3.3 Characteristic functions

Definition

Suppose that ξ and η are real random variables, we call $\zeta = \xi + i\eta$ a complex random variable, where $i^2 = -1$. We call $E\zeta = E\xi + iE\eta$ the expectation of ζ.

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$E\zeta$ possesses properties similar to that of a real mathematical expectation.

Modulus inequality: $|E\zeta| \le E|\zeta|$.

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$$\sqrt{(E\xi)^2 + (E\eta)^2} = \sup_{a^2+b^2=1} \left(aE\xi + bE\eta\right)$$

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取$a = E\xi$, $b = E\eta$得

$$(E\xi)^2+(E\eta)^2\le\sqrt{(E\xi)^2+(E\eta)^2}\cdot E\sqrt{\xi^2+\eta^2}.$$

所以

$$\sqrt{(E\xi)^2+(E\eta)^2}\le E\sqrt{\xi^2+\eta^2}.$$

即

$$|E\zeta|\le E|\zeta|.$$

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Suppose ξ is a real random variable, we call

$$f(t) = Ee^{it\xi}, \quad -\infty < t < \infty$$

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Notice $|e^{it\xi}| = 1$. $Ee^{it\xi}$ exists for all t.

$$f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x).$$

If ξ is a discrete random variable with $P(\xi = x_n) = p_n$, then

$$f(t) = \sum_{n=1}^{\infty} p_n e^{itx_n}, \quad -\infty < t < \infty.$$

If ξ is a continuous random variable with the density function $p(x)$, then

$$f(t) = \int_{-\infty}^{\infty} e^{itx} p(x) dx, \quad -\infty < t < \infty,$$

which is just the Fourier transformation of $p(x)$.

Example

The characteristic function of the degenerate distribution $P(\xi = c) = 1$ is

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$$f(t) = e^{ict}, \qquad -\infty < t < \infty.$$

Example

The characteristic function of the binomial distribution $B(n, p)$ is

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The characteristic function of the binomial distribution $B(n,p)$ is

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\begin{aligned}
f(t) &= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} e^{itk} = \sum_{k=0}^{n} \binom{n}{k} (pe^{it})^k q^{n-k} \\
&= (pe^{it}+q)^n, \quad p+q=1.
\end{aligned}
$$

Example

The characteristic function of the Poisson distribution $P(\lambda)$ is

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$$\begin{aligned} f(t) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} e^{itk} \\ &= \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!} e^{-\lambda} = e^{\lambda(e^{it}-1)}. \end{aligned}$$

Example

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特别地，$U[-1,1]$的特征函数为

$$f(t) = \frac{\sin t}{t}.$$

Example

The characteristic function of the normal distribution $N(a, \sigma^2)$ is

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{itx - \frac{(x-a)^2}{2\sigma^2}} dx$$

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&= e^{iat-\frac{\sigma^2 t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-it\sigma)^2}{2}}dy \\
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&= e^{iat-\frac{\sigma^2t^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-it\sigma)^2}{2}}dy \\
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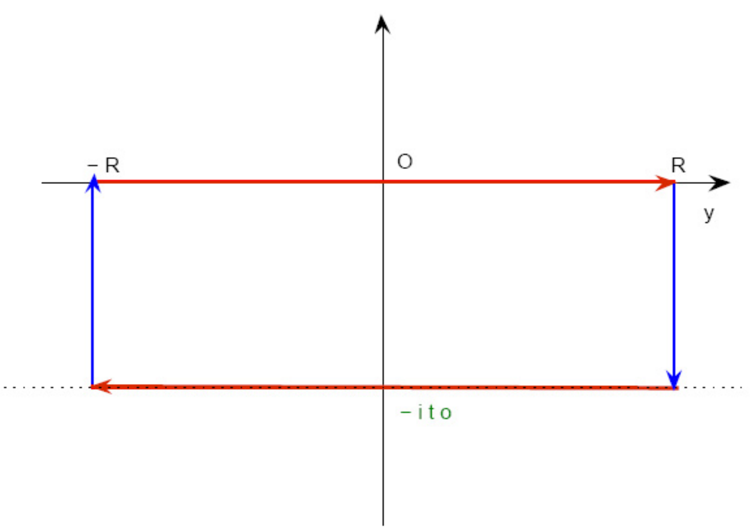
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So it is enough to show that $f_\eta(t) = e^{-\frac{t^2}{2}}$.

It is obvious that

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f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(tx) e^{-\frac{x^2}{2}} dx + i \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sin(tx) e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(tx) e^{-\frac{x^2}{2}} dx.
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f_\eta'(t)
$$

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&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos(tx)e^{-\frac{x^2}{2}}dx.
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So

$$f'_\eta(t) = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)e^{-\frac{x^2}{2}}dx$$

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So

$$\begin{aligned} f_\eta'(t) &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\sin(tx)e^{-\frac{x^2}{2}}dx \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sin(tx)de^{-\frac{x^2}{2}} \\ &= -t\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\cos(tx)e^{-\frac{x^2}{2}}dx = -tf_\eta(t) \end{aligned}$$

Next we need to solve the differential equation

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So

$$f_\eta(t) = e^{-\frac{t^2}{2}}.$$

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f_\eta(t) &= Ee^{it\eta} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(itx)^n}{n!} e^{-\frac{x^2}{2}} dx \\
&= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx
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&= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2}} dx \\
&= \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} E\eta^{2n} = \sum_{n=0}^{\infty} \frac{(it)^{2n}}{(2n)!} (2n-1)!!
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&= \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}E\eta^{2n} = \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}(2n-1)!! \\
&= \sum_{n=0}^{\infty}\frac{(it)^{2n}}{(2n)!}\frac{(2n)!}{2^n n!} = \sum_{n=0}^{\infty}\left(-\frac{t^2}{2}\right)^n\frac{1}{n!} = e^{-\frac{t^2}{2}}.
\end{aligned}
$$

Example

The characteristic function of the Cauchy distribution is

$$f(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx$$

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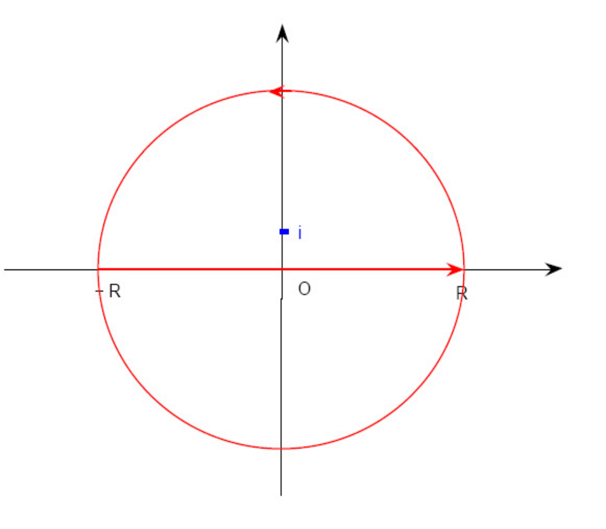
$$f(t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\pi(1+x^2)} dx = e^{-|t|}.$$

In fact, when $t > 0$,

$$\int_{-R}^{R} e^{itx}\frac{1}{\pi(1+x^2)}dx + \int_{semicircle}\frac{e^{itz}}{\pi(1+z^2)}dz$$

$$=2\pi i Res\left(\frac{e^{itz}}{\pi(1+z^2)}\text{ at } i\right)$$

$$=2\pi i(z-i)\frac{e^{itz}}{\pi(1+iz)(1-iz)}\Big|_{z=i} = e^{-t}.$$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$$\left|\int_{semicircle} \frac{e^{itz}}{\pi(1+z^2)}dz\right| \le \int_{semicircle} \frac{1}{\pi(R^2-1)}dz$$

$$= \frac{\pi R}{\pi(R^2-1)} \to 0.$$

3.3.2 Properties

1. $|f(t)| \le f(0) = 1;\ f(-t) = \overline{f(t)}.$

Proof.

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1. $|f(t)| \leq f(0) = 1$; $f(-t) = \overline{f(t)}$.

Proof. Obviously,

$$|f(t)| = \left|\int_{-\infty}^{\infty} e^{itx} dF(x)\right| \leq \int_{-\infty}^{\infty} |e^{itx}| dF(x) = 1$$

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1. $|f(t)| \le f(0) = 1$; $f(-t) = \overline{f(t)}$.

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$$\begin{aligned} f(-t) &= \int_{-\infty}^{\infty} e^{-itx} dF(x) = \int_{-\infty}^{\infty} \overline{e^{itx}} dF(x) \\ &= \overline{\int_{-\infty}^{\infty} e^{itx} dF(x)} = \overline{f(t)}, \end{aligned}$$

as desired.

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& |f(t+h) - f(t)| \\
= \ & |\int_{-\infty}^{\infty} (e^{itx} e^{ihx} - e^{itx}) dF(x)|
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\leq \ & \int_{-\infty}^{\infty} |e^{ihx} - 1| dF(x)
\end{aligned}
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Note that $|e^{ihx}-1|\le 2$ and

$$\begin{aligned}|e^{ihx}-1| &= |e^{i\frac{h}{2}x}||e^{i\frac{h}{2}x}-e^{-i\frac{h}{2}x}| = 2|\sin\frac{hx}{2}| \\ &\le |hx|\le A|h|, \quad \text{when } |x|\le A.\end{aligned}$$

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We have

$$\begin{aligned}|f(t+h)-f(t)| &\le 2\int_{|x|\ge A} dF(x) + A|h|\int_{|x|<A} dF(x) \\ &\le 2\int_{|x|\ge A} dF(x) + |h|A.\end{aligned}$$

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Choose A such that $\int_{|x|\ge A} dF(x) < \epsilon/4$.

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We have

$$\begin{aligned} |f(t+h)-f(t)| &\le 2\int_{|x|\ge A} dF(x) + A|h|\int_{|x|<A} dF(x) \\ &\le 2\int_{|x|\ge A} dF(x) + |h|A. \end{aligned}$$

Choose A such that $\int_{|x|\ge A} dF(x) < \epsilon/4$. And then take $\delta = \varepsilon/(2A)$.

Consequently, $|f(t+h)-f(t)| < \varepsilon$ for all t whenever $|h| < \delta$.

3. $f(t)$ is non-negative definite, i.e., for an arbitrary integer n, any real numbers $t_1, \cdots, t_n$ and complex numbers $\lambda_1, \cdots, \lambda_n$, it follows

$$\sum_{k=1}^{n}\sum_{j=1}^{n} f(t_k - t_j)\lambda_k\overline{\lambda_j} \geq 0.$$

Proof.

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$$
\begin{aligned}
& \sum_{k=1}^{n}\sum_{j=1}^{n} f(t_k - t_j)\lambda_k\overline{\lambda_j} \\
= & \sum_{k=1}^{n}\sum_{j=1}^{n} \int_{-\infty}^{\infty} e^{i(t_k-t_j)x} dF(x)\lambda_k\overline{\lambda_j} \\
= & \int_{-\infty}^{\infty} \left(\sum_{k=1}^{n} e^{it_kx}\lambda_k\right)\left(\sum_{j=1}^{n} e^{-it_jx}\overline{\lambda_j}\right) dF(x)
\end{aligned}
$$

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=&\int_{-\infty}^{\infty}(\sum_{k=1}^{n} e^{it_kx}\lambda_k)\overline{(\sum_{j=1}^{n} e^{it_jx}\lambda_j)}dF(x)\\
=&\int_{-\infty}^{\infty}|\sum_{k=1}^{n} e^{it_kx}\lambda_k|^2dF(x)\geq 0.
\end{aligned}
$$

Bochner-Khinchine Theorem.

The function $f(t)$ is a characteristic function if and only if $f(t)$ is non-negative definite, continuous and $f(0) = 1$.

4. Assume that $\xi_1, \cdots, \xi_n$ are indept., then

$$f_{\xi_1+\xi_2+\cdots+\xi_n}(t) = f_{\xi_1}(t) f_{\xi_2}(t) \cdots f_{\xi_n}(t).$$

(Proof?)

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$$f^{(k)}(0) = i^k E\xi^k.$$

In particular, when $E\xi^2$ exists, $E\xi = -if'(0)$, $E\xi^2 = -f''(0)$, $Var\xi = -f''(0) + [f'(0)]^2$.

Proof. Since

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$$f''(0) = \lim_{h\to 0}\frac{f'(h)+f'(-h)}{2h} = \lim_{h\to 0}\frac{f(h)-2f(0)+f(-h)}{h^2}$$

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&= -\lim_{h\to 0}\int_{-\infty}^{\infty}2\frac{1-\cos hx}{h^2}dF(x).
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注意到, $0\le 2(1-\cos hx)/h^2\le x^2$, 并

且 $\lim_{h\to 0}2(1-\cos hx)/h^2=x^2$ 关于 x 在任一有限区间内一致成立.

因此对任意 $a>0$ 有

$$-f''(0) \;\geq\; \lim_{h\to 0}\int_{-a}^{a} 2\frac{1-\cos hx}{h^2}dF(x)$$

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令 $a\to\infty$ 得 $\int_{-\infty}^{\infty} x^2 dF(x) \leq -f''(0)$, 即 $E\xi^2$ 存在.

现设 $f^{(2k)}(0)$ 存在，同时归纳假设 $E\xi^{2k-2}$ 也存在. 由第一部分结论，$f(t)$ 是 $2k-2$ 次可微的，且

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$$
\begin{aligned}
f^{(2k-2)}(t) =& i^{2k-2}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}x^{2k-2}dF(x) \\
=&(-1)^{k-1}\int_{-\infty}^{\infty} e^{\mathrm{i}tx}x^{2k-2}dF(x).
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记 $G(y)=\int_{-\infty}^{y} x^{2k-2}dF(x)$，其中 $G(\infty)=E\xi^{2k-2}$，
则 $H(y)=G(y)/G(\infty)$ 为分布函数，

$H(y)=G(y)/G(\infty)$的特征函数为

$$
\begin{aligned}
g(t) &= \int_{-\infty}^{\infty} e^{\mathrm{i}ty}dH(y) = \frac{1}{G(\infty)}\int_{-\infty}^{\infty} e^{\mathrm{i}ty}dG(y) \\
&= \frac{1}{G(\infty)}\int_{-\infty}^{\infty} e^{\mathrm{i}ty}y^{2k-2}dF(y) = \frac{(-1)^{k-1}}{G(\infty)}f^{(2k-2)}(t).
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从而 $g''(0)=(-1)^{k-1}f^{(2k)}(0)/G(\infty)$ 存在. 由已证的 $n=2$ 时的结论知

$$
\frac{1}{G(\infty)}\int_{-\infty}^{\infty}x^{2k}dF(x)=\frac{1}{G(\infty)}\int_{-\infty}^{\infty}y^{2}dG(y)
$$

存在. 即 $E\xi^{2k}$ 存在, 结论得证.

6. Let $\eta = a\xi + b$, where a, b are arbitrary constants. Then

$$f_\eta(t) = e^{ibt} f(at).$$

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Proof.

$$Ee^{i(a\xi+b)t} = Ee^{iat\xi} \cdot e^{ibt} = e^{ibt} f(at).$$

Example

Are the following functions characteristic functions of some random variables?

(1) $f(t) = \sin t$;

(2) $f(t) = \ln(e + |t|)$;

(3) $f(t) = 0$ when $t < 0$; $f(t) = 1$ when $t \geq 0$.

Solution……

3.3.3 Inverse formula and uniqueness theorem

Theorem

(Inverse formula) Suppose that $f(t)$ is a c.f. corresponding to cdf $F(x)$. Let x_1, x_2 be two continuity points of $F(x)$, then

$$F(x_2) - F(x_1) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt.$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt$$

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$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
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& \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
= & \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty} \frac{e^{-itx_1} - e^{-itx_2}}{it} e^{itx} dF(x)dt \\
= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-T}^{T} \frac{e^{it(x-x_1)} - e^{it(x-x_2)}}{it} dtdF(x)
\end{aligned}$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\
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= \ & \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-T}^{T} \frac{e^{it(x-x_1)} - e^{it(x-x_2)}}{it} dtdF(x) \\
= \ & \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{e^{it(x-x_1)} - e^{-it(x-x_1)}}{it} - \frac{e^{it(x-x_2)} - e^{-it(x-x_2)}}{it}]dt\}dF(x)
\end{aligned}
$$

Proof. Suppose $\xi \sim F(x)$. Without loss of generality, we assume that $x_1 < x_2$.

$$
\begin{aligned}
& \frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \frac{1}{2\pi}\int_{-T}^{T}\int_{-\infty}^{\infty}\frac{e^{-itx_1}-e^{-itx_2}}{it}e^{itx}dF(x)dt \\
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= & \frac{1}{2\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{e^{it(x-x_1)}-e^{-it(x-x_1)}}{it}-\frac{e^{it(x-x_2)}-e^{-it(x-x_2)}}{it}]dt\}dF(x) \\
= & \frac{1}{\pi}\int_{-\infty}^{\infty}\{\int_{0}^{T}[\frac{\sin t(x-x_1)}{t}-\frac{\sin t(x-x_2)}{t}]dt\}dF(x).
\end{aligned}
$$

$$\triangleq \int_{-\infty}^{\infty} g(x; T, x_1, x_2) dF(x) = Eg(\xi; T, x_1, x_2),$$

where

$$g(\xi; T, x_1, x_2) = \frac{1}{\pi} \int_0^T [\frac{\sin t(x - x_1)}{t} - \frac{\sin t(x - x_2)}{t}] dt.$$

Notice

$$\int_0^T \frac{\sin at}{t} dt = \int_0^{Ta} \frac{\sin t}{t} dt$$

$$\to sgn(a) \int_0^{\infty} \frac{\sin t}{t} dt = \begin{cases} \frac{\pi}{2}, & \text{if } a > 0, \\ 0, & \text{if } a = 0, \\ -\frac{\pi}{2}, & \text{if } a < 0; \end{cases}$$

$\int_0^x \frac{\sin t}{t} dt$ is a bounded function .

It follows that

$$
\begin{aligned}
&\lim_{T\to\infty} g(x;T,x_1,x_2)\\
=&\lim_{T\to\infty}\frac{1}{\pi}\{\int_0^T[\frac{\sin t(x-x_1)}{t}-\frac{\sin t(x-x_2)}{t}]dt\}\\
=&\begin{cases} 0, & x<x_1 \text{ or } x>x_2,\\ \frac{1}{2}, & x=x_1 \text{ or } x=x_2,\\ 1, & x_1<x<x_2. \end{cases}
\end{aligned}
$$

It follows that

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\end{aligned}$$

and

$$|g(x;T,x_1,x_2)|<M.$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx_1}-e^{-itx_2}}{it}f(t)dt \\
= & \lim_{T\to\infty} Eg(\xi;T,x_1,x_2)
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$$\begin{aligned} & \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-itx_1} - e^{-itx_2}}{it} f(t)dt \\ = & \lim_{T\to\infty} Eg(\xi; T, x_1, x_2) = Eg(\xi) \\ = & P(x_1 < \xi < x_2) + \frac{1}{2}\big(P(\xi = x_1) + P(\xi = x_2)\big) \end{aligned}$$

It follows that

$$
\begin{aligned}
& \lim_{T\to\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-itx_1}-e^{-itx_2}}{it} f(t)dt \\
= & \lim_{T\to\infty} Eg(\xi;T,x_1,x_2) = Eg(\xi) \\
= & P(x_1<\xi<x_2)+\frac{1}{2}\big(P(\xi=x_1)+P(\xi=x_2)\big) \\
= & F(x_2)-F(x_1).
\end{aligned}
$$

Theorem

(Uniqueness) A distribution function can be uniquely determined by its characteristic function.

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Proof. By inverse formula, if $y < x$ are continuous points of $F(x)$, then

$$F(x) - F(y) = \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ity} - e^{-itx}}{it} f(t) dt.$$

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Letting $y \to -\infty$ along continuity points of $F(x)$, we have

$$F(x) = \lim_{y\to-\infty} \lim_{T\to\infty} \frac{1}{2\pi} \int_{-T}^{T} \frac{e^{-ity} - e^{-itx}}{it} f(t)dt.$$

Thus it is easy to see that $f(t)$ determines the value of $F(x)$ at its continuity points. As for the discontinuous points, in view of right continuity of $F(x)$, it suffices to take right limits along continuity points. The theorem is proved.

Theorem

(Inverse Fourier transform) Suppose that $f(t)$ is a c.f. and $\int_{-\infty}^{\infty}|f(t)|dt<\infty$, then $F'(x)$ exists and is continuous. Moreover

$$F'(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}f(t)dt.$$

Proof. Since $f(t)$ is absolutely integrable and

$$\left|\frac{e^{-itx}-e^{-ity}}{it}\right| \le |y-x|,$$

it follows that

$$F(y)-F(x) = \lim_{T\to\infty}\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{-itx}-e^{-ity}}{it}f(t)dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}-e^{-ity}}{it}f(t)dt \triangleq H(x,y),$$

whenever x, y are continuous points of $F(x)$.

Also, $H(x, y)$ is a continuous function of (x, y). So, $F(x)$ must be a continuous function and the above equality holds for all x, y.

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$$F'(x) = \lim_{y \to x} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itx} - e^{-ity}}{it(y-x)} f(t) dt$$

Also, $H(x,y)$ is a continuous function of (x,y). So, $F(x)$ must be a continuous function and the above equality holds for all x,y. Now, applying the dominated convergence theorem yields

$$\begin{aligned} F'(x) &= \lim_{y\to x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-itx}-e^{-ity}}{it(y-x)}f(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{y\to x}\frac{e^{-itx}-e^{-ity}}{it(y-x)}f(t)dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}f(t)dt. \end{aligned}$$

For same reason,

$$\lim_{y\to x} F'(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{y\to x} e^{-ity} f(t)dt = F'(x).$$

So, $F'(x)$ is continuous.

Discrete random variables: Assume $P(\xi = k) = p_k, k = 0, 1, 2, \cdots,$ then

$$f(t) = \sum_{k=0}^{\infty} p_k e^{itk}.$$

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$$f(t) = \sum_{k=0}^{\infty} p_k e^{itk}.$$

If $f(t)$ is given, then we can multiply both sides by e^{-itk} and integrate. Noting that

$$\int_0^{2\pi} e^{int} dt = \begin{cases} 2\pi, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

we have

$$p_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-itk} f(t) dt.$$

Example

Show $f(t) = \cos t$ is a characteristic function of some random variable, and find its distribution function.

Solution......

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Solution.......

In general, if $f(t)$ can be written as $\sum a_n e^{ix_n t}$, where $a_n > 0$ and $\sum a_n = 1$, then $f(t)$ is a characteristic function, whose corresponding random variable has distribution sequence $P(\xi = x_n) = a_n$, $n = 1, 2, \cdots$.

Example

If $f(t)$ is a characteristic function of some random variable, show so are $\overline{f(t)}$ and $|f(t)|^2$.

Solution……

3.3.4 Additivity of distribution functions

The additivity, also called regenerativity, means that if ξ and η are independent and follow a common type of distributions, then so do their sum $\xi + \eta$ and the parameter is the sum of parameters of ξ and η.

Example

Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution.

Example

Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution. It is known that the c.f. of ξ_k is $e^{ia_kt-\sigma_k^2t^2/2}$, so the c.f. of $\sum_{k=1}^n \xi_k$ is

$$\prod_{k=1}^n e^{ia_kt-\frac{\sigma_k^2t^2}{2}} = \exp\{i\sum_{k=1}^n a_kt - \frac{\sum_{k=1}^n \sigma_k^2t^2}{2}\}.$$

Example

Suppose that $\xi_1, \cdots, \xi_n$ are indept., and $\xi_k \sim N(a_k, \sigma_k^2)$, $k = 1, \cdots, n$. Find the distribution of $\sum_{k=1}^n \xi_k$.

Solution. It is known that the c.f. of ξ_k is $e^{ia_k t - \sigma_k^2 t^2/2}$, so the c.f. of $\sum_{k=1}^n \xi_k$ is

$$\prod_{k=1}^n e^{ia_k t - \frac{\sigma_k^2 t^2}{2}} = \exp\{i \sum_{k=1}^n a_k t - \frac{\sum_{k=1}^n \sigma_k^2 t^2}{2}\}.$$

Thus $\sum_{k=1}^n \xi_k \sim N\left(\sum_k a_k, \sum_k \sigma_k^2\right)$.

3.3.5 Multivariate characteristic functions

Definition

Suppose the random vector $\xi = (\xi_1, \cdots, \xi_n)'$ has distribution function $F(x_1, \cdots, x_n)$, then its characteristic function is defined by

$$
\begin{aligned}
& f(t_1, \cdots, t_n) = Ee^{i(t_1\xi_1 + \cdots + t_n\xi_n)} \\
= & \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i(t_1x_1 + \cdots + t_nx_n)} dF(x_1, \cdots, x_n).
\end{aligned}
$$

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$$\begin{aligned} f(t_1, \cdots, t_n) &= Ee^{i(t_1\xi_1 + \cdots + t_n\xi_n)} \\ &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{i(t_1x_1 + \cdots + t_nx_n)} dF(x_1, \cdots, x_n). \end{aligned}$$

$$f(\boldsymbol{t}) = Ee^{i\boldsymbol{t}'\boldsymbol{\xi}} = \int_{\boldsymbol{R}^n} e^{i\boldsymbol{t}'\boldsymbol{x}} dF(\boldsymbol{x}),$$

where $\boldsymbol{t} = (t_1, \cdots, t_n)'$, $\boldsymbol{x} = (x_1, \cdots, x_n)'$.

1. The c.f. of $\eta = a_1\xi_1 + \cdots + a_n\xi_n$ is

$$
\begin{aligned}
f_\eta(t) &= Ee^{it\eta} = Ee^{it\sum a_k\xi_k} \\
&= Ee^{i\sum(a_k t)\xi_k} = f(a_1 t, \cdots, a_n t).
\end{aligned}
$$

2. If the c.f. of $(\xi_1, \cdots, \xi_n)'$ is $f(t_1, \cdots, t_n)$, then k-dimensional sub-vector $(\xi_{l_1}, \cdots, \xi_{l_k})'$ has c.f.

$$
f(0, \cdots, 0, t_{l_1}, 0, \cdots, 0, t_{l_k}, 0, \cdots, 0).
$$

3. Assume that ξ_j has c.f. $f_j(t)$, $j = 1, \cdots, n$, then $\xi_1, \cdots, \xi_n$ are indept. iff the c.f. of $(\xi_1, \cdots, \xi_n)'$ is such that

$$f(t_1, \cdots, t_n) = f_1(t_1) \cdots f_n(t_n).$$

4. $(\xi_1, \cdots, \xi_k)'$ and $(\xi_{k+1}, \cdots, \xi_n)'$ are indept. iff the product of their c.f.s is just equal to the c.f. of $(\xi_1, \cdots, \xi_n)'$.