ODE笔记13：一阶偏微分方程

$$F\left(x_1, x_2, \cdots, x_n, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)=0 \qquad (*)$$

若 $u=\varphi\left(x_1, \cdots, x_n\right) \in C^1(D)$，且

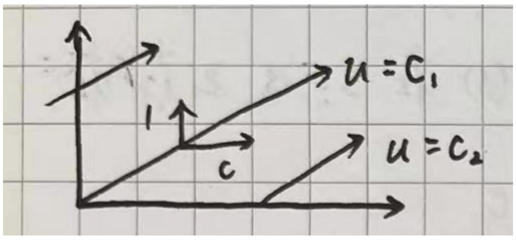
$$F(x_1, \cdots, x_n, \varphi(x_1, \cdots, x_n), \frac{\partial \varphi}{\partial x_1}(x_1, \cdots x_n), \cdots, \frac{\partial \varphi}{\partial x_n}(x_1, \cdots, x_n)) \equiv 0, \ \forall(x_1, \cdots, x_n) \in D.$$

则称 $\varphi\left(x_1, \cdots, x_n\right)$ 为一阶偏微分 $(*)$ 在 D 中解。

传输方程：$u(t, x) \quad u_t+c u_x=0$

$\partial_t u=0 \quad \Longrightarrow \quad u(t, x)=\varphi(x)$ 与 t 无关。

方向导数：左式 $=\begin{pmatrix}1 \\ c\end{pmatrix}\begin{pmatrix}\partial_t \\ \partial_x\end{pmatrix} u$，沿 $\begin{pmatrix}1 \\ c\end{pmatrix}$ 方向的导数为 0 。如图(特征线)：



$u(t, x)$ 沿 $\begin{pmatrix}1 \\ c\end{pmatrix}$ 方向为常数，即 $x-c t=$ 常数。

新坐标：$x^{\prime}=c x+t$, $t^{\prime}=x-c t$

$$u_x=u_{x^{\prime}} c+u_t \cdot 1 \qquad u_t=u_{x^{\prime}} \cdot 1+u_{t^{\prime}} \cdot(-c)$$

$$0=u_t+c u_x=\left(1+c^2\right) u_{x^{\prime}} \quad u=\varphi\left(t^{\prime}\right)=\varphi(x-c t)$$

特征线法：$a u_t+b u_x=f$

特征线：x_0 出发，沿 (a, b) 方向运动，$(t, x)=(t(s), x(s))$，考虑 $u(t(s), x(s))$.

$$\frac{d u(t(s), x(s))}{d s}=\partial_t u \cdot \frac{d t(s)}{d s}+\partial_x u \frac{d x(s)}{d s}=a u_t+b u_x=f$$

$$\begin{cases}\dfrac{d t}{d s}=a & t(0)=0 \\ \dfrac{d x}{d s}=b & x(0)=x_0 \\ \dfrac{d u}{d s}=f & u(0)=\varphi\left(x_0\right)\end{cases} \Longrightarrow \begin{cases}t=t\left(s, x_0\right) \\ x=x\left(s, x_0\right) \\ u=u\left(s, x_0\right)\end{cases} \xrightarrow{\text{隐函数定理}} \begin{cases}s=s(t, x) \\ x_0=x_0(t, x)\end{cases}$$

$u=u\left(s(t, x), x_0(t, x)\right) \quad (t, x)$ 为二元函数。

$$a_1 u_{x_1}+a_2 u_{x_2}+\cdots+a_n u_{x_n}=f \quad \Longleftrightarrow \quad \frac{d x_1}{d s}=a_1, \cdots, \frac{d x_n}{d s}=a_n, \frac{d u}{d s}=f.$$

例1：$u_t+x u_x=x,\left.u\right|_{t=0}=x^3$

解：特征线法：

$$
\begin{cases}
\frac{dt}{ds}=1 & t(0)=0 \quad \Rightarrow t=s \\
\frac{dx}{ds}=x & x(0)=x_0 \quad \Rightarrow x=x_0e^s \\
\frac{du}{ds}=x & u(0)=x_0^3
\end{cases}
$$

$\frac{du}{ds}=x_0e^s$ 且 $u(0)=x_0^3 \quad \Longrightarrow \quad u=x_0^3+\int_0^s x_0e^{s'}ds'=x_0^3+x_0(e^s-1)$

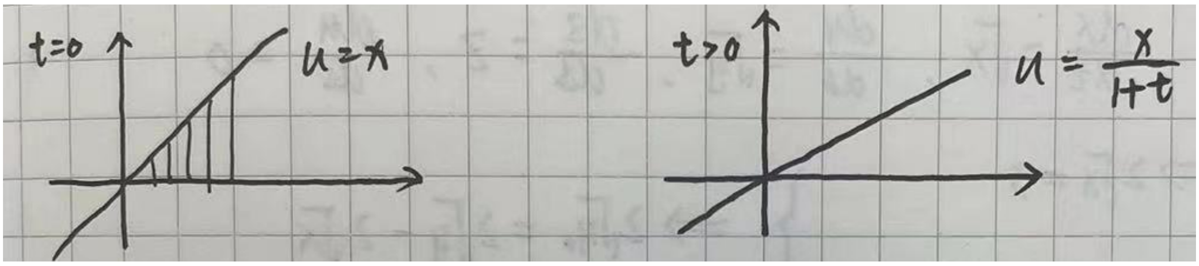
$\therefore u=(xe^{-t})^3+xe^{-t}(e^t-1).$

Burges方程: $\begin{cases} u_t+uu_x=0 \\ u\big|_{t=0}=x \end{cases}$

解:

$$
\begin{cases}
\frac{dt}{ds}=1 & t(0)=0 \quad \Rightarrow t=s \\
\frac{dx}{ds}=u & x(0)=x_0 \\
\frac{du}{ds}=0 & u(0)=x_0 \quad \Rightarrow u=x_0
\end{cases}
$$

$\therefore \begin{cases} \frac{dx}{ds}=x_0 \\ x(0)=x_0 \end{cases} \quad x=x_0+x_0s \quad \Longrightarrow \quad x_0=\frac{x}{1+s}=\frac{x}{1+t} \quad \therefore\ u(t,x)=\frac{x}{1+t}$



例2: $\begin{cases} u_t+uu_x=0 \\ u\big|_{t=0} \begin{cases} 1, & x\leqslant 0 \\ 1-x, & 0<x<1 \\ 0, & x\geqslant 1 \end{cases} \end{cases}$

解:

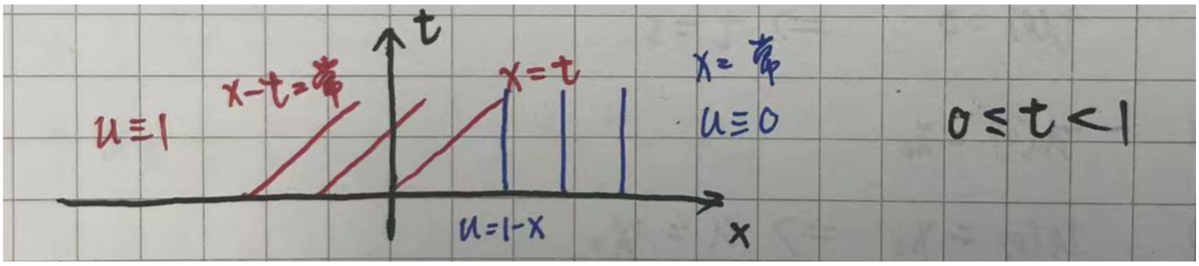
$$
\begin{cases}
\frac{dt}{ds}=1, \quad t(0)=0 \\
\frac{dx}{ds}=u, x(0)=x_0 \\
\frac{du}{ds}=0, u(0)=\begin{cases} 1,\ x_0\leqslant 1 \ \ \Rightarrow \ \ u=1 \\ 1-x_0,\ 0<x_0<1 \ \ \Rightarrow \ \ u=1-x_0 \ \ \Rightarrow \ \ u=0 \\ 0,\ x_0\geqslant 1 \ \ \Rightarrow \ \ u=0 \end{cases}
\end{cases}
$$

(1) $x_0\le 0, \frac{dx}{ds}=1, x(0)=x_0 \quad \Rightarrow x=x_0+s \quad \Rightarrow x_0=x-t\le 0$ 当 $x-t\le 0$ 时, $u\equiv 1$

(2) $0<x_0<1, \frac{dx}{ds}=1-x_0, x(0)=x_0 \quad \Rightarrow \quad x=x_0+(1-x_0)s \quad \Rightarrow \quad x_0=\frac{x-s}{1-s}$

当 $0<\frac{x-1}{1-t}<1$ 时, $u=1-\frac{x-t}{1-t}=\frac{1-x}{1-t}$

(3) $x_0 \geq 1, \frac{dx}{ds} = 0, x(0) = x_0 \quad \Longrightarrow \quad x = x_0$. 当 $x \geq 1, u = 0$



例2： $\begin{cases} \sqrt{x}\dfrac{\partial u}{\partial x} + \sqrt{y}\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0 \\ u\big|_{z=1} = xy \end{cases}$

解：特征线：$\frac{dx}{dt} = \sqrt{x}, \frac{dy}{ds} = \sqrt{y}, \frac{dz}{ds} = z, \frac{du}{ds} = 0$

$$\left.\begin{array}{l} x(0) = 0 \quad \Rightarrow \quad 2\sqrt{x} = s \\ y(0) = y_0 \quad \Rightarrow \quad 2\sqrt{y} - 2\sqrt{y_0} = s \end{array}\right\} \quad \Longrightarrow \quad 2\sqrt{y_0} = 2\sqrt{y} - 2\sqrt{x}$$

$z(0) = z_0 \quad \Rightarrow \quad z = z_0 e^s \quad \Rightarrow \quad z = ze^{-\sqrt{2}x}$

$u(0) = f(y_0, z_0) \Rightarrow u = f(y_0, z_0) = \tilde{f}(2\sqrt{y_0}, \ln z_0) = \tilde{f}(2\sqrt{y} - 2\sqrt{x}, \ln z - 2\sqrt{x})$

$\because z = 1, u = xy = \tilde{f}(2\sqrt{y} - 2\sqrt{x}, -2\sqrt{x}) \triangleq \tilde{f}(\alpha, \beta)$

$$\therefore \begin{cases} \alpha = 2\sqrt{y} - 2\sqrt{x} \Rightarrow y = \left(\frac{\alpha-\beta}{2}\right)^2 \\ \beta = -2\sqrt{x} \Rightarrow x = \left(\frac{\beta}{2}\right)^2 \end{cases}$$

$\therefore \tilde{f}(\alpha, \beta) = xy = \left(\frac{\beta}{2}\right)^2 \left(\frac{\alpha-\beta}{2}\right)^2$

$\therefore u = \tilde{f}(2\sqrt{y} - 2\sqrt{x}, \ln z - 2\sqrt{x}) = \left(\frac{\ln z - 2\sqrt{x}}{2}\right)^2 \cdot \left(\frac{2\sqrt{y} - \ln z}{2}\right)^2$