Solutions of Problems in Evans' Partial Differential Equations

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Chapter 2

FOUR IMPORTANT LINEAR PARTIAL DIFFERENTIAL EQUATIONS

In the following exercises, all given functions are assumed smooth, unless otherwise stated.

1. Write down an explicit formula for a function u solving the initialvalue problem

$$\begin{cases} u_t + b \cdot Du + cu = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Here $c \in \mathbb{R}$ and $b \in \mathbb{R}^n$ are constants.

Solution. Set $z(s) := u(x + sb, t + s)$ for $s \in \mathbb{R}$. Then

$$\begin{aligned} \dot{z}(s) &= Du(x + sb, t + s) \cdot b + u_t(x + sb, t + s) \\ &= -cu(x + sb, t + s) \\ &= -cz(s), \end{aligned}$$

so

$$z(s) = c'(x, t)e^{-cs},$$

where $c'(x,t) \in \mathbb{R}$ is a constant about s.

$$z(-t) = u(x - tb, 0) = g(x - tb) = c'(x,t)e^{ct},$$

so

$$c'(x,t) = e^{-ct}g(x - tb).$$

Hence

$$u(x,t) = z(0) = c'(x,t) = e^{-ct}g(x - tb).$$

2. Prove that Laplace's equation $\Delta u = 0$ is rotation invariant; that is, if O is an orthogonal $n \times n$ matrix and we define

$$v(x) := u(Ox) \quad \left(x \in \mathbb{R}^n\right),$$

then $\Delta v = 0$.

Proof. Set $O = ((o_{ij}))\,, y_i = \sum_{j=1}^n o_{ij}x_j$. Then

$$\begin{aligned} v(x) = v\left(x_1, \ldots, x_n\right) &= u(Ox) = u\left(y_1, \ldots, y_n\right), \\ v_{x_k} &= \sum_{i=1}^n u_{y_i} o_{ik}, \\ v_{x_k x_k} &= \sum_{i=1}^n \sum_{j=1}^n u_{y_i y_j} o_{ik} o_{jk}, \\ \Delta v = \sum_{k=1}^n v_{x_k x_k} &= \sum_{k=1}^n \sum_{i=1}^n \sum_{j=1}^n u_{y_i y_j} o_{ik} o_{jk} \\ &= \sum_{i=1}^n \sum_{j=1}^n \left(u_{y_i y_j} \sum_{k=1}^n o_{ik} o_{jk} \right) = \sum_{i=1}^n u_{y_i y_i} \\ &= 0. \end{aligned}$$

□

3. Modify the proof of the mean-value formulas to show for $n \geq 3$ that

$$u(0) = ⨍_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right) f dx,$$

provided

$$\begin{cases} -\Delta u = f & \text{in } B^0(0,r) \\ \quad u = g & \text{on } \partial B(0,r). \end{cases}$$

Proof. Set

$$\phi(s) := \fint_{\partial B(0,s)} u(x)dS(x) = \fint_{\partial B(0,1)} u(sy)dS(y), \quad 0 < s \le r.$$

Then

$$\begin{aligned}
\phi(r) &= \fint_{\partial B(0,r)} g dS, \\
\lim_{s\to 0} \phi(s) &= u(0), \\
\phi'(s) &= \fint_{\partial B(0,1)} Du(sy)\cdot y dS(y) \\
&= \fint_{\partial B(0,s)} Du(x)\cdot \frac{x}{s} dS(x) \\
&= \fint_{\partial B(0,s)} \frac{\partial u}{\partial \nu} dS(x) \\
&= \frac{s}{n}\fint_{B(0,s)} \Delta u(x)dx \\
&= -\frac{s}{n}\fint_{B(0,s)} f dx. \\
&= -\frac{1}{n\alpha(n)s^{n-1}}\int_{B(0,s)} f dx,
\end{aligned}$$

$$\begin{aligned}
\phi(r) - u(0) &= \int_0^r \phi'(s)ds \\
&= -\frac{1}{n\alpha(n)}\int_0^r \left(s^{1-n}\int_{B(0,s)} f dx\right) ds \\
&= -\frac{1}{n\alpha(n)}\left[\left(\frac{s^{2-n}}{2-n}\int_{B(0,s)} f dx\right)\Bigg|_0^r - \int_0^r \left(\frac{s^{2-n}}{2-n}\int_{\partial B(0,s)} f dS\right) ds\right] \\
&= -\frac{1}{n\alpha(n)}\left[\frac{1}{2-n}\int_{B(0,r)} r^{2-n} f dx - \frac{1}{2-n}\int_0^r \left(\int_{\partial B(0,s)} s^{2-n} f dS\right) ds\right] \\
&= \frac{1}{n(n-2)\alpha(n)}\left(\int_{B(0,r)} r^{2-n} f dx - \int_{B(0,r)} |x|^{2-n} f dx\right) \\
&= \frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)} \left(\frac{1}{r^{n-2}} - \frac{1}{|x|^{n-2}}\right) f dx,
\end{aligned}$$

$$u(0) = \fint_{\partial B(0,r)} g dS + \frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)} \left(\frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}}\right) f dx.$$

□

Remark. If $r=1$, $x \in B^0(0,1)$, then using Theorem 12 in §2.2, we see

$$u(x) = -\int_{\partial B(0,1)} g(y)\frac{\partial G}{\partial \nu}(x,y)dS(y) + \int_{B(0,1)} f(y)G(x,y)dy.$$

$$\frac{\partial G}{\partial \nu}(x,y) = \frac{-1}{n\alpha(n)}\frac{1-|x|^2}{|x-y|^n},$$
$$G = \Phi(y-x) - \Phi(|x|(y-\tilde{x})).$$

Hence

$$u(x) = \frac{1-|x|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(y)}{|x-y|^n}dS(y) + \int_{B(0,1)} f(y)[\Phi(y-x)-\Phi(|x|(y-\tilde{x}))]dy.$$

For arbitrary $r>0$, $\tilde{u}(x) := u(rx)$ solves

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } B^0(0,1) \\ \tilde{u} = \tilde{g} & \text{on } \partial B(0,1), \end{cases}$$

where

$$\tilde{f}(x) := r^2 f(rx),$$
$$\tilde{g}(x) := g(rx).$$

Hence

$$\begin{aligned} u(x) &= \tilde{u}\left(\frac{x}{r}\right) \\ &= \frac{1-\left|\frac{x}{r}\right|^2}{n\alpha(n)}\int_{\partial B(0,1)} \frac{\tilde{g}(y)}{\left|\frac{x}{r}-y\right|^n}dS(y) \\ &\quad + \int_{B(0,1)} \tilde{f}(y)\left[\Phi\left(y-\frac{x}{r}\right) - \Phi\left(\left|\frac{x}{r}\right|\left(y-\widetilde{\left(\frac{x}{r}\right)}\right)\right)\right]dy \\ &= \frac{1-\frac{|x|^2}{r^2}}{n\alpha(n)}\int_{\partial B(0,1)} \frac{g(ry)}{\left|\frac{x}{r}-y\right|^n}dS(y) \\ &\quad + \int_{B(0,1)} r^2 f(ry)\left[\Phi\left(y-\frac{x}{r}\right) - \Phi\left(\frac{|x|}{r}\left(y-\widetilde{\left(\frac{x}{r}\right)}\right)\right)\right]dy \\ &= \frac{1-\frac{|x|^2}{r^2}}{n\alpha(n)}\int_{\partial B(0,r)} \frac{g(z)}{\left|\frac{x}{r}-\frac{z}{r}\right|^n}\frac{1}{r^{n-1}}dS(z) \end{aligned}$$

$$
\begin{aligned}
&+\int_{B(0,r)} r^2 f(z)\left[\Phi\left(\frac{z}{r}-\frac{x}{r}\right)-\Phi\left(\frac{|x|}{r}\left(\frac{z}{r}-\widetilde{\left(\frac{x}{r}\right)}\right)\right)\right]\frac{1}{r^n}dz \\
=&\frac{r^2-|x|^2}{n\alpha(n)r}\int_{\partial B(0,r)}\frac{g(y)}{|x-y|^n}dS(y) \\
&+\int_{B(0,r)} f(y)\left[\Phi(y-x)-\Phi\left(\frac{|x|y}{r}-\frac{rx}{|x|}\right)\right]dy.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
u(0)&=\fint_{\partial B(0,r)} g(y)dS(y)+\int_{B(0,r)} f(x)[\Phi(x)-\Phi(r)]dx \\
&=\fint_{\partial B(0,r)} gdS+\frac{1}{n(n-2)\alpha(n)}\int_{B(0,r)}\left(\frac{1}{|x|^{n-2}}-\frac{1}{r^{n-2}}\right)fdx.
\end{aligned}
$$

4. Give a direct proof that if $u\in C^2(U)\cap C(\bar{U})$ is harmonic within a bounded open set U, then

$$\max_{\bar{U}} u=\max_{\partial U} u.$$

(Hint: Define $u_\varepsilon:=u+\varepsilon|x|^2$ for $\varepsilon>0$, and show u_ε cannot attain its maximum over $\bar{U}$ at an interior point.)

Proof. 1. Assume that there exists an $x_0\in U$ such that

$$\max_{\bar{U}} u_\varepsilon(x)=u_\varepsilon\left(x_0\right).$$

Then

$$Du_\varepsilon\left(x_0\right)=0, D^2u_\varepsilon\left(x_0\right)\leq 0,$$

therefore

$$\Delta u_\varepsilon\left(x_0\right)\leq 0.$$

This is contradict with the fact that

$$\Delta u_\varepsilon=\Delta u+\varepsilon\Delta|x|^2=2n\varepsilon>0.$$

Thus u_ε cannot attain its maximum over $\bar{U}$ at an interior point.

2. Assume that there exists an $x_1 \in U$, such that

$$\max_{\partial U} u(x) < u(x_1) = \max_{\bar{U}} u(x).$$

Let

$$\varepsilon = \frac{1}{\max_{\partial U} |x|^2} \left(\max_{\bar{U}} u(x) - \max_{\partial U} u(x) \right),$$

and set

$$\max_{\partial U} u_\varepsilon(x) = u_\varepsilon(x_2), \quad x_2 \in \partial U.$$

Then

$$\begin{aligned}
\max_{\partial U} u_\varepsilon(x) &= u(x_2) + \frac{|x_2|^2}{\max_{\partial U} |x|^2} \left(\max_{\bar{U}} u(x) - \max_{\partial U} u(x) \right) \\
&\leq u(x_2) + \max_{\bar{U}} u(x) - \max_{\partial U} u(x) \\
&\leq \max_{\bar{U}} u(x) = u(x_1) \\
&\leq u_\varepsilon(x_1).
\end{aligned}$$

The contradiction occurs. Hence

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

□

Remark. There is another way as follows to obtain $\max_{\bar{U}} u = \max_{\partial U} u$ closely after step 1 of the proof. Set $x_0 \in \partial U$ and $\max_{\bar{U}} u_\varepsilon = u_\varepsilon(x_0)$. Then for any $x \in \bar{U}$,

$$u(x) = \lim_{\varepsilon \to 0^+} u_\varepsilon(x) \leq \lim_{\varepsilon \to 0^+} u_\varepsilon(x_0) = u(x_0).$$

5. We say $v \in C^2(\bar{U})$ is subharmonic if

$$-\Delta v \leq 0 \quad \text{in } U.$$

(a) Prove for subharmonic v that

$$v(x) \leq \fint_{B(x,r)} v dy \quad \text{for all } B(x,r) \subset U.$$

(b) Prove that therefore $\max_{\bar{U}} v = \max_{\partial U} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u is harmonic and $v := \phi(u)$. Prove v is subharmonic.

(d) Prove $v := |Du|^2$ is subharmonic, whenever u is harmonic.

Proof. (a) Set

$$\phi(r) := \fint_{\partial B(x,r)} v(y)dS(y) = \fint_{\partial B(0,1)} v(x+rz)dS(z).$$

Then

$$\begin{aligned}
\phi'(r) &= \fint_{\partial B(0,1)} Dv(x+rz)\cdot zdS(z) \\
&= \fint_{\partial B(x,r)} Dv(y)\cdot\frac{y-x}{r}dS(y) \\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}dS(y) \\
&= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} \Delta u(y)dy \geq 0.
\end{aligned}$$

Hence

$$\phi(r) \geq \lim_{t\to 0}\phi(t) = \lim_{t\to 0}\fint_{\partial B(x,t)} v(y)dS(y) = v(x).$$

Therefore

$$\begin{aligned}
\fint_{B(x,r)} vdy &= \frac{1}{\alpha(n)r^n}\int_0^r \left(\int_{\partial B(x,s)} vdS\right)ds \\
&\geq \frac{1}{\alpha(n)r^n}\int_0^r v(x)n\alpha(n)s^{n-1}ds \\
&= \frac{v(x)}{r^n}\int_0^r ns^{n-1}ds \\
&= v(x).
\end{aligned}$$

(b) Suppose there exists a point $x_0 \in U$ with $v(x_0) = M := \max_{\bar{U}} v$. Then for $0 < r < \operatorname{dist}(x_0, \partial U)$,

$$M = v(x_0) \leq \fint_{B(x_0,r)} vdy \leq M,$$

so

$$\fint_{B(x_0,r)} vdy = M.$$

Thus $v(y) = M$ for all $y \in B(x_0, r)$, and $v(x) = M$ for all $x \in \bar{U}$ if U is connected. Hence

$$\max_{\bar{U}} v = \max_{\partial U} v.$$

(c)

$$\begin{aligned}
\frac{\partial v}{\partial x_i} &= \frac{d\phi}{du}\frac{\partial u}{\partial x_i}, \\
\frac{\partial^2 v}{\partial x_i^2} &= \frac{d^2\phi}{du^2}\left(\frac{\partial u}{\partial x_i}\right)^2 + \frac{d\phi}{du}\frac{\partial^2 u}{\partial x_i^2}, \\
-\Delta v &= -\frac{d^2\phi}{du^2}\sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 - 0 \le 0.
\end{aligned}$$

(d)

$$\begin{aligned}
v &= \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j}\right)^2, \\
\frac{\partial v}{\partial x_i} &= \sum_{j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right)^2 = \sum_{j=1}^n 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} = 2\sum_{j=1}^n \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}, \\
\frac{\partial^2 v}{\partial x_i^2} &= 2\sum_{j=1}^n \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}\right) = 2\sum_{j=1}^n \left[\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i^2 \partial x_j}\right], \\
-\Delta v &= -2\sum_{i=1}^n\sum_{j=1}^n \left[\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i^2 \partial x_j}\right] \\
&= -2\sum_{i=1}^n\sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 - 2\sum_{j=1}^n \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2} \\
&= -2\sum_{i=1}^n\sum_{j=1}^n \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 \\
&\le 0.
\end{aligned}$$

Hence v is subharmonic. □

6. Let U be a bounded, open subset of $\mathbb{R}^n$. Prove that there exists a constant C, depending only on U, such that

$$\max_{\bar{U}} |u| \le C(\max_{\partial U} |g| + \max_{\bar{U}} |f|)$$

whenever u is a smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U. \end{cases}$$

(Hint: $-\Delta(u + \frac{|x|^2}{2n}\lambda) \leq 0$, for $\lambda := \max_{\bar{U}} |f|$.)

Proof.

$$\begin{aligned} -\Delta\left(\pm u + \frac{|x|^2}{2n}\lambda\right) &= \mp\Delta u - \frac{\lambda}{2n}\Delta|x|^2 \\ &= \pm f - \lambda \leq 0, \end{aligned}$$

so

$$\begin{aligned} \max_{\bar{U}} \pm u &\leq \max_{\bar{U}}\left(\pm u + \frac{|x|^2}{2n}\lambda\right) \\ &= \max_{\partial U}\left(\pm g + \frac{|x|^2}{2n}\lambda\right) \quad \text{(by (b) of Problem 5)} \\ &\leq \max_{\partial U}|g| + \left(\frac{1}{2n}\max_{\partial U}|x|^2\right)\max_{\bar{U}}|f| \\ &\leq C\left(\max_{\partial U}|g| + \max_{\bar{U}}|f|\right), \end{aligned}$$

where $C = \max\left\{1, \frac{1}{2n}\max_{\partial U}|x|^2\right\}$.

$$\max_{\bar{U}}|u| \in \left\{\max_{\bar{U}} u, \max_{\bar{U}}(-u)\right\},$$

hence

$$\max_{\bar{U}}|u| \leq C\left(\max_{\partial U}|g| + \max_{\bar{U}}|f|\right).$$

□

7. Use Poisson's formula for the ball to prove

$$r^{n-2}\frac{r-|x|}{(r+|x|)^{n-1}}u(0) \leq u(x) \leq r^{n-2}\frac{r+|x|}{(r-|x|)^{n-1}}u(0)$$

whenever u is positive and harmonic in $B^0(0,r)$. This is an explicit form of Harnack's inequality.

Proof. Set $g(x) = u(x)$ for any $x \in \partial B(0,r)$. Then

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x-y|^n} dS(y) \quad \left(x \in B^0(0,r)\right),$$

$$u(0) = \fint_{\partial B(0,r)} g(y)dS(y),$$

$$\begin{aligned} u(x) &\geq \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{(r+|x|)^n} dS(y) \\ &= \frac{r-|x|}{n\alpha(n)r(r+|x|)^{n-1}} \int_{\partial B(0,r)} g(y)dS(y) \\ &= r^{n-2} \frac{r-|x|}{(r+|x|)^{n-1}} u(0), \\ u(x) &\leq \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{(r-|x|)^n} dS(y) \\ &= \frac{r+|x|}{n\alpha(n)r(r-|x|)^{n-1}} \int_{\partial B(0,r)} g(y)dS(y) \\ &= r^{n-2} \frac{r+|x|}{(r-|x|)^{n-1}} u(0). \end{aligned}$$

□

8. Prove Theorem 15 in §2.2.4. (Hint: Since $u \equiv 1$ solves (44) for $g \equiv 1$, the theory automatically implies

$$\int_{\partial B(0,1)} K(x,y)dS(y) = 1$$

for each $x \in B^0(0,1)$.)

Proof. 1. Since $x \mapsto K(x,y)$ is smooth for $x \neq y$, we easily verify as well $u = \int_{\partial B(0,r)} K(x,y)g(y)dS(y) \in C^\infty\left(B^0(0,r)\right)$.

2. For each fixed $x \in B(0,1)$, the mapping $y \mapsto G(x,y)$ is harmonic, except for $y = x$. As $G(x,y) = G(y,x), x \mapsto G(x,y)$ is harmonic, except for $x = y$. Thus $x \mapsto \frac{\partial G(x,y)}{\partial \boldsymbol{\nu}}$ is harmonic for $x \in B^0(0,1), y \in \partial B^0(0,1)$. Therefore

$$\tilde{u}(x) = -\int_{\partial B(0,1)} \tilde{g}(y) \frac{\partial G}{\partial \boldsymbol{\nu}}(x,y)dS(y)$$

is harmonic, and $u(x) = \tilde{u}\left(\frac{x}{r}\right)$ is harmonic.

3. We already know that

$$\int_{\partial B(0,r)} K(x,y)dy = 1. \tag{2.1}$$

Now fix $x^0 \in \partial B(0,r), \varepsilon > 0$. Choose $\delta > 0$ so small that

$$\left|g(y) - g\left(x^0\right)\right| < \varepsilon \quad \text{if } \left|y - x^0\right| < \delta, y \in \partial B(0,r). \tag{2.2}$$

Then if $\left|x - x^0\right| < \frac{\delta}{2}$, $x \in B^0(0,r)$,

$$\begin{aligned}
\left|u(x) - g\left(x^0\right)\right| &= \left|\int_{\partial B(0,r)} K(x,y)\left[g(y) - g\left(x^0\right)\right]dy\right| \\
&\leq \int_{\partial B(0,r)\cap B(x^0,\delta)} K(x,y)\left|g(y) - g\left(x^0\right)\right|dy \\
&\quad + \int_{\partial B(0,r)-B(x^0,\delta)} K(x,y)\left|g(y) - g\left(x^0\right)\right|dy \\
&=: I + J.
\end{aligned} \tag{2.3}$$

Now (2.1), (2.2) imply

$$I \leq \varepsilon \int_{\partial B(0,r)} K(x,y)dy = \varepsilon.$$

Furthermore, if $\left|x - x^0\right| \leq \frac{\delta}{2}$ and $\left|y - x^0\right| \geq \delta$, we have

$$\left|y - x^0\right| \leq |y - x| + \frac{\delta}{2} \leq |y - x| + \frac{1}{2}\left|y - x^0\right|,$$

and so $|y - x| \geq \frac{1}{2}\left|y - x^0\right|$. Thus

$$\begin{aligned}
J &\leq 2\|d\|_{L^\infty} \int_{\partial B(0,r)-B(x^0,\delta)} \frac{r^2 - |x|^2}{n\alpha(n)r} \frac{1}{|x-y|^n} dy \\
&\leq \frac{2\|d\|_{L^\infty}\left(r^2 - |x|^2\right)}{n\alpha(n)r} \int_{\partial B(0,r)-B(x^0,\delta)} \frac{2^n}{|y - x^0|^n} dy \\
&= \frac{2^{n+1}\|d\|_{L^\infty}\left(r^2 - |x|^2\right)}{n\alpha(n)r} \int_{\partial B(0,r)-B(x^0,\delta)} \frac{1}{|y - x^0|^n} dy \\
&\to 0, \quad \text{as } |x| \to r.
\end{aligned}$$

Combining this calculation with estimate (2.3), we deduce $\left|u(x) - g\left(x^0\right)\right| \leq 2\varepsilon$, provided $\left|x - x^0\right|$ is sufficiently small. $\square$

9. Let u be the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ \\ u = g & \text{on } \partial\mathbb{R}^n_+ \end{cases}$$

given by Poisson's formula for the half-space. Assume g is bounded and $g(x) = |x|$ for $x \in \partial\mathbb{R}^n_+, |x| \leq 1$. Show Du is not bounded near $x = 0$. (Hint: Estimate $\frac{u(\lambda e_n)-u(0)}{\lambda}$.)

Proof. We assume that $n \geq 2$.

$$u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|x-y|^n} dy \quad \left(x \in \mathbb{R}^n_+\right),$$

$$u(0) = g(0) = 0,$$

$$u(\lambda e_n) = \frac{2\lambda}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|\lambda e_n - y|^n} dy,$$

$$\begin{aligned}
\frac{u(\lambda e_n) - u(0)}{\lambda} &= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+} \frac{g(y)}{|\lambda e_n - y|^n} dy \\
&= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap B(0,1)} \frac{|y|}{|\lambda e_n - y|^n} dy + \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(0,1)} \frac{g(y)}{|\lambda e_n - y|^n} dy \\
&=: I + J.
\end{aligned}$$

$$\begin{aligned}
\lim_{\lambda \to 0} I &= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap B(0,1)} \lim_{\lambda \to 0} \frac{|y|}{|\lambda e_n - y|^n} dy \\
&= \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ \cap B(0,1)} |y|^{1-n} dy \\
&= \frac{2}{n\alpha(n)} \int_0^1 r^{1-n}(n-1)\alpha(n-1) r^{n-2} dr \\
&= \frac{2(n-1)\alpha(n-1)}{n\alpha(n)} \int_0^1 r^{-1} dr \\
&= \infty.
\end{aligned}$$

$$\begin{aligned}
|J| &\leq \frac{2}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(0,1)} \frac{\|g\|_{L^\infty}}{|y|^n} dy \\
&= \frac{2\|g\|_{L^\infty}}{n\alpha(n)} \int_1^\infty r^{-n}(n-1)\alpha(n-1) r^{n-2} dr \\
&= \frac{2\|g\|_{L^\infty}(n-1)\alpha(n-1)}{n\alpha(n)} \int_1^\infty r^{-2} dr \\
&= \frac{2\|g\|_{L^\infty}(n-1)\alpha(n-1)}{n\alpha(n)}.
\end{aligned}$$

Hence Du is not bounded. □

10. (Reflection principle)

(a) Let U^+ denote the open half-ball $\{x \in \mathbb{R}^n \mid |x| < 1, x_n > 0\}$. Assume $u \in C^2(\overline{U^+})$ is harmonic in U^+, with $u = 0$ on $\partial U^+ \cap \{x_n = 0\}$. Set

$$v(x) := \begin{cases} u(x) & \text{if } x_n \geq 0 \\ -u(x_1, \ldots, x_{n-1}, -x_n) & \text{if } x_n < 0 \end{cases}$$

for $x \in U = B^0(0,1)$. Prove $v \in C^2(U)$ and thus v is harmonic within U.

(b) Now assume only that $u \in C^2(U^+) \cap C(\overline{U^+})$. Show that v is harmonic within U. (Hint: Use Poisson's formula for the ball.)

Proof. (a)

(b) Let

$$w(x) := \frac{1 - |x|^2}{n\alpha(n)} \int_U \frac{v(y)}{|x-y|^n} dS(y),$$

where $x \in U$. According to Theorem 15 in §2.2,

(i) $w \in C^\infty(U)$,

(ii) $\Delta w = 0$ in U, and

(iii) $w = v$ on ∂U.

It is easy to see that $w(x) = 0$ if $x_n = 0$. Thus $v - w = 0$ on ∂U^+, $\partial U^- := U - \overline{U^+}$. Therefore $v - w = 0$ on $\overline{U} = \overline{U^+} \cup \overline{U^-}$ by strong maximum principle. Thus $v = w$ is harmonic within U. □

11. (Kelvin transform for Laplace's equation) The *Kelvin transform* $\mathcal{K}u = \bar{u}$ of a function $u : \mathbb{R}^n \to \mathbb{R}$ is

$$\bar{u}(x) := u(\bar{x})|\bar{x}|^{n-2} = u(x/|x|)|x|^{2-n} \quad (x \neq 0),$$

where $\bar{x} = x/|x|^2$. Show that if u is harmonic, then so is $\bar{u}$. (Hint: First show that $D_x\bar{x}\,(D_x\bar{x})^T = |\bar{x}|^4 I$. The mapping $x \to \bar{x}$ is conformal, meaning angle preserving.)

Proof. (1)

$$(\bar{x})^i = \frac{x^i}{\sum_{j=1}^n x_j^2},$$
$$\frac{\partial(\bar{x})^i}{\partial x_j} = \frac{\delta_{ij}|x|^2 - 2x_i x_j}{|x|^4},$$

where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Set $A = ((a_{ij})) = D_x\bar{x}(D_x\bar{x})^T$. For $D_x\bar{x} = \left(\left(\frac{\partial(\bar{x})^i}{\partial x_j}\right)\right)$,

$$\begin{aligned}
a_{ij} &= \sum_{k=1}^n \frac{\partial(\bar{x})^i}{\partial x_k}\frac{\partial(\bar{x})^j}{\partial x_k} \\
&= \sum_{k=1}^n \frac{\delta_{ik}|x|^2 - 2x_i x_k}{|x|^4}\frac{\delta_{jk}|x|^2 - 2x_j x_k}{|x|^4} \\
&= \frac{1}{|x|^8}\sum_{k=1}^n \left(\delta_{ik}|x|^2 - 2x_i x_k\right)\left(\delta_{jk}|x|^2 - 2x_j x_k\right) \\
&= \frac{1}{|x|^8}\sum_{k=1}^n \left(\delta_{ik}\delta_{jk}|x|^4 - 2\delta_{ik}x_j x_k|x|^2 - 2\delta_{jk}x_i x_k|x|^2 + 4x_i x_j x_k^2\right) \\
&= \frac{1}{|x|^8}\left(\sum_{k=1}^n \delta_{ik}\delta_{jk}|x|^4 - 2x_i x_j|x|^2 - 2x_i x_j|x|^2 + 4x_i x_j|x|^2\right) \\
&= \frac{1}{|x|^4}\sum_{k=1}^n \delta_{ik}\delta_{jk} \\
&= |\bar{x}|^4\delta_{ij}.
\end{aligned}$$

Therefore

$$D_x\bar{x}\,(D_x\bar{x})^T = |\bar{x}|^4 I.$$

(2) Let $y_i = y_i(x_1, \ldots, x_n) = (\bar{x})^i = \frac{x_i}{\sum_{j=1}^n x_j^2}$

$$\frac{\partial y_i}{\partial x_j} = \frac{\delta_{ij}|x|^2 - 2x_i x_j}{|x|^4}.$$

If $i \neq j$,

$$\frac{\partial^2 y_i}{\partial x_j^2} = -2x_i\left[|x|^{-4} + x_j(-4)|x|^{-5}x_j|x|^{-1}\right]$$

$$= -2x_i|x|^{-4} + 8x_i x_j^2 |x|^{-6}.$$

If $i = j$,

$$\begin{aligned}\frac{\partial^2 y_i}{\partial x_j^2} &= -2|x|^{-3}x_i|x|^{-1} - 2\left[2x_i|x|^{-4} + x_i^2(-4)|x|^{-5}x_i|x|^{-1}\right] \\ &= -6x_i|x|^{-4} + 8x_i^3|x|^{-6}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{j=1}^{n} \frac{\partial^2 y_i}{\partial x_j^2} &= [-2(n-1) - 6]x_i|x|^{-4} + 8x_i|x|^{-4} \\ &= (-2n+4)x_i|x|^{-4}.\end{aligned}$$

(3) In this step we will show that

$$\sum_{i=1}^{n} \left(x_i \frac{\partial u}{\partial x_i} + y_i \frac{\partial u}{\partial y_i} \right) = 0.$$

$$\begin{aligned}\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} &= \sum_{i=1}^{n} \left(x_i \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) \\ &= \sum_{i=1}^{n} \left[x_i \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \left(\frac{\delta_{ij}}{|x|^2} - \frac{2x_i x_j}{|x|^4} \right) \right] \\ &= \sum_{j=1}^{n} \left(\frac{1}{|x|^2} \frac{\partial u}{\partial y_j} \sum_{i=1}^{n} x_i \delta_{ij} - \frac{2x_j}{|x|^4} \frac{\partial u}{\partial y_j} \sum_{i=1}^{n} x_i^2 \right) \\ &= \sum_{j=1}^{n} \left(\frac{x_j}{|x|^2} \frac{\partial u}{\partial y_j} - \frac{2x_j}{|x|^2} \frac{\partial u}{\partial y_j} \right) \\ &= -\sum_{j=1}^{n} y_j \frac{\partial u}{\partial y_j} \\ &= -\sum_{i=1}^{n} y_i \frac{\partial u}{\partial y_i}.\end{aligned}$$

Hence

$$\sum_{i=1}^{n} \left(x_i \frac{\partial u}{\partial x_i} + y_i \frac{\partial u}{\partial y_i} \right) = 0.$$

(4)

$$\bar{u}(x) = u\left(\frac{x}{|x|^2}\right)|x|^{2-n} = u\left(y_1, \ldots, y_n\right)|x|^{2-n} = u(y)|x|^{2-n},$$

$$
\begin{aligned}
\frac{\partial \bar{u}}{\partial x_i} &= \left(\sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) |x|^{2-n} + u(y)(2-n)|x|^{1-n} \frac{x_i}{|x|} \\
&= \left(\sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) |x|^{2-n} + (2-n)x_i|x|^{-n}u(y), \\
\frac{\partial^2 \bar{u}}{\partial x_i^2} &= \left[\sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial y_j} \frac{\partial y_j}{\partial x_i} \right) \right] |x|^{2-n} \\
&\quad + \frac{\partial u}{\partial x_i}(2-n)|x|^{-n}x_i \\
&\quad + (2-n)|x|^{-n}u(y) \\
&\quad + (2-n)x_i(-n)|x|^{-n-2}x_i u(y) \\
&\quad + (2-n)x_i|x|^{-n}\frac{\partial u}{\partial x_i} \\
&= |x|^{2-n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_k} \frac{\partial y_k}{\partial x_i} \frac{\partial y_j}{\partial x_i} + \frac{\partial u}{\partial y_j} \frac{\partial^2 y_j}{\partial x_i^2} \right) \\
&\quad + (-2n+4)x_i|x|^{-n}\frac{\partial u}{\partial x_i} \\
&\quad + (2-n)|x|^{-n}u(y)\left(1 - n x_i^2 |x|^{-2}\right) \\
&= |x|^{2-n} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 u}{\partial y_j \partial y_k} \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i} \\
&\quad + |x|^{2-n} \sum_{j=1}^{n} \frac{\partial u}{\partial y_j} \frac{\partial^2 y_j}{\partial x_i^2} \\
&\quad + (-2n+4)x_i|x|^{-n}\frac{\partial u}{\partial x_i} \\
&\quad + (2-n)|x|^{-n}u(y)(1 - n x_i^2 |x|^{-2}),
\end{aligned}
$$

$$\begin{aligned}
\Delta \bar{u} &= \sum_{i=1}^{n} \frac{\partial^2 \bar{u}}{\partial x_i^2} \\
&= |x|^{2-n} \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\frac{\partial^2 u}{\partial y_j \partial y_k} \sum_{i=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial y_k}{\partial x_i} \right) \\
&\quad + |x|^{2-n} \sum_{j=1}^{n} \left(\frac{\partial u}{\partial y_j} \sum_{i=1}^{n} \frac{\partial^2 y_j}{\partial x_i^2} \right) \\
&\quad + (-2n+4)|x|^{-n} \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \\
&\quad + (2-n)|x|^{-n} u(y) \sum_{i=1}^{n} \left(1 - n x_i^2 |x|^{-2}\right) \\
&= |x|^{-2-n} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_i^2} + (-2n+4)|x|^{-2-n} \sum_{j=1}^{n} x_j \frac{\partial u}{\partial y_j} + (-2n+4)|x|^{-n} \sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} \\
&= 0 + (-2n+4)|x|^{-n} \sum_{i=1}^{n} \left(y_i \frac{\partial u}{\partial y_i} + x_i \frac{\partial u}{\partial x_i} \right) \\
&= 0.
\end{aligned}$$

Hence $\bar{u}$ is harmonic. □

12. Suppose u is smooth and solves $u_t - \Delta u = 0$ in $\mathbb{R}^n \times (0, \infty)$.

(a) Show $u_\lambda(x,t) := u(\lambda x, \lambda^2 t)$ also solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) Use (a) to show $v(x,t) := x \cdot Du(x,t) + 2t u_t(x,t)$ solves the heat equation as well.

Proof. (a)

$$\begin{aligned}
(u_\lambda)_t(x,t) &= \lambda^2 u_t(\lambda x, \lambda^2 t), \\
\Delta u_\lambda(x,t) &= \lambda^2 \Delta u(\lambda x, \lambda^2 t).
\end{aligned}$$

Thus

$$\begin{aligned}
(u_\lambda)_t - \Delta u_\lambda &= \lambda^2 (u_t(\lambda x, \lambda^2 t) - \Delta u(\lambda x, \lambda^2 t)) \\
&= 0,
\end{aligned}$$

i.e., $u_\lambda(x,t)$ solves the heat equation for each $\lambda \in \mathbb{R}$.

(b) There are two ways to show it. (a) is not necessary in the second way.

(i) Set

$$v_\lambda(x,t) := \frac{d}{d\lambda}u_\lambda(x,t) = x\cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t).$$

Then

$$\begin{aligned}(v_\lambda)_t(x,t) &= \frac{\partial}{\partial t}\frac{d}{d\lambda}u_\lambda(x,t) = \frac{d}{d\lambda}(u_\lambda)_t(x,t),\\ \Delta v_\lambda(x,t) &= \Delta\left[\frac{d}{d\lambda}u_\lambda(x,t)\right] = \frac{d}{d\lambda}\Delta u_\lambda(x,t),\\ (v_\lambda)_t &= \frac{d}{d\lambda}[(u_\lambda)_t - \Delta u_\lambda] = 0.\end{aligned}$$

Thus

$$v_\lambda(x,t) = x\cdot Du(\lambda x, \lambda^2 t) + 2\lambda t u_t(\lambda x, \lambda^2 t)$$

solves the heat equation. Especially,

$$v_1(x,t) = v(x,t) = x\cdot Du(x,t) + 2tu_t(x,t)$$

solves the heat equation.

(ii)

$$v = \sum_{j=1}^n x_j u_{x_j} + 2tu_t,$$

$$v_{x_i} = \sum_{j=1}^n (\delta_{ij}u_{x_j} + x_j u_{x_i x_j}) + 2tu_{x_i t},$$

$$\begin{aligned}v_{x_i x_i} &= \sum_{j=1}^n (\delta_{ij}u_{x_i x_j} + \delta_{ij}u_{x_i x_j} + x_j u_{x_i x_i x_j}) + 2tu_{x_i x_i t}\\ &= \sum_{j=1}^n (2\delta_{ij}u_{x_i x_j} + x_j u_{x_i x_i x_j}) + 2tu_{x_i x_i t},\end{aligned}$$

$$\begin{aligned}\Delta v &= \sum_{i=1}^n\sum_{j=1}^n (2\delta_{ij}u_{x_i x_j} + x_j u_{x_i x_i x_j}) + 2t\sum_{i=1}^n u_{x_i x_i t}\\ &= 2\Delta u + \sum_{j=1}^n x_j\frac{\partial}{\partial x_j}\Delta u + 2t\frac{\partial}{\partial t}\Delta u.\end{aligned}$$

$$v_t = \sum_{j=1}^n x_j u_{x_j t} + 2u_t + 2tu_{tt}.$$

$$
\begin{aligned}
v_t - \Delta v &= \left(\sum_{j=1}^{n} x_j u_{x_j t} - \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} \Delta u \right) + (2u_t - 2\Delta u) \\
&\quad + \left(2t u_{tt} - 2t \frac{\partial}{\partial t} \Delta u \right) \\
&= \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} (u_t - \Delta u) + 2(u_t - \Delta u) + 2t \frac{\partial}{\partial t} (u_t - \Delta u) \\
&= 0.
\end{aligned}
$$

□

13. Assume $n = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$(*) \qquad v'' + \frac{z}{2} v' = 0.$$

Show that the general solution of $(*)$ is

$$v(z) = c \int_0^z e^{-s^2/4} ds + d.$$

(b) Differentiate $u(x,t) = v(\frac{x}{\sqrt{t}})$ with respect to x and select the constant c properly, to obtain the fundamental solution Φ for $n = 1$. Explain why this procedure produces the fundamental solution. (Hint: What is the initial condition for u?)

Solution. (a) 1. Set $z = \frac{x}{\sqrt{t}}$. Then

$$
\begin{aligned}
u_t &= v' \cdot \left(-\frac{x}{2t\sqrt{t}} \right) = -\frac{z}{2t} v', \\
u_x &= \frac{1}{\sqrt{t}} v', u_{xx} = \frac{1}{t} v'', \\
u_t - u_{xx} &= -\frac{z}{2t} v' - \frac{1}{t} v'' = -\frac{1}{t} \left(v'' + \frac{z}{2} v' \right).
\end{aligned}
$$

Hence $u_t = u_{xx}$ if and only if $v'' + \frac{z}{2} v' = 0$.

2. Set

$$w(z) = v'(z).$$

Then

$$\begin{aligned} w'(z) + \frac{z}{2} w(z) &= 0, \\ v'(z) = w(z) &= ce^{-z^2/4}, \\ v(z) &= c \int_0^z e^{-s^2/4} ds + d. \end{aligned}$$

(b)

$$\begin{aligned} u(x,t) = v\left(\frac{x}{\sqrt{t}}\right) &= c \int_0^{x/\sqrt{t}} e^{-s^2/4} ds + d, \\ u_x(x,t) &= c \frac{1}{\sqrt{t}} e^{-\frac{x^2}{4t}}. \end{aligned}$$

Let $c = \frac{1}{2\sqrt{\pi}}$. Then

$$u_x(x,t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

$$u_{xt} - u_{xxx} = u_{tx} - u_{xxx} = \frac{\partial}{\partial x}(u_t - u_{xx}) = 0,$$

so $u_x(x,t)$ is a solution of the equation $u_t = u_{xx}$.

14. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

where $c \in \mathbb{R}$.

Solution. Assume that $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $f \in C_1^2(\mathbb{R}^n \times [0,\infty))$ and f has compact support. Set

$$v(x,t) = u(x,t)e^{ct}.$$

Then

$$\begin{aligned} v_t &= u_t e^{ct} + cue^{ct}, \\ \Delta v &= (\Delta u)e^{ct}, \\ v_t - \Delta v &= e^{ct}(u_t + cu - \Delta u) = e^{ct} f, \\ v(x,0) &= u(x,0) = g. \end{aligned}$$

Thus $v(x,t)$ is a solution of

$$(*) \qquad \begin{cases} v_t - \Delta v = e^{ct} f & \text{in } \mathbb{R}^n \times (0,\infty) \\ v = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

By Eq. (17) in §2.3,

$$v(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s) dy ds$$

is a solution of $(*)$. Hence

$$u(x,t) = e^{-ct} \left[\int_{\mathbb{R}^n} \Phi(x-y,t) g(y) dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) e^{cs} f(y,s) dy ds \right]$$

is a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t=0\}. \end{cases}$$

15. Given $g : [0,\infty) \to \mathbb{R}$, with $g(0) = 0$, derive the formula

$$u(x,t) = \frac{x}{\sqrt{4\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{\frac{-x^2}{4(t-s)}} g(s) ds$$

for a solution of the initial/boundary-value problem

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u = 0 & \text{on } \mathbb{R}_+ \times \{t=0\} \\ u = g & \text{on } \{x=0\} \times [0,\infty). \end{cases}$$

(Hint: Let $v(x,t) := u(x,t) - g(t)$ and extend v to $\{x<0\}$ by odd reflection.)

Solution. Set

$$v(x,t) = \begin{cases} u(x,t) - g(t) & \text{in } [0,\infty) \times [0,\infty) \\ -u(-x,t) + g(t) & \text{in } (-\infty,0) \times [0,\infty). \end{cases}$$

Then

$$v_t(x,t) = \begin{cases} u_t(x,t) - g'(t) & \text{in } (0,\infty) \times (0,\infty) \\ -u_t(-x,t) + g'(t) & \text{in } (-\infty,0) \times (0,\infty), \end{cases}$$

$$v_x(x,t)=\begin{cases}u_x(x,t) & \text{in } (0,\infty)\times(0,\infty)\\ u_x(-x,t) & \text{in } (-\infty,0)\times(0,\infty),\end{cases}$$

$$v_{xx}(x,t)=\begin{cases}u_{xx}(x,t) & \text{in } (0,\infty)\times(0,\infty)\\ -u_{xx}(-x,t) & \text{in } (-\infty,0)\times(0,\infty),\end{cases}$$

$$\begin{cases} v_t-v_{xx}=\begin{cases}-g'(t) & \text{in } (0,\infty)\times(0,\infty)\\ g'(t) & \text{in } (-\infty,0)\times(0,\infty)\end{cases}\\ \qquad v=0 \text{ on } (\mathbb{R}\times\{t=0\})\cup(\{x=0\}\times(0,\infty)).\end{cases}$$

According to Eq. (13) in §2.3,

$$\begin{aligned} v(x,t)&=\int_0^t\frac{1}{2\sqrt{\pi(t-s)}}\left(\int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}}g'(s)dy-\int_0^\infty e^{-\frac{(x-y)^2}{4(t-s)}}g'(s)dy\right)ds\\ &=\frac{1}{2\sqrt{\pi}}\int_0^t\frac{g'(s)}{\sqrt{t-s}}\left(\int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}}dy-\int_0^\infty e^{-\frac{(x-y)^2}{4(t-s)}}dy\right)ds.\end{aligned}$$

Obviously

$$v(0,t)=0,\quad u(0,t)=g(t).$$

According to the Lemma in §2.3, we have

$$1=\frac{1}{2\sqrt{\pi(t-s)}}\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4(t-s)}}dy,$$

thus

$$\begin{aligned} g(t)&=\left(\int_0^t g'(s)ds\right)\left(\frac{1}{2\sqrt{\pi(t-s)}}\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4(t-s)}}dy\right)\\ &=\frac{1}{2\sqrt{\pi}}\int_0^t\frac{g'(s)}{\sqrt{t-s}}\int_{-\infty}^\infty e^{-\frac{(x-y)^2}{4(t-s)}}dyds.\end{aligned}$$

Therefore for $x\ge 0$,

$$\begin{aligned} u(x,t)&=v(x,t)+g(t)\\ &=\frac{1}{\sqrt{\pi}}\int_0^t\frac{g'(s)}{\sqrt{t-s}}\int_{-\infty}^0 e^{-\frac{(x-y)^2}{4(t-s)}}dyds.\end{aligned}$$

Integrating by parts with respect to the variable s, we get

$$\sqrt{\pi}u(x,t)=\int_{-\infty}^0\int_0^t\frac{g'(s)}{\sqrt{t-s}}e^{-\frac{(x-y)^2}{4(t-s)}}dsdy$$

$$= \int_{-\infty}^{0} \left[\frac{g(s)}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \Big|_0^t - \int_0^t g(s) \frac{\partial}{\partial s} \left(\frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \right) ds \right] dy.$$

$$\frac{g(s)}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \Big|_0^t = 0,$$

$$\begin{aligned}\frac{\partial}{\partial s} \left(\frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \right) &= \frac{1}{2(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \\ &\quad + \frac{1}{\sqrt{t-s}} e^{-\frac{(x-y)^2}{4(t-s)}} \left(-\frac{(x-y)^2}{4} \right) \left(\frac{1}{(t-s)^2} \right) \\ &= \frac{1}{2(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} - \frac{(x-y)^2}{4(t-s)^{5/2}} e^{-\frac{(x-y)^2}{4(t-s)}},\end{aligned}$$

thus

$$\sqrt{\pi} u(x,t) = -\int_0^t \int_{-\infty}^{0} g(s) \left(\frac{1}{2(t-s)^{3/2}} e^{-\frac{(x-y)^2}{4(t-s)}} - \frac{(x-y)^2}{4(t-s)^{5/2}} e^{-\frac{(x-y)^2}{4(t-s)}} \right) dyds.$$

Set $z := x - y$. Then

$$\begin{aligned}\sqrt{\pi} u(x,t) &= -\int_0^t \int_x^{\infty} g(s) \left(\frac{1}{2(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} - \frac{z^2}{4(t-s)^{5/2}} e^{-\frac{z^2}{4(t-s)}} \right) dzds \\ &= -\frac{1}{2} \int_0^t \frac{g(s)}{(t-s)^{3/2}} \int_x^{\infty} e^{-\frac{z^2}{4(t-s)}} dzds + \frac{1}{4} \int_0^t \frac{g(s)}{(t-s)^{5/2}} \int_x^{\infty} z^2 e^{-\frac{z^2}{4(t-s)}} dzds \\ &=: -I + J.\end{aligned}$$

Integrating by parts for J with respect to the variable z, we get

$$\begin{aligned}J &= -\frac{1}{2} \int_0^t \frac{g(s)}{(t-s)^{3/2}} \int_x^{\infty} z \frac{\partial}{\partial z} e^{-\frac{z^2}{4(t-s)}} dzds \\ &= -\frac{1}{2} \int_0^t \frac{g(s)}{(t-s)^{3/2}} \left(z e^{-\frac{z^2}{4(t-s)}} \Big|_x^{\infty} - \int_x^{\infty} e^{-\frac{z^2}{4(t-s)}} dz \right) ds \\ &= \frac{x}{2} \int_0^t \frac{g(s)}{(t-s)^{3/2}} e^{-\frac{z^2}{4(t-s)}} ds + \frac{1}{2} \int_0^t \frac{g(s)}{(t-s)^{3/2}} \int_x^{\infty} e^{-\frac{z^2}{4(t-s)}} dzds \\ &= \frac{x}{2} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds + I.\end{aligned}$$

Hence

$$\sqrt{\pi} u(x,t) = \frac{x}{2} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds,$$

and

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{1}{(t-s)^{3/2}} e^{-\frac{x^2}{4(t-s)}} g(s) ds.$$

Comment. "A-B-A" thought is used in the above solution. The steps are:

(i) Convert this homogeneous initial and boundary value problem to a non-homogeneous zero initial value problem.

(ii) Use Eq. (13) in §2.3 to construct a solution v.

(iii) Calculate $u = v + g$.

(a) Lemma in §2.3,

(b) integrating by parts with respect to s,

(c) set $z = x - y$,

(d) integrating by parts with respect to z.

16. Give a direct proof that if U is bounded and $u \in C_1^2(U_T) \cap C(\bar{U}_T)$ solves the heat equation, then

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

(Hint: Define $u_\varepsilon := u - \varepsilon t$ for $\varepsilon > 0$, and show u_ε cannot attain its maximum over $\bar{U}_T$ at a point in U_T.)

Proof. 1.

$$(u_\varepsilon)_t = u_t - \varepsilon, \quad \Delta u_\varepsilon = \Delta u,$$

so

$$(u_\varepsilon)_t - \Delta u_\varepsilon = u_t - \Delta u - \varepsilon = -\varepsilon < 0$$

on $U_T = U \times (0, T]$.

Assume that there exists a point $(x_0, t_0) \in U_T$, such that

$$\max_{\bar{U}_T} u_\varepsilon(x,t) = u_\varepsilon(x_0, t_0).$$

Then

$$(u_\varepsilon)_t(x_0, t_0) \geq 0 \quad (t = T \text{ is possible}),$$
$$Du_\varepsilon(x_0, t_0) = 0,$$
$$D^2 u_\varepsilon(x_0, t_0) \leq 0,$$
$$\Delta u_\varepsilon(x_0, t_0) \leq 0,$$

$$(u_\varepsilon)_t(x_0,t_0) - \Delta u_\varepsilon(x_0,t_0) \geq 0.$$

This is contradict with the fact that

$$(u_\varepsilon)_t - \Delta u_\varepsilon < 0$$

on U_T. Thus u_ε cannot attain its maximum over $\bar{U}_T$ at a point in U_T.

2. Assume that there exists a point $(x_1,t_1) \in U_T$, such that

$$\max_{\Gamma_T} u(x,t) < u(x_1,t_1) = \max_{\bar{U}_T} u(x,t).$$

Let

$$\varepsilon = \frac{1}{T}\left(\max_{\bar{U}_T} u(x,t) - \max_{\Gamma_T} u(x,t)\right),$$

and set

$$\max_{\Gamma_T} u_\varepsilon(x,t) = u_\varepsilon(x_2,t_2), \quad (x_2,t_2) \in \Gamma_T.$$

Then

$$\begin{aligned}
\max_{\Gamma_T} u_\varepsilon(x,t) &= u(x_2,t_2) - \varepsilon t_2 \\
&= u(x_1,t_1) - (u(x_1,t_1) - u(x_2,t_2)) - \varepsilon t_2 \\
&\leq u(x_1,t_1) - \left(\max_{\bar{U}_T} u(x,t) - \max_{\Gamma_T} u(x,t)\right) - \varepsilon t_2 \\
&= u(x_1,t_1) - \varepsilon T - \varepsilon t_2 \\
&= u(x_1,t_1) - \varepsilon(T + t_2) \\
&\leq u(x_1,t_1) - \varepsilon t_1 \\
&= u_\varepsilon(x_1,t_1).
\end{aligned}$$

The contradiction happens. Hence

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u.$$

$\square$

Remark. There is another way as follows to obtain $\max_{\bar{U}_T} u = \max_{\Gamma_T} u$ closely after step 1 of the proof. Set $x_0 \in \Gamma_T$ and $\max_{\bar{U}_T} u_\varepsilon = u_\varepsilon(x_0)$. Then for any $x \in \bar{U}_T$,

$$u(x) = \lim_{\varepsilon\to 0^+} u_\varepsilon(x) \leq \lim_{\varepsilon\to 0^+} u_\varepsilon(x_0) = u(x_0).$$

17. We say $v \in C_1^2(U_T)$ is a subsolution of the heat equation if

$$v_t - \Delta v \leq 0 \quad \text{in } U_T.$$

(a) Prove for a subsolution v that

$$v(x,t) \leq \frac{1}{4r^n} \iint_{E(x,t;r)} v(y,s) \frac{|x-y|^2}{(t-s)^2} dyds$$

for all $E(x,t;r) \subset U_T$.

(b) Prove that therefore $\max_{\bar{U}_T} v = \max_{\Gamma_T} v$.

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth and convex. Assume u solves the heat equation and $v := \phi(u)$. Prove v is a subsolution.

(d) Prove $v := |Du|^2 + u_t^2$ is a subsolution, whenever u solves the heat equation.

Proof. (a) It doesn't matter if we let $x = t = 0$. Set

$$\phi(r) := \frac{1}{r^n} \iint_{E(r)} v(y,s) \frac{|y|^2}{s^2} dyds$$
$$\psi := -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s} + n \log r.$$

According to the proof of Theorem 3 in Section 2.3, we get

$$\phi'(r) = \frac{1}{r^{n+1}} \iint_{E(r)} -4nv_s(y,s)\psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i}(y,s) y_i dyds.$$

Since $v_t - \Delta v \leq 0$ in U_T,

$$\begin{aligned} \phi'(r) &\geq \frac{1}{r^{n+1}} \iint_{E(r)} -4n\Delta v \psi - \frac{2n}{s} \sum_{i=1}^{n} v_{y_i}(y,s) y_i dyds \\ &= 0 (\text{Again accord to the proof of Theorem 3}). \end{aligned}$$

Thus

$$\begin{aligned} \phi(r) &\geq \lim_{t \to 0} \phi(t) \\ &= 4v(0,0) (\text{Accord to the proof of Theorem for the third time}), \end{aligned}$$

therefore

$$v(0,0) \leq \frac{1}{4r^n} \iint_{E(r)} v(y,s) \frac{|y|^2}{s^2} dyds.$$

(b) Suppose there exists a point $(x_0, t_0) \in U_T$ with $v(x_0, t_0) = M := \max_{\bar{U}_T} v$. Then for all sufficiently small $r > 0$, $E(x_0, t_0; r) \subset U_T$; and we employ (a) to deduce

$$M = v(x_0, t_0) \leq \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} v(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds \leq M,$$

since

$$1 = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds.$$

Thus

$$M = \frac{1}{4r^n} \iint_{E(x_0,t_0;r)} v(y,s) \frac{|x_0 - y|^2}{(t_0 - s)^2} dyds.$$

Consequently

$$v(y,s) = M \text{ for all } (y,s) \in E(x_0, t_0; r).$$

Hence

$$v(x,t) = M \text{ for all } (x,t) \in \bar{U}_{t_0}.$$

$$\max_{\bar{U}_T} v = \max_{\Gamma_T} v$$

for above all.

(c)

$$\begin{aligned}
\frac{\partial v}{\partial t} &= \frac{d\phi}{du} \frac{\partial u}{\partial t}, \\
\frac{\partial^2 v}{\partial x_i^2} &= \frac{d^2\phi}{du^2} \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{d\phi}{du} \frac{\partial^2 u}{\partial x_i^2}, \\
\Delta v &= \frac{d^2\phi}{du^2} \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{d\phi}{du} \Delta u, \\
v_t - \Delta v &= \frac{d^2\phi}{du^2} \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i} \right)^2 + \frac{d\phi}{du} (u_t - \Delta u) \leq 0.
\end{aligned}$$

(d)

$$v = \sum_{j=1}^n \left(\frac{\partial u}{\partial x_j} \right)^2 + \left(\frac{\partial u}{\partial t} \right)^2,$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} &= \sum_{j=1}^{n} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_j}\right)^2 + \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)^2 \\
&= \sum_{j=1}^{n} 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} \\
&= 2\left(\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial t} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}\right) \\
&= 2\left(\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial v}{\partial x_i} &= \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_j}\right)^2 + \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial t}\right)^2 \\
&= \sum_{j=1}^{n} 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} + 2\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x_i \partial t} \\
&= 2\left(\sum_{j=1}^{n} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j} + \frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x_i \partial t}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 v}{\partial x_i^2} &= 2\left\{\sum_{j=1}^{n}\left[\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i^2 \partial x_j}\right] + \left[\left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2 + \frac{\partial u}{\partial t}\frac{\partial^3 u}{\partial x_i^2 \partial t}\right]\right\} \\
&= 2\left\{\left[\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2\right] + \left(\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial^3 u}{\partial x_i^2 \partial x_j} + \frac{\partial u}{\partial t}\frac{\partial^3 u}{\partial x_i^2 \partial t}\right)\right\} \\
&= 2\left\{\left[\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2\right] + \left(\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial x_i^2}\right\},
\end{aligned}
$$

$$
\Delta v = 2\left\{\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2\right] + \left(\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\right)\Delta u\right\},
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t} - \Delta v &= 2\left\{-\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2\right] + \left(\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j} + \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\right)\left(\frac{\partial u}{\partial t} - \Delta u\right)\right\} \\
&= -2\sum_{i=1}^{n}\left[\sum_{j=1}^{n}\left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \left(\frac{\partial^2 u}{\partial x_i \partial t}\right)^2\right] \\
&\le 0.
\end{aligned}
$$

□

18. (Stokes' rule) Assume u solves the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ u = 0, u_t = h & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

Show that $v := u_t$ solves

$$\begin{cases} v_{tt} - \Delta v = 0 & \text{in } \mathbb{R}^n \times (0, \infty) \\ v = h, v_t = 0 & \text{on } \mathbb{R}^n \times \{t = 0\}. \end{cases}$$

This is *Stokes' rule*.

Proof.

$$\begin{aligned} v_{tt} - \Delta v = u_{ttt} - \Delta u_t = (u_{tt})_t - (\Delta u)_t = (u_{tt} - \Delta u)t = 0, \\ v(x,0) = u_t(x,0) = h, \\ v_t(x,0) = u_{tt}(x,0) = \Delta u(x,0) = 0. \end{aligned}$$

□

19.

(a) Show the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y)$$

for arbitrary functions F, G.

(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Use (a) and (b) to rederive d'Alembert's formula.

(d) Under what conditions on the initial data g, h is the solution u a right-moving wave? A left-moving wave?

Solution. (a)

$$\begin{aligned} u_{xy}(x,y) &= 0, \\ u_x(x,y) &= f(x), \\ u(x,y) &= \int f(x)dx = F(x) + G(y). \end{aligned}$$

(b)

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}, \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial t} \\
&= \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}; \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)\frac{\partial \eta}{\partial x} \\
&= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}.
\end{aligned}
$$

Hence

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = -4\frac{\partial^2 u}{\partial \xi \partial \eta},$$

and $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) According to (a) and (b), the solution of $u_{tt} - u_{xx} = 0$ is

$$u(x,t) = F(x+t) + G(x-t).$$

For

$$u = g, u_t = h \text{ on } \mathbb{R} \times \{t = 0\},$$

we have

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x). \end{cases}$$

Thus

$$\begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \int_0^x h(y)dy + C, \end{cases}$$

so

$$
\begin{aligned}
F(x) &= \frac{1}{2}g(x) + \frac{1}{2}\int_0^x h(y)dy + \frac{1}{2}C, \\
G(x) &= \frac{1}{2}g(x) - \frac{1}{2}\int_0^x h(y)dy - \frac{1}{2}C.
\end{aligned}
$$

Hence

$$u(x,t) = F(x+t) + G(x-t)$$

$$= \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \quad (x \in \mathbb{R}, t \geq 0).$$

(d) If $g'(x) = -h(x)$, then $u(x,t) = g(x-t)$ is a right-moving wave. If $g'(x) = h(x)$, then $u(x,t) = g(x+t)$ is a left-moving wave.

20. Assume that for some attenuation function $\alpha = \alpha(r)$ and delay function $\beta = \beta(r) \geq 0$, there exist for all profiles ϕ solutions of the wave equation in $(\mathbb{R}^n - \{0\}) \times \mathbb{R}$ having the form

$$u(x,t) = \alpha(r)\phi(t - \beta(r)).$$

Here $r = |x|$ and we assume $\beta(0) = 0$.

Show that this is possible only if $n = 1$ or 3 , and compute the form of the functions α, β.

(T. Morley, SIAM Review 27 (1985), 69-71)

Proof. If $u(x,t)$ has the form $u(x,t) = \alpha(r)\phi(t-\beta(r))$, then

$$\begin{aligned} u_t(x,t) &= \alpha(r)\phi'(t-\beta(r)), \\ u_{tt}(x,t) &= \alpha(r)\phi''(t-\beta(r)). \end{aligned}$$

And

$$\begin{aligned} u_{x_i}(x,t) &= \alpha'(r)r_{x_i}\phi(t-\beta(r)) - \alpha(r)\phi'(t-\beta(r))\beta'(r)r_{x_i} \\ &= r_{x_i}\alpha'(r)\phi(t-\beta(r)) - r_{x_i}\alpha(r)\beta'(r)\phi'(t-\beta(r)), \\ u_{x_ix_i}(x,t) &= r_{x_ix_i}\alpha'(r)\phi(t-\beta(r)) + r_{x_i}^2\alpha''(r)\phi(t-\beta(r)) \\ &\quad - r_{x_i}^2\alpha'(r)\phi'(t-\beta(r))\beta'(r) \\ &\quad - r_{x_ix_i}\alpha(r)\beta'(r)\phi'(t-\beta(r)) - r_{x_i}^2\alpha'(r)\beta'(r)\phi'(t-\beta(r)) \\ &\quad - r_{x_i}^2\alpha(r)\beta''(r)\phi'(t-\beta(r)) + r_{x_i}^2\alpha(r)\beta'(r)\phi''(t-\beta(r))\beta'(r) \\ &= [r_{x_ix_i}\alpha'(r) + r_{x_i}^2\alpha''(r)]\phi(t-\beta(r)) \\ &\quad - [2r_{x_i}^2\alpha'(r)\beta'(r) + r_{x_ix_i}\alpha(r)\beta'(r) + r_{x_i}^2\alpha(r)\beta''(r)]\phi'(t-\beta(r)) \\ &\quad + r_{x_i}^2\alpha(r)(\beta'(r))^2\phi''(t-\beta(r)), \\ \Delta u(x,t) &= \left[\frac{n-1}{r}\alpha'(r) + \alpha''(r)\right]\phi(t-\beta(r)) \\ &\quad - \left[2\alpha'(r)\beta'(r) + \frac{n-1}{r}\alpha(r)\beta'(r) + \alpha(r)\beta''(r)\right]\phi'(t-\beta(r)) \end{aligned}$$

$$+\alpha(r)(\beta'(r))^2\phi''(t-\beta(r)).$$

Thus

$$\begin{aligned}
u_{tt}(x,t)-\Delta u(x,t) &= -\left[\frac{n-1}{r}\alpha'(r)+\alpha''(r)\right]\phi(t-\beta(r))\\
&\quad+\left[2\alpha'(r)\beta'(r)+\frac{n-1}{r}\alpha(r)\beta'(r)+\alpha(r)\beta''(r)\right]\phi'(t-\beta(r))\\
&\quad+[\alpha(r)-\alpha(r)(\beta'(r))^2]\phi''(t-\beta(r))\\
&=0.
\end{aligned}$$

Because the profile ϕ is arbitrary, we have

$$\frac{n-1}{r}\alpha'(r)+\alpha''(r)=0, \tag{2.4}$$

$$2\alpha'(r)\beta'(r)+\frac{n-1}{r}\alpha(r)\beta'(r)+\alpha(r)\beta''(r)=0, \tag{2.5}$$

$$\alpha(r)-\alpha(r)(\beta'(r))^2=0. \tag{2.6}$$

By Eq. (2.6), $\beta(0)=0$, and $\beta(r)\geq 0$, we have $\beta'(r)=1$, so

$$\beta(r)=\beta(0)+\int_0^r \beta'(s)ds=r.$$

Substitute $\beta'(r)=1$ into Eq. (2.5), we have

$$2\alpha'(r)+\frac{n-1}{r}\alpha(r)=0,$$

therefore

$$\alpha(r)=Cr^{\frac{1-n}{2}},$$

where $C\neq 0$ is a constant. Thus

$$\alpha'(r)=C\frac{1-n}{2}r^{-\frac{n+1}{2}}, \tag{2.7}$$

$$\alpha''(r)=C\frac{n-1}{2}\frac{n+1}{2}r^{-\frac{n+3}{2}}. \tag{2.8}$$

Substitute Eq. (2.7) and (2.8) into Eq. (2.4), we have

$$C\frac{1-n}{2}\frac{n-1}{r}r^{-\frac{n+1}{2}}+C\frac{n-1}{2}\frac{n+1}{2}r^{-\frac{n+3}{2}}=0,$$
$$(n-1)(n-3)=0.$$

Thus $n=1$ or 3. $\square$

21.

(a) Assume $\mathbf{E} = \left(E^1, E^2, E^3\right)$ and $\mathbf{B} = \left(B^1, B^2, B^3\right)$ solve Maxwell's equations

$$\begin{cases} \mathbf{E}_t = \operatorname{curl}\mathbf{B}, \quad \mathbf{B}_t = -\operatorname{curl}\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0. \end{cases}$$

Show

$$\mathbf{E}_{tt} - \Delta\mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta\mathbf{B} = 0.$$

(b) Assume that $\mathbf{u} = \left(u^1, u^2, u^3\right)$ solves the evolution equations of linear elasticity

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda+\mu)D(\operatorname{div}\mathbf{u}) = \mathbf{0} \quad \text{ in } \mathbb{R}^3 \times (0,\infty).$$

Show $w := \operatorname{div}\mathbf{u}$ and $\mathbf{w} :=$ curl $\mathbf{u}$ each solve wave equations, but with differing speeds of propagation.

Proof. (a)

$$\begin{aligned} \mathbf{B}_t &= -\operatorname{curl}\mathbf{E}, \\ \operatorname{curl}\mathbf{B}_t &= -\operatorname{curl}(\operatorname{curl}\mathbf{E}), \\ \frac{\partial}{\partial t}\operatorname{curl}\mathbf{B} &= -D(\operatorname{div}\mathbf{E}) + \Delta\mathbf{E}, \\ \mathbf{E}_{tt} &= \Delta\mathbf{E}, \\ \mathbf{E}_{tt} - \Delta\mathbf{E} &= 0. \end{aligned}$$

$$\begin{aligned} \mathbf{E}_t &= \operatorname{curl}\mathbf{B}, \\ \operatorname{curl}\mathbf{E}_t &= \operatorname{curl}(\operatorname{curl}\mathbf{B}), \\ \frac{\partial}{\partial t}\operatorname{curl}\mathbf{E} &= D(\operatorname{div}\mathbf{B}) - \Delta\mathbf{B}, \\ -\mathbf{B}_{tt} &= -\Delta\mathbf{B}, \\ \mathbf{B}_{tt} - \Delta\mathbf{B} &= 0. \end{aligned}$$

(b)

$$\begin{aligned} w_{tt} &= \frac{\partial^2}{\partial t^2}(\operatorname{div}\mathbf{u}) = \operatorname{div}\mathbf{u}_{tt} \\ &= \operatorname{div}[\mu\Delta\mathbf{u} + (\lambda+\mu)D(\operatorname{div}\mathbf{u})] \end{aligned}$$

$$
\begin{aligned}
&= \mu\,\mathrm{div}(\Delta\mathbf{u}) + (\lambda+\mu)\,\mathrm{div}[D(\mathrm{div}\,\mathbf{u})] \\
&= \mu\Delta(\mathrm{div}\,\mathbf{u}) + (\lambda+\mu)\Delta(\mathrm{div}\,\mathbf{u}) \\
&= (\lambda+2\mu)\Delta(\mathrm{div}\,\mathbf{u}), \\
\Delta w &= \Delta(\mathrm{div}\,\mathbf{u}),
\end{aligned}
$$

thus

$$w_{tt} - (\lambda+2\mu)\Delta w = 0.$$

$$
\begin{aligned}
\mathbf{w}_{tt} &= \frac{\partial^2}{\partial t^2}\,\mathrm{curl}\,\mathbf{u} = \mathrm{curl}\,\mathbf{u}_{tt} \\
&= \mathrm{curl}[\mu\Delta\mathbf{u} + (\lambda+\mu)D(\mathrm{div}\,\mathbf{u})] \\
&= \mu\,\mathrm{curl}(\Delta\mathbf{u}) + (\lambda+\mu)\,\mathrm{curl}[D(\mathrm{div}\,\mathbf{u})] \\
&= \mu\Delta(\mathrm{curl}\,\mathbf{u}) + (\lambda+\mu)0 \\
&= \mu\Delta(\mathrm{curl}\,\mathbf{u}), \\
\Delta\mathbf{w} &= \Delta(\mathrm{curl}\,\mathbf{u}),
\end{aligned}
$$

thus

$$\mathbf{w}_{tt} - \mu\Delta\mathbf{w} = 0.$$

□

22. Let u denote the density of particles moving to the right with speed one along the real line and let v denote the density of particles moving to the left with speed one. If at rate $d > 0$ right-moving particles randomly become left-moving, and vice versa, we have the system of PDE

$$\begin{cases} u_t + u_x = d(v-u) \\ v_t - v_x = d(u-v). \end{cases}$$

Show that both $w := u$ and $w := v$ solve the telegraph equation

$$w_{tt} + 2dw_t - w_{xx} = 0.$$

Proof.

$$u_{tt} + u_{xt} = d(v_t - u_t), \tag{2.9}$$

$$v_{tt} - v_{xt} = d(u_t - v_t), \tag{2.10}$$

$$u_{tx} + u_{xx} = d(v_x - u_x), \tag{2.11}$$

$$v_{tx} - v_{xx} = d(u_x - v_x). \tag{2.12}$$

Substracting Eq. (2.9) by Eq. (2.11), we have

$$\begin{aligned} u_{tt} - u_{xx} &= d(v_t - u_t - v_x + u_x) \\ &= d[d(u-v) + d(v-u) - 2u_t] \\ &= -2du_t. \end{aligned}$$

Thus

$$u_{tt} + 2du_t - u_{xx} = 0.$$

Summing up (2.10) and (2.12), we have

$$\begin{aligned} v_{tt} - v_{xx} &= d(u_t - v_t + u_x - v_x) \\ &= d[d(v-u) + d(u-v) - 2v_t] \\ &= -2dv_t. \end{aligned}$$

Thus

$$v_{tt} + 2dv_t - v_{xx} = 0.$$

□

24. (Equipartition of energy) Let u solve the initial-value problem for the wave equation in one dimension:

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0, \infty) \\ u = g, u_t = h & \text{on } \mathbb{R} \times \{t = 0\}. \end{cases}$$

Suppose g, h have compact support. The kinetic energy is $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dx$ and the potential energy is $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)dx$ Prove

(a) $k(t) + p(t)$ is constant in t,

(b) $k(t) = p(t)$ for all large enough times t.

Proof. (a)

$$k(t) + p(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t) + u_x^2(x,t)dx,$$

$$\begin{aligned}
\frac{d}{dt}[k(t)+p(t)] &= \frac{1}{2}\int_{-\infty}^{\infty} 2u_t(x,t)u_{tt}(x,t) + 2u_x(x,t)u_{xt}(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t) + u_x(x,t)u_{xt}(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t)dx + \int_{-\infty}^{\infty} u_x(x,t)\frac{\partial}{\partial x}u_t(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)u_{tt}(x,t)dx + u_x(x,t)u_t(x,t)|_{x=-\infty}^{x=\infty} \\
&\quad - \int_{-\infty}^{\infty} u_t(x,t)u_{xx}(x,t)dx \\
&= \int_{-\infty}^{\infty} u_t(x,t)[u_{tt}(x,t) - u_{xx}(x,t)]dx \\
&= 0.
\end{aligned}$$

Hence $k(t)+p(t)$ is constant in t.

(b) According to d'Alembert's formula,

$$u(x,t) = \frac{1}{2}[g(x+t)+g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \quad (x \in \mathbb{R}, t \geq 0).$$

Thus

$$\begin{aligned}
u_t(x,t) &= \frac{1}{2}[g'(x+t) - g'(x-t)] + \frac{1}{2}[h(x+t)+h(x-t)], \\
u_x(x,t) &= \frac{1}{2}[g'(x+t) + g'(x-t)] + \frac{1}{2}[h(x+t)-h(x-t)].
\end{aligned}$$

g, h have compact support, thus there exists an $M \geq 0$ such that $\operatorname{spt} g' \cup \operatorname{spt} h \subset [-M, M]$. Let $t > M$.

(1) If $x \leq -t+M$, then $x-t \leq -2t+M < -M$. Thus

$$\begin{aligned}
u_t(x,t) &= \frac{1}{2}g'(x+t) + \frac{1}{2}h(x+t), \\
u_x(x,t) &= \frac{1}{2}g'(x+t) + \frac{1}{2}h(x+t), \\
u_t(x,t) &= u_x(x,t).
\end{aligned}$$

(2) If $-t+M < x < t-M$, then $x+t > M$, $x-t < -M$. Thus

$$u_t(x,t) = u_x(x,t) = 0.$$

(3) If $x \geq t-M$, then $x+t \geq 2t-M > M$. Thus

$$u_t(x,t) = -\frac{1}{2}g'(x-t) + \frac{1}{2}h(x-t),$$

$$u_x(x,t) = \frac{1}{2}g'(x-t) - \frac{1}{2}h(x-t),$$
$$u_t^2(x,t) = u_x^2(x,t).$$

Hence $k(t) = p(t)$ for all large enough times t.

$\square$

Chapter 5

SOBOLEV SPACES

In these exercises U always denotes an open subset of $\mathbb{R}^n$, with a smooth boundary ∂U. As usual, all given functions are assumed smooth, unless otherwise stated.

2. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$\|u\|_{C^{0,\gamma}(U)} \leq \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

Proof.

$$\begin{aligned}
&\|u\|_{C^{0,\gamma}(U)} \\
&= \|u\|_{C(U)} + \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\gamma} \right\} \\
&= \|u\|_{C(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C(U)}^{\frac{\gamma-\beta}{1-\beta}} \\
&\quad + \sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \left(\frac{|u(x)-u(y)|}{|x-y|^\beta} \right)^{\frac{1-\gamma}{1-\beta}} \left(\frac{|u(x)-u(y)|}{|x-y|} \right)^{\frac{\gamma-\beta}{1-\beta}} \right\} \\
&\quad \left(\text{Note that } \gamma = \beta \frac{1-\gamma}{1-\beta} + \frac{\gamma-\beta}{1-\beta}. \right) \\
&\leqslant \|u\|_{C(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C(U)}^{\frac{\gamma-\beta}{1-\beta}} \\
&\quad + \left(\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\beta} \right\} \right)^{\frac{1-\gamma}{1-\beta}} \left(\sup_{\substack{x,y\in U\\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|} \right\} \right)^{\frac{\gamma-\beta}{1-\beta}}
\end{aligned}$$

$$\leqslant \left(\|u\|_{C(U)} + \sup_{\substack{x,y\in U \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|^\beta} \right\} \right)^{\frac{1-\gamma}{1-\beta}}$$

$$\left(\|u\|_{C(U)} + \sup_{\substack{x,y\in U \\ x\neq y}} \left\{ \frac{|u(x)-u(y)|}{|x-y|} \right\} \right)^{\frac{\gamma-\beta}{1-\beta}}$$

(according to the discrete version of Hölder's inequality)

$$= \|u\|_{C^{0,\beta}(U)}^{\frac{1-\gamma}{1-\beta}} \|u\|_{C^{0,1}(U)}^{\frac{\gamma-\beta}{1-\beta}}.$$

□

3. Denote by U the open square $\{x \in \mathbb{R}^2 \mid |x_1| < 1, |x_2| < 1\}$. Define

$$u(x) = \begin{cases} 1 - x_1 & \text{if } x_1 > 0, \quad |x_2| < x_1 \\ 1 + x_1 & \text{if } x_1 < 0, \quad |x_2| < -x_1 \\ 1 - x_2 & \text{if } x_2 > 0, \quad |x_1| < x_2 \\ 1 + x_2 & \text{if } x_2 < 0, \quad |x_1| < -x_2. \end{cases}$$

For which $1 \leq p \leq \infty$ does u belong to $W^{1,p}(U)$?

Solution. Write

$$U_1 = \{x \in U \mid x_1 > 0, |x_2| < x_1\},$$

$$U_2 = \{x \in U \mid x_1 < 0, |x_2| < -x_1\},$$

$$U_3 = \{x \in U \mid x_2 > 0, |x_1| < x_2\},$$

$$U_4 = \{x \in U \mid x_2 < 0, |x_1| < -x_2\};$$

$$D_1 = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 = x_2 \leq 1\},$$

$$D_2 = \{x \in \mathbb{R}^2 \mid -1 \leq x_1 = -x_2 \leq 0\},$$

$$D_3 = \{x \in \mathbb{R}^2 \mid -1 \leq x_1 = x_2 \leq 0\},$$

$$D_4 = \{x \in \mathbb{R}^2 \mid 0 \leq x_1 = -x_2 \leq 1\}.$$

Set

$$v_1(x) = \begin{cases} -1 & \text{if } x \in U_1 \\ 1 & \text{if } x \in U_2 \\ 0 & \text{if } x \in U_3 \\ 0 & \text{if } x \in U_4 \end{cases},$$

$$v_2(x) = \begin{cases} 0 & \text{if } x \in U_1 \\ 0 & \text{if } x \in U_2 \\ -1 & \text{if } x \in U_3 \\ 1 & \text{if } x \in U_4 \end{cases}.$$

For all test functions $\phi \in C_c^\infty(U)$,

$$\begin{aligned}
\int_U u\phi_{x_1}dx &= \int_{U_1}(1-x_1)\phi_{x_1}dx + \int_{U_2}(1+x_1)\phi_{x_1}dx \\
&\quad + \int_{U_3}(1-x_2)\phi_{x_1}dx + \int_{U_4}(1+x_2)\phi_{x_1}dx \\
&= \int_U \phi_{x_1}dx - \int_{U_1} x_1\phi_{x_1}dx + \int_{U_2} x_1\phi_{x_1}dx \\
&\quad - \int_{U_3} x_2\phi_{x_1}dx + \int_{U_4} x_2\phi_{x_1}dx \\
&= \int_{U_1}\phi dx - \int_{\partial U_1} x_1\phi\nu^1 ds - \int_{U_2}\phi dx + \int_{\partial U_2} x_1\phi\nu^1 ds \\
&\quad - \int_{\partial U_3} x_2\phi\nu^1 ds + \int_{\partial U_4} x_2\phi\nu^1 ds \\
&= \int_{U_1}\phi dx - \int_{U_2}\phi dx + \frac{1}{\sqrt{2}}\int_{D_1\cup D_4} x_1\phi ds + \frac{1}{\sqrt{2}}\int_{D_2\cup D_3} x_1\phi ds \\
&\quad - \frac{1}{\sqrt{2}}\int_{D_1\cup D_2} x_1\phi ds - \frac{1}{\sqrt{2}}\int_{D_3\cup D_4} x_1\phi ds \\
&= -\int_U v_1\phi dx + \frac{1}{\sqrt{2}}\int_{\bigcup_{i=1}^4 D_i} x_1\phi ds - \frac{1}{\sqrt{2}}\int_{\bigcup_{i=1}^4 D_i} x_1\phi ds \\
&= -\int_U v_1\phi dx,
\end{aligned}$$

$$\int_U u\phi_{x_2}dx = \int_{U_3}\phi dx - \int_{U_4}\phi dx = -\int_U v_2\phi dx.$$

Thus $u_{x_1} = v_1$, $u_{x_2} = v_2$ in the weak sense. Obviously $v_1, v_2 \in L^p(U)$ for all $1 \le p \le \infty$. Hence u belong to $W^{1,p}(U)$ for all $1 \le p \le \infty$.

4. Assume $n = 1$ and $u \in W^{1,p}(0,1)$ for some $1 \le p < \infty$.

(a) Show that u is equal a.e. to an absolutely continuous function and u' (which exists a.e.) belongs to $L^p(0,1)$.

(b) Prove that if $1 < p < \infty$, then

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'|^p\, dt\right)^{1/p}$$

for a.e. $x, y \in [0,1]$.

Proof. (a) $u \in W^{1,p}(0,1)$, so

$$\int_0^1 u\phi' dx = -\int_0^1 Du\phi dx,$$

for all test functions $\phi \in C_c^\infty(0,1)$, and

$$Du \in L^p(0,1),$$

where Du is the 1^{th}-weak derivative of u. Set

$$v = \int_0^x Du(t)dt, \quad x \in (0,1),$$

which is absolutely continuous. $v' = Du$ a.e., where v' is the derivative of v not in the weak sense, so

$$-\int_0^1 Du\phi dx = -\int_0^1 v'\phi dx = \int_0^1 v\phi' dx \quad \text{(integrating by parts)} \tag{5.1}$$

for all $\phi \in C_c^\infty(0,1)$, thus

$$\int_0^1 (u-v)\phi' dx = 0 = -\int_0^1 0\phi dx,$$

therefore $u = v + c$ a.e., by Problem 11.

The formula (5.1) can be also calculated as follow.

$$\begin{aligned}
\int_0^1 v(x)\phi'(x)dx &= \int_0^1 \int_0^x Du(t)dt\phi'(x)dx \\
&= \int_0^1 Du(t) \int_t^1 \phi'(x)dxdt \\
&= -\int_0^1 Du(t)\phi(t)dt \\
&= -\int_0^1 Du(x)\phi(x)dx
\end{aligned}$$

$u' := v' = Du$ a.e., so $u' \in L^p(0,1)$.

(b) It doesn't matter if we let $x < y$. Set

$$\chi(t) = \begin{cases} 1 & \text{if } t \in [x,y] \\ 0 & \text{if } t \in [0,1] - [x,y]. \end{cases}$$

Then for a.e. $x, y \in [0,1]$,

$$
\begin{aligned}
|u(x)-u(y)| &= \left| \int_x^y u'(t)dt \right| \\
&= \left| \int_0^1 \chi(t)u'(t)dt \right| \\
&\le \int_0^1 |\chi(t)u'(t)|dt \\
&\le \left(\int_0^1 |\chi(t)|^{\frac{p}{p-1}} dt \right)^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p dt \right)^{\frac{1}{p}} \\
&\quad \text{(Hölder's inequality)} \\
&= |x-y|^{1-\frac{1}{p}} \left(\int_0^1 |u'(t)|^p dt \right)^{\frac{1}{p}}.
\end{aligned}
$$

□

5. Let U, V be open sets, with $V \subset\subset U$. Show there exists a smooth function ζ such that $\zeta \equiv 1$ on V, $\zeta = 0$ near ∂U. (Hint: Take $V \subset\subset W \subset\subset U$ and mollify χ_W.)

Proof. Take the open set W such that $V \subset\subset W \subset\subset U$ and

$$2\,\mathrm{dist}(\partial V, \partial W) = \mathrm{dist}(\partial W, \partial U) =: \epsilon.$$

Set

$$\zeta(x) := \begin{cases} \chi_W^\epsilon(x) & \text{if } x \in U \\ 0 & \text{if } x \in \mathbb{R}^n - U. \end{cases}$$

For any $x \in V$,

$$
\begin{aligned}
\zeta(x) &= \int_{B(0,\epsilon)} \eta_\epsilon(y)\chi_W(x-y)dy \\
&= \int_{B(0,\epsilon)} \eta_\epsilon(y)dy \\
&= 1;
\end{aligned}
$$

For any $x \in U - U_\epsilon$,

$$
\begin{aligned}
\zeta(x) &= \int_{B(0,\epsilon)} \eta_\epsilon(y)\chi_W(x-y)dy \\
&= 0.
\end{aligned}
$$

According to Theorem 7 in Appendix C, ζ is smooth on U, thus ζ is smooth on $\mathbb{R}^n$. $\square$

7. Assume that U is bounded and there exists a smooth vector field $\boldsymbol{\alpha}$ such that $\boldsymbol{\alpha} \cdot \boldsymbol{\nu} \geq 1$ along ∂U, where $\boldsymbol{\nu}$ as usual denotes the outward unit normal. Assume $1 \leq p < \infty$.

Apply the Gauss-Green Theorem to $\int_{\partial U} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{\nu} dS$, to derive a new proof of the trace inequality

$$\int_{\partial U} |u|^p dS \leq C \int_U |Du|^p + |u|^p dx$$

for all $u \in C^1(\bar{U})$.

Proof. It doesn't matter to think $|u| \in C^1(\bar{U})$.

$$\begin{aligned}
\int_{\partial U} |u|^p dS &\leq \int_{\partial U} |u|^p \boldsymbol{\alpha} \cdot \boldsymbol{\nu} dS \\
&= \int_U \operatorname{div}(|u|^p \boldsymbol{\alpha}) dx \\
&= \int_U D(|u|^p) \cdot \boldsymbol{\alpha} + |u|^p \operatorname{div} \boldsymbol{\alpha} dx \\
&= \int_U p|u|^{p-1} D|u| \cdot \boldsymbol{\alpha} + |u|^p \operatorname{div} \boldsymbol{\alpha} dx \\
&\leq \int_U p|u|^{p-1} |Du| + |u|^p \operatorname{div} \boldsymbol{\alpha} dx \\
&\leq C \int_U |u|^p + |Du|^p dx.
\end{aligned}$$

In order to get the last step, pay attention to that $\frac{1}{\frac{p}{p-1}} + \frac{1}{p} = 1$ if $p \neq 1$, and then use Young's inequality. $\square$

8. Let U be bounded, with a C^1 boundary. Show that a "typical"function $u \in L^p(U)$ $(1 \leq p < \infty)$ does not have a trace on ∂U. More precisely, prove there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$.

Proof. Take

$$u_n(x) = \max\{0, 1 - n\,\mathrm{dist}(x, \partial U)\}, \quad 1 \le n < \infty,$$

where $u_n \in C(\bar{U}) \cap L^p(U)$. Then

$$\|u_n\|_{L^p(U)} = \left(\int_U |u_n|^p dx\right)^{\frac{1}{p}} \le \left(\int_{U-U_{\frac{1}{n}}} dx\right)^{\frac{1}{p}} \to 0$$

as $n \to \infty$, and

$$\|u_n|_{\partial U}\|_{L^p(\partial U)} = \left(\int_{\partial U} dS\right)^{\frac{1}{p}} > 0$$

is a constant. Thus there exists no constant C such that

$$\|u_n|_{\partial U}\|_{L^p(\partial U)} \le C\|u_n\|_{L^p(U)}.$$

Then there does not exist a bounded linear operator

$$T : L^p(U) \to L^p(\partial U)$$

such that $Tu = u|_{\partial U}$ whenever $u \in C(\bar{U}) \cap L^p(U)$. □

11. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies

$$Du = 0 \quad \text{a.e. in } U.$$

Prove u is constant a.e. in U.

Proof. By Equation (1) in §5.3, we have

$$u^\epsilon_{x_i} = \eta_\epsilon * u_{x_i} \quad \text{in } U_\epsilon, \quad i = 1, \dots, n,$$

where u_{x_i} is in the weak sense. $Du = 0$ a.e. in U, thus $D(u^\epsilon) = 0$ a.e. in U_ϵ. So $u^\epsilon(x) = c_\epsilon$ in U_ϵ, where c_ϵ is a constant. According to Theorem 7 in Appendix C, $u^\epsilon = c_\epsilon \to u$ a.e. as $\epsilon \to 0$, hence u is constant a.e. in U. □

12. Show by example that if we have $\left\|D^h u\right\|_{L^1(V)} \le C$ for all $0 < |h| < \frac{1}{2}\,\mathrm{dist}(V, \partial U)$, it does not necessarily follow that $u \in W^{1,1}(V)$.

Proof. Set

$$U = (-2, 2) \times \cdots \times (-2, 2) \subset \mathbb{R}^n,$$
$$V = (-1, 1) \times \cdots \times (-1, 1) \subset \mathbb{R}^n,$$

and

$$u(x) = \begin{cases} 0, & x \in U \cap (\mathbb{R}^n - \mathbb{R}^n_+), \\ 1, & x \in U \cap \mathbb{R}^n_+. \end{cases}$$

Obviously $u \in L^1(V) \subset L^1_{\mathrm{loc}}(V)$. For all $0 < |h| < \frac{1}{2}\operatorname{dist}(V, \partial U) = \frac{1}{2}$,

$$\begin{aligned} \|D^h u\|_{L^1(V)} &= \int_V \left(\sum_{i=1}^n |D_i^h u|^2 \right)^{\frac{1}{2}} dx \\ &= \int_V |D_n^h u| dx \\ &= 2^{n-1}. \end{aligned}$$

Let $v \in L^1_{\mathrm{loc}}(V)$. Suppose that $u_{x_n} = v$ on V in the weak sense. Then for all $\phi \in C_c^\infty(V)$,

$$\begin{aligned} \int_V v\phi dx &= -\int_V u\phi_{x_n} dx \\ &= -\int_{V \cap \mathbb{R}^n_+} \phi_{x_n} dx \\ &= -\int_{-1}^1 \cdots \int_{-1}^1 \int_0^1 \phi_{x_n} dx_n dx_{n-1} \dots dx_1 \\ &= \int_{-1}^1 \cdots \int_{-1}^1 \phi(x_1, \dots, x_{n-1}, 0) dx_{n-1} \dots dx_1. \end{aligned} \tag{5.2}$$

Choose a sequence $\{\phi_m\}_{m=1}^\infty \subset C_c^\infty(V)$ satisfying $\phi_m(x) = 1$ for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times \cdots \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \{0\}$, and $\phi_m(x) \to 0$ as $m \to \infty$ for all other x. Replace ϕ by ϕ_m in (5.2) and sending $m \to \infty$, we discover

$$\begin{aligned} 0 &= \lim_{m \to \infty} \int_V v\phi_m dx \\ &= \lim_{m \to \infty} \int_{-1}^1 \cdots \int_{-1}^1 \phi_m(x_1, \dots, x_{n-1}, 0) dx_{n-1} \dots dx_n \\ &= 1, \end{aligned}$$

a contradiction. Hence $u \notin W^{1,1}(V)$. □

14. Verify that if $n>1$, the unbounded function $u=\log\log\left(1+\frac{1}{|x|}\right)$ belongs to $W^{1,n}(U)$, for $U=B^0(0,1)$.

Proof.

$$\begin{aligned}\int_U |u(x)|^n dx &= \int_U \left|\log\log\left(1+\frac{1}{|x|}\right)\right|^n dx\\ &= \int_0^1 \left|\log\log\left(1+\frac{1}{r}\right)\right|^n n\alpha(n) r^{n-1}dr\\ &= n\alpha(n)\int_0^1 \left|\log\log\left(1+\frac{1}{r}\right)\right|^n r^{n-1}dr.\end{aligned}$$

By substitution of varibales, we have

$$\lim_{r\to 0^+}\left|\log\log\left(1+\frac{1}{r}\right)\right|^n r^{n-1}=0,$$

so $\int_U |u(x)|^n dx<\infty$, thus

$$u(x)\in L^n(U).$$

By Hölder's inequality, u is locally summable.

Set

$$v_i(x)=\frac{-x_i}{\left[\log\left(1+\frac{1}{|x|}\right)\right](|x|+1)|x|^2},\quad x\in U-\{0\}.$$

Then

$$\begin{aligned}\int_U |v_i(x)|^n dx &= \int_U \frac{1}{\left[\log^n\left(1+\frac{1}{|x|}\right)\right](|x|+1)^n|x|^n}dx\\ &= n\alpha(n)\int_0^1 \frac{1}{\left[\log^n\left(1+\frac{1}{r}\right)\right](r+1)^n r}dr\\ &= n\alpha(n)\int_{\log 2}^{\infty}\frac{1}{s^n(r+1)^{n-1}}ds \quad \left(s=\log\left(1+\frac{1}{r}\right)\right)\\ &\le n\alpha(n)\int_{\log 2}^{\infty}\frac{1}{s^n}ds\\ &<\infty.\end{aligned}$$

For any test function $\phi\in C_c^\infty(U)$ and $\epsilon\in\left(0,\frac{1}{e-1}\right)$,

$$\int_{U-B(0,\epsilon)} u\phi_{x_i}dx=-\int_{U-B(0,\epsilon)} v_i\phi dx+\int_{\partial B(0,\epsilon)} u\phi v^i dS.$$

$$\lim_{\epsilon\to 0}\int_{\partial B(0,\epsilon)} u\phi v^i dS=0,$$

thus we have

$$\int_U u\phi_{x_i} dx = -\int_U v_i \phi dx.$$

Therefore $v_1, \dots, v_n \in L^n(U)$ are weak derivatives of u.

Hence $u \in W^{1,n}(U)$. $\square$

15. Fix $\alpha > 0$ and let $U = B^0(0,1)$. Show there exists a constant C, depending only on n and α, such that

$$\int_U u^2 dx \le C \int_U |Du|^2 dx,$$

provided

$$|\{x \in U \mid u(x) = 0\}| \ge \alpha, \quad u \in H^1(U).$$

Proof. According to Theorem 1 in §5.8 (Poincaré's inequality), we have

$$\|u\|_{L^2(U)} - \|(u)_U\|_{L^2(U)} \le \|u - (u)_U\|_{L^2(U)} \le C(n)\|Du\|_{L^2(U)}.$$

$$\begin{aligned}
\|(u)_U\|_{L^2(U)} &= \left(\int_U (u)_U^2 dx\right)^{\frac{1}{2}} \\
&= \sqrt{\alpha(n)}|(u)_U| \\
&\le \frac{1}{\sqrt{\alpha(n)}} \int_{\operatorname{spt} u} |u| dx \\
&\le \frac{1}{\sqrt{\alpha(n)}} \|1\|_{L^2(\operatorname{spt} U)} \|u\|_{L^2(\operatorname{spt} U)} \quad \text{(Hölder's inequality)} \\
&\le \sqrt{1 - \frac{\alpha}{\alpha(n)}} \|u\|_{L^2(U)}.
\end{aligned}$$

Thus we have

$$\left(1 - \sqrt{1 - \frac{\alpha}{\alpha(n)}}\right) \|u\|_{L^2(U)} \le C(n)\|Du\|_{L^2(U)},$$

$$\int_U u^2 dx \le \frac{C^2(n)}{\left(1 - \sqrt{1 - \frac{\alpha}{\alpha(n)}}\right)^2} \int_U |Du|^2 dx.$$

$\square$

17. (Chain rule) Assume $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose U is bounded and $u \in W^{1,p}(U)$ for some $1 \le p \le \infty$. Show

$$v := F(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F'(u)u_{x_i} \quad (i = 1, \dots, n).$$

Proof. (1) For $1 \leq p < \infty$, there exists fuctions $u_m \in C^\infty(U) \cap W^{1,p}(U)$ such that

$$u_m \to u \text{ in } W^{1,p}(U),$$

according to Theorem 2 in §5.3. Obviously,

$$u_m \to u \text{ a.e. on } U.$$

F and F' are continuous, so $F(u_m) \to F(u)$ and $F'(u_m) \to F'(u)$ a.e. on U.

$$\begin{aligned}
\int_U u\phi_{x_i}dx &= \int_U \lim_{m\to\infty} u_m \phi_{x_i} dx \\
&= \lim_{m\to\infty} \int_U u_m \phi_{x_i} dx \quad \text{([$^+$10] P114. 勒贝格控制收敛定理)} \\
&= -\lim_{m\to\infty} \int_U \frac{\partial u_m}{\partial x_i} \phi dx \\
&= -\int_U \lim_{m\to\infty} \frac{\partial u_m}{\partial x_i} \phi dx \quad \text{([$^+$10] P114. 勒贝格控制收敛定理)}
\end{aligned}$$

for all test functions $\phi \in C_c^\infty(U)$, so $\frac{\partial u_m}{\partial x_i} \to u_{x_i}$ a.e. on U.

$$\begin{aligned}
&\left| F'(u_m)\frac{\partial u_m}{\partial x_i} - F'(u)u_{x_i} \right| \\
=& \left| F'(u_m)\left(\frac{\partial u_m}{\partial x_i} - u_{x_i}\right) + [F'(u_m) - F'(u)]u_{x_i} \right| \\
=& |F'(u_m)| \left| \frac{\partial u_m}{\partial x_i} - u_{x_i} \right| + |F'(u_m) - F'(u)||u_{x_i}|,
\end{aligned}$$

so $F'(u_m)\frac{\partial u_m}{\partial x_i} \to F'(u)u_{x_i}$ a.e. on U. Therefore

$$\begin{aligned}
\int_U F(u)\phi_{x_i}dx &= \int_U \lim_{m\to\infty} F(u_m)\phi_{x_i}dx \\
&= \lim_{m\to\infty} \int_U F(u_m)\phi_{x_i}dx \\
&\quad \text{([$^+$10] P114. 勒贝格控制收敛定理)} \\
&= -\lim_{m\to\infty} \int_U F'(u_m)\frac{\partial u_m}{\partial x_i}\phi dx \\
&= -\int_U \lim_{m\to\infty} F'(u_m)\frac{\partial u_m}{\partial x_i}\phi dx \\
&\quad \text{([$^+$10] P114. 勒贝格控制收敛定理)} \\
&= -\int_U F'(u)u_{x_i}\phi dx.
\end{aligned}$$

Thus a weak derivative of $v = F(u)$ is

$$v_{x_i} = F'(u)u_{x_i} \in L^p(U).$$

$$F(u) - F(0) \le |F(u) - F(0)| \le (\sup F')|u|,$$

thus

$$F(u) \le (\sup F')|u| + F(0),$$

so $F(u) \in L^p(U)$.

Hence

$$v = F(u) \in W^{1,p}(U).$$

□

18. Assume $1 \le p \le \infty$ and U is bounded.

(a) Prove that if $u \in W^{1,p}(U)$, then $|u| \in W^{1,p}(U)$.

(b) Prove $u \in W^{1,p}(U)$ implies $u^+, u^- \in W^{1,p}(U)$, and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \le 0\}, \end{cases}$$
$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \ge 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

(Hint: $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, for

$$F_\varepsilon(z) := \begin{cases} \left(z^2 + \varepsilon^2\right)^{1/2} - \varepsilon & \text{if } z \ge 0 \\ 0 & \text{if } z < 0.) \end{cases}$$

(c) Prove that if $u \in W^{1,p}(U)$, then

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

Proof. (a) According to (b),

$$|u| = u^+ + u^- \in W^{1,p}(U).$$

(b) Obviously $u^+ \in L^p(U)$.

Obviously $F_\varepsilon : \mathbb{R} \to \mathbb{R}$ is C^1, and

$$F_\varepsilon'(z) = \begin{cases} \frac{z}{\sqrt{z^2+\varepsilon^2}} & \text{if } z \geq 0 \\ 0 & \text{if } z < 0 \end{cases}$$

is bounded. According to Problem 17, we have

$$v := F_\varepsilon(u) \in W^{1,p}(U) \quad \text{and} \quad v_{x_i} = F_\varepsilon'(u) u_{x_i} \quad (i = 1, \ldots, n).$$

Thus

$$\begin{aligned}
\int_U u^+ \phi_{x_i} dx &= \int_U \left(\lim_{\varepsilon \to 0} F_\varepsilon(u) \right) \phi_{x_i} dx \\
&= \lim_{\varepsilon \to 0} \int_U F_\varepsilon(u) \phi_{x_i} dx \quad (\text{[}^{+}\text{10] P114. 勒贝格控制收敛定理}) \\
&= -\lim_{\varepsilon \to 0} \int_U F_\varepsilon'(u) u_{x_i} \phi dx \\
&= -\int_U \lim_{\varepsilon \to 0} F_\varepsilon'(u) u_{x_i} \phi dx \quad (\text{[}^{+}\text{10] P114. 勒贝格控制收敛定理}) \\
&= \begin{cases} -\int_U u_{x_i} \phi dx & \text{if } u > 0 \\ 0 & \text{if } u \leq 0. \end{cases}
\end{aligned}$$

Hence, $u^+ \in W^{1,p}(U)$ and

$$Du^+ = \begin{cases} Du & \text{a.e. on } \{u > 0\} \\ 0 & \text{a.e. on } \{u \leq 0\}. \end{cases}$$

$u^- = (-u)^+$, so $u^- \in W^{1,p}(U)$, and

$$Du^- = \begin{cases} 0 & \text{a.e. on } \{u \geq 0\} \\ -Du & \text{a.e. on } \{u < 0\}. \end{cases}$$

(c) $Du = D(u^+ - u^-) = Du^+ - Du^-$, so

$$Du = 0 \text{ a.e. on the set } \{u = 0\}.$$

□

Chapter 6

SECOND-ORDER ELLIPTIC EQUATIONS

In the following exercises we assume the coefficients of the various PDE are smooth and satisfy the uniform ellipticity condition. Also $U \subset \mathbb{R}^n$ is always an open, bounded set, with smooth boundary ∂U.

1. Consider Laplace's equation with potential function c :

$$(*) \qquad -\Delta u + cu = 0,$$

and the divergence structure equation:

$$(**) \qquad -\operatorname{div}(aDv) = 0,$$

where the function a is positive.

(a) Show that if u solves $(*)$ and $w > 0$ also solves $(*)$, then $v := u/w$ solves $(**)$ for $a := w^2$.

(b) Conversely, show that if v solves $(**)$, then $u := va^{1/2}$ solves $(*)$ for some potential c.

Proof. (a)

$$\begin{aligned} v_{x_i} &= \frac{u_{x_i} w - u w_{x_i}}{w^2}, \\ (a v_{x_i})_{x_i} &= (u_{x_i} w - u w_{x_i})_{x_i} \\ &= u_{x_i x_i} w + u_{x_i} w_{x_i} - u_{x_i} w_{x_i} - u w_{x_i x_i} \end{aligned}$$

$$\begin{aligned} &= u_{x_i x_i} w - u w_{x_i x_i}, \\ -\operatorname{div}(aDv) &= -(\Delta u) w + u \Delta w \\ &= -cuw + ucw \\ &= 0. \end{aligned}$$

(b)

$$\begin{aligned} 0 &= \operatorname{div}(aDv) \\ &= Da \cdot Dv + a \operatorname{div}(Dv) \\ &= Da \cdot Dv + a\Delta v, \end{aligned}$$

thus

$$\begin{aligned} \Delta u &= \Delta(va^{1/2}) \\ &= (\Delta v)a^{1/2} + 2Dv \cdot D(a^{1/2}) + v\Delta(a^{1/2}) \\ &= (\Delta v)a^{1/2} + a^{-1/2} Dv \cdot Da + v\Delta(a^{1/2}) \\ &= (\Delta v)a^{1/2} - a^{1/2}\Delta v + v\Delta(a^{1/2}) \\ &= v\Delta(a^{1/2}). \end{aligned}$$

If $c = a^{-1/2}\Delta(a^{1/2})$, then

$$-\Delta u + cu = -v\Delta(a^{1/2}) + v\Delta(a^{1/2}) = 0.$$

□

2. Let

$$Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu.$$

Prove that there exists a constant $\mu > 0$ such that the corresponding bilinear form $B[\cdot,\cdot]$ satisfies the hypotheses of the Lax-Milgram Theorem, provided

$$c(x) \geq -\mu \quad (x \in U).$$

Proof.

$$B[u,v] = \int_U \sum_{i,j=1}^{n} a^{ij} u_{x_i} v_{x_j} + cuvdx$$

for $u, v \in H_0^1(U)$.

$$\begin{aligned}
|B[u,v]| &\leq \int_U \left| \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} \right| dx + \int_U |cuv| dx \\
&\leq \|((a_{ij}))\|_{L^\infty(U)} \int_U |Du||Dv| dx + \|c\|_{L^\infty(U)} \int_U |uv| dx \\
&\text{(for } ((a_{ij})) \text{ is symmetric and positive definite)} \\
&\leq \|((a_{ij}))\|_{L^\infty(U)} \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} + \|c\|_{L^\infty(U)} \|u\|_{L^2(U)} \|v\|_{L^2(U)} \\
&\text{(Hölder's inequality)} \\
&\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)},
\end{aligned}$$

where $\alpha = \|((a_{ij}))\|_{L^\infty(U)} + \|c\|_{L^\infty(U)}$.

There exists a constant $\theta > 0$ such that for all $u \in H_0^1(U)$,

$$\begin{aligned}
\theta \|Du\|^2_{L^2(U)} &= \theta \int_U |Du|^2 dx \\
&\leq \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} dx \quad \text{(uniform ellipticity condition))} \\
&= B[u,u] - \int_U cu^2 dx \\
&\leq B[u,u] + \mu \|Du\|^2_{L^2(U)} \\
&\leq B[u,u] + C\mu \|Du\|^2_{L^2(U)} \quad \text{(Poincaré's inequality)}.
\end{aligned}$$

Let $\mu = \frac{\theta}{2C}$. Then we have $\|Du\|^2_{L^2(U)} \leq \frac{2}{\theta} B[u,u]$. Therefore

$$\begin{aligned}
\|u\|^2_{H_0^1(U)} &= \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} \\
&\leq (C+1) \|Du\|^2_{L^2(U)} \quad \text{(Poincaré's inequality)} \\
&\leq \frac{2(C+1)}{\theta} B[u,u].
\end{aligned}$$

□

3. A function $u \in H_0^2(U)$ is a weak solution of this boundary-value problem for the *biharmonic equation*

$$(*) \qquad \begin{cases} \Delta^2 u = f & \text{in } U \\ u = \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

provided

$$\int_U \Delta u \Delta v dx = \int_U f v dx$$

for all $v \in H_0^2(U)$. Given $f \in L^2(U)$, prove that there exists a unique weak solution of $(*)$.

Proof. 1. Set

$$B[u,v] := \int_U \Delta u \Delta v \mathrm{d}x.$$

Then

$$\begin{aligned}
|B[u,v]| &\leq \int_U |\Delta u||\Delta v| \mathrm{d}x \\
&\leq \|\Delta u\|_{L^2(U)} \|\Delta v\|_{L^2(U)} \quad \text{(Hölder's inequality)} \\
&= \left(\int_U \left(\sum_{i=1}^n u_{x_i x_i} \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_U \left(\sum_{i=1}^n v_{x_i x_i} \right)^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq n \left(\int_U \sum_{i=1}^n u_{x_i x_i}^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_U \sum_{i=1}^n v_{x_i x_i}^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&\leq n \left(\int_U \sum_{|\alpha|=2} |D^\alpha u|^2 \mathrm{d}x \right)^{\frac{1}{2}} \left(\int_U \sum_{|\alpha|=2} |D^\alpha v|^2 \mathrm{d}x \right)^{\frac{1}{2}} \\
&= n \|D^2 u\|_{L^2(U)} \|D^2 v\|_{L^2(U)} \\
&\leq n \|u\|_{H_0^2(U)} \|v\|_{H_0^2(U)}.
\end{aligned}$$

2. Let $u \in C_c^\infty(U)$. According to the Poincaré's inequality in §5.6.1, we have

$$\begin{aligned}
\|u\|_{L^2(U)}^2 &\leq C \|Du\|_{L^2(U)}^2, \\
\|Du\|_{L^2(U)}^2 &= \int_U \sum_{i=1}^n |u_{x_i}|^2 dx \\
&= \sum_{i=1}^n \|u_{x_i}\|_{L^2(U)}^2 \\
&\leq C \sum_{i=1}^n \|D(u_{x_i})\|_{L^2(U)}^2 \\
&= C \sum_{i=1}^n \int_U \sum_{j=1}^n |u_{x_i x_j}|^2 dx
\end{aligned}$$

$$= C\|D^2u\|^2_{L^2(U)}.$$

$$\begin{aligned} B[u,u] &= \int_U |\Delta u|^2 dx = \int_U \sum_{i,j=1}^n u_{x_ix_i}u_{x_jx_j}dx \\ &= \int_U \sum_{i,j=1}^n u_{x_ix_j}u_{x_ix_j}dx = \|D^2u\|^2_{L^2(U)}. \end{aligned}$$

Thus

$$\begin{aligned} \|u\|^2_{H_0^2(U)} &= \int_U |u|^2dx + \int_U |Du|^2dx + \int_U |D^2u|^2dx \\ &= \|u\|^2_{L^2(U)} + \|Du\|^2_{L^2(U)} + \|D^2u\|^2_{L^2(U)} \\ &\le (C+1)\|Du\|^2_{L^2(U)} + \|D^2u\|^2_{L^2(U)} \\ &\le (1+C+C^2)\|D^2u\|^2_{L^2(U)} \\ &= (1+C+C^2)B[u,u], \end{aligned}$$

so

$$\beta\|u\|^2_{H_0^2(U)} \le B[u,u],$$

where $\beta = \frac{1}{1+C+C^2}$.

3. If $u \in H_0^2(U)$, then there exist functions $u_m \in C_c^\infty(U)$ such that $\|u_m - u\|_{H_0^2(U)} \to 0$, according to the definition of $W_0^{k,p}(U)$.

$$|\|u_m\|_{H_0^2(U)} - \|u\|_{H_0^2(U)}| \le \|u_m - u\|_{H_0^2(U)} \to 0,$$

so

$$\|u_m\|_{H_0^2(U)} \to \|u\|_{H_0^2(U)}, \quad \beta\|u_m\|^2_{H_0^2(U)} \to \beta\|u\|^2_{H_0^2(U)};$$

$$\begin{aligned} &|B[u_m,u_m] - B[u,u]| \\ &= \left|\|\Delta u_m\|^2_{L^2(U)} - \|\Delta u\|^2_{L^2(U)}\right| \\ &= \left(\|\Delta u_m\|_{L^2(U)} + \|\Delta u\|_{L^2(U)}\right)\left|\|\Delta u_m\|_{L^2(U)} - \|\Delta u\|_{L^2(U)}\right| \\ &\le C\left|\|\Delta u_m\|_{L^2(U)} - \|\Delta u\|_{L^2(U)}\right| \quad (C \text{ is independent of } m.) \\ &= C\|\Delta u_m - \Delta u\|_{L^2(U)} \\ &= C\|\Delta(u_m - u)\|_{L^2(U)} \quad \text{(by Theorem 1 (ii) in §5.2)} \\ &\le C\|u_m - u\|_{H_0^2(U)} \to 0, \end{aligned}$$

so

$$B[u_m, u_m] \to B[u, u].$$

$$\beta \|u_m\|^2_{H_0^2(U)} \leq B[u_m, u_m],$$

thus

$$\beta \|u\|^2_{H_0^2(U)} \leq B[u, u].$$

4. If $\|u\|_{H_0^2(U)} \leq 1$, then $\|u\|_{L^2(U)} \leq 1$, thus

$$\begin{aligned}
\left| \int_U fu dx \right| &\leq \int_U |fu| dx \\
&\leq \|f\|_{L^2(U)} \|u\|_{L^2(U)} \quad \text{(Hölder's inequality)} \\
&\leq \|f\|_{L^2(U)} < \infty.
\end{aligned}$$

Therefore the linear map

$$\begin{aligned}
A : H_0^2(U) &\to \mathbb{R}, \\
u &\mapsto \int_U fu dx
\end{aligned}$$

is a bounded linear functional.

5. According to Lax-Milgram Theorem, there exists a unique element $u \in H_0^2(U)$ such that

$$B[u, v] = \int_U \Delta u \Delta v dx = \int_U fv dx,$$

thus there exists a unique weak solution of $(*)$. $\square$

4. Assume U is connected. A function $u \in H^1(U)$ is a weak solution of *Neumann's problem*

$$(*) \qquad \begin{cases} -\Delta u = f & \text{in } U \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

if

$$\int_U Du \cdot Dv dx = \int_U fv dx$$

for all $v \in H^1(U)$. Suppose $f \in L^2(U)$. Prove $(*)$ has a weak solution if and only if

$$\int_U f dx = 0.$$

Proof. **Sufficiency.** 1. Set

$$\begin{aligned} A\colon H^1(U) &\to \mathbb{R} \\ u &\mapsto \int_U u dx. \end{aligned}$$

By Hölder's inequality, A is a bounded linear functional, for U is bounded. Therefore $N(A) \subset H^1(U)$ is closed, by Theorem 2 in §8.1 of [19]. So $N(A) \subset H^1(U)$ is a Hilbert space, with the inner product induced from $H^1(U)$.

2. Set

$$\begin{aligned} B\colon N(A) \times N(A) &\to \mathbb{R} \\ (u,v) &\mapsto \int_U Du \cdot Dv dx. \end{aligned}$$

Then for all $u, v \in N(A)$,

$$\begin{aligned} |B[u,v]| &\le \int_U |Du \cdot Dv| dx \\ &\le \int_U |Du||Dv| dx \quad \text{(discrete version of Hölder's inequality)} \\ &\le \|Du\|_{L^2(U)} \|Dv\|_{L^2(U)} \quad \text{(Hölder's inequality)} \\ &\le \|u\|_{N(A)} \|v\|_{N(A)}, \\ \|u\|_{N(A)}^2 &= \|u\|_{L^2(U)}^2 + \|Du\|_{L^2(U)}^2 \\ &\le C\|Du\|_{L^2(U)}^2 \quad \text{(by Theorem 1 in §5.8 (Poincaré's inequality))} \\ &= C \int_U Du \cdot Du dx \\ &= CB[u,u]. \end{aligned}$$

The linear functional

$$\begin{aligned} f\colon N(A) &\to \mathbb{R} \\ v &\mapsto \int_U f v dx \end{aligned}$$

is bounded, by Hölder's inequality. Hence by Lax-Milgram Theorem, there exists a unique element $u_f \in N(A) \subset H^1(U)$ such that

$$B[u_f, v] = \int_U f v dx$$

for all $v \in N(A)$.

3. For all $v \in H^1(U)$,

$$\begin{aligned}
\int_U Du_f \cdot Dv dx &= \int_U Du_f \cdot D[v - (v)_U] dx \\
&= B[u_f, v - (v)_U] \\
&= \int_U f[v - (v)_U] dx \\
&= \int_U fv dx \quad (\int_U f dx = 0).
\end{aligned}$$

Thus u_f is a weak solution of $(*)$. Of course, $u_f + C$ is also a weak solution of $(*)$ for any constant $C \in \mathbb{R}$.

Necessity. Let $v = 1 \in H^1(U)$. Then we get

$$\int_U f dx = 0.$$

□

7. Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + c(u) = f \quad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c : \mathbb{R} \to \mathbb{R}$ is smooth, with $c(0) = 0$ and $c' \geq 0$. Prove $u \in H^2(\mathbb{R}^n)$.

(Hint: Mimic the proof of Theorem 1 in §6.3.1, but without the cutoff function ζ.)

Proof. 1. Set

$$B[u, v] := \int_{\mathbb{R}^n} Du \cdot Dv + c(u) v dx$$

for all $u, v \in H_0^1(\mathbb{R}^n)$. The weak solution $u \in H^1(\mathbb{R}^n)$ has compact support, thus $u \in H_0^1(\mathbb{R}^n)$. For the weak solution u and all $v \in H_0^1(\mathbb{R}^n)$,

$$B[u, v] = \int_{\mathbb{R}^n} Du \cdot Dv + c(u) v dx = \int_{\mathbb{R}^n} fv dx.$$

2. Take $v = -D_k^{-h} D_k^h u \in H_0^1(\mathbb{R}^n)$, with $h \neq 0$, $k \in \{1, \ldots, n\}$. Then

$$B[u, v] = -\int_{\mathbb{R}^n} Du \cdot D_k^{-h} D_k^h Du + c(u) D_k^{-h} D_k^h u dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} D_k^h Du \cdot D_k^h Du dx + \int_{\mathbb{R}^n} D_k^h c(u) \cdot D_k^h u dx \\
&= \int_{\mathbb{R}^n} |D_k^h Du|^2 dx + \int_{\mathbb{R}^n} D_k^h c(u) \cdot D_k^h u dx \\
&= -\int_{\mathbb{R}^n} f D_k^{-h} D_k^h u dx.
\end{aligned}
$$

$$
\begin{aligned}
D_k^h c(u)(x) \cdot D_k^h u(x) &= \frac{c(u(x+he_k)) - c(u(x))}{h} D_k^h u(x) \\
&= c'(\xi) \frac{u(x+he_k) - u(x)}{h} D_k^h u(x) \\
&= c'(\xi) |D_k^h u(x)|^2 \\
&\geq 0
\end{aligned}
$$

for some $\xi \in \mathbb{R}$, therefore

$$
\begin{aligned}
\int_{\mathbb{R}^n} |D_k^h Du|^2 dx &\leq -\int_{\mathbb{R}^n} f D_k^{-h} D_k^h u dx \\
&\leq \epsilon \int_{\mathbb{R}^n} f^2 dx + \frac{1}{4\epsilon} \int_{\mathbb{R}^n} (D_k^{-h} D_k^h u)^2 dx \quad (\epsilon > 0) \\
&\leq \epsilon \int_{\mathbb{R}^n} f^2 dx + \frac{C}{4\epsilon} \int_{\mathbb{R}^n} |D_k^h Du|^2 dx \quad \text{(Theorem 3 (i) in §5.8)}.
\end{aligned}
$$

Thus

$$
\int_{\mathbb{R}^n} |D_k^h Du|^2 dx \leq C \int_{\mathbb{R}^n} f^2 dx < \infty,
$$

for $f \in L^2(\mathbb{R}^n)$.

3. According to Theorem 3 (ii) in §5.8, $u_{x_i x_j}$ exists in the weak sense, and $\|u_{x_i x_j}\|_{L^2(\mathbb{R}^n)} \leq C$, for all $i, j \in \{1, \dots, n\}$. Thus $u \in H^2(\mathbb{R}^n)$, for $u \in H^1(\mathbb{R}^n)$ already. □

8. Let u be a smooth solution of the uniformly elliptic equation $Lu = -\sum_{i,j=1}^n a^{ij}(x) u_{x_i x_j} = 0$ in U. Assume that the coefficients have bounded derivatives.

Set $v := |Du|^2 + \lambda u^2$ and show that

$$
Lv \leq 0 \quad \text{in } U
$$

if λ is large enough. Deduce

$$
\|Du\|_{L^\infty(U)} \leq C(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).
$$

Proof. 1.

$$\begin{aligned}
(|Du|^2)_{x_i x_j} &= (Du \cdot Du)_{x_i x_j} \\
&= 2(Du \cdot (Du)_{x_i})_{x_j} \\
&= 2(Du)_{x_i} \cdot (Du)_{x_j} + 2Du \cdot (Du)_{x_i x_j}, \\
(u^2)_{x_i x_j} &= (2uu_{x_i})_{x_j} \\
&= 2u_{x_i} u_{x_j} + 2uu_{x_i x_j}, \\
0 &= D(-Lu) \\
&= \sum_{i,j=1}^n D(a^{ij} u_{x_i x_j}) \\
&= \sum_{i,j=1}^n (Da^{ij}) u_{x_i x_j} + \sum_{i,j=1}^n a^{ij} (Du)_{x_i x_j}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{1}{2} Lv &= L(|Du|^2 + \lambda u^2) \\
&= -\sum_{i,j=1}^n [a^{ij}(Du)_{x_i} \cdot (Du)_{x_j} + a^{ij} Du \cdot (Du)_{x_i x_j} \\
&\qquad + \lambda a^{ij} u_{x_i} u_{x_j} + \lambda a^{ij} u u_{x_i x_j}] \\
&\leq -\theta |D^2 u|^2 + \sum_{i,j=1}^n Du \cdot (Da^{ij}) u_{x_i x_j} - \lambda \theta |Du|^2 \\
&\leq -\theta |D^2 u|^2 + \sum_{i,j=1}^n \left(\epsilon |Du|^2 + \frac{|Da^{ij}|^2 (u_{x_i x_j})^2}{4\epsilon} \right) - \lambda \theta |Du|^2 \\
&\qquad \text{(by Cauchy's inequality with } \epsilon) \\
&= \left[\left(\sum_{i,j=1}^n \frac{|Da^{ij}|^2}{4\theta} \right) - \lambda \theta \right] |Du|^2 \quad \left(\text{Let } \epsilon = \frac{|Da^{ij}|^2}{4\theta}.\right) \\
&\leq 0 \quad \text{(Let } \lambda \text{ be large enough.).}
\end{aligned}$$

2. $v \in C^\infty(\bar{U})$ for $u \in C^\infty(\bar{U})$. $Lv \leq 0$ in U. Thus by Theorem 1 in §6.4 (Weak maximum principle),

$$\|v\|_{L^\infty(U)} = \|v\|_{L^\infty(\partial U)}.$$

Therefore

$$\begin{aligned}
\|Du\|^2_{L^\infty(U)} &= \||Du|^2\|_{L^\infty(U)} \\
&\le \|v\|_{L^\infty(U)} \\
&= \|v\|_{L^\infty(\partial U)} \\
&\le \lambda(\||Du|^2\|_{L^\infty(\partial U)} + \|u^2\|_{L^\infty(\partial U)}) \\
&= \lambda(\|Du\|^2_{L^\infty(\partial U)} + \|u\|^2_{L^\infty(\partial U)}).
\end{aligned}$$

Thus

$$\begin{aligned}
\|Du\|_{L^\infty(U)} &\le \sqrt{\lambda}\sqrt{\|Du\|^2_{L^\infty(\partial U)} + \|u\|^2_{L^\infty(\partial U)}} \\
&\le \sqrt{\lambda}(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}).
\end{aligned}$$

□

9. Assume u is a smooth solution of $Lu = -\sum_{i,j=1}^n a^{ij}u_{x_ix_j} = f$ in U, $u = 0$ on ∂U, where f is bounded. Fix $x^0 \in \partial U$. A barrier at x^0 is a C^2 function w such that

$$Lw \ge 1 \text{ in } U, \quad w\left(x^0\right) = 0, \quad w \ge 0 \text{ on } \partial U.$$

Show that if w is a barrier at x^0, there exists a constant C such that

$$\left|Du\left(x^0\right)\right| \le C\left|\frac{\partial w}{\partial \nu}\cdot\left(x^0\right)\right|.$$

Proof. Set

$$\begin{aligned}
v_1 &:= u + \|f\|_{L^\infty(U)}w \in C^2(\bar{U}), \\
v_2 &:= u - \|f\|_{L^\infty(U)}w \in C^2(\bar{U}).
\end{aligned}$$

Then

$$\begin{aligned}
Lv_1 &= f + \|f\|_{L^\infty(U)}Lw \ge 0, \\
Lv_2 &= f - \|f\|_{L^\infty(U)}Lw \le 0.
\end{aligned}$$

According to Theorem 1 in §6.4 (Weak maximum principle),

$$\min_{\bar{U}} v_1 = \min_{\partial U} v_1 = v_1(x^0) = 0,$$

$$\max_{\bar{U}} v_2 = \max_{\partial U} v_2 = v_2(x^0) = 0.$$

Thus

$$\begin{aligned}
\frac{\partial v_1}{\partial \nu}(x^0) &= \frac{\partial u}{\partial \nu}(x^0) + \|f\|_{L^\infty(U)} \frac{\partial w}{\partial \nu}(x^0) \leq 0, \\
\frac{\partial v_2}{\partial \nu}(x^0) &= \frac{\partial u}{\partial \nu}(x^0) - \|f\|_{L^\infty(U)} \frac{\partial w}{\partial \nu}(x^0) \geq 0.
\end{aligned}$$

Hence

$$|Du(x^0)| = \left|\frac{\partial u}{\partial \nu}(x^0)\right| \leq \|f\|_{L^\infty(U)} \left|\frac{\partial w}{\partial \nu}(x^0)\right|.$$

The reason of $|Du(x^0)| = \left|\frac{\partial u}{\partial \nu}(x^0)\right|$ is that u is constant on ∂U ($u = 0$ on ∂U). $\square$

10. Assume U is connected. Use (a) energy methods and (b) the maximum principle to show that the only smooth solutions of the Neumann boundary-value problem

$$\begin{cases} -\Delta u = 0 & \text{in } U \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial U \end{cases}$$

are $u \equiv C$, for some constant C.

Proof. (a) Let u be any smooth of this problem. Then

$$0 = -\int_U u \Delta u dx = \int_U Du \cdot Du dx - \int_{\partial U} \frac{\partial u}{\partial \nu} u dS = \int_U |Du|^2 dx.$$

Thus $Du = 0$ on $\bar{U}$. Hence $u = C$.

(b) Assume that u can not attain its maximum over $\bar{U}$ at interior points. Then $u(x) < u(x^0)$ for some point $x^0 \in \partial U$ and all points $x \in U$. According to Hopf's Lemma, $\frac{\partial u}{\partial \nu}(x^0) > 0$, which is a contradiction. Hence u attains its maximum over $\bar{U}$ at an interior point. Thus u is constant within U, according to Theorem 3 (Strong maximum principle) in §6.4. $\square$

11. Assume $u \in H^1(U)$ is a bounded weak solution of

$$-\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} = 0 \quad \text{in } U.$$

Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex and smooth, and set $w = \phi(u)$. Show w is a weak subsolution; that is, $B[w, v] \leq 0$ for all $v \in H_0^1(U), v \geq 0$.

Proof.
$$B[u,v]=\int_U\sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j}dx$$
for all $u\in H^1(U)$, $v\in H_0^1(U)$. For all $v\in C_c^\infty(U)$,
$$\begin{aligned}B[w,v]&=\int_U\sum_{i,j=1}^n a^{ij}[\phi(u)]_{x_i}v_{x_j}dx\\&=\int_U\sum_{i,j=1}^n a^{ij}\phi'(u)u_{x_i}v_{x_j}dx \quad \text{(Problem 17 in §5.10)}\\&=\int_U\sum_{i,j=1}^n a^{ij}u_{x_i}(\phi'(u)v)_{x_j}dx-\int_U\sum_{i,j=1}^n a^{ij}u_{x_i}u_{x_j}\phi''(u)vdx\\&\qquad\text{(Problem 17 in §5.10, and Theorem 1 (iv) in §5.2)}\\&\le 0-\theta\int_U|Du|^2\phi''(u)vdx \quad \text{(Definition of weak solution,}\\&\qquad\text{and uniform ellipticity condition)}\\&\le 0.\end{aligned}$$
□

12. We say that the uniformly elliptic operator
$$Lu=-\sum_{i,j=1}^n a^{ij}u_{x_ix_j}+\sum_{i=1}^n b^iu_{x_i}+cu$$
satisfies the weak maximum principle if for all $u\in C^2(U)\cap C(\bar U)$
$$\begin{cases}Lu\le 0 & \text{in } U\\ u\le 0 & \text{on } \partial U\end{cases}$$
implies that $u\le 0$ in U.

Suppose that there exists a function $v\in C^2(U)\cap C(\bar U)$ such that $Lv\ge 0$ in U and $v>0$ on $\bar U$. Show that L satisfies the weak maximum principle.

(Hint: Find an elliptic operator M with no zeroth-order term such that $w:=u/v$ satisfies $Mw\le 0$ in the region $\{u>0\}$. To do this, first compute $(v^2w_{x_i})_{x_j}$.)

Proof. 1. Defining Mw.
$$w_{x_i}=\left(\frac{u}{v}\right)_{x_i}=\frac{u_{x_i}v-uv_{x_i}}{v^2},$$

$$\begin{aligned}
(v^2 w_{x_i})_{x_j} &= (u_{x_i} v - u v_{x_i})_{x_j} \\
&= u_{x_i x_j} v + u_{x_i} v_{x_j} - u_{x_j} v_{x_i} - u v_{x_i x_j}, \\
-\sum_{i,j=1}^{n} a^{ij} (v^2 w_{x_i})_{x_j} &= -v \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + u \sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} \\
&= vLu - uLv - \sum_{i=1}^{n} b^i (u_{x_i} v - u v_{x_i}) - cvu + cuv \\
&= vLu - uLv - v^2 \sum_{i=1}^{n} b^i w_{x_i} \\
&= Mw - v^2 \sum_{i=1}^{n} b^i w_{x_i},
\end{aligned}$$

where $Mw := vLu - uLv \leq 0$ in the open set $\{u > 0\} \subset U$. Here Mw is defined as a whole, not M acting on w.

2. Studying Mw. For

$$\begin{aligned}
(v^2 w_{x_i})_{x_j} &= 2v v_{x_j} w_{x_i} + v^2 w_{x_i x_j}, \\
Mw &= -\sum_{i,j=1}^{n} a^{ij} (v^2 w_{x_i})_{x_j} + v^2 \sum_{i=1}^{n} b^i w_{x_i} \\
&= -v^2 \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} - 2v \sum_{i,j=1}^{n} a^{ij} v_{x_j} w_{x_i} + v^2 \sum_{i=1}^{n} b^i w_{x_i} \\
&= -v^2 \sum_{i,j=1}^{n} a^{ij} w_{x_i x_j} - v \sum_{i=1}^{n} \left[2 \left(\sum_{j=1}^{n} a^{ij} v_{x_j} \right) + v b^i \right] w_{x_i},
\end{aligned}$$

where

$$M := -v^2 \sum_{i,j=1}^{n} a^{ij} \frac{\partial^2}{\partial x_i \partial x_j} + v \sum_{i=1}^{n} \left[-2 \left(\sum_{j=1}^{n} a^{ij} v_{x_j} \right) + v b^i \right] \frac{\partial}{\partial x_i}$$

is uniformly elliptic.

3. Using Mw and weak maximum principle. According to Theorem 1 (Weak maximum principle) in §6.4,

$$0 < \max_{\{u>0\}} \frac{u}{v} = \max_{\{u>0\}} w = \max_{\partial\{u>0\}} w = 0$$

if $\{u > 0\} \neq \emptyset$. Therefore $\{u > 0\} = \emptyset$, i.e., $u \leq 0$ in U. Hence L satisfies the weak maximum principle. □

13. (Courant minimax principle) Let $L = -\sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j}$, where $((a^{ij}))$ is symmetric. Assume the operator L, with zero boundary conditions, has eigenvalues $0 < \lambda_1 < \lambda_2 \leq \cdots$. Show

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2}=1}} B[u,u] \quad (k = 1, 2, \ldots).$$

Here Σ_{k-1} denotes the collection of $(k-1)$-dimensional subspaces of $H_0^1(U)$.

Remark. $S^\perp$ is the orthogonal complement of S in $H_0^1(U)$ with respect to the $L^2(U)$ inner product.

Lemma 1. Assume that

(i) $U \subset \mathbb{R}^n$ is open and bounded;

(ii) $Lu = -\sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j}$, where $a^{ij} \in C^\infty(\bar{U})$;

(iii) $a^{ij} = a^{ji}$, $i, j = 1, \ldots, n$;

(iv) the uniform ellipticity condition holds;

(v) $\{\lambda_k\}_{k=1}^\infty$ and $\{w_k\}_{k=1}^\infty$ are the eigenvalues and eigenvectors of L, with $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$, and $Lw_k = \lambda_k w_k$, $k = 1, 2, \ldots$.

Then

(6.1)

$$\lambda_k = \min\{B[u,u] \mid u \in H_0^1(U), \|u\|_{L^2(U)} = 1, (u, w_1) = \cdots = (u, w_{k-1}) = 0\}$$

for $k = 1, 2, \ldots$.

Proof of Lemma 1. Let $u \in H_0^1(U)$, $\|u\|_{L^2(U)} = 1$, and $(u, w_1) = \cdots = (u, w_{k-1}) = 0$. We can write

$$u = \sum_{i=k}^\infty d_i w_i,$$

by (8) in §6.5. Then

$$\begin{aligned} B[u,u] &= \sum_{i=k}^\infty d_i^2 \lambda_i \quad \text{(by (6) in §6.5)} \\ &\geq \lambda_k \quad \text{(by (9) in §6.5).} \end{aligned}$$

As $B[w_k, w_k] = \lambda_k$ ((6) in §6.5), we obtain formula (6.1). □

Lemma 2. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of the Hilbert space H and $X = \operatorname{span}\{x_1, \ldots, x_{k-1}\}$ be a $k-1$ dimensional linear subspace of H. Then there exists a vector

$$0 \neq y = \sum_{i=1}^{k} y_i e_i \in X^{\perp}.$$

Proof of Lemma 2. Write

$$x_j = \sum_{i=1}^{\infty} x_{ij} e_i, j = 1, \cdots, k-1$$

and

$$\tilde{x}_j = \sum_{i=1}^{k} x_{ij} e_i \in Y := \operatorname{span}\{e_1, \cdots, e_k\}, j = 1, \cdots, k-1.$$

Then there exists a vector

$$0 \neq y = \sum_{i=1}^{k} y_i e_i \in Y$$

such that

$$(y, \tilde{x}_1) = \cdots = (y, \tilde{x}_{k-1}) = 0.$$

For $j \geq k+1$,

$$(y, e_j) = \left(\sum_{i=1}^{k} y_i e_i, e_j \right) = 0.$$

Therefore

$$(y, x_1) = \cdots = (y, x_{k-1}) = 0.$$

Hence $y \in X^{\perp}$. $\square$

Proof of Problem 13. We obtain

$$\lambda_k \leq \sup_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^{\perp} \\ \|u\|_{L^2(U)} = 1}} B[u, u] \quad (k = 1, 2, \ldots)$$

by Lemma 1.

For any $S \in \Sigma_{k-1}$, there exists a $w = \sum_{i=1}^{k} \alpha_i w_i \in S^{\perp}$ satisfying $\|w\|_{L^2(U)} = 1$, according to Lemma 2. Thus

$$\min_{\substack{u \in S^{\perp} \\ \|u\|_{L^2(U)}}} B[u, u] \leq B[w, w]$$

$$
\begin{aligned}
&= \sum_{i=1}^{k} \alpha_i^2 \lambda_i \quad \text{(by (6) and (7) in §6.5)} \\
&\leq \lambda_k \quad \text{(by (9) in §6.5).}
\end{aligned}
$$

Thus

$$\lambda_k \geq \sup_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B[u,u] \quad (k = 1, 2, \dots).$$

Hence

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min_{\substack{u \in S^\perp \\ \|u\|_{L^2(U)}=1}} B[u,u] \quad (k = 1, 2, \dots).$$

$\square$

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