ODE笔记2：几种函数及恰当方程

齐次函数

$f(x,y)$，若 $\forall\lambda\neq 0$，$f(\lambda x,\lambda y)=\lambda^k f(x,y)$，称$f(x,y)$为$k$**次齐次函数**

齐次方程

$y'=f(x,y)$ ，$f(x,y)$为0次齐次函数

$y'=f(x,y)=f(\lambda x,\lambda y)=f\left(1,\frac{y}{x}\right)=g\left(\frac{y}{x}\right)$，一元函数

例1：$y\prime=\frac{x+y}{x-y}$，为齐次方程。

解：令$u=\frac{y\prime}{x}$

$$\therefore\ u'x+u=\frac{1+u}{1-u},u'x=\frac{1+u^2}{1-u},\frac{1-u}{1+u^2}du=\frac{1}{x}dx$$
$$\Longrightarrow\frac{1}{1+u^2}du-\frac{1}{2}\frac{du^2}{1+u^2}=\frac{1}{x}dx,\arctan u=\frac{1}{2}\ln\left(1+u^2\right)+\ln|x|+c$$
$$\Longrightarrow\arctan\frac{y}{x}=\ln\sqrt{x^2+y^2}+c$$

$y'=\frac{x+y+1}{x-y+2}$ ，换元，$\begin{cases}\xi=x+\frac{3}{2}\\ \eta=y-\frac{1}{2}\end{cases}$，则 $\frac{d\eta}{d\xi}=\frac{\xi+\eta}{\xi-\eta}$

$\Longrightarrow\arctan\frac{y-\frac{1}{2}}{x+\frac{3}{2}}=\ln\sqrt{\left(x+\frac{3}{2}\right)^2+\left(y-\frac{1}{2}\right)^2}+c$

变式：

$(1)y'=x^{k-1}F(\frac{y}{x^k})$

令$u=\dfrac{y}{x^2},y=ux^k\quad u'x^k+kux^{k-1}=x^{k-1}(F(u))\implies u'=\dfrac{1}{x}(F(u)-ku)$

$(2)xy'=F\left(xe^{-y}\right)$

令$u=xe^{-y},y=\ln\dfrac{x}{u}\quad x\cdot\dfrac{u}{x}\left(\dfrac{1}{u}-\dfrac{xu'}{u^2}\right)=F(u)\implies 1-\dfrac{x}{u}u'=F(u)$

$(3)y'=\frac{y}{x}+xF\left(\frac{y}{x}\right)$

令$u'=\frac{y}{x},y=ux\quad u'x+u=u+xF(u),\ u'=F(u)$

$(4)y'=\frac{y}{x+F\left(\frac{y}{y}\right)}$

令$u=\frac{y}{x},y=ux\quad u'x+u=\frac{y}{x+F(u)}\implies u'x=\frac{-uF(u)}{x+F(u)}$

伯努利方程

$y'=p(x)y+q(x)y^\alpha,\ \ \alpha\neq 0,1$

同除y^α，$y^{-\alpha}y'=p(x)y^{1-\alpha}+q(x)$，令$u=y^{1-\alpha}$，则

$\frac{du}{dx}=(1-\alpha)y^{-\alpha}\frac{dy}{dx}=(1-\alpha)(py^{1-\alpha}+q)\implies u'=(1-\alpha)pu+(1-\alpha)q$

$$\Longrightarrow \quad u = e^{\int (1-\alpha)p(x)dx}(c + \int (1-\alpha)q(x)e^{-\int (1-\alpha)p(x)dx}dx)$$

Riccati方程

$$y' + p(x)y + q(x)y^2 = k(x).\ k \not\equiv 0$$

条件：已知一个解$\phi(x)$，$\phi' + p\phi + q\phi^2 = k(x) \implies (y-\phi)' + p(y-\phi) + q\left(y^2 - \phi^2\right) = 0$

令$u = y - \phi$，则$y = u + \phi$，得到$u' + (p + 2\phi q)u + qu^2 = 0$ （伯努利方程！）

例2：$y' + y^2 = \frac{2}{x^2}$

猜一个解：$y = \frac{2}{x}$，记为$\phi(x)$，令$u = y - \frac{2}{x}$

$$u' - \frac{2}{x^2} + u^2 + \frac{4}{x}u + \frac{4}{x^2} = \frac{2}{x^2},\ u' + \frac{4}{x}u + u^2 = 0$$

（1）$u \equiv 0$是解 （2）$u \neq 0, \frac{u'}{u^2} + \frac{4}{xu} + 1 = 0.$ 令$z = \frac{1}{u}$，$z' = \frac{4}{x}z + 1,$

$z = e^{\int \frac{4}{x}dx}(c + \int e^{-\int \frac{4}{x}dx}dx) = cx^4 - \frac{1}{3}x$

$\Longrightarrow \quad y = \frac{2}{x}$ 或 $y = \frac{1}{cx^4 - \frac{1}{3}x} + \frac{2}{x}$

恰当（全微分）方程

$$M(x,y)dx + N(x,y)dy = 0$$

一族函数$F(x,y) = c$，找一个一阶ODE：$\partial_x F dx + \partial_y F dy = 0$

设G为R^2上一个区域，若G上具有一阶连续偏导数$F(x,y)$，s.t.

$$dF(x,y) = M(x,y)dx + N(x,y)dy \qquad (M = \frac{\partial F}{\partial x}, N = \frac{\partial F}{\partial y})$$

则称1阶ODE：$M(x,y)dx + N(x,y)dy = 0$ 是**恰当（全微分）方程。**

$M(x,y)dx + N(x,y)dy = 0$ 是恰当方程 $\iff \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \forall (x,y) \in G$

求F：

（1）$F = \int_{(x_0,y_0)}^{(x,y)} Mdx + Ndy$

$F(x,y) = \int_{x_0}^{x} M(t,y_0)dt + \int_{y_0}^{y} N(x,s)ds = \int_{y_0}^{y} N(x_0,s)ds + \int_{x_0}^{x} M(t,y)dt$

（2）凑：$xdy + ydx \implies d(xy) = 0,\ xy = c$

（3）由$\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$，定$y : F(x,y) = \int M(x,y)dx + c(y)$

$\Longrightarrow \int \frac{\partial}{\partial y}M(x,y)dx + c'(y) = N(x,y)$. x 固定，关于 y 为ODE.解出$c(y)$. [固定x同理]

（4）$\begin{Bmatrix} \frac{\partial F}{\partial x} = M \Rightarrow F = \int Mdx + \Psi(y) \\ \frac{\partial F}{\partial y} = N \Rightarrow F = \int Ndy + \varphi(x) \end{Bmatrix}$ 比较法得出 F

例3：$(3x^2y+8xy^2)dx+(x^3+8x^2y+12y^2)dy=0$

解：（1）$M=3x^2y+8xy^2,\quad N=x^3+8x^2y+12y^2$

$\frac{\partial M}{\partial y}=3x^2+16xy=\frac{\partial N}{\partial x}$，是**恰当方程。**

$$\begin{cases}\frac{\partial F}{\partial x}=M=3x^2y+8xy^2 \quad \overset{y\text{定}}{\Longrightarrow} \quad F=x^3y+4x^2y^2+\psi(y)\\ \frac{\partial F}{\partial y}=N=x^3+8x^2y+12y^2 \quad \overset{x\text{定}}{\Longrightarrow} \quad F=x^3y+4x^2y^2+4y^3+\varphi(x)\end{cases}$$

$\Longrightarrow x^3y+4x^2y^2+4y^3=C,\ C$为任意常数。

（2）公式：

$F(x,y)=\int_0^x M(t,0)dt+\int_0^y N(x,s)ds=\int_0^y(x^3+8x^2s+12s^2)ds=x^3y+4x^2y^2+4y^3$

例4：$\phi(1)=0,\ \frac{2\phi(x)+2x^2}{x^4}ydx+\left(\frac{\phi(x)}{x^2}+\sin y\right)dy=0$ **是恰当方程，求 $\phi(x)$.**

解：$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x},\frac{2\phi(x)+2x^2}{x^4}=\frac{\phi'(x)}{x^2}-\frac{2\phi(x)}{x^3} \quad \Longrightarrow \quad \phi'-2\left(\frac{1}{x}+\frac{1}{x^2}\right)\phi=2$

由公式得：$\phi(x)=e^{\int_1^x 2\left(\frac{1}{t}+\frac{1}{t^2}\right)dt}\left(0+\int_1^x 2e^{-\int_1^t 2\left(\frac{1}{s}+\frac{1}{s^2}\right)ds}dt\right)=-x^2+x^2e^{2-\frac{2}{x}}$

ODE：$\frac{2y}{x^2}e^{2-\frac{2}{x}}dx+\left(e^{2-\frac{2}{x}}-1+\sin y\right)dy=0$　　　猜：$\frac{2y}{x^2}e^{2-\frac{2}{x}}dx+e^{2-\frac{2}{x}}dy=d(e^{2-\frac{2}{x}}y)$

$\Longrightarrow \quad d(e^{2-\frac{2}{x}}y-y-cosy)=0.$　　通解：$e^{2-\frac{2}{x}}y-y-cosy=C$

积分因子法:

$M(x,y)dx+N(x,y)dy=0$ 若非恰当方程，找$\mu(x,y)\neq 0$，s.t. $\mu Mdx+\mu Ndy=0$ 是恰当方程，则称$\mu(x,y)$为其**积分因子。**

$$\frac{\partial(\mu M)}{\partial y}=\frac{\partial(\mu N)}{\partial x},\ \frac{\partial\mu}{\partial y}M+\mu\cdot\frac{\partial M}{\partial y}=\frac{\partial\mu}{\partial x}N+\mu\cdot\frac{\partial N}{\partial x}$$

存在仅含x的积分因子$\mu(x)$　$\Longleftrightarrow$　$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}\triangleq\varphi(x)$

存在仅含y的积分因子$\mu(y)$　$\Longleftrightarrow$　$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}\triangleq\psi(y)$

例5：$(3x^3+y)dx+(2x^2y-x)dy=0$

解：$M=3x^3+y,N=2x^2y-x,\frac{\partial M}{\partial y}=1,\frac{\partial N}{\partial x}=4xy-1,$有$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=-\frac{2}{x}$

$\Longrightarrow \quad \mu(x)=e^{\int-\frac{2}{x}dx}=\frac{1}{x^2}$

原式转化为$\left(3x+\frac{y}{x^2}\right)dx+\left(2y-\frac{1}{x}\right)dy=0,\ 3xdx+2ydy+\left(\frac{y}{x^2}dx-\frac{1}{x}dy\right)=0$

$\Longrightarrow \quad d\left(\frac{3}{2}x^2\right)+dy^2+d\left(-\frac{y}{x}\right)=0,\quad \frac{3}{2}x^2+y^2-\frac{y}{x}=C$

积分因子的性质

μ无穷多：$c\cdot\mu$也是积分因子，$\varphi(F)\mu$也是。（φ连续，$\varphi\neq 0.$）

$dF=\mu Mdx+\mu Ndy=0$，两边同时乘$\varphi(F)$，找$\varphi(F)$的原函数$\int\varphi(F)dF.$

$$d\left(\int\varphi(F)dF\right)=\varphi(F)dF=0=\varphi(F)\mu(Mdx+Ndy)$$

$\varphi'(F)\mu\cdot(Mdx+Ndy)=\varphi'(F)dF=d\varphi(F)$, $\varphi\in C^1$，则$\varphi'(F)\mu$也是积分因子。

例6：$\left(\frac{y}{x}+3x^2\right)dx+\left(1+\frac{x^3}{y}\right)dy=0$

解：$M_y=\frac{1}{x}\quad,N_x=\frac{3x^2}{y}\quad M_y-N_x=\frac{y-3x^2}{xy}$

$$\begin{cases}\frac{y}{x}dx+dy=0 \quad\xrightarrow{\mu_1=x}\quad ydx+xdy=0 \quad xy=c\\ 3x^2dx+\frac{x^3}{y}dy=0 \quad\xrightarrow{\mu_2=y}\quad 3x^2ydx+x^3dy=c \quad x^3y=0\end{cases}$$

$$\Longrightarrow\quad \varphi(xy)\cdot x=\psi(x^2y)\cdot y=\mu \quad\Longrightarrow\quad \mu=x^3y^2 \quad\Longrightarrow\quad x^3y^2\left(\frac{y}{x}dx+dy+3x^2dx+\frac{x^3}{y}dy\right)=0$$

$$\Longrightarrow\quad x^2y^2d(xy)+x^3yd\left(x^3y\right)=0,\quad \frac{1}{3}x^3y^3+\frac{1}{2}\left(x^3y\right)^2=c$$

组合方法

$M_1dx+N_1dy=0$，找$\mu_1\neq 0, F_1(x,y)=c$，积分因子为$\varphi(F_1)\mu_1$，

$M_2dx+N_2dy=0$，找$\mu_2\neq 0, F_2(x,y)=c$，积分因子为$\varphi(F_2)\mu_2$

找公共积分因子：$\mu=\varphi(F_1)\mu_1=\varphi(F_2)\mu_2 \quad\Longrightarrow\quad \mu$为$(M_1+M_2)dx+(N_1+N_2)dy$的积分因子。