2.3 Random vectors

Definition

If random variables $\xi_1(\omega)$, $\xi_2(\omega)$, $\cdots$, $\xi_n(\omega)$ are defined on a common probability space $(\Omega, \mathcal{F}, P)$, then we call

$$\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \xi_2(\omega), \cdots, \xi_n(\omega))$$

an n-dimensional random vector.

$\boldsymbol{\xi}$ is a random vector if and only if for any $B \in \mathscr{B}^n$,

$$\boldsymbol{\xi}^{-1}(B) = \{\omega : \boldsymbol{\xi}(\omega) \in B\} \in \mathcal{F}.$$

$\boldsymbol{\xi}$ is a random vector if and only if for any $B \in \mathscr{B}^n$,

$$\boldsymbol{\xi}^{-1}(B) = \{\omega : \boldsymbol{\xi}(\omega) \in B\} \in \mathcal{F}.$$

In fact, let $\mathscr{C} = \{(-\infty, x_1] \times \cdots \times (-\infty, x_n] : x_1, \cdots, x_n \in \mathcal{R}\}$,
$\mathscr{L} = \{B \in \mathscr{B}^n : \boldsymbol{\xi}^{-1}(B) \in \mathcal{F}\}$.

$\boldsymbol{\xi}$ is a random vector if and only if for any $B \in \mathscr{B}^n$,

$$\boldsymbol{\xi}^{-1}(B) = \{\omega : \boldsymbol{\xi}(\omega) \in B\} \in \mathcal{F}.$$

In fact, let $\mathscr{C} = \{(-\infty, x_1] \times \cdots \times (-\infty, x_n] : x_1, \cdots, x_n \in \mathcal{R}\}$, $\mathscr{L} = \{B \in \mathscr{B}^n : \boldsymbol{\xi}^{-1}(B) \in \mathcal{F}\}$. Note

$$\{\xi_1 \le x_1\} \bigcap \cdots \bigcap \{\xi_n \le x_n\} \in \mathcal{F}.$$

So

$$\mathscr{C} \subset \mathscr{L} \subset \mathscr{B}^n.$$

It is easily proved that $\mathscr{L}$ is a σ-field.

So

$$\mathscr{C} \subset \mathscr{L} \subset \mathscr{B}^n.$$

It is easily proved that $\mathscr{L}$ is a σ-field. It follows that

$$\sigma(\mathscr{C}) \subset \mathscr{L} \subset \mathscr{B}^n.$$

So

$$\mathscr{C} \subset \mathscr{L} \subset \mathscr{B}^n.$$

It is easily proved that $\mathscr{L}$ is a σ-field. It follows that

$$\sigma(\mathscr{C}) \subset \mathscr{L} \subset \mathscr{B}^n.$$

But $\sigma(\mathscr{C}) = \mathscr{B}^n$. Hence $\mathscr{L} = \mathscr{B}^n$.

2.3.1 Discrete random vectors

If a random vector takes only a finitely many or countably many pairs of values, then we call it a discrete random vector.

The vector's probability distribution

Example

There are two white balls and three black balls in a box. We draw two balls out of the box consecutively, one at a time. Suppose that ξ represents the number of white balls in the first draw, and η the number of white balls on the second draw. Calculate the joint probability distribution either (1) with replacement or (2) without replacement.

$\xi \backslash \eta$	0	1
0	$\frac{3}{5} \cdot \frac{3}{5}$	$\frac{3}{5} \cdot \frac{2}{5}$
1	$\frac{2}{5} \cdot \frac{3}{5}$	$\frac{2}{5} \cdot \frac{2}{5}$

$\xi \backslash \eta$	0	1
0	$\frac{3}{5}\cdot\frac{3}{5}$	$\frac{3}{5}\cdot\frac{2}{5}$
1	$\frac{2}{5}\cdot\frac{3}{5}$	$\frac{2}{5}\cdot\frac{2}{5}$

$\xi \backslash \eta$	0	1
0	$\frac{3}{5}\cdot\frac{2}{4}$	$\frac{3}{5}\cdot\frac{2}{4}$
1	$\frac{2}{5}\cdot\frac{3}{4}$	$\frac{2}{5}\cdot\frac{1}{4}$

The joint distribution array of a 2-dimensional discrete random vector is written as

$$P(\xi = x_i, \eta = y_j) = p_{ij}, \quad i, j = 1, 2, \cdots$$

The joint distribution array of a 2-dimensional discrete random vector is written as

$$P(\xi = x_i, \eta = y_j) = p_{ij}, \quad i, j = 1, 2, \cdots$$

$\xi\diagdown\eta$	y_1	y_2	$\cdots$	y_j	$\cdots$	$(\xi)\ p_{i\cdot}$
x_1	p_{11}	p_{11}	$\cdots$	p_{1j}	$\cdots$	$p_{1\cdot}$
x_2	p_{21}	p_{22}	$\cdots$	p_{2j}	$\cdots$	$p_{2\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_i	p_{i1}	p_{i2}	$\cdots$	p_{ij}	$\cdots$	$p_{i\cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(\eta)\ p_{\cdot j}$	$p_{\cdot 1}$	$p_{\cdot 2}$	$\cdots$	$p_{\cdot j}$	$\cdots$	1

Properties:

$$p_{ij} \geq 0, i, j = 1, 2, \cdots; \quad \sum_i \sum_j p_{ij} = 1.$$

Properties:

$$p_{ij} \geq 0, i, j = 1, 2, \cdots ; \quad \sum_i \sum_j p_{ij} = 1.$$

$$P((\xi, \eta) \in B^2) = \sum_{(x_i, y_j) \in B^2} p_{ij}, \quad \forall B^2 \in \mathcal{B}^2.$$

The marginal distributions:

$$\begin{aligned} P(\xi = x_i) &= \sum_{j=1}^{\infty} P(\xi = x_i, \eta = y_j) \\ &= \sum_{j=1}^{\infty} p_{ij} =: p_{i\cdot}, \quad i = 1, 2, \cdots , \end{aligned}$$

$$\begin{aligned} P(\eta = y_j) &= \sum_{i=1}^{\infty} P(\xi = x_i, \eta = y_j) \\ &= \sum_{i=1}^{\infty} p_{ij} =: p_{\cdot j}, \quad j = 1, 2, \cdots , \end{aligned}$$

Example

Calculate the marginal distributions in Example

Example

Calculate the marginal distributions in Example

$\xi \backslash \eta$	0	1	$p_{i\cdot}$
0	$\frac{3}{5}\cdot\frac{3}{5}$	$\frac{3}{5}\cdot\frac{2}{5}$	$\frac{3}{5}$
1	$\frac{2}{5}\cdot\frac{3}{5}$	$\frac{2}{5}\cdot\frac{2}{5}$	$\frac{2}{5}$
$p_{\cdot j}$	$\frac{3}{5}$	$\frac{2}{5}$	

Example

Calculate the marginal distributions in Example

$\xi \backslash \eta$	0	1	$p_{i\cdot}$
0	$\frac{3}{5}\cdot\frac{3}{5}$	$\frac{3}{5}\cdot\frac{2}{5}$	$\frac{3}{5}$
1	$\frac{2}{5}\cdot\frac{3}{5}$	$\frac{2}{5}\cdot\frac{2}{5}$	$\frac{2}{5}$
$p_{\cdot j}$	$\frac{3}{5}$	$\frac{2}{5}$	

$\xi \backslash \eta$	0	1	$p_{i\cdot}$
0	$\frac{3}{5}\cdot\frac{2}{4}$	$\frac{3}{5}\cdot\frac{2}{4}$	$\frac{3}{5}$
1	$\frac{2}{5}\cdot\frac{3}{4}$	$\frac{2}{5}\cdot\frac{1}{4}$	$\frac{2}{5}$
$p_{\cdot j}$	$\frac{3}{5}$	$\frac{2}{5}$	

The joint distribution array of an n-dimensional discrete random vector is

$$P(\xi_1 = x_1(i_1), \xi_2 = x_2(i_2), \cdots, \xi_n = x_n(i_n)) = p_{i_1 i_2 \cdots i_n},$$

where $i_1, i_2, \cdots, i_n = 1, 2, \cdots$.

2.3.2 Joint distribution functions

Definition

Let $(\xi_1, \cdots, \xi_n)$ be a random vector. Its joint distribution function is defined to be

$$F(x_1, \cdots, x_n) = P(\xi_1 \le x_1, \cdots, \xi_n \le x_n)$$

for any $(x_1, x_2, \cdots, x_n) \in \mathbf{R^n}$.

For the 2-dimensional random vector (ξ, η), distribution function is

$$F(x,y) = P(\xi \le x, \eta \le y).$$

For rectangle region $I: a_1 < x \le b_1, a_2 < y \le b_2$,

$$\begin{aligned} P((\xi,\eta) \in I) =& F(b_1,b_2) - F(a_1,b_2) \\ & - F(b_1,a_2) + F(a_1,a_2). \end{aligned}$$

Properties of the bivariate distribution function:

1. Monotonically non-decreasing in each argument;
2. Right continuous in each argument;
3. For any (x, y),

$$F(x, -\infty) = 0,\ F(-\infty, y) = 0,\ F(\infty, \infty) = 1.$$

4. For any $a_1 < b_1, a_2 < b_2$,

$$F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \geq 0.$$

Marginal distribution functions:

The distribution function of ξ is

$$\begin{aligned} F_\xi(x) =& P(\xi \le x, -\infty < \eta < \infty) \\ =& F(x, \infty), \quad x \in \mathbf{R}. \end{aligned}$$

The distribution function of η is

$$F_\eta(y) = F(\infty, y), \quad y \in \mathbf{R}.$$

2.3.3 Continuous random vectors

2.3.3 Continuous random vectors

Definition

If there exists a nonnegative integrable function $p(x_1, \cdots, x_n)$ such that the distribution function $F(x_1, \cdots, x_n)$ can be written as

$$F(x_1, \cdots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} p(y_1, \cdots, y_n) dy_1 \cdots dy_n,$$

then we call F a distribution of continuous type, and call p a joint probability density function.

p satisfies the following conditions:

1. $p(x_1, \cdots, x_n) \geq 0;$
2. $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(y_1, \cdots, y_n) dy_1 \cdots dy_n = 1.$$

p satisfies the following conditions:

1. $p(x_1, \cdots, x_n) \geq 0$;
2. $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} p(y_1, \cdots, y_n) dy_1 \cdots dy_n = 1.$$

For any continuity point of $p(x_1, \cdots, x_n)$,

$$\frac{\partial^n F}{\partial x_1 \cdots \partial x_n} = p(x_1, \cdots, x_n).$$

$\xi(\omega) = (\xi_1(\omega), \cdots, \xi_n(\omega))$:

$$P(\xi(\omega) \in B_n) = \int \cdots \int_{(x_1, \cdots, x_n) \in B_n} p(x_1, \cdots, x_n) dx_1 \cdots x_n$$

for any Borel set B_n.

Marginal density function:

Suppose that (ξ, η) has pdf $p(x, y)$ and df $F(x, y)$, then the marginal distribution function of ξ is as follows:

$$F_\xi(x) = F(x, +\infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} p(u, v)dudv$$
$$= \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} p(u, v)dv\right)du \triangleq \int_{-\infty}^{x} p_\xi(u)du.$$

So, the pdf of ξ is

$$p_\xi(u) = \int_{-\infty}^{+\infty} p(u, v)dv$$

Similarly, η is also a continuous random variable with the density function

$$p_\eta(v) = \int_{-\infty}^{\infty} p(u, v) du.$$

$p_\xi(u)$ and $p_\eta(v)$ are by definition the marginal densities of (ξ, η) $(p(x, y))$.

Example

Suppose that a random vector (ξ, η) has the density function as follows:

$$p(x, y) = \begin{cases} Ae^{-2(x+y)}, & x > 0, y > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(1) Determine the constant A;

(2) Find the distribution function;

(3) Find the marginal densities;

(4) Find $P(\xi < 1, \eta < 2)$;

(5) Find $P(\xi + \eta < 1)$.

Solution. (1) From the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy = 1,$$

it follows that

Solution. (1) From the fact that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)dxdy = 1,$$

it follows that

$$1 = \int_{0}^{\infty}\int_{0}^{\infty} Ae^{-2(x+y)}dxdy = \frac{A}{4},$$

which implies $A = 4$.

(2) The distrbution function of (ξ, η) is

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}p(u,v)dudv.$$

(2) The distrbution function of (ξ, η) is

$$F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} p(u, v) du dv.$$

When $x \leq 0$ or $y \leq 0$, $p(x, y) = 0$, so $F(x, y) = 0$;

(2) The distrbution function of (ξ, η) is

$$F(x,y)=\int_{-\infty}^{x}\int_{-\infty}^{y}p(u,v)dudv.$$

When $x\le 0$ or $y\le 0$, $p(x,y)=0$, so $F(x,y)=0$; When $x>0$ and $y>0$, we have

$$\begin{aligned}F(x,y) &= \int_0^x\int_0^y 4e^{-2(u+v)}dudv \\ &= (1-e^{-2x})(1-e^{-2y}).\end{aligned}$$

So

$$F(x,y)=\begin{cases}(1-e^{-2x})(1-e^{-2y}), & x>0, y>0,\\ 0, & \text{otherwise.}\end{cases}$$

(3) The marginal distribution function of ξ is

$$F_\xi(x) = F(x, \infty) = \begin{cases} 1 - e^{-2x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

(3) The marginal distribution function of ξ is

$$F_\xi(x) = F(x, \infty) = \begin{cases} 1 - e^{-2x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

So the marginal density function of ξ is

$$p_\xi(x) = F'_\xi(x) = \begin{cases} 2e^{-2x}, & x > 0, \\ 0, & x \le 0. \end{cases}$$

Similarly, the marginal distribution function of η is

$$F_\eta(y) = F(\infty, y) = \begin{cases} 1 - e^{-2y}, & y > 0, \\ 0, & y \le 0, \end{cases}$$

and the marginal density function of η is

$$p_\eta(y) = F_\eta'(y) = \begin{cases} 2e^{-2y}, & y > 0 \\ 0, & y \le 0, \end{cases}$$

(4)

$$P(\xi < 1, \eta < 2) = F(1-0, 2-0) = F(1,2) = (1-e^{-2})(1-e^{-4})$$

(4)

$P(\xi < 1, \eta < 2) = F(1-0, 2-0) = F(1,2) = (1-e^{-2})(1-e^{-4})$

(5)

$$P(\xi + \eta < 1) = \iint\limits_{x+y<1} p(x,y)dxdy$$

(4)
$P(\xi < 1, \eta < 2) = F(1-0, 2-0) = F(1,2) = (1-e^{-2})(1-e^{-4})$
(5)

$$P(\xi + \eta < 1) = \iint\limits_{x+y<1} p(x,y)dxdy$$

$$= \iint\limits_{x+y<1, x>0, y>0} 4e^{-2(x+y)}dxdy$$

$$= \int_0^1 \left(\int_0^{1-x} 4e^{-2(x+y)}dy \right) dx = 1 - 3e^{-2}.$$

Two typical continuous random vectors.

1. The n-dimensional uniform distribution

The n- dimensional uniform distribution has the following density function

$$p(x_1, \cdots, x_n) = \begin{cases} A, & (x_1, \cdots, x_n) \in G, \\ 0, & \text{otherwise} . \end{cases}$$

where G is a Borel set in $\mathbf{R}^{\mathbf{n}}$. It immediately follows that $A = 1/S_G$, where S_G is the measure of G (as G is a 2 or 3-dimensional region, S_G is its area or volume)

Example

Suppose that (ξ, η) obeys the uniform distribution in the unit disk $x^2 + y^2 \leq 1$. Find its marginal densities.

Solution. The joint density of ξ and η is

$$p(x,y)=\begin{cases}\frac{1}{\pi}, & x^2+y^2\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

Solution. The joint density of ξ and η is

$$p(x,y)=\begin{cases}\frac{1}{\pi}, & x^2+y^2\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

The marginal density of ξ is

$$p_\xi(x)=\int_{-\infty}^{\infty} p(x,y)dy.$$

Solution. The joint density of ξ and η is

$$p(x,y)=\begin{cases}\frac{1}{\pi}, & x^2+y^2\le 1,\\ 0, & \text{otherwise.}\end{cases}$$

The marginal density of ξ is

$$p_\xi(x)=\int_{-\infty}^{\infty}p(x,y)dy.$$

It is obvious that $p_\xi(x)=0$ as $|x|>1$.

When $|x| \leq 1$,

$$p_\xi(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2},$$

When $|x| \le 1$,

$$p_\xi(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi}\sqrt{1-x^2},$$

Hence

$$p_\xi(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2}, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

Similarly,

$$p_\eta(y) = \begin{cases} \frac{2}{\pi}\sqrt{1-y^2}, & |y| \leq 1, \\ 0, & |y| > 1. \end{cases}$$

2. The n-dimensional normal distribution

Suppose that $\boldsymbol{\Sigma} = (\sigma_{ij})$ is an $n \times n$ positive definite symmetric matrix. Let $|\boldsymbol{\Sigma}|$ be its determinant, and $\boldsymbol{\Sigma}^{-1}$ its inverse. Let $\boldsymbol{x} = (x_1, \cdots, x_n)'$, $\boldsymbol{\mu} = (\mu_1, \cdots, \mu_n)'$. Call

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\}.$$

an n-dimensional normal density function.

Proof of $\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} = 1$:

Proof of $\int \cdots \int p(\boldsymbol{x})d\boldsymbol{x} = 1$: First, we consider the special case that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, where $\boldsymbol{I}$ is an $n \times n$ identical matrix.

Proof of $\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} = 1$: First, we consider the special case that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, where $\boldsymbol{I}$ is an $n \times n$ identical matrix. In this case

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{x}\} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}.$$

Proof of $\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} = 1$: First, we consider the special case that $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}$, where $\boldsymbol{I}$ is an $n \times n$ identical matrix. In this case

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\boldsymbol{x}'\boldsymbol{x}\} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}.$$

So

$$\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} = \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}} dx_i = 1.$$

Now, for the general case, there is an $n \times n$ positive definite symmetric matrix $\boldsymbol{B}$ such that $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}$. Then $\boldsymbol{\Sigma}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{B}^{-1}$ and $|\boldsymbol{B}| = |\boldsymbol{\Sigma}|^{1/2}$.

Now, for the general case, there is an $n \times n$ positive definite symmetric matrix $\boldsymbol{B}$ such that $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}$. Then $\boldsymbol{\Sigma}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{B}^{-1}$ and $|\boldsymbol{B}| = |\boldsymbol{\Sigma}|^{1/2}$. Let $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})$. Then $\boldsymbol{y}' = (\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{B}^{-1}$, and so

$$\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} \;=\; \int \cdots \int \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{-\frac{1}{2}\boldsymbol{y}'\boldsymbol{y}\}|\boldsymbol{B}|d\boldsymbol{y}$$

Now, for the general case, there is an $n \times n$ positive definite symmetric matrix $\boldsymbol{B}$ such that $\boldsymbol{\Sigma} = \boldsymbol{B}\boldsymbol{B}$. Then $\boldsymbol{\Sigma}^{-1} = \boldsymbol{B}^{-1}\boldsymbol{B}^{-1}$ and $|\boldsymbol{B}| = |\boldsymbol{\Sigma}|^{1/2}$. Let $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})$. Then $\boldsymbol{y}' = (\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{B}^{-1}$, and so

$$\begin{aligned}
\int \cdots \int p(\boldsymbol{x}) d\boldsymbol{x} &= \int \cdots \int \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\{-\frac{1}{2}\boldsymbol{y}'\boldsymbol{y}\} |\boldsymbol{B}| d\boldsymbol{y} \\
&= \int \cdots \int \frac{1}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\boldsymbol{y}'\boldsymbol{y}\} d\boldsymbol{y} \\
&= \prod_{i=1}^{n} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y_i^2/2} dy_i = 1.
\end{aligned}$$

Special cases.

If $\boldsymbol{\Sigma} = \boldsymbol{I}$, $\boldsymbol{\mu} = \mathbf{0}$, where $\boldsymbol{I}$ is an $n \times n$ identical matrix, then

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}} \exp\{-\frac{1}{2}\sum_{i=1}^{n} x_i^2\}.$$

It is called an n-dimensional standard normal density.

For $n = 1$, set $\boldsymbol{\Sigma} = \sigma^2$ and $\boldsymbol{\mu} = \mu$. Then

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\{-\frac{(x-\mu)^2}{2\sigma^2}\},$$

which is just the 1-dimensional normal density function.

For $n = 2$, set

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & r\sigma_1\sigma_2 \\ r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix},$$

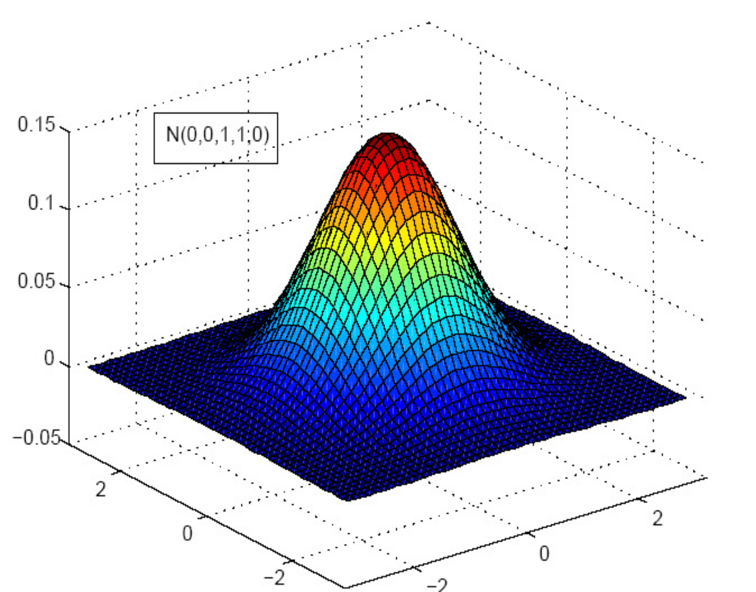
where $\sigma_1, \sigma_2 > 0, |r| < 1$. Then $|\boldsymbol{\Sigma}| = \sigma_1^2\sigma_2^2(1 - r^2)$,

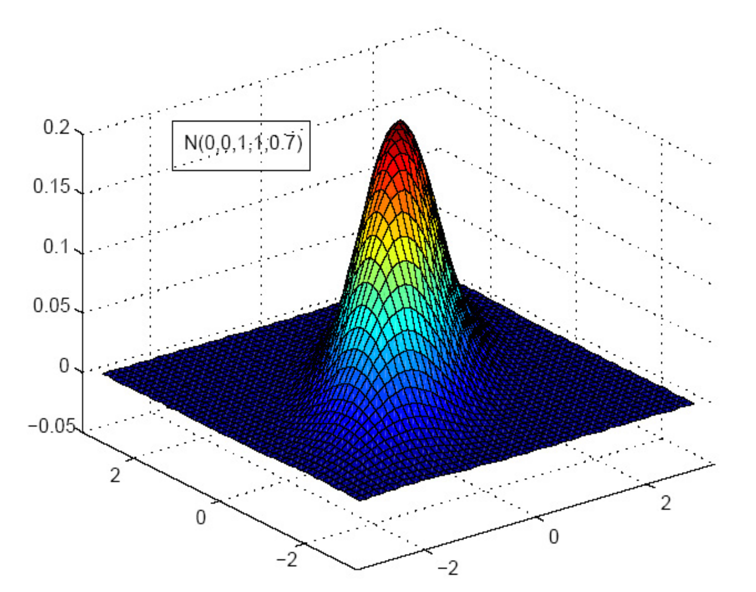
$$\begin{aligned}\boldsymbol{\Sigma}^{-1} =& \frac{1}{|\boldsymbol{\Sigma}|}\begin{pmatrix} \sigma_2^2 & -r\sigma_1\sigma_2 \\ -r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix} \\ =& \frac{1}{1 - r^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{r}{\sigma_1\sigma_2} \\ -\frac{r}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}.\end{aligned}$$

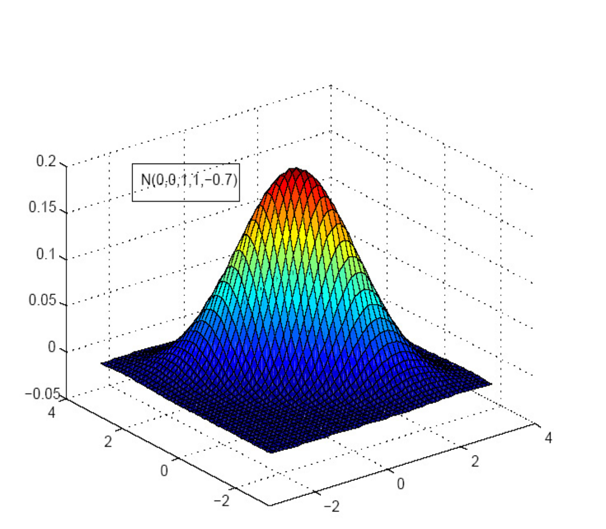
Also, set $\boldsymbol{x} = (x, y), \boldsymbol{\mu} = (\mu_1, \mu_2)$. Then

$$
\begin{aligned}
& p(x,y) \\
= & \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\Big\{ -\frac{1}{2(1-r^2)} \\
& \times \Big[\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2r(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\Big]\Big\},
\end{aligned}
$$

and simply write $(\xi, \eta) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$


N(0,0,1,1;0)
0.15
0.1
0.05
0
−0.05
2
0
−2
2
0
−2


N(0,0,1,1,0.7)
0.2
0.15
0.1
0.05
0
−0.05
2
0
−2
2
0
−2


N(0,0,1,1,−0.7)
0.2
0.15
0.1
0.05
0
−0.05
4
2
0
−2
−2
0
2
4

Marginal distribution: Some simple computation gives

$$
\begin{aligned}
p(x,y) =& \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\} \\
&\times \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-r^2}}\exp\left\{-\frac{\left[y-\mu_2-\frac{r\sigma_2}{\sigma_1}(x-\mu_1)\right]^2}{2\sigma_2^2(1-r^2)}\right\},
\end{aligned}
$$

Marginal distribution: Some simple computation gives

$$
\begin{aligned}
p(x,y) =& \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\} \\
& \times \frac{1}{\sqrt{2\pi}\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{[y-\mu_2-\frac{r\sigma_2}{\sigma_1}(x-\mu_1)]^2}{2\sigma_2^2(1-r^2)}\right\},
\end{aligned}
$$

Hence the marginal density of ξ is

$$
p_\xi(x) = \int_{-\infty}^{\infty} p(x,y)dy = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}.
$$

So $\xi \sim N(\mu_1, \sigma_1^2)$. Similarly, $\eta \sim N(\mu_2, \sigma_2^2)$.

Example

Suppose that (ξ, η) has the joint density function

$$p(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} (1 + \sin(xy)),$$

where $-\infty < x, y < +\infty$. Find its marginal distributions.

Solution.

$$p_\xi(x) = \int_{-\infty}^{\infty} p(x,y)dy$$

Solution.

$$
\begin{aligned}
p_\xi(x) &= \int_{-\infty}^{\infty} p(x,y)dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&\quad + \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \sin(xy) dy
\end{aligned}
$$

Solution.

$$
\begin{aligned}
p_\xi(x) &= \int_{-\infty}^{\infty} p(x,y)dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \\
&\quad + \frac{1}{2\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \sin(xy)dy \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.
\end{aligned}
$$

Solution.

$$\begin{aligned} p_\xi(x) &= \int_{-\infty}^{\infty} p(x,y)dy \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &\quad + \frac{1}{2\pi}e^{-\frac{x^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \sin(xy)dy \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty. \end{aligned}$$

Similarly,

$$p_\eta(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}, \qquad -\infty < y < \infty.$$

思考题: 1. 构造一个n维分布, 使得它本身不是正态分布, 但其任意$k(1 \le k \le n-1)$维分布是正态分布.

2. 设$f(x) > 0$为密度函数, 可微. 假设对任意的$\boldsymbol{x} = (x_1, \ldots, x_n)$, 函数

$$L(\theta; \boldsymbol{x}) = f(x_1 - \theta) \cdots f(x_n - \theta),$$

在$\theta_0 = \frac{1}{n}\sum_{i=1}^n x_i$处取得极大值. 证明$f(x)$是正态密度.

Properties:

$$(\xi, \eta) \sim N(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, r)$$
$$\Leftrightarrow \left(\frac{\xi - \mu_1}{\sigma_1}, \frac{\eta - \mu_2}{\sigma_2} \right) \sim N(0, 0, 1, 1, r).$$

Proof. "$\Rightarrow$":

$$
\begin{aligned}
& P\left(\frac{\xi-\mu_1}{\sigma_1} \leq x, \frac{\eta-\mu_2}{\sigma_2} \leq y\right) \\
= \quad & P(\xi \leq \mu_1 + x\sigma_1, \eta \leq \mu_2 + y\sigma_2)
\end{aligned}
$$

Proof. "$\Rightarrow$":

$$
\begin{aligned}
& P\left(\frac{\xi-\mu_1}{\sigma_1}\le x, \frac{\eta-\mu_2}{\sigma_2}\le y\right) \\
= \; & P(\xi\le \mu_1+x\sigma_1, \eta\le \mu_2+y\sigma_2) \\
= \; & \int_{-\infty}^{\mu_1+x\sigma_1}\int_{-\infty}^{\mu_2+y\sigma_2} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\right. \\
& \times\left.\left[\frac{(u-\mu_1)^2}{\sigma_1^2}-\frac{2r(u-\mu_1)(v-\mu_2)}{\sigma_1\sigma_2}+\frac{(v-\mu_2)^2}{\sigma_2^2}\right]\right\} du dv
\end{aligned}
$$

Proof. "$\Rightarrow$":

$$
\begin{aligned}
& P\left(\frac{\xi-\mu_1}{\sigma_1}\le x, \frac{\eta-\mu_2}{\sigma_2}\le y\right) \\
= \ & P(\xi\le \mu_1+x\sigma_1, \eta\le \mu_2+y\sigma_2) \\
= \ & \int_{-\infty}^{\mu_1+x\sigma_1}\int_{-\infty}^{\mu_2+y\sigma_2}\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}}\exp\left\{-\frac{1}{2(1-r^2)}\right. \\
& \times\left.\left[\frac{(u-\mu_1)^2}{\sigma_1^2}-\frac{2r(u-\mu_1)(v-\mu_2)}{\sigma_1\sigma_2}+\frac{(v-\mu_2)^2}{\sigma_2^2}\right]\right\}dudv \\
= \ & \int_{-\infty}^{x}\int_{-\infty}^{y}\frac{1}{2\pi\sqrt{1-r^2}}\exp\left\{-\frac{s^2-2rst+t^2}{2(1-r^2)}\right\}dtds
\end{aligned}
$$

(by letting $s=(u-\mu_1)/\sigma_1, \ \ t=(v-\mu_2)/\sigma_2$)

Proof. "$\Rightarrow$":

$$
\begin{aligned}
& P\left(\frac{\xi-\mu_1}{\sigma_1}\le x, \frac{\eta-\mu_2}{\sigma_2}\le y\right) \\
= \;& P(\xi \le \mu_1 + x\sigma_1, \eta \le \mu_2 + y\sigma_2) \\
= \;& \int_{-\infty}^{\mu_1+x\sigma_1}\int_{-\infty}^{\mu_2+y\sigma_2} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \exp\left\{-\frac{1}{2(1-r^2)}\right. \\
& \times \left.\left[\frac{(u-\mu_1)^2}{\sigma_1^2} - \frac{2r(u-\mu_1)(v-\mu_2)}{\sigma_1\sigma_2} + \frac{(v-\mu_2)^2}{\sigma_2^2}\right]\right\} dudv \\
= \;& \int_{-\infty}^{x}\int_{-\infty}^{y} \frac{1}{2\pi\sqrt{1-r^2}} \exp\left\{-\frac{s^2-2rst+t^2}{2(1-r^2)}\right\} dtds \\
& (\text{ by letting } s=(u-\mu_1)/\sigma_1, \;\; t=(v-\mu_2)/\sigma_2)
\end{aligned}
$$

"$\Leftarrow$": Similarly.