2.4 Independence of random variables

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Definition

Suppose that the joint distribution sequence of a discrete random vector (ξ, η) satisfies

$$P(\xi = x_i, \eta = y_j) = P(\xi = x_i)P(\eta = y_j),$$
$$i, j = 1, 2, \cdots,$$

then we call ξ and η mutually independent.

$$p_{ij} = p_{i\cdot} \cdot p_{\cdot j}, \qquad i, j = 1, 2, \cdots .$$

For any x and y,

$$\begin{aligned} P(\xi \le x, \eta \le y) &= \sum_{x_i \le x} \sum_{y_j \le y} P(\xi = x_i, \eta = y_j) \\ &= \sum_{x_i \le x} P(\xi = x_i) \sum_{y_j \le y} P(\eta = y_j) \\ &= P(\xi \le x) P(\eta \le y). \end{aligned}$$

That is,

$$F(x, y) = F_\xi(x) F_\eta(y). \tag{2.62}$$

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$$
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P(\xi \le x, \eta \le y) &= \sum_{x_i \le x} \sum_{y_j \le y} P(\xi = x_i, \eta = y_j) \\
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\end{aligned}
$$

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On the contrary,

$$
(2.62) \Longrightarrow P(\xi = x_i, \eta = y_j) = P(\xi = x_i) P(\eta = y_j)
$$

Definition

Suppose that $F(x, y)$, $F_\xi(x)$ and $F_\eta(y)$ are the joint distribution function and marginal distribution functions of (ξ, η) respectively. If

$$F(x, y) = F_\xi(x) F_\eta(y), \quad \forall x, y,$$

$$(i.e., \quad P(\xi \le x, \eta \le y) = P(\xi \le x) P(\eta \le y), \quad \forall x, y)$$

then we say ξ and η are independent.

Theorem

Suppose that $p(x,y)$, $p_\xi(x)$ and $p_\eta(y)$ are the joint density function and marginal density functions of (ξ,η) respectively. Then ξ and η are independent if and only if

$$p(x,y) = p_\xi(x)p_\eta(y).$$

Proof. For any x, y, it follows

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$$\begin{aligned}
& F(x,y) = F_\xi(x)F_\eta(y) \\
\Leftrightarrow\ & \int_{-\infty}^{x}\int_{-\infty}^{y} p(u,v)dudv = \int_{-\infty}^{x} p_\xi(u)du \int_{-\infty}^{y} p_\eta(v)dv
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\Leftrightarrow\ & p(x,y) = p_\xi(x)p_\eta(y).
\end{aligned}
$$

This is the desired conclusion.

Example

Suppose $(\xi, \eta) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$. Find out the necessary and sufficient condition for ξ, η to be independent.

Solution. Note that $\xi \sim N(\mu_1, \sigma_1^2)$ and $\eta \sim N(\mu_2, \sigma_2^2)$. By definition,

$$\xi, \eta \text{ are independent } \Leftrightarrow p(x,y) = p_\xi(x)p_\eta(y)$$

Solution. Note that $\xi \sim N(\mu_1, \sigma_1^2)$ and $\eta \sim N(\mu_2, \sigma_2^2)$. By definition,

$$
\begin{aligned}
& \xi, \eta \text{ are independent } \Leftrightarrow p(x, y) = p_\xi(x) p_\eta(y) \\
\Leftrightarrow \quad & \frac{1}{\sqrt{2\pi}\sigma_1} \exp\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\} \\
& \times \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-r^2}} \exp\{-\frac{[y-\mu_2-\frac{r\sigma_2}{\sigma_1}(x-\mu_1)]^2}{2\sigma_2^2(1-r^2)}\} \\
& = \frac{1}{2\pi\sigma_1\sigma_2} \exp\{-\frac{1}{2}[\frac{(x-\mu_1)^2}{\sigma_1^2} + \frac{(y-\mu_2)^2}{\sigma_2^2}]\} \\
\Leftrightarrow \quad & r = 0.
\end{aligned}
$$

n random variables:

Definition

Suppose that $F(x_1, \cdots, x_n)$, $F_1(x_1), \cdots, F_n(x_n)$ are joint distribution function and marginal distribution functions of $\xi_1, \cdots, \xi_n$, then we call them mutually independent if

$$F(x_1, \cdots, x_n) = F_1(x_1) \cdots F_n(x_n).$$

$$\Big(i.e., \ P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \cdots P(\xi_n \le x_n), \quad \forall x_1, \cdots, x_n \Big)$$

Corollary

If $\xi_1, \cdots, \xi_n$ are mutually independent, then so are any r random variables $(2 \le r < n)$.

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Proof. By the definition of the independence of $\xi_1, \cdots, \xi_n$, we have for all $x_1, \cdots, x_n$,

$$P(\xi_1 \le x_1, \cdots, \xi_n \le x_n) = P(\xi_1 \le x_1) \cdots P(\xi_n \le x_n).$$

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It follows that

$$\begin{aligned} & P(\xi_{i_1} \le x_{i_1}, \cdots, \xi_{i_r} \le x_{i_r}) \\ = & P(\xi_{i_1} \le x_{i_1}) \cdots P(\xi_{i_r} \le x_{i_r}), \quad \forall x_{i_1}, \cdots, x_{i_r}. \end{aligned}$$

思考题: 1. Find n random variables (or vectors) $\xi_1, \ldots, \xi_n$, such that they are not independent, but every r $(2 \le r < n)$ of them are independent.

2. Suppose F is a distribution function. Find n random variables (or vectors) $\xi_1, \ldots, \xi_n$, such that $\xi_i \sim F$, $i = 1, \ldots, n$, $\xi_1, \ldots, \xi_n$ are not independent, but every r $(2 \le r < n)$ of them are independent.

- $\xi_1, \cdots, \xi_n$ are indept. iff (if and only if)

$$P(\xi_1 \in B_1, \cdots, \xi_n \in B_n) = P(\xi_1 \in B_1) \cdots P(\xi_n \in B_n)$$

for any $B_1, \cdots, B_n \in \mathscr{B}$.

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for any $B_1, \cdots, B_n \in \mathscr{B}$.

- An n-dimensional $\boldsymbol{\xi}$ and an m-dimensional $\boldsymbol{\eta}$ are indept. iff

$$P(\boldsymbol{\xi} \in A, \boldsymbol{\eta} \in B) = P(\boldsymbol{\xi} \in A)P(\boldsymbol{\eta} \in B),$$

for all $A \in \mathscr{B}^n, B \in \mathscr{B}^m$, equivalently,

$$P(\boldsymbol{\xi} \leq \boldsymbol{x}, \boldsymbol{\eta} \leq \boldsymbol{y}) = P(\boldsymbol{\xi} \leq \boldsymbol{x})P(\boldsymbol{\eta} \leq \boldsymbol{y}), \ \forall x \in \mathcal{R}^n, \in \mathcal{R}^m.$$

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- If two random vectors are independent, then so are their sub-vectors.

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Suppose that ξ is a constant a, show ξ and η are independent for any random variable η.

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Proof Let B_1 and B_2 be two Borel sets. We want to prove

$$P(\xi \in B_1, \eta \in B_2) = P(\xi \in B_1)P(\eta \in B_2). \qquad (*)$$

If $a \notin B_1$, then $P(\xi \in B_1) = 0$ and

$$P(\xi \in B_1, \eta \in B_2) \leq P(\xi \in B_1) = 0.$$

$(*)$ is true.

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$(*)$ is true.
If $a \in B_1$, then $P(\xi \in B_1) = 1$ and

$$\begin{aligned} P(\xi \in B_1, \eta \in B_2) =& P(\eta \in B_2) - P(\xi \notin B_1, \eta \in B_2) \\ =& P(\eta \in B_2). \end{aligned}$$

$(*)$ is also true.

思考题: Suppose that X and Y are independent, $X+Y$ and X have the same distribution. Prove $P(Y=0)=1$.

Infinite many of random variables:

Definition

Suppose that $\{\xi_n; n \geq 1\}$ is a sequence of random variables. Then we call them mutually independent if for each n, $\xi_1, \ldots, \xi_n$ are independent.

思考题: Find a sequence $\{\xi_i; i \geq 1\}$ of random variables such that every two of them are independent, but every three of them are not independent.