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$$E(\xi_1 \cdots \xi_n) = E\xi_1 \cdots E\xi_n.$$

4. If $\xi \le \eta$ and the exceptions of ξ and η exist, then $E\xi \le E\eta$.

Example

Suppose that

$$P(\xi = m) = \frac{\binom{M}{m}\binom{N-M}{n-m}}{\binom{N}{n}}, \quad m = 0, 1, \cdots, n.$$

$(n \le M \le N)$. Find $E\xi$.

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$$E\xi = E\left[\sum_{i=1}^n \xi_i\right] = \sum_{i=1}^n E\xi_i = \frac{nM}{N}.$$

Example

Suppose that $\xi_1, \cdots, \xi_n$ are independent identically distributed positive random variables with a common density function $f(x)$. Show for any $1 \le k \le n$,

$$E\frac{\xi_1 + \cdots + \xi_k}{\xi_1 + \cdots + \xi_n} = \frac{k}{n}.$$

Proof.

Proof. Notice that $\frac{\xi_k}{\xi_1+\cdots+\xi_n}$ is positive and

$$E\frac{\xi_k}{\xi_1+\cdots+\xi_n}$$
$$=\int_0^\infty\cdots\int_0^\infty\frac{x_k}{x_1+\ldots+x_n}f(x_1)\cdots f(x_n)dx_1\cdots dx_n$$

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= & \int_0^\infty \cdots \int_0^\infty \frac{y_1}{y_1+\ldots+y_n} f(y_1)\cdots f(y_n)dy_1\cdots dy_n
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Example

A grove of 52 trees is arranged in a circular fashion. If a total of 15 chipmunks (花栗鼠) live in these tress, show that there is a group of 7 consecutive trees that together house at least 3 chipmunks.

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解：给定树j,记它连同它旁边按顺时针方向排列的6颗构成一个邻域U_j，生活在U_j中的chipmunks个数记为Y_j.

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解: 给定树j,记它连同它旁边按顺时针方向排列的6颗构成一个邻域U_j, 生活在U_j中的chipmunks个数记为Y_j. 我们只要证明$EY_j > 2\ \forall j$, 就说明了存在一个树,使得至少有3个chipmunks生活在此树的邻域中.

事实上，如果$Y_j \le 2\ \forall j$，那么$\sum_{j=1}^{52} Y_j \le 2 * 52$.

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这与

$$\sum_{j=1}^{52} EY_j > 2*52$$

矛盾.

下面求EY_j.

$$令\ \ X_i = \begin{cases} 1, & \text{if chipmunk i live in } U_j, \\ 0, & \text{otherwise.} \end{cases}$$

则$Y_j = \sum_{i=1}^{15} X_i$.

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$$EX_i = P(X_i = 1) = \frac{7}{52}.$$

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$$所以EY_j=\sum_{i=1}^{15}EX_i=\frac{105}{52}>2.$$

Example

Make a census on some kind of disease in a community with large population. Now check blood for N citizens in two ways: (1) each person each time, so need check N times. (2) check the mixture of bloods of a group of k people. If the outcome reports no virus, that means all these k people are not of this disease; while if the outcome reports virus, then each person from this group is checked again, so k people need check $k+1$ times in this way. Which way may decrease the number of checks?

Solution. Consider the second way. Denote by ξ the number of times each person needs check in a group of k people in the second way. Then

$$\xi = \begin{cases} 1/k, & \text{none of } k \text{ people is sick} \\ (k+1)/k, & \text{at least one of } k \text{ people is sick.} \end{cases}$$

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So $P(\xi = \frac{1}{k}) = (1-p)^k$, $P(\xi = 1 + \frac{1}{k}) = 1 - (1-p)^k$. Hence

$$\begin{aligned} E\xi &= \frac{1}{k}(1-p)^k + (1+\frac{1}{k})(1-(1-p)^k) \\ &= 1 - (1-p)^k + \frac{1}{k}. \end{aligned}$$

Properties of expectations (continue)

Corollary

Suppose $|\xi| \le \eta$, $E\eta < \infty$. *Then* $E\xi$ *exists and* $|E\xi| \le E|\xi| \le E\eta$.

Proof. For $M > 0$, let $\xi_M = |\xi|$ if $|\xi| \le M$, and 0 if $|\xi| > M$. Then $0 \le \xi_M \le M$. By Property 1, $0 \le E\xi_M \le M$.

Proof. For $M > 0$, let $\xi_M = |\xi|$ if $|\xi| \le M$, and 0 if $|\xi| > M$. Then $0 \le \xi_M \le M$. By Property 1, $0 \le E\xi_M \le M$. So $E\xi_M$, $E\eta$ exist, and $\xi_M \le \eta$. It follows that

$$\int_{-M}^{M} |x| dF_\xi(x) = E\xi_M \le E\eta$$

by Property 4.

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by Property 4. Hence $E|\xi| = \int_{-\infty}^{\infty} |x| dF_\xi(x) \le E\eta < \infty$. Finally, since $-|\xi| \le \xi \le |\xi|$, by Property 4 we have $-E|\xi| \le E\xi \le E|\xi|$. The proof is completed.

Corollary

Let $p > 1$. *If* $E|\xi|^p$ *exists, then* $E|\xi|$ *exists.*

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Proof. Since $|\xi| \le 1 + |\xi|^p$, $E|\xi| \le 1 + E|\xi|^p$.

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$$P(|\xi| \geq \epsilon) \leq \frac{E|\xi|}{\epsilon}, \quad \forall \epsilon > 0.$$

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In fact, let

$$\eta = \begin{cases} 1, & \text{if } |\xi| \geq \epsilon, \\ 0, & \text{for otherwise.} \end{cases} \qquad \text{Then} \quad \eta \leq \frac{|\xi|}{\epsilon}.$$

By Property 4, we have

$$P(|\xi| \geq \epsilon) = E\eta \leq E\left[\frac{|\xi|}{\epsilon}\right] = \frac{E|\xi|}{\epsilon}.$$

Andrey Andreyevich Markov (June 1856 –July 1922)

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In fact, the "only if" part is obvious. For the "if" part, by Property 5 we have

$$P(|\xi| \geq \epsilon) = 0 \text{ for all } \epsilon > 0.$$

So $P(|\xi| > 0) = 0$.

Convergence theorems

7. (*Monotone convergence theorem*). If $0 \le \xi_n(\omega) \nearrow \xi(\omega)$, then

$$\lim_{n\to\infty} E\xi_n = E\xi. \qquad (*)$$

If $0 \le \xi_n(\omega) \searrow 0$, and $E\xi_n$s are finite, then

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8. (*Dominated convergence theorem*). If $\xi_n(\omega) \to \xi(\omega)$, $|\xi_n| \leq \eta$ and $E\eta < \infty$, then (*) holds.

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8. (*Dominated convergence theorem*). If $\xi_n(\omega) \to \xi(\omega)$, $|\xi_n| \le \eta$ and $E\eta < \infty$, then (*) holds.

9. (*Bounded convergence theorem*). If $\xi_n(\omega) \to \xi(\omega)$ and $|\xi_n| \le M < \infty$, then (*) holds.

证明： 先证明有界收敛定理.

首先，已知$|\xi_n| \leq M$, $|\xi| \leq M$. 由性质1, $E\xi_n$, $E\xi$存在，并且$|E\xi_n - E\xi| \leq E|\xi_n - \xi|$.

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首先，已知$|\xi_n| \le M$, $|\xi| \le M$. 由性质1, $E\xi_n$, $E\xi$存在，并且$|E\xi_n - E\xi| \le E|\xi_n - \xi|$. 对任给的$\epsilon > 0$，记$A_n = \{|\xi_n - \xi| > \epsilon\}$. 在$A_n^c$上有$|\xi_n - \xi| \le \epsilon$. 而在$A_n$上有$|\xi_n - \xi| \le 2M$.

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首先，已知$|\xi_n| \le M$, $|\xi| \le M$. 由性质1, $E\xi_n$, $E\xi$存在，并且$|E\xi_n - E\xi| \le E|\xi_n - \xi|$. 对任给的$\epsilon > 0$, 记$A_n = \{|\xi_n - \xi| > \epsilon\}$. 在$A_n^c$上有$|\xi_n - \xi| \le \epsilon$. 而在$A_n$上有$|\xi_n - \xi| \le 2M$. 所以

$$|\xi_n - \xi| \le \epsilon + 2MI_{A_n}.$$

因此

$$|E\xi_n - E\xi| \le E|\xi_n - \xi| \le \epsilon + 2MP(A_n).$$

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$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n) = 0.$$

所以

$$\limsup_{n\to\infty} |E\xi_n - E\xi| \le \limsup_{n\to\infty} E|\xi_n - \xi| \le \epsilon.$$

由$\epsilon > 0$的任意性得

$$\lim_{n\to\infty} |E\xi_n - E\xi| = \lim_{n\to\infty} E|\xi_n - \xi| = 0.$$

现在证明单调收敛定理. 设$0 \le \xi_n(\omega) \nearrow \xi(\omega)$. 对任意的$M > 0$,
令$\eta_n = \xi_n I\{|\xi_n| \le M\}$, $\eta = \xi I\{|\xi| \le M\}$. 则$\eta_n \le \xi_n$, $\eta \le \xi$,

$$\eta_n(\omega) \to \eta(\omega), \quad \forall \omega$$

并且$0 \le \eta_n \le M$. 由有界收敛定理和数学期望的单调性知,

$$\lim_{n\to\infty} E\xi_n \ge \lim_{n\to\infty} E\eta_n = E\eta = \int_0^M x dF_\xi(x).$$

令$M \to \infty$得

$$\lim_{n\to\infty} E\xi_n \ge E\xi.$$

如果$E\xi = \infty$, 则结论已经得证.

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$$\eta_n(\omega) \to \eta(\omega), \quad \forall \omega$$

并且$0 \le \eta_n \le M$. 由有界收敛定理和数学期望的单调性知,

$$\lim_{n\to\infty} E\xi_n \ge \lim_{n\to\infty} E\eta_n = E\eta = \int_0^M x dF_\xi(x).$$

令$M \to \infty$得

$$\lim_{n\to\infty} E\xi_n \ge E\xi.$$

如果$E\xi = \infty$, 则结论已经得证. 如果$E\xi < \infty$, 则由于$\xi_n \le \xi$, 由单调性得$E\xi_n \le E\xi$. 所以

$$\lim_{n\to\infty} E\xi_n = E\xi.$$

下设$0 \le \xi_n(\omega) \searrow 0$, $E\xi_n$存在, 这时

$$0 \le \xi_1 - \xi_n \nearrow \xi_1.$$

所以

$$E(\xi_1 - \xi_n) \to E\xi_1.$$

所以

$$E\xi_n \to 0.$$

最后证明控制收敛定理. 记

$$\eta_n = \sup_{m\geq n} |\xi_m - \xi|.$$

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则$0 \leq \eta_n(\omega) \searrow 0$. 另一方面，由于$0 \leq \eta_n \leq 2\eta$，所以$0 \leq E\eta_n \leq 2E\eta < \infty$.

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$$\eta_n = \sup_{m \geq n} |\xi_m - \xi|.$$

则$0 \leq \eta_n(\omega) \searrow 0$．另一方面，由于$0 \leq \eta_n \leq 2\eta$，所以$0 \leq E\eta_n \leq 2E\eta < \infty$．由单调收敛定理，

$$E\eta_n \to 0.$$

而

$$|E\xi_n - E\xi| \leq E|\xi_n - \xi| \leq E\eta_n.$$

因此

$$\lim_{n \to \infty} E\xi_n = E\xi.$$

The proofs of the Basic properties of expectations

Basic properties of expectations of discrete random variables

Property 1 (*Absolute integrability*): Suppose ξ is a discrete random variable. Then $E\xi$ is finite if and only if $E|\xi| < \infty$. Further

$$E\xi = E\xi^+ - E\xi^-, \quad E|\xi| = E\xi^+ + E\xi^-.$$

Property 2 (*Linearity*): Suppose ξ and η are discrete random variables. If $E\xi$ and $E\eta$ exist, then

$$E(a\xi + b\eta) = aE\xi + bE\eta.$$

Property 3 (*Monotonicity*): Suppose ξ and η are discrete random variables. If $\xi \le \eta$ and the expectations of ξ and η exits, then $E\xi \le E\eta$.

Property 4 (*Modulus inequality*): Suppose ξ and η are discrete random variables. If $|\xi| \le \eta$ and the expectation $E\eta$ exists, then $E\xi$ exists and $|E\xi| \le E|\xi| \le E\eta$.

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Proof. Write $\xi = \sum_i x_i I_{A_i}$, $\eta = \sum_j y_j I_{B_j}$, where $x_i, y_j \ge 0$.

Property 4 (*Modulus inequality*): Suppose ξ and η are discrete random variables. If $|\xi| \leq \eta$ and the expectation $E\eta$ exists, then $E\xi$ exists and $|E\xi| \leq E|\xi| \leq E\eta$.

Proof. Write $\xi = \sum_i x_i I_{A_i}$, $\eta = \sum_j y_j I_{B_j}$, where $x_i, y_j \geq 0$. Then

$$\eta = \sum_{i,j} y_j I_{A_i B_j}, |\xi| = \sum_{i,j} |x_i| I_{A_i B_j}.$$

So on the event $A_i B_j$, $y_j \geq |x_i|$.

Property 4 (*Modulus inequality*): Suppose ξ and η are discrete random variables. If $|\xi| \leq \eta$ and the expectation $E\eta$ exists, then $E\xi$ exists and $|E\xi| \leq E|\xi| \leq E\eta$.

Proof. Write $\xi = \sum_i x_i I_{A_i}$, $\eta = \sum_j y_j I_{B_j}$, where $x_i, y_j \geq 0$. Then

$$\eta = \sum_{i,j} y_j I_{A_i B_j}, |\xi| = \sum_{i,j} |x_i| I_{A_i B_j}.$$

So on the event $A_i B_j$, $y_j \geq |x_i|$. Therefore, $y_j I_{A_i B_j} \geq |x_i| I_{A_i B_j}$. By Property 3, $y_j P(A_i B_j) \geq |x_i| P(A_i B_j)$.

so

$$\begin{aligned}\infty > E\eta = \sum_j y_j P(B_j) &= \sum_{i,j} y_j P(A_i B_j) \\ &\ge \sum_{i,j} |x_i| P(A_i B_j) = \sum_i |x_i| P(A_i).\end{aligned}$$

Hence $E\xi$ exists and

$$\begin{aligned}|E\xi| =& \left|\sum_i x_i P(A_i)\right| \le \sum_i |x_i| P(A_i) \\ =& E|\xi| \le E\eta < \infty.\end{aligned}$$

Corollary

Suppose ξ and η are discrete random variables, $a \le \xi \le b$. Then $E\xi$ exists and

$$a \le E\xi \le b.$$

Properties of Mathematical expectation for general random variables

For a random variable ξ. Define

$$\xi^{(m)} = \frac{k}{2^m} \quad \text{if} \quad \frac{k}{2^m} < \xi \leq \frac{k+1}{2^m}.$$

Then

- If $\xi \geq 0$, then $0 \leq \xi_m \nearrow \xi$ and $0 \leq \xi - \xi^{(m)} \leq \frac{1}{2^m}$.
- In general, $|\xi - \xi^{(m)}| \leq \frac{1}{2^m}$.

Theorem

$E\xi$ exists if and only if $E\xi^{(m)}$ exists for one m (and then all m). Furthermore,

$$E\xi = \lim_{m\to\infty} E\xi^{(m)}$$

$$\left| E\xi - E\xi^{(m)} \right| \le \frac{1}{2^m}.$$

Suppose ξ has cdf $F(x)$. Write $x_{m,k}=\frac{k}{2^m}$. Then

$$\xi^{(m)}=\sum_{k=-\infty}^{\infty}x_{m,k}I\{x_{m,k}<\xi\le x_{m,k+1}\},$$

$$\begin{aligned}E|\xi^{(m)}|&=\sum_{k=-\infty}^{\infty}|x_{m,k}|P(x_{m,k}<\xi\le x_{m,k+1})\\&=\sum_{k=-\infty}^{\infty}|x_{m,k}|\Delta F(x_{m,k})\\&=\sum_{k=-\infty}^{\infty}\int_{x_{m,k}<x\le x_{m,k+1}}|x_{m,k}|dF(x),\end{aligned}$$

where $\Delta F(x_{m,k})=F(x_{m,k+1})-F(x_{m,k})$.

For $x_{m,k} < x \le x_{m,k+1}$, we have $\Big||x_{m,k}| - |x|\Big| \le \frac{1}{2^m}$. So,

$$\begin{aligned} & \int_{-\infty}^{\infty} |x| dF(x) - \frac{1}{2^m} \\ \le & \sum_{k=-\infty}^{\infty} |x_{m,k}| \Delta F(x_{m,k}) \\ = & \sum_{k=-\infty}^{\infty} \int_{x_{m,k} < x \le x_{m,k+1}} |x_{m,k}| dF(x) \\ \le & \int_{-\infty}^{\infty} |x| dF(x) + \frac{1}{2^m}. \end{aligned}$$

So, $E\xi$ exists if and only if $E\xi^{(m)}$ exits.

Similarly,

$$E\xi^{(m)} = \sum_{k=-\infty}^{\infty} x_{m,k} P(x_{m,k} < \xi \le x_{m,k+1})$$
$$= \sum_{k=0}^{\infty} x_{m,k} \Delta F(x_{m,k})$$

and

$$\int_{-\infty}^{\infty} x dF(x) - \frac{1}{2^m} \le E\xi^{(m)} \le \int_{-\infty}^{\infty} x dF(x) + \frac{1}{2^m}.$$

The proof is completed.

3.1.5 Basic properties of expectations

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1. (*Positivity*). If $0 \le \xi$, then
$$E\xi \ge 0.$$
If $a \le \xi \le b$, then
$$a \le E\xi \le b.$$

Proof. Assume $0 \le \xi$, then $0 \le \xi^{(m)}$.

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$$0 \le E\xi^{(m)} \nearrow E\xi.$$

Proof. Assume $0 \leq \xi$, then $0 \leq \xi^{(m)}$. When $E\xi^{(m)} = \infty$, $E\xi = +\infty \geq 0$. When $E\xi^{(m)} < \infty$, then $E\xi$ is finite, and

$$0 \leq E\xi^{(m)} \nearrow E\xi.$$

Assume $a \leq \xi \leq b$, then $a - \frac{1}{2^m} \leq \xi^{(m)} \leq b$. So, $E\xi^{(m)}$ exists and

$$a - \frac{1}{2^m} \leq E\xi^{(m)} \leq b.$$

Letting $m \to \infty$ yields $a \leq E\xi \leq b$.

2. (*Linearity*). $E\xi$ and $E\eta$ exist $\Longrightarrow$

$$E(a\xi + b\eta) = aE\xi + bE\eta.$$

Proof of Property (2). Note $|X - X^{(m)}| \le \frac{1}{2^m}$. So

$$\begin{aligned}
&\left|(a\xi + b\eta)^{(m)} - (a\xi^{(m)} + b\eta^{(m)})\right| \\
\le &\left|(a\xi + b\eta)^{(m)} - (a\xi + b\eta)\right| + \left|(a\xi^{(m)} + b\eta^{(m)}) - (a\xi + b\eta)\right| \\
\le &\left|(a\xi + b\eta)^{(m)} - (a\xi + b\eta)\right| + |a| \cdot |\xi^{(m)} - \xi| + |b| \cdot |\eta^{(m)} - \eta| \\
\le &\frac{1 + |a| + |b|}{2^m}.
\end{aligned}$$

Proof of Property (2). Note $|X - X^{(m)}| \le \frac{1}{2^m}$. So

$$\begin{aligned}
&\left|(a\xi + b\eta)^{(m)} - (a\xi^{(m)} + b\eta^{(m)})\right| \\
\le &\left|(a\xi + b\eta)^{(m)} - (a\xi + b\eta)\right| + \left|(a\xi^{(m)} + b\eta^{(m)}) - (a\xi + b\eta)\right| \\
\le &\left|(a\xi + b\eta)^{(m)} - (a\xi + b\eta)\right| + |a| \cdot |\xi^{(m)} - \xi| + |b| \cdot |\eta^{(m)} - \eta| \\
\le &\frac{1 + |a| + |b|}{2^m}.
\end{aligned}$$

It follows that

$$\left|E\left[(a\xi + b\eta)^{(m)}\right] - (aE\xi^{(m)} + bE\eta^{(m)})\right| \le \frac{1 + |a| + |b|}{2^m}.$$

Taking the limit $m \to \infty$ completes the proof.

3. Suppose that ξ and η are independent, and expectations $E\xi$ and $E\eta$ exists. Then

$$E\xi\eta = E\xi E\eta.$$

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Proof. Write

$$x_i = \frac{i}{2^m}.$$

The possible values of $\xi^{(m)}\eta^{(m)}$ are those $x_i x_j$s.

$$E(\xi^{(m)}\eta^{(m)}) = \sum_l z_l P(\xi^{(m)}\eta^{(m)} = z_l)$$

$$= \sum_l z_l \sum_{i,j:x_ix_j=z_l} P(\xi^{(m)} = x_i, \eta^{(m)} = x_j)$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x_i x_j P(\xi^{(m)} = x_i, \eta^{(m)} = x_j)$$

$$= \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} x_i x_j P(\xi^{(m)} = x_i) P(\eta^{(m)} = x_j)$$

$$= \sum_{i=-\infty}^{\infty} x_i P(\xi^{(m)} = x_i) \sum_{j=-\infty}^{\infty} x_j P(\eta^{(m)} = x_j) = E\xi^{(m)} E\eta^{(m)}.$$

Similarly,

$$
\begin{aligned}
&\sum_l |z_l| P(\xi^{(m)}\eta^{(m)} = z_l) \\
=&\sum_l |z_l| \sum_{i,j: x_i x_j = z_l} P(\xi^{(m)} = x_i, \eta^{(m)} = x_j) \\
=&\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |x_i x_j| P(\xi^{(m)} = x_i, \eta^{(m)} = x_j) \\
=&\sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} |x_i x_j| P(\xi^{(m)} = x_i) P(\eta^{(m)} = x_j) \\
=&\sum_{i=-\infty}^{\infty} |x_i| P(\xi^{(m)} = x_i) \sum_{j=-\infty}^{\infty} |x_j| P(\eta^{(m)} = x_j) < \infty.
\end{aligned}
$$

So, $E(\xi^{(m)}\eta^{(m)})$ exists and

$$E(\xi^{(m)}\eta^{(m)}) = E\xi^{(m)}E\eta^{(m)} \to E\xi E\eta.$$

On the other hand,

$$|(\xi\eta)^{(m)} - \xi\eta| \le \frac{1}{2^m},$$

$$\begin{aligned} \xi^{(m)}\eta^{(m)} - \xi\eta =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi(\eta^{(m)} - \eta) \\ =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi^{(m)}(\eta^{(m)} - \eta) \\ & + (\xi - \xi^{(m)})(\eta^{(m)} - \eta), \end{aligned}$$

$$|\xi^{(m)}\eta^{(m)} - \xi\eta| \le \frac{1}{2^m}|\eta^{(m)}| + \frac{1}{2^m}|\xi^{(m)}| + \frac{1}{2^m}\frac{1}{2^m}.$$

On the other hand,

$$|(\xi\eta)^{(m)} - \xi\eta| \leq \frac{1}{2^m},$$

$$\begin{aligned}\xi^{(m)}\eta^{(m)} - \xi\eta =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi(\eta^{(m)} - \eta) \\ =& (\xi^{(m)} - \xi)\eta^{(m)} + \xi^{(m)}(\eta^{(m)} - \eta) \\ &+ (\xi - \xi^{(m)})(\eta^{(m)} - \eta),\end{aligned}$$

$$|\xi^{(m)}\eta^{(m)} - \xi\eta| \leq \frac{1}{2^m}|\eta^{(m)}| + \frac{1}{2^m}|\xi^{(m)}| + \frac{1}{2^m}\frac{1}{2^m}.$$

It follows that

$$|\xi^{(m)}\eta^{(m)} - (\xi\eta)^{(m)}| \leq \frac{1}{2^m}|\eta^{(m)}| + \frac{1}{2^m}|\xi^{(m)}| + \frac{2}{2^m}.$$

Note that $\xi^{(m)}\eta^{(m)}$, $(\xi\eta)^{(m)}$, $|\eta^{(m)}|$, $|\xi^{(m)}|$ are discrete random variables, and $E[\xi^{(m)}\eta^{(m)}]$, $E[|\eta^{(m)}|]$, $E[|\xi^{(m)}|]$ exist.

So $E[(\xi\eta)^{(m)}]$ exists and

$$
\begin{aligned}
&|E[\xi^{(m)}\eta^{(m)}] - E[(\xi\eta)^{(m)}]| \\
\le & \frac{1}{2^m}E[|\eta^{(m)}|] + \frac{1}{2^m}E[|\xi^{(m)}|] + \frac{2}{2^m} \\
\le & \frac{1}{2^m}E[|\eta|] + \frac{1}{2^m}E[|\xi|] + \frac{4}{2^m} \to 0.
\end{aligned}
$$

Hence, $E[\xi\eta]$ exists and

$$\begin{aligned} E[\xi\eta] &= \lim_{m\to\infty} E[(\xi\eta)^{(m)}] \\ &= \lim_{m\to\infty} E[\xi^{(m)}\eta^{(m)}] \\ &= \lim_{m\to\infty} E[\xi^{(m)}]E[\eta^{(m)}] \\ &= E[\xi]E[\eta]. \square \end{aligned}$$