1. (1) 检验数最小的入基变量能让目标函数的下降率最优。

不是尺度不变的。正尺度变换相当于该变量所对应的一列乘一个系数，这样会改变目标函数的下降率。

(2) 选择检验数为负且绝对值最大的变量，同时选择 $\min\left\{\frac{b_i}{a_{ij}} : a_{ij}>0\right\}$ 对应的元素作为枢轴元。这个方法是尺度不变的，因为 $d\cdot r_j\cdot\min\left\{\frac{b_i}{d\,a_{ij}} : a_{ij}>0\right\} = r_j\cdot\min\left\{\frac{b_i}{a_{ij}} : a_{ij}>0\right\}$

2. 等价于

$$
\begin{aligned}
\min\quad & -2x_1-3x_2+x_3+12x_4\\
\text{s.t.}\quad & -2x_1-9x_2+x_3+9x_4+x_5=0\\
& \tfrac{1}{3}x_1+x_2-\tfrac{1}{3}x_3-2x_4+x_6=0\\
& x_1,\cdots,x_6\geq 0
\end{aligned}
$$

$$
\begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 & 0\\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0\\ -2 & -3 & 1 & 12 & 0 & 0 & 0 \end{pmatrix} \rightarrow
$$

$$
\begin{pmatrix} 1 & 0 & -2 & -9 & 1 & 9 & 0\\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0\\ -1 & 0 & 0 & 6 & 0 & 3 & 0 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & 0 & -2 & -9 & 1 & 9 & 0\\ 0 & 1 & \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0\\ 0 & 0 & -2 & -3 & 1 & 12 & 0 \end{pmatrix} \rightarrow
\begin{pmatrix} 1 & 9 & 1 & 0 & -2 & -9 & 0\\ 0 & 1 & \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0\\ 0 & 3 & -1 & 0 & 0 & 6 & 0 \end{pmatrix}
$$

$$
\begin{pmatrix} 1 & 9 & 1 & 0 & -2 & -9 & 0\\ -\frac{1}{3} & -2 & 0 & 1 & \frac{1}{3} & 1 & 0\\ 1 & 12 & 0 & 0 & -2 & -3 & 0 \end{pmatrix} \rightarrow
\begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 & 0\\ -\frac{1}{3} & -2 & 0 & 1 & \frac{1}{3} & 1 & 0\\ 0 & 6 & 0 & 3 & -1 & 0 & 0 \end{pmatrix} \rightarrow
\begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 & 0\\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0\\ -2 & -3 & 1 & 12 & 0 & 0 & 0 \end{pmatrix}
$$

Bland 规则：

$$
\begin{pmatrix} -2 & -9 & 1 & 9 & 1 & 0 & 0\\ \frac{1}{3} & 1 & -\frac{1}{3} & -2 & 0 & 1 & 0\\ -2 & -3 & 1 & 12 & 0 & 0 & 0 \end{pmatrix} \rightarrow
\begin{pmatrix} 0 & -3 & -1 & -3 & 1 & 6 & 0\\ 1 & 3 & -1 & -6 & 0 & 3 & 0\\ 0 & 3 & -1 & 0 & 0 & 6 & 0 \end{pmatrix}
$$

解无界.

3. (1) 原问题等价于

$$
\begin{aligned}
\min\quad & (c^T,-c^T,0)\cdot(x^+,x^-,z)\\
\text{s.t.}\quad & x^+-x^-+Az=Ab\\
& (x^+,x^-,z)^T\geq 0
\end{aligned}
$$

对应对偶规划：

$$
\begin{aligned}
\max\quad & wAb\\
\text{s.t.}\quad & w(I,-I,A)\leq(c^T,-c^T,0)
\end{aligned}
$$

(2) 对偶规划：

$$
\begin{aligned}
\max\quad & w_1 b_L-w_2 b_L+w_3 l-w_4 u\\
\text{s.t.}\quad & w_1 A-w_2 A+w_3-w_4\leq c^T\\
& w_1,\cdots,w_4\geq 0
\end{aligned}
$$

4.(1) 其对偶规划为:

$$\begin{aligned} \max \quad & 3w_1 + 4w_2 \\ \text{s.t.} \quad & w_1 + 2w_2 \le 2 \\ & 2w_1 - w_2 \le 3 \\ & w_1 + 3w_2 \le 4 \\ & w_1, w_2 \ge 0 \end{aligned} \Rightarrow \begin{aligned} \min \quad & -3w_1 - 4w_2 \\ \text{s.t} \quad & w_1 + 2w_2 + w_3 = 2 \\ & 2w_1 - w_1 + w_4 = 3 \\ & w_1 + 3w_2 + w_5 = 4 \\ & w_1, \cdots, w_5 \ge 0 \end{aligned}$$

$$\begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 2 \\ 2 & -1 & 0 & 1 & 0 & 3 \\ 1 & 3 & 0 & 0 & 1 & 4 \\ -3 & -4 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 1 \\ \frac{5}{2} & 0 & \frac{1}{2} & 1 & 0 & 4 \\ -\frac{1}{2} & 0 & -\frac{3}{2} & 0 & 1 & 1 \\ -1 & 0 & 2 & 0 & 0 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} - & - & - & - & - & - \\ 1 & 0 & \frac{1}{5} & \frac{2}{5} & 0 & \frac{8}{5} \\ - & - & - & - & - & - \\ 0 & 0 & \frac{11}{5} & \frac{2}{5} & 0 & \frac{28}{5} \end{pmatrix}$$

$\Rightarrow$ 最优值为 $\frac{28}{5}$.

(2) 对偶规划:

$$\begin{aligned} \max \quad & -6w_1 + 4w_2 + 3w_3 \\ \text{s.t.} \quad & -w_1 + w_2 \le 3 \\ & -w_1 + w_3 \le 2 \\ & -w_1 - w_2 - w_3 \le 1 \\ & w_1, w_2, w_3 \ge 0 \end{aligned} \Rightarrow \begin{aligned} \min \quad & 6w_1 - 4w_2 - 3w_3 \\ \text{s.t.} \quad & -w_1 + w_2 + w_4 = 3 \\ & -w_1 + w_3 + w_5 = 2 \\ & -w_1 - w_2 - w_3 + w_6 = 1 \\ & w_1, w_2, w_3 \ge 0 \end{aligned}$$

$$\begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 0 & 1 & 0 & 1 & 0 & 2 \\ -1 & -1 & -1 & 0 & 0 & 1 & 1 \\ 6 & -4 & -3 & 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 0 & 1 & 0 & 1 & 0 & 2 \\ -2 & 0 & -1 & 1 & 0 & 1 & 4 \\ 2 & 0 & -3 & 4 & 0 & 0 & 12 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 1 & 0 & 1 & 0 & 0 & 3 \\ -1 & 0 & 1 & 0 & 1 & 0 & 2 \\ -3 & 0 & 0 & 1 & 1 & 1 & 6 \\ -1 & 0 & 0 & 4 & 3 & 0 & 18 \end{pmatrix}$$

$\Rightarrow$ 无解

5. (1) $c = (c_1, c_2, \cdots, c_n)^T$. 令 $\|c\|_2 = \sqrt{\sum_{i=1}^{n} c_i^2}$

取 $x = (-\|c\|_2, -\|c\|_2, \cdots, -\|c\|_2)$, $y = x - c$ 即可

(2) 取 $\bar{x} = -x$, $\bar{y} = -y$. 则 $\bar{x}, \bar{y} \ge 0$

$$\begin{aligned} \min \quad & (-a^T, b^T)\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \\ \text{s.t.} \quad & \begin{pmatrix} -I & I \\ I & -I \end{pmatrix}\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \ge \begin{pmatrix} c \\ -c \end{pmatrix}, \ \bar{x}, \bar{y} \ge 0 \end{aligned} \iff \begin{aligned} \max \quad & w\begin{pmatrix} c \\ -c \end{pmatrix} \\ \text{s.t} \quad & w\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \le (-a^T, b^T) \end{aligned}$$

(P) 有有限最优解等价于 (D) 有有限最优解 $\Longleftrightarrow$ $a \leq b$

(3) (D) 的最优解为 $w_i = \begin{cases} a_i, & c_i \leq 0 \\ b_i, & c_i > 0 \end{cases}$

由互补松弛定理，$x_i(w_i - a_i) = y_i(w_i - b_i) = 0$

$\Rightarrow$ (P) 的最优解为 $(x_i, y_i) = \begin{cases} (c_i, 0), & c_i \leq 0 \\ (0, -c_i), & c_i > 0 \end{cases}$