t)|\leq Ae^{a|x|^2}$，则

$$\max_{R^n\times[0,T]} u\leq\sup_{R^n} g$$

证明：令 $v(x,t)=u(x,t)-Ft$.

$$\begin{cases}\mathcal{L}v=\mathcal{L}u-F\leq 0\\ v(x,0)=u(x,0)=g(x)\end{cases} \quad \Rightarrow \max_{R^n\times[0,T]}(u(x,t)-Ft)\leq\sup_{R^n} g$$

定理：

$$\begin{cases}\partial_t u-\Delta u=f & x\in R^n\text{ , }t\in[0,T]\\ u(x,0)=g & x\in R^n\end{cases} \quad \text{的解有无穷多个.}$$

$$n=1\text{ , }\varphi(t)=\begin{cases}e^{-\frac{1}{t^2}}\text{ , }t\neq 0\\ 0\text{ , }t=0\end{cases} \quad \text{此时 } \varphi(t)\in C^\infty \text{ 但 } \varphi\notin C^\omega\text{ (解析)}$$

$$\underline{\varphi(x)=\varphi(0)+\varphi'(0)x+\frac{\varphi''(0)}{2}x^2+\cdots+\frac{\varphi^{(n)}(0)}{n!}x^n+\cdots} \quad \text{错误}\times$$

$$\text{令 } u(x,t)=\begin{cases}\sum\limits_{n=0}^{\infty}\varphi^{(n)}(t)\frac{x^{2n}}{(2n)!}\text{ , }t\neq 0\\ 0\text{ , }t=0\end{cases} \quad \text{有：}$$

$$(1)\lim_{t\to 0}u(x,t)=0 \qquad (2)\partial_x^2 u=\sum_{n=1}^{\infty}\varphi^{(n)}(t)\frac{x^{2n-2}}{(2n-2)!}=\sum_{n=0}^{\infty}\varphi^{(n+1)}(t)\frac{x^{2n}}{(2n)!}=\partial_t u$$