2.2 Distribution functions and continuous random variables

2.2.1 Distribution functions

1. Definitions

There is no distribution sequence for other types of random variables. If all possible values of a random variable consists of an interval, then we are not able to enumerate all these values and their probabilities.

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$$P(a < \xi \leq b) = P(\xi \leq b) - P(\xi \leq a).$$

We only need to the probability of the type

$$P(\xi \leq x).$$

Definition

Let ξ be a random variable on a probability space $(\Omega, \mathcal{F}, P)$, We define its distribution function (sometimes, cumulative distribution function (CDF)) as

$$F(x) = P(\xi \le x), \qquad -\infty < x < \infty.$$

Having distribution function, the probability $P(\xi(\omega) \in B)$ can be expressed in term of it for any Borel set B. For example:

$$P(a < \xi \leq b) = F(b) - F(a)$$

$$\begin{aligned} P(a < \xi \le b) &= F(b) - F(a) \\ P(\xi < a) &= \lim_{b \to a-0} P(\xi \le b) = F(a-0); \end{aligned}$$

$$
\begin{aligned}
P(a<\xi\le b) &= F(b)-F(a) \\
P(\xi<a) &= \lim_{b\to a-0} P(\xi\le b)=F(a-0); \\
P(\xi=a) &= P(\xi\le a)-P(\xi<a) \\
&= F(a)-F(a-0);
\end{aligned}
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P(\xi < a) &= \lim_{b \to a-0} P(\xi \le b) = F(a-0); \\
P(\xi = a) &= P(\xi \le a) - P(\xi < a) \\
&= F(a) - F(a-0); \\
P(\xi > a) &= 1 - P(\xi \le a) = 1 - F(a);
\end{aligned}$$

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P(a<\xi\le b) &= F(b)-F(a) \\
P(\xi<a) &= \lim_{b\to a-0} P(\xi\le b)=F(a-0); \\
P(\xi=a) &= P(\xi\le a)-P(\xi<a) \\
&= F(a)-F(a-0); \\
P(\xi>a) &= 1-P(\xi\le a)=1-F(a); \\
P(a<\xi<b) &= P(\xi<b)-P(\xi\le a) \\
&= F(b-0)-F(a).
\end{aligned}
$$

Example

Suppose that a random variable ξ is distributed as Bernoulli distribution:

$$\begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}, \quad p, q > 0, \quad p + q = 1.$$

Determine its distribution function $F(x)$, and calculate $P(-1 < \xi < 0.5)$.

Solution. When $x < 0$, $P(\xi \leq x) = 0$ (null event);

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when $x \geq 1$,

$$P(\xi \leq x) = P(\xi = 0) + P(\xi = 1) = q + p = 1.$$

Hence we obtain the following distribution function:

$$F(x) = \begin{cases} 0, & x < 0, \\ q, & 0 \le x < 1, \\ 1, & x \ge 1. \end{cases}$$

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Next, let us compute $P(-1 < \xi < 0.5)$.

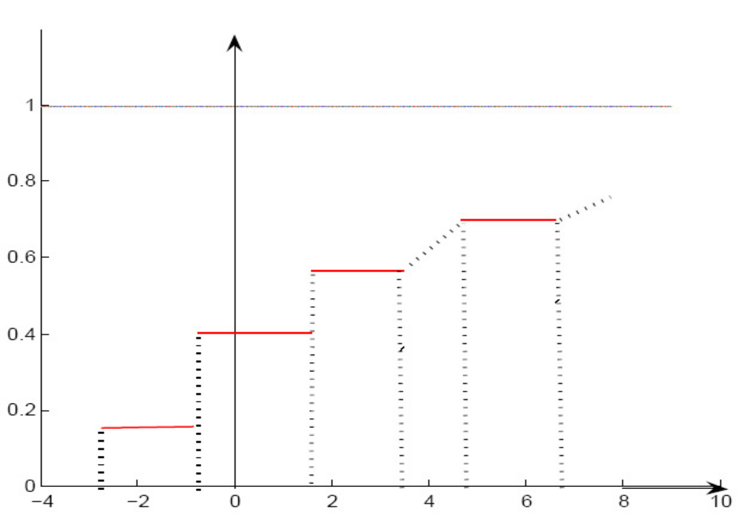
$$P(-1 < \xi < 0.5) = F(0.5 - 0) - F(-1) = q.$$

Suppose that ξ has the distribution sequence

$$\begin{pmatrix} x_1 & x_2 & \cdots & x_n & \cdots \\ p(x_1) & p(x_2) & \cdots & p(x_n) & \cdots \end{pmatrix},$$

where $x_1 < x_2 < \cdots < x_k < \cdots$, then

$$F(x) = \sum_{k: x_k \le x} p(x_k) = \begin{cases} 0, & x < x_1, \\ p(x_1), & x_1 \le x < x_2, \\ \cdots & \cdots \\ \sum_{i \le k} p(x_i), & x_k \le x < x_{k+1}, \\ \cdots & \cdots \end{cases}$$


1
0.8
0.6
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0.2
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−4
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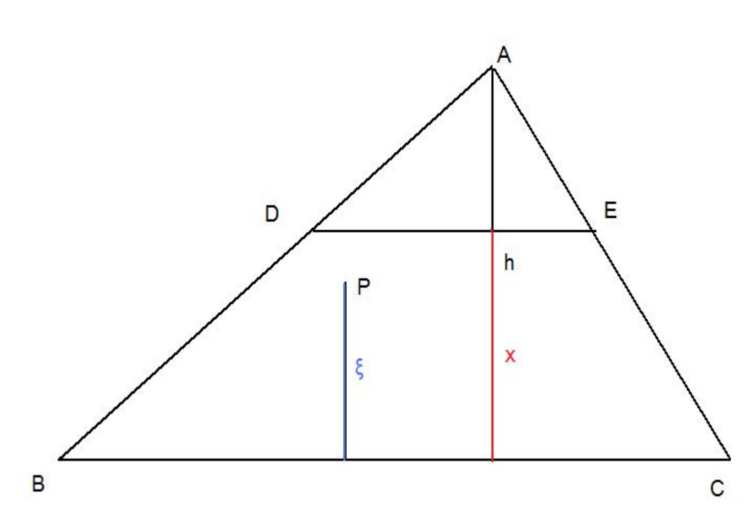
0
2
4
6
8
10

Example

Choose randomly a point P in a triangle ΔABC, let ξ be the distance from P to edge BC. Calculate the distribution function of ξ.

Solution.



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$$P(\xi \le x) = \frac{\text{area of } DBCE}{\text{area of } \triangle ABC} = 1 - \left(1 - \frac{x}{h}\right)^2.$$

The distribution function is

$$F(x)=\begin{cases}0, & x<0,\\ 1-\left(1-\frac{x}{h}\right)^2, & 0\le x<h,\\ 1, & x\ge h.\end{cases}$$

2. Properties Three fundamental properties:

1. $F(x)$ is monotonic non-decreasing in x, that is, if $a < b$ then $F(a) \le F(b)$;
2. $\lim_{x\to-\infty} F(x) = 0, \lim_{x\to+\infty} F(x) = 1$;
3. $F(x)$ is right continuous, that is $F(x+0) = F(x)$.

Proof. (1) If $a < b$, then $\{\xi \le a\} \subset \{\xi \le b\}$.

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$$\{\xi \le -(n+1)\} \subset \{\xi \le -n\}, \quad \bigcap_{n=1}^{\infty}\{\xi \le -n\} = \emptyset.$$

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$$= P\left(\lim_{n\to\infty}\{\xi \le n\}\right) = P\left(\bigcup_{n=1}^{\infty}\{\xi \le n\}\right)$$

$$= P(\Omega) = 1.$$

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=\ &P(\bigcap_{n=1}^{\infty}\{\xi\le x+1/n\})\\
=\ &P(\xi\le x)=F(x).
\end{aligned}$$

Similarly,

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$$= \lim_{n\to\infty} P(\xi \le x - 1/n)$$

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= \ & P(\bigcup_{n=1}^{\infty}\{\xi \le x-1/n\}) \\
= \ & P(\xi < x).
\end{aligned}
$$

Example

Suppose a random variable has the distribution function as follows:

$$F(x) = \begin{cases} 0, & x \le -1, \\ a + b \arcsin x, & -1 < x \le 1, \\ 1, & x > 1. \end{cases}$$

Find constants a, b.

Solution. From the fact that

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it follows that

$$a - b\pi/2 = 0 \quad \text{and} \quad a + b\pi/2 = 1.$$

Hence $a = 1/2$, $b = 1/\pi$.

Theorem

Let $F(x)$ be a real function satisfying Properties (1), (2) and (3). Then there is an unique probability measure P_F: $\mathscr{B} \to [0, 1]$, such that

$$P_F\Big((-\infty, x]\Big) = F(x) \ \ \forall x.$$

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On the probability space $(\boldsymbol{R}, \mathscr{B}, P_F)$, define $X(r) = r$, $r \in \boldsymbol{R}$. Then X is a r.v. variable with

$$P_F(X \le x) = P_F\Big((-\infty, x]\Big) = F(x).$$

So F is the distribution function of X.

2.2.2 Continuous random variables and density functions

Definition

Suppose that a random variable ξ takes all values of an interval (finite or infinite), and that there exists a non-negative integrable function $p(x)$ such that the distribution function $F(x)$ can be written as

$$F(x) = \int_{-\infty}^{x} p(y)dy, \quad -\infty < x < \infty.$$

Then ξ is called a continuous random variable, and $p(x)$ is called the probability density function (PDF) of ξ, or more simply its density function. $F(x)$, having the above property, is said to be absolutely continuous.

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$$P(\xi \in B) = \int_B p(x)dx, \quad B \in \mathcal{B}$$

3

$$
\begin{aligned}
P(\xi = c) =& F(c) - F(c-0) \\
=& \lim_{h\to 0^+} \int_{c-h}^{c} p(y)dy = 0.
\end{aligned}
$$

The density function possesses the following properties.

- $p(x) \geq 0$,
- $\int_{-\infty}^{\infty} p(x)dx = 1$.

On the contrary, if a function defined in $(-\infty, \infty)$ satisfies these two properties then it can be considered to be a density function of some random variable.

$$F(x) = \begin{cases} 0, & x < 0 \\ (1+x)/2, & 0 \le x < 1, \\ 1, & x \ge 1 \end{cases}$$

is a distribution function. But it is neither discrete nor continuous (discontinuous).

$$F(x) = \begin{cases} 0, & x < 0 \\ (1+x)/2, & 0 \le x < 1, \\ 1, & x \ge 1 \end{cases}$$

is a distribution function. But it is neither discrete nor continuous (discontinuous). In fact, it is a mixture of $F_1(x)$ and $F_2(x)$:

$$F(x) = \frac{F_1(x) + F_2(x)}{2},$$

where F_1 is a degenerate distribution at $x = 0$ and $F_2(x)$ is uniform on $[0, 1]$.

2.2.3 Typical continuous random variables

1. The uniform distribution $\xi \sim U(a,b)$

$$p(x) = \begin{cases} 1/(b-a), & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

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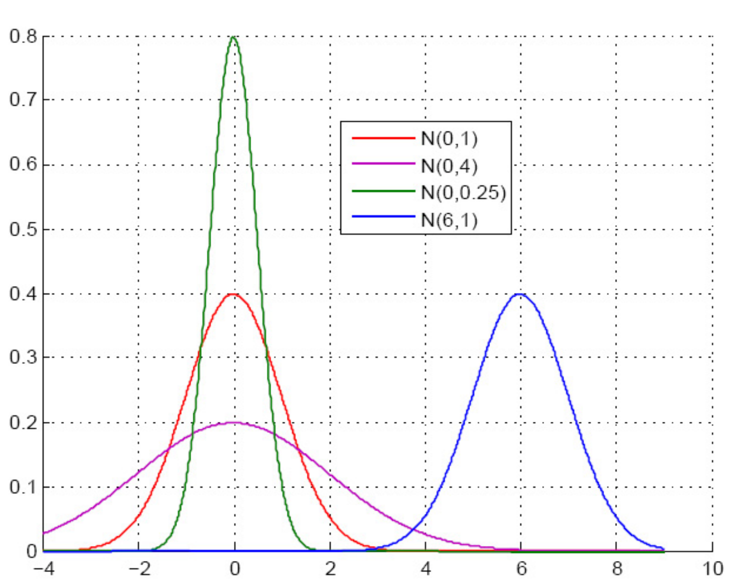
Distribution function:

$$F(x) = \begin{cases} 0, & x < a, \\ (x-a)/(b-a), & a \le x < b, \\ 1, & x \ge b. \end{cases}$$

2. The normal distribution $\xi \sim N(\mu, \sigma^2)$, where $-\infty < \mu < \infty, \sigma > 0$.

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty,$$

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt, \quad -\infty < x < \infty.$$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8
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N(0,1)
N(0,4)
N(0,0.25)
N(6,1)

Johann Carl Friedrich Gauss (April 1777–February 1855)

Verifying $\int p(x)dx = 1$: Let

$$I =: \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

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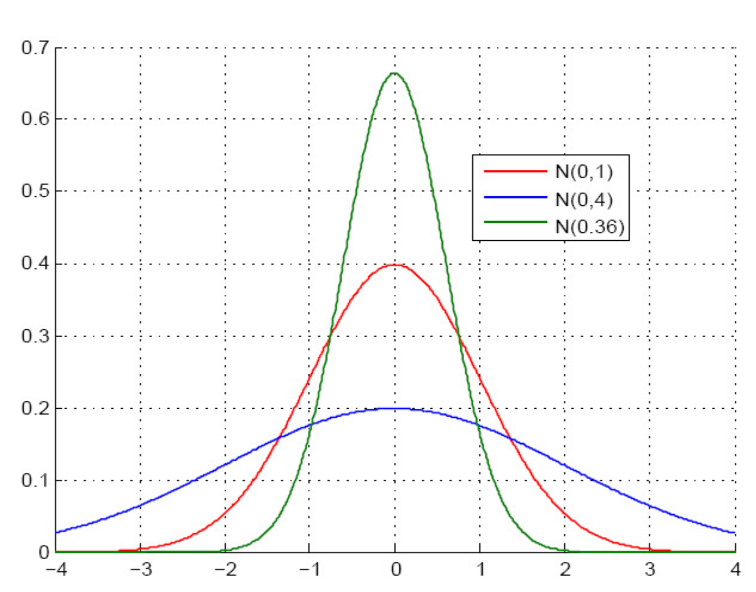
I^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dxdy \\
&= \int_0^{\infty}\int_0^{2\pi} \frac{1}{2\pi} e^{-\frac{r^2}{2}} rdrd\theta = -e^{-\frac{r^2}{2}}\Big|_0^{\infty} = 1.
\end{aligned}
$$

Standard normal distribution: $N(0,1)$

$$\varphi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\quad \Phi(x)=\int_{-\infty}^{x}\varphi(t)dt,$$

$$-\infty<x<\infty.$$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N(0,1)
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Table of normal distribution.

(1) $\xi \sim N(0,1)$.

- $x > 0$: ......
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So

$$P(\xi \leq x) = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

Example

Let $\xi \sim N(0,1)$.

(1) Find $P(-1 < \xi < 3)$;

(2) Suppose $P(\xi < \lambda) = 0.9755$, find λ.

Solution. (1)

$$
\begin{aligned}
& P(-1 < \xi < 3) \\
= \; & \Phi(3) - \Phi(-1)
\end{aligned}
$$

Solution. (1)

$$
\begin{aligned}
& P(-1 < \xi < 3) \\
= \; & \Phi(3) - \Phi(-1) = \Phi(3) + \Phi(1) - 1
\end{aligned}
$$

Solution. (1)

$$\begin{aligned} & P(-1 < \xi < 3) \\ = \ & \Phi(3) - \Phi(-1) = \Phi(3) + \Phi(1) - 1 \\ = \ & 0.9987 + 0.8413 - 1 = 0.8400. \end{aligned}$$

Solution. (1)

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(2) Note that $\Phi(\lambda)=0.9755$, which lies between $\Phi(1.96)=0.9750$ and $\Phi(1.98)=0.9762$. By using a linear interpolation,

$$\begin{aligned} \lambda \approx & 1.96+\frac{\Phi(\lambda)-\Phi(1.96)}{\Phi(1.98)-\Phi(1.96)}\cdot(1.98-1.96) \\ \approx & 1.968. \end{aligned}$$

Example

Suppose that $\xi \sim N(2, 9)$, calculate $P(5 < \xi < 20)$.

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$$
\begin{aligned}
& P(5<\xi<20) \\
= \; & P\left(\frac{5-2}{3}<\frac{\xi-2}{3}<\frac{20-2}{3}\right) \\
= \; & P(1<\eta<6)=\Phi(6)-\Phi(1) \\
\approx \; & 1-0.8413=0.1587.
\end{aligned}
$$

Example

Suppose that $\xi \sim N(\mu, \sigma^2)$, find $P(|\xi - \mu| < \sigma)$, $P(|\xi - \mu| < 2\sigma)$ and $P(|\xi - \mu| < 3\sigma)$.

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Similarly,

$$P(|\xi - \mu| < 2\sigma) = P(|\eta| < 2) \approx 0.9545,$$

$$P(|\xi - \mu| < 3\sigma) = P(|\eta| < 3) \approx 0.9973.$$

Some important values of $\Phi(x)$

x	1	2	3	4
$\Phi(x)$	0.841345	0.977250	0.998650	0.999968
$2\Phi(x)-1$	0.6826895	0.9544997	0.9973002	0.9999367

$\Phi(x)$	0.9000	0.9500	0.9750	0.9900	0.9950
x	1.2816	1.6449	1.9600	2.3263	2.5758

Example

某人被控告为一个新生儿的父亲. 此案鉴定人作证时指出:母亲的怀孕的天数(即从受孕到婴儿出生的时间)近似地服从正态分布, 参数为$\mu = 270$, $\sigma^2 = 100$. 有证据表明: 被告在孩子出生前290天出国, 而于出生前240天才回来. ???

解： 用X表示怀孕期的天数，如果被告是孩子的父亲，那么$X \geq 290$或者$X \leq 240$，

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\begin{aligned}
& P(X \geq 290 \text{ or } X \leq 240) \\
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\end{aligned}
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& P(X \geq 290 \text{ or } X \leq 240) \\
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= \;& P\left(\frac{X-270}{10} \geq 2\right) + P\left(\frac{X-270}{10} \leq -3\right)
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= \ & 1 - \Phi(2) + 1 - \Phi(3) \\
= \ & 1 - 0.9772 + 1 - 0.9987 = 0.0241.
\end{aligned}
$$

Example

There are two ways by bus from a city's southern district to a train station located in the city's northern area. The first route is shorter, but the ride encounters heavy traffic, so the time τ required is $N(50, 100)$; the second ride is a bit longer, but unexpected traffic jams seldom occur and the time τ required is $N(60, 16)$.

1. If one has 70 minutes, then what way should be chosen?
2. What is it about if one has 65 minutes ?

Solution.

Solution. (1) For the first route, the probability one can arrive at the train station within 70 minutes is

$$P(\tau \le 70) = \Phi\left(\frac{70-50}{\sqrt{100}}\right) = \Phi(2) = 0.9772;$$

Solution. (1) For the first route, the probability one can arrive at the train station within 70 minutes is

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For the second route, the probability one can arrive at the train station within 70 minutes is

$$P(\tau \le 70) = \Phi\left(\frac{70-60}{\sqrt{16}}\right) = \Phi(2.5) = 0.9938.$$

So, it is better to take the second route.

(2) For the first route, the probability one can arrive at the train station within 65 minutes is

$$P(\tau \leq 65) = \Phi\left(\frac{65-50}{\sqrt{100}}\right) = \Phi(1.5) = 0.9332;$$

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For the second route, the probability one can arrive at the train station within 65 minutes is

$$P(\tau \le 65) = \Phi\left(\frac{65-60}{\sqrt{16}}\right) = \Phi(1.25) = 0.8944.$$

So, it is better to take the first route.

3. The Exponential distribution

Density function:

$$p(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \qquad (\lambda > 0).$$

Distribution function:

3. The Exponential distribution

Density function:

$$p(x)=\begin{cases}\lambda e^{-\lambda x}, & x\geq 0,\\ 0, & x<0.\end{cases}\qquad (\lambda>0).$$

Distribution function:

$$F(x)=\begin{cases}1-e^{-\lambda x}, & x\geq 0,\\ 0, & x<0.\end{cases}$$

The exponential distribution possesses a memoryless property. Suppose $\xi \sim E(\lambda)$. Then for any $s, t > 0$,

$$P(\xi > s + t | \xi > s) = P(\xi > t).$$

Proof.

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Proof. $P(\xi > t) = 1 - F(t) = e^{-\lambda t}$.

$$P(\xi > s+t | \xi > s) = \frac{P(\xi > s+t, \xi > s)}{P(\xi > s)}$$

$$\begin{aligned} P(\xi > s+t|\xi > s) &= \frac{P(\xi > s+t, \xi > s)}{P(\xi > s)} \\ &= \frac{P(\xi > s+t)}{P(\xi > s)} \end{aligned}$$

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P(\xi > s+t | \xi > s) &= \frac{P(\xi > s+t, \xi > s)}{P(\xi > s)} \\
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&= \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}} \\
&= e^{-\lambda t}
\end{aligned}
$$

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Hazard rate functions

Suppose $X > 0$ (life time of some item) have distribution function F and density p. The hazard rate (sometimes called the failure rate) function $\lambda(t)$ of F is defined by

$$\lambda(t) = \frac{p(t)}{\overline{F}(t)}, \quad \text{where } \overline{F} = 1 - F.$$

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Suppose $X > 0$ (life time of some item) have distribution function F and density p. The hazard rate (sometimes called the failure rate) function $\lambda(t)$ of F is defined by

$$\lambda(t) = \frac{p(t)}{\overline{F}(t)}, \quad \text{where } \overline{F} = 1 - F.$$

Conversely,

$$F(t) = 1 - \exp\left\{-\int_0^t \lambda(s)ds\right\}.$$

$$P(X \in (t, t+dt) | X > t)$$

$$
\begin{aligned}
& P(X \in (t, t+dt) | X > t) \\
= & \frac{P(X \in (t, t+dt), X > t)}{P(X > t)} \\
= & \frac{P(X \in (t, t+dt))}{P(X > t)}
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\end{aligned}
$$

$\lambda(t)$ represents the conditional probability intensity that a t-unit-old item will fail.

For exponential distribution $E(\lambda)$, the hazard function is

$$\begin{aligned}\lambda(t) =& \frac{p(t)}{\overline{F}(t)} \\ =& \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda.\end{aligned}$$

Example

One often hears that the death rate of a person who smokes is, at each age, twice that of a nonsmoker. What does this mean?

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If $\lambda_s(t)$ denotes the hazard rate of a smoker of age t and $\lambda_n(t)$ denotes the hazard rate of a nonsmoker of age t, then

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One often hears that the death rate of a person who smokes is, at each age, twice that of a nonsmoker. What does this mean?

If $\lambda_s(t)$ denotes the hazard rate of a smoker of age t and $\lambda_n(t)$ denotes the hazard rate of a nonsmoker of age t, then

$$\lambda_s(t) = 2\lambda_n(t).$$

$$
\begin{aligned}
p_{non}^{A\to B} &=: P(\text{A-year-old nonsmoker reaches age B}) \\
&= P\Big(\text{nonsmoker's lifetime} > B \\
&\qquad \big|\text{nonsmoker's lifetime} > A\Big) \\
&= \frac{1-F_{non}(B)}{1-F_{non}(A)} = \frac{\exp\left\{-\int_0^B \lambda_n(t)dt\right\}}{\exp\left\{-\int_0^A \lambda_n(t)dt\right\}} \\
&= \exp\left\{-\int_A^B \lambda_n(t)dt\right\}
\end{aligned}
$$

$$
\begin{aligned}
p_s^{A\to B} &=: P(\text{A-year-old smoker reaches age B}) \\
&= P(\text{smoker's lifetime} > B \,|\, \text{smoker's lifetime} > A) \\
&= \exp\left\{-\int_A^B \lambda_s(t)dt\right\} \\
&= \exp\left\{-2\int_A^B \lambda_n(t)dt\right\} \\
&= \left[\exp\left\{-\int_A^B \lambda_n(t)dt\right\}\right]^2 = (p_{non}^{A\to B})^2
\end{aligned}
$$

Suppose

$$\lambda_n(t) = \frac{1}{30}, \quad 50 \le t \le 60.$$

Then

$$p_{non}^{50\to 60} = e^{-1/3} \approx 0.7165,$$

$$p_{s}^{50\to 60} = e^{-2/3} \approx 0.5134.$$

4. The Weibull distribution

Density function:

$$p(x) = \begin{cases} \frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} \exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right\}, & x \geq \mu, \\ 0, & x < \mu. \end{cases}.$$

Distribution function:

4. The Weibull distribution

Density function:

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Distribution function:

$$F(x) = \begin{cases} 1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right\}, & x \ge \mu, \\ 0, & x < \mu. \end{cases}$$

For Weibull distribution, the hazard function is

$$\lambda(t) = \frac{p(t)}{\overline{F}(t)} = \frac{\alpha}{\sigma}\left(\frac{t-\mu}{\sigma}\right)^{\alpha-1}.$$

威布尔[Ernst Hjalmar Waloddi Weibull (18 June 1887 – 12 October 1979) was a Swedish engineer, scientist, and mathematician]在1939年研究物质材料的强度时首先提出了这一类分布.

The Weibull distribution is widely used, in the field of life phenomena, as distribution of the lifetime of some object, particularly when the "weakest link" model is appropriate for the object. That is, consider an object consisting of many parts and suppose that the object experiences death (failure) when any of its parts fail. Under these conditions, it has been shown that a Weibull distribution provides a close approximation to the distribution of the lifetime of the item.

5. 帕累托(Pareto)分布

若随机变量ξ的密度函数为

$$p(x)=\begin{cases}\alpha x_0^{\alpha}x^{-(\alpha+1)}, & x>x_0,\\ 0, & x\le x_0,\end{cases}$$

则称ξ服从帕累托分布，其中参数$x_0>0$, $\alpha>0$.
意大利经济学家帕累托首先引入这一分布来描述一个国家中家庭年收入的分布.

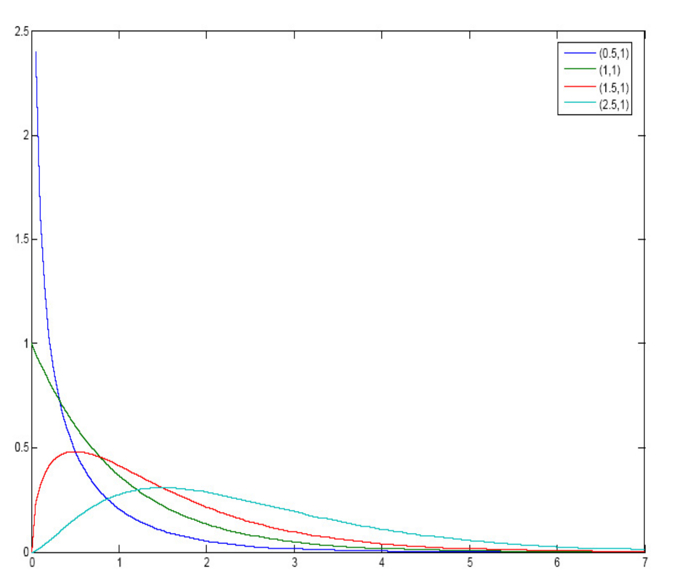
Vilfredo Federico Damaso Pareto (July 1848– August 1923)

6. The Gamma distribution

$\xi \sim \Gamma(\lambda, r)$: Density function

$$p(x) = \begin{cases} \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases} \qquad (\lambda > 0, r > 0)$$

where $\Gamma(r)$ is the first type of Euler integral. When r is an integer, we call it an Erlang distribution. $r = 1$: an exponential distribution.


(0.5,1)
(1,1)
(1.5,1)
(2.5,1)
0
0.5
1
1.5
2
2.5
0
1
2
3
4
5
6
7

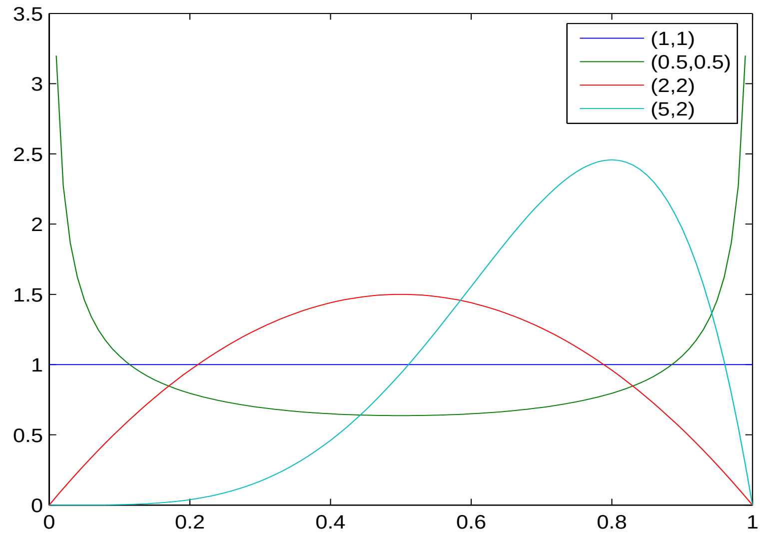
7. The Beta distribution

若随机变量ξ的密度函数为

$$p(x)=\begin{cases}\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}, & 0\le x\le 1,\\ 0, & \text{其它},\end{cases}$$

则称ξ服从参数为a和b的β分布，记作$\xi\sim\beta(a,b)$，其中$a,b>0$，
$B(a,b)=\int_0^1 x^{a-1}(1-x)^{b-1}dx$ 是有名的β积分，并
且$B(a,b)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$.

当$a=b=1$时，贝塔分布就是区间$[0,1]$上的均匀分布. 贝塔分布可以用来为取值在有限区间上的随机现象建模.


(1,1)
(0.5,0.5)
(2,2)
(5,2)
3.5
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

8 Cauchy distribution

$$p(x) = \frac{1}{\pi}\frac{1}{1+(x-\theta)^2}, \quad -\infty < x < \infty.$$

More generally,

$$p(x) = \frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\theta}{\sigma}\right)^2}, \quad -\infty < x < \infty.$$

Augustin-Louis Cauchy (August 1789 – May 1857))