PDE 大作业

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1、记 $L = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{ij} + \sum_{i=1}^{n} b_i(x)\partial_i - c(x)$ 为一致椭圆算子.

首先证明关于 L 的弱极值原理：若 $u \in C^2(\Omega)\cap C(\bar\Omega)$ 且 $Lu \geq 0$, $x\in\Omega$. 则

$$\max_{\bar\Omega} u \leq \max_{\partial\Omega} u^+ ,\quad u^+ = \max\{0, u\}$$ （即若最大值非负，一定在边界取到）

证明：在 Ω 内 $Lu \geq 0$. $\forall \varepsilon>0$, 令 $w(x) = u(x) + \varepsilon e^{\mu x_1}$. μ 待定. （x_1 为 x 的第一个分量）

计算 $Lw = Lu + \varepsilon L(e^{\mu x_1}) = Lu + \varepsilon e^{\mu x_1}(a_{11}\mu^2 + b_1\mu - c)$.

由 $(a_{ij})_{n\times n}$ 具有正定性，故 $a_{11} \geq \lambda > 0$. 而 b_1, c 有界，故可取 μ 充分大，使 $a_{11}\mu^2 + b_1\mu - c \geq 1$.

固定 μ，有 $Lw > 0$, $x\in\Omega$.

此时 $\max_{\bar\Omega} w \leq \max_{\partial\Omega} w^+$. 则 $\max_{\bar\Omega} u \leq \max_{\bar\Omega} w \leq \max_{\partial\Omega} w^+ \leq \max_{\partial\Omega} u^+ + \varepsilon \max_{\partial\Omega} e^{\mu x_1}$.

令 $\varepsilon\to 0$ 即得 $\max_{\bar\Omega} u \leq \max_{\partial\Omega} u^+$.

下证比较原理：令 $u = u_1 - u_2$. 则 u 满足 $\begin{cases} Lu \geq 0, & x\in\Omega \\ u \leq 0, & x\in\partial\Omega \end{cases}$

由弱极值原理，Ω 上有 $u\leq 0$. 则 $(u_1 - u_2)|_\Omega \leq 0$. □

Hopf 原理：B 为 R^n 中开球，$u(x)\in C^2(B)\cap C(\bar B)$. $x_0\in\partial B$

(1) $Lu \geq 0$, $x\in B$

(2) $u(x) < u(x_0)$, $x\in B$, $u(x_0)\geq 0$. 那么 $\frac{\partial u}{\partial \nu}(x_0) > 0$

证明：不妨令 B 为 $B_1(0)$. 令 $w(x) = e^{-\mu|x|^2} - e^{-\mu}$, $v(x) = u(x) - u(x_0) + \varepsilon w(x)$. μ, ε 待定正常数.

考虑 $D = B_1\setminus B_{\frac12}$. $Lw = e^{-\mu|x|^2}\left(4\mu^2\sum_{i,j=1}^{n} a_{ij}x_ix_j - 2\mu\sum_{i=1}^{n}(a_{ii} + b_ix_i) + c\right) + ce^{-\mu} \geq$

$e^{-\mu|x|^2}\left(4\mu^2\sum_{i,j=1}^{n} a_{ij}x_ix_j - 2\mu\sum_{i=1}^{n}(a_{ii} + b_ix_i) + c\right)$

由 $(a_{ij})_{n\times n}$ 正定，可取充分大的 μ，使 $4\mu^2\sum_{i,j=1}^{n} a_{ij}x_ix_j - 2\mu\sum_{i=1}^{n}(a_{ii}+b_ix_i) + c \geq 0$. $x\in D$

固定 μ，则 $Lw\geq 0$, $x\in D$. 从而 $Lv = Lu + \varepsilon Lw \geq 0$. $x\in D$.

考虑 ∂D. ① $x\in\partial B_1$, $v(x) = u(x) - u(x_0) \leq 0$

② $x\in\partial B_{\frac12}$, $w(x) = e^{-\frac{\mu}{4}} - e^{-\mu} \leq 1$.

$\min_{x\in\partial B_{\frac12}} (u(x_0) - u(x)) > 0$. 取 $\varepsilon < \min_{x\in\partial B_{\frac12}} (u(x_0) - u(x))$，则 $v(x) = u(x) - u(x_0) + \varepsilon w(x) < 0$.

$\begin{cases} Lv \geq 0, & x\in D \\ v \leq 0, & x\in\partial D \end{cases}$ 由极值原理，$v\leq 0$, $x\in D$. $v(x_0) = \max_{\bar D} v$. 故 $\frac{\partial v}{\partial \nu}(x_0) \geq 0$. 即

$\frac{\partial u}{\partial \nu}(x_0) \geq -\varepsilon\frac{\partial w}{\partial \nu}(x_0) = \varepsilon\cdot\frac{2\mu|x|^2}{R}e^{-\mu|x|^2} > 0$. □

2、反证：若 $u \nleq 0$，则 $\exists x_1 \in \Omega$，$u(x_1) < 0$，由连续性知 $\exists$ 开球 $B_\varepsilon(x_1)$，$\forall x \in B_\varepsilon(x_1)$，$u(x) < 0$

而 $Lu \geq 0$ 恒成立，由 ε 的任意性，令其满足 $\exists x_0 \in \partial B_\varepsilon(x_1)$，有 $u(x_0) = 0$

$$\begin{cases} Lu \geq 0, & x \in B_\varepsilon(x_1) \\ u(x) < u(x_0) = 0, & x \in B_\varepsilon(x_1) \end{cases}$$
由 Hopf 引理知 $\frac{\partial u}{\partial \nu}(x_0) > 0$.

那么必然存在一点 $x_2 \in B_1(x_0)$（考虑 x_2 在上述单位方向导数的同方向即可），有 $u(x_2) > 0$，矛盾。从而不存在合适的 x_0 满足 $u(x_0) = 0$ $\Rightarrow u < 0$.

3、对于 w，构造一个特定的算子：$M = \sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{ij} + \sum_{i=1}^{n}\left(b_i(x) - \sum_{j=1}^{n}\frac{2}{w}a_{ij}(x)\,w_j\right)\partial_i$.

容易验证 M 满足对应的 Hopf 引理（与 L 类似）.

令 $v = \frac{u}{w}$，则 $v \in C^2(\Omega) \cap C(\bar{\Omega})$. v 在 $\partial\Omega$ 中的 x_0 取到非负最大值 $\frac{u}{w}(x_0)$

考虑 Mv，计算得 $Mv = M\left(\frac{u}{w}\right) = \frac{Lu}{w} - \frac{uLw}{w^2} \geq 0$，$x \in \Omega$

由 Hopf 引理知 $\frac{\partial}{\partial \nu}\left(\frac{u}{w}\right)(x_0) > 0$.

4、构造向量内积 $\langle \xi, \eta \rangle_A = \sum a_{ij}\xi_i\eta_j$，$\langle \xi, \xi \rangle_A = \|\xi\|_A^2 \leq \Lambda|\xi|^2$

由 Cauchy - Schwarz 不等式：$|\langle \xi, \eta \rangle_A| \leq \|\xi\|_A \|\eta\|_A \leq \Lambda|\xi|\cdot|\eta|$

$Lu = 0$，则 $0 = \int_\Omega \eta Lu\,dx = -\sum_{j=1}^{n}\int_\Omega \eta\,(A\cdot Du)_j\,dx = -\sum_{j=1}^{n}\int_\Omega \eta\,(A\cdot Du)_j\,dx + \sum_{j=1}^{n}\int_\Omega \eta_j\,(A\cdot Du)_j\,dx$

$= \int_\Omega \langle Dv, Du \rangle_A\,dx$，其中 $v|_{\partial\Omega} = 0$.

取 $v = \eta^2 u$，得 $0 = \int_\Omega \langle Du, Dv \rangle_A dx = \int_\Omega \langle Du, \eta^2 Du + 2\eta u D\eta \rangle_A\,dx$

$= \int_\Omega \eta^2 \langle Du, Du \rangle_A\,dx + 2\int_\Omega \eta u \langle Du, D\eta \rangle_A\,dx$

$\therefore \lambda\int_\Omega \eta^2|Du|^2 dx \leq \int_\Omega \eta^2\langle Du, Du\rangle_A\,dx = -2\int_\Omega \eta u\langle Du, D\eta\rangle_A\,dx \leq 2\int_\Omega \eta|u||\langle Du, D\eta\rangle_A|\,dx$

$\leq 2\Lambda\int_\Omega |\eta|\cdot|u|\cdot|Du|\cdot|D\eta|\,dx$.

由 Cauchy - Schwarz 不等式：$2|\eta|\cdot|u|\cdot|Du|\cdot|D\eta| \leq \varepsilon\eta^2\cdot|Du|^2 + \frac{u^2|D\eta|^2}{\varepsilon}$

取 $\varepsilon = \frac{\lambda}{2\Lambda}$，得 $\lambda\int_\Omega \eta^2|Du|^2 dx \leq \left(\Lambda\int_\Omega \varepsilon\eta^2|Du|^2 dx\right) + \left(\Lambda\int_\Omega \frac{u^2|Du|}{\varepsilon}dx\right) = \frac{\lambda}{2}\int_\Omega \eta^2|Du|^2 dx + \frac{2\Lambda^2}{\lambda}\int_\Omega u^2|Du|^2 dx$ $\Rightarrow \int_\Omega \eta^2|Du|^2 dx \leq \frac{4\Lambda^2}{\lambda^2}\int_\Omega u^2|D\eta|^2 dx$.

5、构造 $\eta(x) = \begin{cases} 1, & |x| < r \\ \frac{R-|x|}{R-r}, & r \leq |x| < R \\ 0, & |x| \geq R \end{cases}$
则 $\int_{B_r(0)} |Du|^2 dx \leq \int_{B_R(0)} \eta^2|Du|^2 dx \leq \frac{4\Lambda^2}{\lambda^2}\int_{B_R(0)} |D\eta|^2 u^2 dx$

$\leq \frac{4\Lambda^2}{\lambda^2(R-r)^2}\int_{B_R(0)} u^2 dx \leq \frac{4\Lambda^2}{\lambda^2(R-r)^2}\int_\Omega u^2 dx$

6. (1) 由 $|D(uv)|^2 = |uDv + vDu|^2 \le 2u^2|Dv|^2 + 2v^2|Du|^2$

$\Rightarrow \int_{B_R} |D(uv)|^2 dx \le 2\int_{B_R} (u^2|Dv|^2 + v^2|Du|^2)\, dx \le 2(1+\frac{4\Lambda^2}{\lambda^2})\int_{B_R} |Dv|^2 u^2 dx$

由 Poincare 不等式：$\exists C(n)>0$，有 $\int_{B_R} (uv)^2 dx \le \frac{r^2}{C(n)} \int_{B_R} |D(uv)|^2 dx$

考虑截断函数：$v(x) = \begin{cases} 1, & |x| < \frac{R}{2} \\ \frac{R-|x|}{R-\frac{R}{2}}, & \frac{R}{2} \le |x| < R \\ 0 & |x| \ge R \end{cases}$，则 $\int_{B_{\frac{R}{2}}} u^2 dx \le \int_{B_R} (uv)^2 dx \le \frac{R^2}{2C(n)} \cdot (1+\frac{4\Lambda^2}{\lambda^2})$

$\int_{B_R} |Dv|^2 u^2 dx \le \frac{2}{C(n)}(1+\frac{4\Lambda^2}{\lambda^2}) \int_{B_R \setminus B_{\frac{R}{2}}} u^2 dx$

$\Rightarrow \int_{B_R} u^2 dx = \int_{B_{\frac{R}{2}}} u^2 dx + \int_{B_R \setminus B_{\frac{R}{2}}} u^2 dx \ge \left(1 + \frac{C(n)}{2}(1+\frac{4\Lambda^2}{\lambda^2})^{-1}\right) \int_{B_{\frac{R}{2}}} u^2 dx$

取 $\theta = \dfrac{1}{1+\frac{C(n)}{2}(1+\frac{4\Lambda^2}{\lambda^2})^{-1}}$，即得 $\int_{B_{\frac{R}{2}}} u^2 dx \le \theta \int_{B_R} u^2 dx$ （右边用 Du 替换 u 即得相同结果）

(2) 与 (1) 类似，只需对截断函数进行修改即可。考虑：$v(x) = \begin{cases} 1, & |x| < r \\ \frac{R-|x|}{R-r}, & r \le |x| < R \\ 0, & |x| \ge R \end{cases}$

其余步骤与 (1) 类似，可得结果。

作业2：广义函数

1. (1) 由 $(e^{-|x|})^\wedge = \frac{2}{\sqrt{2\pi}(1+\xi^2)}$，设 $g(x)=\frac{1}{1+x^2}$，则 $\hat{f}(x)=\left[\frac{1}{a^2}g(\frac{x}{a})\right]^\wedge(\xi)$

$$= \frac{1}{a^2}[g(\tfrac{x}{a})]^\vee(-\xi) = \frac{1}{a^2}\cdot a\cdot \check{g}(-a\xi) = \frac{1}{a}\check{g}(-a\xi) = \underbrace{\frac{\sqrt{2\pi}}{2a}e^{-a|\xi|}}$$

(2) $f(x)=\ln|x|$，考虑欧拉常数 γ： $\gamma+\ln x = \int_0^x \frac{1-\cos t}{t}dt - \int_x^\infty \frac{\cos t}{t}dt$

由 $F(1)=\gamma$，$F'(x)=\frac{1}{x}$，有 $\gamma+\ln x = \int_0^1 \frac{1-\cos(xt)}{t}dt - \int_1^\infty \frac{\cos(xt)}{t}dt$

$\Rightarrow \gamma+\ln|x| = \int_0^\infty \frac{I_{[0,1]}(t)-\cos(xt)}{t}dt$，$\forall x\neq 0$ 记 $T(x)=\gamma+\ln|x|$.

$$\langle T,\phi\rangle = \int_{\mathbb{R}}(\gamma+\ln|x|)\phi(x)dx = \int_0^\infty \frac{2I_{[0,1]}(t)\hat\phi(0)-\hat\phi(t)-\hat\phi(-t)}{2t}dt$$

其中 $\hat\phi(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty \phi(x)e^{-ixt}dx$，易证 $\hat{\hat\phi}(t)=\phi(-t)$. 考虑 $\hat\phi$：

$$\langle T,\hat\phi\rangle = \int_0^\infty \frac{2I_{[0,1]}(t)\phi(0)-\phi(-t)-\phi(t)}{2t}dt = \int_0^1\frac{2\phi(0)-\phi(t)-\phi(-t)}{2t}dt - \int_1^\infty \frac{\phi(t)+\phi(-t)}{2t}dt$$

$$= \frac{1}{2}\int_0^\infty(\ln t)(\phi'(t)-\phi'(-t))dt = \frac{1}{2}\int_{\mathbb{R}}\operatorname{sign}(t)\ln|t|\phi'(t)dt = -\pi\langle pf\tfrac{1}{|\omega|},\phi\rangle$$

$\Rightarrow \hat{T} = -\pi\, pf\frac{1}{|\omega|}$，其中 $pf\frac{1}{|\omega|} = D(\operatorname{sign}(x)\log|x|)$ 为柯西主值.

$\therefore (\gamma+\ln|x|)^\wedge = \sqrt{2\pi}\gamma\delta + (\ln|x|)^\wedge \quad \Rightarrow (\ln|x|)^\wedge = \underbrace{-\pi\, pf\frac{1}{|\omega|} - \sqrt{2\pi}\gamma\delta(\omega)}$.

(3) $F(xe^{-\pi y^2}) = \frac{1}{2\pi}\int_{\mathbb{R}^2} xe^{-\pi y^2}\cdot e^{-i(x\xi_1+y\xi_2)}dxdy = \frac{1}{2\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty x\cdot e^{-ix\xi_1}\cdot e^{-\pi y^2-iy\xi_2}dxdy =$

$F(x)\cdot F(e^{-\pi y^2}) = (-1)\cdot\sqrt{2\pi}\cdot\delta'(\xi_1)\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{\xi_2^2}{4\pi}} = -\delta'(\xi_1)\cdot e^{-\frac{\xi_2^2}{4\pi}}$

(4) 与(3)同理. $F(\delta'(x)e^{-\frac{y^2}{2}}) = F(\delta'(x))\cdot F(e^{-\frac{y^2}{2}}) = F(F(-\frac{x}{\sqrt{2\pi}}))\cdot F(e^{-\frac{y^2}{2}}) =$

$\frac{1}{\sqrt{2\pi}}\xi_1\cdot e^{-\frac{\xi_2^2}{2}}$.

2. (1) $(\partial_t^2-\partial_x^2)u=\delta(x,t)$. 关于 x 进行 Fourier 变换：

$\hat{u}_{tt}+\lambda^2\hat{u}=\delta(t)$. 假设边界条件为：$u|_{t=0}=\varphi(x)$，$u_t|_{t=0}=\psi(x)$，则有：

$\hat{u}|_{t=0}=\hat\varphi(\lambda)$，$\hat{u}_t|_{t=0}=\hat\psi(\lambda)$. 容易得到：

$$\hat{u}(\lambda,t) = \hat\varphi(\lambda)\cos(\lambda t) + \frac{1}{\lambda}\hat\psi(\lambda)\sin(\lambda t) + \int_0^t \frac{1}{\lambda}\delta(\tau)\sin(\lambda(t-\tau))d\tau$$

t 为参数，作逆 Fourier 变换：$u(x,t)=\frac{1}{2}[\varphi(x+t)-\varphi(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\psi(\xi)d\xi + \frac{1}{2}\int_0^t\int_{x-(t-\tau)}^{x+(t-\tau)}$

$\delta(\xi,\tau)d\xi d\tau$. 为基本解.

$\Rightarrow$ 解为 $u*0 = \frac{1}{2}[\varphi(x+t)-\varphi(x-t)]+\frac{1}{2}\int_{x-t}^{x+t}\psi(\xi)d\xi$. 为达朗贝尔公式.

(2) $-\Delta u+u=\delta(x)$. 作 Fourier 变换：$(1+|\xi|^2)Fu=(2\pi)^{-\frac{n}{2}}$. $Fu=\frac{(2\pi)^{-\frac{n}{2}}}{1+|\xi|^2}$

$\Rightarrow E(x)=F^{-1}\left(\frac{(2\pi)^{-\frac{n}{2}}}{1+|\xi|^2}\right)$ 对于一维情况，$n=1$，有 $(e^{-|x|})^\wedge=\frac{2}{\sqrt{2\pi}(1+\xi^2)}$

$\therefore F^{-1}\left(\frac{(2\pi)^{-\frac{1}{2}}}{1+|\xi|^2}\right)=\frac{1}{2}e^{-|x|}$ 即基本解为 $\frac{1}{2}e^{-|x|}$. 该结果对 n 维同样成立.

3. $\|uv\|_{H^s}^2=\int_{R^n}(1+|\xi|^2)^s|\widehat{uv}(\xi)|^2d\xi=\frac{1}{(2\pi)^n}\int_{R^n}(1+|\xi|^2)^s|\hat{u}(\xi)*\hat{v}(\xi)|^2d\xi=\frac{1}{(2\pi)^n}\int_{R^n}(1+|\xi|^2)\cdot$
$|\int_{R^n}\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta|^2d\xi=\frac{1}{(2\pi)^n}\int_{R^n}\left(\int_{R^n}(1+|\xi|^2)^{\frac{s}{2}}\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta\right)^2d\xi$

由 $(1+|\xi|^2)^{\frac{s}{2}}=(1+|\xi-\eta+\eta|^2)^{\frac{s}{2}}\le(4+2|\xi-\eta|^2+2|\eta|^2)^{\frac{s}{2}}\le(2+2|\xi-\eta|^2)^{\frac{s}{2}}+(2+2|\eta|^2)^{\frac{s}{2}}\le 2^{\frac{s}{2}}\cdot$
$\left((1+|\xi-\eta|^2)^{\frac{s}{2}}+(1+|\eta|^2)^{\frac{s}{2}}\right)$.

原式 $\le\frac{1}{(2\pi)^n}\cdot 2^{\frac{s}{2}}\int_{R^n}\left(\int_{R^n}(1+|\xi-\eta|^2)^{\frac{s}{2}}\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta+\int_{R^n}(1+|\eta|^2)^{\frac{s}{2}}\hat{u}(\xi-\eta)\hat{v}(\eta)d\eta\right)^2d\xi\le$
$\frac{1}{(2\pi)^n}2^{\frac{s}{2}}\int_{R^n}\int_{R^n}(1+|\xi-\eta|^2)^s\hat{u}(\xi-\eta)^2\hat{v}(\eta)^2d\eta d\xi+\frac{1}{(2\pi)^n}2^{\frac{s}{2}}\int_{R^n}\int_{R^n}(1+|\eta|^2)^s\hat{u}(\xi-\eta)^2\hat{v}(\eta)^2d\eta d\xi+$
$\frac{1}{(2\pi)^n}2^{\frac{s}{2}}\int_{R^n}\int_{R^n}2(1+|\xi-\eta|^2)^{\frac{s}{2}}(1+|\eta|^2)^{\frac{s}{2}}\hat{u}(\xi-\eta)^2\hat{v}(\eta)^2d\eta d\xi\le C^2\left(\|u\|_{H^s(R^n)}\|v\|_{L^\infty(R^n)}+\|u\|_{L^\infty(R^n)}\|v\|_{H^s(R^n)}\right)^2$

4. (1) $u\in D'(\Omega)$. 则 $\forall K$紧$\subseteq R^n$, $\exists C, N\ge 0$, st $|\langle u,\phi\rangle|\le C\sum_{|\alpha|\le N}\sup_K|D^\alpha\phi|$. ($\forall\phi\in D(\Omega)$. $\mathrm{supp}\,\phi\subseteq K$)

① 若 $0\notin\mathrm{supp}\,\phi$, 则 $\forall x\in R^n\setminus\{0\}$, $\exists$邻域 W. st u在W中为0. 即 $\forall\psi\in C_c^\infty(W)$, 有 $\langle u,\psi\rangle=0$

$\forall x\in K$, 都存在满足条件的 W 由有限覆盖定理知 $\exists$有限个 W_i, 满足 $\mathrm{supp}\,\phi\subset(\cup W_i)$

$\therefore 0\notin\mathrm{supp}\,\phi$, $\langle u,\phi\rangle=0$

② 若 $0\in\mathrm{supp}\,\phi$. 选取一个K紧集使得 $0\in K$. 不失一般性, 令 $K=\bar{B}_r$. 若 $\mathrm{supp}\,\phi\not\subset\bar{B}_r$, 可取截断函数$\eta$使得 $\mathrm{supp}\,\phi\eta\in\bar{B}_r$ 且 $\eta|_{B_r}=1$, $\eta|_{\partial B_r}=0$, $|\partial^\alpha\eta|\le\frac{2}{r}$, $\langle u,\phi\rangle=\langle u,\eta\phi\rangle$

$\Rightarrow\langle u,\phi\rangle\le C\sum_{|\alpha|\le N_0}\sup_K|\partial^\alpha(\eta\phi)|\le C\cdot C_{N_0}^{\frac{N_0}{2}}\frac{2}{r}\sup_{K,\,|\alpha|\le N_0}|\partial^\alpha\phi|=0$ (取N足够大)

(2) 由(1)中结论, $\exists N_0\ge 0$, $\forall\phi$有 $\partial^\alpha\phi(0)=0$ ($|\alpha|\le N_0$) $\langle u,\phi\rangle=0$

$\forall\phi\in C_c^\infty(R^n)$, 对$\phi$进行拉格朗日余项的泰勒展开.

$\phi(x)=\sum_{|\alpha|\le N_1}\frac{1}{\alpha!}D^\alpha f(0)x^\alpha+\sum_{|\alpha|=N_1+1}\frac{1}{\alpha!}D^\alpha f(\theta x)x^\alpha$

当 $N_1=N_0+1$ 时, $\frac{1}{\alpha!}D^\alpha f(0)x^\alpha$ 满足 $\forall N_0$ 阶偏导, 当 $x=0$ 时, 取 0

$\langle u,\phi\rangle=\sum_{|\alpha|\le N}\frac{1}{\alpha!}D^\alpha f(0)\langle u,x^\alpha\rangle=\sum_{|\alpha|\le N_1}\frac{\langle\partial^\alpha\delta,f\rangle}{\alpha!}\langle u,x^\alpha\rangle(-1)^{|\alpha|}$

$\Rightarrow u=\sum_{|\alpha|\le N_1}\frac{\langle u,x^\alpha\rangle}{\alpha!}(-1)^{|\alpha|}\cdot\partial^\alpha\delta=\sum_{\alpha\in N}C_\alpha\cdot\partial^\alpha\delta$.

5、设 ω 为 Ω 中开集，且有紧闭包，Γ 是 Ω 上 Laplace 方程的基本解，在分部意义下有 $-\Delta\Gamma=\delta$.

取 $f\in C_0^\infty(\Omega;[0,1])$，$f|_\omega\equiv 1$. 记 $w(x,y)=\Delta f(y)\Gamma(x-y)$ $(x\in\omega,\ y\in\Omega)$，$\omega$ 良定.

由于 $w(x,\cdot)\in C_0^\infty(\Omega)$，可定义 $v(x):=\langle u,w(x,\cdot)\rangle$，$x\in\omega$，则 $v\in C^\infty(\omega)$.

对 $g\in C_0^\infty(\omega)$，$\langle v,g\rangle=\int_\omega\langle u,w(x,\cdot)\rangle g(x)dx=\langle u,\int_\omega w(x,\cdot)g(x)dx\rangle$

对 $y\in\omega$，有 $\int_\omega w(x,y)g(x)dx=\Delta f(y)\int_\omega\Gamma(x-y)g(x)dx=\Delta f(y)(\Gamma*g)(y)=\Delta(f\cdot\Gamma*g)-f(y)$ $(\Delta\Gamma*g)(y)-2Df(y)\cdot D(\Gamma*g)(y)=\Delta(f\cdot\Gamma*g)(y)+g(y)$

最后一个等号成立是因为 $f|_\omega=1$，$Df|_\omega=0$，$-\Delta*g=\delta*g=g$.

$\Rightarrow\langle v,g\rangle=\langle\Delta u,f(\Gamma*g)\rangle+\langle u,g\rangle=\langle u,g\rangle$

$\therefore u|_\omega=v\in C^\infty(\omega)$. 根据 ω 的任意性，$u\in C^\infty(\Omega)$.