Notes on Evans' Partial Differential Equations

张钰奇

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Part I

REPPRESENTATION FORMULAS FOR SOLUTIONS

Chapter 2

FOUR IMPORTANT LINEAR PARTIAL DIFFERENTIAL EQUATIONS

2.1 TRANSPORT EQUATION

2.1.1 Initial-value problem

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ORIGINAL 1. we deduce

$$u(x,t) = g(x - tb) \quad (x \in \mathbb{R}^n, t \geq 0). \tag{3}$$

It is easy to check that

$$\begin{aligned} u_t + b \cdot Du &= (Dg)(x - tb) \cdot (-b) + b \cdot (Dg)(x - tb) \\ &= 0, \end{aligned}$$

and

$$u(x, 0) = g(x).$$

2.2 LAPLACE'S EQUATION

2.2.1 Fundamental solution.

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ORIGINAL 1.

DEFINITION. *A C^2 function u satisfying (1) is called a harmonic function.*

If n is even, then

$$f(x) = \sum_{i=1}^{n} (-1)^i x_i^2$$

is harmonic.

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ORIGINAL 1.

DEFINITION. *The function*

$$\Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & (n = 2) \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & (n \geq 3), \end{cases} \tag{6}$$

defined for $x \in \mathbb{R}^n, x \neq 0$, is the fundamental solution of Laplace's equation.

Verifying the fundamental solution If $n = 2$,

$$\Phi_{x_i} = -\frac{x_i}{2\pi|x|^2};$$

If $n \geq 3$,

$$\begin{aligned} \Phi_{x_i} &= \frac{(2-n)x_i}{n(n-2)\alpha(n)|x|^n} \\ &= -\frac{x_i}{n\alpha(n)|x|^n}. \end{aligned}$$

Thus

$$\Phi_{x_i} = -\frac{x_i}{n\alpha(n)|x|^n}$$

for all $n \geq 2$. Therefore

$$\begin{aligned}
\Phi_{x_i x_i} &= -\frac{1}{n\alpha(n)} \frac{|x|^n - n|x|^{n-2}x_i^2}{|x|^{2n}} \\
&= -\frac{1}{n\alpha(n)} \left(\frac{1}{|x|^n} - \frac{nx_i^2}{|x|^{n+2}} \right) \\
&= -\frac{1}{n\alpha(n)|x|^n} \left(1 - \frac{nx_i^2}{|x|^2} \right),
\end{aligned}$$

and

$$\Delta \Phi = 0.$$

Integrating the fundamental solution If $n = 2$, then

$$\begin{aligned}
\int_{B(0,r)} \Phi(x)dx &= -\frac{1}{2\pi} \int_{B(0,r)} \log|x|dx \\
&= -\frac{1}{2\pi} \int_0^r (\log s) 2\pi s ds \\
&= -\int_0^r s \log s ds \\
&= -\frac{1}{2} r^2 \log r + \frac{1}{4} r^2.
\end{aligned}$$

If $n \geq 3$, then

$$\begin{aligned}
\int_{B(0,r)} \Phi(x)dx &= \frac{1}{n(n-2)\alpha(n)} \int_{B(0,r)} \frac{1}{|x|^{n-2}} dx \\
&= \frac{1}{n(n-2)\alpha(n)} \int_0^r \frac{n\alpha(n)s^{n-1}}{s^{n-2}} ds \\
&= \frac{1}{n-2} \frac{s^2}{2} \Big|_0^r \\
&= \frac{r^2}{2(n-2)}.
\end{aligned}$$

Hence

$$\int_{B(0,r)} \Phi(x)dx = \begin{cases} -\frac{1}{2} r^2 \log r + \frac{1}{4} r^2 & n = 2 \\ \frac{r^2}{2(n-2)} & n \geq 3. \end{cases}$$

ORIGINAL 2. Observe also that we have the estimates

$$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}, |D^2\Phi(x)| \leq \frac{C}{|x|^n} \quad (x \neq 0) \tag{7}$$

for some constant $C > 0$.

If $i \neq j$,

$$\Phi_{x_i x_j} = -\frac{x_i}{n\alpha(n)}(-n)|x|^{-n-1}\frac{x_j}{|x|} = \frac{x_i x_j}{\alpha(n)|x|^{n+2}}.$$

So

$$\Phi_{x_i x_j} = -\frac{1}{n\alpha(n)|x|^n}\left(\delta_{ij} - \frac{n x_i x_j}{|x|^2}\right),$$

where $\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$ Hence

$$\begin{aligned} D^2\Phi(x) : D^2\Phi(x) &= \sum_{i=1}^n \sum_{j=1}^n \Phi_{x_i x_j}^2 \\ &= \frac{1}{n^2\alpha^2(n)|x|^{2n}} \sum_{i=1}^n \sum_{j=1}^n \left(\delta_{ij}^2 - \frac{2n\delta_{ij}x^i x_j}{|x|^2} + \frac{n^2 x_i^2 + x_j^2}{|x|^4}\right) \\ &= \frac{1}{n^2\alpha^2(n)|x|^{2n}}(n - 2n + n^2) \\ &= \frac{n-1}{n\alpha^2(n)|x|^{2n}}, \end{aligned}$$

$$|D^2\Phi(x)| = \frac{\sqrt{n-1}}{\sqrt{n}\alpha(n)|x|^n}.$$

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ORIGINAL 1. We have

$$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy = \int_{\mathbb{R}^n} \Phi(y)f(x-y)dy; \tag{9}$$

$$\int_{\mathbb{R}^n} \Phi(x-y)f(y)dy = \int_{\mathbb{R}^n} \Phi(z)f(x-z)dz = \int_{\mathbb{R}^n} \Phi(y)f(x-y)dy.$$

ORIGINAL 2. But

$$\frac{f(x+he_i-y) - f(x-y)}{h} \to f_{x_i}(x-y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$,

$f_{x_i}(x)$ is uniformly continuous on $\mathbb{R}^n$, thus for any $\varepsilon > 0$, there exists a positive number $\delta > 0$, independent of x, such that $|f_{x_i}(x+he_i) - f_{x_i}(x)| < \varepsilon$, if $|h| < \delta$. According to Lagrange's mean value theorem,

$$\frac{f(x+he_i-y)-f(x-y)}{h} = f_{x_i}(x+\xi e_i - y),$$

where $0 < \xi h$. Therefore if $|h| < \delta$,

$$\left|\frac{f(x+he_i-y)-f(x-y)}{h} - f_{x_i}(x-y)\right| = |f_{x_i}(x+\xi e_i - y) - f_{x_i}(x-y)| < \varepsilon,$$

where δ is independent of $x-y$. Hence

$$\frac{f(x+he_i-y)-f(x-y)}{h} \to f_{x_i}(x-y)$$

uniformly on $\mathbb{R}^n$ as $h \to 0$.

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ORIGINAL 1. Now

$$|I_\varepsilon| \le \|D^2 f\|_{L^\infty(\mathbb{R}^n)} \int_{B(0,\varepsilon)} |\Phi(y)|dy \le \begin{cases} C\varepsilon^2 |\log \varepsilon| & (n=2) \\ C\varepsilon^2 & (n \ge 3). \end{cases} \tag{12}$$

$$|\Delta_x f(x-y)| \le \sum_{i=1}^n |f_{x_i x_i}(x-y)| \le n|D^2 f(x-y)| \le n\|D^2 f\|_{L^\infty(\mathbb{R}^n)}.$$

If $n = 2$,

$$\begin{aligned}\int_{B(0,\varepsilon)} |\Phi(y)|dy &= -\frac{1}{2\pi}\int_{B(0,\varepsilon)} \log|y|dy = -\frac{1}{2\pi}\int_0^\varepsilon \log r \cdot 2\pi r dr \\ &= -\int_0^\varepsilon r\log r dr = -\frac{1}{2}r^2\left(\log r - \frac{1}{2}\right)\Big|_0^\varepsilon \\ &= -\frac{1}{2}\varepsilon^2\left(\log\varepsilon - \frac{1}{2}\right) = -\frac{1}{2}\varepsilon^2\log\varepsilon + \frac{1}{4}\varepsilon^2;\end{aligned}$$

If $n \ge 3$,

$$\begin{aligned}\int_{B(0,\varepsilon)} |\Phi(y)|dy &= \frac{1}{n(n-2)\alpha(n)}\int_{B(0,\varepsilon)} \frac{1}{|y|^{n-2}}dy \\ &= \frac{1}{n(n-2)\alpha(n)}\int_0^\varepsilon \frac{1}{r^{n-2}}\cdot n\alpha(n)r^{n-1}dr = \frac{1}{n-2}\int_0^\varepsilon r dr \\ &= \frac{r^2}{2(n-2)}\Big|_0^\varepsilon = \frac{\varepsilon^2}{2(n-2)}.\end{aligned}$$

ORIGINAL 2.

$$+\int_{\partial B(0,\varepsilon)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y) dS(y)$$

The $\frac{\partial f}{\partial \nu}(x-y)$ here should be understood as $\frac{\partial g}{\partial \nu}(y)$, where $g(y) = f(x-y)$.

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ORIGINAL 1. We sometimes write

$$-\Delta\Phi = \delta_0 \quad \text{in } \mathbb{R}^n,$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0.

Since

$$D\Phi(x) = -\frac{(x_1, \ldots, x_n)}{n\alpha(n)|x|^n}$$

for all integers $n \geq 2$,

$$\begin{aligned}
\int_{\partial B(0,r)} -D\Phi \cdot \nu dS &= \int_{\partial B(0,r)} |D\Phi| dS \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,r)} \frac{1}{|x|^{n-1}} dS \\
&= \frac{1}{n\alpha(n)} \frac{n\alpha(n) r^{n-1}}{r^{n-1}} \\
&= 1.
\end{aligned}$$

2.2.2 Mean-value formulas.

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ORIGINAL 1.

$$\phi(r) := ⨍_{\partial B(x,r)} u(y) dS(y) = ⨍_{\partial B(0,1)} u(x+rz) dS(z).$$

$$\begin{aligned}
⨍_{\partial B(x,r)} u(y) dS(y) &= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(x,r)} u(y) dS(y) \\
&= \frac{1}{n\alpha(n) r^{n-1}} \int_{\partial B(0,1)} u(x+rz) r^{n-1} dS(z) \\
&= \frac{1}{n\alpha(n)} \int_{\partial B(0,1)} u(x+rz) dS(z) \\
&= ⨍_{\partial B(0,1)} u(x+rz) dS(z).
\end{aligned}$$

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ORIGINAL 1.

THEOREM (Estimates on derivatives)**.** *Assume u is harmonic in U. Then*

$$|D^{\alpha} u\left(x_{0}\right)| \leq \frac{C_{k}}{r^{n+k}}\|u\|_{L^{1}\left(B\left(x_{0}, r\right)\right)} \tag{18}$$

for each ball $B\left(x_{0}, r\right) \subset U$ and each multiindex α of order $|\alpha|=k$.

Here

$$C_{0}=\frac{1}{\alpha(n)}, C_{k}=\frac{\left(2^{n+1} n k\right)^{k}}{\alpha(n)} \quad(k=1, \ldots). \tag{19}$$

If $k=1$, then

$$|u_{x_{i}}(x_{0})| \leq\left(\frac{2}{r}\right)^{n+1} \frac{n}{\alpha(n)}\|u\|_{L^{1}\left(B\left(x_{0}, r\right)\right)}.$$

2.2.3 Properties of harmonic functions.

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ORIGINAL 1. By calculations similar to those in (20), we establish that

$$|D^{\alpha} u(x_{0})| \leq \frac{n k}{r}\|D^{\beta} u\|_{L^{\infty}\left(\partial B\left(x_{0}, \frac{r}{k}\right)\right)}.$$

$$\begin{aligned}
|D^{\alpha} u(x_{0})| &= \left|⨍_{B\left(x_{0}, \frac{r}{k}\right)} D^{\alpha} u(x) d x\right| \\
&= \left|\frac{k^{n}}{\alpha(n) r^{n}} \int_{\partial B\left(x_{0}, \frac{r}{k}\right)} D^{\beta} u \cdot \nu_{i} d S\right| \\
&\leq \frac{n k}{r}\|D^{\beta} u\|_{L^{\infty}\left(\partial B\left(x_{0}, \frac{r}{k}\right)\right)}.
\end{aligned}$$

ORIGINAL 2. Combining the two previous estimates yields the bound

$$|D^{\alpha} u(x_{0})| \leq \frac{(2^{n+1} n k)^{k}}{\alpha(n) r^{n+k}}\|u\|_{L^{1}\left(B\left(x_{0}, r\right)\right)}. \tag{21}$$

$$
\begin{aligned}
|D^\alpha u(x_0)| &\leq \frac{nk}{r} \frac{(2^{n+1}n(k-1))^{k-1}}{\alpha(n)\left(\frac{k-1}{k}r\right)^{n+k-1}} \|u\|_{L^1(B(x_0,r))} \\
&= \frac{(2^{n+1}nk)^k}{\alpha(n)r^{n+k}} \frac{k^n}{2^{n+1}(k-1)^n} \|u\|_{L^1(B(x_0,r))} \\
&\leq \frac{(2^{n+1}nk)^k}{\alpha(n)r^{n+k}} \|u\|_{L^1(B(x_0,r))}.
\end{aligned}
$$

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ORIGINAL 1. Since $\Phi(x) \to 0$ as $|x| \to \infty$ for $n \leq 3$, $\tilde{u} := \int_{\mathbb{R}^n} \Phi(x-y)f(y)dy$ is a bounded solution of $-\Delta u = f$ in $\mathbb{R}^n$.

2.2.4 Green's function.

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ORIGINAL 1. The function

$$K(x,y) := \frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n} \qquad (x \in \mathbb{R}^n_+, y \in \partial\mathbb{R}^n_+)$$

is *Poisson's kernel* for $\mathbb{R}^n_+$, and (33) is *Poisson's formula.*

If $n = 1$, then

$$K(x,y) = \frac{2x}{2} \frac{1}{x} = 1.$$

If $n = 2$, then

$$K(x,y) = \frac{2x_2}{2\pi} \frac{1}{|x-y|^2} = \frac{x_2}{\pi |x-y|^2}.$$

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ORIGINAL 1. Thus $x \mapsto -\frac{\partial G}{\partial y_n}(x,y) = K(x,y)$ is harmonic for $x \in \mathbb{R}^n_+$, $y \in \partial\mathbb{R}^n_+$.

$$
\begin{aligned}
\Delta_x \left(-\frac{\partial G}{\partial y_n}(x,y) \right) &= -\sum_{i=1}^n \frac{\partial^3 G}{\partial x_i^2 \partial y_n}(x,y) \\
&= -\frac{\partial}{\partial y_n} \sum_{i=1}^n \frac{\partial^2 G}{\partial x_i^2}(x,y)
\end{aligned}
$$

$$= 0.$$

ORIGINAL 2.

$$\leq \frac{2^{n+2}\|g\|_{L^\infty} x_n}{n\alpha(n)} \int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} |y - x^0|^{-n} dy$$

$$\begin{aligned}
\int_{\partial\mathbb{R}^n_+ - B(x^0,\delta)} |y - x^0|^{-n} dy &= \int_\delta^\infty r^{-n}(n-1)\alpha(n-1)r^{n-2} dr \\
&= (n-1)\alpha(n-1) \int_\delta^\infty r^{-2} dr \\
&= (n-1)\alpha(n-1)\frac{1}{\delta}
\end{aligned}$$

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ORIGINAL 1.

$$= \frac{-1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{(|x||y - \tilde{x}|)^n} = -\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x - y|^n}$$

if $y \in \partial B(0,1)$.

LEMMA. *If $x, y \in \mathbb{R}^n$, and $x, y \neq 0$, then*

$$\left| \frac{x}{|x|} - |x|y \right| = \left| \frac{y}{|y|} - |y|x \right|.$$

Proof.

$$\left| \frac{x}{|x|} - |x|y \right|^2 = 1 - 2x \cdot y + |x|^2|y|^2 = \left| \frac{y}{|y|} - |y|x \right|^2.$$

□

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ORIGINAL 1. We change variables to obtain Poisson's formula

$$u(x) = \frac{r^2 - |x|^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{|x - y|^n} dS(y) \quad (x \in B^0(0,r)).$$

If $x = 0$, then

$$\begin{aligned}
u(x) &= \frac{r^2}{n\alpha(n)r} \int_{\partial B(0,r)} \frac{g(y)}{r^n} dS(y) \\
&= \fint_{\partial B(0,r)} g(y) dS(y).
\end{aligned}$$

2.2.5 Energy methods.

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ORIGINAL 1. and so an integration by parts shows

$$0 = -\int_U w\Delta w dx = \int_U |Dw|^2 dx.$$

$$w|_{\partial U} = u|_{\partial U} - \tilde{u}|_{\partial U} = g - g = 0,$$

$$\begin{aligned} -\int_U w\Delta w dx &= \int_U Dw \cdot Dw dx - \int_{\partial U} \frac{\partial w}{\partial \nu} w dS \text{ (by Theorem 3. (ii) in C.2.)} \\ &= \int_U |Dw|^2 dx. \end{aligned}$$

2.3 HEAT EQUATION

2.3.1 Fundamental solution.

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ORIGINAL 1. Let us insert (4) into (1) and thereafter compute

$$\alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) = 0 \tag{5}$$

for $y := t^{-\beta} x$.

$$u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right) = \frac{1}{t^\alpha} v(y),$$

$$\begin{aligned} u_t &= -\alpha t^{-(\alpha+1)} v\left(\frac{x}{t^\beta}\right) + t^{-\alpha} Dv\left(\frac{x}{t^\beta}\right) \cdot x(-\beta) t^{-\beta-1} \\ &= -\alpha t^{-(\alpha+1)} v(y) - \beta t^{-(\alpha+1)} y \cdot Dv(y), \end{aligned}$$

$$\begin{aligned} \Delta u &= \frac{1}{t^\alpha} \frac{1}{t^{2\beta}} \Delta v\left(\frac{x}{t^\beta}\right) \\ &= t^{-(\alpha+2\beta)} \Delta v(y), \end{aligned}$$

$$\begin{aligned} -u_t + \Delta u &= \alpha t^{-(\alpha+1)} v(y) + \beta t^{-(\alpha+1)} y \cdot Dv(y) + t^{-(\alpha+2\beta)} \Delta v(y) \\ &= 0. \end{aligned}$$

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ORIGINAL 1. Thereupon (6) becomes

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0,$$

for $r = |y|$, $' = \frac{d}{dr}$.

$$\begin{aligned} v(y) &= w(|y|) = w(r), \\ Dv &= \frac{w'}{r}(y_1, \dots, y_n), \\ y \cdot Dv &= r w'. \end{aligned}$$

$$\begin{aligned} v_{y_i} &= \frac{w' y_i}{r}, \\ v_{y_i y_i} &= w'' \frac{y_i^2}{r^2} + \frac{w'}{r} - w' \frac{y_i^2}{r^3}, \\ \Delta v &= w'' + \frac{n w'}{r} - \frac{w'}{r} \\ &= w'' + \frac{n-1}{r} w'. \end{aligned}$$

ORIGINAL 2. But then for some constant b

$$w = b e^{-\frac{r^2}{4}}. \tag{7}$$

$$\begin{aligned} w = b e^{-\frac{r^2}{4}}, w' &= -\frac{1}{2} b r e^{-\frac{r^2}{4}}, \\ \lim_{r\to\infty} \frac{1}{2} r^n w &= \lim_{r\to\infty} \frac{1}{2} b r^n e^{-\frac{r^2}{4}} = 0, \\ \lim_{r\to\infty} r^{n-1} w' &= \lim_{r\to\infty} -\frac{1}{2} b r^n e^{-\frac{r^2}{4}} = 0. \end{aligned}$$

ORIGINAL 3.

DEFINITION. *The function*

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} & (x \in \mathbb{R}^n, t > 0) \\ 0 & (x \in \mathbb{R}^n, t < 0) \end{cases}$$

is called the fundamental solution of the heat equation.

If $n = 1$, then

$$\Phi(x,t) := \begin{cases} \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} & (x \in \mathbb{R}, t > 0) \\ 0 & (x \in \mathbb{R}, t < 0) \end{cases}.$$

The graph of this function is in the NB file.

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ORIGINAL 1. Interpretation of fundamental solution. In view of Theorem 1 we sometimes write

$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

δ_0 denoting the Dirac measure on $\mathbb{R}^n$ giving unit mass to the point 0 .

$$\begin{cases} \Phi_t(x-y,t) - \Delta\Phi(x-y,t) = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi(x-y,t) = \delta_y(x) & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{t = 0\}, \end{cases}$$

$$g(x) = \int_{\mathbb{R}^n} g(y)\delta_y(x)dy,$$

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)dy.$$

2.3.2 Mean-value formula.

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ORIGINAL 1. Write $E(r) = E(0,0;r)$ and set

$$\begin{aligned} \phi(r) &:= \frac{1}{r^n} \iint_{E(r)} u(y,s)\frac{|y|^2}{s^2} dyds \\ &= \iint_{E(1)} u(ry, r^2 s)\frac{|y|^2}{s^2} dyds. \end{aligned} \tag{20}$$

$$E(r) = \left\{(y,s) \in \mathbb{R}^{n+1} \mid s \le 0, \Phi(-y,-s) \ge \frac{1}{r^n}\right\}.$$

Let $y = ry'$, $s = r^2 s'$, then

$$\begin{aligned}
\Phi(-y,-s) &= \Phi(-ry', -r^2 s') \\
&= \frac{1}{[4\pi(-r^2 s')]^{n/2}} e^{-\frac{|-ry'|^2}{4(-r^2 s')}} \\
&= \frac{1}{r^n} \frac{1}{[4\pi(-s')]^{n/2}} e^{-\frac{|y'|^2}{4(-s')}} \\
&= \frac{1}{r^n} \Phi(-y', -s')
\end{aligned}$$

for $-s > 0$. Thus $(y,s) \in E(r)$ if and only if $(y', s') \in E(1)$. Hence

$$\begin{aligned}
\frac{1}{r^n} \iint_{E(r)} u(y,s) \frac{|y|^2}{s^2} dyds &= \frac{1}{r^n} \iint_{E(1)} u(ry', r^2 s') \frac{|ry'|^2}{(r^2 s')^2} r^{n+2} dy' ds' \\
&= \iint_{E(1)} u(ry', r^2 s') \frac{|y'|^2}{(s')^2} dy' ds' \\
&= \iint_{E(1)} u(ry, r^2 s) \frac{|y|^2}{s^2} dyds.
\end{aligned}$$

ORIGINAL 2. We compute

$$\begin{aligned}
\phi'(r) &= \iint_{E(1)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2r u_s \frac{|y|^2}{s} dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2 u_s \frac{|y|^2}{s} dyds \\
&=: A + B.
\end{aligned}$$

$$\begin{aligned}
\phi'(r) &= \iint_{E(1)} \sum_{i=1}^{n} u_{y_i}(ry', r^2 s') y_i' \frac{|y'|^2}{(s')^2} + 2r u_s(ry', r^2 s') \frac{|y'|^2}{s'} dy' ds' \\
&= \iint_{E(r)} \left(\sum_{i=1}^{n} u_{y_i}(y,s) \frac{y_i}{r} \frac{|y/r|^2}{(s/r^2)^2} + 2r u_s(y,s) \frac{|y/r|^2}{s/r^2} \right) \frac{1}{r^{n+2}} dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i}(y,s) y_i \frac{|y|^2}{s^2} + 2 u_s(y,s) \frac{|y|^2}{s} dyds \\
&= \frac{1}{r^{n+1}} \iint_{E(r)} \sum_{i=1}^{n} u_{y_i} y_i \frac{|y|^2}{s^2} + 2 u_s \frac{|y|^2}{s} dyds.
\end{aligned}$$

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ORIGINAL 1. Also, let us introduce the useful function

$$\psi := -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} + n\log r \tag{21}$$

and observe $\psi = 0$ on $\partial E(r)$, since $\Phi(y,-s) = r^{-n}$ on $\partial E(r)$.

$$\Phi(y,-s) = (-4\pi s)^{-\frac{n}{2}} e^{\frac{|y|^2}{4s}} = r^{-n}$$

on $\partial E(r)$, so

$$\log\Phi(y,-s) = -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s} = -n\log r.$$

Therefore $\psi = 0$ on $\partial E(r)$.

ORIGINAL 2. We utilize (21) to write

$$\begin{aligned} B &= \frac{1}{r^{n+1}}\iint_{E(r)} 4u_s \sum_{i=1}^n y_i\psi_{y_i}\, dyds \\ &= -\frac{1}{r^{n+1}}\iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^n u_{sy_i} y_i \psi\, dyds; \end{aligned}$$

there is no boundary term since $\psi = 0$ on $\partial E(r)$.

1.

$$\begin{aligned} \psi_{y_i} &= \frac{1}{4s}2|y|\frac{y_i}{|y|} = \frac{y_i}{2s}, \\ \sum_{i=1}^n y_i\psi_{y_i} &= \frac{|y|^2}{2s}, \\ B &= \frac{1}{r^{n+1}}\iint_{E(r)} 2u_s\frac{|y|^2}{s}dyds \\ &= \frac{1}{r^{n+1}}\iint_{E(r)} 4u_s\sum_{i=1}^n y_i\psi_{y_i}dyds. \end{aligned}$$

2.

$$\begin{aligned} 4u_s\sum_{i=1}^n y_i\psi_{y_i} &= 4u_s y\cdot D\psi \\ &= 4y\cdot(u_s D\psi) \\ &= 4y\cdot[D(u_s\psi) - \psi Du_s] \\ &= 2D|y|^2\cdot D(u_s\psi) - 4\psi y\cdot Du_s, \end{aligned}$$

$$\iint_{E(r)} 2D|y|^2 \cdot D(u_s\psi)dyds = -\iint_{E(r)} 4nu_s\psi dyds \quad \text{(by Theorem 3 in C.2)},$$

hence

$$\begin{aligned} B &= \frac{1}{r^{n+1}} \iint_{E(r)} 4u_s \sum_{i=1}^{n} y_i \psi_{y_i} dyds \\ &= -\frac{1}{r^{n+1}} \iint_{E(r)} 4nu_s\psi + 4\sum_{i=1}^{n} u_{sy_i} y_i \psi dyds. \end{aligned}$$

ORIGINAL 3. Thus ϕ is constant, and therefore

$$\phi(r) = \lim_{t\to 0} \phi(t) = u(0,0) \left(\lim_{t\to 0} \frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds \right) = 4u(0,0),$$

as

$$\frac{1}{t^n} \iint_{E(t)} \frac{|y|^2}{s^2} dyds = \iint_{E(1)} \frac{|y|^2}{s^2} dyds = 4.$$

We omit the details of this last computation.

1.

$$\begin{aligned} E(1) &= \{(y,s) \in \mathbb{R}^{n+1} \mid \Phi(y,-s) \geq 1\}. \\ \Phi(y,-s) &= \frac{1}{(-4\pi s)^{\frac{n}{2}}} e^{\frac{1}{4s}|y|^2}, \\ \log \Phi(y,-s) &= -\frac{n}{2} \log(-4\pi s) + \frac{|y|^2}{4s}. \end{aligned}$$

Thus

$$E(1) = \left\{ (y,s) \in \mathbb{R}^{n+1} \mid |y| \leq \sqrt{2ns\log(-4\pi s)} \right\}.$$

Let

$$s(t) := -\frac{1}{4\pi} e^{-t^2}, \quad t \in [0, +\infty).$$

Then

$$E(1) = \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; |y| \leq \sqrt{\frac{n}{2\pi}} t e^{-\frac{1}{2}t^2} \right\}.$$

2.

$$\begin{aligned} \iint_{E(1)} \frac{|y|^2}{s^2} dyds &= \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} ds \int_{B\left(0, \sqrt{2ns\log(-4\pi s)}\right)} |y|^2 dy \\ &= \int_0^{+\infty} 16\pi^2 e^{2t^2} \frac{t}{2\pi} e^{-t^2} dt \int_{B\left(0, \sqrt{\frac{n}{2\pi}} te^{-\frac{1}{2}t^2}\right)} |y|^2 dy \end{aligned}$$

$$
\begin{aligned}
&= 8\pi \int_0^{+\infty} te^{t^2}\,dt \int_0^{\sqrt{\frac{n}{2\pi}}te^{-\frac{1}{2}t^2}} r^2 n\alpha(n) r^{n-1}\,dr \\
&= 8\pi n\alpha(n) \int_0^{+\infty} te^{t^2}\,dt\, \frac{r^{n+2}}{n+2}\Big|_0^{\sqrt{\frac{n}{2\pi}}te^{-\frac{1}{2}t^2}} \\
&= \frac{8\pi n\alpha(n)}{n+2} \int_0^{+\infty} te^{t^2}\,dt \left(\sqrt{\frac{n}{2\pi}}te^{-\frac{1}{2}t^2}\right)^{n+2} \\
&= \frac{4n^{\frac{n}{2}+2}\alpha(n)}{(2\pi)^{\frac{n}{2}}(n+2)} \int_0^{+\infty} t^{n+3} e^{-\frac{n}{2}t^2}\,dt \\
&= -\frac{4n^{\frac{n}{2}+1}\alpha(n)}{(2\pi)^{\frac{n}{2}}(n+2)} \int_0^{+\infty} t^{n+2} \frac{d}{dt} e^{-\frac{n}{2}t^2}\,dt \\
&= \frac{4n^{\frac{n}{2}+1}\alpha(n)}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} t^{n+1} e^{-\frac{n}{2}t^2}\,dt \\
&= -\frac{4n^{\frac{n}{2}}\alpha(n)}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} t^{n} \frac{d}{dt} e^{-\frac{n}{2}t^2}\,dt \\
&= \frac{4n^{\frac{n}{2}+1}\alpha(n)}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} t^{n-1} e^{-\frac{n}{2}t^2}\,dt \\
&= 4\left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{n}{2}|x|^2}\,dx \\
&= 4\left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\frac{n}{2}x_i^2}\,dx \\
&= 4\left(\frac{n}{2\pi}\right)^{\frac{n}{2}} \left(\frac{2\pi}{n}\right)^{\frac{n}{2}} \\
&= 4.
\end{aligned}
$$

Now we give a simpler way calculating $\iint_{E(1)} \frac{|y|^2}{s^2}\,dyds$.

1.

$$
\begin{aligned}
E(1) &= \{(y,s) \in \mathbb{R}^{n+1} \mid \Phi(y,-s) \geq 1\}. \\
\Phi(y,-s) &= \frac{1}{(-4\pi s)^{\frac{n}{2}}} e^{\frac{1}{4s}|y|^2}, \\
\log \Phi(y,-s) &= -\frac{n}{2}\log(-4\pi s) + \frac{|y|^2}{4s}.
\end{aligned}
$$

Thus

$$
E(1) = \left\{(y,s) \in \mathbb{R}^{n+1} \mid |y| \leq \sqrt{2ns\log(-4\pi s)}\right\}.
$$

Let

$$
s(t) := -\frac{1}{4\pi}e^{-t}, \quad t \in [0,+\infty).
$$

Then

$$E(1) = \left\{ (y,s) \in \mathbb{R}^{n+1} \;\middle|\; |y| \le \sqrt{\frac{n}{2\pi} t^{\frac{1}{2}} e^{-\frac{t}{2}}} \right\}.$$

2.

$$\begin{aligned}
\iint_{E(1)} \frac{|y|^2}{s^2} dyds &= \int_{-\frac{1}{4\pi}}^{0} \frac{1}{s^2} ds \int_{B\left(0,\sqrt{2ns\log(-4\pi s)}\right)} |y|^2 dy \\
&= \int_0^{+\infty} 16\pi^2 e^{2t} \frac{t}{4\pi} e^{-t} dt \int_{B\left(0,\sqrt{\frac{n}{2\pi}t^{\frac{1}{2}}e^{-\frac{t}{2}}}\right)} |y|^2 dy \\
&= 4\pi \int_0^{+\infty} e^t dt \int_0^{\sqrt{\frac{n}{2\pi}t^{\frac{1}{2}}e^{-\frac{t}{2}}}} r^2 n\alpha(n) r^{n-1} dr \\
&= 4\pi n\alpha(n) \int_0^{+\infty} e^t dt \left. \frac{r^{n+2}}{n+2} \right|_0^{\sqrt{\frac{n}{2\pi}t^{\frac{1}{2}}e^{-\frac{t}{2}}}} \\
&= \frac{4\pi n\alpha(n)}{n+2} \int_0^{+\infty} e^t dt \left(\sqrt{\frac{n}{2\pi} t^{\frac{1}{2}} e^{-\frac{t}{2}}} \right)^{n+2} \\
&= \frac{n^{\frac{n}{2}+2}}{2^{\frac{n}{2}-1}(n+2)\Gamma\left(\frac{n}{2}+1\right)} \int_0^{+\infty} t^{\frac{n}{2}+1} e^{-\frac{n}{2}t} dt \\
&= \frac{n^{\frac{n}{2}+2}}{2^{\frac{n}{2}-1}(n+2)\Gamma\left(\frac{n}{2}+1\right)} \int_0^{+\infty} \left(\frac{2t}{n}\right)^{\frac{n}{2}+1} e^{-t} \frac{2}{n} dt \\
&= \frac{8}{(n+2)\Gamma\left(\frac{n}{2}+1\right)} \int_0^{+\infty} t^{\left(\frac{n}{2}+2\right)-1} e^{-t} dt \\
&= \frac{8}{(n+2)\Gamma\left(\frac{n}{2}+1\right)} \Gamma\left(\frac{n}{2}+2\right) \\
&= 4.
\end{aligned}$$

Part II

THEORY FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Chapter 5

SOBOLEV SPACES

5.2 SOBOLEV SPACES

5.2.1 Weak derivatives.

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ORIGINAL 1.

DEFINITION. *Suppose* $u, v \in L^1_{\text{loc}}(U)$ *and* α *is a multiindex. We say that* v *is the* α^{th}*-weak partial derivative of* u*, written*

$$D^\alpha u = v,$$

provided

$$\int_U u D^\alpha \phi dx = (-1)^{|\alpha|} \int_U v \phi dx$$

for all test functions $\phi \in C_c^\infty(U)$*.*

If $D^\alpha u = v$, and $u = \tilde{u}$ a.e. in U, then $D^\alpha \tilde{u} = v$.

5.2.2 Definition of Sobolev spaces.

p. 259

ORIGINAL 1.

DEFINITION. *If $u \in W^{k,p}(U)$, we define its* norm *to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p dx\right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

$$\|u\|^2_{H^k(U)} = \sum_{|\alpha|\le k} \int_U |D^\alpha u|^2 dx = \sum_{i=0}^{k} \|D^i u\|^2_{L^2(U)}.$$

5.2.3 Elementary properties.

p. 261

ORIGINAL 1. 2. Assertions (ii) and (iii) are easy, and the proofs are omitted.

Proof. (ii) $\lambda u + \mu v \in L^1_{\mathrm{loc}}(U)$. For all $\phi \in C_c^\infty(U)$,

$$\begin{aligned} \int_U (\lambda u + \mu v) D^\alpha \phi dx &= \lambda \int_U u D^\alpha \phi dx + \mu \int_U v D^\alpha \phi dx \\ &= \lambda(-1)^{|\alpha|} \int_U (D^\alpha u)\phi dx \mu(-1)^{|\alpha|} \int_U (D^\alpha v)\phi dx \\ &= (-1)^{|\alpha|} \int_U (\lambda D^\alpha u + \mu D^\alpha v)\phi dx. \end{aligned}$$

Thus

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v \in L^p(U).$$

Hence $\lambda u + \mu v \in W^{k,p}(U)$.

(iii) $u \in L^1_{\mathrm{loc}}(V)$. For all $\phi \in C_c^\infty(V) \subset C_c^\infty(U)$,

$$\begin{aligned} \int_V u D^\alpha \phi dx &= \int_U u D^\alpha \phi dx \\ &= (-1)^{|\alpha|} \int_U (D^\alpha u)\phi dx \\ &= (-1)^{|\alpha|} \int_V (D^\alpha u)\phi dx. \end{aligned}$$

$D^\alpha u \in L^p(V)$. Hence $u \in W^{k,p}(V)$. $\square$

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ORIGINAL 1. Thus

$$\int_V |u_m^\varepsilon(x) - u_m(x)|\, dx \leq \varepsilon \int_{B(0,1)} \eta(y) \int_0^1 \int_V |Du_m(x - \varepsilon t y)|\, dx dt dy$$
$$\leq \varepsilon \int_V |Du_m(z)|\, dz.$$

Set

$$\operatorname{spt} u_m + \varepsilon t y := \{x \in V | x - \varepsilon t y \in \operatorname{spt} u_m\}.$$

Then

$$\int_V |Du_m(x - \varepsilon t y)| dx = \int_{\operatorname{spt} u_m + \varepsilon t y} |Du_m(x - \varepsilon t y)| dx$$
$$= \int_{\operatorname{spt} u_m} |Du_m(z)| dz$$
$$= \int_V |Du_m(z)| dz.$$

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ORIGINAL 1. 7. We next employ assertion (5) with $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and use a standard diagonal argument to extract a subsequence $\{u_{m_l}\}_{l=1}^\infty \subset \{u_m\}_{m=1}^\infty$ satisfying

$$\limsup_{l,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

Let $\delta = 1$. Then there exists a subsequence $\{u_{m_j}^{(1)}\}_{j=1}^\infty$ of $\{u_m\}_{m=1}^\infty$ such that

$$\limsup_{j,k\to\infty} \left\| u_{m_j}^{(1)} - u_{m_k}^{(1)} \right\|_{L^q(V)} \leq 1.$$

For the sequence $\{u_{m_j}^{(i)}\}_{j=1}^\infty$, let $\delta = \frac{1}{i+1}$. Then there exists a subsequence $\{u_{m_j}^{(i+1)}\}_{j=1}^\infty$ of $\{u_{m_j}^{(i)}\}_{j=1}^\infty$ such that

$$\limsup_{j,k\to\infty} \left\| u_{m_j}^{(i+1)} - u_{m_k}^{(i+1)} \right\|_{L^q(V)} \leq \frac{1}{i+1}.$$

Let $u_{m_l} = u_{m_l}^{(l)}$. Then $\{u_{m_l}\}_{l=1}^\infty$ satisfies

$$\limsup_{j,k\to\infty} \|u_{m_l} - u_{m_k}\|_{L^q(V)} = 0.$$

5.8 ADDITIONAL TOPICS

5.8.1 Difference quotients

a. Difference quotients and $\mathbf{W^{1,p}}$.

p. 292

ORIGINAL 1. Consequently

$$\int_V \left|D^h u\right|^p dx \leq C \sum_{i=1}^n \int_V \int_0^1 \left|Du\left(x+the_i\right)\right|^p dtdx$$

$$\begin{aligned}
\left|D^h u\right|^p &= \left(\sum_{i=1}^n \left|D_i^h u\right|^2\right)^{\frac{p}{2}} \\
&\leq C \sum_{i=1}^n \left|D_i^h u\right|^p \\
&= C \sum_{i=1}^n \left|\int_0^1 \left|Du\left(x+the_i\right)\right| dt\right|^p \\
&\leq C \sum_{i=1}^n \int_0^1 \left|Du\left(x+the_1\right)\right|^p dt \qquad \text{(Hölder's inequality).}
\end{aligned}$$

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ORIGINAL 1. Thus

$$\int_V \left|D^h u\right|^p dx \leq C \int_U |Du|^p dx.$$

$$\int_V |Du(x+the_i)|^p dx \leq \int_U |Du|^p dx,$$

thus

$$\int_V |D^h u|^p dx \leq C \int_U |Du|^p dx.$$

ORIGINAL 2. Choose $i = 1, \ldots, n, \phi \in C_c^\infty(V)$, and note for small enough h that

$$\int_V u(x) \left[\frac{\phi\left(x+he_i\right)-\phi(x)}{h}\right] dx = -\int_V \left[\frac{u(x)-u\left(x-he_i\right)}{h}\right] \phi(x) dx;$$

that is,

$$\int_V u\left(D_i^h \phi\right) dx = -\int_V \left(D_i^{-h} u\right) \phi dx.$$

This is the "integration-by-parts" formula for difference quotients.

It is sufficient to prove that

$$\int_V u(x)\phi(x+he_i)dx = \int_V u(x-he_i)\phi(x)dx.$$

Take an open set $W \subset\subset V$ such that $\phi(x) = 0$ on $V - W$. Set $W - he_i := \{x \in \mathbb{R}^n \mid x + he_i \in W\} \subset V$. Then

$$\begin{aligned}
\int_V u(x)\phi(x+he_i)dx &= \int_{W-he_i} u(x)\phi(x+he_i)dx \\
&= \int_W u(y-he_i)\phi(y)dy \\
&= \int_V u(y-he_i)\phi(y)dy \\
&= \int_V u(x-he_i)\phi(x)dx.
\end{aligned}$$

5.9 Other spaces of fuctions

5.9.1 The space $\mathbf{H}^{-1}$

p. 299

ORIGINAL 1.

NOTATION. *We write "$f = f^0 - \sum_{i=1}^n f^i_{x_i}$" whenever (1) holds.*

If $f^1, \dots, f^n, v \in C^1(\bar{U})$, then

$$\begin{aligned}
\langle f, v\rangle &= \int_U f^0 v + \sum_{i=1}^n f^i v_{x_i} dx \\
&= \int_U f^0 v dx + \sum_{i=1}^n \int_U f^i v_{x_i} dx \\
&= \int_U f^0 v dx - \sum_{i=1}^n \int_U f^i_{x_i} v dx \\
&= \int_U \left(f^0 - \sum_{i=1}^n f^i_{x_i}\right) v dx
\end{aligned}$$

$$
\begin{aligned}
&= \left(f^0 - \sum_{i=1}^{n} f^i_{x_i}, v \right)_{L^2(U)} \\
&= \left\langle f^0 - \sum_{i=1}^{n} f^i_{x_i}, v \right\rangle.
\end{aligned}
$$

Chapter 6

SECOND-ORDER ELLIPTIC EQUATIONS

6.1 DEFINITIONS

6.1.1 Elliptic equations.

p. 312

ORIGINAL 1. We say that the PDE $Lu = f$ is in divergence form if L is given by (2) and is in nondivergence form provided L is given by (3).

$\sum_{i,j=1}^{n}(a^{ij}u_{x_i})_{x_j}$ is the divergence of

$$\sum_{i=1}^{n}(a^{i1}u_{x_i}, \ldots, a^{in}u_{x_i})$$

if u is smooth.

p. 314

ORIGINAL 1. (i) *The* bilinear form $B[\ ,\ ]$ *associated with the divergence form elliptic operator L defined by* (2) *is*

$$B[u,v] := \int_U \sum_{i,j=1}^{n} a^{ij}u_{x_i}v_{x_j} + \sum_{i=1}^{n} b^i u_{x_i} v + cuvdx \tag{8}$$

for $u, v \in H_0^1(U)$.

Set

$$b^i = 0, \quad c \geq 0.$$

Then for all $u, v \in H_0^1(U)$,

(i) $B[u,v] = B[v,u]$;

(ii) the mapping $(\ ,v)$ is linear;

(iii) $B[u,u] = \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} u_{x_j} + cu^2 dx \geq \int_U \theta |Du|^2 + cu^2 dx \geq 0$;

(iv) $B[u,u] = 0$ if and only if $u = 0$. (Recall that we identify functions in $H_0^1(U)$ which agree a.e.) In fact, if $u \neq 0$, $B[u,u] \geq \int_U \theta |Du|^2 + cu^2 dx > 0$, by Problem 11 in §5.10.

Hence $B[\ ,\ ]$ is an inner product on $H_0^1(U)$.

6.2 EXISTENCE OF WEAK SOLUTIONS

6.2.1 Lax-Milgram Theorem.

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ORIGINAL 1. This inequality easily implies (4).

THEOREM 1. *Let X, Y be two normed linear space. If the linear map $A\colon X \to Y$ satisfies $\beta\|u\| \leq \|Au\|$ for some constant $\beta > 0$, then*

(i) A is injective,

(ii) $R(A)$ is closed, if in addition X is a Banach space and A is continuous.

Proof. (i)

$$\beta\|u_1 - u_2\| \leq \|A(u_1 - u_2)\| = \|Au_1 - Au_2\|.$$

(ii) Let y be a limit point of $R(A)$. Then there exists a sequence $\{y_n\}_{k=1}^\infty \subset R(A)$, such that $y_k \neq y_l$ if $k \neq l$ and $\lim_{k\to\infty} y_k = y$. Set $Ax_k = y_k$. Then $\beta\|x_k - x_l\| \leq \|Ax_k - Ax_l\| = \|y_k - y_l\|$. Thus $\{x_k\}_{k=1}^\infty$ is a Cauchy sequence. Therefore $\lim_{k\to\infty} x_k = x$ for some $x \in X$. Hence $y = Ax \in R(A)$, since A is continuous. Now we have seen that $R(A)$ is closed. $\square$

ORIGINAL 2. We return now to the specific bilinear form $B[\ ,\ ]$, defined in §6.1.2 by the formula

$$B[u,v] = \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuvdx$$

for $u, v \in H_0^1(U)$, and try to verify the hypothesis of the Lax-Milgram Theorem.

$H^1(U)$ is a Hilbert space (see Examples in §D.2). Thus $H_0^1(U)$, the closure of $C_c^\infty(U)$ in $H^1(U)$, is complete, i.e., $H_0^1(U)$ is a Hilbert space.

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ORIGINAL 1. Indeed, from our choice of γ and the energy estimates from §6.2.2 we note that if (13) holds, then

$$\beta \|u\|^2_{H_0^1(U)} \leq B_\gamma[u,u] = (g,u) \leq \|g\|_{L^2(U)} \|u\|_{L^2(U)} \leq \|g\|_{L^2(U)} \|u\|_{H_0^1(U)},$$

so that (18) implies

$$\|Kg\|_{H_0^1(U)} \leq C \|g\|_{L^2(U)} \quad (g \in L^2(U))$$

for some appropriate constant C.

$$\|Kg\|_{H_0^1(U)} = \|\gamma L_\gamma^{-1} g\|_{H_0^1(U)} = \|\gamma u\|_{H_0^1(U)} = \gamma \|u\|_{H_0^1(U)} \leq \frac{\gamma}{\beta} \|g\|_{L^2(U)}.$$

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ORIGINAL 1. Consequently we see the PDE (24) has a unique weak solution for each $f \in L^2(U)$ if and only if (28) holds.

The PDE (24) has a unique weak solution for each $f \in L^2(U)$ if and only if

$$\frac{\gamma}{\gamma + \lambda} \notin \sigma_p(K),$$

thus if and only if

$$\lambda \notin \Sigma := \left\{ \frac{\gamma}{\eta} - \gamma \,\middle|\, \eta \in \sigma_p(K) - 0 \right\}.$$

6.3 REGULARITY

6.3.1 Interior regularity.

p. 328

ORIGINAL 2. Since $u \in H^2_{\text{loc}}(U)$, we can integrate by parts:

$$B[u,v] = (Lu, v).$$

$$\begin{aligned} B[u,v] &= \int_U \sum_{i,j=1}^n a^{ij} u_{x_i} v_{x_j} + \sum_{i=1}^n b^i u_{x_i} v + cuv dx \\ &= \int_U - \sum_{i,j=1}^n (a^{ij} u_{x_i})_{x_j} v + \sum_{i=1}^n b^i u_{x_i} v + cuv dx \\ &= \int_U (Lu) v dx \\ &= (Lu, v). \end{aligned}$$

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ORIGINAL 1. Here we used the formulas

$$\int_U v D_k^{-h} w dx = - \int_U w D_k^h v dx$$

Here $v = a^{ij} u_{x_i}$, $w = \zeta^2 D_k^h u$.

THEOREM. *Let v, w are two summable functions on the open set $U \subset \mathbb{R}^n$. If $w = 0$ on $U - \bar{W}$, where the open set $W \subset \bar{W} \subset U$, then*

$$\int_U v D_k^{-h} w dx = - \int_U (D_k^h v) w dx$$

for all $k \in 1, \ldots, n$ and all small enough $|h| > 0$, where $D_k^{-h} w(x) := 0$ if $x \in U$, $x - he_k \notin U$.

Proof. It is sufficient to prove that

$$\int_U v(x) w(x - he_k) dx = \int_U v(x + he_k) w(x) dx.$$

Set $W + he_k := \{x \in \mathbb{R}^n \mid x - he_k \in W\} \subset U$. Then

$$\begin{aligned}\int_U v(x)w(x-he_k)dx &= \int_{W+he_k} v(x)w(x-he_k)dx \\ &= \int_W v(x+he_k)w(x)dx \\ &= \int_U v(x+he_k)w(x)dx.\end{aligned}$$

□

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ORIGINAL 1. Furthermore we see from (5) that

$$|A_2| \le C \int_U \zeta |D_k^h Du||D_k^h u| + \zeta |D_k^h Du||Du| + \zeta |D_k^h u||Du| dx,$$

for some appropriate constant C.

ζ is smooth on $\mathbb{R}^n$, so ζ_{x_j} is bounded on $\bar{U}$.

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ORIGINAL 1. with the estimate

$$\|u\|_{H^2(V)} \le C \left(\|f\|_{L^2(U)} + \|u\|_{H^1(U)} \right). \tag{23}$$

According to Theorem 3(ii) in §5.8.2,

$$\|D^2 u\|_{L^2(V)}^2 \le C \int_U f^2 + u^2 + |Du|^2 dx.$$

Then

$$\begin{aligned}\|u\|_{H^2(V)}^2 &= \|u\|_{H^1(V)}^2 + \|D^2 u\|_{L^2(V)}^2 \\ &= \|u\|_{H^1(U)}^2 + C(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2) \\ &= C(\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2), \\ \|u\|_{H^2(V)} &\le C\sqrt{\|f\|_{L^2(U)}^2 + \|u\|_{H^1(U)}^2} \\ &\le C(\|f\|_{L^2(U)} + \|u\|_{H^1(U)}).\end{aligned}$$

ORIGINAL 2. Now set $v = \zeta^2 u$ in identity (9) and perform elementary calculations, to discover

$$\int_U \zeta^2 |Du|^2 dx \leq C \int_U f^2 + u^2 dx$$

$$\sum_{i,j=1}^n \int_U a^{ij} u_{x_i} (\zeta^2 u)_{x_j} dx = \int_U \left(f - \sum_{i=1}^n b^i u_{x_i} - cu \right) \zeta^2 u dx,$$

$$\begin{aligned}\int_U \sum_{i,j=1}^n \zeta^2 a^{ij} u_{x_i} u_{x_j} dx = \int_U &- \left(\sum_{i,j=1}^n 2a^{ij} \zeta_{x_j} \cdot \zeta u_{x_i} \cdot u \right) + \zeta^2 fu \\ &- \left(\sum_{i=1}^n b^i \zeta \cdot \zeta u_{x_i} \cdot u \right) - c\zeta^2 u^2 dx.\end{aligned}$$

$$\sum_{i,j=1}^n \zeta^2 a^{ij} u_{x_i} u_{x_j} \geq \theta \zeta^2 |Du|^2,$$

$$\begin{aligned}&- \left(\sum_{i,j=1}^n 2a^{ij} \zeta_{x_j} \cdot \zeta u_{x_i} \cdot u \right) + \zeta^2 fu - \left(\sum_{i=1}^n b^i \zeta \cdot \zeta u_{x_i} \cdot u \right) - c\zeta^2 u^2 \\ &\leq C \left(\epsilon \zeta^2 |Du|^2 + \frac{u^2}{\epsilon} \right) + (f^2 + u^2) + C \left(\epsilon \zeta^2 |Du|^2 + \frac{u^2}{\epsilon} \right) + Cu^2 \\ &\leq C\epsilon \zeta^2 |Du|^2 + C(f^2 + u^2).\end{aligned}$$

Thus

$$\theta \int_U \zeta^2 |Du|^2 dx \leq C\epsilon \int_U \zeta^2 |Du|^2 dx + C \int_U f^2 + u^2 dx.$$

Let $\epsilon = \frac{\theta}{2C}$. Then we have

$$\int_U \zeta^2 |Du|^2 dx \leq C \int_U f^2 + u^2 dx.$$

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ORIGINAL 1. 1. We will establish (27), (28) by induction on m, the case $m = 0$ being Theorem 1 above.

Set $m = 0$. For any $V \subset\subset U$, choose a specific open set $W \subset \mathbf{R}^n$ such that $V \subset\subset W \subset\subset U$. Then $a^{ij} \in C^1(W)$, $b^i, c \in L^\infty(W)$, $f \in L^2(W)$, and $u \in H^1(W)$ (by Theorem 1(iii) in §5.2) is a weak solution of the elliptic PDE $Lu = f$ in W. Now according to Theorem 1, $u \in H^1_{\text{loc}}(W)$, $u \in H^2_{\text{loc}}(U)$ and

$$\|u\|_{H^2(V)} \leq C \left(\|f\|_{L^2(W)} + \|u\|_{L^2(W)} \right) \leq C \left(\|f\|_{L^2(U)} + \|u\|_{L^2(U)} \right),$$

with the constant C depending only on V, U, and the coefficients of L.

6.4 MAXIMUM PRINCIPLES

6.4.1 Weak maximum principle.

p. 346

ORIGINAL 1. So (5) and (6) are incompatible, and we have a contradiction.

Here we proved that u can't attain its maximum over $\bar{U}$ in U, which is stronger than $\max_{\bar{U}} u = \max_{\partial U} u$.

ORIGINAL 2. Then according to steps 1 and 2 above $\max_{\bar{U}} u^{\epsilon} = \max_{\partial U} u^{\epsilon}$.

According to steps 1 and 2, u can't attain its maximum over $\bar{U}$ in U.

ORIGINAL 3. Let $\epsilon \to 0$ to find $\max_{\bar{U}} u = \max_{\partial U} u$.

Assume that there exists an $x^1 \in U$, such that

$$\max_{\partial U} u(x) < u(x^1) = \max_{\bar{U}} u(x).$$

Let

$$\epsilon = \frac{1}{\max_{\partial U} e^{\lambda x_1}} \left(\max_{\bar{U}} u(x) - \max_{\partial U} u(x) \right),$$

and set

$$\max_{\partial U} u^{\epsilon}(x) = u^{\epsilon}(x^2), \quad x^2 \in \partial U.$$

Then

$$\begin{aligned}
\max_{\partial U} u^{\epsilon}(x) &= u(x^2) + \frac{e^{\lambda (x^2)_1}}{\max_{\partial U} e^{\lambda x_1}} \left(\max_{\bar{U}} u(x) - \max_{\partial U} u(x) \right) \\
&\leq u(x^2) + \max_{\bar{U}} u(x) - \max_{\partial U} u(x) \\
&\leq \max_{\bar{U}} u(x) = u(x^1) \\
&< u^{\epsilon}(x^1).
\end{aligned}$$

The contradiction occurs. Hence

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

6.4.2 Strong maximum principle.

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ORIGINAL 1.

Remark. So in particular, if $Lu = 0$ in U, then

$$\max_{\bar{U}} |u| = \max_{\partial U} |u|.$$

$$-\max_{\partial U} u^- \leq \min_{\bar{U}} u \leq \max_{\bar{U}} u \leq \max_{\partial U} u^+,$$

thus

$$\max_{\bar{U}} |u| \leq \max\left\{\max_{\partial U} u^+, \max_{\partial U} u^-\right\} = \max_{\partial U} |u|.$$

ORIGINAL 2.

LEMMA (Hopf's Lemma)**.** *Assume $u \in C^2(U) \cap C^1(\bar{U})$ and*

$$c \equiv 0 \quad \text{in } U.$$

Suppose further

$$Lu \leq 0 \quad \text{in } U$$

and there exists a point $x^0 \in \partial U$ such that

$$u(x^0) > u(x) \quad \text{for all } x \in U.$$

Assume finally that U satisfies the interior ball condition at x^0; that is, there exists an open ball $B \subset U$ with $x^0 \in \partial B$

(i) *Then*

$$\frac{\partial u}{\partial \nu}(x^0) > 0,$$

where ν is the outer unit normal to B at x^0.

(ii) *If*

$$c \geq 0 \quad \text{in } U,$$

the same conclusion holds provided

$$u(x^0) \geq 0.$$

Let us prove (i) by (ii).

If $u(x^0) < 0$, let $\tilde{u} := u - u(x^0)$. Then

(i) $\tilde{u} \in C^2(U) \cap C^1(\bar{U})$,

(ii) $L\tilde{u} = Lu \leq 0$ in U,

(iii) $\tilde{u}(x^0) > \tilde{u}(x)$ for all $x \in U$, and

(iv) $\tilde{u}(x^0) = 0 \geq 0$.

Thus by (ii)

$$\frac{\partial u}{\partial \nu}(x^0) = \frac{\partial \tilde{u}}{\partial \nu}(x^0) > 0.$$

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ORIGINAL 1. We calculate for $k, l = 1, \ldots, n$ that

$$w_{x_k x_l} = \sum_{i,j=1}^{n} 2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l} + R,$$

where the remainder term R, resulting from derivatives falling upon the coefficients, satisfies an estimate of the form

$$|R| \leq \epsilon |D^2 v|^2 + C(\epsilon)|Dv|^2$$

for each $\epsilon > 0$.

$$\begin{aligned}
w_{x_k} &= \sum_{i,j=1}^{n} (a^{ij}_{x_k} v_{x_i} v_{x_j} + a^{ij} v_{x_i x_k} v_{x_j} + a^{ij} v_{x_i} v_{x_j x_k}) \\
&= \sum_{i,j=1}^{n} (a^{ij}_{x_k} v_{x_i} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j}), \\
w_{x_k x_l} &= \sum_{i,j=1}^{n} (a^{ij}_{x_k x_l} v_{x_i} v_{x_j} + a^{ij}_{x_k} v_{x_i x_l} v_{x_j} + a^{ij}_{x_k} v_{x_i} v_{x_j x_l} \\
&\quad + 2a^{ij}_{x_l} v_{x_i x_k} v_{x_j} + 2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l}) \\
&= \sum_{i,j=1}^{n} (a^{ij}_{x_k x_l} v_{x_i} v_{x_j} + 2a^{ij}_{x_k} v_{x_i x_l} v_{x_j} \\
&\quad + 2a^{ij}_{x_l} v_{x_i x_k} v_{x_j} + 2a^{ij} v_{x_i x_k x_l} v_{x_j} + 2a^{ij} v_{x_i x_k} v_{x_j x_l}),
\end{aligned}$$

$$
\begin{aligned}
R &= \sum_{i,j=1}^{n} (a^{ij}_{x_k x_l} v_{x_i} v_{x_j} + 2a^{ij}_{x_k} v_{x_i x_l} v_{x_j} + 2a^{ij}_{x_l} v_{x_i x_k} v_{x_j}) \\
&\leq C|Dv|^2 + \epsilon|D^2 v|^2 + C(\epsilon)|Dv|^2 \\
&= \epsilon|D^2 v|^2 + C(\epsilon)|Dv|^2.
\end{aligned}
$$

p. 353

ORIGINAL 1. Hence

$$
0 \leq \zeta^4 \left(-\sum_{k,l=1}^{n} a^{kl} w_{x_k x_l} + \sum_{k=1}^{n} b^k w_{x_k} \right) + \hat{R}, \tag{32}
$$

where the remainder term $\hat{R}$, which comprises terms for which derivatives fall upon the cutoff function ζ, satisfies the estimate

$$
|\hat{R}| \leq C(\zeta^2 w + \zeta^3 |Dw|).
$$

$$
\begin{aligned}
(\zeta^4 w)_{x_k} &= 4\zeta^3 \zeta_{x_k} w + \zeta^4 w_{x_k}, \\
(\zeta^4 w)_{x_k x_l} &= 12\zeta^2 \zeta_{x_l} \zeta_{x_k} w + 4\zeta^3 \zeta_{x_k x_l} w + 4\zeta^3 \zeta_{x_k} w_{x_l} + 4\zeta^3 \zeta_{x_l} w_{x_k} + \zeta^4 w_{x_k x_l},
\end{aligned}
$$

thus

$$
\begin{aligned}
&- \sum_{k,l=1}^{n} a^{kl} (\zeta^4 w)_{x_k x_l} + \sum_{k=1}^{n} b^k (\zeta^4 w)_{x_k} \\
&= -12\zeta^2 w \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k} \zeta_{x_l} - 4\zeta^3 w \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k x_l} - 8\zeta^3 \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k} w_{x_l} \\
&\quad - \zeta^4 \sum_{k,l=1}^{n} a^{kl} w_{x_k x_l} + 4\zeta^3 w \sum_{k=1}^{n} b^k \zeta_{x_k} + \zeta^4 \sum_{k=1}^{n} b^k w_{x_k} \\
&= \zeta^4 \left(-\sum_{k,l=1}^{n} a^{kl} w_{x_k x_l} + \sum_{k=1}^{n} b^k w_{x_k} \right) - 12\zeta^2 w \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k} \zeta_{x_l} \\
&\quad - 4\zeta^3 w \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k x_l} - 8\zeta^3 \sum_{k,l=1}^{n} a^{kl} \zeta_{x_k} w_{x_l} + 4\zeta^3 w \sum_{k=1}^{n} b^k \zeta_{x_k}.
\end{aligned}
$$

p. 358

ORIGINAL 1. But then according to (8), $\mu_k = d_k \lambda_k^{1/2}$; and so the series (8) in fact converges also in $H_0^1(U)$.

$$
\begin{aligned}
B[u, w_k] = B\left[\sum_{i=1}^{\infty} \mu_i \frac{w_i}{\lambda_i^{1/2}}, w_k\right] = \mu_k \lambda_k^{1/2}, \\
(u, w_k) = \left(\sum_{i=1}^{\infty} d_i w_i, w_k\right) = d_k, \\
B[u, w_k] = B[w_k, u] = \lambda_k (w_k, u) = \lambda_k (u, w_k),
\end{aligned}
$$

thus

$$
\begin{aligned}
\mu_k \lambda_k^{1/2} &= \lambda_k d_k, \\
\mu_k &= d_k \lambda_k^{1/2}.
\end{aligned}
$$

Part IV

APPENDICES

Chapter 7

APPENDIX A: NOTATION

7.1 A.3. Notation for functions.

Notation for derivatives.

p. 701

ORIGINAL 1. (v) $\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = \mathrm{tr}(D^2 u) =$ *Laplacian* of u.

Set $v(x) = u(cx)$, where $c \in \mathbb{R}$ is a constant. Then

$$\begin{aligned} Dv(x) &= cDu(cx), \\ \Delta v(x) &= c^2 \Delta u(cx). \end{aligned}$$

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ORIGINAL 1. (v) $\|Du\|_{L^p(U)} = \||Du|\|_{L^p(U)}$.
$\|D^2 u\|_{L^p(U)} = \||D^2 u|\|_{L^p(U)}$.

$$\begin{aligned} \|Du\|_{L^2(U)}^2 = \||Du|\|_{L^2(U)}^2 &= \left\| \left(\sum_{i=1}^{n} |u_{x_i}|^2 \right)^{\frac{1}{2}} \right\|_{L^2(U)}^2 \\ &= \int_U \sum_{i=1}^{n} |u_{x_i}|^2 dx = \sum_{i=1}^{n} \int_U |u_{x_i}|^2 dx \\ &= \sum_{i=1}^{n} \|u_{x_i}\|_{L^2(U)}^2. \end{aligned}$$

$$
\begin{aligned}
\|D^2u\|^2_{L^2(U)} = \||D^2u|\|^2_{L^2(U)} &= \left\| \left(\sum_{|\alpha|=2} |D^\alpha u|^2 \right)^{\frac{1}{2}} \right\|^2_{L^2(U)} \\
&= \int_U \sum_{i,j=1}^{n} |u_{x_i x_j}|^2 dx = \sum_{i,j=1}^{n} \int_U |u_{x_i x_j}|^2 dx \\
&= \sum_{i,j=1}^{n} \|u_{x_i x_j}\|^2_{L^2(U)}.
\end{aligned}
$$

Chapter 8

APPENDIX C: CALCULUS

8.1 C.5. Convolution and smoothing

p. 713

ORIGINAL 1.

Consider a family of smooth, bounded regions $U(\tau) \subset \mathbb{R}^n$ that depend smoothly upon the parameter $\tau \in \mathbb{R}$. Write v for the velocity of the moving boundary $\partial U(\tau)$ and ν for the outward pointing unit normal.

THEOREM 6 (Differentiation formula for moving regions)**.** *If $f = f(x, \tau)$ is a smooth function, then*

$$\frac{d}{d\tau}\int_{U(\tau)} f dx = \int_{\partial U(\tau)} f\mathbf{v}\cdot\nu dS + \int_{U(\tau)} f_\tau dx.$$

If $U(\tau) = B^0(y, \tau)$ and $\tau \neq 0$, this formula can be proved as follows.

$$\begin{aligned}
&\frac{d}{d\tau}\int_{B(y,\tau)} f(x,\tau)dx \\
&= \frac{d}{d\tau}\int_{B(0,1)} f(y+\tau z,\tau)\tau^n dz \\
&= \int_{B(0,1)} Df(y+\tau z,\tau)\cdot z\tau^n + f_\tau(y+\tau z,\tau)\tau^n + f(y+\tau z,\tau)n\tau^{n-1}dz \\
&= \int_{B(y,\tau)} Df(x,\tau)\cdot\frac{x-y}{\tau}dx + \int_{B(y,\tau)} f(x,\tau)n\tau^{-1}dx + \int_{B(y,\tau)} f_\tau(x,\tau)dx \\
&= \int_{B(y,\tau)} Df(x,\tau)\cdot\frac{D(|x-y|^2)}{2\tau}dx + \int_{B(y,\tau)} f(x,\tau)\frac{\Delta(|x-y|^2)}{2\tau}dx
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{B(y,\tau)} f_\tau(x,\tau)dx \\
&= \int_{\partial B(y,\tau)} f(x,\tau)dS + \int_{B(y,\tau)} f_\tau(x,\tau)dx.
\end{aligned}
$$

p. 714

ORIGINAL 1. That is,

$$f^\epsilon(x) = \int_U \eta_\epsilon(x-y)f(y)dy = \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy$$

for $x \in U_\epsilon$.

$$
\begin{aligned}
\int_U \eta_\epsilon(x-y)f(y)dy &= \int_{B(x,\epsilon)} \eta_\epsilon(x-y)f(y)dy \\
&= \int_{B(0,\epsilon)} \eta_\epsilon(z)f(x-z)dz \\
&= \int_{B(0,\epsilon)} \eta_\epsilon(y)f(x-y)dy.
\end{aligned}
$$

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