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Its c.f. is

$$f(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}),$$

i.e.,

$$f(t_1, \cdots, t_n) = \exp(i\sum_{k=1}^{n} a_k t_k - \frac{1}{2}\sum_{l=1}^{n}\sum_{s=1}^{n} b_{ls} t_l t_s).$$

Proof. Write $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}'$ ($\boldsymbol{L} = \boldsymbol{B}^{1/2}$). Let $\boldsymbol{\eta} = \boldsymbol{L}^{-1}(\boldsymbol{\xi} - \boldsymbol{a})$. Then by Theorem 2 in §2.5, the pdf of $\boldsymbol{\eta}$ is

$$p_{\boldsymbol{\eta}}(\boldsymbol{y}) = p(\boldsymbol{x})|\boldsymbol{L}| \quad \big(\text{where} \quad \boldsymbol{x} = \boldsymbol{L}(\boldsymbol{y} + \boldsymbol{a})\big)$$

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& p_{\boldsymbol{\eta}}(\boldsymbol{y}) = p(\boldsymbol{x})|\boldsymbol{L}| \ \ \Big(\text{where} \quad \boldsymbol{x} = \boldsymbol{L}(\boldsymbol{y} + \boldsymbol{a})\Big) \\
= \ & \frac{1}{(\sqrt{2\pi})^{n/2}} \exp\{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})'(\boldsymbol{L}')^{-1}\boldsymbol{L}^{-1}(\boldsymbol{x} - \boldsymbol{a})\}
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Proof. Write $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}'$ ($\boldsymbol{L} = \boldsymbol{B}^{1/2}$). Let $\boldsymbol{\eta} = \boldsymbol{L}^{-1}(\boldsymbol{\xi} - \boldsymbol{a})$. Then by Theorem 2 in §2.5, the pdf of $\boldsymbol{\eta}$ is

$$\begin{aligned} p_{\boldsymbol{\eta}}(\boldsymbol{y}) &= p(\boldsymbol{x})|\boldsymbol{L}| \ \big(\text{where} \quad \boldsymbol{x} = \boldsymbol{L}(\boldsymbol{y} + \boldsymbol{a})\big) \\ &= \frac{1}{(\sqrt{2\pi})^{n/2}} \exp\{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{a})'(\boldsymbol{L}')^{-1}\boldsymbol{L}^{-1}(\boldsymbol{x} - \boldsymbol{a})\} \\ &= \frac{1}{(\sqrt{2\pi})^{n/2}} \exp\{-\frac{1}{2}\boldsymbol{y}'\boldsymbol{y}\} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\{-\frac{y_i^2}{2}\}, \end{aligned}$$

i.e., $\eta_1, \cdots, \eta_n$ i.i.d. $\sim N(0,1)$. From Property 3′ in §3.3 it follows that

$$f_{\boldsymbol{\eta}}(\boldsymbol{t}) = \prod_{i=1}^{n} e^{-\frac{t_i^2}{2}} = \exp\{-\frac{1}{2}\boldsymbol{t}'\boldsymbol{t}\}.$$

Also $\boldsymbol{\xi} = \boldsymbol{L}\boldsymbol{\eta} + \boldsymbol{a}$. It follows that

$$\begin{aligned} f(\boldsymbol{t}) &= Ee^{i\boldsymbol{t}'\boldsymbol{\xi}} = e^{i\boldsymbol{t}'\boldsymbol{a}} Ee^{i\boldsymbol{t}'\boldsymbol{L}\boldsymbol{\eta}} = e^{i\boldsymbol{t}'\boldsymbol{a}} Ee^{i(\boldsymbol{L}'\boldsymbol{t})'\boldsymbol{\eta}} \\ &= e^{i\boldsymbol{t}'\boldsymbol{a}} \exp\{-\frac{1}{2}(\boldsymbol{L}'\boldsymbol{t})'(\boldsymbol{L}'\boldsymbol{t})\} \\ &= e^{i\boldsymbol{t}'\boldsymbol{a}} \exp\{-\frac{1}{2}\boldsymbol{t}'\boldsymbol{L}\boldsymbol{L}'\boldsymbol{t})\} \\ &= e^{i\boldsymbol{t}'\boldsymbol{a}} \exp\{-\frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}\} \\ &= \exp\{i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}\}. \end{aligned}$$

When $\boldsymbol{B}$ is non-negative definite,

$$f(\boldsymbol{t}) = \exp(i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t})$$

is also a c.f.. In fact, Write $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}'$, if $\boldsymbol{\eta} = N(\boldsymbol{0}, \boldsymbol{I}_{n\times n})$, then the c.f. of $\boldsymbol{\xi} = \boldsymbol{L}\eta + \boldsymbol{a}$ is $f(\boldsymbol{t})$.

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We call the corresponding distribution a singular normal distribution or a degenerate normal distribution. When the rank of $\boldsymbol{B}$ is r $(r < n)$, it is actually only a distribution in r dimensional subspace.

3.4.2 Properties

1. Any sub-vector $(\xi_{l_1},\cdots,\xi_{l_k})'$ of $\boldsymbol{\xi}$ also follows normal distribution as $N(\tilde{\boldsymbol{a}},\tilde{\boldsymbol{B}})$, where $\tilde{\boldsymbol{a}}=(a_{l_1},\cdots,a_{l_k})'$, $\tilde{\boldsymbol{B}}$ is a $k\times k$ matrix consisting of elements in both $l_1,\cdots,l_k$ rows and $l_1,\cdots,l_k$ columns in $\boldsymbol{B}$. $N(\boldsymbol{a},\boldsymbol{B})$ has expected value $\boldsymbol{a}$, covariance matrix $\boldsymbol{B}$.

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Proof. In the cf of $\boldsymbol{\xi}$: $f_{\boldsymbol{\xi}}(\boldsymbol{t}) = \exp\left\{i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}\right\}$, setting all t_j except $t_{l_1}, \cdots, t_{l_k}$ to be 0 yields the cf of $(\xi_{l_1}, \cdots, \xi_{l_k})'$: $\exp\left\{i\tilde{\boldsymbol{t}}'\tilde{\boldsymbol{a}} - \frac{1}{2}\tilde{\boldsymbol{t}}'\tilde{\boldsymbol{B}}\tilde{\boldsymbol{t}}\right\}$.

2. $N(\boldsymbol{a}, \boldsymbol{B})$ has expected value $\boldsymbol{a}$, covariance matrix B.

(Method a) Write $\boldsymbol{B}=\boldsymbol{L}\boldsymbol{L}'$. Suppose $\boldsymbol{\eta}\sim N(\boldsymbol{0},\boldsymbol{I}_{n\times n})$, then the c.f. of $\boldsymbol{\xi}=:\boldsymbol{L}\boldsymbol{\eta}+\boldsymbol{a}\sim N(\boldsymbol{a},\boldsymbol{B})$.

(Method a) Write $\boldsymbol{B} = \boldsymbol{L}\boldsymbol{L}'$. Suppose $\boldsymbol{\eta} \sim N(\boldsymbol{0}, \boldsymbol{I}_{n\times n})$, then the c.f. of $\boldsymbol{\xi} =: \boldsymbol{L}\boldsymbol{\eta} + \boldsymbol{a} \sim N(\boldsymbol{a}, \boldsymbol{B})$.

It is obvious that

$$E\boldsymbol{\eta} = \boldsymbol{0}, \quad Var\boldsymbol{\eta} = \boldsymbol{I}.$$

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$$E\boldsymbol{\xi}=\boldsymbol{L}E\boldsymbol{\eta}+\boldsymbol{a},$$

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Proof. (Method b) If $\boldsymbol{B}$ is non-singular, the proof is already given in Section 3.2. When $\boldsymbol{B}$ is singular, suppose $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$, $\boldsymbol{\eta} \sim N(\mathbf{0}, \boldsymbol{I})$, $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are independent.

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$$
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& = \exp\left\{ i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'(\boldsymbol{B} + \boldsymbol{I})\boldsymbol{t} \right\}.
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Hence

$$E\boldsymbol{\xi} = \boldsymbol{a} \quad \text{and} \quad Var\boldsymbol{\xi} = \boldsymbol{B}.$$

3. $\xi_1, \cdots, \xi_n$ with joint normal distribution are mutually independent iff they are pairwise uncorrelated.

Proof. Suppose $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{B} = (b_{ij})$. Then the cf of $\boldsymbol{\xi}$ is

$$f(\boldsymbol{t}) = \exp\left\{ i \sum_{k=1}^{n} a_k t_k - \frac{1}{2} \sum_{l=1}^{n} \sum_{s=1}^{n} b_{ls} t_l t_s \right\},$$

and then the cf of ξ_k is $f_k(t) = \exp\{i a_k t - \frac{1}{2} b_{kk} t^2\}$. So

$$\begin{aligned} & \xi_1, \cdots, \xi_n \text{ are independent} \\ \Longleftrightarrow\ & f(t_1, \cdots, t_n) = f_1(t_1) \cdots f_n(t_n) \\ \Longleftrightarrow\ & \boldsymbol{t}' \boldsymbol{B} \boldsymbol{t} = \sum_{j=1}^{n} b_{jj} t_j^2 \\ \Longleftrightarrow\ & b_{ij} = 0, \quad i, j = 1, 2 \cdots, n, \quad i \neq j \\ \Longleftrightarrow\ & \xi_i \text{ and } \xi_j \text{ are uncorrelated}, \ \ i, j = 1, 2 \cdots, n, \ i \neq j. \end{aligned}$$

4. Suppose $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_n) \sim N(\boldsymbol{a}, \boldsymbol{B})$, $\boldsymbol{C} = (c_{ij})_{m\times n}$ is an $m \times n$ matrix, then

$$\boldsymbol{\eta} = \boldsymbol{C}\boldsymbol{\xi} + \boldsymbol{\mu} \sim N(\boldsymbol{C}\boldsymbol{a} + \boldsymbol{\mu}, \boldsymbol{C}\boldsymbol{B}\boldsymbol{C}'),$$

an m-dimensional normal distribution.

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Proof.

$$f_{\boldsymbol{\eta}}(\boldsymbol{t}) = Ee^{i\boldsymbol{t}'(\boldsymbol{C}\boldsymbol{\xi}+\boldsymbol{\mu})} = e^{i\boldsymbol{t}'\boldsymbol{u}}Ee^{i(\boldsymbol{C}'\boldsymbol{t})'\boldsymbol{\xi}}$$

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5. $\boldsymbol{\xi}$ is normally distributed iff any linear combination of its components follows normal distributions. Specifically, let $\boldsymbol{l} = (l_1, \cdots, l_n)'$ be any n dimensional real vector, then

$$\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B}) \Leftrightarrow \zeta = \boldsymbol{l}'\boldsymbol{\xi} \sim N(\boldsymbol{l}'\boldsymbol{a}, \boldsymbol{l}'\boldsymbol{B}\boldsymbol{l})$$

$$\Leftrightarrow \quad \zeta = \sum_{j=1}^{n} l_j \xi_j \sim N(\sum_{j=1}^{n} l_j a_j, \sum_{j=1}^{n}\sum_{k=1}^{n} l_j l_k b_{jk})$$

Proof."$\Longrightarrow$" (A special case of Property 4). Actually,

$$f_\zeta(t) = Ee^{it\boldsymbol{l}'\boldsymbol{\xi}}$$

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$$\begin{aligned} f_\zeta(t) &= Ee^{it\boldsymbol{l}'\boldsymbol{\xi}} = \exp\left\{i(t\boldsymbol{l})'\boldsymbol{a} - \frac{1}{2}(t\boldsymbol{l})')\boldsymbol{B}(t\boldsymbol{l})\right\} \\ &= \exp\left\{it(\boldsymbol{l}'\boldsymbol{a}) - \frac{1}{2}t^2\boldsymbol{l}'\boldsymbol{B}\boldsymbol{l}\right\}. \end{aligned}$$

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So $\zeta = \boldsymbol{l}'\boldsymbol{\xi} \sim N(\boldsymbol{l}'\boldsymbol{a}, \boldsymbol{l}'\boldsymbol{B}\boldsymbol{l})$.

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So $\zeta = \boldsymbol{l}'\boldsymbol{\xi} \sim N(\boldsymbol{l}'\boldsymbol{a}, \boldsymbol{l}'\boldsymbol{B}\boldsymbol{l})$.

"$\Longleftarrow$" First, by assumption, each ξ_k is normal. So its mean and variance exists, and then $Cov\{\xi_k, \xi_j\}$ exists.

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"$\Longleftarrow$" First, by assumption, each ξ_k is normal. So its mean and variance exists, and then $Cov\{\xi_k, \xi_j\}$ exists. Denote $\boldsymbol{a} = E\boldsymbol{\xi}$ and $\boldsymbol{B} = Var\boldsymbol{\xi}$. We want to show that $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$.

For any $\boldsymbol{t}$, let $\zeta = \boldsymbol{t}'\boldsymbol{\xi}$. By assumption, ζ is normal.

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$$\zeta \sim N(\boldsymbol{t}'\boldsymbol{a}, \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}).$$

Hence

$$f_{\boldsymbol{\xi}}(\boldsymbol{t}) = Ee^{i\boldsymbol{t}'\boldsymbol{\xi}} = f_{\zeta}(1)$$

For any $\boldsymbol{t}$, let $\zeta = \boldsymbol{t}'\boldsymbol{\xi}$. By assumption, ζ is normal. On the other hand, $E\zeta = \boldsymbol{t}'E\boldsymbol{\xi} = \boldsymbol{t}'\boldsymbol{a}$ and $Var\zeta = \boldsymbol{t}'(Var\boldsymbol{\xi})\boldsymbol{t} = \boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}$. It follows that

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Hence

$$\begin{aligned} f_{\boldsymbol{\xi}}(\boldsymbol{t}) &= Ee^{i\boldsymbol{t}'\boldsymbol{\xi}} = f_\zeta(1) \\ &= \exp\left\{ i\boldsymbol{t}'\boldsymbol{a} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t} \right\}. \end{aligned}$$

So, $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$.

6. Assume that $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1', \boldsymbol{\xi}_2')'$, where $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are k and $n-k$-dimensional sub-vectors of $\boldsymbol{\xi}$ respectively, and

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}.$$

Then $\boldsymbol{\xi}_1 \sim N(\boldsymbol{a}_1, \boldsymbol{B}_{11})$, $\boldsymbol{\xi}_2 \sim N(\boldsymbol{a}_2, \boldsymbol{B}_{22})$; and, $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are independent if and only if $\boldsymbol{B}_{12} = \boldsymbol{0}$ (resp. $\boldsymbol{B}_{21} = \boldsymbol{0}$), i.e., $Cov\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\} = E\big[(\boldsymbol{\xi}_1 - E\boldsymbol{\xi}_1)(\boldsymbol{\xi}_2 - E\boldsymbol{\xi}_2)'\big] = \boldsymbol{0}$.

Proof. The first of conclusion is obvious. And also, it is obvious that, if ξ_1 and ξ_2 are independent, then

$$\boldsymbol{B}_{12} = E(\boldsymbol{\xi}_1 - E\boldsymbol{\xi}_1)E(\boldsymbol{\xi}_2 - E\boldsymbol{\xi}_2)' = \mathbf{0}.$$

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Conversely, if $\boldsymbol{B}_{12} = \mathbf{0}$ and $\boldsymbol{B}_{21} = \mathbf{0}$, then

$$f_{\xi}(\boldsymbol{t}) = \exp\left\{i\boldsymbol{a}'\boldsymbol{t} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}\right\}$$

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$$\begin{aligned} f_{\xi}(\boldsymbol{t}) &= \exp\left\{i\boldsymbol{a}'\boldsymbol{t} - \frac{1}{2}\boldsymbol{t}'\boldsymbol{B}\boldsymbol{t}\right\} \\ &= \exp\left\{i\boldsymbol{a}_1'\boldsymbol{t}_1 + i\boldsymbol{a}_2'\boldsymbol{t}_2 - \frac{1}{2}\boldsymbol{t}_1'\boldsymbol{B}_{11}\boldsymbol{t}_1 - \frac{1}{2}\boldsymbol{t}_2'\boldsymbol{B}_{22}\boldsymbol{t}_2\right\} \end{aligned}$$

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7. Assume that $\boldsymbol{\xi} \sim N(\boldsymbol{a}, \boldsymbol{B})$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1', \boldsymbol{\xi}_2')'$, where $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are k and $n-k$-dimensional sub-vectors of $\boldsymbol{\xi}$ respectively,

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{pmatrix}$$

is positive definite and $\boldsymbol{\xi}_1 \sim N(\boldsymbol{a}_1, \boldsymbol{B}_{11})$, $\boldsymbol{\xi}_2 \sim N(\boldsymbol{a}_2, \boldsymbol{B}_{22})$. Then conditioning on $\boldsymbol{\xi}_1 = \boldsymbol{x}_1$, the conditional distribution of $\boldsymbol{\xi}_2$ is a normal distribution

$$N(\boldsymbol{a}_2 + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{x}_1 - \boldsymbol{a}_1), \boldsymbol{B}_{22} - \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{12}).$$

思路:

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$$\xi_2 = a_2 + C(\xi_1 - a_1) + \eta$$

即

$$\eta = \xi_2 - a_2 - C(\xi_1 - a_1).$$

为使得η与ξ_1独立，只要η与ξ_1不相关.

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$$\boldsymbol{\xi}_2 = \boldsymbol{a}_2 + \boldsymbol{C}(\boldsymbol{\xi}_1 - \boldsymbol{a}_1) + \boldsymbol{\eta}$$

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$$\boldsymbol{\eta} = \boldsymbol{\xi}_2 - \boldsymbol{a}_2 - \boldsymbol{C}(\boldsymbol{\xi}_1 - \boldsymbol{a}_1).$$

为使得$\boldsymbol{\eta}$与$\boldsymbol{\xi}_1$独立，只要$\boldsymbol{\eta}$与$\boldsymbol{\xi}_1$不相关. 而

$$\begin{aligned} Cov\{\boldsymbol{\eta}, \boldsymbol{\xi}_1\} =& E(\boldsymbol{\xi}_2 - \boldsymbol{a}_2)(\boldsymbol{\xi}_1 - \boldsymbol{a}_1)' - \boldsymbol{C} Var\{\boldsymbol{\xi}_1\} \\ =& \boldsymbol{B}_{21} - \boldsymbol{C}\boldsymbol{B}_{11}. \end{aligned}$$

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故要取

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Proof. Let

$$\boldsymbol{\eta} = \boldsymbol{\xi}_2 - \boldsymbol{a}_2 - \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{\xi}_1 - \boldsymbol{a}_1).$$

Then $(\boldsymbol{\xi}_1, \boldsymbol{\eta})$ is still normal random vector, and $\boldsymbol{\xi}_2 = \boldsymbol{a}_2 + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{\xi}_1 - \boldsymbol{a}_1) + \boldsymbol{\eta}$.

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Then $(\boldsymbol{\xi}_1, \boldsymbol{\eta})$ is still normal random vector, and $\boldsymbol{\xi}_2 = \boldsymbol{a}_2 + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{\xi}_1 - \boldsymbol{a}_1) + \boldsymbol{\eta}$. It is easily seen that $E\boldsymbol{\eta} = \mathbf{0}$ and

$$\begin{aligned}
Var\boldsymbol{\eta} =& Var\boldsymbol{\xi}_2 - 2\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}Cov\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2\} + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}Var\boldsymbol{\xi}_1(\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1})' \\
=& \boldsymbol{B}_{22} - 2\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{12} + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{11}(\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1})' \\
=& \boldsymbol{B}_{22} - \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{12} \widehat{=} \boldsymbol{\Sigma}.
\end{aligned}$$

It follows that $\boldsymbol{\eta} \sim N(\mathbf{0}, \boldsymbol{\Sigma})$.

Also,

$$E\boldsymbol{\eta}(\boldsymbol{\xi}_1-\boldsymbol{a}_1)'=\boldsymbol{B}_{21}-\boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}\boldsymbol{B}_{11}=\boldsymbol{0}.$$

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It follows that

$$\begin{aligned}\boldsymbol{\xi}_2\big|_{\boldsymbol{\xi}_1=\boldsymbol{x}_1} =& \boldsymbol{a}_2 + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{x}_1 - \boldsymbol{a}_1) + \boldsymbol{\eta}\big|_{\boldsymbol{\xi}_1=\boldsymbol{x}_1} \\ \sim& N(\boldsymbol{a}_2 + \boldsymbol{B}_{21}\boldsymbol{B}_{11}^{-1}(\boldsymbol{x}_1 - \boldsymbol{a}_1), \boldsymbol{\Sigma}).\end{aligned}$$

Example

Suppose $\xi_1, \ldots, \xi_n$ be i.i.d. normal $N(\mu, \sigma^2)$ random variables. Let

$$\overline{\xi} = \frac{\sum_{k=1}^n \xi_k}{n}, \quad \widehat{\sigma}^2 = \frac{1}{n}\sum_{k=1}^n (\xi_k - \overline{\xi})^2.$$

Show that $\overline{\xi}$ and $\widehat{\sigma}^2$ are independent.

Proof. Since $(\overline{\xi}, \xi_1 - \overline{\xi}, \ldots, \xi_n - \overline{\xi})$ is a linear transform of the normal vector $(\xi_1, \ldots, \xi_n)$, so it is also a normal vector.

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$$\begin{aligned} Cov\{\overline{\xi}, \xi_k - \overline{\xi}\} =& Cov\{\overline{\xi}, \xi_k\} - Var\{\overline{\xi}\} \\ =& \frac{1}{n}\sigma^2 - \frac{1}{n}\sigma^2 = 0. \end{aligned}$$

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Hence $\overline{\xi}$ and $(\xi_1 - \overline{\xi}, \ldots, \xi_n - \overline{\xi})$ are independent.

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Example

Assume $\boldsymbol{\xi} = (\xi_1, \xi_2)' \sim N(a_1, a_2, \sigma^2, \sigma^2, r)$, prove $\eta_1 = \xi_1 + \xi_2$ and $\eta_2 = \xi_1 - \xi_2$ are independent, and find respective distributions of η_1, η_2.

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\begin{aligned}
Var\eta_1 &= Var\xi_1 + Var\xi_2 + 2Cov\{\xi_1, \xi_2\} \\
&= 2\sigma^2 + 2r\sigma\sigma = 2\sigma^2(1 + r),
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Solution. Since (η_1, η_2) is a linear transform of (ξ_1, ξ_2), so (η_1, η_2) follows a normal distribution. Also, $E\eta_1 = a_1 + a_2$, $E\eta_2 = a_1 - a_2$,

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Solution. Since (η_1, η_2) is a linear transform of (ξ_1, ξ_2), so (η_1, η_2) follows a normal distribution. Also, $E\eta_1 = a_1 + a_2$, $E\eta_2 = a_1 - a_2$,

$$\begin{aligned} Var\eta_1 &= Var\xi_1 + Var\xi_2 + 2Cov\{\xi_1, \xi_2\} \\ &= 2\sigma^2 + 2r\sigma\sigma = 2\sigma^2(1+r), \\ Var\eta_2 &= Var\xi_1 + Var\xi_2 - 2Cov\{\xi_1, \xi_2\} \\ &= 2\sigma^2 - 2r\sigma\sigma = 2\sigma^2(1-r), \\ Cov\{\eta_1, \eta_2\} &= Var\xi_1 - Var\xi_2 = 0. \end{aligned}$$

So η_1 and η_2 are independent, and $\eta_1 \sim N(a_1 + a_2, 2\sigma^2(1+r))$, $\eta_2 \sim N(a_1 - a_2, 2\sigma^2(1-r))$.

Example

设$X_1, X_2, \cdots, X_n$为i.i.d.$N(\mu, \sigma^2)$变量.

$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$和$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$, 则

(1) $\overline{X} \sim N(\mu, \sigma^2/n)$;

(2) $\frac{(n-1)S^2}{\sigma^2} = \frac{1}{\sigma^2}\sum_{i=1}^n (X_i - \overline{X})^2 \sim \chi^2(n-1)$;

$S^2 \sim Gamma(\frac{n-1}{2}, \frac{n-1}{2\sigma^2})$;

(3) $\overline{X}$与S^2独立.

证明: 记$\boldsymbol{X}=(X_1,X_2,\cdots,X_n)'$. 则$\boldsymbol{X}$服从$n$维正态分布. 构造一个正交矩阵$A=(a_{ij})_{n\times n}$使得第一行的元素都为$1/\sqrt{n}$:

证明: 记$\boldsymbol{X} = (X_1, X_2, \cdots, X_n)'$. 则$\boldsymbol{X}$服从$n$维正态分布. 构造一个正交矩阵$A = (a_{ij})_{n\times n}$使得第一行的元素都为$1/\sqrt{n}$:

$$A = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2\cdot 1}} & \frac{-1}{\sqrt{2\cdot 1}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & \frac{-2}{\sqrt{3\cdot 2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{pmatrix}$$

作线性变换$\boldsymbol{Y} = A\boldsymbol{X}$, 即

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ \frac{1}{\sqrt{2\cdot 1}} & \frac{-1}{\sqrt{2\cdot 1}} & 0 & \cdots & 0 \\ \frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & \frac{-2}{\sqrt{3\cdot 2}} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}.$$

则$\boldsymbol{Y}$也服从n维正态分布,

且$Y_1 = (X_1 + X_2 + \cdots + X_n)/\sqrt{n} = \sqrt{n}\overline{X}$.

可得

$$
\begin{aligned}
E\boldsymbol{Y} =& AE\boldsymbol{X} = A(\mu, \mu, \cdots, \mu)' \\
=& \begin{pmatrix}
\frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\
\frac{1}{\sqrt{2\cdot 1}} & \frac{-1}{\sqrt{2\cdot 1}} & 0 & \cdots & 0 \\
\frac{1}{\sqrt{3\cdot 2}} & \frac{1}{\sqrt{3\cdot 2}} & \frac{-2}{\sqrt{3\cdot 2}} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \cdots & \frac{-(n-1)}{\sqrt{n(n-1)}}
\end{pmatrix}
\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \mu \\
=& (\sqrt{n}\mu, 0, \cdots, 0)',
\end{aligned}
$$

$$
Var\boldsymbol{Y} = A(Var\boldsymbol{X})A' = A(\sigma^2 I)A' = \sigma^2 I.
$$

因此，$Y_1, Y_2, \cdots, Y_n$相互独立，$Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$，$Y_k \sim N(0, \sigma^2)$，$k = 2, \cdots, n$.

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另一方面,

$$\sum_{i=1}^{n} Y_i^2 = \boldsymbol{Y}'\boldsymbol{Y} = \boldsymbol{X}'A'A\boldsymbol{X} = \boldsymbol{X}'\boldsymbol{X} = \sum_{i=1}^{n} X_i^2$$

$$= \sum_{i=1}^{n} (X_i - \overline{X})^2 + n\overline{X}^2 = (n-1)S^2 + Y_1^2.$$

所以

$$(n-1)S^2=\sum_{i=2}^n Y_i^2 \text{ 与 } \overline{X}=Y_1/\sqrt{n} \text{ 独立},$$

并且

$$(n-1)S^2/\sigma^2=\sum_{i=2}^n (Y_i/\sigma)^2 \sim \chi^2(n-1),$$

$$\overline{X}=Y_1/\sqrt{n}\sim N(\mu,\sigma^2/n),$$

结论得证.

由上面的结论知道

$$U=\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\sim N(0,1).$$

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$$U = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1).$$

将上面的σ^2用S^2代替后，是否仍然服从$N(0,1)$分布?

可知

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1),$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1),$$

且两者相互独立，

可知

$$\frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1),$$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1),$$

且两者相互独立，因此

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} = \frac{\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)S^2}{\sigma^2}/(n-1)}} \sim t(n-1).$$

Example

设$X_1, X_2, \cdots, X_n$为i.i.d.$N(\mu, \sigma^2)$变量.
$\overline{X} = \frac{1}{n}\sum_{i=1}^n X_i$和$S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \overline{X})^2$,, 则

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{\sqrt{n}(\overline{X} - \mu)}{S} \sim t(n-1).$$