ODE笔记7：线性微分方程组

令 $\vec{X}=\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, F=\begin{pmatrix}f_1\\ \vdots\\ f_n\end{pmatrix}$ $\quad \begin{cases}\frac{d\vec{X}}{dt}=\vec{F}(t,\vec{X})\\ \vec{X}(t_0)=\vec{X}_0\end{cases}\ (*) \quad \Longrightarrow \quad \begin{cases}\frac{dx_1}{dt}=f_1\left(t,x_1,x_2,\ldots x_n\right)\\ \frac{dx_2}{dt}=f_2\left(t,x_1,x_2,\ldots x_n\right)\\ \vdots\\ \frac{dx_n}{dt}=f_n\left(t,x_1,x_2,\ldots x_n\right)\end{cases}$

$\|\vec{X}\|=\sqrt{x_1^2+\cdots+x_n^2}$, $\frac{d\|\vec{X}\|}{dt}$?

$R=\left\{(t,\vec{x}),|t_0-t|\leqslant a,\left\|\vec{X}-\vec{X}_0\right\|\leqslant b\right\} \quad F\in C(R)$

$\Longrightarrow$ $(*)$ 在 $[t_0,t_0+\alpha]$ 上 $\exists$ 解 $\vec{X}(t),\quad \alpha=\min\left\{a,\frac{b}{M}\right\},M=\max\limits_R\|F\|$

$L_p:\left\|F\left(t,\vec{x}_1\right)-F\left(t,\vec{x}_2\right)\right\|\leqslant L\left\|\vec{x}_1-\vec{x}_2\right\|$

首次积分：

$\vec{X}=F(t,\vec{X})\quad(*)$ 若函数 $\psi(t,\vec{X})$ 连续，不恒等于常数。若 $\vec{X}=\vec{\phi}(t)$ 为 $(*)$ 解，有 $\psi(t,\vec{\phi})\equiv C$。则称其为 $(*)$ 的**一个首次积分**。

n 阶ODEs存在 n 个独立的首次积分，$\psi_i(t,\vec{X})=C_i,i=1,2,\ldots,n \quad\Longrightarrow\quad \frac{\partial(\psi_1,\psi_2,\ldots,\psi_n)}{\partial(x_1,x_2,\ldots,x_n)}\neq 0$

例1： $\begin{cases}\dfrac{dx}{dt}=y-x\left(x^2+y^2-1\right) & (1)\\ \dfrac{dy}{dt}=-x-y\left(x^2+y^2-1\right) & (2)\end{cases}$

解：$y\cdot(1)-x\cdot(2)\quad\Longrightarrow\quad y\frac{dx}{dt}-x\frac{dy}{dt}=x^2+y^2\quad\Longrightarrow\quad\frac{-yx'+xy'}{x^2+y^2}=-1\quad\Longrightarrow\quad\arctan\frac{y}{x}=-t+c_1$

$x\cdot(1)+y\cdot(2)\quad\Longrightarrow\quad xx'+yy'=\left(-x^2+y^2\right)\left(x^2+y^2-1\right)$, $\frac{\frac{1}{2}\left(x^2+y^2\right)'}{(x^2+y^2)(x^2+y^2-1)}=-1\quad\Longrightarrow\quad\ln\left|\frac{x^2+y^2-1}{x^2+y^2}\right|=-2t+$

$\frac{r^2-1}{r^2}=\tilde{C}_2e^{-2t}$, $r=\frac{1}{\sqrt{1-\tilde{C}_2e^{-2t}}}$，极坐标如下：

$$\begin{cases}x=r\cos\theta=\frac{\cos(-t+c_1)}{\sqrt{1-\tilde{c}_2e^{-2t}}}\\ y=r\sin\theta=\frac{\sin(-t+c_1)}{\sqrt{1-\tilde{c}_2e^{-2t}}}\end{cases}$$

Hamilton系统：

$p(t),q(t).\quad H(p,q)=C$ 为**Hamilton守恒量**。

$\begin{cases}\frac{dp}{dt}=H_q\\ \frac{dq}{dt}=-H_p\end{cases}\quad \frac{d}{dt}H(p(t),q(t))=H_p\frac{dp}{dt}+H_q\frac{dq}{dt}=H_pH_q+H_q(-H_p)=0$

考虑例1：$\frac{dx}{dt}=y-x\left(x^2+y^2-1\right)=H_y\quad\Longrightarrow\quad H=\frac{1}{2}y^2-x(x^2y+\frac{1}{3}y^3-y)+\varphi(x)$

$\frac{dy}{dt}=-x-y\left(x^2+y^2-1\right)=-H_x\quad\Longrightarrow\quad H=\frac{1}{2}x^2+y(\frac{1}{3}x^3+y^2x-x)+\psi(y)$

$\therefore H$ 不是Hamilton系统。

$\begin{cases}\vec{X}'=A\vec{X}+B,A,B\in C(I)\\ \vec{X}(t_0)=\vec{X}_0\end{cases}$ $\exists!$ 解，$|\vec{X}_1,\ldots,\vec{X}_n|=\begin{vmatrix}x_{11}&\cdots&x_{1n}\\ \vdots&&\vdots\\ x_{n1}&\cdots&x_{nn}\end{vmatrix}$

例2： $\begin{cases} \vec{X}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{X} \\ \vec{X}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$

解：$\vec{X}_n(t) = \vec{X}_0 + \int_0^t A\vec{X}_{n-1}(t)ds$

picard迭代序列： $\vec{X}_1(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}$

$\vec{X}_2(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s \\ 1 \end{pmatrix} ds = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 1 \\ -s \end{pmatrix} ds = \begin{pmatrix} t \\ 1 - \frac{1}{2}t^2 \end{pmatrix}$

$\vec{X}_3(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} s \\ 1 - \frac{1}{2}s^2 \end{pmatrix} ds = \begin{pmatrix} t - \frac{1}{3!}t^3 \\ 1 - \frac{1}{2!}t^2 \end{pmatrix}$

$\ldots\ldots \quad \longrightarrow \vec{X} = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

性质：（1）t_0 时刻，$\vec{X}(t_0) = 0 \quad \Longrightarrow \quad \vec{X}(t) \equiv 0, t \in I$

（2）$A \in C(I) \quad \Longrightarrow \quad \vec{X}(t)$ 在 I 上存在

（3）**叠加原理：** 若 $\vec{X}_1, \vec{X}_2$ 为 I 上解 $\quad \Longrightarrow \quad c_1\vec{X}_1(t) + c_2\vec{X}_2$ 也是解。

引理1.1：

若 $\vec{X}_1(t), \ldots, \vec{X}_n(t)$ 为 $\vec{X}' = A\vec{X}$ 的 n 个解，$t \in I$，则 $\vec{X}_1(t), \ldots, \vec{X}_n(t)$ 在 I 上：

线性无关 $\quad \Longleftrightarrow \quad W(t) = |\vec{X}_1(t), \ldots, \vec{X}_n(t)| \neq 0, \forall t \in I.$

线性相关 $\quad \Longleftrightarrow \quad W(t) = 0, \forall t \in I.$

证明：$\Longrightarrow$: $W(t) \neq 0, \forall t \in I$，记 $\Phi = (\vec{X}_1, \ldots, \vec{X}_n)_{n \times n}$，则 $\Phi\vec{c} = 0, \exists\,!\,\vec{c} = 0$，故

$c_1\vec{X}_1(t) + \ldots + c_n\vec{X}_n = 0 \iff c_1 = \ldots = c_n = 0$，从而 $\vec{X}_1, \ldots, \vec{X}_n$ 线性无关。

$\Longleftarrow$: 逆否命题：若 $\exists\, t_0 \in I, st.\ W(t_0) = 0$，则 $\vec{X}_1, \ldots, \vec{X}_n$ 在 I 上线性相关。

$\therefore \Phi\vec{c} = 0, \exists$ 非零解 $\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$。令 $Y(t) = c_1\vec{X}_1(t) + \ldots + c_n\vec{X}_n(t), Y(t_0) = 0.$ 目标：$\forall t \in I, Y(t) = 0.$

由叠加原理：$\begin{cases} Y' = AY \\ Y(t_0) = 0 \end{cases} \quad \Longrightarrow \quad \exists\,!\, Y(t) \equiv 0, \forall t \in I.$

定理1.2：

n阶线性ODEs：$\vec{X}' = A\vec{X}$，所有解构成一个n维线性空间，称 $\vec{X}_1(t), \ldots, \vec{X}_n(t), t \in I$ 为 $\vec{X}' = A\vec{X}$ 的**基本解组。**
$\Phi(t) = (\vec{X}_1(t), \ldots, \vec{X}_n(t))$ 称为**基解矩阵。**

通解：$\vec{X}(t) = \Phi(t) \cdot \vec{c}$ 特解：$\vec{X}(t) = \Phi(t) \cdot \Phi^{-1}(t_0)\vec{X}_0$

性质（1）：$P = (P_1, \ldots, P_n)$ 为可逆矩阵 $\quad \Longrightarrow \quad \Phi P = (\Phi P_1, \ldots, \Phi P_n)$ 为基解矩阵 $\therefore \Phi P_1, .., \Phi P_n$ 为LODEs解。

性质（2）：若另一个基解矩阵 Ψ，则 $\exists P$ 可逆，st $\Psi(t) = \Phi(t) \cdot P$

Liouville公式：$W(t) = W(t_0)e^{\int_{t_0}^t trAds}$

$\vec{X}_1 = \begin{pmatrix} x_{11}(t) \\ \vdots \\ x_{n1}(t) \end{pmatrix}, \vec{X}_1' = A\vec{X}_1 = \begin{pmatrix} \sum a_{1k}x_{k1} \\ \vdots \\ \sum a_{nk}x_{k1} \end{pmatrix}, x_{ij}'(t) = \sum_{k=1}^n a_{ik}x_{kj}$

$W' = |\vec{X}_1'\vec{X}_2 \ldots \vec{X}_n| + \ldots + |\vec{X}_1\vec{X}_2 \ldots \vec{X}_n'| = \begin{vmatrix} x_{11}' & x_{12}' & \cdots & x_{1n}' \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{vmatrix} + \ldots + \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1}' & x_{n2}' & \cdots & x_{nn}' \end{vmatrix}$

$$=\begin{vmatrix}\sum a_{1k}x_{k1} & \sum a_{1k}x_{k2} & \cdots & \sum a_{1k}x_{kn}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}+\begin{vmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \vdots\\ \sum a_{nk}x_{k1} & \sum a_{nk}x_{k2} & \cdots & \sum a_{nk}x_{kn}\end{vmatrix}$$

$$=\sum_{k=1}^{n}a_{1k}\begin{vmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}+\sum_{k=1}^{n}a_{2k}\begin{vmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}+\ldots+\sum_{k=1}^{n}a_{nk}\begin{vmatrix}x_{11} & x_{12} & \cdots & x_{1n}\\ x_{21} & x_{22} & \cdots & x_{2n}\\ \vdots & \vdots & & \vdots\\ x_{n1} & x_{n2} & \cdots & x_{nn}\end{vmatrix}$$

$$=a_{11}W+a_{22}W+\ldots+a_{nn}W=trA\cdot W$$

ch4: $\begin{cases}\vec{X}'=A\vec{X}+B,A,B\in C(I)\\ \vec{X}(t_0)=\vec{X}_0\end{cases}$ Picard $\Longrightarrow \exists\,!\vec{X}(t)$

(1) $\vec{X}(t)=\vec{X}_0+\int_{t_0}^{t}(A(s)\vec{X}(s)+B(s))ds$

(2) $\vec{X}_0(t)=\vec{X}_0,\vec{X}_n(t)=\vec{X}_0+\int_{t_0}^{t}(A(s)\vec{X}_{n-1}(s)+B(s))ds$

(3) $\{\vec{X}_n\}$ 为Cauchy列，$t\in I$.

(4) $\vec{X}_n(t)\rightrightarrows\vec{X}(t),t\in I$，验证 $\vec{X}(t)$ 为解。

齐次： $\vec{X}'=A\vec{X}$，通解：$\vec{X}=c_1\vec{X}_1(t)+\ldots+c_n\vec{X}_n(t)=\Phi\cdot\vec{c}$

非齐次： $\vec{X}'=A\vec{X}+B$，通解：$\vec{X}=c_1\vec{X}_1(t)+\ldots+c_n\vec{X}_n(t)+\vec{X}^*(t)$，其中 $\vec{X}^*(t)$ 为 $\vec{X}'=A\vec{X}+B$ 的一个特解。

$\Longrightarrow$ **线性非齐次方程的解 = 线性齐次方程的通解 + 线性非齐次方程的一个特解**

常数变易法：

设 $(*)$ 的解 $\vec{X}(t)=\Phi(t)\vec{U}(t)$

$$\begin{cases}\Phi'\vec{U}+\Phi\vec{U}'=A\Phi\vec{U}+\vec{B}\\ \Phi'=A\Phi\Rightarrow\Phi'\vec{U}=A\Phi\vec{U}\end{cases}\quad\Rightarrow\quad\Phi\vec{U}'=B\,,\ \vec{U}'=\Phi^{-1}B\quad\Rightarrow\quad\vec{U}(t)=\vec{c}+\int\Phi^{-1}Bdt$$

$\therefore\vec{X}(t)=\Phi(t)(\vec{c}+\int\Phi^{-1}Bdt)$

例3：$\vec{X}'=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\vec{X}+\begin{pmatrix}0\\ e^{2t}\end{pmatrix}$

(1) $\Phi(t)=e^{\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}t}=e^{2t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}$

(2) 常数变易法：设 $\vec{X}=e^{2t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\vec{u}$ 代入方程组，有：$e^{2t}\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\vec{u}'=\begin{pmatrix}0\\ e^{2t}\end{pmatrix}\quad\Longrightarrow\quad\begin{cases}u_1=c_1-\frac{1}{2}t^2\\ u_2=c_2+t\end{cases}$

$$\Longrightarrow\quad\vec{X}(t)=\Phi(t)\vec{u}=e^{2t}\begin{pmatrix}c_1+c_2t+\frac{1}{2}t^2\\ c_2+t\end{pmatrix}$$

当 $\int_{t_0}^{t}A(s)dsA(t)=A(t)\int_{t_0}^{t}A(s)ds$，记 $D(t)=\int_{t_0}^{t}A(s)ds$，有：$\vec{X}'=A\vec{X},\vec{X}(t)=e^{\int_{t_0}^{t}A(s)ds}\vec{X}_0$

(1) 验证

(2) Picard迭代：

$$\vec{X}_0(t)=\vec{X}_0$$
$$\vec{X}_1=\vec{X}_0+\int_{t_0}^{t}A(s)\vec{X}_0ds=\vec{X}_0+D(t)\vec{X}_0$$
$$\vec{X}_2(t)=\vec{X}_0+\int_{t_0}^{t}A(s)(\vec{X}_0+D(t)\vec{X}_0)ds=\vec{X}_0+D\vec{X}_0+\frac{D^2}{2}\vec{X}_0$$
$$\cdots\cdots\Longrightarrow\vec{X}_n(t)=\vec{X}_0+D\vec{X}_0+\cdots+\frac{D^n}{n!}\vec{X}_0$$

当 $n \to +\infty$ 时，有 $\vec{X}(t) = e^{D(t)} \cdot \vec{X}_0$

$\vec{X}' = A\vec{X}$ 通解：$e^{At} \cdot \vec{c}$，$e^{At} = Pe^{Jt}P^{-1} = P \begin{pmatrix} e^{J_1 t} & & \\ & \ddots & \\ & & e^{J_s t} \end{pmatrix} P^{-1}, A = PJP^{-1}, e^{J_k t} = e^{\lambda_k t} \begin{pmatrix} 1 & t & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ & & & 1 \end{pmatrix}$

基解矩阵 $\Phi(t) = Pe^{Jt}$，通解 $\vec{X}(t) = \Phi(t) \cdot \vec{c}$，直接求 P。

（1）A 单根：特征值 $\lambda_1, \ldots, \lambda_n$，特征向量 $\gamma_1, \ldots, \gamma_n$，则：

$$P = (\gamma_1, \ldots, \gamma_n), J = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}, \Phi(\gamma_1, \ldots, \gamma_n) e^{\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} t} = (\gamma_1 e^{\lambda_1 t}, \ldots, \gamma_n e^{\lambda_n t})$$

（*）特征值 $\lambda_1, \ldots, \lambda_s, \alpha_1 \pm i\beta_1, \ldots, \alpha_h \pm i\beta_h$，其中 $s + 2h = n$，特征向量：$\gamma_1, \ldots, \gamma_s, m_1, \overline{m}_1, \ldots, m_h, \overline{m}_h$，则 n 个线性无关

解为 $\gamma_1 e^{\lambda_1 t}, \ldots, \gamma_s e^{\lambda_s t}, Re(m_1 e^{(\alpha_1 \pm i\beta_1)t}), Im(m_1 e^{(\alpha_1 \pm i\beta_1)t}), \ldots, Re(m_h e^{(\alpha_h \pm i\beta_h)t}), Im(m_h e^{(\alpha_h \pm i\beta_h)t})$，以上 n

个线性无关解组成 $\Phi(t)$.

$\vec{X}' = A\vec{X}$ 有形如 $\vec{\gamma} e^{\lambda t}$ 的解 $\implies$ $\lambda\vec{\gamma} = A\vec{\gamma}$.

（2）A 重根：特征值 $\lambda_1, \ldots, \lambda_s$，重数 $n_1, \ldots, n_s, n_1 + \ldots + n_s = n, A = PJP^{-1}, e^{At} = Pe^{Jt}P^{-1}, \Phi = Pe^{Jt} = (\vec{X}_1, \ldots, \vec{X}_n) =$

$$(\gamma_1, \ldots, \gamma_n) \begin{pmatrix} e^{\lambda_1 t} \begin{pmatrix} 1 & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & \ddots & \vdots \\ & & 1 \end{pmatrix} & & \\ & \ddots & \\ & & e^{J_s t} \end{pmatrix} = (\underbrace{\gamma_1 e^{\lambda_1 t}, \gamma_1 t e^{\lambda_1 t} + \gamma_2 e^{\lambda_2 t}, \ldots, \frac{\gamma_1 t^{n_1 - 1} e^{\lambda_1 t}}{(n_1 - 1)!} + \ldots + \gamma_{n_1} e^{\lambda_{n_1} t}}_{\text{第1块：} e^{J_1 t}}, \ldots \ldots)$$

总结：（1）矩阵 A，特征值为 λ，n 重。

（2）$(A - \lambda E)^n \gamma = 0$ 有 $\gamma_1, \ldots, \gamma_n$ 线性无关解，$\vec{X}_i = e^{\lambda t}(\gamma_i + (A - \lambda E)\gamma_i t + \ldots + (A - \lambda E)^{n-1} \gamma_i \frac{t^{n-1}}{(n-1)!})$

（3）$\Phi = [\vec{X}_1, \ldots, \vec{X}_n]$

计算 e^{At}：

（1）$e^{At} = \Phi(t)\Phi^{-1}(0)$

（2）$A: \lambda, n$ 重根，$e^{At} = e^{(A - \lambda E)t} \cdot e^{\lambda E t} = e^{\lambda E}(E + (A - \lambda E)t + \ldots + \frac{(A - \lambda E)^{n-1}}{(n-1)!} t^{n-1})$

计算 $\vec{X}' = A\vec{X} + B$：

（1）$\vec{X}(t) = A\vec{X} + B, \vec{X}(t) = e^{At} \cdot \vec{c} + e^{At} \int e^{-At} B(t) dt$

（2）取 $\vec{X}(0) = \vec{X}_0$，

$$\vec{X}(t) = e^{At} \cdot \vec{X}_0 + e^{At} \int_0^t e^{-At} B(s) ds = e^{At} \vec{X}_0 + \int_0^t e^{A(t-s)} B(s) ds$$

或：

$$\vec{X}(t) = \Phi(t) \cdot \vec{c} + \Phi(t) \int \Phi^{-1}(s) B(s) dt = \Phi(t)\Phi^{-1}(0)\vec{X}_0 + \int_0^t \Phi(t-s)\Phi^{-1}(0) B(s) ds$$

例4： $A=\begin{pmatrix}\alpha & \beta\\ -\beta & \alpha\end{pmatrix}$

解：（1）特征值：$\lambda_1=\alpha+\beta i,\lambda_2=\alpha-\beta i$

（2）特征向量：$(A-\lambda_1 E)\gamma_1=0,\gamma_1=\begin{pmatrix}1\\ i\end{pmatrix}$，同理 $\gamma_2=\begin{pmatrix}1\\ -i\end{pmatrix}$

$\therefore \vec{X}_1=\gamma_1 e^{\lambda_1 t}=\begin{pmatrix}1\\ i\end{pmatrix}\cdot e^{\alpha t}(\cos\beta t+i\sin\beta t)$，$\vec{X}_2=\gamma_1 e^{\lambda_1 t}=\begin{pmatrix}1\\ -i\end{pmatrix}\cdot e^{\alpha t}(\cos\beta t-i\sin\beta t)$

$$Re\vec{X}_1=\frac{\vec{X}_1+\vec{X}_2}{2},Im\vec{X}_1=\frac{\vec{X}_1-\vec{X}_2}{2i}\quad\Longrightarrow\quad\Phi(t)=(Re\vec{X}_1,Im\vec{X}_1)=e^{\alpha t}\begin{pmatrix}\cos\beta t & \sin\beta t\\ -\sin\beta t & \cos\beta t\end{pmatrix}=e^{At}$$

例5： $A=\begin{pmatrix}2 & 1\\ -1 & 4\end{pmatrix}$

解：$\lambda_1=\lambda_2=3\quad(A-3E)^2\gamma=0,\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\gamma=0_{2\times 2}\cdot\gamma=0$，取 $\gamma_1=\begin{pmatrix}1\\ 0\end{pmatrix},\gamma_2=\begin{pmatrix}0\\ 1\end{pmatrix}$，那么：

$$\vec{X}_1=e^{3t}(\begin{pmatrix}1\\ 0\end{pmatrix}+\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}t)=e^{3t}\begin{pmatrix}1-t\\ -t\end{pmatrix},\ \vec{X}_2=e^{3t}(\begin{pmatrix}0\\ 1\end{pmatrix}+\begin{pmatrix}-1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix}t)=e^{3t}\begin{pmatrix}t\\ 1+t\end{pmatrix}$$

$$\Longrightarrow\quad\Phi(t)=[\vec{X}_1,\vec{X}_2]=e^{3t}\begin{pmatrix}1-t & t\\ -t & 1+t\end{pmatrix}$$

例6： $A=\begin{pmatrix}-3 & 1 & & & \\ & -3 & 1 & & \\ & & -3 & & \\ & & & -3 & \\ & & & & -3\end{pmatrix}$

解：$e^{At}=e^{-3t}(E+(A-\lambda E)t+\ldots+\frac{(A-\lambda E)^4}{4!}t^4)$

$$=e^{-3t}(E+\begin{pmatrix}0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0\end{pmatrix}t+\begin{pmatrix}0 & 0 & 1 & & \\ & 0 & 0 & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 0\end{pmatrix}\frac{t^2}{2})=e^{-3t}\begin{pmatrix}1 & t & \frac{t^2}{2} & & \\ & 1 & t & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{pmatrix}$$