§4.2 Convergence in probability and weak law of large numbers

Consider the event A in random trial E. Suppose the probability of occurring A is p $(0 < p < 1)$. Now we experiment independently n times—n-fold Bernoulli trial. Let

$$\xi_i = \begin{cases} 1, & \text{A occurs at the } i\text{-th trial,} \\ 0, & \text{A does not occur at the } i\text{-th trial,} \end{cases}$$

$1 \leq i \leq n$. Then $P(\xi_i = 1) = p$, $P(\xi_i = 0) = 1 - p$. Let $S_n = \Sigma_{i=1}^n \xi_i$. Then

$$\frac{S_n}{n} = F_n(A) - - - \text{the frequency of } A.$$

What does

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It is nature to hope that the probability to appear $\{|S_n/n - p| \geq \varepsilon\}$ could be as smaller as possible when n is large enough.

Weak laws of large numbers

Theorem

(Bernoulli) Let $\{\xi_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables with* $P(\xi_n = 1) = p$, $P(\xi_n = 0) = 1 - p$, $0 < p < 1$. *Put* $S_n = \sum_{i=1}^{n} \xi_i$. *Then we have*

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) \to 0, \quad \forall \epsilon > 0,$$

i.e., for any $\epsilon > 0$ *and* $\delta > 0$*, there is a* $N = N(\epsilon, \delta)$ *such that*

$$P\left(\left|\frac{S_n}{n} - p\right| \geq \epsilon\right) < \delta, \quad \textit{for all } n \geq N.$$

Proof. By the Chebyshev inequality,

$$P\left(\left|\frac{S_n}{n}-p\right|\geq\epsilon\right)$$

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$$=\frac{1}{\epsilon^2}\frac{np(1-p)}{n^2}=\frac{1}{\epsilon^2}\frac{p(1-p)}{n}\to 0.$$

Theorem

(Chebyshev) Let $\{\xi_n, , n \ge 1\}$ *be a sequence of independent (or pairwise correlated) random variables defined on the probability space* $(\Omega, \mathcal{F}, P)$ *with* $E\xi_n = \mu_n$ *and* $Var\xi_n = \sigma_n^2$*. If* $\sum_{k=1}^n \sigma_k^2/n^2 \longrightarrow 0$*, then* $\{\xi_n, n \ge 1\}$ *obeys the weak law of large numbers, i.e.,*

$$P\left(\left| \frac{1}{n} \sum_{k=1}^{n} \xi_k - \frac{1}{n} \sum_{k=1}^{n} \mu_k \right| \ge \epsilon \right) \to 0, \quad \forall \epsilon > 0.$$

Using the Chebyshev inequality, we have

$$
\begin{aligned}
& P(|\frac{1}{n}\sum_{k=1}^{n}(\xi_k-\mu_k)|\geq\varepsilon) \\
\leq\ & P\left(\left|\frac{1}{n}\sum_{k=1}^{n}\xi_k-E\frac{1}{n}\sum_{k=1}^{n}\xi_k\right|\geq\varepsilon\right)\leq\frac{1}{\varepsilon^2}Var(\frac{1}{n}\sum_{k=1}^{n}\xi_k) \\
=\ & \frac{1}{\varepsilon^2n^2}\sum_{k=1}^{n}\sigma_k^2\longrightarrow 0\ ;\ \text{as}\quad n\longrightarrow\infty.
\end{aligned}
$$

The proof is complete.

Example 8. Suppose that $\xi_k \sim \begin{pmatrix} k^s & -k^s \\ 0.5 & 0.5 \end{pmatrix}$, where $s < 1/2$ is a constant, and $\{\xi_k, k \geq 1\}$ is indept.. Prove that $\{\xi_k, k \geq 1\}$ obeys the weak LLN.

Proof.

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Proof. We have $E\xi_k = 0$, $Var\xi_k = k^{2s}$. When $s < 1/2$,

$$\frac{1}{n^2}\sum_{k=1}^{n} Var\xi_k = \frac{1}{n^2}\sum_{k=1}^{n} k^{2s} < \frac{1}{n^2}\sum_{k=1}^{n} n^{2s} = n^{2s-1} \longrightarrow 0.$$

In addition, $\{\xi_k, k \geq 1\}$ is also independent,

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In addition, $\{\xi_k, k \geq 1\}$ is also independent, so $\{\xi_k, k \geq 1\}$ obeys the Chebyshev LLN, i.e.,

$$P\left(\left|\frac{1}{n}\sum_{k=1}^n \xi_k\right| \geq \epsilon\right) \to 0, \quad \forall \epsilon > 0.$$

Corollary

Let $\{\xi_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables defined* $(\Omega, \mathcal{F}, P)$ *with* $Var(\xi_1) < \infty$. *Let* $E\xi_1 = \mu$, $S_n = \Sigma_{k=1}^n \xi_k$. *Then* $\{\xi_n, n \geq 1\}$ *obeys the weak LLN, i.e.,*

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0 \quad as \quad n \to \infty, \quad \forall \epsilon > 0.$$

Theorem

(Khinchine) Let $\{\xi_n, n \geq 1\}$ *be a sequence of independent and identically distributed random variables defined* $(\Omega, \mathcal{F}, P)$ *with* $E|\xi_1| < \infty$. *Let* $E\xi_1 = \mu$, $S_n = \Sigma_{k=1}^n \xi_k$. *Then* $\{\xi_n, n \geq 1\}$ *obeys the weak LLN, i.e.,*

$$P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) \to 0 \quad as \quad n \to \infty, \quad \forall \epsilon > 0.$$

Proof : Let $M_k = \sqrt{k}$,

$$\eta_k = \xi_k I\{|\xi_k| \le M_k\} = \begin{cases} \xi_k, & \text{if } |\xi_k| \le M_k, \\ 0, & \text{if } |\xi_k| > M_k, \end{cases}$$

$$\zeta_k = \xi_k - \eta_k = \xi_k I\{|\xi_k| > M_k\} = \begin{cases} 0, & \text{if } |\xi_k| \le M_k, \\ \xi_k, & \text{if } |\xi_k| > M_k. \end{cases}.$$

Then

$$
\begin{aligned}
& P\left(\frac{|\sum_{k=1}^n(\eta_k - E\eta_k)|}{n} \ge \epsilon/2\right) \\
\le & \frac{4}{\epsilon^2 n^2}\sum_{k=1}^n Var\left(\eta_k\right) \le \frac{4}{\epsilon^2 n^2}\sum_{k=1}^n E\eta_k^2 \\
= & \sum_{k=1}^n \frac{4}{\epsilon^2 n^2} E[\xi_k^2 I\{|\xi_k| \le M_k\}] \le \frac{4}{\epsilon^2} E\left[\frac{\xi_1^2 I\{|\xi_1| \le M_n\}}{n}\right] \\
\le & \frac{4M_n}{n\epsilon^2} E\left[|\xi_1|\right] \to 0.
\end{aligned}
$$

$$P\left(\frac{|\sum_{k=1}^n(\zeta_k - E\zeta_k)|}{n} \geq \epsilon/2\right) \leq \frac{2}{\epsilon n} E\left|\sum_{k=1}^n(\zeta_k - E\zeta_k)\right|$$

$$\leq \frac{2}{\epsilon n}\sum_{k=1}^n E|\zeta_k - E\zeta_k| \leq 2\frac{2}{\epsilon n}\sum_{k=1}^n E|\zeta_k| \leq \frac{4}{\epsilon}\frac{1}{n}\sum_{k=1}^n E[|\xi_1| I\{|\xi_1| > M_k\}] \to 0.$$

Hence,

$$P\left(\frac{|\sum_{k=1}^n(\xi_k - E\xi_k)|}{n} \geq \epsilon\right)$$

$$\leq P\left(\frac{|\sum_{k=1}^n(\zeta_k - E\zeta_k)|}{n} \geq \epsilon/2\right) + P\left(\frac{|\sum_{k=1}^n(\eta_k - E\eta_k)|}{n} \geq \epsilon/2\right)$$

$$\to 0 \quad \text{as } n \to \infty.$$

Corollary

Let $\{\xi_n, n \ge 1\}$ be a sequence of pairwise independent and identically distributed random variables defined $(\Omega, \mathcal{F}, P)$ with $E|\xi_1| < \infty$. Let $E\xi_1 = \mu$, $S_n = \sum_{k=1}^n \xi_k$. Then $\{\xi_n, n \ge 1\}$ obeys the weak LLN, i.e.,

$$P\left(\left|\frac{S_n}{n} - \mu\right| \ge \epsilon\right) \to 0 \quad as \quad n \to \infty, \quad \forall \epsilon > 0.$$

4.2.1 Convergence in probability

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Definition

Suppose ξ and $\{\xi_n, n \geq 1\}$, are defined on the same probability space $(\Omega, \mathcal{F}, P)$. If for any $\varepsilon > 0$

$$\lim_{n\to\infty} P(|\xi_n - \xi| \geq \varepsilon) = 0,$$

or equivalently $\lim_{n\to\infty} P(|\xi_n - \xi| < \varepsilon) = 1$, then we say that ξ_n converges to ξ in probability, written $\xi_n \xrightarrow{P} \xi$.

Throwing a dot in $[0,1]$ randomly, the dot is located any point in $[0,1]$ with the same possibility. Let ω denote the location of dot and define

$$\xi(\omega)=\begin{cases}1, & \omega\in[0,0.5],\\ 0, & \omega\in(0.5,1],\end{cases}\qquad \eta(\omega)=\begin{cases}0, & \omega\in[0,0.5],\\ 1, & \omega\in(0.5,1].\end{cases}$$

Then ξ and η have the same distribution function

$$F(x)=\begin{cases}0, & x<0,\\ \frac{1}{2}, & 0\le x<1,\\ 1, & x\ge 1.\end{cases}$$

If we define $\xi_n=\xi$, for $n\ge 1$, then $\xi_n\xrightarrow{d}\eta$, but $|\xi_n-\eta|\equiv 1$.

1. Suppose ξ and $\{\xi_n, n \geq 1\}$ are random variables defined on the probability space $(\Omega, \mathcal{F}, P)$.
(1) If $\xi_n \xrightarrow{P} \xi$, then $\xi_n \xrightarrow{d} \xi$.
(2) If $\xi_n \xrightarrow{d} c$, where c is a constant, then $\xi_n \xrightarrow{P} c$.

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(1) If $\xi_n \xrightarrow{P} \xi$, then $\xi_n \xrightarrow{d} \xi$.
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Proof. (1) Let F and F_n be the cdfs of ξ and ξ_n respectively, and let x be a continuity point of F.

For any $\varepsilon > 0$,

$$
\begin{aligned}
(\xi_n \le x) &= (\xi_n \le x, |\xi_n - \xi| < \varepsilon) \\
&\quad +(\xi_n \le x, |\xi_n - \xi| \ge \varepsilon) \\
&\subset (\xi \le x + \varepsilon) \cup (|\xi_n - \xi| \ge \varepsilon).
\end{aligned}
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&\subset (\xi \le x + \varepsilon) \cup (|\xi_n - \xi| \ge \varepsilon).
\end{aligned}
$$

Thus

$$
F_n(x) \le F(x+\varepsilon) + P(|\xi_n - \xi| \ge \varepsilon).
$$

Since $\xi_n \xrightarrow{P} \xi$ as $n \longrightarrow \infty$, we obtain

$$P(|\xi_n - \xi| \geq \varepsilon) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

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Thus

$$\limsup_{n\to\infty} F_n(x) \leq F(x+\varepsilon).$$

Similarly

$$(\xi \le x) \subset (\xi_n \le x + \varepsilon) \cup (|\xi - \xi_n| \ge \varepsilon)$$

and thus

$$F(x) \le F_n(x + \varepsilon) + P(|\xi_n - \xi| \ge \varepsilon).$$

Similarly

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So

$$F(x - \varepsilon) \le F_n(x) + P(|\xi_n - \xi| \ge \varepsilon).$$

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So

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Thus

$$F(x - \varepsilon) \le \liminf_{n \to \infty} F_n(x).$$

We conclude that

$$F(x-\varepsilon) \le \liminf_{n\to\infty} F_n(x) \le \limsup_{n\to\infty} F_n(x) \le F(x+\varepsilon).$$

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Letting $\epsilon \to 0$ yields

$$\lim_{n\to\infty} F_n(x) = F(x).$$

That is

$$\xi_n \xrightarrow{d} \xi.$$

(2) If $\xi_n \xrightarrow{d} c$, then

$$\lim_{n\to\infty} F_n(x) = \begin{cases} 0, & x < c, \\ 1, & x > c. \end{cases}$$

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Hence for any $\varepsilon > 0$,

$$\begin{aligned} & P(|\xi_n - c| > \varepsilon) \\ = \; & P(\xi_n > c + \varepsilon) + P(\xi_n < c - \varepsilon) \\ = \; & 1 - P(\xi_n \le c + \varepsilon) + P(\xi_n < c - \varepsilon) \\ = \; & 1 - F_n(c + \varepsilon) + F_n(c - \varepsilon - 0) \\ \to \; & 0 \quad (n \to \infty). \end{aligned}$$

Example

(Khinchine LLN) Let $\{\xi_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables defined $(\Omega, \mathcal{F}, P)$ with $E|\xi_1| < \infty$. Let $E\xi_1 = \mu$, $S_n = \Sigma_{k=1}^n \xi_k$. Then

$$\frac{S_n}{n} \xrightarrow{P} \mu.$$

Example

(Khinchine LLN) Let $\{\xi_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables defined $(\Omega, \mathcal{F}, P)$ with $E|\xi_1| < \infty$. Let $E\xi_1 = \mu$, $S_n = \sum_{k=1}^n \xi_k$. Then

$$\frac{S_n}{n} \xrightarrow{P} \mu.$$

Proof. Since the limit μ is a constant, it is sufficient to show that

$$\frac{S_n}{n} \xrightarrow{d} \mu.$$

Let $f(t)$ and $f_n(t)$ be the c.f.s of ξ_1 and S_n/n respectively.

Let $f(t)$ and $f_n(t)$ be the c.f.s of ξ_1 and S_n/n respectively. Since $\{\xi_n, n \geq 1\}$ is i.i.d., we have $f_n(t) = (f(t/n))^n$. Moreover, from the Taylor expansion formula, we have

$$f(t) = 1 + i\mu t + o(t) \quad \text{as} \quad t \longrightarrow 0,$$

since $E\xi_1 = \mu$.

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$$f(t) = 1 + i\mu t + o(t) \quad \text{as} \quad t \longrightarrow 0,$$

since $E\xi_1 = \mu$. For every $t \in \boldsymbol{R}$,

$$f(t/n) = 1 + i\frac{\mu t}{n} + o(\frac{1}{n}) \quad \text{as} \quad n \to \infty,$$

$$f_n(t) = (1 + i\frac{\mu t}{n} + o(\frac{1}{n}))^n \to e^{i\mu t}.$$

Let $f(t)$ and $f_n(t)$ be the c.f.s of ξ_1 and S_n/n respectively. Since $\{\xi_n, n \geq 1\}$ is i.i.d., we have $f_n(t) = (f(t/n))^n$. Moreover, from the Taylor expansion formula, we have

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$$f_n(t) = (1 + i\frac{\mu t}{n} + o(\frac{1}{n}))^n \to e^{i\mu t}.$$

By the inverse limit theorem in we know that $S_n/n \xrightarrow{d} \mu$. So, we have $S_n/n \xrightarrow{P} \mu$. The proof is complete.

2. Let $\{\xi, \xi_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Prove that
(1) If $\xi_n \xrightarrow{P} \xi$, $\xi_n \xrightarrow{P} \eta$, then $P(\xi = \eta) = 1$.
(2) If $\xi_n \xrightarrow{P} \xi$, f is the continuous function on $(-\infty, \infty)$, then $f(\xi_n) \xrightarrow{P} f(\xi)$.

2. Let $\{\xi, \xi_n, n \geq 1\}$ be a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. Prove that
(1) If $\xi_n \xrightarrow{P} \xi$, $\xi_n \xrightarrow{P} \eta$, then $P(\xi = \eta) = 1$.
(2) If $\xi_n \xrightarrow{P} \xi$, f is the continuous function on $(-\infty, \infty)$, then $f(\xi_n) \xrightarrow{P} f(\xi)$.

In general, if $\boldsymbol{\xi}_n =: (\xi_{n,1}, \ldots, \xi_{n,m}) \xrightarrow{P} \boldsymbol{\xi} := (\xi_1, \ldots, \xi_m)$ (i.e., $\|\boldsymbol{\xi}_n - \boldsymbol{\xi}\| \to 0$, or equivalently, $\xi_{n,k} \to \xi_k$, $k = 1, \ldots, m$) and $f(\boldsymbol{x})$ is a m-dimensional continuous function, then

$$f(\boldsymbol{\xi}_n) \xrightarrow{P} f(\boldsymbol{\xi}).$$

Proof. (1) For any $\varepsilon > 0$, we have

$$(|\xi - \eta| \geq \varepsilon) \subseteq (|\xi_n - \xi| \geq \frac{\varepsilon}{2}) \cup (|\xi_n - \eta| \geq \frac{\varepsilon}{2}).$$

Thus

$$P(|\xi - \eta| \geq \varepsilon) \leq P(|\xi_n - \xi| \geq \frac{\varepsilon}{2}) + P(|\xi_n - \eta| \geq \frac{\varepsilon}{2}).$$

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Note that the left side of the above inequality is independent of n and $\xi_n \xrightarrow{P} \xi$, $\xi_n \xrightarrow{P} \eta$ as $n \longrightarrow \infty$. Therefore $P(|\xi - \eta| \geq \varepsilon) = 0$.

Proof. (1) For any $\varepsilon > 0$, we have

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Thus

$$P(|\xi-\eta|\geq\varepsilon)\leq P(|\xi_n-\xi|\geq\frac{\varepsilon}{2})+P(|\xi_n-\eta|\geq\frac{\varepsilon}{2}).$$

Note that the left side of the above inequality is independent of n and $\xi_n\xrightarrow{P}\xi$, $\xi_n\xrightarrow{P}\eta$ as $n\longrightarrow\infty$. Therefore $P(|\xi-\eta|\geq\varepsilon)=0$. Furthermore,

$$\begin{aligned}P(|\xi-\eta|>0) &= P(\bigcup_{n=1}^{\infty}(|\xi-\eta|\geq\frac{1}{n}))\\ &\leq \sum_{n=1}^{\infty}P(|\xi-\eta|\geq\frac{1}{n})=0,\end{aligned}$$

i.e., $P(\xi=\eta)=1$.

(2). For any $\epsilon > 0$ and $M > 0$, choose $0 < \delta < M/2$ such that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \epsilon$ whenever $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta, \|\boldsymbol{x}\| \leq M, \|\boldsymbol{y}\| \leq M$.

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$$
\begin{aligned}
& \{|f(\boldsymbol{x}) - f(\boldsymbol{y})| \ge \epsilon\} \\
\subset & \{\|\boldsymbol{x} - \boldsymbol{y}\| \ge \delta\} \bigcup \{\|\boldsymbol{x}\| > M\} \bigcup \{\|\boldsymbol{y}\| > M\} \\
\subset & \{\|\boldsymbol{x} - \boldsymbol{y}\| \ge \delta\} \bigcup \{\|\boldsymbol{y}\| > M/2\}.
\end{aligned}
$$

(2). For any $\epsilon > 0$ and $M > 0$, choose $0 < \delta < M/2$ such that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| < \epsilon$ whenever $\|\boldsymbol{x} - \boldsymbol{y}\| < \delta, \|\boldsymbol{x}\| \le M, \|\boldsymbol{y}\| \le M$. Then

$$\begin{aligned}
&\{|f(\boldsymbol{x}) - f(\boldsymbol{y})| \ge \epsilon\} \\
\subset &\{\|\boldsymbol{x} - \boldsymbol{y}\| \ge \delta\} \bigcup \{\|\boldsymbol{x}\| > M\} \bigcup \{\|\boldsymbol{y}\| > M\} \\
\subset &\{\|\boldsymbol{x} - \boldsymbol{y}\| \ge \delta\} \bigcup \{\|\boldsymbol{y}\| > M/2\}.
\end{aligned}$$

So,

$$\begin{aligned}
&P\left(|f(\boldsymbol{\xi}_n) - f(\boldsymbol{\xi})| \ge \epsilon\right) \\
\le &P\left(\|\boldsymbol{\xi}_n - \boldsymbol{\xi}\| \ge \delta\right) + P(\|\boldsymbol{\xi}\| > M/2) \to 0,
\end{aligned}$$

as $n \to \infty$ and then $M \to \infty$.

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    1. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} \eta$, then $\xi_n \pm \eta_n \xrightarrow{P} \xi \pm \eta$;

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   2. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} \eta$, then $\xi_n \eta_n \xrightarrow{P} \xi\eta$;

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   2. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} \eta$, then $\xi_n \eta_n \xrightarrow{P} \xi\eta$;
   3. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} c$, where c is a constant, both η_n and c are not 0, then $\xi_n / \eta_n \xrightarrow{P} \xi / c$;

3. We have
   1. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} \eta$, then $\xi_n \pm \eta_n \xrightarrow{P} \xi \pm \eta$;
   2. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} \eta$, then $\xi_n \eta_n \xrightarrow{P} \xi\eta$;
   3. If $\xi_n \xrightarrow{P} \xi, \eta_n \xrightarrow{P} c$, where c is a constant, both η_n and c are not 0, then $\xi_n/\eta_n \xrightarrow{P} \xi/c$;
   4. If $\xi_n \xrightarrow{d} \xi, \eta_n \xrightarrow{P} c$, where c is a constant, then $\xi_n + \eta_n \xrightarrow{d} \xi + c$, $\eta_n\xi_n \xrightarrow{d} c\xi$.

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$$\left|\eta_n^{-1} - c^{-1}\right| = \frac{|\eta_n - c|}{|\eta_n||c|} < \frac{\epsilon \frac{1}{2} c^2}{\frac{1}{2}|c| \cdot |c|} = \epsilon.$$

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It follows that

$$P\left(\left|\eta_n^{-1} - c^{-1}\right| \geq \epsilon\right) \leq P\left(|\eta_n - c| \geq \delta\right) \to 0.$$

For (4), it suffices to show that for any bounded continuous function $g(x, y)$ we have

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For (4), it suffices to show that for any bounded continuous function $g(x,y)$ we have

$$Eg(\xi_n,\eta_n)\to Eg(\xi,c). \qquad (*)$$

If fact, choosing $g(x,y)=e^{it(x+y)}$ and $g(x,y)=e^{itxy}$ yields

$$Ee^{it(\xi_n+\eta_n)}\to Ee^{it(\xi+c)},\ Ee^{it(\xi_n\eta_n)}\to Ee^{it(c\xi)},$$

respectively, which completes the proof by the inverse limit theorem.

Now, suppose $g(x, y)$ is a continuous function with $|g(x, y)| \leq M$, then it is uniformly continuous in any bounded area. So for any given $\epsilon > 0$ and any $A > 0$ there exist a $\delta = \delta(A, \epsilon, g) > 0$ such that $|g(\xi_n, \eta_n) - g(\xi_n, c)| \leq \epsilon$ whenever $|\eta_n - c| \leq \delta$ and $|\xi_n| \leq A$.

Then

$$
\begin{aligned}
& |Eg(\xi_n,\eta_n) - Eg(\xi,c)| \\
\le\ & |Eg(\xi_n,\eta_n) - Eg(\xi_n,c)| + |Eg(\xi_n,c) - Eg(\xi,c)| \\
\le\ & E\left[|g(\xi_n,\eta_n) - g(\xi_n,c)|\right] + |Eg(\xi_n,c) - Eg(\xi,c)|
\end{aligned}
$$

Then

$$
\begin{aligned}
& |Eg(\xi_n,\eta_n)-Eg(\xi,c)| \\
\le\ & |Eg(\xi_n,\eta_n)-Eg(\xi_n,c)|+|Eg(\xi_n,c)-Eg(\xi,c)| \\
\le\ & E\left[|g(\xi_n,\eta_n)-g(\xi_n,c)|\right]+|Eg(\xi_n,c)-Eg(\xi,c)| \\
\le\ & \epsilon+2MP(|\eta_n-c|>\delta) \\
& +|Eg(\xi_n,c)-Eg(\xi,c)|+2MP(|\xi_n|>A).
\end{aligned}
$$

The second term will converge to zero because $\eta_n \xrightarrow{P} c$. The third will also converge to zero because $\xi_n \xrightarrow{d} \xi$ and $g(x,c)$ is a continuous function of x.

For the fourth term, we can choose A such that $\pm A$ is continuous points of the distribution function of ξ. Then $2MP(|\xi_n| > A)$ will converges to

$$2MP(|\xi| > A),$$

which can be smaller than the given $\epsilon > 0$ if A is large enough. Finally, by the arbitrariness of ϵ, $(*)$ is proved.

4. $\xi_n \xrightarrow{P} \xi$ if and only if

$$E\frac{|\xi_n - \xi|}{1 + |\xi_n - \xi|} \longrightarrow 0.$$

Proof. Let $F_n(x)$ denote the distribution function of $\xi_n - \xi$. Sufficiency: We have

$$P(|\xi_n - \xi| > \varepsilon) = \int_{|x|>\epsilon} dF_n(x) = \int_{|x|>\epsilon} \frac{1+|x|}{|x|} \frac{|x|}{1+|x|} dF_n(x)$$

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&\le \int_{|x|>\epsilon} \frac{1+\varepsilon}{\varepsilon} \frac{|x|}{1+|x|} dF_n(x)
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Proof. Let $F_n(x)$ denote the distribution function of $\xi_n - \xi$. Sufficiency: We have

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That is $\xi_n \xrightarrow{P} \xi$.

Necessity: For any $\varepsilon > 0$,

$$
\begin{aligned}
E\frac{|\xi_n - \xi|}{1 + |\xi_n - \xi|} &= \int_{-\infty}^{+\infty} \frac{|x|}{1+|x|} dF_n(x) \\
&= \int_{|x|<\varepsilon} \frac{|x|}{1+|x|} dF_n(x) + \int_{|x|\geq\varepsilon} \frac{|x|}{1+|x|} dF_n(x)
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&\le \frac{\varepsilon}{1+\varepsilon}+\int_{|x|\ge\varepsilon}dF_n(x) \\
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\end{aligned}
$$

Necessity: For any $\varepsilon > 0$,

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Since $\xi_n \xrightarrow{P} \xi$, first letting $n\to\infty$ and then letting $\varepsilon\to 0$ yield

$$E\frac{|\xi_n-\xi|}{1+|\xi_n-\xi|}\longrightarrow 0.$$

Let

$$\rho(\xi,\eta) = E\frac{|\xi-\eta|}{1+|\xi-\eta|}.$$

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Theorem

$\rho(\cdot,\cdot)$ satisfies

- $\rho(\xi,\eta)=0$ *if and only if* $P(\xi=\eta)=1$;
- $\rho(\xi,\eta)=\rho(\eta,\xi)$;
- $\rho(\xi,\tau)\le\rho(\xi,\eta)+\rho(\eta,\tau)$.

Let

$$\mathfrak{R} = \{\xi : \xi \text{ is a random variable on } (\Omega, \mathcal{F}, P)\}.$$

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Let

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Theorem

- $(\mathfrak{R}, \rho)$ *is a metric space;*
- $(\mathfrak{R}, \rho) = (\mathfrak{R}, \xrightarrow{P})$;
- $(\mathfrak{R}, \rho)$ *is complete, i.e.,* $\xi_n - \xi_m \xrightarrow{P} 0$ *as* $n, m \to \infty$ *if and only if there exists a random variable* ξ *such that* $\xi_n \xrightarrow{P} \xi$.

5. (*Dominated convergence theorem*) Suppose $\xi_n \xrightarrow{P} \xi$, $P(|\xi_n| \le \eta) = 1$ and $E\eta < \infty$. Then

$$E\xi_n \to E\xi.$$

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$$E\xi_n \to E\xi.$$

Proof. First, we have $P(|\xi| \le \eta) = 1$. In fact, for any $\epsilon > 0$,

$$\begin{aligned} P(|\xi| > \eta + \epsilon) &= P(|\xi| > \eta + \epsilon, |\xi_n - \xi| < \epsilon) \\ &\quad + P(|\xi| > \eta + \epsilon, |\xi_n - \xi| \ge \epsilon) \\ &\le P(|\xi_n - \xi| \ge \epsilon) \to 0, \end{aligned}$$

which implies $P(|\xi| \le \eta) = 1$.

Now, for any $\epsilon > 0$ and $M > 0$, we have

$$\begin{aligned}
|\xi_n - \xi| &\le \epsilon + |\xi_n - \xi| I\{|\xi_n - \xi| \ge \epsilon\} \\
&\le \epsilon + 2\eta I\{|\xi_n - \xi| \ge \epsilon\} \\
&\le \epsilon + 2M I\{|\xi_n - \xi| \ge \epsilon\} + 2\eta I\{\eta \ge M\} \quad a.s..
\end{aligned}$$

Now, for any $\epsilon > 0$ and $M > 0$, we have

$$\begin{aligned} |\xi_n - \xi| &\le \epsilon + |\xi_n - \xi| I\{|\xi_n - \xi| \ge \epsilon\} \\ &\le \epsilon + 2\eta I\{|\xi_n - \xi| \ge \epsilon\} \\ &\le \epsilon + 2M I\{|\xi_n - \xi| \ge \epsilon\} + 2\eta I\{\eta \ge M\} \;\; a.s.. \end{aligned}$$

For any $\epsilon > 0$, choose $M > 0$ large enough such that

$$E\eta I\{\eta \ge M\} = \int_{y \ge M} y dF_\eta(y) < \epsilon/4.$$

Then choose N large enough such that

$$P(|\xi_n - \xi| \ge \epsilon) < \epsilon/(4M), \; n \ge N.$$

Then for $n \geq N$,

$$\begin{aligned} & |E\xi_n - E\xi| \leq E|\xi_n - \xi| \\ \leq \ & \epsilon + 2MP(|\xi_n - \xi| \geq \epsilon) + 2E\eta I\{\eta \geq M\} < 2\epsilon. \end{aligned}$$

The applications of LLN

Example

Let $\{\xi_k, k \geq 1\}$ be a sequence of i.i.d. random variables with $E\xi_k = \mu$ and $Var\xi_k = \sigma^2$. Let

$$\bar{\xi}_n = \frac{1}{n}\sum_{k=1}^{n}\xi_k, \quad \widehat{\sigma}_n^2 = \frac{1}{n}\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)^2.$$

Prove that $\widehat{\sigma}_n^2 \xrightarrow{P} \sigma^2$ and find the asymptotic distribution of $\sqrt{n}\frac{\bar{\xi}_n-\mu}{\widehat{\sigma}_n}$.

Proof.

$$
\begin{aligned}
\widehat{\sigma}_n^2 &= \frac{1}{n}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2 \\
&= \frac{1}{n}\sum_{k=1}^{n}\left((\xi_k-\mu)-(\bar{\xi}_n-\mu)\right)^2 \\
&= \frac{1}{n}\sum_{k=1}^{n}(\xi_k-\mu)^2-(\bar{\xi}_n-\mu)^2.
\end{aligned}
$$

By the Khinchine weak LLN, we have $\bar{\xi}_n \xrightarrow{P} \mu$. Thus $\bar{\xi}_n-\mu \xrightarrow{P} 0$.

Moreover, since $\{(\xi_k-\mu)^2, k\ge 1\}$ is i.i.d. and $E(\xi_k-\mu)^2 = Var\xi_k = \sigma^2$, $\{(\xi_k-\mu)^2, k\ge 1\}$ also obeys the Khinchine weak LLN, i.e. $\sum_{k=1}^n(\xi_k-\mu)^2/n \xrightarrow{P} \sigma^2$. Hence $\widehat{\sigma}_n^2 \xrightarrow{P} \sigma^2$.

Moreover, since $\{(\xi_k-\mu)^2, k\ge 1\}$ is i.i.d. and $E(\xi_k-\mu)^2 = Var\xi_k = \sigma^2$, $\{(\xi_k-\mu)^2, k\ge 1\}$ also obeys the Khinchine weak LLN, i.e. $\sum_{k=1}^n(\xi_k-\mu)^2/n \xrightarrow{P} \sigma^2$. Hence $\widehat{\sigma}_n^2 \xrightarrow{P} \sigma^2$. By the Lindeberg-Lévy central limit theorem,

$$\sqrt{n}\frac{\bar{\xi}_n-\mu}{\sigma} = \frac{\sum_{k=1}^n(\xi_k-\mu)}{\sqrt{n\sigma^2}} \xrightarrow{d} N(0,1).$$

Hence

$$\sqrt{n}\frac{\bar{\xi}_n-\mu}{\widehat{\sigma}_n} = \frac{\sigma}{\widehat{\sigma}_n}\cdot\sqrt{n}\frac{\bar{\xi}_n-\mu}{\sigma} \xrightarrow{d} N(0,1).$$

Example Prove that for any $q > p > 0$,

$$\lim_{n\to\infty}\int_0^1\cdots\int_0^1 \frac{x_1^q+\cdots+x_n^q}{x_1^p+\cdots+x_n^p}dx_1\cdots dx_n = \frac{p+1}{q+1}.$$

Example Prove that for any $q > p > 0$,

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Proof. Let $\{\xi_i\}$ i.i.d. $\sim U(0,1)$, and let

$$\eta_n = \frac{\xi_1^q + \cdots + \xi_n^q}{\xi_1^p + \cdots + \xi_n^p}.$$

Then $0 \le \eta_n \le 1$ and

$$\int_0^1 \cdots \int_0^1 \frac{x_1^q + \cdots + x_n^q}{x_1^p + \cdots + x_n^p} dx_1 \cdots dx_n = E\eta_n.$$

On the other hand, by WLLN,

$$\frac{1}{n}\sum_{k=1}^{n}\xi_k^q \xrightarrow{P} E\xi_1^q = \frac{1}{q+1}$$
$$\frac{1}{n}\sum_{k=1}^{n}\xi_k^p \xrightarrow{P} E\xi_1^p = \frac{1}{p+1}.$$

So,

$$\eta_n \xrightarrow{P} \frac{E\xi_1^q}{E\xi_1^p} = \frac{p+1}{q+1}.$$

Hence

$$\int_0^1 \cdots \int_0^1 \frac{x_1^q + \cdots + x_n^q}{x_1^p + \cdots + x_n^p} dx_1 \cdots dx_n = E\eta_n \to \frac{p+1}{q+1}.$$

Convergence in mean of order r :

Definition

Let $r > 0$, ξ and $\{\xi_n, n \geq 1\}$ be random variables defined on $(\Omega, \mathcal{F}, P)$ with $E|\xi|^r < \infty$ and $E|\xi_n|^r < \infty$. If

$$E|\xi_n - \xi|^r \longrightarrow 0,$$

then we say that $\{\xi_n, n \geq 1\}$ converges in mean of order r to ξ, denoted by $\xi_n \xrightarrow{L_r} \xi$.

$$\xi_n \xrightarrow{L_r} \xi \quad \Rightarrow \quad \xi_n \xrightarrow{P} \xi.$$

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$$\xi_n \xrightarrow{L_r} \xi \quad \not\Leftarrow \quad \xi_n \xrightarrow{P} \xi.$$

Definition

Define ξ_n by $P(\xi_n = n) = 1/\log(n+3)$, $P(\xi_n = 0) = 1 - 1/\log(n+3)$, $n = 1, 2, \cdots$. It is easy to know $\xi_n \xrightarrow{P} 0$, but for any $0 < r < \infty$,

$$E|\xi_n|^r = \frac{n^r}{\log(n+3)} \longrightarrow \infty.$$

That is, $\xi_n \xrightarrow{L_r} 0$ does not hold true.

Theorem

Suppose $\xi_n \xrightarrow{P} \xi$, $P(|\xi_n| \le \eta) = 1$, *and* $E\eta^r < \infty$. *Then*

$$\xi_n \xrightarrow{L_r} \xi.$$

Theorem

Suppose $\xi_n \xrightarrow{P} \xi$, $P(|\xi_n| \le \eta) = 1$, *and* $E\eta^r < \infty$. *Then*

$$\xi_n \xrightarrow{L_r} \xi.$$

Proof. Since $\xi_n \xrightarrow{P} \xi$, so $P(|\xi| \le \eta) = 1$. Then

$$P(|\xi_n - \xi|^r \le 2^r\eta^r) = 1, \quad |\xi_n - \xi|^r \xrightarrow{P} 0.$$

Theorem

Suppose $\xi_n \xrightarrow{P} \xi$, $P(|\xi_n| \le \eta) = 1$, *and* $E\eta^r < \infty$. *Then*

$$\xi_n \xrightarrow{L_r} \xi.$$

Proof. Since $\xi_n \xrightarrow{P} \xi$, so $P(|\xi| \le \eta) = 1$. Then

$$P(|\xi_n - \xi|^r \le 2^r\eta^r) = 1, \quad |\xi_n - \xi|^r \xrightarrow{P} 0.$$

By the dominated convergence theorem,

$$E[|\xi_n - \xi|^r] \to 0.$$

附录:

Theorem

Suppose $E[|\xi_n|^r] < \infty$, $E[|\xi|^r] < \infty$. *Then* $\xi_n \xrightarrow{L_r} \xi$ *if and only if*

$$\xi_n \xrightarrow{P} \xi$$

and

$$\lim_{M\to\infty} \sup_n E[|\xi_n|^r I\{|\xi_n| \geq M\}] = 0.$$

$$(\textit{uniformly integrable}).$$

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$$\xi_n \xrightarrow{P} \xi \implies \xi_{n,M} \xrightarrow{P} \xi_M.$$

Proof. For the "if part", for any $M>0$, let $\xi_{n,M}=(-M)\vee\xi_n\wedge M$, $\xi_M=(-M)\vee\xi\wedge M$.

$$\xi_n\xrightarrow{P}\xi \implies \xi_{n,M}\xrightarrow{P}\xi_M.$$

By the dominated convergence theorem,

$$E[|\xi_{n,M}|^r]\overset{n\to\infty}{\longrightarrow}E[|\xi_M|^r],\quad E[|\xi_{n,M}-\xi_M|^r]\overset{n\to\infty}{\longrightarrow}0.$$

Hence

$$E[|\xi|^r] = \lim_{M\to\infty} E[|\xi_M|^r]$$
$$= \lim_{M\to\infty} \lim_{n\to\infty} E[|\xi_{n,M}|^r] \le \limsup_n E[|\xi_n|^r] < \infty.$$

Hence

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Note $|\xi - \xi_M| = 0$ if $|\xi| \le M$, and $= |\xi| - M$ if $|\xi| > M$. So

$$\begin{aligned}|\xi_n - \xi| \le & |\xi_{n,M} - \xi_M| + |\xi_n - \xi_{n,M}| + |\xi - \xi_M| \\ \le & |\xi_{n,M} - \xi_M| + |\xi_n| I\{|\xi_n| \ge M\} + |\xi| I\{|\xi| \ge M\}.\end{aligned}$$

$$|\xi_n - \xi|^r \le 3^r\Big(|\xi_{n,M} - \xi_M|^r + |\xi_n|^r I\{|\xi_n| \ge M\} + |\xi|^r I\{|\xi| \ge M\}\Big).$$

$$|\xi_n - \xi|^r \le 3^r\big(|\xi_{n,M} - \xi_M|^r + |\xi_n|^r I\{|\xi_n| \ge M\} + |\xi|^r I\{|\xi| \ge M\}\big).$$

So,

$$\begin{aligned}
&\limsup_{n\to\infty} E[|\xi_n - \xi|^r] \\
\le& \lim_{M\to\infty} \limsup_{n\to\infty} 3^r \Big\{ E[|\xi_{n,M} - \xi_M|^r] \\
&\qquad + E[|\xi_n|^r I\{|\xi_n| \ge M\}] + E[|\xi|^r I\{|\xi| \ge M\}] \Big\} \\
=&0.
\end{aligned}$$

For the "only if part", it is obvious that $\xi_n \xrightarrow{L_r} \xi \implies \xi_n \xrightarrow{P} \xi$.

Hence

$$E[|\xi_{n,M} - \xi_M|^r] \to 0.$$

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It follows that

$$E\Big[\big|(\xi_n - \xi_{n,M}) - (\xi - \xi_M)\big|^r\Big] \to 0.$$

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Note $|\xi - \xi_M| = 0$ if $|\xi| \le M$, and $= |\xi| - M$ if $|\xi| > M$. So

$$\begin{aligned}
&|\xi_n - \xi_{n,M}| \\
\le & |\xi - \xi_M| + \big|(\xi_n - \xi_{n,M}) - (\xi - \xi_M)\big| \\
\le & |\xi| I\{|\xi| \ge M\}| + \big|(\xi_n - \xi_{n,M}) - (\xi - \xi_M)\big|,
\end{aligned}$$

$$
\begin{aligned}
|\xi_n - \xi_{n,M}|^r \leq & 2^r |\xi| I\{|\xi| \geq M\}|^r \\
& + 2^r \Big|(\xi_n - \xi_{n,M}) - (\xi - \xi_M)\Big|^r.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
& \limsup_{n\to\infty} E\left[|\xi_n - \xi_{n,M}|^r\right] \\
\leq & 2^r E\left[|\xi|^r I\{|\xi| \geq M\}|\right] \to 0, \quad M \to \infty.
\end{aligned}
$$

Note again $|\xi_n - \xi_{n,M}| = 0$ if $|\xi_n| \le M$, and $= |\xi_n| - M$ if $|\xi_n| > M$.

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$$|\xi_n - \xi_{n,M}| \ge \frac{1}{2}|\xi_n| \quad \text{if } |\xi_n| \ge 2M.$$

That is

$$|\xi_n| I\{|\xi_n| \ge 2M\} \le 2|\xi_n - \xi_{n,M}|.$$

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It follows that

$$\lim_{M\to\infty} \limsup_{n\to\infty} E\left[|\xi_n|^r I\{|\xi_n| \ge 2M\}\right] = 0.$$

Hence, for any $\epsilon > 0$, there exist $M_0 > 0$ and $N \geq 1$ such that

$$E[|\xi_n|^r I\{|\xi_n| \geq M\}] \leq \epsilon \quad \text{for all } M \geq M_0, n \geq N.$$

Hence, for any $\epsilon > 0$, there exist $M_0 > 0$ and $N \geq 1$ such that

$$E[|\xi_n|^r I\{|\xi_n| \geq M\}] \leq \epsilon \quad \text{for all } M \geq M_0, n \geq N.$$

For each $n = 1, \ldots, N$, there exists an M_n such that

$$E[|\xi_n|^r I\{|\xi_n| \geq M\}] \leq \epsilon \quad \text{for all } M \geq M_n.$$

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For each $n = 1, \ldots, N$, there exists an M_n such that

$$E[|\xi_n|^r I\{|\xi_n| \geq M\}] \leq \epsilon \quad \text{for all } M \geq M_n.$$

Choose $M^* = \max\{M_0, M_1, \cdots, M_N\}$, then

$$\sup_n E[|\xi_n|^r I\{|\xi_n| \geq M\}] \leq \epsilon \quad \text{for all } M \geq M^*.$$